Optimal control of investment, premium and deductible for a non-life insurance company

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Abstract

A risk-averse insurance company controls its reserve, modelled as a perturbed Cramér-Lundberg process, by choice of both the premium p and the deductible K offered to potential customers. The surplus is allocated to financial investment in a riskless and a basket of risky assets potentially correlating with the insurance risks and thus serving as a partial hedge against these. Assuming customers differ in riskiness, increasing p or K reduces the number of customers $n(p, K)$ and increases the arrival rate of claims per customer $\lambda(p, K)$ through adverse selection, with a combined negative effect on the aggregate arrival rate $n(p, K)\lambda(p, K)$. We derive the optimal premium rate, deductible, investment strategy, and dividend payout rate (consumption by the owner-manager) maximizing expected discounted life-time utility of intermediate consumption under the assumption of constant absolute risk aversion. Closed-form solutions are provided under specific assumptions on the distributions of size and frequency claims.

JEL classification: G11, G22, C60, D82

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1 Introduction

The optimal choice by an insurance company of strategies for dividend payout, investment in safe and risky financial assets, and reinsurance arrangements have been studied heavily by means of stochastic control, see, e.g., Schmidli (2008). A further control variable at the company's disposal is the premium. In the non-life insurance literature, this is typically set as the net premium, i.e., arrival rate times expected size of claims, with an added safety loading based on the expected value principle, standard deviation principle, variance principle, or similar. As an alternative to these premium principles, Asmussen et al. (2013) consider the premium as a direct control, chosen optimally to balance the trade-off between resulting portfolio size and average profit per customer. Analysis of the individual customer's decision problem of whether or not to insure at a given premium delivers both the dependence of portfolio size on premium and, as in Rothschild and Stiglitz (1976), a further adverse selection effect on the riskiness of the average customer. Thus, while a higher premium increases revenue per customer and reduces portfolio size, it leads in addition to a higher average claim arrival rate. Thøgersen (2016) extends this analysis by investigating the dependence of the optimal premium on a fixed deductible, taken as given rather than controlled by the company. In the present paper, we consider the combined stochastic control problem faced by an insurance company simultaneously choosing premium, deductible, safe as well as risky financial investments, and the resulting residual dividend payout or consumption flow. The optimization criterion considered is the expected discounted life-time utility from this latter flow. Our work provides a unifying framework, combining and generalizing both the literature on optimal investment of the insurance company's surplus and the optimal premium control literature.

We consider an infinitely-lived non-life insurance company that collects premiums from policyholders in continuous time, and covers net (of deductible) claims arriving according to a compound Poisson process. The contract offered is characterized by the premium rate, $p_t > 0$, and the deductible, $K_t \geq 0$. The company decides on both p_t and K_t to control its exposure to the insurance risk stemming from its contractual obligations, taking into account the effects on the size of the insurance portfolio, $n(p_t, K_t)$, and the average arrival rate of claims, $\lambda(p_t, K_t)$. Further, the company has access to a risk-free asset and a basket of risky assets for investment of the surplus from its insurance activities. Thus, it can enhance its risk management by partially hedging its insurance risk exposure through a financial portfolio that exploits any correlation between unexpected variations in surplus from the insurance business and returns to assets in the investment opportunity set. Correlation between the company's assets and liabilities can arise due to common dependence on the economic and natural environment, including systemic risk and contagion between the financial and insurance sectors. Despite the resulting hedging opportunity, the market remains incomplete, since the risk arising from insurance claims cannot be completely eliminated under any investment strategy. For simplicity, we assume that all financial assets offer constant expected rates of return, volatilities, and correlations through time.

An early contribution to the premium control literature was by Martin-Löf (1983), who proposed a linear feedback rule, setting the premium as a linear function of the surplus (reserve) and expected claims. Vandebroek and Dhaene (1990) showed that this is close to the optimal policy in a linear-quadratic control problem with

the premium as control and the surplus as state. These papers did not consider the dependence of portfolio size and claim arrival rate on the choice of premium, and the linear nature of the premium policy stemmed from an assumption that the quadratic objective penalized deviations in premiums and surplus levels from otherwise unspecified target levels assumed to derive from ruin theory. Taylor (1986) adopted a demand function specification for portfolio size as function of the premium, with price elasticity depending on average (or market) premium. Several subsequent studies have extended the deterministic discrete-time framework of Taylor. For example, Emms et al. (2007) consider a stochastic continuous-time generalization, and Pantelous and Passalidou (2017) consider a stochastic demand function (portfolio size) in discrete time. However, none of these papers allows for adverse selection effects of changes in premium on the riskiness of customers. Højgaard (2002) allows the claim arrival rate or intensity to depend in an unspecified manner on a safety loading, and hence the premium, assuming the expected value principle. In effect, the optimization is with respect to the arrival rate, but this is assumed proportional to portfolio size, and not separated from the average arrival rate per customer, i.e., adverse selection is still not accommodated. Joint determination of mutually consistent portfolio size and arrival rate requires analysis of the customer's problem, with endogenous derivation of (rather than assumption about) the demand function and average customer risk, following Asmussen et al. (2013), who assume no deductible, and the extension to a fixed deductible in Thøgersen (2016). In the insurance industry, a deductible may be introduced for a variety of reasons (see, e.g., Burnecki et al., 2005). First, it avoids handling costs associated with a large number of small claims; second, it prevents customers from making claims by sharing the cost of claims with the company; third, it allows the company to reduce the premium rate. We focus on a further strategic role of the deductible, namely, the insurance company can control the size and risk composition of its portfolio through the choice of (p_t, K_t), jointly. Thus, we contribute to the optimal premium control literature by adding optimal deductible control, as well as financial asset investment and hedging opportunities.

The second strand of literature that we build on is that on optimal investment in safe and risky assets of an insurance company's surplus. Browne (1995) allowed correlation between the financial and insurance risks, as in our case, and derived the optimal portfolio strategy of an insurance company taking the premium rate as given, and facing an insurance risk process described by the diffusion approximation to the classical Cramér-Lundberg model. Hipp and Plum (2000) and Yang and Zhang (2005) studied the corresponding asset allocation problems in which the surplus from the insurance business is instead modelled by a compound Poisson or jump-diffusion process. Using dynamic programming techniques, they obtained closed-form expressions for the optimal investment policy. Zheng et al. (2016) consider robust optimal portfolio and reinsurance arrangements for an ambiguity-averse insurance company. Zhou et al. (2017) consider optimal investment and premium control, but assume an unspecified monotone relation between safety loading and claim arrival rate, as in Højgaard (2002), i.e., no adverse selection, and they rely on a diffusion approximation to the surplus. With the exception of this paper, the investment literature takes the premium as given, and none of the papers considers a deductible. We contribute to the investment literature by adding the premium and deductible as further controls, including the resulting adverse selection effects, and recognizing the compound

Poisson component of the surplus process.

We assume that the surplus is driven by a perturbed Cramér-Lundberg process that in addition to the premium flow and compound Poisson process includes a diffusion component, following Gerber (1970). This perturbation captures any uncertainty about premium income and additional (beyond the compound Poisson process) uncertainty about aggregate claims (see Dufresne and Gerber, 1991, Furrer and Schmidli, 1994, and Schmidli, 1995), possibly stemming from variations in administrative expenses, operational costs, economic environment, or business conditions more generally. Specifically, we allow for correlation between the insurance and financial risk components of the company's balance sheet through this perturbation term. There are good reasons to expect such correlation to matter for insurance companies. Besides general co-movements of administrative and operational costs with the state of the economy and thus asset prices through the business cycle, correlation between insurance claims and equities can arise due to the occurrence or anticipation of natural disasters, epidemics, financial crises, economic recession, etc. Examples abound, e.g., Achleitner et al. (2002) consider the September 11, 2001, attack on the World Trade Center and document strong correlation impacts on the relation between underwriting and investment risks faced by property and casualty insurers, stemming from the simultaneous shock to their assets and liabilities. During the financial crisis of 2008, claims covered by credit insurance increased dramatically, while financial markets tumbled. Global warming and climate change are expected to increase the incidence of storms, droughts, floods, and thus property claims, and at the same time lead to economic and social instability, hence inducing contagion between insurance and financial markets (Hainaut, 2017). A currently relevant example is the ongoing coronavirus COVID-19 pandemic, which is generating massive claims from travel insurance, event cancellation, and health policy holders, while simultaneously depressing hotel, restaurant, tourism, entertainment, transportation, manufacturing, and many other economic sectors, and thus asset prices. Ward and Zurbruegg (2000) document co-movement between the insurance sector and the aggregate economy, and Billio et al., 2012 provide statistical evidence of systemic risk and correlation between the insurance and financial sectors. To accommodate such correlation in our model, we assume that the prices of financial assets available to the insurance company are driven by a vector of geometric Brownian motions, along the lines of Merton (1971), that are potentially correlated with the insurance surplus through the perturbation term, and we investigate the risk management implications of such correlation.

The insurance company's objective function deserves some mention. We consider expected discounted life-time utility from the running consumption or dividend flow extracted from the insurance surplus and financial wealth process, assuming that the instantaneous utility exhibits constant absolute risk aversion (CARA), following Merton (1969, 1971). This reflects the idea that the company pays dividends to a risk averse owner, who then consumes the running dividend, and that the company acts in the owner's interest, including the financial investment decisions. An alternative would be for the company to maximize expected discounted dividends, without regard to risk aversion, and let the owner handle the optimization of the consumption path, using the financial markets to smooth the dividend stream. For example, in the diffusion case, with dividends as the only control, Radner and Shepp (1996) and Asmussen and Taksar (1997) considered the maximization of expected

dividends discounted at a constant rate. However, separation of company ownership and management actually requires that the company maximize shareholder value. With the insurance surplus correlated with financial markets, this is given by expected dividends discounted instead at a rate appropriately adjusted for systematic or market risk, as in the Merton (1973) intertemporal capital asset pricing model (CAPM), with a risk premium given by a quantity of risk, reflecting the correlation, multiplied by the market price of risk, reflecting the risk aversion of the representative agent. In the present case, the representative agent corresponds to the owner, and rather than looking for a beta-adjustment in the discount rate for dividends along these lines, we consider directly the optimization of the owner's expected discounted life-time utility, thus collapsing ownership and control, with risk aversion reflected through the utility function rather than the discount rate. In economics and finance, expected discounted CARA utility of running (intermediate) consumption has been used widely, e.g., in applications to precautionary saving and incomplete markets, see Kimball and Mankiw (1989), Caballero (1990), Svensson and Werner (1993), and Wang (2004, 2006, 2009). In insurance, alternative criterions have been considered, e.g., minimization of ruin probability in the premium control literature, Asmussen et al. (2013), and in the investment literature either maximization of CARA utility of terminal wealth (as opposed to running consumption), Yang and Zhang (2005), or minimization of ruin probability, Hipp and Plum (2000). We do not explicitly consider ruin, but focus instead on maximization of the expected discounted utility stream with respect to premium, deductible, and investment.

Following Asmussen et al. (2013), we consider the possibility that potential customers are heterogeneous, and not equally risky to the insurance company. In particular, they exhibit different arrival rates of claims. In a complete information version of the model, the individual customer knows its own arrival rate. However, there is asymmetric information between the customer and the insurance company, and the latter only knows the distribution of arrival rates across customers. This feature of the approach accommodates adverse selection. In the sequel, we consider an incomplete or partial information extension of the model, in which individual customers are uncertain about their own arrival rates. Using these assumptions in the analysis of the customer's problem, we endogenously derive mutually consistent functional forms of $n(p_t, K_t)$ and $\lambda(p_t, K_t)$, which enter into the company's optimization problem.

We find that the optimal contract to be offered by the insurance company is characterized by a premium rate and a deductible that are inversely related and constant through time in our setting. Optimal investment in risky assets is similarly constant, and includes both a speculative component and a hedge against insurance risk. The optimal consumption (dividend) rate is not constant through time, but fluctuates with wealth in an affine fashion, with marginal propensity to consume out of current wealth given by the interest rate, and the level of consumption depending in addition on the discount rate and risk aversion of the company, as well as on the risk characteristics of potential customers. The optimal premium exceeds the expected net premium, which would correspond to standard insurance pricing without safety loading, or to the expected reservation premium of hypothetical risk-neutral customers. It even exceeds the expected reservation premium after accounting for the risk aversion of customers, even though this level is shown to replace the average or market premium of Taylor (1986) and Emms et al. (2007) in the demand function. The excess of the optimal pre-

mium beyond the risk-adjusted expected reservation premium depends on the relative valuation of net claims by company and customer, thus reflecting that the company is risk averse, too, and exerts market power to protect itself against risky customers, i.e., the adverse selection aspect. In the particular case of exponentially distributed arrival rates of claims, we provide a closed form expression for the optimal premium rate, as function of the optimal deductible, and a semi-closed form characterization of the optimal deductible. For exponentially distributed claim sizes, optimal premium and deductible are both given in closed form, as are optimal financial investments and dividends (consumption). For more general situations, we provide theoretical characterizations and numerical examples to illustrate the optimal controls. We investigate how the optimal premium and deductible interact with investment and consumption decisions. In a sensitivity analysis, we explore the impact on the four optimal controls of variations in the parameters governing systemic risk and contagion in our model, i.e., the correlation between the surplus from the insurance business and the financial market risks, and the degree of perturbation to the surplus process. For the special case of a fixed deductible, we provide a closed-form expression for the optimal premium, thus complementing the result by Thøgersen (2016) of existence for ruin probability minimization, without explicit solution for the optimal premium.

The paper is organized as follows. The optimization problem faced by the insurance company is introduced in Section 2. Section 3 studies the optimal premium, deductible, investment, and dividend strategies maximizing expected discounted utility of intermediate consumption. The closed-form solution to the company's Hamilton-Jacobi-Bellman equation is derived, and the conditions for the verification theorem guaranteeing the existence of an optimal strategy are established. Section 4 characterizes the optimal premium and deductible for the case of random arrival rates of claims, with complete information on the customer's side, and asymmetric information between customer and insurance company. Section 5 presents some results for the extension to partial information on the customer's side. The average arrival rate of claims for this case is derived in the Appendix. Section 6 concludes.

2   The model

Consider a company that at time $t = 0$ allocates an initial endowment $w_0 > 0$ between a risk-free asset and a bundle of risky assets, and starts an insurance business by selling insurance contracts. Thereafter, at each instant $t > 0$, the company (i) receives premiums from policyholders; (ii) pays to policyholders when claims occur; (iii) rebalances portfolio holdings by buying or (short-)selling units of its financial assets; and (iv) consumes part of its wealth. Such consumption may be viewed as dividend payments to the owner-manager. The company is therefore subject to insurance risk arising from the insurance policies written and market risk arising from its investment strategy. Throughout, we assume that the financial market is frictionless, and therefore all securities can be traded continuously over time without transaction costs or taxes.

For a mathematical formulation of the problem faced by the company, let $(\Omega, \mathbb{P}, \mathcal{F})$ be a complete probability space endowed with a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$, representing the information available to the company at time t. There are $I \geq 1$ risky assets with

price processes S_t^i, $i = 1, \ldots, I$. The price of the risk-free asset, or money market account, is denoted by S_t^0. For each $i = 0, 1, \ldots, I$, we denote by θ_t^i the number of units of asset i held by the company at time $t \geq 0$. Thus, the financial wealth of the company at time t is defined by the market value of the financial portfolio,

$$W_t^{\boldsymbol{\theta}} := \boldsymbol{\theta}_t \cdot \mathbf{S}_t = \sum_{i=0}^{I} \theta_t^i S_t^i, \quad t \geq 0,$$

where $\mathbf{S}_t = \left(S_t^0, \ldots, S_t^I\right)^\top$ and $\boldsymbol{\theta}_t = (\theta_t^0, \ldots, \theta_t^I)^\top$. The cumulative gains/losses to the financial investment portfolio associated with trading strategy $\boldsymbol{\theta}_t$ are therefore given by

$$G_t^{\boldsymbol{\theta}} := \int_0^t \boldsymbol{\theta}_\tau \cdot \mathrm{d}\mathbf{S}_\tau, \quad t \geq 0.$$

In addition, the company holds an insurance portfolio. At each instant $t \geq 0$, the portfolio includes as many policies as the company can attract in the insurance market, given the premium rate $p_t \geq 0$ and deductible $K_t \geq 0$. Let $n(p_t, K_t)$ represent the number of customers in the insurance portfolio from a total population of N potential customers. The arrival rate of claims from a typical insured individual is assumed to be a positive-valued measurable function of $p_t \geq 0$ and $K_t \geq 0$, denoted $\lambda(p_t, K_t)$. Hence, p_t and K_t are the quantities the company can use to control its exposure to insurance risk. As each of the $n(p_t, K_t)$ customers pays the premium p_t, the cumulative gains/losses to the insurance portfolio are given by the reserve or surplus process

$$R_t^{p,K} := \int_0^t n(p_s, K_s) p_s \, ds - X_t^{p,K}, \quad t \geq 0, \tag{2.1}$$

where $X_t^{p,K}$ is the aggregate insurance risk process, reflecting customer claims, as specified below.

A strategy $(\boldsymbol{\theta}, p, K)$ is said to be *self-financing* if $w_0 + G_t^{\boldsymbol{\theta}} + R_t^{p,K} \geq W_t^{\boldsymbol{\theta}}$, for all $t \geq 0$. The company pays out any residual wealth $w_0 + G_t^{\boldsymbol{\theta}} + R_t^{p,K} - W_t^{\boldsymbol{\theta}} \geq 0$ as dividends, for consumption by the owner-manager. We say that a self-financing strategy $(\boldsymbol{\theta}, p, K)$ is *admissible* if the cumulative consumption process

$$C_t^{\boldsymbol{\theta},p,K} := w_0 + G_t^{\boldsymbol{\theta}} + R_t^{p,K} - W_t^{\boldsymbol{\theta}}, \quad t \geq 0,$$

is differentiable with respect to $t \geq 0$. The process $c_t^{\boldsymbol{\theta},p,K} := \frac{\mathrm{d}}{\mathrm{d}t} C_t^{\boldsymbol{\theta},p,K}$ is the instantaneous consumption rate. Focusing on admissible strategies, the budget constraint of the company can be written in differential form as

$$\mathrm{d}W_t^{\boldsymbol{\theta},p,K} = \boldsymbol{\theta}_t \cdot \mathrm{d}\mathbf{S}_t + n(p_t, K_t) p_t \, \mathrm{d}t - \mathrm{d}X_t^{p,K} - c_t^{\boldsymbol{\theta},p,K} \, \mathrm{d}t, \quad W_0^{\boldsymbol{\theta},p,K} = w_0. \tag{2.2}$$

Throughout, we assume the following dynamics for the financial and insurance markets. First, on the financial side, the price processes of the ith risky asset, S_t^i, and the risk-free asset, S_t^0, are governed by the standard Black-Scholes model,

$$\mathrm{d}S_t^i = S_t^i \left[\mu^i \, \mathrm{d}t + \sum_{j=1}^{I} \sigma^{ij} \, \mathrm{d}B_t^j \right], \quad S_0^i > 0, \quad i = 1, \ldots, I, \tag{2.3}$$

$$\mathrm{d}S_t^0 = r S_t^0 \, \mathrm{d}t, \quad S_0^0 = 1, \tag{2.4}$$

where $\mathbf{B}_t = (B_t^1, \ldots, B_t^I)^\top$ is an I-dimensional vector of independent standard Brownian motions with respect to $\mathbb{F}$; $r > 0$ is the continuously compounded risk-free rate; $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^I)^\top \in \mathbb{R}^I$ is the vector of expected returns to the risky assets; and $\boldsymbol{\sigma} = (\sigma^{ij})_{1 \leq i,j \leq I} \in \mathbb{R}^{I \times I}$ is the volatility or diffusion matrix, with associated variance-covariance matrix $\boldsymbol{\Sigma} := \boldsymbol{\sigma}\boldsymbol{\sigma}^\top$.

Secondly, on the insurance side, given a progressively measurable process $(p, K) = \{(p_t, K_t)\}_{t \geq 0}$ with values in $\mathbb{R}_+^2$, representing premium and deductible choices, the aggregate insurance risk process $X_t^{p,K}$ is given by the jump-diffusion process

$$\mathrm{d}X_t^{p,K} = n(p_t, K_t) b \,\mathrm{d}\overline{B}_t + \mathrm{d}J_t^{p,K}, \quad X_0 = 0, \tag{2.5}$$

with $(J_t^{p,K})_{t \geq 0}$ the compound Poisson process

$$J_t^{p,K} = \sum_{m=1}^{N_t^{p,K}} (Y_m - K_{\tau_m})^+ \tag{2.6}$$

aggregating the net (of deductible) claims $(Y_m - K_{\tau_m})^+$ to be paid. Here, $(N_t^{p,K})_{t \geq 0}$ is an adapted point process with intensity $n(p_t, K_t)\lambda(p_t, K_t)$ counting the total number of claims, τ_m are the jump times of $N^{p,K}$, and $Y_m \in (0, \infty)$ are i.i.d. random claim sizes with distribution $F(\mathrm{d}y)$, independent of $(N_t^{p,K})_{t \geq 0}$. For concreteness, we focus in (2.6) on the case of a *fixed amount deductible*. Other types include franchise, proportional, limited proportional, and disappearing deductibles (see, e.g., Burnecki et al., 2005). The perturbation process $\overline{B}_t$ is a one-dimensional standard Brownian motion with respect to $\mathbb{F}$, accounting for additional uncertainty in the aggregate insurance risk process and assumed independent of the number and sizes of claims. The coefficient $b \geq 0$ in (2.5) captures the magnitude of the perturbation per insurance contract. Finally, $\langle B^i, \overline{B} \rangle_t = \rho^i t$, where $\rho^i \in (-1, 1)$ is the correlation coefficient between the log-price of the ith risky asset and the perturbation shocks to the aggregate insurance risk process.

Systemic or contagion risk corresponds to the perturbed case, $b > 0$, combined with negative insurance-finance correlation, $\rho^i < 0$, i.e., a positive shock to $\overline{B}$ (crisis, pandemic, etc.) reduces reserves, while a negative shock to B^i reduces the ith asset price, for a loss on both the insurance and finance sides of the business. For modelling purposes, the perturbation process can be easily constructed as

$$\overline{B} := \boldsymbol{\rho} \cdot \mathbf{B} + \sqrt{1 - \|\boldsymbol{\rho}\|^2}\, B^{I+1},$$

for $\boldsymbol{\rho} = (\rho^1, \ldots, \rho^I)^\top$ a correlation vector satisfying $\|\boldsymbol{\rho}\|^2 = \sum_{i=1}^I (\rho^i)^2 \leq 1$, and B^{I+1} a one-dimensional standard Brownian motion with respect to $\mathbb{F}$, independent of $\mathbf{B}$. In the unperturbed case, $b = 0$, the definition in (2.5) reduces the company's insurance business surplus (2.1) to the classical Cramér-Lundberg process, at least for (p_s, K_s) constant. On the other hand, for $b > 0$, the company can construct investment strategies in the financial markets to manage the risks arising from the insurance activities. However, the latter cannot be completely eliminated, because market incompleteness remains, even for $\|\boldsymbol{\rho}\|^2 = 1$, due to the unspanned risks arising from the insurance claims. In this case, the modelling of the surplus corresponds to a perturbed Cramér-Lundberg process.

Remark 2.1. In an insurance portfolio with n outstanding (independent) contracts and zero deductible, the total number of claims up to time t is given by

$$\sum_{l=1}^{n} \sum_{m=1}^{N_t^l} Y_m^l,$$

where Y_m^l are i.i.d. random variables with distribution $F(\mathrm{d}y)$, and N^l independent Poisson processes with arrival rates λ^l, say, with N_t^l representing the number of claims by customer l up to time t. It is straightforward to show that this sum of independent compound Poisson processes has the same distribution as a compound Poisson process with arrival rate $n\lambda$ and claims with distribution $F(\mathrm{d}y)$, where $\lambda = (1/n)\sum_{l=1}^{n}\lambda^l$ is the average arrival rate across customers, see, e.g., Proposition 3.3.4 in Mikosch (2004). Since our goal is to maximize the expected value of a function of the wealth process, this equivalence in law justifies the use of the compound Poisson process with arrival rate $n\lambda$ as a model of total claims in the aggregate risk process. ∎

For each $t \geq 0$, let $A_t^i = \theta_t^i S_t^i$ denote the amount of financial wealth invested in the ith risky asset. The remaining financial wealth is invested in the risk-free asset. Using $\mathbf{A}_t = (A_t^1, \ldots, A_t^I)^\top$ as control variable, the budget constraint (2.2) transforms into the controlled stochastic differential equation (SDE)

$$\mathrm{d}W_t^{c,p,K,\boldsymbol{A}} = [rW_t - c_t]\,\mathrm{d}t + \mathbf{A}_t^\top \left[(\boldsymbol{\mu} - r\underline{\mathbf{1}})\,\mathrm{d}t + \boldsymbol{\sigma}\,\mathrm{d}\mathbf{B}_t\right]$$
$$+ n\,(p_t, K_t)\left(p_t\,\mathrm{d}t - b\,\mathrm{d}\overline{B}_t\right) - \int_{\mathbb{R}_+} (y - K_t)^+ N^{p,K}(\mathrm{d}y, \mathrm{d}t)\,, \quad W_0^{c,p,K,\boldsymbol{A}} = w_0\,, \quad (2.7)$$

where $\underline{\mathbf{1}} = (1, \ldots, 1)^\top \in \mathbb{R}^I$. Here, $N^{p,K}(\mathrm{d}y, \mathrm{d}t)$ denotes a (random) jump measure with compensator $n(p, K)\lambda(p, K)F(\mathrm{d}y)\,\mathrm{d}t$.

A control policy described by the process $(c, p, K, \mathbf{A})$ with values in $\mathbb{R}_+ \times \mathbb{R}_+^2 \times \mathbb{R}^I$ at time $t \geq 0$ is an *admissible policy* if it is adapted to $\mathbb{F}$ and the expected discounted lifetime utility derived from consumption,

$$V(c, p, K, \mathbf{A}) := \mathbb{E}\left[\int_0^\infty e^{-\delta t} U(c_t)\,\mathrm{d}t \,\Big|\, W_0 = w_0\right]\,, \tag{2.8}$$

subject to the budget constraint (2.7), is finite for all initial wealth levels w_0. Here, company (owner-manager) preferences are characterized by (δ, η), with $\delta > 0$ the discount rate, i.e., the subjective rate of time preference, or impatience, and $\eta = -U''/U' > 0$ the constant absolute risk aversion (CARA) coefficient, with $U(x) = -\frac{1}{\eta}e^{-\eta x}$ the instantaneous utility (see Pratt, 1964). This utility function plays a prominent role in insurance mathematics and actuarial practice, as it ensures that the principle of "zero utility" yields a fair premium, independent of the level of reserves (see Goovaerts et al., 1990, Section II.6). In what follows, we denote the set of admissible policies by $\mathcal{A}$. The objective of the company is to find the strategy $(c, p, K, \mathbf{A})$ that maximizes $V(c, p, K, \mathbf{A})$ over all admissible policies.

3 The Hamilton-Jacobi-Bellman equation

The optimal time-homogeneous value function is written as

$$\vartheta(w_0) = \sup_{(c,p,K,\mathbf{A})\in\mathcal{A}} V(c, p, K, \mathbf{A})\,,$$

with $V(\cdot)$ given in (2.8). The associated transversality condition is

$$\lim_{T \to \infty} e^{-\delta T} \mathbb{E}[\vartheta(W_T)] = 0 \,.$$

The problem is amenable to analysis using stochastic dynamic programming and the appropriate form of Itô's lemma for jump-diffusion processes, together with standard time-homogeneity arguments for optimal Markov policies for infinite-horizon problems. If $\vartheta(\cdot)$ is sufficiently differentiable and satisfies the transversality condition, then, for any level of financial wealth $w \in \mathbb{R}$, it satisfies the non-linear second-order integro-differential equation, usually referred to as Hamilton-Jacobi-Bellman (HJB) equation (see Section III.7 of Fleming and Soner, 2006, and Sennewald, 2007),

$$-\delta\vartheta(w) + \sup_{(c,p,K,\mathbf{A}) \in \mathcal{A}} \left\{ U(c) + [\mathcal{L}^{c,p,K,\mathbf{A}}\vartheta](w) \right\} = 0 \,, \tag{3.1}$$

where $\mathcal{L}^{c,p,K,\mathbf{A}}$ is the operator

$$[\mathcal{L}^{c,p,K,\mathbf{A}}\vartheta](w) = \left[rw + \mathbf{A}^\top(\boldsymbol{\mu} - r\mathbf{1}) + n(p,K)p - c \right]\vartheta'(w)$$
$$+ \frac{1}{2}\left[\left\| \boldsymbol{\sigma}^\top \mathbf{A} \right\|^2 + n(p,K)^2 b^2 - 2n(p,K)b\mathbf{A}^\top\boldsymbol{\sigma}\boldsymbol{\rho} \right]\vartheta''(w)$$
$$+ n(p,K)\lambda(p,K)\left[\int_0^\infty \vartheta(w - (y-K)^+)\, F(\mathrm{d}y) - \vartheta(w) \right].$$

Assuming $F(\mathrm{d}y)$ is absolutely continuous, we conjecture that a solution to the HJB equation (3.1) takes the form

$$\vartheta(w) = \beta e^{-r\eta w} \,, \tag{3.2}$$

with $\beta < 0$. Finding an interior solution to the HJB equation for this conjecture requires maximizing

$$U(c) + cr\eta\beta e^{-r\eta w} \tag{3.3}$$

over $c \in \mathbb{R}_+$, and

$$r\eta[n(p,K)p + \mathbf{A}^\top(\boldsymbol{\mu} - r\mathbf{1})]$$
$$- \frac{(r\eta)^2}{2}\left[\left\| \boldsymbol{\sigma}^\top \mathbf{A} \right\|^2 + n(p,K)^2 b^2 - 2n(p,K)b\mathbf{A}^\top\boldsymbol{\sigma}\boldsymbol{\rho} \right]$$
$$- n(p,K)\lambda(p,K)q(K) \tag{3.4}$$

over $(p, K, \mathbf{A}) \in \mathbb{R}_+^2 \times \mathbb{R}^I$, with

$$q(K) := \mathbb{E}\left[e^{r\eta(Y-K)^+} \right]$$
$$= e^{-r\eta K} \int_K^\infty e^{r\eta y}\, F(\mathrm{d}y) - \bar{F}_Y(K)$$
$$= e^{-r\eta K} \mathbb{E}\left[e^{r\eta Y} \,|\, Y \geq K \right] \mathbb{P}(Y \geq K) - \bar{F}_Y(K) \tag{3.5}$$

representing a certain expected valuation of net claims $(Y-K)^+$, using the notation that $F(\mathrm{d}y)$ has density $f_Y(y)$, cumulative distribution function (c.d.f.) $F_Y(y) = \mathbb{P}(Y \leq y)$, and complementary c.d.f. $\bar{F}_Y(y) = 1 - F_Y(y)$. We assume throughout

that $\mathbb{E}[e^{r\eta Y}] < \infty$, so that $q(K)$ is well defined for all $K \geq 0$. Note that (3.4) is independent of the level of wealth w and strictly concave in $\mathbf{A}$.

The first-order necessary optimality conditions for maximization with respect to c and $\mathbf{A}$ are, respectively,

$$e^{-\eta c} + r\eta\beta e^{-r\eta w} = 0\,, \tag{3.6}$$

$$r\eta(\mu_i - r) - (r\eta)^2([\sigma\sigma^\top A]_i - nb[\sigma\rho]_i) = 0\,, \quad i = 1,\ldots,I\,. \tag{3.7}$$

Solving (3.6) for c yields the candidate for optimal consumption $\hat{c}_t = \hat{c}(W_t)$,

$$\hat{c}(w) := rw - \frac{1}{\eta}\ln(-\beta r\eta)\,, \tag{3.8}$$

with $\hat{c}(w)$ a stationary Markov control policy. Thus, optimal consumption is affine in wealth, reflecting interest income, and positive as long as wealth remains above the suitable lower bound $W_t > \frac{1}{r\eta}\ln(-\beta r\eta)$ for all t (see Merton, 1969).

Solving the system (3.7) yields candidates for optimal amounts of financial wealth to be invested in risky assets,

$$\hat{\mathbf{A}} = \mathbf{\Sigma}^{-1}\left[\frac{1}{r\eta}(\boldsymbol{\mu} - r\mathbf{\underline{1}}) + n(p,K)b\boldsymbol{\sigma}\boldsymbol{\rho}\right]\,.$$

For $\boldsymbol{\sigma}$ invertible, this simplifies to

$$\hat{\mathbf{A}} = (\boldsymbol{\sigma}^{-1})^\top\left[\frac{1}{r\eta}\boldsymbol{\psi} + n(p,K)b\boldsymbol{\rho}\right]\,, \tag{3.9}$$

where $\boldsymbol{\psi} = \boldsymbol{\sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{\underline{1}})$ is the I-vector of market prices of risk. Without systemic or contagion risk, i.e., if $b = 0$ or $\boldsymbol{\rho} = 0$, (3.9) reduces to the speculative or myopic demand for risky assets, cf. Merton (1969). In this case, there is complete separation between the financial and insurance markets, and thus the insurance risk is completely unspanned. In the perturbed case with insurance–finance correlation, $b > 0$ and $0 < \|\boldsymbol{\rho}\|^2 \leq 1$, optimal demand for risky assets involves an additional hedging component, the second term in (3.9), proportional to the size of the insurance portfolio, $n(p,K)$, as the company is using the the financial markets to manage its exposure to insurance risk.

Substitution of the optimal investment strategy (3.9) into (3.4) produces the partially maximized or profile Hamiltonian criterion for (p,K),

$$Q(p,K) := \frac{1}{2}\|\boldsymbol{\psi}\|^2 + r\eta n(p,K)\left(b\boldsymbol{\rho}^\top\boldsymbol{\psi} + p\right)$$
$$- \frac{1}{2}\left(r\eta bn(p,K)\right)^2\left(1 - \|\boldsymbol{\rho}\|^2\right) - n(p,K)\lambda(p,K)q(K)\,. \tag{3.10}$$

Thus, the optimal premium p and deductible K are obtained as the interior solution to the maximization of (3.10) over $(p,K) \in \mathbb{R}_+^2$.

Proposition 3.1. *Suppose that the diffusion matrix $\boldsymbol{\sigma}$ is invertible and that there exists a pair of constants $(\hat{p}, \hat{K})$ that maximizes $Q(p,K)$ in (3.10). Then the strategy $(\hat{c}(\cdot), \hat{p}, \hat{K}, \hat{\mathbf{A}})$ is optimal, with $\hat{\mathbf{A}}$ given by (3.9), also constant, and $\hat{c}(w)$ given by (3.8), with*

$$\hat{\beta} = -\frac{1}{r\eta}\exp\left(1 - \frac{1}{r}\left[\delta + Q(\hat{p}, \hat{K})\right]\right) < 0\,.$$

Proof. The wealth process $\hat{W}$ controlled by the strategy $(\hat{c}(w), \hat{p}, \hat{K}, \hat{\mathbf{A}})$ is a jump-diffusion process with differential

$$
\mathrm{d}\hat{W}_t = \frac{1}{\eta} \ln(-\beta r \eta)\, \mathrm{d}t + \hat{\mathbf{A}}^\top [(\boldsymbol{\mu} - r\underline{\mathbf{1}})\, \mathrm{d}t + \boldsymbol{\sigma}\, \mathrm{d}\mathbf{B}_t]
$$
$$
+ n(\hat{p}, \hat{K})(\hat{p}\, \mathrm{d}t - b\, \mathrm{d}\overline{B}_t) - \int_{\mathbb{R}_+} (y - \hat{K})^+ N^{\hat{p}, \hat{K}}(\mathrm{d}y, \mathrm{d}t)\,.
$$

Note that $\mathbb{E}[\vartheta(\hat{W}_{t-})]$ is finite for all $t \geq 0$ since $\mathbb{E}[e^{-r\eta Y}] \leq 1$. Applying Itô's lemma, compensating the integrals with respect to $N^{\hat{p}, \hat{K}}(\mathrm{d}y, \mathrm{d}t)$, and taking expected values, we get that $f(t) := \mathbb{E}[\vartheta(\hat{W}_t)]$ satisfies the ordinary differential equation (ODE)

$$
f'(t) = -\Big[r \ln(-r\eta\beta) + Q(\hat{p}, \hat{K}) \Big] f(t), \quad f(0) = \vartheta(w_0)\,.
$$

That is, $\mathbb{E}[\vartheta(\hat{W}_{t-})] = \vartheta(w_0) \exp(-[r \ln(-r\eta\beta) + Q(\hat{p}, \hat{K})]t)$. Inserting the guess (3.2) together with the strategy $(\hat{c}(w), \hat{p}, \hat{K}, \hat{\mathbf{A}})$ into the HJB equation (3.1), we get

$$
-\delta + r - r \ln(-r\eta\beta) - Q(\hat{p}, \hat{K}) = 0\,,
$$

yielding the optimal value of $\hat{\beta}$. Moreover, the transversality condition

$$
e^{-\delta T} \mathbb{E}[\vartheta(\hat{W}_T)] = e^{-rT} \vartheta(w_0) \to 0, \quad \text{as } T \to \infty
$$

holds. Thus, the conditions for the Verification Theorem linking the solution to the HJB equation (3.1) with sufficient conditions for existence of optimal strategies are satisfied, and the desired result follows (see Theorem 9.1 in Section III.9 of Fleming and Soner, 2006). ∎

The joint determination of optimal financial asset holdings, $\hat{\mathbf{A}}$, as in the investment literature, and optimal premium rate, $\hat{p}$, as in the premium control literature, represents a unifying framework. As a further generalization, the optimal deductible, $\hat{K}$, is characterized, as well, and the optimal consumption (dividend payout) rate $\hat{c}(w)$ follows. Under further conditions, closed-form solutions are given in Section 4.

By Proposition 3.1, if there exists a pair (p, K) that maximizes the profile Hamiltonian $Q(p, K)$ in (3.10), then the optimal premium, deductible, and investments in risky assets, $(\hat{p}, \hat{K}, \hat{\mathbf{A}})$, are independent of the level of financial wealth, and constant through time. Therefore, and as a result of the CARA assumption, the fraction of wealth invested in risky assets falls as the company accumulates more wealth, with all wealth invested in the money market account in the limit $W_t \to \infty$. Since $Q(p, K)$ does not depend on the company's discount rate, δ, neither do $(\hat{p}, \hat{K}, \hat{\mathbf{A}})$. Optimal consumption (3.8) is not constant, but fluctuates with income from interest on wealth. Although the level depends on the discount rate, through $\hat{\beta}$ from Proposition 3.1, the marginal propensity to consume depends on neither δ nor wealth, $\hat{c}'(w) = r$.

Remark 3.1. The mean returns $\boldsymbol{\mu}$ and volatility matrix $\boldsymbol{\sigma}$ in the dynamics of the risky assets (2.3) could be functions of the price processes, in which case $\mathbf{S}_t$ remains a Markov process, and the verification theorem still applies, with the HJB equation modified accordingly. In this case, optimal investments $\hat{\mathbf{A}}$, premium rate $\hat{p}$, and deductible $\hat{K}$ are feedback controls, depending on the prices of the risky assets. ∎

To be able to access the calculus, we henceforth treat the insurance exposure $n(p, K)$ as continuous and differentiable in (p, K), which should apply to a reasonable degree of approximation for moderate to large market size, N. Accordingly, $n(p, K)/N$ represents the fraction of potential customers purchasing insurance. In this case, the first-order necessary optimality conditions for maximizing $Q(p, K)$ in (3.10) are

$$\frac{\partial n(p, K)}{\partial p}\left[r\eta\left(p + b\left[\boldsymbol{\rho}^\top\boldsymbol{\psi} - r\eta b(1 - \|\boldsymbol{\rho}\|^2)n(p, K)\right]\right) - \lambda(p, K)q(K)\right]$$
$$+ n(p, K)\left[r\eta - q(K)\frac{\partial\lambda(p, K)}{\partial p}\right] = 0, \quad (3.11)$$

$$\frac{\partial n(p, K)}{\partial K}\left[r\eta\left(p + b\left[\boldsymbol{\rho}^\top\boldsymbol{\psi} - r\eta b(1 - \|\boldsymbol{\rho}\|^2)n(p, K)\right]\right) - \lambda(p, K)q(K)\right]$$
$$+ n(p, K)\left[r\eta\lambda(p, K)\bar{F}_Y(K) + q(K)\left(r\eta\lambda(p, K) - \frac{\partial\lambda(p, K)}{\partial K}\right)\right] = 0, \quad (3.12)$$

using $q'(K) = -r\eta(1 - F_Y(K) + q(K))$ for $q(K)$ from (3.5). In general, it is difficult to state conditions under which a critical point $(\hat{p}, \hat{K})$ satisfying the first order conditions (3.11) and (3.12) indeed maximizes $Q(p, K)$. The main difficulty lies in the characterization of the effect of the control variables on the trade-off between portfolio size and profit per customer. Therefore, the next step is to identify appropriate forms of $n(p, K)$ and $\lambda(p, K)$.

4 Random claim arrival rates

The result from Proposition 3.1 yields optimal investment and consumption, provided an optimal premium and deductible pair maximizing the profile Hamiltonian $Q(p, K)$ in (3.10) can be found. Further analysis of the existence and properties of such a pair $(\hat{p}, \hat{K})$ requires determining suitable forms of the portfolio size $n(p, K)$, the claim arrival rate $\lambda(p, K)$, and the expected net claim valuation function $q(K)$ in (3.5), all of which enter $Q(p, K)$. We derive these three functions from a simple model of the behavior of individual customers, thus ensuring that they are mutually consistent. First, we consider directly the decision problem faced by each potential customer of whether or not to insure at the terms (p, K) offered by the company. Next, we assume that customers are not equally risky. Specifically, while each knows its own claim (casualty) arrival rate, Λ, the insurance company does not have this information. It only knows the distribution across customers, and therefore treats the arrival rate of each potential customer as a random variable. This approach, following ideas from Asmussen et al. (2013) and Thøgersen (2016), delivers the three functions required, thus paving the way for an analysis of optimal $(\hat{p}, \hat{K})$, and by implication optimal investment and consumption, too. The asymmetric information between customer and insurance company implies adverse selection. Because Λ is assumed known to the customer, we say that the customer has complete information. This is relaxed in Section 5.

The individual customer's problem is whether or not to insure at terms (p, K). Without insurance, claims Y_m arriving at rate Λ must be paid by the customer.

With insurance, premiums are paid at rate p per period, and claims are only paid up to the deductible, i.e., payments are given by the uninsurable risks $\min\{Y_m, K\}$, again arriving at rate Λ. Thus, the flow of Y_m at rate Λ is compared to the flows of $\min\{Y_m, K\}$ at rate Λ and p. Writing the additional payments that the customer is liable for if uninsured, compared to if insured, as $Y_m - \min\{Y_m, K\} = (Y_m - K)^+$, i.e., the net (of deductible) claims, it is reasonable to compare p to the flow of $(Y_m - K)^+$ arriving at rate Λ. Specifically, when a claim Y_m arrives, the customer always pays the portion $\min\{Y_m, K\}$, whether insured or not, and the issue is whether the customer or the insurance company should cover the remainder, $(Y_m - K)^+$. The customer insures if doing so is preferred over paying the net claims as they arrive, at rate Λ.

Under risk neutrality, the customer insures if this choice is the cheapest, on average, i.e., if $p < p_0(K) := \Lambda \mathbb{E}\left[(Y - K)^+\right]$. Thus, the reservation premium, i.e., the maximum premium that a customer is willing to pay for insurance with a given deductible, is for a risk-neutral customer given by the arrival rate, Λ, times

$$\mathbb{E}\left[(Y - K)^+\right] = \int_K^\infty yF(\mathrm{d}y) - K\bar{F}_Y(K), \tag{4.1}$$

the expected net claim.

Under risk aversion, the reservation premium is naturally higher. In the sequel, we consider specific examples of how to allow for risk aversion on the part of the customer. In these cases, the reservation premium takes the more general form $p_a(K) = \Lambda a(K)$, with $a(K)$ the customer's risk-adjusted expected net claim. Thus, a potential customer buys insurance if and only if

$$p < p_a(K) = \Lambda a(K). \tag{4.2}$$

This reservation premium policy summarizes customer behavior. Regarding company behavior, we first analyze this for arbitrary $a(K)$, then turn to specific forms of this.

While the insurance company does not know each potential customer's value of Λ, the individual arrival rate of claims, it knows the distribution of this in the population. A standard approach in insurance is to let the company merely provide risk sharing, i.e., as Λ is unknown to the company, the premium charged to customers is given by the net premium, namely, expected frequency times size of net claims, $\mathbb{E}[\Lambda]\mathbb{E}\left[(Y - K)^+\right] = \mathbb{E}[p_0(K)]$, also recognized as the expected risk-neutral reservation premium. A safety loading can be added to this. Allowing for risk averse customers leads to an expected reservation premium $\mathbb{E}[p_a(K)] = \mathbb{E}[\Lambda]a(K)$ instead, based on (4.2). For this to be considered as a candidate premium to be charged to customers, viability of the insurance market requires it exceed the net premium, i.e., $a(K) > \mathbb{E}\left[(Y - K)^+\right]$. It does for a risk averse customer. However, $\mathbb{E}[p_a(K)]$ is only adjusted for customer risk aversion. It ignores both risk aversion on the part of the insurance company and adverse selection, i.e., that the company knows the form of the insurance participation constraint (4.2), and takes into account that only the riskier (higher Λ) potential customers will insure.

For concreteness, let Λ be i.i.d. in the population, with absolutely continuous distribution function $G(z)$, and density function $g(z)$. For given premium rate p and deductible K, the company attracts

$$n(p, K) = N\mathbb{P}\left(\Lambda > \alpha(K)p\right) = N\left[1 - G\left(\alpha(K)p\right)\right] \tag{4.3}$$

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customers out of the total population of size N, where $\alpha(K) := 1/a(K)$. The average arrival rate of claims across insured customers is

$$\lambda(p, K) = \mathbb{E}\left[\Lambda \,|\, \Lambda > \alpha(K)p\right] = \frac{\mathbb{E}\left[\Lambda \mathbb{1}_{\{\Lambda > \alpha(K)p\}}\right]}{1 - G(\alpha(K)p)} = \frac{\int_{\alpha(K)p}^{\infty} z\, G(\mathrm{d}z)}{1 - G(\alpha(K)p)}, \tag{4.4}$$

and the aggregate arrival rate for the collective insurance portfolio is given by the product of (4.3) and (4.4),

$$n(p, K)\lambda(p, K) = N \int_{\alpha(K)p}^{\infty} zg(z)\, dz\,.$$

Henceforth, we assume that the risk-adjusted expected net claim $a(K)$ is differentiable and decreasing, $a'(K) < 0$. This holds trivially under risk neutrality, $a(K) = \mathbb{E}\left[(Y - K)^+\right]$, and is natural under risk aversion, cf. Example 4.1 below. Thus, $\alpha'(K) > 0$, so $\alpha(K)p$ is increasing in both p and K. It follows that while the portfolio size $n(p, K)$ in (4.3) is decreasing in both p and K, the average claim rate $\mathbb{E}\left[\Lambda \,|\, \Lambda > \alpha(K)p\right]$ in (4.4) is increasing in both. In particular,

$$\frac{\partial \lambda(p, K)}{\partial p} > 0\,, \text{ and } \frac{\partial \lambda(p, K)}{\partial K} > 0\,.$$

This is adverse selection, cf. Rothschild and Stiglitz (1976). Only the riskier customers will continue to purchase insurance if premium or deductible is raised. Thus, the company faces a trade-off in the maximization of (3.10). To illustrate, consider the unperturbed case, $b = 0$, i.e.,

$$Q(p, K) = \frac{1}{2}\|\psi\|^2 + r\eta n(p, K)p - n(p, K)\lambda(p, K)q(K)\,. \tag{4.5}$$

An increase in p or K reduces portfolio size $n(p, K)$, and this multiplies on $r\eta p - \lambda(p, K)q(K)$, with $\lambda(p, K)$ increasing in p and K due to adverse selection, $q(K)$ either increasing or decreasing in K, and the sign of the derivatives of (4.5) therefore indeterminate with respect to both p and K without further restrictions. The derivatives of $n(p, K)\lambda(p, K)$ with respect to both p and K can be signed, but still not those of (4.5), due to the presence of $q(K)$ and the product $n(p, K)p$. This leaves scope for finding interior maximizers. Of course, in the perturbed case, $b > 0$, the analysis is further complicated.

For general perturbation of the reserve process, $b \geq 0$, the company's first order conditions (3.11) and (3.12) reduce to

$$r\eta \bar{G}(\alpha(K)p) + \alpha g(\alpha(K)p)\left\{\alpha(K)q(K)p\right.$$

$$\left. + \left(1 - \|\boldsymbol{\rho}\|^2\right)\left(r\eta b\right)^2 N\bar{G}(\alpha(K)p) - r\eta\left[b\boldsymbol{\rho}^\top\boldsymbol{\psi} + p\right]\right\} = 0\,,$$

$$g(\alpha(K)p)\alpha(K)'p\left\{\alpha(K)q(K)p + \left(1 - \|\boldsymbol{\rho}\|^2\right)\left(r\eta b\right)^2 N\bar{G}(\alpha(K)p)\right.$$

$$\left. - r\eta\left[b\boldsymbol{\rho}^\top\boldsymbol{\psi} + p\right]\right\} + r\eta\lambda(p, K)\bar{G}(\alpha(K)p)\left[\bar{F}_Y(K) + q(K)\right] = 0\,,$$

with $\bar{G} = 1 - G$, and where we have used the partial derivatives

$$\frac{\partial n(p, K)}{\partial p} = -Ng(\alpha(K)p)\alpha(K) < 0,$$

$$\frac{\partial \left(n(p, K)\lambda(p, K)\right)}{\partial p} = -N\alpha(K)^2 g(\alpha(K)p)p < 0,$$

$$\frac{\partial n(p, K)}{\partial K} = -Ng(\alpha(K)p)\alpha'(K)p < 0,$$

$$\frac{\partial \left(n(p, K)\lambda(p, K)\right)}{\partial K} = -N\alpha(K)\alpha'(K)g(\alpha(K)p)p^2 < 0.$$

(4.6)

Example 4.1 (Reservation premium using certainty equivalent). Following Asmussen et al. (2013), the reservation premium under risk aversion, $p_a(K) = \Lambda a(K)$ in (4.2), may be determined by assuming that the risk-adjusted expected net claim $a(K)$ is given by the certainty equivalent of the net claim $(Y - K)^+$, as opposed to the expected net claim (4.1). Recall that for a strictly increasing and strictly convex cost function φ, the certainty equivalent Z_φ of a random variable Z satisfies $\varphi(Z_\varphi) = \mathbb{E}\left[\varphi(Z)\right]$. By Jensen's inequality, $\mathbb{E}\left[\varphi(Z)\right] > \varphi(\mathbb{E}\left[Z\right])$, and thus $\varphi(Z_\varphi) > \varphi(\mathbb{E}\left[Z\right])$. As φ is increasing, we have $Z_\varphi > \mathbb{E}\left[Z\right]$. Thus, with this specification, $a(K) = (Y - K)_\varphi^+ > \mathbb{E}\left[(Y - K)^+\right]$, i.e., the risk adjustment increases the reservation premium beyond the net premium. In more detail,

$$a(K) = \varphi^{-1}\left(\mathbb{E}[\varphi((Y - K)^+)]\right) = \varphi^{-1}\left(\int_K^\infty \varphi(y - K)\,F(\mathrm{d}y) + \varphi(0)F_Y(K)\right),$$

and if φ is differentiable, with $\lim_{y\to\infty} \varphi(y)f_Y(y) = 0$, we obtain

$$\varphi'(a(K))a'(K) = \frac{d}{dK}\varphi(a(K)) = -\int_K^\infty \varphi'(y - K)f_Y(y)\,dy,$$

using implicit differentiation and Leibniz's rule. As $\varphi'(\cdot) > 0$, this verifies the maintained assumption $a'(K) < 0$ in the certainty equivalence case, and thus $\alpha'(K) > 0$, so the derivatives of portfolio size and aggregate claim rate with respect to premium and deductible are signed as in (4.6). A concrete specification is given by the quadratic cost function $\varphi(z) = \varphi_0 + \varphi_2 z^2$, with $\varphi_2 > 0$. In this case,

$$Z_\varphi = \left(\mathbb{E}\left[Z^2\right]\right)^{1/2} = \left(\left(\mathbb{E}\left[Z\right]\right)^2 + \mathrm{Var}\left[Z\right]\right)^{1/2} > \mathbb{E}\left[Z\right],$$

as long as Z is non-degenerate. The risk-adjusted expected net claim is

$$a(K) = \left(\mathbb{E}[(Y - K)^2 \mathbb{1}_{\{Y \geq K\}}]\right)^{1/2}$$
$$= \left(\int_K^\infty y^2\,F(\mathrm{d}y) - 2K\int_K^\infty y\,F(\mathrm{d}y) + K^2[1 - F_Y(K)]\right)^{1/2}, \quad (4.7)$$

and the customer's reservation premium is $p_a(K) = \Lambda a(K)$, cf. (4.2). Further, using (4.1),

$$\alpha'(K) = a(K)^{-3}\left[\int_K^\infty yF(\mathrm{d}y) - K\bar{F}_Y(K)\right]$$
$$= \alpha(K)^3 \mathbb{E}\left[(Y - K)^+\right], \quad (4.8)$$

i.e., proportional to the expected net claim. Asmussen et al. (2013) considered the case $K = 0$, no deductible. ∎

Example 4.2 (Reservation premium using principle of equivalent utility). As an alternative to the certainty equivalent method, risk aversion may be introduced into the customer's problem using the principle of equivalent utility. According to this, the customer's reservation premium equates the expected utilities of wealth with and without insurance (see Gerber and Pafum, 1998). In this case, assuming the customer may invest wealth in the risk-free asset at interest rate r, and has preferences over financial wealth of CARA type with constant absolute risk aversion $\zeta > 0$, Thøgersen (2016) approximates the reservation premium as

$$
\begin{aligned}
p_{\tilde{a}}(K) \approx \Lambda \tilde{a}(K) &= \Lambda \left(\mathbb{E}[(Y-K)^+] + \frac{r\zeta}{2} \mathbb{E}[(Y-K)^2 \mathbb{1}_{\{Y \geq K\}}] \right) \\
&= \Lambda \left(\mathbb{E}[(Y-K)^+] + \frac{r\zeta}{2} a(K)^2 \right). \qquad (4.9)
\end{aligned}
$$

Thus, the risk-adjusted expected net claim $a(K)$ from the certainty equivalence principle, (4.7), is replaced by $\tilde{a}(K)$. Instead of adjusting the expected net claim up, using the cost function φ as in the certainty equivalence case, a second-moment term is added to it, in analogy with the variance premium principle. In fact, the added term is proportional to the square of the certainty equivalent, $a(K)$, from Example 4.1. ∎

In further analysis below, we use the certainty equivalence approach from Example 4.1, along with assumptions on the distributions characterizing the size and frequency of claims, Y and Λ, to derive explicit formulae for portfolio size, claim arrival rate, and expected net claim valuation. A parallel analysis could be carried out based on the principle of equivalent utility from Example 4.2, or other approaches to risk aversion in the customer's problem.

4.1 Exponentially distributed arrival rate

A plausible assumption, e.g., in automobile or fire insurance (cf. Bühlmann and Gisler, 2005, and Denuit et al., 2007) is that Λ is exponentially distributed across customers, with density function $g(z) = \frac{1}{\lambda_0} e^{-\frac{z}{\lambda_0}}$, so that $\mathbb{E}[\Lambda] = \lambda_0$. Henceforth, we adopt this assumption. Thus, (4.3) and (4.4) become

$$
n(p, K) = N e^{-\frac{\alpha(K)}{\lambda_0} p}, \text{ and } \lambda(p, K) = \lambda_0 + \alpha(K)p. \qquad (4.10)
$$

The demand function is the negative exponential, with premium elasticity $\varepsilon(p, K) = \partial \log n(p, K) / \partial \log p = \alpha(K)p/\lambda_0$, and $\lambda(p, K) = \lambda_0(1 + \varepsilon(p, K))$. Taylor (1986) and Emms et al. (2007) considered the negative exponential demand function, as well, with elasticity proportional to $p/\bar{p}$, interpreting $\bar{p}$ as an average (or market) premium, and not accommodating adverse selection. In contrast, the arrival rate or average riskiness of customers $\lambda(p, K)$ in our analysis increases with the premium in an affine fashion, with slope $\alpha(K)$. Thus, the adverse selection adjustment of the arrival rate, relative to the standard model with constant rate λ_0, is given precisely by the term $\alpha(K)p$ in (4.10), or $\varepsilon(p, K)$ in the equivalent multiplicative representation. Further, (4.10) shows that the demand function and the arrival rate are tied in a subtle manner via $\alpha(K)$ in our analysis. In the demand function, $\bar{p}$ is replaced by $\lambda_0/\alpha(K) = \mathbb{E}[\lambda_0]a(K) = \mathbb{E}[p_a(K)]$, the expected reservation premium for a risk

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averse customer, cf. (4.2). However, this cannot be interpreted as an average market premium. We show next that the optimizing company charges more, due to its own risk aversion and adverse selection.

Let $K^\star := \sup \{K \geq 0 : \alpha(K)q(K) \geq r\eta\}$, with $q(K)$ the expected net claim valuation function from (3.5), and define

$$m(K) := -\frac{\alpha'(K)}{\alpha(K)^2}r\eta + [q(K) + \bar{F}_Y(K)][2r\eta - q(K)\alpha(K)]. \qquad (4.11)$$

We have the following result.

Theorem 4.1. *Assume that $b = 0$, and that $\alpha(K)$ and $q(K)$ satisfy*

$$q(K), \ \alpha(K)q(K) \to 0 \ \ as \ K \to \infty. \qquad (4.12)$$

If there exists a unique positive $\hat{K} \geq K^\star$ such that $m(\hat{K}) = 0$, with $m(K) < 0$ for $K > \hat{K}$, then the pair $(\hat{p}, \hat{K})$ is optimal, with

$$\hat{p} = \frac{r\eta\lambda_0}{\alpha(\hat{K})[r\eta - q(\hat{K})\alpha(\hat{K})]}. \qquad (4.13)$$

Proof. Under the assumptions on Λ and b, the profile Hamiltonian (4.5) reduces to

$$Q(p, K) = \frac{1}{2}\|\psi\|^2 + Ne^{-\frac{\alpha(K)p}{\lambda_0}}\left([r\eta - \alpha(K)q(K)]p - \lambda_0 q(K)\right). \qquad (4.14)$$

For some $K > 0$ sufficiently large, $Q(p, K)$ is strictly positive for all $p \geq 0$, so if $\hat{K}$ maximizes $Q(p, K)$, then $r\eta > \alpha(\hat{K})q(\hat{K})$. Thus, it suffices to maximize $Q(p, K)$ over $p \geq 0$ and $K \geq 0$ satisfying

$$r\eta > \alpha(K)q(K). \qquad (4.15)$$

Differentiating with respect to p, we get

$$\frac{\partial Q(p, K)}{\partial p} = Ne^{-\frac{\alpha(K)p}{\lambda_0}}\kappa(p, K),$$

with

$$\kappa(p, K) := r\eta - \frac{\alpha(K)}{\lambda_0}[r\eta - \alpha(K)q(K)]p.$$

Therefore, given $K > 0$ satisfying (4.15), the premium rate

$$p(K) = \frac{r\eta\lambda_0}{\alpha(K)[r\eta - q(K)\alpha(K)]}$$

satisfies the first-order condition $\partial Q(p, K)/\partial p = 0$. Since $\partial Q(0, K)/\partial p = Nr\eta > 0$, and $\kappa(\cdot, K)$ decreases strictly to $-\infty$ as $p \to +\infty$, $Q(\cdot, K)$ is strictly increasing on $[0, p(K)]$ and decreasing on $[p(K), +\infty)$. This implies that the critical point $p(K)$ in fact maximizes $Q(\cdot, K)$, so the profile criterion for the remaining variable K is $Q(p(K), K)$. Differentiating this yields

$$\frac{\partial Q(p(K), K)}{\partial K} = N\exp\left(-\frac{r\eta}{r\eta - \alpha(K)q(K)}\right)\frac{r\eta}{r\eta - \alpha(K)q(K)}m(K),$$

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i.e., $m(\hat{K}) = 0$ implies that $\hat{K}$ is a critical point, so $m(\cdot)$ is an essential profile gradient, in this sense. From the definition of K^* and the assumptions on $\hat{K}$, it follows that $\partial Q(p(K), K)/\partial K$ is positive on the interval $(K^*, \hat{K})$ and negative for $K > \hat{K}$. Hence, $\hat{K}$ maximizes $Q(p(K), K)$, and the desired result follows from Proposition 3.1. ∎

Thus, given an optimal deductible $\hat{K}$, the optimal premium $\hat{p}$ in this case admits the closed form representation (4.13), and the optimal deductible the semi-closed form characterization $m(\hat{K}) = 0$. Optimal consumption (dividend) and investment follow from (3.8) and (3.9), respectively, using Proposition 3.1.

Using that $\alpha(K) = a(K)^{-1}$, it is possible to interpret the optimal premium (4.13) in more detail. Thus, since (4.15) applies at the optimum, we have

$$\hat{p} = \frac{1}{1 - q(\hat{K})/[r\eta a(\hat{K})]} \lambda_0 a(\hat{K}) > \lambda_0 a(\hat{K}) > \lambda_0 \mathbb{E}\left[\left(Y - \hat{K}\right)^+\right]. \qquad (4.16)$$

The last expression in (4.16) is the net premium, without safety loading, evaluated at the optimal deductible, $\hat{K}$, as selected by the company. It coincides with $\mathbb{E}[p_0(\hat{K})]$, the expected reservation premium for a risk-neutral customer. The last inequality in (4.16) reflects the fact that the expected reservation premium for a risk averse customer, $\mathbb{E}[p_a(\hat{K})] = \lambda_0 a(\hat{K})$, is higher. This is the factor replacing the average or market premium $\bar{p}$ from Taylor (1986) and Emms et al. (2007) in our demand function (4.10). Nevertheless, the optimal premium $\hat{p}$ in our setting is even higher, as reflected in the first inequality in (4.16). The expected reservation premium for a risk averse customer is scaled up further by a factor exceeding unity, by (4.15). This additional adjustment captures the feature that beside the customers, the company (owner-manager) is risk averse, too, and accounts for customer behavior in the optimization. Thus, the scaling factor involves company risk aversion, η, and, in particular, the relative valuation by company and customers of net claims, $q(\hat{K})/a(\hat{K})$. If customers are very risk averse, so that the denominator dominates, and $q(\hat{K})/[r\eta a(\hat{K})]$ is close to zero, then the scaling factor is near unity, and (4.16) shows that the company sets a premium basically determined by the very risk averse customers' expected reservation premium. In contrast, if the company is relatively more risk averse, and hence more concerned about future net claims, reflected in a high value of $q(\hat{K})$ from (3.5), then the scaling factor can be larger. In this case, the company exerts its market power to charge more than customers' expected reservation premium, thus protecting itself from the riskier customers, i.e., the adverse selection aspect.

Both (4.11) and (4.13) involve the functions $\alpha(K)$ and $q(K)$, which in turn depend on the claim size distribution. We consider illustrative examples of this.

Example 4.3 (Gamma distributed claim size). Consider the case that the random claim Y_m follows a gamma distribution with density

$$F(\mathrm{d}y) = \frac{\gamma^\theta}{\Gamma(\theta)} y^{\theta-1} e^{-\gamma y} \,\mathrm{d}y \,,$$

with shape parameter $\theta > 0$, (inverse) scale parameter γ, and where $\Gamma(x)$ is the gamma function. Hence, $\mathbb{E}[Y_m] = \theta/\gamma$ and $\mathrm{Var}[Y_m] = \theta/\gamma^2$. We subsequently as-

sume $\gamma > r\eta$. In this case, the truncated moments are

$$\int_K^\infty e^{r\eta y} F(\mathrm{d}y) = \left(\frac{\gamma}{\gamma - r\eta}\right)^\theta \frac{\Gamma(\theta, (\gamma - r\eta)K)}{\Gamma(\theta)}, \tag{4.17}$$

$$\int_K^\infty y^n F(\mathrm{d}y) = \frac{\Gamma(n + \theta, \gamma K)}{\gamma^n \Gamma(\theta)}, \tag{4.18}$$

with $\Gamma(s, x)$ the upper incomplete gamma function. Using (4.18) with $n = 1$ and 0 in (4.1) yields the expected net claim

$$\mathbb{E}\left[(Y - K)^+\right] = \frac{\Gamma(1 + \theta, \gamma K)}{\gamma \Gamma(\theta)} - K \frac{\Gamma(\theta, \gamma K)}{\Gamma(\theta)}.$$

By the recurrence relation $\Gamma(s + 1, x) = s\Gamma(s, x) + x^s e^{-x}$, this is compactly recast as

$$\mathbb{E}\left[(Y - K)^+\right] = \frac{1}{\gamma \Gamma(\theta)} \left[\Gamma(\theta, \gamma K)(\theta - \gamma K) + (\gamma K)^\theta e^{-\gamma K}\right]. \tag{4.19}$$

Using the certainty equivalent based on the quadratic cost function φ from Example 4.1 for risk adjustment, substitution of (4.18) with $n = 2, 1, 0$ into (4.7) allows calculating the customer's risk-adjusted expected net claim explicitly,

$$a(K) = \left\{\frac{\Gamma(2 + \theta, \gamma K)}{\gamma^2 \Gamma(\theta)} - 2K \frac{\Gamma(1 + \theta, \gamma K)}{\gamma \Gamma(\theta)} + K^2 \frac{\Gamma(\theta, \gamma K)}{\Gamma(\theta)}\right\}^{1/2}. \tag{4.20}$$

By one more iteration in the recurrence relation, $\alpha(K) = a(K)^{-1}$ satisfies

$$\alpha(K) = \left\{\frac{1}{\gamma^2 \Gamma(\theta)} \left[\Gamma(\theta, \gamma K)[(\theta - \gamma K)^2 + \theta] + e^{-\gamma K}(\gamma K)^\theta[1 + \theta - \gamma K]\right]\right\}^{-1/2}. \tag{4.21}$$

Thus, the risk adjustment amounts to the difference between $a(K)$, i.e., the reciprocal of (4.21), and (4.19). For the derivative of $\alpha(K)$, combination of (4.8) and (4.19) yields

$$\alpha'(K) = \frac{\alpha(K)^3}{\gamma \Gamma(\theta)} \left[\Gamma(\theta, \gamma K)(\theta - \gamma K) + (\gamma K)^\theta e^{-\gamma K}\right] > 0,$$

and $a'(K) < 0$, reconfirming the maintained assumption. Moreover, substituting (4.17) into (3.5), the company's expected net claim valuation is given explicitly as

$$q(K) = e^{-r\eta K} \left(\frac{\gamma}{\gamma - r\eta}\right)^\theta \frac{\Gamma(\theta, (\gamma - r\eta)K)}{\Gamma(\theta)} - \frac{\Gamma(\theta, \gamma K)}{\Gamma(\theta)}. \tag{4.22}$$

With $\alpha(K)$ and $q(K)$ from (4.21) and (4.22) in hand, substitution in (4.11) yields a concrete characterization $m(\hat{K}) = 0$ of the optimal deductible $\hat{K}$, and in (4.13) a closed-form solution for $\hat{p}$, as function of $\hat{K}$. ∎

A number of numerical illustrations are provided in this and the following sections, using parameter values chosen with a time unit of one year in mind. Figure 1 sheds light on the behavior of the functions $Q(p, K)$ and $m(K)$ in Theorem 4.1, i.e., the profile Hamiltonian (4.14) and essential gradient (4.11), for gamma distributed claims, as in Example 4.3. Two cases are considered, with common mean claim, $\mathbb{E}[Y] = \theta/\gamma =$
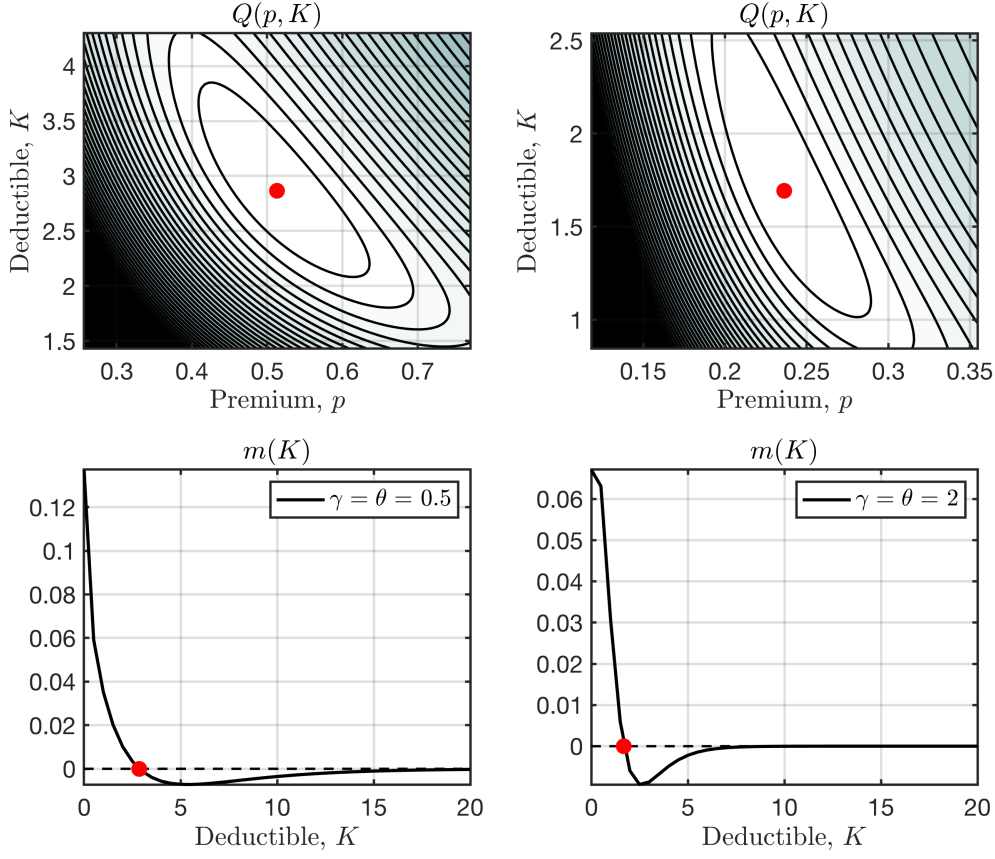
Figure 1. Theorem 4.1 for gamma distributed claims. The figure shows the contours of the profile Hamiltonian $Q(p, K)$ and the essential gradient $m(K)$ for arrival rate of claims $\Lambda \sim \text{Exp}(0.5)$. Left exhibits are for $Y \sim \text{Gamma}(0.5, 0.5)$, and right exhibits for $Y \sim \text{Gamma}(2, 2)$. A dot indicates the critical point $(\hat{p}, \hat{K}) = (p(\hat{K}), \hat{K})$, at which $m(\hat{K}) = 0$. The remaining parameters are $(r, \eta, \lambda_0, N) = (0.05, 3, 0.5, 100)$.

1, but different claim size dispersions. Left exhibits are for $Y \sim \text{Gamma}(0.5, 0.5)$, which implies $\text{Var}[Y] = \theta/\gamma^2 = 2$, and right exhibits for $Y \sim \text{Gamma}(2, 2)$, so $\text{Var}[Y] = 0.5$. In addition to θ and γ, the functions Q and m involve the interest rate, r, set at 5%, and company risk aversion, η, set at 3. Further, Q involves λ_0, set at 0.5, i.e., two years between casualties at mean frequency, and N, normalized at 100, so that $n(p, K)$ represents the percentage of customers insuring. The parameter values correspond to those used in the sensitivity analysis in Section 4.3. For these values, conditions (4.12) and (4.15) are satisfied, with $\alpha(K)$ from (4.21) and $q(K)$ from (4.22). Further, $K^\star = 0.22$ in the low claim size dispersion case, $K^\star = -\infty$ in the high. Although Q in (4.14) involves the market prices of risk, $\boldsymbol{\psi}$, these affect neither the choice of p and K, nor the contours in Figure 1. A dot in the figure indicates the critical point $(\hat{p}, \hat{K}) = (p(\hat{K}), \hat{K})$ maximizing $Q(p, K)$, hence implying $m(\hat{K}) = 0$. The optimal premium rate and deductible are $(\hat{p}, \hat{K}) = (0.51, 2.87)$ in the high claim dispersion case, and $(0.23, 1.69)$ in the low. Thus, greater uncertainty about size of future claims leads the insurance company to increase both premium and deductible.

From the figure, Q is well behaved and, in particular, m has a unique zero. The orientation of the axes of the approximately elliptic contours indicates that p and K are negatively related around the optimum, i.e., lower deductible implies more insurance and hence higher premium. The risk-adjusted expected net claim $a(\hat{K})$ based on (4.21) is 0.74 in the high claim dispersion case, and 0.33 in the low. This is compared to expected net claims of 0.23 and 0.09, based on (4.19), i.e., the risk adjustment amounts to 122% and 267% in the high and low claim dispersion cases, respectively. For comparison, the alternative $\tilde{a}(\cdot)$ specification in (4.9), based on the principle of equivalent utility, combines the already calculated expected net claims with and without risk adjustment, amounting to 0.27 and 0.10 in the high and low dispersion cases, i.e., much less risk adjustment, assuming the same CARA coefficient for customers as for the company, $\zeta = \eta = 3$. The expected reservation premiums for a risk-neutral customer are 0.12 and 0.05 for the two claim dispersion cases, compared to 0.37 and 0.17 if accounting for customer risk aversion, henceforth focussing on the certainty equivalence case, with $a(\hat{K})$ based on (4.21). Thus, the latter premiums, which replace $\bar{p}$ from the literature in the demand function, are higher than the premiums under risk neutrality and no safety loading, but lower than the optimal $\hat{p}$, which in addition accounts for risk aversion of the company, by 38% and 35% for the two dispersion levels. Adjustments for both company and customer risk aversion (first and second inequality in (4.16)) constitute large portions of the premium, with the latter adjustment largest in the example.

Company net claim valuations $q(\hat{K})$ are 0.031 and 0.014 in the two dispersion cases, and although clearly measured on a different scale than the customer valuations $a(\hat{K})$ of 0.74 and 0.33 (i.e., utility rather than monetary units), the ratios $q(\hat{K})/a(\hat{K})$ matter for optimal pricing, and (4.15) is satisfied in each case. Substituting $a(\hat{K})$ and $\hat{p}$ in the demand function (4.10) shows that 25% and 24% of potential customers insure in the two cases, i.e., in the example, the company adjusts the contract offered so as to maintain a roughly constant portfolio size, regardless of customer risk. The relatively modest share of potential customers purchasing insurance is indicative of the market power exercised by the optimizing company.

The calculations concern the unperturbed case, $b = 0$, as in Theorem 4.1, i.e., insurance controls $(\hat{p}, \hat{K})$ are chosen separately from risky asset investments. Nevertheless, $(\hat{p}, \hat{K})$ matter for optimal consumption (dividend payout) in (3.8) through the optimized $Q(\hat{p}, \hat{K})$ in $\hat{\beta}$ from Proposition 3.1, and the latter further involves the company's discount rate, δ. To illustrate, set $\delta = 5\%$. Further, $Q(\hat{p}, \hat{K})$ includes ψ, and we consider for simplicity $I = 1$ risky asset, with expected return $\mu = 8\%$ and volatility $\sigma = 20\%$. In this case, the market price of risk is 0.15, and optimal investment in the risky asset (3.9) is 5.0. Substitution of $Q(\hat{p}, \hat{K})$ into Proposition 3.1 yields $\hat{\beta} = -1.85 \times 10^{-8}$ and -0.0015 in the high and low dispersion cases. If, for example, initial wealth is $w_0 = 100$, then the optimal consumption path (3.8) starts out at 11.57 in the high and 7.79 in the low claim size dispersion case. The computations illustrate the general point, namely, we provide a unifying framework, with optimal investment, premium, deductible, and payout determined jointly.

Figure 2 illustrates the impact of insurance risk perturbation b, using numerical analysis of $Q(p, K)$ from (3.10) to go beyond Theorem 4.1. Focusing on the high claim size dispersion case, $\text{Var}[Y] = 2$, with the remaining parameters as before (except that δ does not enter Q), the optimal premium and deductible are
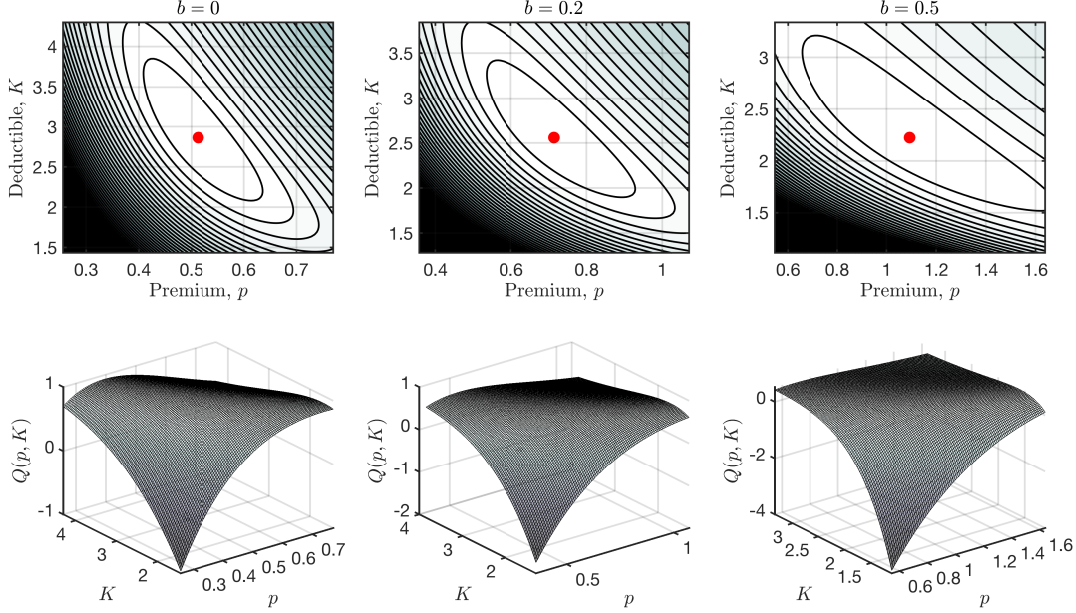
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Figure 2. The $Q(p, K)$ function. The figure shows the function $Q(p, K)$ and its contours for different degrees of insurance risk perturbation b. The arrival rate of claims is $\Lambda \sim \text{Exp}(0.5)$, and claim sizes $Y \sim \text{Gamma}(0.5, 0.5)$. A dot indicates the critical point $(\hat{p}, \hat{K})$ maximizing $Q(p, K)$. The remaining parameters are $(r, \eta, I, \mu, \sigma, \rho, \lambda_0, N) = (0.05, 3, 1, 0.08, 0.2, -0.5, 0.5, 100)$.

$(\hat{p}, \hat{K}) = (0.51, 2.87)$ for $b = 0$ (as in the previous figure), $(0.71, 2.56)$ for $b = 0.2$, and $(1.09, 2.22)$ for $b = 0.5$. Here, the results for $b > 0$ require a value for the insurance-finance correlation, and we set $\rho = -0.5$, thus illustrating a systemic risk or contagion case, i.e., insurance risk and financial market losses tend to coincide. The results in the figure show that in case of increased perturbation, i.e., higher b, the company optimally uses the financial markets to hedge the portion of the insurance risk that does not depend on the deductible K, so K can be lowered (note that axes differ), and the premium increased, reflecting both the better insurance product (lower deductible) and the greater uncertainty faced by the company regarding its insurance business. Numerical results (not in the figure) show that this response pattern applies for all non-zero $\rho \in (-1, 1)$. Further, optimal risky investment (3.9) indicates short-selling, at -3.41 for $b = 0.2$ and -5.51 for $b = 0.5$, in spite of the long myopic or speculative position 5.0, shared with the unperturbed case. Thus, hedging demand is negative, at -10.51 and -8.41 in the high and low contagion cases, reflecting the negative insurance-finance correlation. As before, the optimal consumption path could similarly be characterized, for given values of δ and w_0. In effect, the calculations illustrate the general Proposition (3.1), and the interaction between investment, premium, and deductible in optimum.

Example 4.4. (Exponentially distributed claims). For $\theta = 1$ in Example 4.3, the

truncated moments of the claim size distribution reduce to

$$\int_K^\infty e^{r\eta y} F(\mathrm{d}y) = \frac{\gamma e^{-(\gamma - r\eta)K}}{\gamma - r\eta} \,, \tag{4.23}$$

$$\int_K^\infty y^n F(\mathrm{d}y) = \frac{n!}{\gamma^n} e^{-\gamma K} \sum_{j=0}^n \frac{(\gamma K)^j}{j!} \,. \tag{4.24}$$

Substituting (4.24) for $n = 1, 0$ in (4.1) yields the expected net claim $\mathbb{E}\left[(Y - K)^+\right] = e^{-\gamma K}/\gamma$. Using the certainty equivalent based on the quadratic cost function φ from Example 4.1, the customer's risk-adjusted expected net claim $a(K)$, the reciprocal $\alpha(K) = a(K)^{-1}$, and its derivative can be found from

$$\alpha(K) = \frac{\gamma}{\sqrt{2}} e^{\frac{\gamma}{2}K} \,, \text{ and } \alpha'(K) = \frac{\gamma^2}{2\sqrt{2}} e^{\frac{\gamma}{2}K} \,. \tag{4.25}$$

From (4.23) and (3.5), the company's expected net claim valuation satisfies

$$q(K) + \bar{F}_Y(K) = e^{-r\eta K} \int_K^\infty e^{r\eta y} F(\mathrm{d}y) = \frac{\gamma e^{-\gamma K}}{\gamma - r\eta} \,,$$

which together with (4.25) implies that

$$q(K)\alpha(K) = \left[\frac{\gamma e^{-\gamma K}}{\gamma - r\eta} - e^{-\gamma K}\right] \frac{\gamma}{\sqrt{2}} e^{\frac{\gamma}{2}K} = r\eta \frac{\gamma e^{-\frac{\gamma}{2}K}}{\sqrt{2}(\gamma - r\eta)} \longrightarrow 0 \text{ as } K \to \infty,$$

and hence condition (4.12) holds. Inserting these calculations in (4.11) yields the essential gradient

$$m(K) = r\eta e^{-\frac{3\gamma}{2}K} \left\{ -\frac{1}{\sqrt{2}} e^{\gamma K} + \frac{2\gamma}{\gamma - r\eta} e^{\frac{\gamma}{2}K} - \frac{\gamma^2}{\sqrt{2}(\gamma - r\eta)^2} \right\} \,.$$

Using the change of variables $k = e^{\frac{\gamma}{2}K}$, the condition $m(K) = 0$ is equivalent to the quadratic equation $B_2 k^2 + B_1 k + B_0 = 0$, with coefficients

$$B_2 = -\frac{1}{\sqrt{2}} \,, \quad B_1 = \frac{2}{1 - r\eta/\gamma} \,, \quad B_0 = -\frac{1}{\sqrt{2}(1 - r\eta/\gamma)^2} \,,$$

and the two real-valued roots

$$k_{(\pm)} = \frac{1}{1 - r\eta/\gamma} (\sqrt{2} \pm 1) \,.$$

It follows that $k_{(+)} > 1$, and thus $K_{(+)} = 2\ln k_{(+)}/\gamma > 0$. Moreover, in this case we have

$$K^* = \begin{cases} 0 \,, & \text{if } \gamma > r\eta/(1 - 1/\sqrt{2}), \\ \frac{2}{\gamma} \ln\left[\frac{1}{\sqrt{2}(1 - r\eta/\gamma)}\right] \,, & \text{otherwise,} \end{cases}$$

which is only strictly less than $K_{(+)}$. Since the parabola $B_2 k^2 + B_1 k + B_0$ opens downward, the critical point $K_{(+)}$ is the optimal deductible, as it satisfies the conditions of Theorem 4.1. ∎

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For the exponential distribution case from Example 4.4, we have the following sharpening of Theorem 4.1.

Corollary 4.1. *Assume that the arrival rate Λ and size Y_m of claims have exponential distributions with densities $g(z) = \frac{1}{\lambda_0} e^{-\frac{z}{\lambda_0}}$ and $f_Y(y) = \gamma e^{-\gamma y}$, with $\gamma > r\eta$, and that customers' risk-adjusted expected net claim is given by the certainty equivalent $(y - K)_\varphi^+$ based on the quadratic loss function $\varphi(y) = \varphi_0 + \varphi_2 y^2$. Then, the deductible*

$$\hat{K} = \frac{2}{\gamma} \ln \left[\frac{1 + \sqrt{2}}{1 - r\eta/\gamma} \right]$$

and the premium rate

$$\hat{p} = \lambda_0 \frac{2(1 - r\eta/\gamma)}{\gamma(1 + \sqrt{2})}$$

are optimal.

Proof. Inserting

$$\alpha(K_{(+)}) = \frac{\gamma}{\sqrt{2}} k_{(+)} = \frac{\gamma(1 + \sqrt{2})}{\sqrt{2}(1 - r\eta/\gamma)}$$

and

$$r\eta - \alpha(K_{(+)})q(K_{(+)}) = r\eta \frac{1 + \sqrt{2}}{2 + \sqrt{2}}$$

into (4.13) yields the optimal premium rate. The desired result follows from Theorem 4.1. ∎

The corollary illustrates that our approach can deliver closed-form solutions for both optimal premium and optimal deductible, and thus for optimal investments and consumption (dividends), too, using Proposition 3.1. The analytical results make the nature of the optimal policy very transparent. By the corollary,

$$\hat{p} = \frac{\lambda_0}{\gamma/2} e^{-\frac{\gamma}{2} \hat{K}}, \tag{4.26}$$

showing that

$$\frac{\partial \hat{p}}{\partial \hat{K}} = -\lambda_0 e^{-\frac{\gamma}{2} \hat{K}} = -\frac{\gamma}{2} \hat{p} < 0, \tag{4.27}$$

i.e., premium and deductible are inversely related at the optimum, with a higher deductible (less insurance) commanding a lower premium. This analytical result is consistent with the indications from the contour diagrams in Figures 1 and 2 for more general cases.

The explicit expressions in the present case allow an analytical investigation of parameter dependence. First, while the optimal deductible $\hat{K}$ depends on the company's interest rate r and risk aversion η, as well as the parameter γ governing the claim size distribution, which is natural, it does not depend on the mean claim frequency λ_0. In contrast, the optimal premium $\hat{p}$ depends on all four parameters, $(r, \eta, \lambda_0, \gamma)$. Thus, the company sets the deductible, in essence a truncation of the claim distribution, as a function of the latter, and of company parameters (r, η), but not of the frequency of

claims, which is instead compensated for via the optimal premium, in a proportional fashion. Therefore, we have

$$\frac{\partial \hat{K}}{\partial \lambda_0} = 0\,, \text{ and } \frac{\partial \hat{p}}{\partial \lambda_0} = \frac{\hat{p}}{\lambda_0} > 0\,, \tag{4.28}$$

i.e., more accidents imply higher premium, not higher deductible.

Secondly, the optimal premium only depends on the company parameters (r, η) through their effect on the optimal deductible, as evident from (4.26). Defining the parameter function $\phi = 2/\left[\gamma(\gamma - r\eta)\right] > 0$, we have

$$\frac{\partial \hat{K}}{\partial r} = \eta\phi > 0\,, \text{ and } \frac{\partial \hat{K}}{\partial \eta} = r\phi > 0\,, \tag{4.29}$$

i.e., the company protects itself through higher deductible in case of higher risk aversion or borrowing cost. Using (4.27), (4.29), and the chain rule, we immediately get

$$\frac{\partial \hat{p}}{\partial r} = -\frac{\gamma}{2}\hat{p}\eta\phi < 0\,, \text{ and } \frac{\partial \hat{K}}{\partial \eta} = -\frac{\gamma}{2}\hat{p}r\phi < 0\,, \tag{4.30}$$

of opposite sign, compared to (4.29), reflecting the negative trade-off between premium and deductible. Further, by direct differentiation of $\hat{K}$ in Corollary 4.1,

$$\frac{\partial \hat{K}}{\partial \gamma} = -\frac{1}{\gamma}(\hat{K} + r\eta\phi) < 0\,, \tag{4.31}$$

i.e., higher γ implies lower mean $1/\gamma$ and variance $1/\gamma^2$ of claim sizes, and thus lower deductible.

Finally, as expected, the effect of the claim size distribution on the optimal premium is more complicated, since the reduced exposure (lower mean and dispersion) points to lower premium, and the reduced deductible (4.31) to higher. Specifically, in (4.26), it is necessary to account both for the direct effect of γ on $\hat{p}$, and for the indirect effect of γ via $\hat{K}$. The combined result is

$$\frac{\partial \hat{p}}{\partial \gamma} = \lambda_0 \frac{2(1 - 2r\eta/\gamma)}{\gamma^2(1 + \sqrt{2})}\,, \tag{4.32}$$

and although the maintained assumption $\gamma > r\eta$ guarantees a positive premium $\hat{p}$ in Corollary 4.1, this does not necessarily imply $\gamma > 2r\eta$. Consequently, as the only one of the eight relevant parameter derivatives of $(\hat{p}, \hat{K})$, the sensitivity of the premium with respect to the claim size distribution in (4.32) can take on either sign, depending on parameter values. If $\gamma > 2r\eta$, e.g., for sufficiently low risk aversion η, the company must compensate for the reduced deductible (better insurance offered) in (4.31) through an increased premium, reflecting the negative premium-deductible trade-off in (4.27). However, if $r\eta < \gamma < 2r\eta$, e.g., for high risk aversion, the company values the improvement in claim distribution (reduction in mean and dispersion) so much that it lowers premium and deductible in tandem.

As optimal investment and consumption are given in closed form, too, their dependence on parameters (the same four, as well as the discount rate δ and parameters of the investment opportunity set) can be investigated along similar lines, although this is left out here, for space considerations.

4.2 Fixed deductible

Our framework accommodates the special case of a fixed deductible, taken as given and not controlled by the company. The explicit sufficient conditions for the existence of an optimal premium rate are summarized in the following result, for arbitrary degree of insurance risk perturbation, $b \geq 0$.

Theorem 4.2. *Let the deductible be fixed at $K = \overline{K}$. Suppose that $\alpha(\overline{K})$ and $q(\overline{K})$ satisfy $r\eta > \alpha(\overline{K})q(\overline{K})$, and that*

$$1 + \frac{b\alpha(\overline{K})}{\lambda_0} \left[r\eta bN(1 - \|\boldsymbol{\rho}\|^2) - \boldsymbol{\rho}^\top \boldsymbol{\psi} \right] > 0. \tag{4.33}$$

Then there exists a unique $\hat{p} > 0$ for which $L(\hat{p}) = 0$, with

$$L(p) := r\eta \left[1 - \frac{b\alpha(\overline{K})}{\lambda_0} \boldsymbol{\rho}^\top \boldsymbol{\psi} \right] - \frac{\alpha(\overline{K})}{\lambda_0} (r\eta - \alpha(\overline{K})q(\overline{K}))p$$

$$+ \frac{\alpha(\overline{K})}{\lambda_0} (r\eta b)^2 (1 - \|\boldsymbol{\rho}\|^2) N e^{-\frac{\alpha(\overline{K})}{\lambda_0} p},$$

and this premium rate $\hat{p}$ is optimal.

Proof. Consider $Q(p, K)$ in (3.10) as a function of p, only, and note that $Q'(p, \overline{K}) = n(p)L(p)$. Condition (4.33) implies that $L(0) > 0$, and $L(p)$ decreases strictly to $-\infty$ as $p \to +\infty$. Thus, there exists a unique $\hat{p}$ such that $L(\hat{p}) = 0$, which is also a critical point of $Q(p, \overline{K})$, and it is in fact optimal, since $L(p)$ (and hence $Q'(p, \overline{K})$) is positive if and only if $p < \hat{p}$. ∎

Remark 4.1. In the unperturbed case, $b = 0$, and under the assumptions of Theorem 4.2, the optimal premium rate is given by (4.13) with $\hat{K}$ replaced by $\overline{K}$. In case of gamma distributed claims, as in Example 4.3, this provides a closed-form solution for the optimal premium rate, using (4.21) for $\alpha(\cdot)$ and (4.22) for $q(\cdot)$. This complements the result in Thøgersen (2016) on existence of an optimum premium rate for given fixed deductible under minimization of ruin probability, but with no closed-form solution for the premium. ∎

Figure 3 illustrates the properties of the essential gradient $L(p)$ (i.e., $L(p) = 0$ implies $Q'(p, \overline{K}) = 0$) from Theorem 4.2 for different degrees of insurance risk perturbation b and different values of the fixed deductible $\overline{K}$. Claim sizes follow the high dispersion case $Y \sim \text{Gamma}(0.5, 0.5)$ considered in the previous numerical examples, and the values of the remaining parameters follow those examples, too. A dot indicates the optimal premium $\hat{p}$, satisfying $L(\hat{p}) = 0$. From the figure, a lower deductible implies a better insurance contract, hence more customers and higher premium, a demand side effect. Stronger perturbation implies greater uncertainty about the reserve process, and the insurance company protects itself by charging a higher premium, i.e., a supply side effect.

4.3 Sensitivity analysis

Here, we consider the systemic risk or contagion case in more detail. We provide a numerical illustration of the impact of the correlation between the financial and
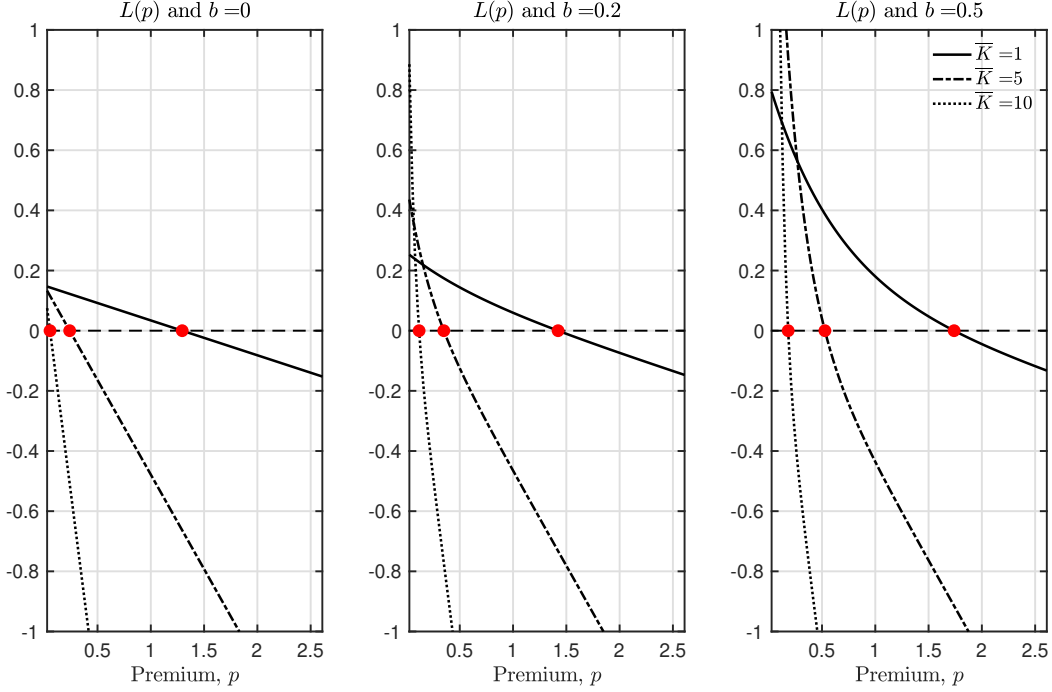
Figure 3. The $L(p)$ function. The figure shows the essential gradient $L(p)$ for different degrees of insurance risk perturbation, b, and different values of the fixed deductible, $\overline{K}$. A dot indicates the optimal premium $\hat{p}$, i.e., the critical point satisfying $L(\hat{p}) = 0$. The arrival rate of claims is $\Lambda \sim \text{Exp}(0.5)$, and claim sizes $Y \sim \text{Gamma}(0.5, 0.5)$. The remaining parameters are $(r, \eta, I, \mu, \sigma, \rho, \lambda_0, N) = (0.05, 3, 1, 0.08, 0.2, -0.5, 0.5, 100)$.

insurance sides of the business, ρ, and the degree of perturbation to the reserve, b, on the optimal premium rate, deductible, asset portfolio allocation, and consumption (dividend) rate. We focus on the high claim size dispersion case from the previous numerical examples, and set the remaining parameters as in those.

Figure 4 shows the results. Naturally, in the unperturbed case, $b = 0$, none of the four controls depends on ρ, as indicated by the solid horizontal line in each exhibit. Next, the negative relation between optimal premium and deductible is confirmed. For given ρ, stronger insurance risk perturbation, b, implies greater uncertainty about a portion of the reserve that is unrelated to the deductible and can be hedged, so the deductible can be lowered, and the premium increased, reflecting both the lower deductible (more insurance) and greater uncertainty. The magnitude of the impact of such an increase in perturbation on the optimal premium and deductible is reduced as the hedging opportunities offered by the financial markets increase, i.e., as ρ increases in magnitude, hence inducing an (inverted) U-shape of optimal (p) K as function of ρ, for given b. The financial hedge is established through a lower position in the risky asset (selling short, if necessary) for $\rho < 0$, i.e., allocating more of the financial wealth to the money market account in this insurance-finance contagion case, and a higher position for $\rho > 0$ (borrowing in the money market, if necessary). Thus, risky asset investments are monotonic in ρ, and these hedging demands are more pronounced (steeper in ρ) for stronger perturbation, b. At $\rho = 0$, hedging demands vanish, regardless the value of b, and investments reduce to the myopic or speculative demand $\hat{\mathbf{A}} = 5.0$, cf. (3.9), so the curves corresponding to different b intersect at this level for $\rho = 0$.
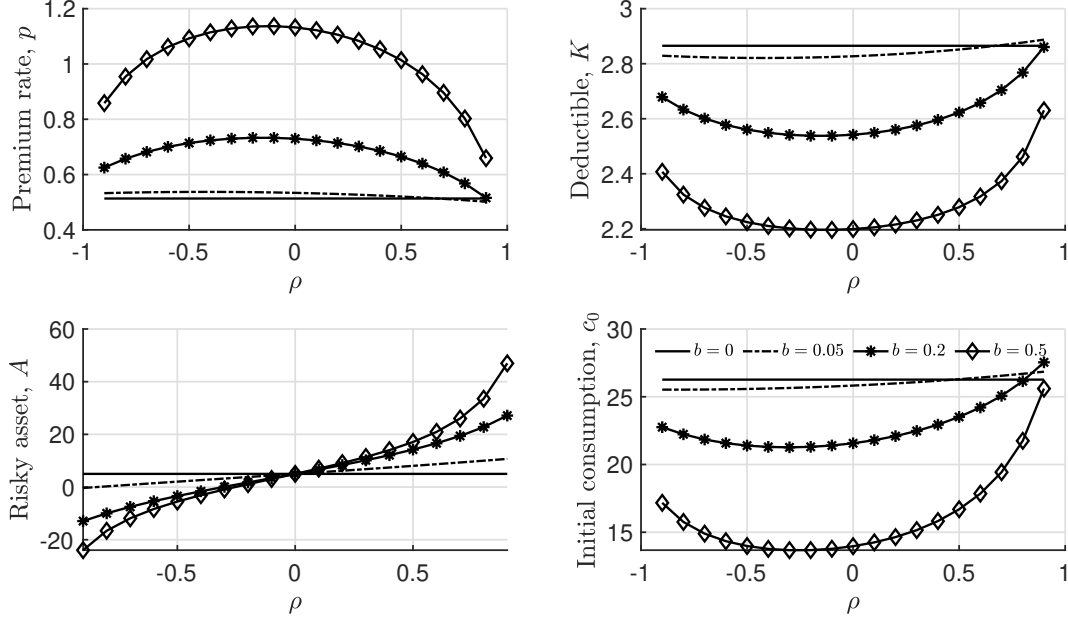
Figure 4. Optimal policies. The figure shows the optimal premium rate, p, deductible, K, risky asset investment, A, and initial consumption, c_0, for different values of the insurance-finance correlation, ρ, and degree of insurance risk perturbation, b. The arrival rate of claims is $\Lambda \sim \text{Exp}(0.5)$, and claim sizes $Y \sim \text{Gamma}(0.5, 0.5)$. The remaining parameters are $(r, \delta, \eta, I, \mu, \sigma, \lambda_0, N) = (0.05, 0.05, 3, 1, 0.08, 0.2, 0.5, 100)$.

Finally, as expected, the response of optimal consumption to variation in (ρ, b) is more complicated. The figure shows the initial consumption rate, $c_0 = \hat{c}(w_0)$. For moderate to strong perturbation ($b \geq 0.2$ in the figure), initial consumption is minimal at a certain level of the insurance-finance correlation, $\rho_c(b)$, say, around -0.25 in the figure, corresponding well with the negative systemic risk or contagion scenario for ρ considered in Figures 1 and 2. As ρ increases beyond $\rho_c(b)$, and the company invests larger amounts in the risky asset (reduces the short positions used to hedge insurance risk), consumption increases, because $\mu > r$, i.e., the equity premium is positive, and the investment strategy now earns a higher expected return (cf. Merton, 1971). For given ρ, consumption is lower for stronger perturbation b, at least for b above a certain level, $b_c(\rho)$, say (e.g., for $b \geq 0.05$ at $\rho = 0.5$ in the figure). However, the increase in the consumption (dividend) rate with the magnitude of ρ (more precisely, the magnitude of the deviation in ρ from $\rho_c(b)$) is greater if hedging opportunities are more important, as captured by the perturbation or contagion parameter b.

5   Customers with partial information

The results in the previous section assume that the customers have complete information in the sense that they know with certainty their frequency of casualty occurrences. In practice, however, customers can be overly optimistic or pessimistic regarding own risk. Following Asmussen et al. (2013), we allow for this possibility by assuming that customers assessment of their own arrival rate takes the form ΛS, with S a random variable independent of Λ, representing the optimism/pessimism of the

customer. The decision by a potential customer of whether or not to buy insurance is now based upon ΛS rather than the true Λ, so customers have partial information, only. Assuming the distribution of S is i.i.d. across customers and inverse gamma, $1/S$ has a gamma distribution, with density $f_{1/S}(x) = (\zeta^\tau/\Gamma(\tau))x^{\tau-1}e^{-\zeta x}$, with shape parameter τ and (inverse) scale parameter ζ. Finite mean, $\mathbb{E}\left[S\right] < \infty$, requires $\tau > 1$, and finite variance, $\text{Var}\left[S\right] < \infty$, requires $\tau > 2$. We assume $\tau > 1$, i.e., customers can be considered optimistic or pessimistic on average according to whether $\mathbb{E}\left[S\right]$ is below or above unity. Then, for a given premium rate, p, and deductible K, the company attracts

$$n(p, K) = N\mathbb{P}(\Lambda S > \alpha(K)p) = N\mathbb{P}\left(\Lambda > \frac{\alpha(K)p}{S}\right) = N\left(\frac{\lambda_0\zeta}{\lambda_0\zeta + \alpha(K)p}\right)^\tau \quad (5.1)$$

customers, using that $\mathbb{P}\left(\Lambda > \frac{x}{S}\right) = \mathbb{E}\left[e^{-\frac{x}{\lambda_0 S}}\right]$. Thus, the demand function is now generalized hyperbolic, as opposed to the negative exponential demand function in (4.10). Under partial information, the average arrival rate of claims is given by

$$\lambda(p, K) = \mathbb{E}[\Lambda \mid \Lambda S > \alpha(K)p] = \lambda_0 + \frac{\lambda_0\alpha(K)\tau p}{\lambda_0\zeta + \alpha(K)p}, \quad (5.2)$$

in which the last term, excess expected arrivals beyond λ_0, is a combined adverse selection and partial information effect (see the Appendix for the derivation). Under the maintained assumption that the customer's risk-adjusted expected net claim $a(K)$ is differentiable with $a'(K) < 0$, it follows that

$$\frac{\partial n(p, K)}{\partial p} = -\frac{\tau\alpha(K)n(p, K)}{\lambda_0\zeta + \alpha(K)p} < 0, \quad \frac{\partial n(p, K)}{\partial K} = -\frac{\tau\alpha'(K)n(p, K)p}{\lambda_0\zeta + \alpha(K)p} < 0,$$

$$\frac{\partial \lambda(p, K)}{\partial p} = \frac{\lambda_0^2\alpha(K)\tau\zeta}{(\lambda_0\zeta + \alpha(K)p)^2} > 0, \quad \frac{\partial \lambda(p, K)}{\partial K} = \frac{\lambda_0^2\alpha'(K)\tau\zeta p}{(\lambda_0\zeta + \alpha(K)p)^2} > 0.$$

Similarly to the complete information case, the number of customers, $n(p, K)$, falls with increases in the premium rate and the deductible, while the average frequency of claims, $\lambda(p, K)$, increases due to adverse selection. Using these results, it is straightforward to show that the expected arrival rate of claims from the aggregate insurance portfolio, $n(p, K)\lambda(p, K)$, will again decline as the premium rate or deductible is raised.

Using these partial derivatives and factoring out $n(p, K)$, the company's first order conditions (3.11) and (3.12) reduce to

$$r\eta\left(\lambda_0\zeta + \alpha(K)p\right)^2 - r\eta\tau\alpha(K)\left(\lambda_0\zeta + \alpha(K)p\right)\left[b\boldsymbol{\rho}^\top\boldsymbol{\psi} + p\right]$$
$$+ \left(1 - \|\boldsymbol{\rho}\|^2\right)\left(r\eta b\right)^2\tau\alpha(K)n(p, K)\left(\lambda_0\zeta + \alpha(K)p\right)$$
$$+ q(K)\lambda_0\alpha(K)^2\tau(\tau + 1)p = 0, \quad (5.3)$$
$$- r\eta\tau\alpha'(K)p\left(\lambda_0\zeta + \alpha(K)p\right)\left[b\boldsymbol{\rho}^\top\boldsymbol{\psi} + p\right]$$
$$+ \left(1 - \|\boldsymbol{\rho}\|^2\right)\left(r\eta b\right)^2\tau\alpha'(K)n(p, K)p\left(\lambda_0\zeta + \alpha(K)p\right)$$
$$+ r\eta\lambda_0\left(\bar{F}_Y(K) + q(K)\right)\left(\lambda_0\zeta + \alpha(K)p\right)\left[\lambda_0\zeta + \alpha(K)(1 + \tau)p\right]$$
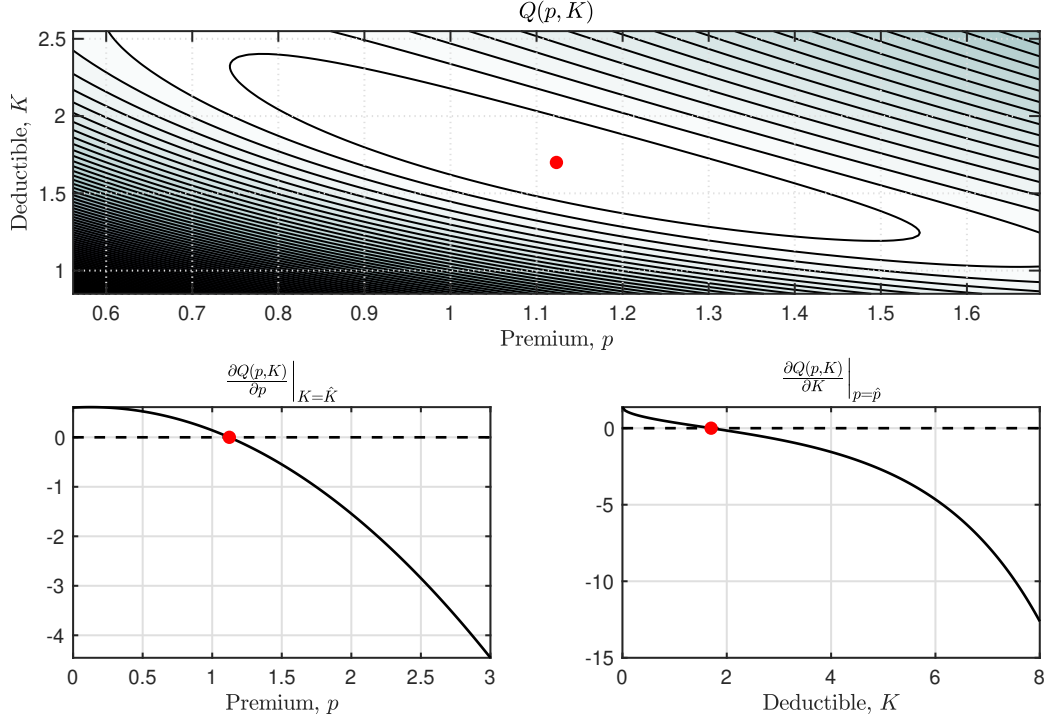$$+ \alpha(K)\alpha'(K)q(K)\lambda_0(1 + \tau)\tau p^2 = 0. \quad (5.4)$$

Figure 5. The $Q(p, K)$ function under partial information. The top exhibit shows contours of the profile Hamiltonian $Q(p, K)$. The bottom exhibits illustrate the first order conditions with respect to the premium rate p and deductible K. Dots indicate the critical points $\hat{p}$ and $\hat{K}$ for which $\partial Q(\hat{p}, K)/\partial p = \partial Q(p, \hat{K})/\partial K = 0$. The arrival rate of claims is $\Lambda \sim \text{Exp}(0.5)$, the customer's partial information variable $S \sim \text{Inverse-Gamma}(4, 4)$, and claims sizes $Y \sim \text{Gamma}(0.5, 0.5)$. The remaining parameters are $(r, \eta, b, \lambda_0, N) = (0.05, 3, 0, 0.5, 100)$.

Figure 5 illustrates the properties of the profile Hamiltonian $Q(p, K)$ from (3.10), now using (5.1) for portfolio size $n(p, K)$ and (5.2) for average arrival rate $\lambda(p, K)$. The figure further illustrates the associated first order conditions (5.3) and (5.4). Customers' beliefs about their own claim arrival rate are characterized by $S \sim$ Inv-Gamma$(4, 4)$, implying that $\mathbb{E}[S] = 1.33$, i.e., a case of pessimistic customers, on average, and $\text{Var}[S] = 0.88$. The values of the remaining parameters can be found in the caption below the figure. As in the complete information case, $\alpha(K)$ follows from (4.21), and $q(K)$ from (4.22). A dot indicates the optimal premium $\hat{p}$ and deductible $\hat{K}$ maximizing $Q(p, K)$. The optimal values are $(\hat{p}, \hat{K}) = (1.12, 1.70)$. Again, optimal asset holdings and consumption follow from Proposition 3.1. Compared to the left exhibits of Figure 1, i.e., the corresponding complete information case, where $(\hat{p}, \hat{K}) = (0.51, 2.87)$, then on average, pessimistic customers in the partial information extension are liable to purchase more insurance, thus leading the optimizing company to increase the premium, and at the same time reduce the deductible, consistent with the overall negative premium-deductible trade-off.

The analysis illustrates the flexibility and generality of the approach to accommodate alternative portfolio size and claim frequency specifications, and still obtain jointly optimal premium, deductible, investment, and consumption (dividend) controls.

5.1 Fixed deductible

Consider again the special case in which the company does not control the deductible, which is fixed at $K = \overline{K}$. In this case, the first order condition (5.3) can be compactly written as

$$M(p, \overline{K}) := M_2(\overline{K})p^2 + M_1(\overline{K})p + M_0(\overline{K}) + \frac{(r\eta b)^2(1 - \|\boldsymbol{\rho}\|^2)(\lambda_0 \zeta)^\tau N\tau\alpha(\overline{K})}{(\lambda_0\zeta + \alpha(\overline{K})p)^{\tau-1}} = 0 \,,$$

(5.5)

with

$$M_2(\overline{K}) := r\eta\alpha(\overline{K})^2(1 - \tau) \,,$$
$$M_1(\overline{K}) := \lambda_0\alpha(\overline{K})[\alpha(\overline{K})q(\overline{K})\tau(1 + \tau) + r\eta\zeta(2 - \tau)] - \alpha(\overline{K})^2\tau r\eta b\boldsymbol{\rho}^\top\boldsymbol{\psi} \,,$$
$$M_0(\overline{K}) := r\eta\lambda_0\zeta[\lambda_0\zeta - \alpha(\overline{K})\tau b\boldsymbol{\rho}^\top\boldsymbol{\psi}] \,.$$

We have the following result, for arbitrary perturbation to the reserve process, $b \geq 0$.

Theorem 5.1. *Suppose that*

$$\lambda_0\zeta + r\eta b^2(1 - \|\boldsymbol{\rho}\|^2)N\tau\alpha(K) > \alpha(K)\tau b\boldsymbol{\rho}^\top\boldsymbol{\psi} \,,$$

(5.6)

and that there exists a unique $\hat{p} > 0$ such that $M(\hat{p}, \overline{K}) = 0$. Then $\hat{p}$ is optimal.

Proof. Consider $Q(p, \overline{K})$ in (3.10) and $M(p, \overline{K})$ in (5.5) as functions only of p. If $M(\hat{p}, \overline{K}) = 0$, then $\hat{p}$ is a critical point of $Q(p, \overline{K})$. Moreover, $M(0, \overline{K}) > 0$ under (5.6). Then $M(p, \overline{K})$, and hence $Q'(p, \overline{K})$, is positive only for $p \leq \hat{p}$, which implies that $\hat{p}$ is optimal. ∎

Remark 5.1. If $\hat{p} > -\frac{1}{2}\left(\frac{M_1(\overline{K})}{M_2(\overline{K})}\right)$, then $M'(p, \overline{K})$ is negative for $p > \hat{p}$, and so is $M(p, \overline{K})$. ∎

Since $M_0(\overline{K})$ and $M_1(\overline{K})$ involve the market price of risk $\boldsymbol{\psi}$ for $b > 0$, so does the optimal premium $\hat{p}$ in Theorem 5.1. Again, optimal investment and consumption follow from Proposition 3.1. Finally, for $b = 0$, the optimal premium admits a closed-form representation, as stated in the following result.

Corollary 5.1. *Assume $b = 0$. Then $M(p, \overline{K})$ in (5.5) is a quadratic equation in p. Under the assumptions of Theorem 5.1, the optimal premium rate is*

$$\hat{p}(\overline{K}) = \frac{\lambda_0}{2r\eta\alpha(\overline{K})(1 - \tau)}\left[\widetilde{M_1}(\overline{K}) + \sqrt{\widetilde{M_1}(\overline{K})^2 + 4(r\eta\zeta)^2(\tau - 1)}\right] \,,$$

with $\widetilde{M_1}(\overline{K}) = \alpha(\overline{K})q(\overline{K})\tau(1 + \tau) + r\eta\zeta(2 - \tau)$.

For gamma distributed claims, $\alpha(K)$ and $q(K)$ are given by (4.21) and (4.22), so in this case, for given fixed deductible $\overline{K}$, the corollary provides the closed-form solution for the optimal premium, $\hat{p}$, and hence for optimal investment and consumption (dividend), too. For general claim size distributions, $\alpha(K) = a(K)^{-1}$ and $q(K)$ in the corollary are determined from (4.7) and (3.5).

6 Conclusion

Systemic risk and contagion between financial and insurance markets is a very realistic and relevant phenomenon, and should be accounted for in the management of insurance companies. Our analysis provides a unifying framework for financial investment and premium control, with simultaneous optimal choice of the deductible, as well as optimization of the path for consumption, or dividend payout to the owner-manager, assuming the latter exhibits constant absolute risk aversion. Optimal risky asset holdings reflect both speculative financial investments and a hedge against insurance risk. Insurance demand reflects risk aversion on the customer side, too, as well as adverse selection, i.e., an increase in either premium or deductible reduces portfolio size and increases the riskiness of the average customer. Premium rate and deductible are inversely related around the optimum. The optimal premium exceeds the net premium, namely, expected frequency time size of net (of deductible) claims, which would correspond to standard insurance pricing, i.e., risk sharing, but without safety loading, and also to the expected reservation premium of hypothetical risk-neutral customers. The optimal premium further exceeds the higher expected reservation premium of customers after appropriately accounting for their risk aversion, although this is the level replacing the average or market premium from the literature in the demand function. The extent of the excess of the optimal premium beyond customers' expected risk-adjusted reservation premium reflects the company's risk aversion, the relative valuation of net claims by company and customers, and the market power exerted by the company to protect itself against risky customers, i.e., the adverse selection aspect. Closed-form solutions for optimal investment, premium, deductible, and payout are obtained under specific assumptions on the distributions of size and frequency of claims.

Overall, the results should encourage insurance companies to manage risks stemming from the financial and insurance sides of their businesses in an active manner, and jointly. Future research could investigate (i) alternative utility functions, e.g., constant relative risk aversion; (ii) separation of ownership and control of the insurance company; (iii) accounting explicitly for the possibility of ruin, possibly using minimization of ruin probability as an alternative criterion; (iv) different types of deductibles, e.g., the proportional deductible; (v) using the principle of equivalent utility rather than the certainty equivalent for the reservation premium, cf. Examples 4.1 and 4.2; and (vi) competition from other insurance companies. Although beyond the scope of the present paper, all these variations offer exciting avenues for further investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

Appendix

Average arrival rate of claims under partial information

Let $1/S$ have a gamma distribution with density $\frac{\zeta^\tau}{\Gamma(\tau)}y^{\tau-1}e^{-\zeta y}$ and let Λ be exponentially distributed with $\mathbb{E}[\Lambda] = \lambda_0$. The law of total probability yields

$$\mathbb{E}[\Lambda \mid S\Lambda > x] = \frac{\mathbb{E}[\Lambda \mathbf{1}_{\{S\Lambda>x\}}]}{\mathbb{P}(S\Lambda > x)} = \frac{\int_0^\infty \mathbb{E}[\Lambda \mathbf{1}_{\{\Lambda>x/S\}} \mid S = s]f_S(s)\,ds}{\mathbb{P}(S\Lambda > x)}.$$

Using the moment generating function of $1/S$, we get

$$\mathbb{P}(S\Lambda > x) = \mathbb{P}\left(\Lambda > \frac{x}{S}\right) = \mathbb{E}\left[e^{-\frac{x}{\lambda_0}\frac{1}{S}}\right] = \left(\frac{\zeta}{\zeta + \frac{x}{\lambda_0}}\right)^\tau.$$

and thus the demand function (5.1). Now observe that

$$\begin{aligned}
\mathbb{E}[\Lambda \mathbf{1}_{\{\Lambda>x/S\}} \mid S = s] &= \mathbb{E}\left[\Lambda \mathbf{1}_{\{\Lambda>\frac{x}{s}\}}\right] \\
&= \mathbb{E}\left[\Lambda \mid \Lambda > \frac{x}{s}\right]\mathbb{P}\left(\Lambda > \frac{x}{s}\right) \\
&= \left(\lambda_0 + \frac{x}{s}\right)e^{-\frac{x}{\lambda_0 s}}.
\end{aligned}$$

It follows that

$$\int_0^\infty \mathbb{E}[\Lambda \mathbf{1}_{\{\Lambda>x/S\}} \mid S = s]f_S(s)\,ds = \mathbb{E}\left[\left(\lambda_0 + \frac{x}{S}\right)e^{-\frac{x}{\lambda_0}\frac{1}{S}}\right].$$

Using the change-of-variable $z = (\zeta + \frac{x}{\lambda_0})y$ and the identity $\Gamma(\tau+1) = \tau\Gamma(\tau)$, we get

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{S}e^{-\frac{x}{\lambda_0}\frac{1}{S}}\right] &= \frac{\zeta^\tau}{\Gamma(\tau)}\int_0^\infty ye^{-\frac{x}{\lambda_0}y}y^{\tau-1}e^{-\zeta y}\,dy \\
&= \frac{\zeta^\tau}{\Gamma(\tau)}\frac{1}{(\zeta + \frac{x}{\lambda_0})^{1+\tau}}\int_0^\infty z^\tau e^{-z}\,dz \\
&= \frac{\tau\zeta^\tau}{(\zeta + \frac{x}{\lambda_0})^{1+\tau}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}[\Lambda \mid S\Lambda > x] &= \lambda_0 + x\frac{\tau\zeta^\tau}{(\zeta + \frac{x}{\lambda_0})^{1+\tau}}\left(\frac{\zeta}{\zeta + \frac{x}{\lambda_0}}\right)^{-\tau} \\
&= \lambda_0 + \frac{x\tau}{\zeta + \frac{x}{\lambda_0}},
\end{aligned}$$

which is the average arrival rate of claims in (5.2).

References

ACHLEITNER, P. M., J. H. BIEBEL, AND D. WICHELS (2002): "Does WTC Matter for the Investment Policy of P/C Insurance Companies?" *The Geneva Papers on Risk and Insurance-Issues and Practice*, 27, 275–282.

ASMUSSEN, S., B. J. CHRISTENSEN, AND M. TAKSAR (2013): "Portfolio Size as Function of the Premium: Modelling and Optimization," *Stochastics*, 85, 575–588.

ASMUSSEN, S. AND M. TAKSAR (1997): "Controlled Diffusion Models for Optimal Dividend Pay-out," *Insurance: Mathematics and Economics*, 20, 1–15.

BILLIO, M., M. GETMANSKY, A. W. LO, AND L. PELIZZON (2012): "Econometric Measures of Connectedness and Systemic Risk in the Finance and Insurance sectors," *Journal of Financial Economics*, 104, 535–559.

BROWNE, S. (1995): "Optimal Investment Policies for a Firm with a Random Risk Process: Exponential Utility and Minimizing the Probability of Ruin," *Mathematics of Operations Research*, 20, 937–958.

BÜHLMANN, H. AND A. GISLER (2005): *A Course in Credibility Theory and its Applications*, Berlin: Springer.

BURNECKI, K., J. NOWICKA-ZAGRAJEK, AND A. WYLOMANSKA (2005): *Pure Risk Premiums under Deductibles*, Berlin: Springer.

CABALLERO, R. J. (1990): "Consumption Puzzles and Precautionary Savings," *Journal of Monetary Economics*, 25, 113–136.

DENUIT, M., X. MARÉCHAL, S. PITREBOIS, AND J.-F. WALHIN (2007): *Actuarial Modelling of Claim Counts: Risk Classification, Credibility and Bonus-malus Systems*, Chichester: John Wiley & Sons.

DUFRESNE, F. AND H. U. GERBER (1991): "Risk Theory for the Compound Poisson Process that is Perturbed by Diffusion," *Insurance: Mathematics and Economics*, 10, 51–59.

EMMS, P., S. HABERMAN, AND I. SAVOULLI (2007): "Optimal Strategies for Pricing General Insurance," *Insurance: Mathematics and Economics*, 40, 15–34.

FLEMING, W. H. AND H. M. SONER (2006): *Controlled Markov Processes and Viscosity Solutions*, New York: Springer.

FURRER, H. J. AND H. SCHMIDLI (1994): "Exponential Inequalities for Ruin Probabilities of Risk Processes Perturbed by Diffusion," *Insurance: Mathematics and Economics*, 15, 23–36.

GERBER, H. U. (1970): "An Extension of the Renewal Equation and its Application in the Collective Theory of Risk," *Scandinavian Actuarial Journal*, 1970, 205–210.

GERBER, H. U. AND G. PAFUM (1998): "Utility Functions: From Risk Theory to Finance," *North American Actuarial Journal*, 2, 74–91.

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GOOVAERTS, M. J., R. KAAS, A. VAN HEERWAARDEN, AND T. BAUWELINCKX (1990): *Effective Actuarial Methods*, Amsterdam: North-Holland.

HAINAUT, D. (2017): "Contagion Modeling Between the Financial and Insurance Markets with Time Changed Processes," *Insurance: Mathematics and Economics*, 74, 63–77.

HIPP, C. AND M. PLUM (2000): "Optimal Investment for Insurers," *Insurance: Mathematics and Economics*, 27, 215–228.

HØJGAARD, B. (2002): "Optimal Dynamic Premium Control in Non-life Insurance. Maximizing Dividend Pay-outs," *Scandinavian Actuarial Journal*, 4, 225 – 245.

KIMBALL, M. S. AND N. G. MANKIW (1989): "Precautionary Saving and the Timing of Taxes," *Journal of Political Economy*, 97, 863–879.

MARTIN-LÖF, A. (1983): "Premium Control in an Insurance System: An Approach Using Linear Control Theory," *Scandinavian Actuarial Journal*, 1983, 1–27.

MERTON, R. C. (1969): "Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case," *The Review of Economics and Statistics*, 51, 247–57.

——— (1971): "Optimum Consumption and Portfolio Rules in a Continuous-Time Model," *Journal of Economic Theory*, 3, 373–413.

——— (1973): "An Intertemporal Capital Asset Pricing Model," *Econometrica*, 41, 867–87.

MIKOSCH, T. (2004): *Non-Life Insurance Mathematics: An Introduction With Stochastic Processes*, Berlin: Springer.

PANTELOUS, A. A. AND E. PASSALIDOU (2017): "Optimal Strategies for a Non-linear Premium-Reserve Model in a Competitive Insurance Market," *Annals of Actuarial Science*, 11, 1–19.

PRATT, J. W. (1964): "Risk Aversion in the Small and in the Large," *Econometrica*, 32, 122–136.

RADNER, R. AND L. SHEPP (1996): "Risk vs. Profit Potential: A Model for Corporate Strategy," *Journal of Economic Dynamics and Control*, 20, 1373–1393.

ROTHSCHILD, M. AND J. STIGLITZ (1976): "Equilibrium in Competitive Insurance Markets: An Essay on the Economics of Imperfect Information," *The Quarterly Journal of Economics*, 90, 629–649.

SCHMIDLI, H. (1995): "Cramer-Lundberg Approximations for Ruin Probabilities of Risk Processes Perturbed by Diffusion," *Insurance: Mathematics and Economics*, 16, 135–149.

——— (2008): *Stochastic Control in Insurance*, London: Springer.

SENNEWALD, K. (2007): "Controlled Stochastic Differential Equations under Poisson Uncertainty and with Unbounded Utility," *Journal of Economic Dynamics and Control*, 31, 1106 – 1131.

Svensson, L. E. O. and I. M. Werner (1993): "Nontraded Assets in Incomplete Markets: Pricing and Portfolio Choice," *European Economic Review*, 37, 1149–1168.

Taylor, G. C. (1986): "Underwriting Strategy in a Competitive Insurance Environment," *Insurance: Mathematics and Economics*, 5, 59–77.

Thøgersen, J. (2016): "Optimal Premium as a Function of the Deductible: Customer Analysis and Portfolio Characteristics," *Risks*, 4, 42.

Vandebroek, M. and J. Dhaene (1990): "Optimal Premium Control in a Non-life Insurance Business," *Scandinavian Actuarial Journal*, 1990, 3–13.

Wang, N. (2004): "Precautionary Saving and Partially Observed Income," *Journal of Monetary Economics*, 51, 1645–1681.

——— (2006): "Generalizing the Permanent-Income Hypothesis: Revisiting Friedman's Conjecture on Consumption," *Journal of Monetary Economics*, 53, 737–752.

——— (2009): "Optimal Consumption and Asset Allocation with Unknown Income Growth," *Journal of Monetary Economics*, 56, 524–534.

Ward, D. and R. Zurbruegg (2000): "Does Insurance Promote Economic Growth? Evidence from OECD Countries," *The Journal of Risk and Insurance*, 67, 489–506.

Yang, H. and L. Zhang (2005): "Optimal Investment for Insurer with Jump-Diffusion Risk Process," *Insurance: Mathematics and Economics*, 37, 615–634.

Zheng, X., J. Zhou, and Z. Sun (2016): "Robust Optimal Portfolio and Proportional Reinsurance for an Insurer under a CEV Model," *Insurance: Mathematics and Economics*, 67, 77–87.

Zhou, M., K. C. Yuen, and C.-C. Yin (2017): "Optimal Investment and Premium Control in a Nonlinear Diffusion model," *Acta Mathematicae Applicatae Sinica, English Series*, 33, 945–958.

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