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Abstract

This paper shows that different states of the financial system command a different effect in worsening financial conditions on economic vulnerability. As soon as financial conditions start deteriorating, the economic outlook becomes more pessimistic and uncertain. No increase in macroeconomic uncertainty is expected when financial conditions worsen from an already tighter than usual situation. We also find that past information on GDP growth is paramount to study and predict economic vulnerability. Both these findings have relevant forecasting and policymaking implications, and persist once we consider other measures of the real economic activity.

From a methodological perspective, we carry out the analysis under a novel approach which relies on the state of the art in dynamic modelling of multiple quantiles. The proposed methodology exploits the entire information of past GDP growth, can accommodate a state dependent effect of financial conditions and allows for statistical inference under the standard quasi maximum likelihood setting.

Keywords: Economic vulnerability, macro-financial linkages, Growth-at-Risk, score driven models.
JEL: C32, C53, E32, E44

1. Introduction

The Great Depression, the Great Recession and the recent COVID-19 outbreak taught us a harsh lesson: few negative shocks can wipe out years of stable economic growth. For instance, once the Great Recession was over, the US had to wait five quarters to see their GDP back at pre-crisis levels. Evidently, policymakers have to keep track of the risk of a sudden large shock to economic growth. But how can they sense whether an economic meltdown looms ahead?

Financial markets may provide an answer to such question. Indeed, there exists a large evidence that financial variables predict the *expected level* of future real economic activity.[1] However, while

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[1] Examples of such *point predictive* ability are: Estrella and Hardouvelis (1991) and Gilchrist and Zakrajšek (2012) for the credit market, Bernanke and Blinder (1992) for the predictive power of the money market, and Culp et al. (2018), Faccini et al. (2019) and Cremers et al. (2020) for what concerns the market for options.

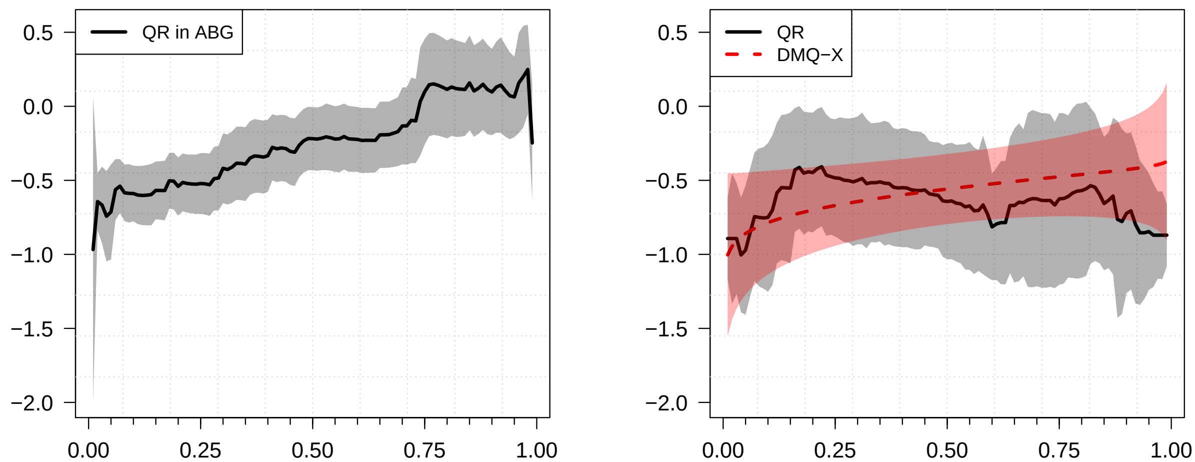
economists are increasingly aware that the time has come to go beyond *point predictions* and to focus on *distributional forecasts*, the literature has devoted much less attention to the impact of financial variables on the *expected distribution* of future economic activity. This is a great deficiency, as it is clear that distributional forecasts are paramount for a proper handling of economic vulnerability, by which we mean the risk of a sudden and large negative shock to the real economy.

The quest for macroeconomic density forecasts is not a brand new one. In the United Sates, survey-based density forecasts have been popular since the late sixties, see Diebold et al. (1997) among others. Model based density forecasts for the inflation rate can be traced back to the fan charts introduced by Bank of England in 1997 (Britton et al., 1998). Among others, Jore et al. (2010) and Clark (2011) used Vector Autoregressions to produce density forecasts of macroeconomic variables. In a recent contribution, Adrian et al. (2019) developed a ready-to-use methodology to predict macroeconomic densities from financial variables. Their approach focuses on the distribution of future GDP growth conditional on current economic and financial conditions. The former are represented by the latest growth rate of GDP, while the Chicago FED National Financial Conditions Index (NFCI) subsumes the latter. The NFCI gauges the conditions of the US financial system by pooling together 105 financial indicators. Higher values of the index imply a more distressed market environment, with positive (negative) values representing tighter (looser) financial conditions. As such, the NFCI classifies the state of financial system by construction: a positive (negative) value implies that the economy is undergoing a period of financial distress. Adrian et al. (2019) study the relation between the two variables through a two-step approach: the first step uses linear quantile regressions (Koenker and Bassett, 1978) to pin down the relation between current economic and financial conditions, and the conditional quantiles of GDP growth; the second one smooths the resulting quantiles with the Skew-t distribution of Azzalini (1985). Using this approach, the authors find that financial conditions have a heterogeneous effect on the left and on the right tail of the distribution of GDP growth, with worsened financial conditions increasing the chances of a macroeconomic shock. We summarize such finding in Panel (a) of Figure 1, which depicts the quantile regression coefficient associated to the NFCI for quantile levels: $\tau = 1\%, \ldots, 99\%$.[2] As in the case of standard linear regression, these coefficients convey the marginal effect of financial conditions on the conditional τ-quantile of GDP growth.

Adrian et al. (2019) claim that financial conditions have a starkly different effect on the left (low values of τ) and on the right tail (high values of τ) of the conditional distribution of GDP growth. In this paper, we show that *such* asymmetric behaviour is not present: the results of Panel (a) of Figure 1 are solely driven by omitting relevant variables. Indeed, Angrist et al. (2006) showed that quantile regression coefficients becomes biased when relevant variables are excluded, thus behaving like ordinary least squares regression coefficients.[3] When we consider past relevant information, results

[2]Panel a of Figure 1 shows β_2 in the model $q_t^y(\tau) = \beta_0 + \beta_1 y_{t-1} + \beta_2 x_{t-1}$, while Panel b reports δ in $q_t^y(\tau) = \beta_0 + \sum_{h=1}^{j} \beta_h y_{t-h} + \sum_{h=1}^{j} \gamma_h y_{t-h}^2 + \delta x_{t-1}$. For both models, y_t is the quarter-on-quarter growth rate of real GDP at time t.

[3]The omitted variable issue is relevant provided that one is interested in the distribution of $Y_t | X_{t-1}, \mathcal{F}_{t-1}$, where



(a) Information set of Adrian et al. (2019).

(b) Extended information set.

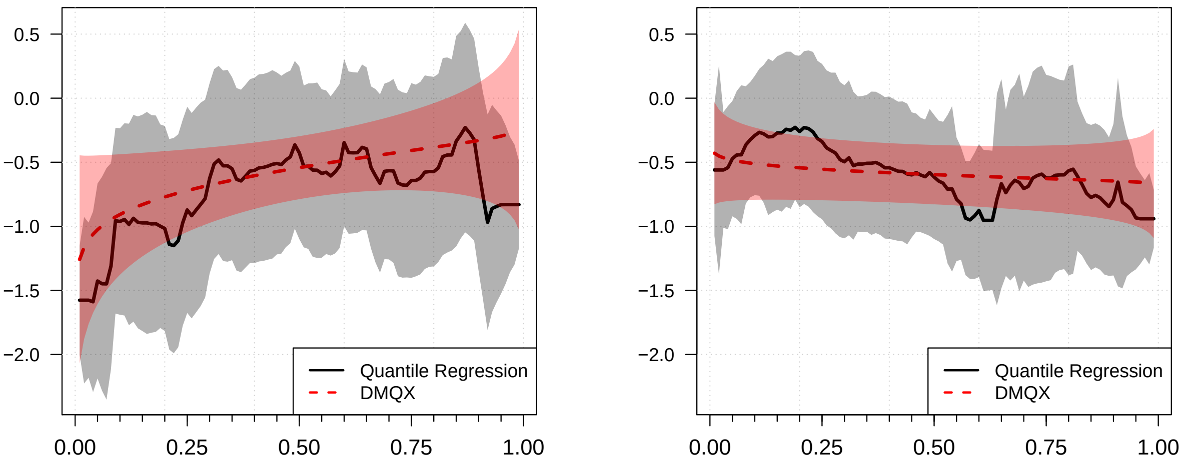
Figure 1: Left panel: coefficient of the NFCI in the quantile regression of Adrian et al. (2019) with associated 95% confidence bands. Right panel: coefficient of the NFCI in a quantile regression incorporating past information of Y (black solid line) with associated 95% confidence bands (black shaded area). For both panels, the horizontal axis reports the probability levels between 1% and 99%.

turn out to be those reported in Panel (b) of Figure 1: the effect of past NFCI becomes homogeneous across different probability levels. Results from Panel (b) are obtained including eighteen lags of Y_t and Y_t^2 rather than Y_{t-1} in the quantile regression design.[4] Biases in the estimates of Adrian et al. (2019) are primary due to omitting lagged values of Y_t^2, which turn out to be correlated with past financial conditions. The fact that results depend both on past realizations and on past variability remarks the need to include all the past information on GDP growth rate.

The first finding of this paper is that, on average, financial conditions do not have an asymmetric effect on the conditional distribution of GDP growth. However, the second empirical contribution is to prove that such a heterogeneous response does exist when we let the relation between the financial system and future GDP growth depend on the state of the former. The left (right) Panel of Figure

$\mathcal{F}_{t-1}$ represents all information on past GDP growth, and not in that of $Y_t|X_{t-1}, Y_{t-1}$, which includes only the most recent value of GPD growth in the conditioning set. Regarding this point, we note that time series analysis focuses on the distribution associated to the first random variable $Y_t|X_{t-1}, \mathcal{F}_{t-1}$, as it coincides with the one step ahead predictive distribution. Finally, note that the two random variables have the same distribution only if current GPD growth conditional on past NFCI solely depends on its mos recent realization, something which the presence of economic cycles makes hard to believe.

[4]The number of lagged quarters corresponds to the average length of an economic cycle between 1945 and 2020, according to the NBER business cycle dating committee. Similar results arise when we use the filtered variance from an AR(2)-GARCH(1,1) in place of the lags of Y_t^2 as well as when we include higher order moments or change the number of included lags.



(a) Loose financial conditions.

(b) Tight financial conditions.

Figure 2: Left panel: overall effect of the NFCI in a state dependent quantile regression (black solid line) and in the DMQ-X (red dashed line) when financial conditions are loose. Right panel: overall effect of the NFCI in a state dependent quantile regression (black solid line) and in our model (red dashed line) when financial conditions are tight. For both panels, the horizontal axis reports the probability levels between 1% and 99% while shaded areas are 95% confidence bands.

2 shows the expected effect of the NFCI on the conditional quantiles of GDP growth when financial conditions are looser (tighter) than usual. The black solid lines report results for state dependent quantile regression, while red dashed ones are obtained under the econometric framework of this paper.[5] When financial conditions are looser than usual, i.e. the NFCI is negative, an increase in the index makes the leftmost quantiles decrease more than the rightmost ones. Hence, as soon as the financial system *starts* becoming more distressed, policymakers should be worried about an ongoing buildup of macroeconomic downside risk. The asymmetric effect is reversed, though much less relevant, when financial conditions are already tighter than usual, i.e. the NFCI is greater than zero.

For policymakers' interventions to be as timely and efficient as possible, decisions must rely on a methodology which captures both the entire history of GDP growth and the state dependent asymmetry in the effect of financial conditions. In this regard, we make a substantial methodological contribution by carrying out the analysis under a statistical framework which accommodates both features while making no parametric assumption on the conditional distribution of GDP growth.

The proposed method, which we call DMQ-X, combines the dynamic multiple quantile (DMQ)

[5]Here, the quantile regressions include eighteen lagged values of Y_t and Y_t^2, and model the state dependency through the absolute value of the NFCI. Hence, Figure 2 depicts $\delta+\eta$ in the model $q_t^y(\tau) = \beta_0 + \sum_{h=1}^{j} \beta_h y_{t-h} + \sum_{h=1}^{j} \gamma_h y_{t-h}^2 + \delta x_{t-1} + \eta \left| x_{t-1} \right|$ when $x_{t-1} \geq 0$ and $\delta - \eta$ otherwise.

model of Catania and Luati (2019) with the dynamic additive quantile (DAQ) model of Gouriéroux and Jasiak (2008). The DMQ relies on a generalization of the score driven approach of Creal et al. (2013) and Harvey (2013), and has been proved to outperform standard dynamic quantiles models at predicting tail quantiles. We amend the DMQ by introducing a set of exogenous variables through a linear combination of base-line quantile functions as in Gouriéroux and Jasiak (2008).[6] Hence, the DMQ part subsumes the entire history of GDP growth while the impact of financial conditions is left to the DAQ, whose flexibility can accommodate the observed state dependent asymmetry. The proposed approach enjoys a flexible dynamics for the quantiles, has a parsimonious parametrization and satisfies the condition of non-crossing quantiles *in finite samples*. Finally, we derive conditions under which the estimator of the model is consistent and asymptotically normally distributed. The second property makes statistical inference readily available.

Besides the influential work of Adrian et al. (2019), other papers have considered the link between current financial conditions and the entire distribution of real GDP growth. Using different methods resting on different assumptions, the results lead to somehow contradictory conclusions. Giglio et al. (2016) find that systemic risk measures exhibit a predictive power on the lower quantiles of future GDP growth. Plagborg-Møller et al. (2020) carry on an extensive empirical analysis and conclude that: moments of order higher than one are poorly predicted, financial variables provide limited information on distributional forecasts and the approach of Adrian et al. (2019) implies a large uncertainty on the estimated moments and tail risk measures. Brownlees and Souza (2020) show that a standard GARCH model outperforms the quantile regression of Adrian et al. (2019) when predicting macroeconomic downside risk. Delle Monache et al. (2021) model the conditional distribution of GDP growth as a Skew-t whose parameters vary over time according to a score driven mechanism; they find that financial indicators, especially those pertaining to non-financial leverage, have a predictive power on macroeconomic downside risk. Chavleishvili and Manganelli (2019), Chavleishvili et al. (2020) and Carriero et al. (2020) obtain conclusions in line with Adrian et al. (2019) and Delle Monache et al. (2021) using either structural quantile based vector-autoregressions (VARs) or Bayesian VARs. We find that past information on GDP growth is the key driver of the time variation in its conditional distribution: for most of the sample, moments filtered with and without financial conditions are hardly distinguishable. Consequently, including financial conditions does not return large point/density forecasting gains for much of the sample. It does it at the onset, and during, the 2007-2009 recession. The gain is larger for a model that can accommodate the aforementioned state dependent asymmetric effect. Things change when it comes to predict tail risk: including financial conditions always returns more accurate forecasts. Remarkably, a model without financial conditions but which includes the entire history of GPD growth outperforms the approach of Adrian et al. (2019) at forecasting tail risk, a further hint of the importance of a proper conditioning when assessing economic vulnerability.

The remainder of the paper is organized as follows. Section 2 introduces the econometric framework; in Section 3, we carry out a full sample analysis which shows the importance of

[6]The choice of such functions is unrelated to the conditional distribution of Y_t, about which we remain agnostic.

conditioning on all past information, as well as detailing the consequences of the state dependent asymmetric effect reported in Figure 2; Section 4 analyses the importance of past information and of the state dependent asymmetric effect when it comes to point, density and tail risk forecasts; in Section 5, we conduct a robustness check where we show that the conditionally asymmetric effect is a stylized fact of economic indicators other than the real GDP; Section 6 concludes the paper. Appendix A is a technical appendix which contains all relevant theoretical results. Appendix B reports a simulation exercise which shows the finite sample properties of the DMQ-X methodology.

2. Econometric framework

In this section, we outline the main features of the econometric framework of the paper; derivations and further relevant details are in Section A of the Technical Appendix. In what follows, y_t denotes the growth rate of real GDP between quarter $t-1$ and quarter t while x_t stands for the NFCI at time t. Capital letters Y_t and X_t indicate the corresponding random variables and $F_{t|t-1}$ labels the cumulative distribution function of Y_t given the *whole* history of GDP growth and the *latest* realization of the NFCI. Formally, such function is defined as $F_{t|t-1}(y) \coloneqq P(Y_t \leq y | \mathcal{F}_{t-1})$ where $\mathcal{F}_{t-1} \coloneqq \{\sigma(Y_{t-s},\, s=1,\ldots,t) \cup \sigma(X_{t-1})\}$. We obtain estimates of $F_{t|t-1}$ as a by-product of estimating J quantiles of the distribution of interest, where we define the τ_j-quantile as

$$q_t^{\tau_j} = F_{t|t-1}^{-1}(\tau_j),$$

for $\tau_j \in (0,1)$ a probability level.

In this paper, we use a semiparametric approach to model J quantiles of the conditional distribution of GDP growth. The proposed approach combines the Dynamic Multiple Quantile (DMQ) model of Catania and Luati (2019) with the Dynamic Additive Quantile (DAQ) model of Gouriéroux and Jasiak (2008), and writes each quantile as the sum of two quantile functions

$$q_t^{\tau_j} = \mathrm{DMQ}_t^{\tau_j} + q_{x,t}^{\tau_j}, \tag{1}$$

where $\mathrm{DMQ}_t^{\tau_j}$ contains the quantiles implied by the DMQ and $q_{x,t}^{\tau_j}$ is the DAQ part which accounts for the effect of X_{t-1}. Equation (1) encompasses many possible specifications for $q_{x,t}^{\tau_j}$, each more suitable to a certain application. We let the NFCI enter the model through the following specification:

$$q_{x,t}^{\tau_j} = \mu\, x_{t-1} + \left(|x_{t-1} - \omega^+| \mathbb{1}(x_{t-1} \geq 0) + |x_{t-1} - \omega^-| \mathbb{1}(x_{t-1} < 0)\right) \sigma Q(\tau_j; \boldsymbol{\alpha}) \tag{2}$$

where $(\mu, \omega^+, \omega^-) \in \mathbb{R}^3$, $\sigma \geq 0$ and Q is a standard location-scale quantile function parametrized by

$\boldsymbol{\alpha} \in \mathbb{R}^n$.[7] Equation (2) implies that the marginal effect of the NFCI depends on its latest value:

$$\begin{aligned}
\frac{\partial}{\partial x_{t-1}} q_t^{\tau_j} &= \mu + s\left(x_{t-1} - \omega^+\right) \sigma Q\left(\tau_j; \boldsymbol{\alpha}\right) \quad \text{if } x_{t-1} > 0, \\
\frac{\partial}{\partial x_{t-1}} q_t^{\tau_j} &= \mu + s\left(x_{t-1} - \omega^-\right) \sigma Q\left(\tau_j; \boldsymbol{\alpha}\right) \quad \text{if } x_{t-1} < 0,
\end{aligned} \tag{3}$$

where $s(x) = 1$ if $x \geq 0$ and minus one otherwise. Figure 2 plots the expected values (red dashed lines) of these derivatives using the estimates obtained in Section 3. The parameters ω^+ and ω^- are two thresholds that determine the presence and the *shape* of the asymmetric effect found in Figure 2. Indeed, rejecting the null hypothesis that the two parameters coincide corresponds to rejecting the null hypothesis of no state dependence. Moreover, their size and sign determine the degree of heterogeneity in the state dependent asymmetric response. The parameter μ appears in both derivatives and conveys the first order effect of a shock in the NFCI: when x_{t-1} increases of one unit all the quantiles of Y_t will shift at least by μ. Finally, σ controls the heterogeneity in the response of tail and core quantiles, while the parameters of Q seize possible higher order effects in excess of those induced by ω^+ and ω^-.

The model, which we call DMQ-X, is defined by $d = 4 + 4 + n$ parameters, where the first four parameters characterize the baseline DMQ and n is the number of parameters specific to Q, e.g. one in the case of a Student's-t distribution. For instance, nine parameters are required to model J quantiles when Q is set to the quantile function of a Student's t distribution. Such number is much smaller than what required by any model where the amount of parameters increases with respect to the number of quantiles, e.g. the MQ-CAViaR of White et al. (2015). The limited number of parameters is also due to a targeting scheme, discussed in Section A.2 of the Technical Appendix and in line with the variance targeting of Engle and Mezrich (1996) and Engle (2002).

Concerning the DMQ part, Catania and Luati (2019) model the spacing between a reference quantile and the precedent (subsequent) $\lfloor J/2 \rfloor$ as the sum of negative (positive) processes. Both the reference quantile and the spacings have a dynamics whose updates rely on a quasi-score driven mechanism. The latter transfers past information to all the regions of the conditional distribution, thus enriching the signal available to tail quantiles, which are otherwise poorly estimated, as it is the case for quantile regressions and for dynamic models such as the MQ-CAViaR. We refer to Section A.1 in the Technical Appendix for a thorough explanation of the DMQ.

Estimation of the DMQ-X relies on a two-stage quasi maximum likelihood approach. To improve the quality of the estimation, we reduce the dimensionality of the parameter space by targeting the quantiles of the unconditional distribution of quarter-on-quarter GDP growth. We estimate the latter

[7] Both the Normal, the Student's t and their skewed counterparts à la Azzalini (1985) are location-scale random variables. Therefore, such class of distributions is large enough to accommodate asymmetric and fat-tailed effects of the exogenous variable.

from a realization of the time series of interest. Such procedure results into the first stage estimator:

$$\hat{\boldsymbol{\theta}}_{1,T} := \underset{\boldsymbol{\theta}_1 \in \boldsymbol{\Theta}_1}{\arg\max}\, L\left(\boldsymbol{\theta}_1; y_t\right), \tag{4}$$

where L is the relevant likelihood function. We define the second stage estimator as the minimizer of the multiple check loss function:[8]

$$\hat{\boldsymbol{\theta}}_{2,T} := \underset{\boldsymbol{\theta} \in \boldsymbol{\Theta}_2}{\arg\min} \sum_{t=1}^{T} \sum_{j=1}^{J} \rho_{\tau_j}\left(y_t - q_t^{\tau_j}\left(\boldsymbol{\theta}_2\right)\right), \tag{5}$$

for ρ_{τ_j} the check function associated with the τ_j-quantile: $\rho_{\tau_j}(a) = a\left(\tau_j - \mathbb{1}\{a<0\}\right)$. The following theorems establish consistency and asymptotic normality of the two-stage estimator $\hat{\boldsymbol{\theta}}_T := \left(\hat{\boldsymbol{\theta}}_{1,T}, \hat{\boldsymbol{\theta}}_{2,T}\right)'$.

Theorem 1 (Strong consistency). *Let $\boldsymbol{\theta}_0$ be the true parameter vector. Then, under assumptions 1-7 in the Technical Appendix, $\hat{\boldsymbol{\theta}}_T \overset{a.s.}{\to} \boldsymbol{\theta}_0$.*

Theorem 2 (Asymptotic Normality). *Let $\boldsymbol{\theta}_0$ be the true parameter vector. Then, under assumptions 1-10 in the Technical Appendix,*

$$\sqrt{T}\left(\hat{\boldsymbol{\theta}}_{2,T} - \boldsymbol{\theta}_{2,0}\right) \xrightarrow{d} \mathcal{N}\left(\mathbf{0}, Q_0^{-1} V_0 Q_0^{-1}\right),$$

with Q_0 and V_0 reported in equation (28) in the Technical Appendix.

We prove both theorems in Section A.3 of the Technical Appendix. The proofs rely on extending the results of Catania et al. (2020), which are in turn based on White (1994). The same section reports consistent estimators of Q_0 and V_0.

Prediction

In the proposed approach, the time $t+1$ quantiles are fully determined by the information set at time t. Hence, one step ahead predictions are readily available. For any forecast horizon $h > 1$, we can use Equation (1) to write the h-step ahead prediction $\hat{q}_{t+h|t}^{\tau_j} := \mathbb{E}\left[q_{t+h}^{\tau_j} \middle| \mathcal{F}_t\right]$ as

$$\hat{q}_{t+h|t}^{\tau_j} = \widehat{\mathrm{DMQ}}_{t+h|t}^{\tau_j} + \hat{q}_{x,t+h|t}^{\tau_j} \tag{6}$$

[8] We solve the optimization problem in equation 5 using the Differential Evolution algorithm of Storn and Price (1997), which we implement with the R package *DEoptim* of Mullen et al. (2009).

where $\widehat{\mathrm{DMQ}}^{\tau_j}_{t+h|t} := \mathbb{E}\left[\mathrm{DMQ}^{\tau_j}_{t+h}\middle|\mathcal{F}_t\right]$ and $\hat{q}^{\tau_j}_{x,t+h|t} := \mathbb{E}\left[q^{\tau_j}_{x,t+h|t}\middle|\mathcal{F}_t\right]$ are the time t forecasts for the DMQ and for the DAQ part, respectively. Section A.1 of the Technical Appendix reports closed form expressions for the DMQ part. For $\hat{q}^{\tau_j}_{x,t+h|t}$, we have that

$$\hat{q}^{\tau_j}_{x,t+h|t} = \mu\,\mathbb{E}_{\mathrm{t}}\left[X_{t+h-1}\right]+\mathbb{E}_{\mathrm{t}}\left[\left(|X_{t+h-1}-\omega^{+}|\mathbb{1}\left(X_{t+h-1}\geq 0\right)+|X_{t+h-1}-\omega^{-}|\mathbb{1}\left(X_{t+h-1}<0\right)\right)\right]\sigma Q\left(\tau_j;\boldsymbol{\alpha}\right). \tag{7}$$

In principle, equation (7) requires specifying a model for X_t, $|X_t-\omega^{+}|$ and $|X_t-\omega^{-}|$. Alternatively, a widely used approach to multi-step ahead forecasts is the direct forecast approach, whereby:

$$\hat{q}^{\tau_j}_{x,t+h|t} = \mu x_{t-1} + \left(|x_{t-1}-\omega^{+}|\mathbb{1}\left(x_{t-1}\geq 0\right)+|x_{t-1}-\omega^{-}|\mathbb{1}\left(x_{t-1}<0\right)\right)\sigma Q\left(\tau_{j;}\boldsymbol{\alpha}\right),$$

so that the information set at time t suffices to predict $q^{\tau_j}_{x,t+h|t}$. We refer to Marcellino et al. (2006) for a comparison of direct and iterated forecasts in the context of autoregressive methods.

3. Assessing vulnerability

In this section, we use the DMQ-X methodology to analyse the relation between financial conditions and economic vulnerability in light of the preliminary findings of Section 1. We consider quarterly realizations of the growth rate of (real) GDP and of the Chicago FED National Financial Conditions Index (NFCI). The NFCI is a common factor extracted from a panel of 105 financial indicators. The index gauges US financial conditions as implied by multiple markets (debt, equity and the money market), as well as by the traditional and the shadow banking system; details of its construction are in Brave and Butters (2012). Higher values of the NFCI point to a more distressed financial system, with positive (negative) values corresponding to tighter (looser) than usual financial conditions. Though the FED Chicago releases the NFCI on a weekly basis, we follow Adrian et al. (2019) and consider its average value over a quarter. Figure 3 plots the annualized quarter-on-quarter growth rate of real GDP (blue dashed line) along with the NFCI (red solid line); both time series are standardized and range between Q1:1971 and Q4:2019. The growth rate of GDP fluctuates more, and more erratically, than the NFCI. However, peaks in the NFCI correspond to negative spikes in the growth rate: a hint of a correlation between extreme values of the two. Starting from this insight, we use the DMQ-X to examine the relation between the NFCI at time t and the growth rate of GDP between time t and $t+1$, where the unit of time is quarters.

We fit the DMQ-X to the entire sample by considering $J = 99$ probability levels and using the median as the reference quantile.[9] We let the NFCI enter the model through the quantile function of the Skewed Normal distribution of Azzalini (1985), whose only free parameter is the slant coefficient α_{sn}, which governs the skewness of the effects induced by the baseline quantile function. Lastly, we

[9]The idea of fixing the median and modelling the related quantile spacings was first proposed by Granger (2010). Catania and Luati (2019) showed that the model is not so sensitive to the choice of the reference quantile.

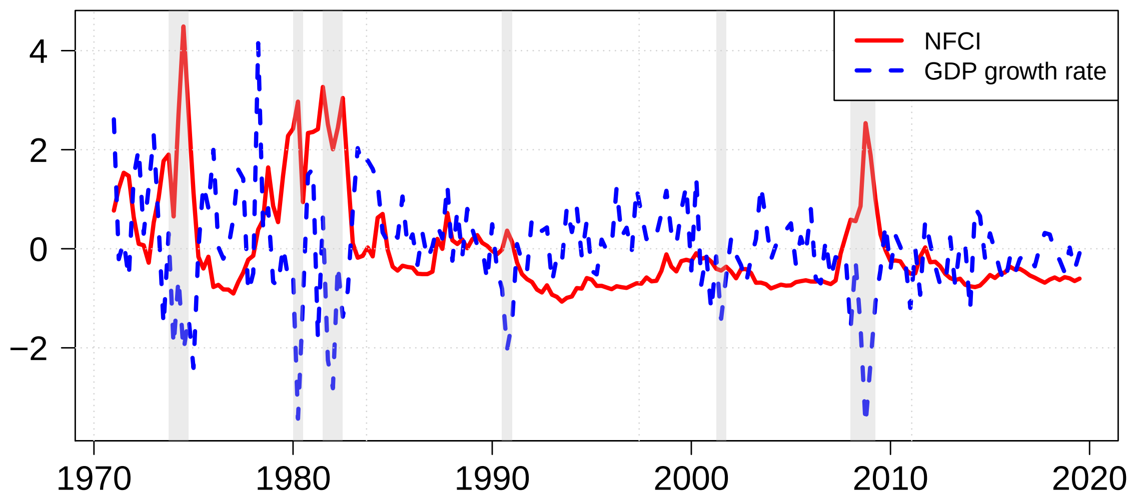


Figure 3: Quarterly realizations of the NFCI (red solid line) and of the average annualized GDP growth rate (blue dashed line, standardized) between Q1:1971 and Q4:2019. Gray shaded areas are official NBER recession dates.

assume that the unconditional distribution of GPD growth is the Skew-t of Azzalini (1985), thus adopting the parametric targeting approach described in Section A.2 of the Technical Appendix.

Table 1 reports the estimated parameters for the DMQ-X along with the asymptotic standard errors, in parenthesis. The first four columns pertain to the DMQ part: β (ϕ) measures the persistence of the reference quantile (quantile spacings) and α (γ) the impact of the (respectively) score-driven innovations. The persistence parameters are large, suggesting the importance of including more lags of past economic conditions when modelling the conditional distribution of GDP growth.

Financial conditions exhibit a negative and significant first order effect on the conditional distribution of GDP growth, as implied by a statistically significant μ. At the same time, we so not reject the null hypothesis of no skewness in the effects induced by the baseline quantile function, i.e. $\mathbb{H}_0 : \alpha_{\text{sn}} = 0$, at any level of significance. Finally, we consider a Wald test for the null hypothesis $\mathbb{H}_0 : \omega^- = \omega^+$; the resulting t-statistics and p-value are 2.623 and 0.008, respectively, thus pointing towards rejection of $\mathbb{H}_0$ at the 99% confidence level. Such test is particularly important as it testifies the presence of a state dependent effect of financial conditions on the distribution of GDP growth. These estimates for the DAQ part return the effects depicted by the red dashed lines of Figure 2: the expected impact of an increase in the NFCI is remarkably asymmetric when financial conditions are loose; such heterogeneous response of the quantiles is slightly reversed and not so prominent when financial conditions are tighter than usual.

The absence of a stark overall asymmetric effect implies that the variation in the conditional distribution of GDP growth is mainly driven by its past. This is evident if we look at Figure 4, which shows the expected value, standard deviation, skewness and kurtosis of GDP growth either conditional on current financial conditions and the entire information of GDP growth (red solid line)

DMQ part				DAQ part				
β	α	ϕ	γ	μ	σ	α_{sn}	ω^+	ω^-
0.985	0.065	0.756	0.239	−0.586	0.230	−7.671	1.310	−1.254
(0.029)	(0.025)	(0.134)	(0.090)	(0.118)	(0.139)	(116.5)	(0.538)	(0.591)

Table 1: Estimated parameters for the DMQ-X with $J = 99$ quantiles along with asymptotic standard errors (in parenthesis). The estimation sample runs from Q1:1971 to Q4:2019 and comprises of quarterly realizations of the NFCI and of the standardized growth rate of real GDP.

or when only the latter is considered (blue dashed line, obtained through a DMQ with no DAQ part).[10] For most of the sample, the moments implied by the two approaches closely mimic each other. Discrepancies cluster around periods of economic downturn, especially those preceded by a financial crisis, e.g. the most recent Global Financial Crisis.[11]

The findings based on the DMQ-X methodology sharply contrast with those obtained using the two-step approach of Adrian et al. (2019), which we report in Figure 5. If one does not consider financial conditions, using only the most recent realization of GDP growth makes it impossible to pin down a reasonable dynamics for moments higher than the first. When financial conditions are included, the standard deviation completely absorbs the behaviour of the NFCI, which it perfectly mirrors, as revealed by a closer inspection of Figures 3 and 5. Finally, we note that simple quantile regression techniques are not suited to filter higher order moments, no matter whether financial conditions are considered.

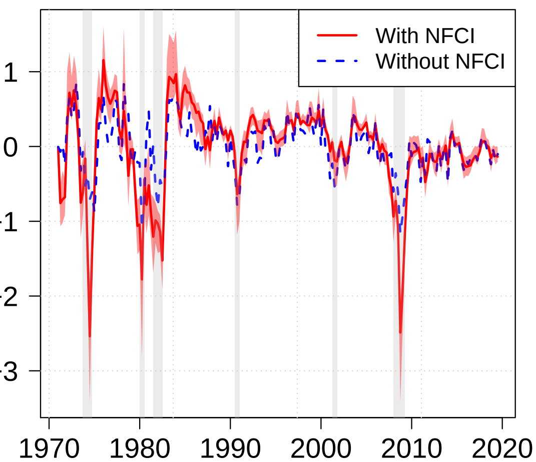
Figures 2 (in the introduction) and 4 also reveal the consequences on the conditional distribution of GDP of the state dependent relation between financial and economic conditions. The first plot tells us that an increase in the NFCI will lower the expectations on future economic growth (both red dashed lines are centred around the estimate of μ, which is negative), no matter the state of the financial system. At the same time, such increase will lead to a higher variance when financial conditions are loose while, on average, it will shrink the distribution towards the more pessimistic expectation

[10] At each point in time, we obtain the k-th conditional moment of Y as

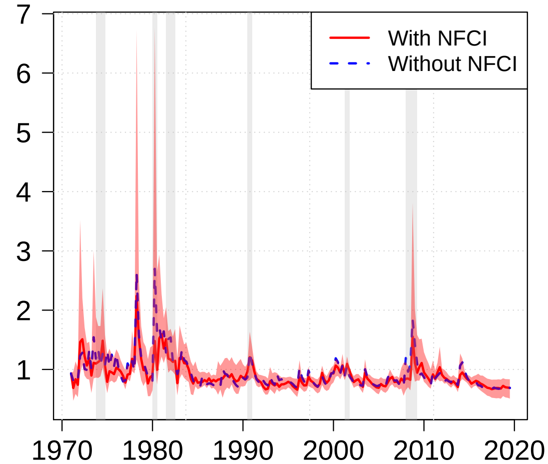
$$\mathbb{E}\left[Y_t^k \middle| \mathcal{F}_{t-1}\right] = \int_{\mathbb{R}} y^k dF_{t|t-1}(y) \approx \tau_1 \left\{q_{1,t}\right\}^k + (1-\tau_J)\left\{q_{J,t}\right\}^k + \sum_{j=1}^{J-1} (\tau_{j+1}-\tau_j)\left[\frac{q_{j+1,t}+q_{j,t}}{2}\right]^k,$$

where equality holds in the limit $J \to \infty$, $\tau_1 \downarrow 0$ and $\tau_J \uparrow 1$.

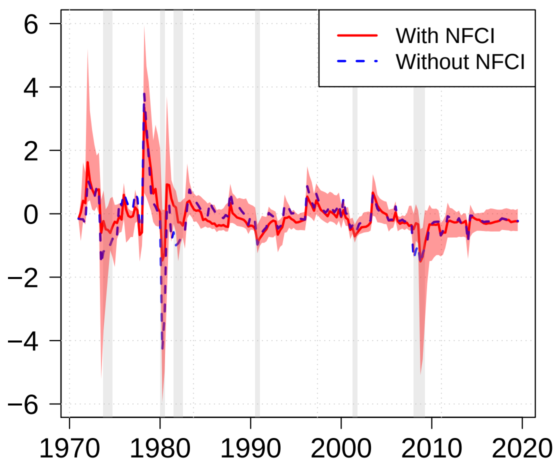
[11] In an unreported analysis, we obtained similar results by extracting the moments from the quantiles implied by a quantile regression including eighteen lags of y_t and y_t^2 and the lagged NFCI. Such similarity corroborates our presumption that the time varying conditional distribution of GDP growth is mostly driven by its past, once we condition on the right information set.



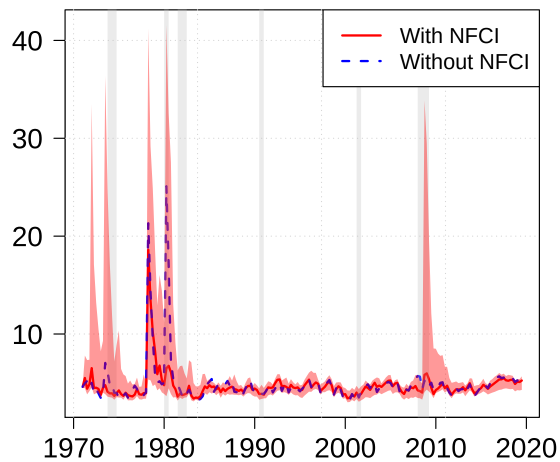
(a) Expected value.



(b) Standard deviation.



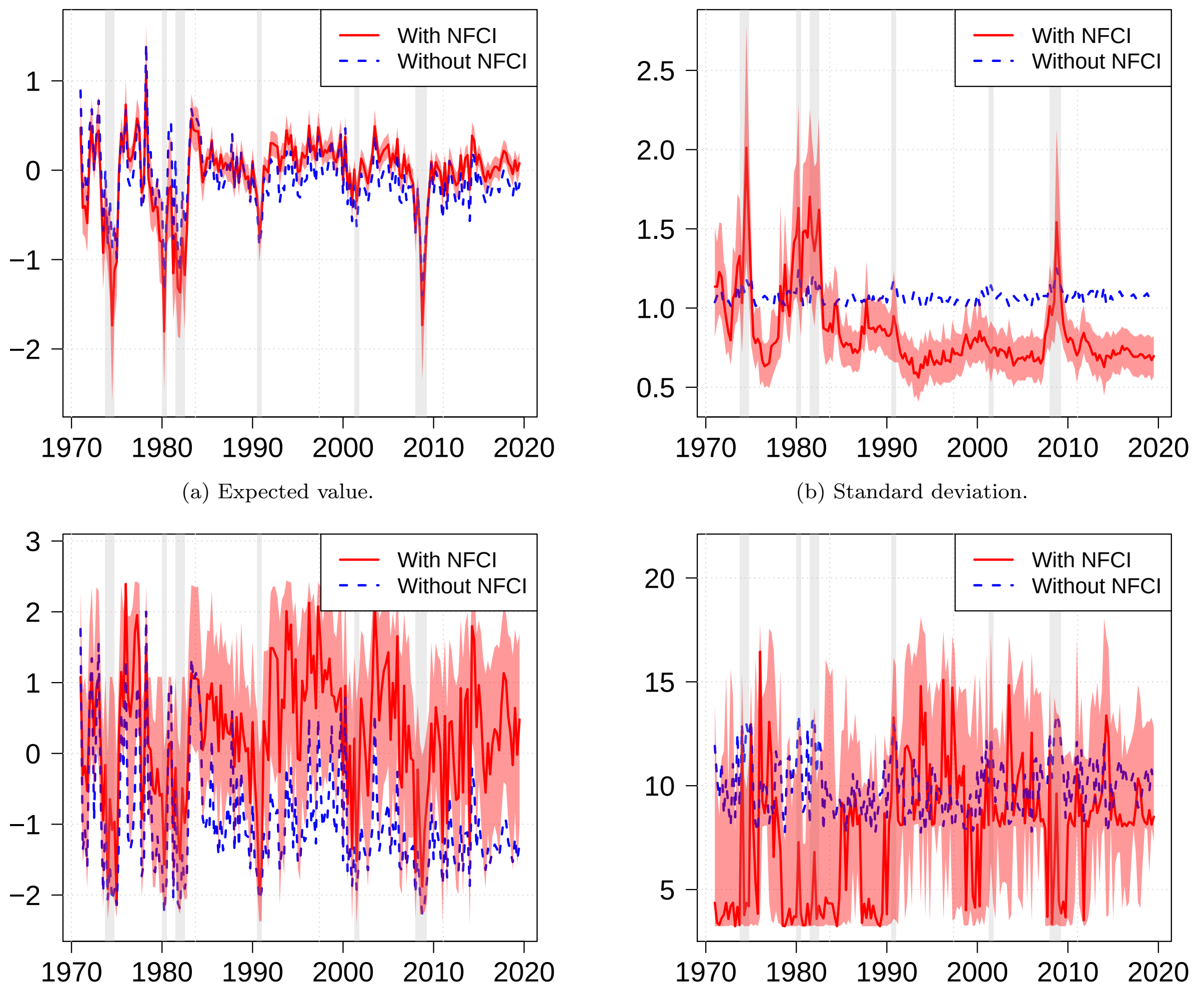
(c) Skewness.



(d) Kurtosis.

Figure 4: Expected value, standard deviation, skewness and kurtosis filtered with the DMQ-X (red solid line) and the DMQ (blue dashed line) between Q1:1971 and Q4:2019. Gray shaded areas are official NBER recession dates, while light red ones are 95% confidence bands for the DMQ-X based quantities. All time series refer to standardized GDP growth rate.

when financial conditions are already tighter than usual. This is apparent from Panel b of Figure 4: considering financial conditions *and* past GDP information results in a lower standard deviation than when only the latter are included; this is particularly evident amidst economic downturns. To remark such effect, Figure 6 plots the conditional density implied either at the end of Q2:2007 and Q3:2007 (Panel a) or right after Q2:2008 and Q3:2008 (Panel b). The first two dates mark the first



(a) Expected value.

(b) Standard deviation.

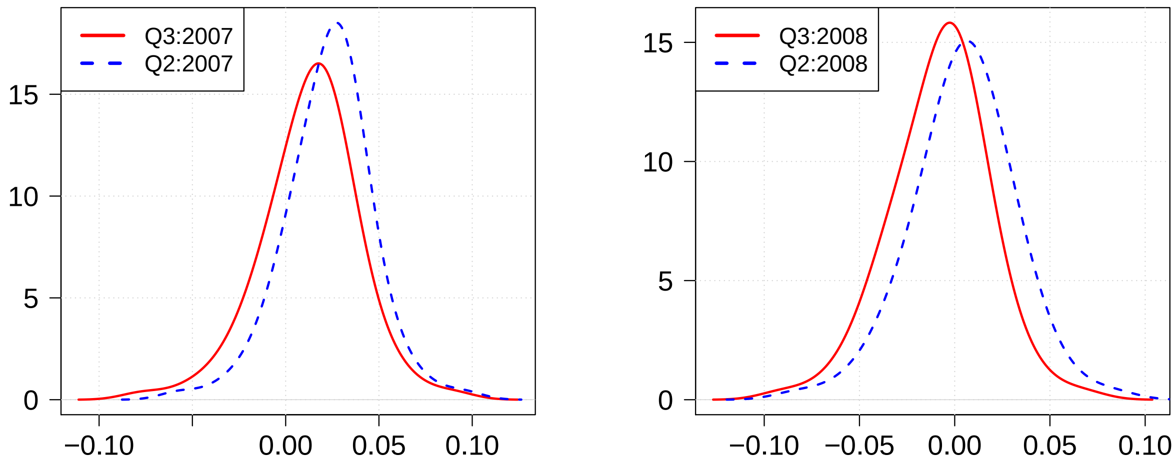
(c) Skewness.

(d) Kurtosis.

Figure 5: Expected value, standard deviation, skewness and kurtosis filtered through 99 quantile regressions including either lagged GDP growth rate and the NFCI (red solid line) or lagged growth only (blue dashed line) as predictors. Moments refer to the time window Q1:1971 to Q4:2019. Gray shaded areas are official NBER recession dates, while light red ones are 95% confidence bands for the model with GDP growth and the NFCI. All the moments refer to the standardized time series of GDP growth rate.

appreciable increase in the NFCI, which moved from -0.64 to -0.11. On the other hand, Q3:2008 includes September 13, 2008: the day Lehman Brothers filed for bankruptcy. Such event triggered a spike in the NFCI, which was already well above zero.

When the NFCI increases from a negative value, the central tendency of the conditional



(a) Q2:2007 to Q3:2007.

(b) Q2:2008 to Q3:2008.

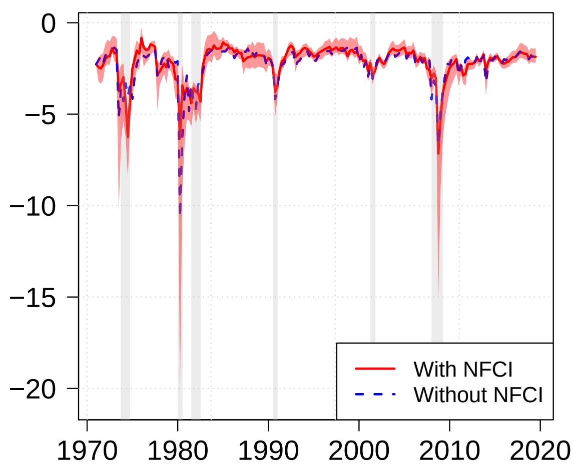
Figure 6: Left panel: one-quarter ahead conditional density of GDP growth at the end of Q2:2007 (blue dashed line) and of Q3:2007 (red solid line). Right panel: same quantities at the end of Q2:2008 and of Q3:2008. The horizontal axis reports GDP growth rate in *non-standardized* units.

distribution of GDP growth shifts to the left and its dispersion increases (the new density is less peaked than the previous one). Though the location effect is invariant to the state, the right Panel of Figure 6 shows the reverse consequence for the dispersion: an increase in the NFCI makes the new density more peaked. Such feature has relevant policymaking implications: on the one hand, it shows that as soon as financial conditions start deteriorating the economic outlook becomes negative and less certain; on the other, it implies that when financial conditions are already tighter than usual, their *point signal* becomes more certain. As a last remark, this state dependent relation between the NFCI and quarter-on-quarter changes in the real economic activity holds true for macroeconomic variables other than real GDP. We show such robustness in Section 5, where we repeat the above exercise for industrial production, real per capita consumption, total nonfarm payroll, the unemployment rate and the Chicago FED National Activity Index.

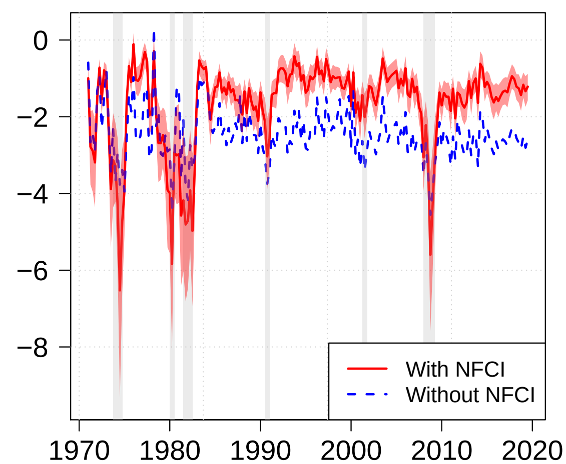
Finally, we study how proper measures of macroeconomic tail risk behave under a statistical framework which fully incorporates past information and that accommodates the observed state dependent effect. To this extent, we consider the time series of *macroeconomic Expected Shortfall*:

$$\mathrm{ES}_{\alpha,t} \coloneqq \frac{1}{\alpha}\int_0^\alpha Q_{t|t-1}(u)\,du, \tag{8}$$

where $Q_{t|t-1}$ is the quantile function at time t conditional on the information at time $t-1$. Such metrics conveys the expected value of GDP growth provided that it is below its $100 \times \alpha\%$ quantile (in what follows, $\alpha = 0.05$). Figure 7 plots this risk measure both under the methodology of this paper (left panel) and under the one of Adrian et al. (2019) (right panel). Similarly to what happened in



(a) DMQ-X and DMQ.



(b) Approach of Adrian et al. (2019).

Figure 7: Left panel: Macroeconomic expected shortfall filtered with the DMQ-X (red solid line) and the DMQ (blue dashed line). Right panel: same quantity with the model of Adrian et al. (2019) with (red solid line) and without (blue dashed line) financial conditions. Grey shaded areas are official NBER recession dates, while light red ones are 95% confidence bands for the models with financial conditions. All metrics refer to the standardized time series of GDP growth rate.

Figure 4, past GDP growth is the key driver of time variation in the expected shortfall, provided that we do not restrict ourselves to its latest realization. The similarity between the time series in Panel (a) is so striking that it necessary to conduct a more in depth analysis in order to ascertain the impact of financial conditions on tail risk measures. Such investigation will be one of the cornerstones of the next section, where we carry out an extensive prediction exercise.

4. Predicting economic vulnerability

In this section, we study the relevance of the findings of Section 3 when it comes macroeconomic point, density and tail risk forecasts . We postulate that if past information is the primary driver of the time variation in the distribution of GDP growth, then it will also be the main source of predictive power, at least in normal times. At the same time, the consequences of a state dependent relation between financial conditions and the economic activity suggest that the former will play a major role at the onset of a recession.

We split the dataset of Section 3 into two sub-samples: the first one ranges from Q1:1971 to Q4:1992, while the second sample spans the remaining time window, i.e. Q1:1993 to Q4:2019. As a first step, we estimate the DMQ-X over the entire first dataset and use it to predict the distribution of GDP growth rate between Q4:1992 and Q1:1993. Then, we incorporate the observation for Q1:1993 into the *estimation sample* and predict GDP growth for the successive quarter. We repeat this procedure until exhaustion of the second sample.

To assess the importance of financial conditions, we compare the results of the DMQ-X approach with those of a baseline DMQ. Moreover, we benchmark the two dynamic models against the two-steps methodology Adrian et al. (2019), which we label with ABG-X if the NFCI is included and with ABG otherwise. Table 2 presents four error metrics related to point, density and tail-risk forecasting.[12] In all the panels, we report the relevant error metrics for models in the rows as a fraction of the same quantity for models in the columns. Hence, values smaller than one imply a superior performance of the former. Numbers in brackets are the (bias corrected, see Harvey et al., 1997) test statistics for the Diebold and Mariano (DM) test of equal accuracy of the forecasts.

The upper left panel contains the mean squared forecast errors (MSFE) of the point prediction $\mathbb{E}\left[Y_{t+1}|\,\mathcal{F}_t\right]$. Conditioning on the entire history of GDP growth implies a lower MSFE, no matter whether financial conditions are included. This is apparent from the second row, where we see that a baseline DMQ has a lower MSFE than both approaches of Adrian et al. (2019). Considering the NFCI improves the point predictive power, irrespectively of how we include previous realizations of GDP growth. However, accounting for the state dependent relation between financial conditions and the business cycle enhances such improvement. Indeed, the DMQ-X has an MSFE which is 88.6% of the DMQ, while including financial conditions in the framework of Adrian et al. (2019) only lowers the MSFE by 6%.

In the upper and lower right panels, we read two (average) metrics of density forecast accuracy: the weighted Continuous Ranked Probability Score (wCRPS) and the Continuous Ranked Probability Score (CRPS), see Gneiting and Ranjan (2011). Both measures give us the probabilistic forecast accuracy of a model; the CRPS assigns the same importance to the whole density, while we can tune the wCRPS to give more emphasis to a certain region of the distribution. Given our interest in the

[12]The four error metrics are the mean squared forecast error (MSFE), the weighed continuously ranked probability score (wCRPS), the continuously ranked probability score (CRPS) and the loss function of Fissler et al. (2016) (FZ). They are given by

$$
\begin{aligned}
\mathrm{MSFE} &= \frac{1}{T}\sum_{t=1}^{T}\left(y_t-\hat{y}_t\right)^2,\\
\mathrm{wCRPS} &= 2\int_0^1\left(\mathbb{1}\left\{y_t\le \hat{q}_t^{\tau_j}\right\}-\tau_j\right)\left(q_t^{\tau_j}-y_t\right)\left(1-\tau_j\right)^2 d\tau_j,\\
\mathrm{CRPS} &= 2\int_0^1\left(\mathbb{1}\left\{y_t\le q_t^{\tau_j}\right\}-\tau_j\right)\left(q_t^{\tau_j}-y_t\right) d\tau_j,\\
\mathrm{FZ} &= \frac{1}{\tau_j\widehat{\mathrm{ES}}_t^{\tau_j}}\mathbb{1}\left\{y_t\le \hat{q}_t^{\tau_j}\right\}\left(y_t-\hat{q}_t^{\tau_j}\right)+\frac{\hat{q}_t^{\tau_j}}{\widehat{\mathrm{ES}}_t^{\tau_j}}+\log\left(-\widehat{\mathrm{ES}}_t^{\tau_j}\right)-1,
\end{aligned}
$$

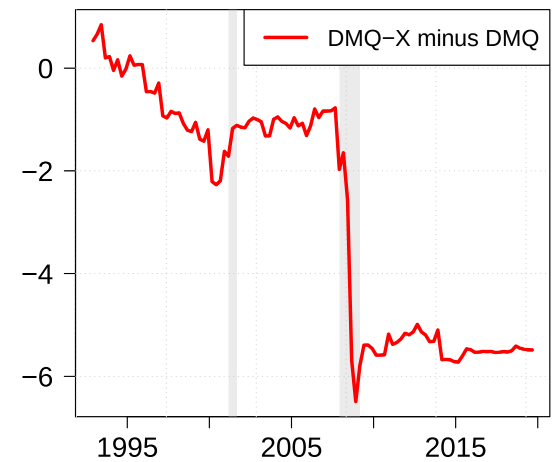
where y_t and $\hat{y}_t$ are the observed and predicted value of Y at time t, respectively; $\hat{q}_t^{\tau_j}$ is the predicted τ_j-quantile and $\widehat{\mathrm{ES}}_t^{\tau_j}$ the predicted macroeconomic expected shortfall for the same probability level.

	MSFE			wCRPS		
	DMQ	ABG-X	ABG	DMQ	ABG-X	ABG
DMQ-X	0.886	0.874	0.825	0.949	0.974	0.902
	(-1.280)	(**-2.413**)	(-1.557)	(-1.320)	(-1.013)	(-1.920)
DMQ		0.987	0.931		1.027	0.951
		(-0.161)	(-1.574)		(0.555)	(-1.810)
ABG-X			0.943			0.918
			(-0.589)			(-1.456)
	FZ Loss			**CRPS**		
	DMQ	ABG-X	ABG	DMQ	ABG-X	ABG
DMQ-X	0.726	0.708	0.516	0.952	0.971	0.923
	(-1.671)	(**-2.236**)	(-1.883)	(-1.507)	(**-2.270**)	(-1.692)
DMQ		0.975	0.710		1.020	0.971
		(-1.623)	(-1.661)		(0.653)	(-1.453)
ABG-X			0.729			0.952
			(0.932)			(-1.167)

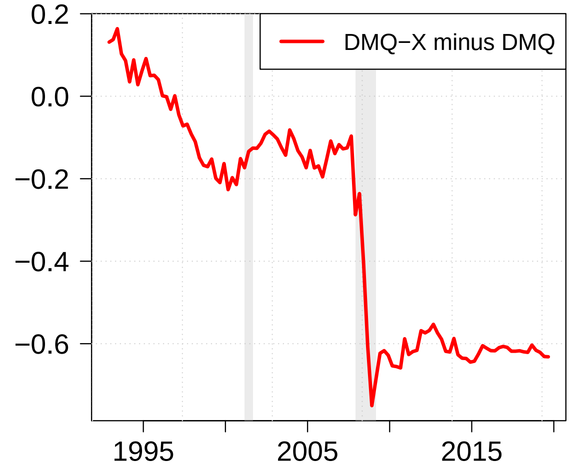
Table 2: In clockwise order: mean squared forecast error, weighted Continuously Ranked Probability Score, Continuously Ranked Probability Score and FZ loss. All numbers are the metrics for models in the rows expressed as a fraction of the same quantity for models in the columns. Numbers in parenthesis are the test statistics for the test of Diebold and Mariano (1995). Bold numbers stand for rejection of the null hypothesis of equal forecast accuracy at the 95% confidence level; gray shaded cells denote rejection at the 90% confidence level.

left tail, we weigh more the leftmost quantiles. When financial conditions are not considered, using the whole past of GDP growth delivers improved density forecasts; such predictions are more accurate at the 10% level of significance, when one places more importance on the left tail of the distribution. Including the NFCI enhances the probabilistic predictive power, no matter whether we consider the DMQ-X or the ABG-X methodology. Such refinement is larger when one moves from the ABG to the ABG-X specification. This is not a surprising finding, as Section 3 showed that past GDP growth accounts for much of the time variation in the conditional distribution of real output. Thus, properly accounting for it reduces the need to look at financial conditions.

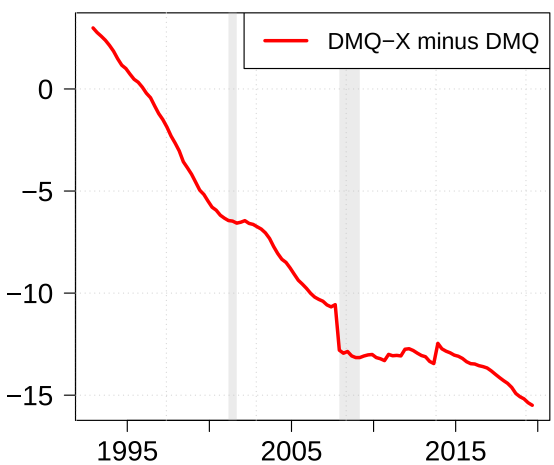
Finally, the lower left panel shows the average FZ loss of Fissler et al. (2016). This loss function measures the accuracy of jointly predicting the α-Growth-at-Risk, i.e. the worst loss in output that one expects to encounter in $\alpha \times 100\%$ of the considered time periods (here, quarters), and the macroeconomic expected shortfall at the same probability level α. As such, it gauges the predictive power on tail risk. Analogously to the MSFE case, past realizations of GDP growth are a key driver



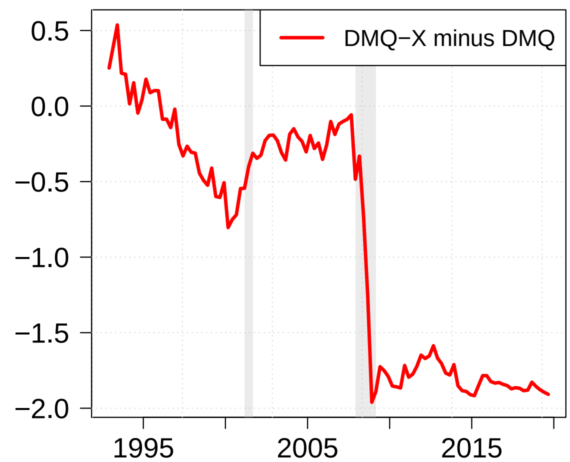
(a) Squared forecast error.



(b) Weighted continuously ranked probability score.



(c) FZ Loss function.



(d) Continuously ranked probability score

Figure 8: Cumulative sum of the difference between different forecasting metrics. From the upper left panel, in clockwise order: squared forecast error, wCRPS, CRPS and the FZ loss function.

of tail risk: the DMQ approach delivers a lower MSFE than the methodology of Adrian et al. (2019) and its forecasts are significantly more accurate at least at the 90% confidence level. At the same time, financial conditions turn out to be paramount when it comes to forecasting tail risk: both the methodology of this paper and the one of Adrian et al. (2019) return much lower average values than their *GDP-only* counterparts. However, while the forecasts implied by the DMQ-X are more accurate than those of DMQ at the 10% level of significance, no such conclusion can be drawn for the forecasts of the ABG-X and of the ABG methodology.

As a last remark, the state dependent asymmetric effect of financial conditions plays a key role in point, density and tail risk forecasting. Indeed, the DMQ-X methodology implies a lower average loss than the ABG-X one, no matter which metrics we look at. Moreover, capturing the state dependency returns point, density (as implied by the CRPS) and tail risk forecasts which are significantly more accurate at least at the 95% confidence level.

To better assess the relevance of the NFCI in forecasting GDP growth, Figure 8 shows the time series of the running sum of the difference in the four prediction losses between the DMQ-X and the DMQ.[13] For what regards point and density forecast (Panels a, b and d), financial conditions do not make such a difference over most of the sample. However, they do make it as soon as financial conditions start deteriorating: the first drop in the three plots corresponds to the forecast made on Q3:2007 for Q4:2007, which is the first date where we observe an increase in the NFCI. Financial conditions are still informative when a recession has already hit the economy, as shown by the second major drop occurring at the forecasts made on Q3:2008. This is in line with the policymaking considerations we made in Section 3: initial signals of distressed financial conditions prelude worsened and less certain economic conditions; at the same time, a further deterioration reduces the uncertainty around the (negative) point prediction. Things change dramatically for the FZ Loss. Financial conditions improve the forecast accuracy over the entire sample. As for the other metrics, we can observe a substantial drop right at the onset of the 2007-2009 recession: a further hint of the anticipative power of financial conditions on economic vulnerability.

5. Robustness with respect to the measure of real economic activity

In this section, we check whether the state dependent asymmetric effect stylized fact also characterizes other measures of the real economic activity. To this extent, we consider quarterly time series of five macroeconomic variables: industrial production (IP), real personal consumption expenditures per capita (PCE), total nonfarm payroll (TNP), the unemployment rate (UR) and the Chicago FED national activity index (CFNAI). The choice of these series is not arbitrary and is likely to guarantee a neat picture of the economy. As a matter of fact, industrial production and per capita consumption gauge the state of the supply and of the demand side in the market for goods and services. At the same time, total nonfarm payroll and the unemployment rate proxy for the healthiness of the labour market. Finally, the Chicago FED defines the CFNAI as "a single summary measure of a common factor in these (85) national economic data". The CFNAI takes value zero when the economy is expanding at its historical average while positive (negative) values point to an above (below) average growth; more details on the index are in Brave (2009).

As in Section 3, we use the DMQ-X methodology to study the relation between the NFCI and each of the five variables.[14] For industrial production, per capita consumption and total nonfarm

[13] To be precise, each plot shows $L(t) = \sum_{s=0}^{t} (e_{1,s} - e_{2,s})$, where $e_{1,s}$ and $e_{2,s}$ are the time s losses for the DMQ-X and for the DMQ, respectively.

[14] Similarly to what done in Section 1, we used a set of quantile regressions to graphically confirm the validity of this

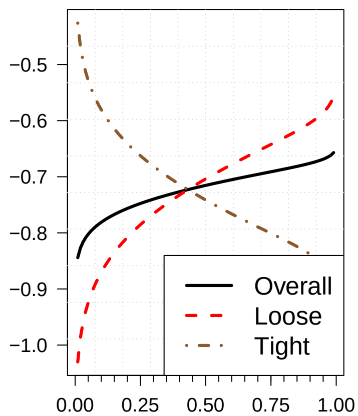
	IP	PCE	TNP	UR	CFNAI
μ	-0.723	-0.451	-0.474	0.756	-0.666
	(0.095)	(0.148)	(0.070)	(0.109)	(0.124)
σ	0.106	0.190	0.135	0.175	0.181
	(0.052)	(0.107)	(0.089)	(0.112)	(0.124)
α_{sn}	-5.660	-8.113	1.581	9.779	-9.113
	(25.505)	(104.944)	(1.429)	(121.328)	(171.926)
ω^+	3.581	3.005	1.107	0.356	2.212
	(2.037)	(1.351)	(0.323)	(0.422)	(0.968)
ω^-	-4.252	-1.023	-4.167	-1.252	-1.019
	(2.552)	(1.235)	(2.125)	(0.645)	(0.684)
$\mathbb{H}_0$	1.756	1.716	2.304	2.117	2.300

Table 3: Estimated parameters for the DAQ part of a DMQ-X with $J = 99$ quantiles along with asymptotic standard errors (in parenthesis). The last row reports the Wald test statistics for the null hypothesis $\mathbb{H}_0 : \omega^+ = \omega^-$. The estimation sample runs from Q1:1971 to Q4:2019 and includes quarterly realizations of the NFCI and of the relevant transformation for one of: industrial production (IP), real personal consumption expenditures per capita (PCE), total nonfarm payroll (TNP), the unemployment rate (UR) and the Chicago FED national activity index (CFNAI).

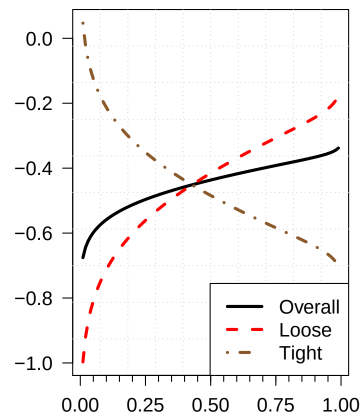
payroll we consider the quarter-on-quarter annualized growth rate, as we did with real GDP. For the unemployment rate, we look at the change in its level between two subsequent quarters. Finally, the definition of the CFNAI implies no need to transform the index. Table 3 shows the estimated coefficients for the DAQ part along with their asymptotic standard errors (in parenthesis). No matter which quantity we look at, an increase in the NFCI has a negative and statistically significant first order effect on the real economic activity. In analogy to what seen for GDP growth, we never reject the null hypothesis of no higher order effects induced by Q, i.e. $\mathbb{H}_0 : \alpha_{\text{sn}} = 0$. Remarkably, we always reject the null hypothesis of no state-dependent effect either at the 10% or at the 5% level of significance. Hence, there is evidence of a state dependent relation between (current) financial and (future) economic conditions, no matter how we measure the latter. Such finding holds true both for the labour market and for that of goods and services.

Having established that macro-financial dynamics are state dependent, we would like to see how such non-linearity plays out for different economic variables. To this extent, Figure 9 shows the expected effects of the NFCI on the conditional quantiles of each of the five variables. The black solid lines denote the *overall expected effect* while the red dashed (brown dotted-dashed) ones depict

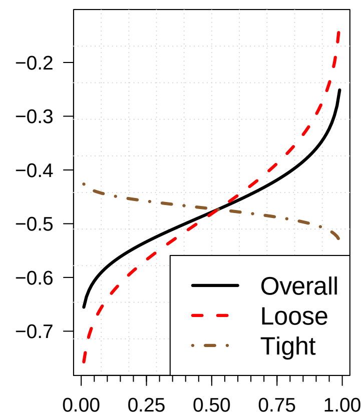
robustness check. Results were always in line with those of Figure 9.



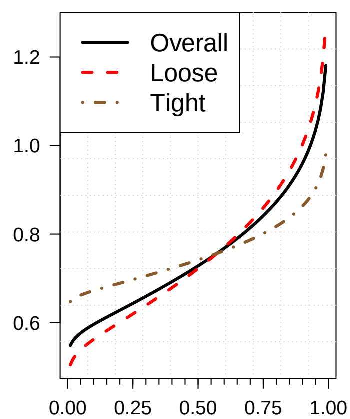
(a) Industrial production.



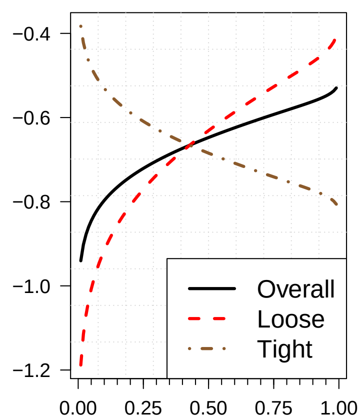
(b) Real consumption per capita.



(c) Total nonfarm payroll.



(d) Unemployment rate.



(e) CFNAI.

Figure 9: Overall (black solid lines) and state dependent (red dashed or brown dotted-dashed lines, per legend) expected effect of the NFCI on the conditional quantiles of: industrial production, real personal consumption expenditure per capita, total nonfarm payroll, unemployment rate and the CFNAI. The horizontal axis reports 99 probability levels between 1% and 99%.

the expected effect when financial conditions are loose (tight). Results for industrial production, consumption, total nonfarm payroll and the NFCAI closely mirror those for GDP growth: as financial conditions start deteriorating, the outlook on future economic activity becomes more negative and less certain; conversely, we expect the distribution to shrink towards its central tendency when financial conditions become increasingly tight. Such counteracting effect implies that the *overall response* is less asymmetric than when financial conditions are loose. The unemployment rate has a slightly different behaviour, as worsening financial conditions imply a lower expected value and a higher variance, no matter the state of the financial system. The increase on the variance is less pronounced when financial conditions are already tighter than usual. Combining two similar effects imply that worsened financial conditions have an overall asymmetric effect on the conditional quantiles of the unemployment rate.

6. Conclusions

In this paper, we uncovered a state dependent relation between financial conditions and economic vulnerability. Specifically, different states of the financial system imply a different impact of worsening financial conditions on the future distribution of GDP growth. As soon as financial conditions start deteriorating, the economic outlook becomes more pessimistic and less certain. On the other hand, when financial conditions worsen from an already tighter than usual situation the outlook becomes more pessimistic but there is no increase in the related uncertainty. Such finding is shared by other measures of the real economic activity, which characterize the market for goods and services (industrial production and real consumption per capita), the market for labour (total nonfarm payroll and the unemployment rate) and which gauge the overall growth rate of the economy (the CFNAI). We also detailed the importance of incorporating past information on GDP growth when assessing economic vulnerability: its omission makes the estimated effect of worsening financial conditions biased. A full sample analysis showed that past information drives most of the time variation in the conditional distribution of real output, as implied by negligible differences between the moments filtered with and without considering financial conditions. Finally, a prediction exercise stressed the relevance of considering both the entire past and a state dependent relation when using financial conditions to predict macroeconomic downside risk.

Both the full sample analysis and the prediction exercise relied on a novel methodology which allowed us to consistently model the dynamics multiple quantiles when exogenous variables are considered. The proposed approach does not make any assumption on the conditional distribution of the variable of interest and it enjoys many desirable properties, such as a flexible structure and a parsimonious parametrization.

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A. Technical Appendix

A.1. *A primer on the DMQ of Catania and Luati (2019)*

This section gives an overview of the dynamic multiple quantiles (DMQ) model of Catania and Luati (2019). Consider a probability space $\mathcal{P} = (\Omega, \mathcal{F}, P)$, and a real valued stationary and ergodic stochastic process $Y = \{Y_t\}_{t\in\mathbb{Z}}$. We equip $\mathcal{P}$ with the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in\mathbb{Z}}$, where $\mathcal{F}_t \coloneqq \sigma(\{Y_s;\ s \leq t\})$ is the σ-field spanned by Y up to, and including, time t. Finally, we assume that Y has finite first moment.

Let $F_{t|t-1}$ denote the cumulative distribution function (cdf) of Y_t given $\mathcal{F}_{t-1}$ and, for $j \in \{1, \ldots, J\}$, let $\tau_j \in (0,1)$ be a probability level such that there exists a finite $d_t^{\tau_j} \in \mathbb{R}$ defined as

$$d_t^{\tau_j} \coloneqq \inf\left\{y_t \in \mathbb{R} \,|\, F_{t|t-1}(y_t) \geq \tau_j\right\},$$

and called τ_j-quantile of Y_t given the filtration at time $t-1$.

Catania and Luati (2019) model the spacing between a reference quantile and the precedent (subsequent) $(J/2-1)$ as the sum of some negative (positive) processes. In this way, the quantile for level τ_j reads:

$$d_t^{\tau_j} = \begin{cases} d_t^{\tau_{j^*}} - \sum_{i=j}^{j^*-1} \eta_{i,t}, & \text{if } \tau_j < \tau_{j^*}, \\ d_t^{\tau_{j^*}}, & \text{if } \tau_j = \tau_{j^*}, \\ d_t^{\tau_{j^*}} + \sum_{i=j^*+1}^{j} \eta_{i,t}, & \text{if } \tau_j > \tau_{j^*}, \end{cases} \tag{9}$$

where the reference quantile has dynamics

$$d_t^{\tau_{j^*}} = \bar{d}^{\tau_{j^*}}(1-\beta) + \alpha u_{t-1}^{\tau_{j^*}} + \beta d_{t-1}^{\tau_{j^*}}, \tag{10}$$

and η_j is the following positive process:

$$\begin{aligned} \eta_{j,t} &= \exp(\xi_{j,t}), \\ \xi_{j,t} &= \bar{\xi}_j(1-\phi) + \gamma u_{t-1}^{\tau_j} + \phi \xi_{j,t-1}. \end{aligned} \tag{11}$$

To reduce the dimensionality of the estimation problem, the parameters $\bar{\xi}_j$ and $\bar{d}^{\tau_{j^*}}$ are set by targeting the unconditional quantiles of Y. In this way, the only free-parameters are $\boldsymbol{\theta}_1 = (\alpha, \beta, \phi, \gamma)$ with ϕ and β bounded by one, and α and γ different from zero. The vector $\mathbf{u} = (u^{\tau_1}, \ldots, u^{\tau_{j^*}}, \ldots, u^{\tau_J})$ contains J martingale difference sequences (MDS) arising from the gradient of the multiple check loss function $\mathcal{L}$:

$$\mathcal{L}(\boldsymbol{\theta}; Y_t) = \sum_{j=1}^{J} \rho_{\tau_j}\left(Y_t - d_t^{\tau_j}(\boldsymbol{\theta})\right), \tag{12}$$

where $\boldsymbol{\theta}$ is the parameter vector of the DMQ and ρ_{τ_j} is the τ_j-quantile check function: $\rho_{\tau_j}(a) = a(\tau_j - \mathbb{1}\{a < 0\})$. Such intuition is closely related to the score-driven approach pioneered by Creal et al. (2013) and Harvey (2013). As a matter of fact, Koenker and Machado (1999), Poiraud-Casanova and Thomas-Agnan (2000) and Kotz et al. (2012) showed the equivalence between the negative of $\mathcal{L}$ and the log-likelihood of J independent random variables whose density relates to that of the asymmetric Laplace distribution. Thus, the negative of $\mathcal{L}$ becomes a quasi log-likelihood function and we can work with a quasi-score driven dynamics without imposing any assumption on the conditional distribution of Y. From equation (12), the terms in $\mathbf{u}$ become

$$u_t^{\tau_j} \propto \frac{\partial}{\partial \xi_{j,t}} \sum_{k=1}^{J} \rho_{\tau_k}\left(y_t - d_t^{\tau_k}\right), \qquad u_t^{\tau_{j^*}} \propto \frac{\partial}{\partial d_t^{\tau_{j^*}}} \sum_{k=1}^{J} \rho_{\tau_k}\left(y_t - d_t^{\tau_k}\right). \tag{13}$$

Note how $d_t^{\tau_k} = f\left(\left(d_t^{\tau_{j^*}}, \boldsymbol{\xi}_t\right), \mathbf{z}_t\right)$, where $\mathbf{z}_t$ is the vector of hitting variables with i-th element: $z_{i,t} = \mathbb{1}\left\{Y_t < d_t^{\tau_i}\right\} - \tau_i$. Applying the chain rule to the expression for $d_t^{\tau_k}$ we get:

$$u_t^{\tau_j} \propto \sum_{k=1}^{J} \frac{\partial d_t^{\tau_k}}{\partial \xi_{j,t}} z_{k,t}, \qquad u_t^{\tau_{j^*}} \propto \sum_{k=1}^{J} z_{k,t}. \tag{14}$$

Which implies the case-by-case expression,

$$u_t^{\tau_j} = \begin{cases} a_j^{-1} \sum_{i=1}^{j} z_{i,t}, & \text{if } j < j^*, \\ a_j^{-1} \sum_{i=1}^{J} z_{i,t}, & \text{if } j = j^*, \\ -a_j^{-1} \sum_{i=j}^{J} z_{i,t}, & \text{if } j > j^*, \end{cases} \tag{15}$$

As noted by Christoffersen (1998), the elements of $\{\mathbf{z}_t\}_{t\in\mathcal{T}}$ form a sequence of i.i.d., zero mean random variables. Hence, each $\left\{u_t^{\tau_j}\right\}_{t\in\mathcal{T}}$ is a sequence of i.i.d. random variables with zero mean and variance

$$\mathbb{E}_{t-1}\left[\left(u_t^{\tau_j}\right)^2\right] = \frac{\bar{\omega}_j}{a_j^2},$$

where

$$\bar{\omega}_j = \begin{cases} \sum_{k=1}^{j} \tau_k\left(1-\tau_k\right) + \sum_{l_1=1}^{j} \sum_{l_1 \neq l_2} \min\left(\tau_{l_1}, \tau_{l_2}\right)\left(1 - \max\left(\tau_{l_1}, \tau_{l_2}\right)\right), & \text{if } j < j^*, \\ \sum_{k=1}^{J} \tau_k\left(1-\tau_k\right) + \sum_{l_1=1}^{J} \sum_{l_1 \neq l_2} \min\left(\tau_{l_1}, \tau_{l_2}\right)\left(1 - \max\left(\tau_{l_1}, \tau_{l_2}\right)\right), & \text{if } j = j^*, \\ \sum_{k=j}^{J} \tau_k\left(1-\tau_k\right) + \sum_{l_1=j}^{J} \sum_{l_1 \neq l_2} \min\left(\tau_{l_1}, \tau_{l_2}\right)\left(1 - \max\left(\tau_{l_1}, \tau_{l_2}\right)\right), & \text{if } j > j^*, \end{cases}$$

and we set $a_j = \sqrt{\bar{\omega}_j}$ so that $\mathbb{E}_{t-1}\left[\left(u_t^{\tau_j}\right)^2\right] \equiv 1$.

Finally, a word on the h-step ahead forecasts. For $h > 1$, Catania and Luati (2019) find the following closed form expression for the reference quantile:

$$\hat{d}_{t+h|t}^{\tau_{j^*}} := \mathbb{E}_t\left[d_{t+h}^{\tau_{j^*}}\right] = \bar{d}^{\tau_{j^*}}\left(1-\beta\right) \sum_{s=0}^{h-2} \beta^s + \beta^{h-1} d_{t+1}^{\tau_{j^*}}. \tag{16}$$

For the remaining quantiles, the h-step ahead forecast reads

$$\hat{d}_{t+h|t}^{\tau_j} = \begin{cases} \hat{d}_{t+h|t}^{\tau_{j+1}} - \mathbb{E}_t\left[\eta_{j,t+h}\right], & \text{if } j < j^*, \\ \\ \hat{d}_{t+h|t}^{\tau_{j-1}} + \mathbb{E}_t\left[\eta_{j,t+h}\right], & \text{if } j > j^*, \end{cases} \tag{17}$$

where $\mathbb{E}_t\left[\eta_{j,t+h}\right]$ is given by

$$\mathbb{E}_t\left[\eta_{j,t+h}\right] = \begin{cases} \omega_{j,t+h} \prod_{s=0}^{h-2} \exp\left\{-a_j^{-1}\gamma\phi^s \sum_{l=1}^{j} \tau_l\right\} \sum_{l=0}^{j} h\left(\tau_l\right) \exp\left\{a_j^{-1}\gamma\phi^s\left(j-l\right)\right\}, & \text{if } j < j^*, \\ \\ \omega_{j,t+h} \prod_{s=0}^{h-2} \exp\left\{a_j^{-1}\gamma\phi^s \sum_{l=j}^{J} \tau_l\right\} \sum_{l=j-1}^{J} g\left(\tau_l\right) \exp\left\{-a_j^{-1}\gamma\phi^s\left(J-l\right)\right\}, & \text{if } j > j^*, \end{cases} \tag{18}$$

with: $\omega_{j,t+h} = \exp\left\{\bar{\xi}_j\left(1-\phi\right)\sum_{s=0}^{h-2}\phi^s\right\} \exp\left\{\phi^{h-1}\xi_{j,t+1}\right\}$ and

$$h\left(\tau_l\right) = \begin{cases} \tau_1, & \text{if } l = 0, \\ \tau_{l+1} - \tau_l, & \text{if } 0 < l < j, \\ 1 - \tau_j, & \text{if } l = j, \end{cases} \qquad g\left(\tau_l\right) = \begin{cases} \tau_j, & \text{if } l = j-1, \\ \tau_{l+1} - \tau_l, & \text{if } j-1 < l < J, \\ 1 - \tau_J, & \text{if } l = J. \end{cases}$$

A.2. Quantile targeting

While introducing the DMQ-X, we mentioned that we reduce the number of free parameters by targeting the unconditional quantiles of Y. Such approach, which is in line with the variance targeting of Engle and Mezrich (1996) and Engle (2002), has been shown to improve the maximum likelihood estimation of the model. Recall that, in the DMQ-X

$$q_t^{\tau_j} = \text{DMQ}_t^{\tau_j} + q_{x,t}^{\tau_j},$$

for $\text{DMQ}_t^{\tau_j}$ equivalent to $d_t^{\tau_j}$ in Equation (9). Then, using the notation of Section A.1, we use such targeting scheme to fix the values of $\bar{d}^{\tau_{j^*}}$ and $\bar{\xi}_j$.

Let $\bar{\mathbf{q}} = \left(\bar{q}_1, \ldots, q_{j^*}, \ldots, \bar{q}_J\right)$ be the J unconditional quantiles of Y associated to the J probability levels of the DMQ-X and define $\bar{q}_x^{\tau_j} := \mathbb{E}\left[q_{x,t}^{\tau_j}\right]$. For the reference quantile, we want that

$$\bar{q}_{j^*} = \mathbb{E}\left[q_t^{\tau_{j^*}}\right] = \mathbb{E}\left[d_t^{\tau_{j^*}}\right] + \bar{q}_x^{\tau_{j^*}}.$$

We obtain the first addend as a limiting case of equation (16):

$$\lim_{h\to\infty} \hat{d}_{t+h|t}^{\tau_{j^*}} = \bar{d}^{\tau_{j^*}}.$$

The second expectation is given by

$$\bar{q}_x^{\tau_{j^*}} = \mu\mathbb{E}\left[X_{t-1}\right] + \mathbb{E}\left[\left(\left|X_{t-1} - \omega^+\right|\mathbb{1}\left(X_{t-1} \geq 0\right) + \left|X_{t-1} - \omega^-\right|\mathbb{1}\left(X_{t-1} < 0\right)\right)\right]\sigma Q\left(\tau_{j^*}; \boldsymbol{\alpha}\right).$$

Thus, we obtain the following expression for $\bar{d}^{\tau_{j^*}}$:

$$\bar{d}^{\tau_{j^*}} = \bar{q}_{j^*} - \bar{q}_x^{\tau_{j^*}}. \tag{19}$$

For the collection of $\bar{\xi}_j$ we start from the unconditional quantile spacings

$$\overline{\Delta}_j \coloneqq \begin{cases} \bar{q}_{j+1} - \bar{q}_j, & \text{if } j < j^*, \\ \bar{q}_j - \bar{q}_{j-1}, & \text{if } j > j^*, \end{cases} \tag{20}$$

and match them with the expected value of those implied by the DMQ-X (call them, Δ^{d}):

$$\overline{\Delta}_j = \mathbb{E}\left[\Delta_{j,t}^{\mathrm{d}}\right] = \begin{cases} \mathbb{E}\left[\eta_{j,t}\right] + \left(\bar{q}_x^{\tau_{j+1}} - \bar{q}_x^{\tau_j}\right), & \text{if } j < j^*, \\ \mathbb{E}\left[\eta_{j,t}\right] + \left(\bar{q}_x^{\tau_j} - \bar{q}_x^{\tau_{j-1}}\right), & \text{if } j > j^*. \end{cases} \tag{21}$$

Starting from Equation (18), we can write

$$\mathbb{E}\left[\eta_{j,t}\right] = \begin{cases} \exp\left\{\bar{\xi}_j - \frac{\gamma}{a_j(1-\phi)} \sum_{l=1}^{j} \tau_l\right\} \prod_{s=0}^{\infty} \left(\sum_{l=0}^{j} h\left(\tau_l\right) \exp\left\{a_j^{-1} \gamma \phi^s \left(j-l\right)\right\}\right), & \text{if } j < j^*, \\ \exp\left\{\bar{\xi}_j + \frac{\gamma}{a_j(1-\phi)} \sum_{l=j}^{J} \tau_l\right\} \prod_{s=0}^{\infty} \left(\sum_{l=j-1}^{J} g\left(\tau_l\right) \exp\left\{-a_j^{-1} \gamma \phi^s \left(J-l\right)\right\}\right), & \text{if } j > j^*. \end{cases} \tag{22}$$

Substituting Equation 22 into the rightmost part of Equation 21, taking logarithms and rearranging terms gives:

$$\bar{\xi}_j = \begin{cases} \log\left(\widehat{\Delta}_j\right) + \frac{\gamma}{a_j(1-\phi)} \sum_{l=1}^{j} \tau_l - \sum_{s=0}^{\infty} \log\left(\sum_{l=0}^{j} h\left(\tau_l\right) \exp\left\{a_j^{-1} \gamma \phi^s \left(j-l\right)\right\}\right), & \text{if } j < j^*, \\ \log\left(\widehat{\Delta}_j\right) - \frac{\gamma}{a_j(1-\phi)} \sum_{l=j}^{J} \tau_l - \sum_{s=0}^{\infty} \log\left(\sum_{l=j-1}^{J} g\left(\tau_l\right) \exp\left\{-a_j^{-1} \gamma \phi^s \left(J-l\right)\right\}\right), & \text{if } j > j^*, \end{cases} \tag{23}$$

where

$$\widehat{\Delta}_j \coloneqq \begin{cases} \overline{\Delta}_j - \left(\bar{q}_x^{\tau_{j+1}} - \bar{q}_x^{\tau_j}\right), & \text{if } j < j^*, \\ \overline{\Delta}_j - \left(\bar{q}_x^{\tau_j} - \bar{q}_x^{\tau_{j-1}}\right), & \text{if } j > j^*. \end{cases}$$

The definition of $\widehat{\Delta}_j$ imposes the following condition on the values of $(\sigma, \boldsymbol{\alpha})$:

$$\sigma < \frac{1}{m}\left[\min_{j=2,\ldots,J}\left(\frac{\bar{q}_j - \bar{q}_{j-1}}{Q\left(\tau_j; \boldsymbol{\alpha}\right) - Q\left(\tau_{j-1}; \boldsymbol{\alpha}\right)}\right)\right] \tag{24}$$

where

$$m \coloneqq \mathbb{E}\left[\left(|X_{t-1} - \omega^+| \mathbb{1}\left(X_{t-1} \geq 0\right) + |X_{t-1} - \omega^-| \mathbb{1}\left(X_{t-1} < 0\right)\right)\right].$$

In practice, we can estimate the unconditional quantiles of Y from an observed time series $(y_1, \ldots, y_T)$ in two ways: either we fix $\bar{\mathbf{q}}$ to the corresponding J empirical quantiles or we use the observed quantities to estimate the unconditional distribution of Y, which we assume to have a certain parametric specification, and then target the quantiles of such distribution. Catania et al. (2020) compare the two targeting procedures and find that they deliver similar results in large samples.

A.3. Asymptotic properties

In this appendix, we prove strong consistency and asymptotic normality of the estimator defined in Section 2. Throughout the section, we shall work on the complete probability space $(\Omega, \mathcal{F}, P)$ which we endow with the filtration $\mathbb{F}$. The notation $L^n(P)$ denotes finite n-th absolute over the measure space $(\Omega, \mathcal{F}, P)$.

Assumptions

We now list the assumptions needed to prove consistency and asymptotic normality of the Two-stage quasi maximum likelihood estimator of Section 2

Assumption 1 (Correct specification). *The model is correctly specified and has true parameter* $\boldsymbol{\theta}_0 \in \boldsymbol{\Theta}$ *such that* $q_t^{\tau_j}(\boldsymbol{\theta}_0)$ *is a quantile of* Y_t *for the probability level* $\tau_j \in (0,1)$*.*

Assumption 2 (Data generating process). $\widetilde{\mathbf{Y}} = \{Y_t, X_t\}_{t\in\mathbb{Z}}$ *is a stationary and ergodic stochastic process which generates the filtration* $\mathbb{F}$*. Such filtration enriches the complete probability space* $(\Omega, \mathcal{F}, P)$*. At any* $t \in \mathbb{Z}$ *we have that* $Y_t \in L^1(P)$*,* $X_t \in L^2(P)$ *and* $P(X_t = 0) = 0$*.*

Assumption 3 (Quantile process). $q_t^{\tau_j}(\boldsymbol{\theta})$ *is given by equations* (1)*,* (2)*,* (9)*,* (10) *and* (11) *for* $\boldsymbol{\theta} \in \boldsymbol{\Theta}$*.*

Assumption 4 (Parameter set). $\boldsymbol{\Theta} = \boldsymbol{\Theta}_1 \times \boldsymbol{\Theta}_2$ *is compact.* $|\beta| < 1$*,* $|\phi| < 1$*, and* σ *and* $\boldsymbol{\alpha}$ *satisfy equation* (24)*. Furthermore,* $\alpha \neq 0$ *and* $\gamma \neq 0$*.*

Assumption 5 (Regularity of the functions). *For all* $t \in \mathbb{Z}$ *the random variable* $Y_t|\,\mathcal{F}_{t-1}$ *admits a density* $f_{t|t-1}$*. Such function is Lipschitz continuous on its domain and* $f_{t|t-1}\left(q_t^{\tau_j}(\boldsymbol{\theta}_0)\right) > 0$*, for any* j *and* t*. The quantile function* Q *is at least* $\mathcal{C}^2$ *on* $\boldsymbol{\Theta}$*.*

Assumption 6 (Identifiability). $\boldsymbol{\theta}_{2,0}$ *lies in the interior of* $\boldsymbol{\Theta}_2$*. For every* $v > 0$ *there exists* $\delta_v > 0$ *such that for any* $\boldsymbol{\theta}_2 \in \boldsymbol{\Theta}_2$ *with* $\|\boldsymbol{\theta}_2 - \boldsymbol{\theta}_{2,0}\| > v$ *it holds that* $P\left(\cup_{j=1,\dots,J}\left|q_t^{\tau_j}(\boldsymbol{\theta}) - q_t^{\tau_j}(\boldsymbol{\theta}_0)\right| > \delta_v\right) > 0$*.*

Assumption 7 (Consistency of the first stage estimator). $\hat{\boldsymbol{\theta}}_1$ *is a consistent estimator of* $\boldsymbol{\theta}_{1,0}$*.*

Assumption 8 (Positive definiteness of relevant matrices). *The matrices* Q_0 *and* R_0 *are invertible. Furthermore,* $V_{0,11} = \mathbb{E}\left[s_t(\boldsymbol{\theta}_{1,0})\, s_t(\boldsymbol{\theta}_{1,0})'\right]$ *and* $V_{0,2} = \mathbb{E}\left[\eta_t(\boldsymbol{\theta}_0)\, \eta_t(\boldsymbol{\theta}_0)'\right]$ *are positive definite, and* $V_{0,22} - V_{0,12}' V_{0,11}^{-1} V_{0,12}$ *is invertible for* $V_{0,12} = \mathbb{E}\left[s_t(\boldsymbol{\theta}_{1,0})\, \eta_t(\boldsymbol{\theta}_0)'\right]$*. All the quantities are defined in the proof of Theorem 2.*

Assumption 9 (Relevant sequences). *There exist a set of deterministic sequences* $\left\{c_T^{(i)}\right\}_{i=1,\dots,5}$ *and a sequence of probabilities* $\{\pi_T\}$ *such that* $c_T^{(i)} = o\left(T^{\frac{1}{2}}\right)$ *and* $\pi_T = o_p\left(T^2\right)$*, for* $i = 1, \dots, 5$*.*

Assumption 10 (First stage estimator). *The quantity* $\underset{j=1,\dots,J}{\max}\ \underset{\boldsymbol{\theta}\in\boldsymbol{\Theta}}{\sup} \left|\nabla_{\boldsymbol{\theta}_1} \tilde{q}_t^{\tau_j}(\boldsymbol{\theta})\right|$ *is finite and* $s(\boldsymbol{\theta}_1)$ *admits a first order asymptotic expansion.*

Proof of Theorem 1 (Consistency)

We claim that if Assumptions 1 to 7 hold, $\hat{\boldsymbol{\theta}}_T$ is a strongly consistent estimator of $\boldsymbol{\theta}_0$, the true parameter. The claim holds true as long as the proposed model satisfies the four assumptions of Corollary 3.10 from White (1994). These four assumptions are Assumptions 2.1, 2.3, 3.1 and 3.5 of the same reference. Assumption 2 is equivalent to Assumption 2.1. Assumption 2.3 requires the objective function to be measurable in Y_t for any $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ as well as almost surely continuous in $\boldsymbol{\theta}$. The multiple check loss function satisfies both requirements, provided that Assumption 3 and Assumption 5 hold. Let us define

$$\varphi_t \coloneqq \varphi\left(y_t, \mathbf{q}_t\left(\boldsymbol{\theta}\right)\right) = \sum_{j=1}^{J} \rho_{\tau_j}\left(y_t - q_t^{\tau_j}\left(\boldsymbol{\theta}\right)\right), \tag{25}$$

Assumption 3.1 requires that $\bar{\varphi} \coloneqq \mathbb{E}\left[\varphi_t\right]$ exists, is finite, is continuous over $\boldsymbol{\Theta}$ and that the sequence $\left\{\varphi_t\right\}_{t\in\mathcal{T}}$ obeys either a weak or a strong uniform law of large numbers (ULNN). From Theorem A.2.2 of White (1994) (the theorem first appeared in Ranga Rao, 1962), the last two conditions are satisfied if there exists a random variable D with finite first moment and such that

$$P\left(\left|\varphi_t\right| \leq D\right) = 1, \qquad \forall \boldsymbol{\theta} \in \boldsymbol{\Theta}.$$

First, notice that

$$\left|\varphi_t\right| = \left|\sum_{j=1}^{J} \rho_{\tau_j}\left(y_t - q_t^{\tau_j}\left(\boldsymbol{\theta}\right)\right)\right| \leq \sum_{j=1}^{J} \left|y_t - q_t^{\tau_j}\left(\boldsymbol{\theta}\right)\right| \left|z_{j,t}\right| < \sum_{j=1}^{J} \left|y_t - q_t^{\tau_j}\left(\boldsymbol{\theta}\right)\right|, \tag{26}$$

for $z_{j,t} \coloneqq \mathbb{1}\left\{Y_t < q_t^{\tau_j}\right\} - \tau_j$ and bounded by one. The rightmost term has an upper bound too:

$$\sum_{j=1}^{J} \left|y_t - q_t^{\tau_j}\left(\boldsymbol{\theta}\right)\right| \leq \sum_{j=1}^{J} \left|y_t\right| + \sum_{j=1}^{J} \left|q_t^{\tau_j}\left(\boldsymbol{\theta}\right)\right| \leq J\left(\left|y_t\right| + U\right), \tag{27}$$

where

$$U \coloneqq \max_{j=1,\dots,J} \sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \left|q_t^{\tau_j}\left(\boldsymbol{\theta}\right)\right|.$$

Thus, we have found a positive random variable $D \coloneqq J\left(\left|Y_t\right| + U\right)$ which is a.s. greater or equal than $\left|\varphi_t\right|$. For Theorem A.2.2. to hold, D must be integrable. Y_t is integrable by assumption, while for U we have that

$$\begin{aligned}
\max_{j=1,\dots,J} \sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \left|q_t^{\tau_j}\left(\boldsymbol{\theta}\right)\right| &= \max_{j=1,\dots,J} \sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \left|\mathrm{DMQ}_{j,t}\left(\boldsymbol{\theta}\right) + q_{x,t}^{\tau_j}\left(\boldsymbol{\theta}\right)\right| \leq \\
&\leq \max_{j=1,\dots,J} \sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \left|\mathrm{DMQ}_{j,t}\left(\boldsymbol{\theta}\right)\right| + \max_{j=1,\dots,J} \sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \left|q_{x,t}^{\tau_j}\left(\boldsymbol{\theta}\right)\right| = \\
&= S_1 + S_2.
\end{aligned}$$

Catania and Luati (2019) show finiteness of S_1. For what regards S_2, we have that

$$\begin{aligned}\left|q_{x,t}^{\tau_j}(\boldsymbol{\theta})\right| = &\left|\mu X_{t-1} + \left\{|X_{t-1}-\omega^+|\mathbb{1}(X_{t-1}\geq 0) + |X_{t-1}-\omega^-|\mathbb{1}(X_{t-1}<0)\right\}\sigma Q(\tau_j;\boldsymbol{\alpha})\right| \leq \\ &\leq |\mu X_{t-1}| + \left(|X_{t-1}-\omega^+| + |X_{t-1}-\omega^-|\right)\sigma\left|Q(\tau_j;\boldsymbol{\alpha})\right| =: \delta(j,\boldsymbol{\theta}).\end{aligned}$$

Which implies that

$$S_2 := \max_{j=1,\dots,J}\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}}\left|q_{x,t}^{\tau_j}(\boldsymbol{\theta})\right| \leq \max_{j=1,\dots,J}\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}}\delta(j,\boldsymbol{\theta}).$$

For any $j=\{1,\dots,J\}$, δ is a continuous function of $\boldsymbol{\theta}$ which we maximize on a compact subspace. Therefore, the supremum over $\boldsymbol{\Theta}$ is a maximum. Let $\left(\boldsymbol{\theta}_*,\tilde{j}\right)$ be the maximizer of δ on $\boldsymbol{\Theta}\times\{1,\dots,J\}$. Then $\delta\left(\boldsymbol{\theta}_*,\tilde{j}\right)$ is linear in $|X_t|$ and has finite expected value thanks to the exogenous variable being square-integrable (Assumption 2). Thus we get,

$$|\varphi_t| \leq J\left(|Y_t| + S_1 + \delta\left(\boldsymbol{\theta}_*,\tilde{j}\right)\right).$$

Since the upper bound is integrable, Theorem A.2.2 of White (1994) implies that $\bar{\varphi}$ exists finite and continuous over $\boldsymbol{\Theta}$. Moreover, a strong ULLN for $\{\varphi_t\}_{t\in\mathcal{T}}$ applies. Finally, Assumption 3.5 of White (1994) requires the first stage estimator to be consistent (Assumption 7), and that $\lim_{T\to\infty}\mathbb{E}\left[\frac{1}{T}\sum_{t=1}^T\varphi\left(Y_t,\mathbf{q}_t\left((\boldsymbol{\theta}_{1,0},\boldsymbol{\theta}_2)\right)\right)\right]$ is continuous on $\boldsymbol{\Theta}$ uniformly in T and admits a unique maximizer $\boldsymbol{\theta}_{2,0}$. For the continuity, we note that

$$\left|\frac{1}{T}\sum_{t=1}^T\varphi\left(Y_t,\mathbf{q}_t(\boldsymbol{\theta})\right)\right| \leq \frac{1}{T}\left\{\sum_{t=1}^T|Y_t\tau_j| + \sum_{t=1}^T\left|q_t^{\tau_j}(\boldsymbol{\theta})\tau_j\right| + \sum_{t=1}^T|Y_t| + \sum_{t=1}^T\left|q_t^{\tau_j}(\boldsymbol{\theta})\right|\right\}.$$

In the limit, the right hand side is both bounded and independent of T. The second property is due to stationarity and ergodicity of Y. Hence, Theorem A.2.2 of White (1994) implies the desired regularity. Lastly, White et al. (2015) show that Assumption 6 implies identifiability of the second step estimator. This concludes the proof.

Proof of Theorem 2 (Asymptotic Normality)

We claim that if Assumptions 1 to 10 hold

$$\sqrt{T}\left(\hat{\boldsymbol{\theta}}_{2,0}-\boldsymbol{\theta}_{2,0}\right)\xrightarrow{d}\mathcal{N}\left(0,Q_0^{-1}V_0Q_0^{-1}\right),$$

where

$$\begin{aligned}Q_0 &= \sum_{j=1}^J\mathbb{E}\left[f_{t|t-1}\left(q_t^{\tau_j}(\boldsymbol{\theta}_0)\right)\nabla_{\boldsymbol{\theta}_2}\tilde{q}_t^{\tau_j}(\boldsymbol{\theta}_0)\nabla_{\boldsymbol{\theta}_2}\tilde{q}_t^{\tau_j}(\boldsymbol{\theta}_0)'\right],\\ V_0 &= \mathbb{E}\left[\zeta_t(\boldsymbol{\theta}_0)\zeta_t(\boldsymbol{\theta}_0)'\right],\end{aligned} \tag{28}$$

with $\zeta_t(\boldsymbol{\theta}_0) = \eta(\boldsymbol{\theta}_0) - R_0\Omega^{-1}s_t(\boldsymbol{\theta}_{1,0})$, $s_t(\boldsymbol{\theta}_{1,0}) = \nabla_{\boldsymbol{\theta}_1} L(\boldsymbol{\theta}_1)$, $\Omega = \mathbb{E}\left[\nabla_{\boldsymbol{\theta}_1} s_t(\boldsymbol{\theta}_{1,0})\right]$,

$$\eta(\boldsymbol{\theta}_0) = \sum_{j=1}^{J} \nabla_{\boldsymbol{\theta}_2} \tilde{q}_t^{\tau_j}(\boldsymbol{\theta}_0)\left(\tau_j - \mathbb{1}\left(y_t \leq \tilde{q}_t^{\tau_j}(\boldsymbol{\theta}_0)\right)\right)$$

$$R_0 = \sum_{j=1}^{J} \mathbb{E}\left[f_{t|t-1}\left(q_t^{\tau_j}(\boldsymbol{\theta}_0)\right)\nabla_{\boldsymbol{\theta}_2}\tilde{q}_t^{\tau_j}(\boldsymbol{\theta}_0)\nabla_{\boldsymbol{\theta}_1}\tilde{q}_t^{\tau_j}(\boldsymbol{\theta}_0)'\right]$$

The proof consists in verifying Assumptions 2.1, 2.3, 3.1, 3.2', 3.6, 3.7(a), 3.8, 3.9 and 6.5 of Theorem 6.11 in White (1994). We already showed the first three conditions in the proof of Theorem 1. Assumption 3.2' is equivalent to our Assumption 6. Assumptions 3.6, 3.7(a) and 3.8 are delicate as they require the objective function to be at least $\mathcal{C}^2$ on $\boldsymbol{\Theta}$. We will verify such assumptions on the smooth function

$$\lambda(\boldsymbol{\theta}) = \mathbb{E}\left[\nabla_{\boldsymbol{\theta}}\varphi\left(Y_t, \tilde{\mathbf{q}}_t(\boldsymbol{\theta})\right)\right] = \mathbb{E}\left[\sum_{j=1}^{J}\nabla_{\boldsymbol{\theta}}\tilde{q}_t^{\tau_j}(\boldsymbol{\theta})\left(\tau_j - \mathbb{1}\left(Y_t \leq \tilde{q}_t^{\tau_j}(\boldsymbol{\theta})\right)\right)\right] \tag{29}$$

where $\tilde{q}_t^{\tau_j}$ is a smooth counterpart of the j-th quantile filtered by the DMQ-X. To construct such variable, we first follow Engle and Manganelli (2004) and define an approximation to $z_{j,t} = \mathbb{1}\left\{Y_t < q_t^{\tau_j}\right\} - \tau_j$ which is at least C^2 on $\boldsymbol{\Theta}$. Let $\left\{c_T^{(1)}\right\}$ be a deterministic sequence converging to 0, the smoothed version of $z_{j,t}$ is given by

$$\tilde{z}_{j,t} \coloneqq \left(1 + \exp\left\{\frac{y_t - \tilde{q}_t^{\tau_j}}{c_T^{(1)}}\right\}\right)^{-1} - \tau_j. \tag{30}$$

Similarly, we define C^2 equivalents to $|X_{t-1} - \omega^+|$ and $|X_{t-1} - \omega^-|$. To this extent, we first note that

$$\left|X_{t-1} - \omega^+\right| = \left(X_{t-1} - \omega^+\right)s\left(X_{t-1} - \omega^+\right) = \left(X_{t-1} - \omega^+\right)\left\{\mathbb{1}\left(X_{t-1} \geq \omega^+\right) - \mathbb{1}\left(X_{t-1} < \omega^+\right)\right\},$$

where s denotes the sign function. We then approximate the two indicator functions with

$$\tilde{a}_{t,1}^{+} \coloneqq \left(1 + \exp\left\{\frac{\omega^+ - x_{t-1}}{c_T^{(2)}}\right\}\right)^{-1}, \qquad \tilde{a}_{t,2}^{+} \coloneqq \left(1 + \exp\left\{\frac{x_{t-1} - \omega^+}{c_T^{(3)}}\right\}\right)^{-1}. \tag{31}$$

In the same way, we approximate $|X_{t-1} - \omega^-|$ through the smooth functions

$$\tilde{a}_{t,1}^{-} \coloneqq \left(1 + \exp\left\{\frac{\omega^- - x_{t-1}}{c_T^{(4)}}\right\}\right)^{-1}, \qquad \tilde{a}_{t,2}^{-} \coloneqq \left(1 + \exp\left\{\frac{x_{t-1} - \omega^-}{c_T^{(5)}}\right\}\right)^{-1}. \tag{32}$$

Finally, we let $\tilde{q}_t^{\tau_j}$, $\tilde{u}_{j,t}$ and $\tilde{q}_{x,t}^{\tau_j}$ denote the smoothed counterparts of $q_t^{\tau_j}$, $u_{j,t}$ and $q_{x,t}^{\tau_j}$, respectively. Such processes are fully characterized by the just defined smooth counterparts and by the triplet $\left(y_1, x_1, q_1^{\tau_j}\right)$.

By construction, the filter $\tilde{q}_t^{\tau_j}(\boldsymbol{\theta})$ is twice continuously differentiable and $\mathbb{E}\left[\left|\nabla \tilde{q}_t^{\tau_j}\right|^n\right] < \infty$ as long as $\mathbb{E}\left[|X_t|^n\right] < \infty$, as we show in Appendix A.4. The last property implies that $\lambda(\boldsymbol{\theta}) = \mathbb{E}\left[\tilde{\eta}_t(\boldsymbol{\theta})\right] < \infty$ which is equivalent to Assumption 3.7(a) of White (1994). Following the same steps as White et al. (2015), we obtain that

$$\nabla_{\boldsymbol{\theta}_2} \lambda(\boldsymbol{\theta}) = \nabla_{\boldsymbol{\theta}_2} \mathbb{E}\left[\sum_{j=1}^{J} \nabla_{\boldsymbol{\theta}_2} \tilde{q}_t^{\tau_j}(\boldsymbol{\theta})\left(\tau_j - \mathbb{1}\left(Y_t \leq \tilde{q}_t^{\tau_j}(\boldsymbol{\theta})\right)\right)\right] = -\tilde{Q}_0 + O\left(|\boldsymbol{\theta}_2 - \boldsymbol{\theta}_{2,0}|\right),$$

where

$$\tilde{Q}_0 = \sum_{j=1}^{J} \mathbb{E}\left[f_{t|t-1}\left(\tilde{q}_t^{\tau_j}(\boldsymbol{\theta}_0)\right) \nabla_{\boldsymbol{\theta}_2} \tilde{q}_t^{\tau_j}(\boldsymbol{\theta}_0) \nabla_{\boldsymbol{\theta}_2} \tilde{q}_t^{\tau_j}(\boldsymbol{\theta}_0)'\right],$$

which is bounded thanks to Assumption 5 and to the boundedness of the relevant derivatives, which we prove in A.4. Hence, Theorem A.2.2 of White (1994) implies that our objective function satisfies Assumption 3.6 and 3.8 of the same reference, at least with respect to $\boldsymbol{\theta}_2$. For what regards the first step parameter, we can write

$$\nabla_{\boldsymbol{\theta}_1} \lambda(\boldsymbol{\theta}) = \nabla_{\boldsymbol{\theta}_1} \mathbb{E}\left[\sum_{j=1}^{J} \nabla_{\boldsymbol{\theta}_2} \tilde{q}_t^{\tau_j}(\boldsymbol{\theta})\left(\tau_j - \mathbb{1}\left(Y_t \leq \tilde{q}_t^{\tau_j}(\boldsymbol{\theta})\right)\right)\right] = -\tilde{R}_0 + O\left(|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_{1,0}|\right),$$

for

$$\tilde{R}_0 = \sum_{j=1}^{J} \mathbb{E}\left[f_{t|t-1}\left(\tilde{q}_t^{\tau_j}(\boldsymbol{\theta}_0)\right) \nabla_{\boldsymbol{\theta}_2} \tilde{q}_t^{\tau_j}(\boldsymbol{\theta}_0) \nabla_{\boldsymbol{\theta}_1} \tilde{q}_t^{\tau_j}(\boldsymbol{\theta}_0)'\right],$$

which is bounded thanks to Assumption 6 and 10. As before, boundedness of $\tilde{R}_0$ implies that our objective function satisfies Assumption 3.6 and 3.8 also with respect to $\boldsymbol{\theta}_1$. Assumption 8 coincides with Assumption 3.9 of White (1994).

Finally, we shall prove that the double array $T^{-\frac{1}{2}}\left(\nabla_{\boldsymbol{\theta}_1} l\left(Y_t, \boldsymbol{\theta}_{1,0}\right), \nabla_{\boldsymbol{\theta}_2} \varphi\left(Y_t, \tilde{\mathbf{q}}_t(\boldsymbol{\theta})\right)\right)'$ satisfies a central limit theorem (CLT), with l being the time t likelihood of the assumed unconditional distribution. Catania et al. (2020) prove that a CLT for martingale difference sequences (Theorem 5.24 in White, 2001) holds thanks to Assumptions 9 and 10. As our proof would be identical to theirs, we omit it.

It remains to prove that the difference between $q_t^{\tau_j}(\boldsymbol{\theta})$ and $\tilde{q}_t^{\tau_j}(\boldsymbol{\theta})$ is asymptotically negligible. The proof is identical to that of Catania and Luati (2019) and relies on the probability sequence of Assumption 9. We omit it and refer the reader to their paper.

A.4. Useful quantities and their properties

Gradient of the smoothed variables

Let us consider the smooth variables from equations (30), (31) and (32). For the sake of simplicity, we will assume (without loss of generality) that the five deterministic sequences of Assumption 9 all coincide with c_T. The partial derivatives w.r.t. the i-th entry of $\boldsymbol{\theta}_2$ read

$$\begin{aligned}
&\partial_{\boldsymbol{\theta}_{2,i}} \tilde{z}_{j,t} = k_{j,t,T}\, \partial_{\boldsymbol{\theta}_{2,i}} \tilde{q}_t^{\tau_j}(\boldsymbol{\theta}), \quad \partial_{\boldsymbol{\theta}_{2,i}} \tilde{a}_{t,1}^{+} = k_{t,1}^{+} \mathbb{1}\left(\boldsymbol{\theta}_i = \omega^{+}\right), \\
&\partial_{\boldsymbol{\theta}_{2,i}} \tilde{a}_{t,2}^{+} = k_{t,2}^{+} \mathbb{1}\left(\boldsymbol{\theta}_i = \omega^{+}\right), \quad \partial_{\boldsymbol{\theta}_{2,i}} \tilde{a}_{t,1}^{-} = k_{t,1}^{-} \mathbb{1}\left(\boldsymbol{\theta}_i = \omega^{-}\right), \\
&\partial_{\boldsymbol{\theta}_{2,i}} \tilde{a}_{t,2}^{-} = k_{t,2}^{-} \mathbb{1}\left(\boldsymbol{\theta}_i = \omega^{-}\right)
\end{aligned}$$

where

$$k_{j,t,T} = \frac{\exp\left\{c_T^{-1}\left(y_t - \tilde{q}_t^{\tau_j}(\boldsymbol{\theta})\right)\right\}}{c_T\left(1+\exp\left\{c_T^{-1}\left(y_t - \tilde{q}_t^{\tau_j}(\boldsymbol{\theta})\right)\right\}\right)^2}, \quad k_{t,1}^{+} = \frac{\exp\left\{c_T^{-1}\left(\omega^{+} - x_{t-1}\right)\right\}}{c_T\left(1+\exp\left\{c_T^{-1}\left(\omega^{+} - x_{t-1}\right)\right\}\right)^2},$$

$$k_{t,2}^{+} = -\frac{\exp\left\{c_T^{-1}\left(x_{t-1} - \omega^{+}\right)\right\}}{c_T\left(1+\exp\left\{c_T^{-1}\left(x_{t-1} - \omega^{+}\right)\right\}\right)^2}, \quad k_{t,1}^{-} = \frac{\exp\left\{c_T^{-1}\left(\omega^{-} - x_{t-1}\right)\right\}}{c_T\left(1+\exp\left\{c_T^{-1}\left(\omega^{-} - x_{t-1}\right)\right\}\right)^2},$$

$$k_{t,2}^{-} = -\frac{\exp\left\{c_T^{-1}\left(x_{t-1} - \omega^{-}\right)\right\}}{c_T\left(1+\exp\left\{c_T^{-1}\left(x_{t-1} - \omega^{-}\right)\right\}\right)^2}$$

and $\tilde{q}_t^{\tau_j}(\boldsymbol{\theta})$ is the smoothed j-th quantile of the DMQ-X at time t. Combining equation (1) and equation (9), we get

$$\tilde{q}_t^{\tau_j} = \begin{cases} \tilde{d}_t^{\tau_{j^*}} - \sum_{i=j}^{j^*-1} \tilde{\eta}_{i,t} + \tilde{q}_{x,t}^{\tau_j}, & \text{if } \tau_j < \tau_{j^*}, \\ \tilde{d}_t^{\tau_{j^*}} + \tilde{q}_{x,t}^{\tau_j}, & \text{if } \tau_j = \tau_{j^*}, \\ \tilde{d}_t^{\tau_{j^*}} + \sum_{i=j^*+1}^{j} \tilde{\eta}_{i,t} + \tilde{q}_{x,t}^{\tau_j}, & \text{if } \tau_j > \tau_{j^*}, \end{cases} \tag{33}$$

where $\tilde{d}_t^{\tau_{j^*}}$, $\tilde{\eta}_{i,t}$ and $\tilde{q}_{x,t}^{\tau_j}$ are the smoothed counterparts of $d_t^{\tau_{j^*}}$, $\eta_{i,t}$ and $\tilde{q}_{x,t}^{\tau_j}$, respectively. With equation (33) in mind, the i-th partial derivative of the (smoothed) j-th quantile reads

$$\tilde{q}_{i,t}^{\tau_j}(\boldsymbol{\theta}) \coloneqq \partial_{\boldsymbol{\theta}_{2,i}} \tilde{q}_t^{\tau_j}(\boldsymbol{\theta}) = \tilde{d}_{i,t}^{\tau_{j^*}}(\boldsymbol{\theta}) - b_j \sum_{k=l}^{r} \tilde{\eta}_{k,t}(\boldsymbol{\theta})\, \tilde{\xi}_{i,k,t}(\boldsymbol{\theta}) + \tilde{q}_{i,x,t}^{\tau_j}$$

where $b_j = \{\mathbb{1}(\tau_j < \tau_{j^*}) - \mathbb{1}(\tau_j > \tau_{j^*})\}$, $\{l, r\} = \{j, j^*-1\}$ if $j < j^*$ and $\{l, r\} = \{j^*+1, j\}$ if $j > j^*$, and the subscript i always denotes the partial derivative with respect to the i-th entry of $\boldsymbol{\theta}_2$.

Using equations (10) and (14), we can write the i-th partial derivative of the reference quantile as

$$\tilde{d}_{i,t}^{\tau_{j^*}} = \bar{d}^{\tau_{j^*}} \partial_{\boldsymbol{\theta}_{2,i}}(1-\beta) + \left(\partial_{\boldsymbol{\theta}_{2,i}} \alpha\right) \sum_{m=1}^{J} \tilde{z}_{m,t-1} + \alpha \sum_{m=1}^{J} k_{m,t,T} \tilde{q}_{i,t}^{\tau_m} + \left(\partial_{\boldsymbol{\theta}_{2,i}} \beta\right) \tilde{d}_{t-1}^{\tau_{j^*}} + \beta \tilde{d}_{i,t-1}^{\tau_{j^*}} + \tilde{q}_{i,x,t}^{\tau_{j^*}}, \tag{34}$$

where we have omitted the dependence on $\boldsymbol{\theta}$ of $\tilde{d}_{i,t}^{\tau_j*}$, $\tilde{q}_{i,t}^{\tau_m}$, $\tilde{q}_{t-1}^{\tau_j*}$ and $\tilde{q}_{i,x,t}^{\tau_j*}$ to enhance readability. From equation (11) the partial derivative $\tilde{\xi}_{i,k,t}(\boldsymbol{\theta})$ reads

$$\tilde{\xi}_{i,j,t} = \bar{\xi}_j \partial_{\boldsymbol{\theta}_{2,i}} (1-\phi) + \left(\partial_{\boldsymbol{\theta}_{2,i}} \gamma\right) \sum_m \tilde{z}_{m,t-1} + \gamma \sum_m k_{m,t,T} \tilde{q}_{i,t}^{\tau_m} + \left(\partial_{\boldsymbol{\theta}_{2,i}} \phi\right) \tilde{\xi}_{j,t-1} + \phi \tilde{\xi}_{i,j,t}, \tag{35}$$

where the sums run over the truncated domains of $\{l, r\}$. Finally, the i-th partial derivative of $\tilde{q}_{x,t}^{\tau_j}$ is

$$\tilde{q}_{i,x,t}^{\tau_j} = x_t \mathbb{1}(\boldsymbol{\theta}_{2,i} = \mu) + Q(\tau_j; \boldsymbol{\alpha}) \mathbb{1}(\boldsymbol{\theta}_{2,i} = \sigma) S_t + \sigma Q_i(\tau_j; \boldsymbol{\alpha}) S_t + \sigma Q(\tau_j; \boldsymbol{\alpha}) \left(\partial_{\boldsymbol{\theta}_{2,i}} S_t\right), \tag{36}$$

with

$$S_t := \left[\left(x_t - \omega^+\right)\left(a_{t,1}^+ - a_{t,2}^+\right) \mathbb{1}(x_t \geq 0) + \left(x_t - \omega^-\right)\left(a_{t,1}^- - a_{t,2}^-\right) \mathbb{1}(x_t < 0)\right]$$

and

$$\begin{aligned} \partial_{\boldsymbol{\theta}_{2,i}} S_t = &\left(a_{t,2}^+ - a_{t,1}^+\right) \mathbb{1}(x_t \geq 0) \mathbb{1}\left(\boldsymbol{\theta}_{2,i} = \omega^+\right) + \left(x - \omega^+\right)\left(k_{t,1}^+ - k_{t,2}^+\right) \mathbb{1}\left(\boldsymbol{\theta}_i = \omega^+\right) \mathbb{1}(x_t \geq 0) + \\ &\left(a_{t,2}^- - a_{t,1}^-\right) \mathbb{1}(x_t < 0) \mathbb{1}\left(\boldsymbol{\theta}_{2,i} = \omega^-\right) + \left(x - \omega^-\right)\left(k_{t,1}^- - k_{t,2}^-\right) \mathbb{1}\left(\boldsymbol{\theta}_i = \omega^-\right) \mathbb{1}(x_t < 0). \end{aligned}$$

Boundedness of the derivatives

We now want to show that for any positive integer n if X_t belongs to $L^n(P)$, then $\mathbb{E}\left[\left|\tilde{q}_{i,t}^{\tau_j}\right|^n\right] < \infty$ for all $j = 1, \ldots, J$ and $t = 1, \ldots, T$. To this extent, let $\delta_{i,j,t}$ denote the i-th partial derivative of the DMQ part. Then we have that

$$\mathbb{E}\left[\left|\tilde{q}_{i,t}^{\tau_j}\right|^n\right] = \mathbb{E}\left[\left|\delta_{i,j,t} + \tilde{q}_{i,x,t}^{\tau_j}\right|^n\right] \leq \mathbb{E}\left[\left(|\delta_{i,j,t}| + \left|\tilde{q}_{i,x,t}^{\tau_j}\right|\right)^n\right] = \sum_{k=0}^{n} \binom{n}{k} \mathbb{E}\left[|\delta_{i,j,t}|^k \left|\tilde{q}_{i,x,t}^{\tau_j}\right|^{n-k}\right]. \tag{37}$$

Catania and Luati (2019) establish an upper bound on $|\delta_{i,j,t}|$ for any j. Such bound is a real number, which is constant over time and depends on the value of k in $|\delta_{i,j,t}|^k$. Hence, we can write the bound on $|\delta_{i,j,t}|^k$ as $u_{j,k}$ and obtain:

$$\sum_{k=0}^{n} \binom{n}{k} \mathbb{E}\left[|\delta_{i,j,t}|^k \left|\tilde{q}_{i,x,t}^{\tau_j}\right|^{n-k}\right] \leq n! \sum_{k=0}^{n} u_{j,k} \mathbb{E}\left[\left|\tilde{q}_{i,x,t}^{\tau_j}\right|^{n-k}\right].$$

It remains to show that finiteness of the n-th absolute moment of X_t implies finiteness of $\mathbb{E}\left[\left|\tilde{q}_{i,x,t}^{\tau_j}\right|^{n-k}\right]$. From equation (36), we have that

$$\begin{aligned} \left|\tilde{q}_{i,x,t}^{\tau_j}\right| &= \left|x_t \mathbb{1}(\boldsymbol{\theta}_{2,i} = \mu) + Q(\tau_j; \boldsymbol{\alpha}) \mathbb{1}(\boldsymbol{\theta}_{2,i} = \sigma) S_t + \sigma Q_i(\tau_j; \boldsymbol{\alpha}) S_t + \sigma Q(\tau_j; \boldsymbol{\alpha}) \left(\partial_{\boldsymbol{\theta}_{2,i}} S_t\right)\right| \leq \\ &\leq |x_t| + |Q(\tau_j; \boldsymbol{\alpha})| |S_t| + \sigma |Q_i(\tau_j; \boldsymbol{\alpha})| |S_t| + \sigma |Q(\tau_j; \boldsymbol{\alpha})| \left|\partial_{\boldsymbol{\theta}_{2,i}} S_t\right|. \end{aligned} \tag{38}$$

For S_t we have that,

$$|S_t| \leq \left|x_t - \omega^+\right| + \left|x_t - \omega^-\right|,$$

and the right hand side has finite expectation provided that X_t is at least absolutely integrable. Note how the above inequality holds because both $\left(a^+_{t,1} - a^+_{t,2}\right)$ and $\left(a^-_{t,1} - a^-_{t,2}\right)$ are bounded by one. Finally, we see that

$$\left|\partial_{\boldsymbol{\theta}_{2,i}} S_t\right| \leq 2 + \left|x_t - \omega^+\right| \left|k^+_{t,1} - k^+_{t,2}\right| + \left|x_t - \omega^-\right| \left|k^-_{t,1} - k^-_{t,2}\right|,$$

where both $\left(k^+_{t,1} - k^+_{t,2}\right)$ and $\left(k^-_{t,1} - k^-_{t,2}\right)$ are bounded, so that the right hand side has finite expectation as long as X_t is at least absolutely integrable. Taking the $(n-k)$-th power and its expectation we get

$$\mathbb{E}\left[\left|\tilde{q}^{\tau_j}_{i,x,t}\right|^{n-k}\right] \leq \mathbb{E}\left[\left(2 + \left|X_t - \omega^+\right| \left|k^+_{t,1} - k^+_{t,2}\right| + \left|X_t - \omega^-\right| \left|k^-_{t,1} - k^-_{t,2}\right|\right)^{n-k}\right].$$

Applying again the binomial theorem, the above expression is finite as long as $\mathbb{E}\left[|X_t|^{n-k}\right] < \infty$. Thus, if $Q \in C^2(\boldsymbol{\Theta})$ finiteness of $\mathbb{E}\left[|X_t|^n\right]$ suffices for $\mathbb{E}\left[\left|\tilde{q}^{\tau_j}_{i,t}\right|^n\right] < \infty$.

B. Simulation study

This appendix presents a simulation exercise which evaluates the finite sample properties of the estimator introduced in Section 2. This study consider the exogenous variable

$$X = \{X_t;\, 0 \le t \le T\}\,, \qquad X_t \overset{i.i.d.}{\sim} \mathcal{N}\,(0,1)\,.$$

X influences the quantiles of Y_t through the quantile function of the Skewed Normal distribution of Azzalini (1985). The estimation relies on a parametric targeting approach and Y is assumed to follow a Skewed Normal distribution. The first step parameter vector reads

$$\boldsymbol{\theta}_1 = (\mu_1, \sigma_1, \alpha_{\text{sn},1})' \in \boldsymbol{\Theta}_1, \tag{39}$$

while that of the DMQ-X is given by

$$\boldsymbol{\theta}_2 = \left(\alpha, \beta, \gamma, \phi, \mu, \sigma, \omega^+, \omega^-, \alpha_{\text{sn},2}\right)', \in \boldsymbol{\Theta}_2. \tag{40}$$

In both cases, α_{sn} is the *slant* parameter, governing the skewness of the Skewed Normal distribution: a positive (negative) α implies a right (left) skewed distribution. The parameter space is

$$\boldsymbol{\Theta} = \boldsymbol{\Theta}_1 \times \boldsymbol{\Theta}_2,$$

with $\boldsymbol{\Theta}_1 = \mathbb{R}^2 \times \mathbb{R}_{++}$ and $\boldsymbol{\Theta}_2 = \mathbb{R}^6 \times (-1,1)^2 \times \mathbb{R}_{++}$.

The analysis repeats $M = 500$ times the following two-step procedure: simulate T observations $Y_m = (Y_{m,1}, \ldots, Y_{m,T})$ from the true model; use the sequence $\{Y_{m,t}, X_t\}$ to estimate the DMQ-X. Point 1 relies on an inverse-CDF approach to simulate $Y_{m,t}$ given the information set at time $t-1$. More precisely, it simulates $J = 99$ quantiles and uses them to approximate the conditional CDF of Y_m: $F_{t|t-1}$. Then, the inverse of $F_{t|t-1}$ is applied to a random draw from a Uniform distribution on $(0,1)$. This exercise considers four sample sizes: small $T = 250$, medium-small $T = 500$, medium-large $T = 1000$ and large $T = 2000$.[15] The true model is identified by the parameters:

$$\begin{aligned}\boldsymbol{\theta}_1 &= (0, 1, -1)\,,\\ \boldsymbol{\theta}_2 &= (0.1, 0.9, 0.2, 0.7, 0, 0.2, 1, -1, -3)\,,\end{aligned}$$

which are in line with what found in the empirical analysis.

Table 4 reports the mean estimate, along with the bias and the root mean squared error, for the nine parameters of the second step. As the sample size increases, the estimated parameters shrink towards the true ones. This is evident from the strictly decreasing root mean squared errors. Notably, for all parameters but $\alpha_{\text{sn},2}$ such metrics is a tiny fraction of the true value already for $T = 250$: a remarkable property when one wishes to conduct inference in small sample, like macro-financial ones. For what regards the bias, all estimates are close to the true ones already for $T = 250$, with the most far apart being the one for $\alpha_{\text{sn},2}$ with a relative bias of 10%. Results suggest the validity of our econometric framework even when the observed series is as short as those considered in the paper.

[15] Unreported results with $T = 5000$ and $T = 10000$ were in line with the reported ones and confirmed the validity of the asymptotic analysis in Section A.3.

	$T = 250$			$T = 500$		
	Mean	Bias	RMSE	Mean	Bias	RMSE
β	0.834	-0.066	0.130	0.866	-0.034	0.079
α	0.104	0.004	0.037	0.103	0.003	0.026
γ	0.696	-0.004	0.123	0.698	-0.002	0.095
ϕ	0.186	-0.014	0.052	0.190	-0.010	0.040
μ	0.001	0.001	0.043	0.001	0.001	0.030
σ	0.212	0.012	0.072	0.191	-0.009	0.050
$\alpha_{sn,2}$	-2.768	0.232	9.950	-3.529	-0.529	9.039
ω^+	1.046	0.046	0.406	0.939	-0.061	0.308
ω^-	-0.953	0.047	0.536	-0.902	0.098	0.429

	$T = 1000$			$T = 2000$		
	Mean	Bias	RMSE	Mean	Bias	RMSE
β	0.886	-0.014	0.040	0.896	-0.004	0.025
α	0.101	0.001	0.018	0.099	-0.001	0.012
γ	0.692	-0.008	0.076	0.696	-0.004	0.056
ϕ	0.195	-0.005	0.030	0.194	-0.006	0.022
μ	0.000	0.000	0.020	0.000	0.000	0.015
σ	0.186	-0.014	0.042	0.180	-0.020	0.037
$\alpha_{sn,2}$	-2.815	0.185	7.570	-3.145	-0.145	5.874
ω^+	0.946	-0.054	0.212	0.915	-0.085	0.180
ω^-	-0.910	0.090	0.291	-0.940	0.060	0.156

Table 4: Mean estimate, bias and root mean squared error for the parameters of the second step estimator.

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