DEPARTMENT OF ECONOMICS
AND BUSINESS ECONOMICS
AARHUS UNIVERSITY

CREATES
Center for Research in Econometric Analysis of Time Series

Optimal Asset Allocation for Commodity Sovereign Wealth Funds

Alfonso A. Irarrazabal, Lin Ma and
Juan Carlos Parra-Alvarez

CREATES Research Paper 2020-10

Department of Economics and Business Economics
Aarhus University
Fuglesangs Allé 4
DK-8210 Aarhus V
Denmark

Email: oekonomi@au.dk
Tel: +45 8716 5515

Optimal Asset Allocation for Commodity Sovereign Wealth Funds*

Alfonso A. Irarrazabal† Lin Ma‡ Juan Carlos Parra-Alvarez§

August 2020

Abstract

This paper studies the dynamic asset allocation problem faced by an infinitively-lived commodity-based sovereign wealth fund under incomplete markets. Since the non-tradable stream of commodity revenues is finite, the optimal consumption and investment strategies are time dependent. Using data from the Norwegian Petroleum Fund, we find that the optimal demand for equity should decrease gradually from 60 to 40 percent over the next 60 years. However, the solution is particularly sensitive to the correlation between oil and stock price changes. We also estimate wealth-equivalent welfare losses, relative to the optimal rule, when following alternative suboptimal investment rules.

JEL Classification: *E21, G11, G23, Q32.*

Keywords: *Optimal asset allocation, sovereign wealth fund, commodities, income risk, suboptimal investments.*

*We appreciate the comments and suggestions from Juan Pablo Nicolini, Knut Einar Rosendahl and Olvar Bergland and seminar participants at Norges Bank, Norwegian University of Life Sciences, Central Bank of Chile and the Ministry of Finance of Norway. We also thank Yacine Aït-Sahalia for providing us with Matlab code to perform the ML estimations.

†BI Norwegian Business School; Email address: alfonso.irarrazabal@bi.no

‡CICERO, Center for International Climate Research; Email address: lin.ma.econ@gmail.com

§Aarhus University; CREATES; Danish Finance Institute; Email address: jparra@econ.au.dk

1 Introduction

Sovereign wealth funds (SWFs hereafter) are institutional investors that engage in long-run investment strategies with the objective to ensure a gradual transfer of wealth across generations. Although these investment funds have existed for decades, there has been a significant increase of SWFs since 2000. The source of income of most SWFs comes from commodity revenues and/or the accumulation of foreign exchange reserves. As of 2019, there were 48 different commodity-based SWFs in the world administering US$4 trillion in assets (SWF Institute, 2020), corresponding to US$1,163 billion more than the estimated size of hedge funds worldwide (Statista, 2019), and to 5 percent of the global investment industry (Fages et al., 2019). Since commodity prices are extremely volatile (cf. Deaton and Laroque, 1992), investors face the challenge to design optimal investment strategies that help them manage the associated income risk. To the extent that commodity revenues are correlated with stock prices, investors have the possibility to hedge this volatility away by adjusting their exposure to stocks.

In this paper, we study the optimal consumption-investment decision of oil-based SWFs when the risk from its volatile revenues is only partially hedgeable due to market incompleteness. To do so, we use an otherwise standard strategic asset allocation model with stochastic income similar to those in Bodie et al. (1992), Heaton and Lucas (1997), Viceira (2001), Campbell and Viceira (2002), Cocco et al. (2005), and Munk and Sørensen (2010). However, since most SWFs are set up by countries interested in sustaining a standard of living for all future generations, we assume that the fund's planning horizon is infinite, while the commodity revenues are received only for a finite number of periods. In order to distinguish the effects of atemporal risk aversion from those due to intertemporal substitution, we assume that the preferences of the SWF's manager over intermediate consumption are recursive as in Duffie and Epstein (1992a,b). In turn, this will allow us to reconcile high risk taking induced by large risk premiums with a low tolerance for volatile consumption. Moreover, all of the uncertainty in the income stream is assumed to come from the stochastic behavior of oil prices.

To solve the dynamic portfolio problem faced by the SWF we invoke the principle of optimality (Bellman, 1957) and split the planning horizon according to the time of depletion of the commodity. The resulting subproblems (before and after depletion) are then solved backwards in time. First, we solve an infinite horizon problem in which the SWF does not receive any income flow. The optimal value function, as well as the levels of investment and consumption for this problem, can be obtained in closed form. Using the resulting indirect utility as a terminal condition, we then solve a finite horizon problem in which the SWF receives a stream of stochastic commodity revenues for a fixed period of time. Similar to the case of asset allocation over the life cycle with uncertain labor income, we find that the SWF's optimal investment profile is time dependent. The optimal demand for the risky asset includes an intertemporal hedging component that depends on the correlation between risky assets and income. Negative (positive) values of this correlation result in a decreasing (increasing) demand for stocks over time that converges to a long-run value equal to the (leveraged) Sharpe ratio of the risky asset.

To assess the degree of market incompleteness, we use monthly data on the S&P500 index and the WTI price of crude oil from 1973 to 2019 to estimate the correlation between oil and stock price changes using a continuous-time vector autoregression model following the maximum likelihood framework in Ait-Sahalia (2002, 2008). In line with previous results in the literature, we find statistical evidence of a time-varying correlation. Consistent with conventional wisdom, and similar to the evidence reported in Jones and Kaul (1996), Sadorsky (1999), Gorton and Rouwenhorst (2006), Lee and Chiou (2011) and Bhardwaj et al. (2015), we find a negative, but low, average correlation of -7 percent for the period 1973-2007. However, at the outset of the financial crisis that led to the Great Recession, the correlation becomes positive and high. More specifically, for the period 2008-2019 we estimate a statistically significant correlation coefficient that reaches a value of around 30 percent by the end of 2018. Similar results have been documented in Filis et al. (2011), Buyuksahin and Robe (2014), Bernanke (2016), Lombardi and Ravazzolo (2016) and Datta et al. (2018), who have argued that this phenomenon could be the result of a generalized weakening of global aggregate demand, the growth in the commodity-market activity, or the zero lower bound on nominal interest rates. Consequently, we conclude that oil income is not fully spanned by traded assets and hence its associated risk cannot be fully hedged through the financial markets. Although similar limits to diversification have be documented for long-term investors with stochastic labor income (see Campbell, 1996, Davis and Willen, 2000, Campbell and Viceira, 2002), our estimates suggest that oil-based SWFs face a covariance structure between asset returns and income that is diametrically different: low correlation, and large volatility of income that exceeds that of stock prices. Therefore, in contrast to the case of uncertain labor income with no liquidity/investment constraints, it is no longer possible to accurately approximate the optimal investment strategy for oil-based SWF investors using the assumption of complete markets (cf. Bick et al., 2009 and Munk and Sørensen, 2010).

In the presence of stochastic oil revenues and market incompleteness, the finite horizon component of the model does not admit a closed-form solution. Therefore, we resort to numerical methods to approximate the optimal consumption and investment decisions. In particular, we use the state space reduction of Duffie et al. (1997), and the corresponding finite difference representation introduced in Munk and Sørensen (2010) which we implement numerically using the method described in Gomez (2019). We calibrate our model to match salient features of the Norwegian Government Pension Fund Global (GPFG), popularly known as the Petroleum Fund. For a low and negative correlation between oil and stock prices, similar to that observed between 1973-2007, our quantitative results indicate that the SWF should allocate around 60 percent of its financial wealth into stocks at the beginning of the planning horizon. Under the maintained assumption of a fixed amount of oil reserves, the fund should thereafter decrease its position in the risky asset monotonically until it reaches a value of 40 percent 60 years later once the natural resource has been fully depleted. This initial overshooting, relative to its long-run value, is the result of two complementary effects: (i) a wealth or leverage effect from the capitalized value of the future stream of commodity revenues, and (ii) a positive hedging demand that accounts for 20 percent of the total demand for stocks. This additional demand is primarily driven by the high volatility of oil prices, and not by their correlation with the risky asset. If on the contrary, the correlation is positive, the model implies a large recomposition of the investment portfolio with a large fraction of wealth allocated into the risk-free asset. In particular, with a correlation coefficient of 30 percent, we find an initial allocation into the risky asset of around 30 percent that should increase monotonically towards its long-run value of 40 percent. In either case, the optimal investment strategy goes hand-in-hand with a gradual injection of oil revenues into the economy as measured by a relatively constant optimal consumption-to-wealth ratio that fluctuates between 2 and 3 percent per year. This consumption pattern is consistent with the fiscal rule (*handlingsregelen*) introduced by the Norwegian parliament in 2001 with the objective to spend oil revenues in a gradual and controlled way that helps preventing any undesirable overheating of the economy and/or the occurrence of a Dutch disease. Lindset and Mork (2019) have recently shown that such a smooth path is consistent with the government's desire for smoothness in taxes and public expenditures. As a corollary to our quantitative experiments we conclude that if the correlation between stock and oil prices remains positive and large in the near future, the Norwegian GPFG should consider lowering its exposure to equity. Our simulations suggest that the current mandate on the stock/bond mix is not compatible with an investment strategy that exploits all the diversification possibilities in an optimal way, and instead exposes the fund to otherwise hedgeable risks.

Our results relate to a number of recent contributions in the study of asset allocation for oil-based SWFs. Scherer (2011) studies the portfolio problem of a SWF fund that must decide how to allocate its oil revenues into different asset classes. Through the lens of a standard mean-variance analysis (Markowitz, 1952), and assuming that the value of the oil resources relative to the government's aggregate wealth is constant over time, he finds that the optimal demand for risky assets includes a hedging demand component that is a function of the oil wealth-to-financial wealth ratio, and of the correlation between oil price changes and asset returns. He shows that, when the set of investment opportunities includes assets that correlate negatively with oil prices, the SWF should then decrease its position in the risky asset as the oil reserves decrease. However, his approach abstracts from the optimal consumption-saving decision, and from the implications that the finite nature of the commodity revenues might have on the optimal investment strategy[1]. Closer to our approach is the work by van den Bremer et al. (2016). They extend the work in van den Bremer and van der Ploeg (2013) to study the role played by non-tradable commodity assets in the optimal consumption and investment decisions of an infinitively lived SWF under both complete and incomplete markets. They also conclude that the optimal investment profile of a SWF should take into account the amount of underground wealth through a hedging demand component. Moreover, they show that any undiversifiable risk should be alleviated by an increase of precautionary savings against current consumption. However, in contrast to our paper, they implicitly assume that the initial endowment of oil reserves are never depleted, and as a consequence the fund receives a stream of oil income to perpetuity. Additionally, in order to study the implications of their model under incomplete markets they approximate the optimal allocations by assuming that consumption is a linear function of wealth, a result that only holds if markets are complete.

Next, we evaluate the welfare costs of not following an investment strategy that optimally exploits the intertemporal hedging opportunities available to the SWF. This exercise is motivated by the investment mandate given to the Norwegian GPFG according to which the equity/bond mix in the aggregate portfolio is fixed. Currently, the fund's administrator (The Norges Bank Investment Management, NBIM) is allowed to invest between 60 to 80 percent of its wealth in equities. What are the consequences of deviating from the optimal investment strategy? Associated with a given suboptimal policy, we answer this question by introducing a measure of wealth-equivalent welfare compensation. The latter is defined as the percentage of additional initial financial wealth that the government would need to transfer to the portfolio administrator in order to achieve the same indirect utility or welfare that could be otherwise obtained by following the optimal investment strategy. In particular, we consider two different suboptimal investment profiles: (i) a constant investment share, and (ii) and ad-hoc deterministic rule that fixes the equity holdings in every period equal to the median optimal investment share. Using our benchmark calibration with a negative correlation between stock and oil prices, we find that following a strategy that fixes the position in equities at 70 percent (the midpoint of the current mandate of the Norwegian GPFG) would require a wealth compensation equivalent to 12.5% more of the initial endowment. An alternative interpretation of this result, is that following a constant investment rule leads to significant welfare losses. We show that these losses can be considerably reduced by implementing instead a time-varying, but ad-hoc, investment rule. For practical purposes, this policy may be considered as a

[1]Using monthly data for the period 1997-2008, the author does not find any significant correlation between the nominal returns on the MSCI World index and the nominal price of crude oil. Therefore, he concludes that global equities do not provide a hedge against fluctuations in oil prices. In the face of a similar insignificant correlation, Døskeland (2007) proposed to use a cointegration approach to identify the long-term relation between financial assets and non-tradable assets. When applied to the Norwegian case, he finds a similar result: the government should increase its current (initial) holdings in equity and reduce it over time.

second-best policy in an environment with institutional constraints that prevent the SWF investor to hedge commodity fluctuations periodically.

We are not the first to report large welfare losses from the implementation of suboptimal policies. Campbell and Viceira (1999), find that failing to hedge in the presence of time-varying risk-premia leads to large welfare losses relative to the optimal policy, specially for mildly risk averse investors with positive positions in equity. Similarly, Gomes (2007) and Larsen and Munk (2012) report considerable utility losses from ignoring the intertemporal hedging opportunities for investors facing interest rate risk and stock volatility risk. Finally, Bick et al. (2009) study the welfare losses incurred by an investor with stochastic labor income that uses the investment rule that would prevail under complete markets when markets are in fact incomplete. They find that the losses of following this misspecified suboptimal policy is at most 14% of the initial total wealth when the true correlation between income and equities is zero, and drops to 3.2% if the correlation is 60 percent.

The remainder of the paper proceeds as follows. Section 2 provides a brief introduction and description of the Norwegian GPFG, with a particular focus on the institutional framework it faces and the investment strategy followed since its inception. Section 3 formalizes the optimal allocation problem faced by a commodity-based SWF and provides economic intuition behind the optimal consumption and investment policies when the oil income is both spanned and unspanned by the financial market. In Section 4 we discuss the calibration of the model, and discuss the optimal allocations when markets are incomplete. We also study the sensibility of the optimal policies to changes in the correlation between stock and oil prices, and the investor's coefficient of relative risk aversion. Section 5 studies the welfare costs of following suboptimal policies, and Section 6 concludes.

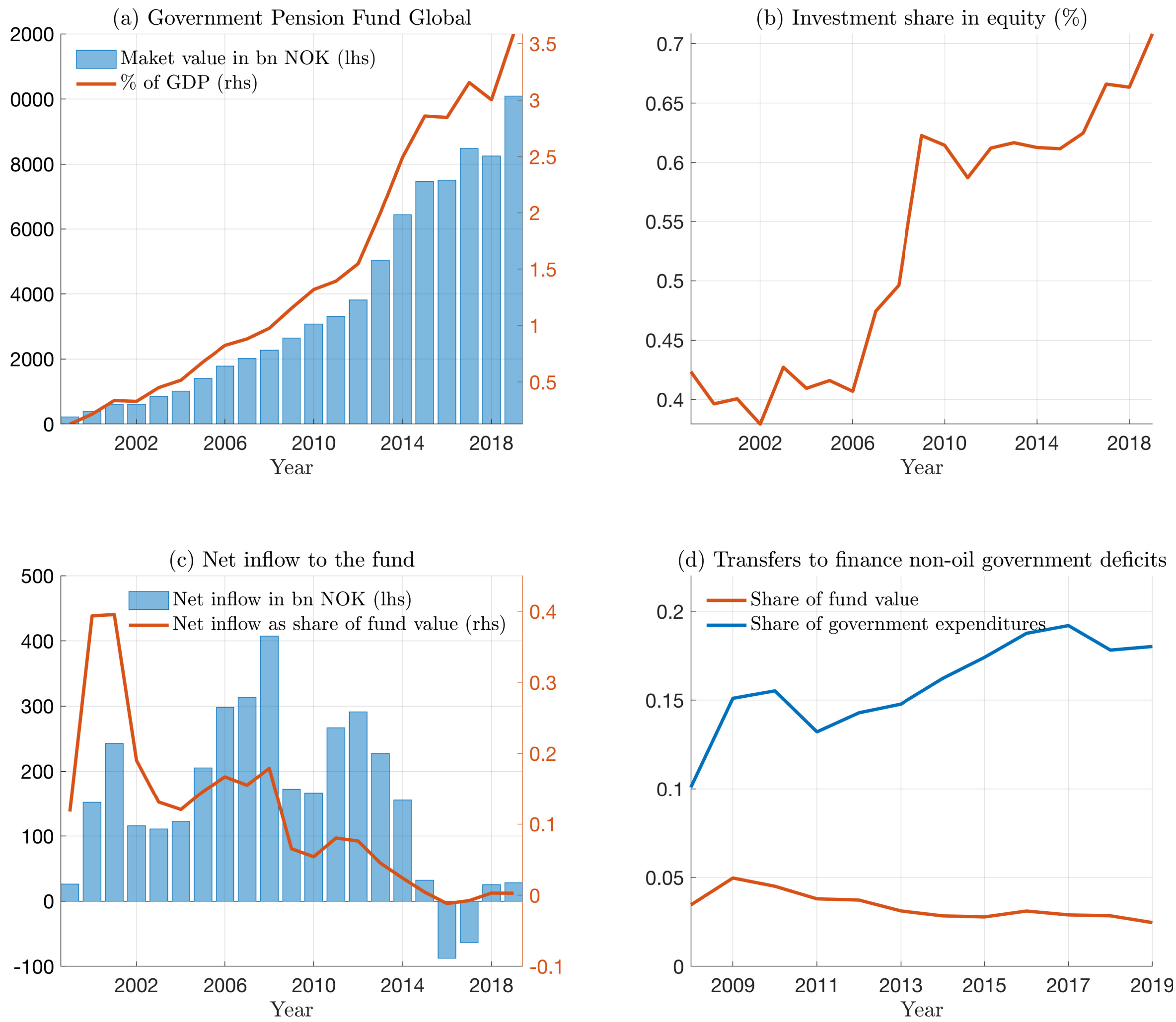
2 The Norwegian Sovereign Wealth Fund

Norway has one of the world largest established SWFs, the Government Pension Fund Global (GPFG). In 2019, the market value of the GPFG amounted to US$1,148 billion, nearly 3.5 times the real GDP of mainland Norway[2], and about 26% larger in market value than the China Investment Corporation which, according to the Sovereign Wealth Fund Institute, is the second biggest SWF. Panel (a) in Figure 2.1 shows the uninterrupted accumulation of wealth for the period 1998-2019.

The GPFG was created by the Norwegian Parliament in 1990 under the Government Pension Fund Act in order to ensure a long-term responsible management of the revenues generated from oil-related activities. Specifically, the objective of the fund is to manage the financial challenges posed by an aging population and to serve as a countercyclical

[2]According to Statistics Norway, mainland Norway refers to all the domestic production activity with the exception of exploration of crude oil and natural gas, transport via pipelines and ocean transport.

Figure 2.1. Main facts about the GPFG (1998-2019). Panel (a) plots the market value of the GPFG in billions of NOK and as a fraction of mainland real GDP; Panel (b) plots the fraction of the GPFG's market value invested in equities; Panel (c) plots the total capital inflows (net of managements costs) to the fund associated with oil revenues in billions of NOK and as a fraction of the GPFG's market value of wealth; Panel (d) plots the transfers from the GPFG to the Central Government as a fraction of the GPFC's market value and as a fraction of the total budgeted expenditures for the period 2006-2019.

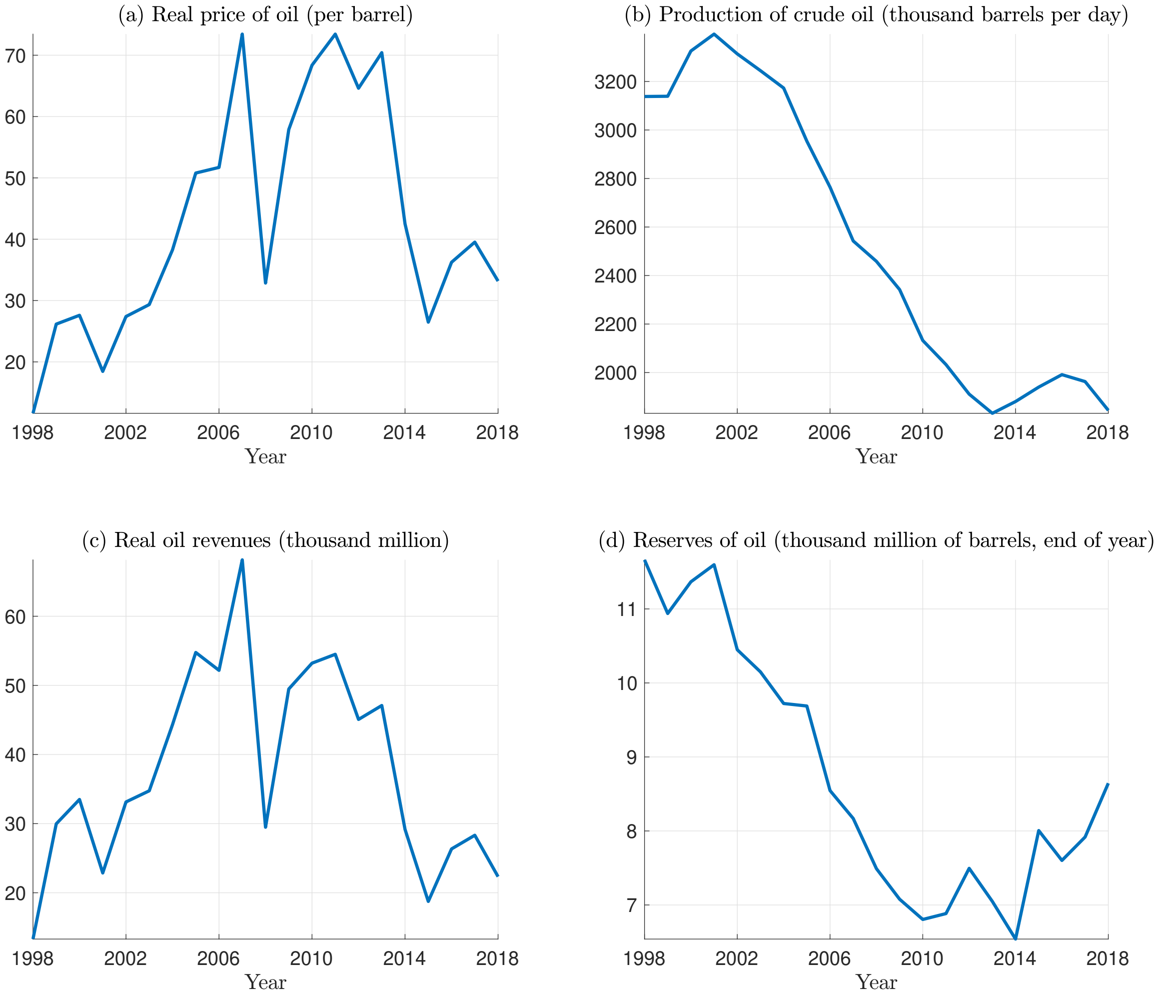


Source: Norges Bank Investment Management, Statistics Norway and Ministry of Finance.

fiscal tool that can be used to neutralize declines in the price of oil and in the economic activity in general. The Ministry of Finance has the overall responsibility for the fund's management. Accordingly, it issues a set of guidelines that are executed by the fund's Executive Board who defines an investment policy that is implemented by the portfolio manager. The Ministry's guidelines delimit the types of risks that the SWF can take, and the Executive Board acts consequently by setting up an asset allocation strategy that distributes the fund's wealth into different asset classes.

In 1998 the Norges Bank Investment Management (NBIM) was created to administer the fund's portfolio. The NBIM receives oil revenues in the form of transfers from the government and combines them with the fund's own accumulated wealth to implement the

Figure 2.2. Main components of oil revenues (1998-2018). Panel (a) plots the real price of oil per barrel in U.S. dollars as measured by the WTI price index and deflated by the U.S. CPI base 2000=100 (P_t). Panel (b) plots the production of crude oil in Norway* (Q_t). Panel (c) plots the total oil revenues in U.S. dollars from oil production ($P_t \times Q_t$). Panel (d) plots the total proved reserves of underground oil available at the end of a given year**.



Source: BP Statistical Review of World Energy (BP, 2019)

Notes: *Includes crude oil, shale oil, oil sands, condensates and NGLs. Excludes liquid fuels from other sources such as biomass and derivatives of coal and natural gas. **Correspond to quantities that geological and engineering information indicate with reasonable certainty can be recovered in the future from known reservoirs under existing economic and operating conditions.

requirements defined by the Executive Board. Panel (b) in Figure 2.1 shows the effective share of wealth invested in equity markets over the period 1998-2019. This investment profile mimics the management guidelines set by the Ministry of Finance, who in 1998 set a cap on the amount of wealth invested in stocks to 40%. In 2007 this cap was increased such that the equity portfolio constitutes between 60% and 80% of the total portfolio. The remaining fraction of wealth is distributed between fixed income (20-40%) and real estate assets (up to 7%).

The GPFG is set up such that two types of revenues are transferred to NBIM directly: government oil revenues and the fund's return. Panel (c) shows the annual (net) inflows to the NBIM for the period 1998-2019. Throughout the period the net transfers to the fund have decreased as a fraction of the GPFG's total wealth. As shown in Figure 2.2, this behavior is consistent with three factors: i) the drop in the average real price of oil observed from 2006, Panel (a); ii) the decline in the production of crude oil in Norway, Panel (b); and iii) the fall in the proved reserves of oil, Panel (c). All of these factors have led to a reduced operating surplus from extraction activities.

In addition to the NBIM funding sources, the Ministry of Finance established in 2001 a fiscal spending rule that stipulates how to transfer the oil income and its associated returns back to the Norwegian economy in a smooth and controlled way. These transfers from NBIM to the Central Government budget should follow the expected real return on the fund and must be directed to finance non-oil fiscal budget deficits. According to NBIM (2016): "*The spending rule is not a legal requirement, but rather a political economic yardstick which secures the original fund objectives and strengthens the inter-generational perspective.*" The transfer rule was initially set at 4% and in February of 2017 it was reduced to 3%. Panel (d) in Figure 2.1 plots a realized measure of the transfer rule computed as the value of the non-oil deficits budgeted by the Ministry of Finance as a fraction of the GPFG's market value. For the period 2006-2019, the transfers to the Central Government averaged 3.4% of the fund's wealth, and have become an important source of funding of the government total expenditures: in 2018 these transfers represented 18% of the total government spending.

3 The allocation problem of a SWF investor

This section describes the problem faced by a price-taking commodity SWF manager. Time is assumed to evolve continuously. Our framework is a stylized representation of the decision problem faced by the Executive Board of the Norwegian GPFG (the fund's manager) introduced in Section 2. We focus on the optimal asset allocation decisions made by the fund's manager who is also required to ensure a smooth stream of transfers to the government conditional on a decreasing and finite path of commodity revenues. The manager's planning horizon is assumed to be infinite in order to capture the long-term objective of building financial wealth that ensures sustained transfers for future generations. However, the fund's revenues that originate from oil-related activities are only available for a finite period of time due to the exhaustible character of nonrenewable natural resources.

3.1 Description of the model

Investment opportunities. The fund's manager has costless access to two tradable assets. A money market account with a constant, continuously compounded, real return r (risk-free bond), and a risky asset with dividends continuously reinvested (stock market index) with price S_t that evolves over time according to a geometric Brownian motion

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma_S S_t \mathrm{d}Z_{S,t}, \tag{3.1}$$

where μ denotes the asset's constant instantaneous return, $\sigma_S > 0$ is the constant volatility, and $Z_{S,t}$ is a standard Brownian motion. Thus, we assume that the manager's investment opportunity set is constant. Let $\alpha_t \in [0, 1]$ denote the fraction of financial wealth invested in the risky asset at time t, while the remainder, $1-\alpha_t$, is invested in the risk-free asset.

Oil income. Assuming zero exploration (and discoveries) of new reserves, the availability of the natural resource Q_t decreases over time at a constant extraction rate $\kappa_Q > 0$

$$\mathrm{d}Q_t = -\kappa_Q Q_t \mathrm{d}t, \tag{3.2}$$

until full depletion. We denote by $\hat{T} < \infty$ the time of depletion, which is assumed to be know with certainty by the SWF manager at the beginning of the planning horizon. The price per unit of the commodity is assumed to follow a geometric Brownian motion with drift rate κ_P, and constant volatility $\sigma_P > 0$,

$$\mathrm{d}P_t = \kappa_P P_t \mathrm{d}t + \sigma_P P_t \left(\rho_{PS}\mathrm{d}Z_{S,t} + \sqrt{1-\rho_{PS}^2}\mathrm{d}Z_{P,t}\right), \tag{3.3}$$

where $Z_{P,t}$ is a standard Brownian motion independent of $Z_{S,t}$, and $\rho_{PS} \in [-1, 1]$ denotes the instantaneous correlation between the return to the risky asset and the growth in the commodity price.

Let $Y_t = P_t Q_t$ denote the SWF's continuous stream of non-negative income. As long as the resource is not yet depleted, Itô's Lemma implies that the fund's revenues follow

$$\mathrm{d}Y_t \;=\; \kappa Y_t \mathrm{d}t + \sigma_P Y_t \left(\rho_{PS}\mathrm{d}Z_{S,t} + \sqrt{1-\rho_{PS}^2}\mathrm{d}Z_{P,t}\right) \quad \forall t \leq \hat{T}, \tag{3.4}$$

where $\kappa = (\kappa_P - \kappa_Q)$ represents the expected income growth. After the natural resource has been depleted it follows that $Y_t = 0$ for all $t > \hat{T}$. Equation (3.4) assumes that all the uncertainty in the oil income arises from the exogenous variation in the commodity price set in the world markets.

In the particular case where $|\rho_{PS}| = 1$, financial markets are complete and all the uncertainty in the oil income process is spanned by the stock price process. In other words, the stream of revenues can be perfectly replicated by some trading strategy in the financial markets and hence, valued as a traded asset. If $|\rho_{PS}| < 1$, the markets are said to be incomplete and the income risk is not fully spanned by financial markets.

Preferences. We assume that the manager displays recursive preferences as first proposed by Kreps and Porteus (1978), Epstein and Zin (1989) and Weil (1990), and extended to continuous time by Duffie and Epstein (1992a,b). This allow us to disentangle the effects that risk aversion and intertemporal substitution have separately on the optimal investment and consumption decisions. In particular, the preferences of the fund's manager are given by

$$V_t = \mathbb{E}_t \left[\int_t^{\infty} f(C_s, V_s) \mathrm{d}s \right], \tag{3.5}$$

where $f(C_s, V_s)$ is a normalized aggregator of current consumption, C_s, and continuation utility, V_s. In its more general form, the aggregator is defined as

$$f(C, V) = \beta\theta V \left\{ \left[\frac{C}{\left[(1-\gamma) V\right]^{\frac{1}{1-\gamma}}} \right]^{1-\frac{1}{\psi}} - 1 \right\}, \tag{3.6}$$

where $\beta > 0$ is the rate of time preference, $\gamma > 0$ denotes the coefficient of relative risk aversion (RRA) towards atemporal bets, $\psi > 0$ denotes the elasticity of intertemporal substitution (EIS), so that $1/\psi$ can be understood as aversion towards intertemporal fluctuations, and $\theta = (1-\gamma)/(1-\psi^{-1})$. The normalized aggregator exhibits the property that for $\gamma > 1/\psi$, the investor prefers early over later resolution of uncertainty. The ability to separate the investor's risk aversion from her aversion to intertemporal substitution is important in order to generate a smooth path for the consumption-to-wealth ratio that emulates the intergenerational concerns of the government without affecting the short-term allocation strategies that can be achieved through portfolio diversification. If $\gamma = 1/\psi$ it follows that $\theta = 1$ which makes the recursion in (3.6) linear, and the preferences in (3.5) collapse to the usual time-additive utility model. In this case it is no longer possible to separately investigate how the manager's optimal portfolio is affected by her attitudes towards risk without affecting at the same time intertemporal choices.

The manager's decision problem. Let W_t denote the fund's financial wealth at time t, i.e., the value of the portfolio of financial assets held at time t. Given an initial endowment of financial wealth, $W_0 = w > 0$, and a value for oil revenues, $Y_0 = y > 0$, the fund's manager chooses a consumption plan $\{C_t\}_{t=0}^{\infty}$ and an investment strategy $\{\alpha_t\}_{t=0}^{\infty}$ that maximize the present discounted value of her non-expected recursive utility subject to the evolution of wealth and income. We further assume that the fund is able to continuously rebalance its portfolio, and does not face restrictions on borrowing or short sales. Throughout it is assumed that the manager knows the stochastic processes that drive the oil and stock prices. More specifically, for all $t \in [0, \infty)$ she solves

$$J(w, y) = \max_{\{C_t, \alpha_t\}_{t=0}^{\infty}} V_t \tag{3.7}$$

subject to

$$\mathrm{d}W_t = (\mu_{P,t}W_t - C_t + Y_t)\,\mathrm{d}t + \sigma_{P,t}W_t\mathrm{d}Z_{S,t}, \tag{3.8}$$

$$\mathrm{d}Y_t = \kappa Y_t\mathrm{d}t + \sigma_P Y_t\left(\rho_{PS}\mathrm{d}Z_{S,t} + \sqrt{1-\rho_{PS}^2}\mathrm{d}Z_{P,t}\right) \quad \forall t < \hat{T}, \tag{3.9}$$

where J_t is the problem's value function, and where we have defined

$$\mu_{P,t} = \alpha_t\,(\mu - r) + r, \text{ and } \sigma_{P,t} = \alpha_t\sigma_S,$$

to be the instantaneous expected return and volatility per unit of financial wealth on the composite portfolio held by the SWF. Note that for all $t \geq \hat{T}$ it follows that $Y_t = 0$.

3.2 Solution under complete and incomplete markets

To solve the problem faced by the SWF's manager we invoke the principle of optimality to break the infinite horizon problem in (3.7) into two subproblems according to the known time of depletion of the commodity

$$J\,(w, y) = \max_{\{C_t,\alpha_t\}_{t=0}^{\hat{T}}} \mathbb{E}_0\left[\int_0^{\hat{T}} f(C_t, J_t)\mathrm{d}t + J\,(W_{\hat{T}})\right].$$

Thus for $t < \hat{T}$, the period in which the fund receives an income stream from oil activities, the manager solves a finite horizon problem with terminal utility given by $J\,(W_{\hat{T}})$, where $W_{\hat{T}}$ represents the fund's financial wealth at time of depletion $\hat{T}$. For $t \geq \hat{T}$, the manager no longer receives oil income and faces the following infinite horizon allocation problem

$$J\,(W_{\hat{T}}) = \max_{\{C_t,\alpha_t\}_{t=\hat{T}}^{\infty}} \mathbb{E}_{\hat{T}}\left[\int_{\hat{T}}^{\infty} f(C_s, J_s)\mathrm{d}s\right].$$

Therefore, our solution strategy consists of solving the allocation problem in two stages. In the first stage we compute the optimal consumption and investment policies that will prevail for $t \geq \hat{T}$. The second stage uses the value of the optimal program at time $\hat{T}$ found in the first stage as a terminal condition to solve for the optimal allocations for all $t < \hat{T}$.

First stage. After the nonrenewable resource has been fully depleted it follows that $Y_t = 0$ for all t, and the problem faced by the fund's manager corresponds to a standard infinite horizon asset allocation problem with constant investment opportunities and complete financial markets (cf. Svensson, 1989). As shown in Appendix A, the *Hamilton-Jacobi-Bellman* (HJB) equation for this problem is

$$0 = \max_{\{C,\alpha\}}\left\{f\,(C, J) + J_W\,[rW + (\mu - r)\,\alpha W - C] + \frac{1}{2}\sigma_S^2 J_{WW}\,(\alpha W)^2\right\},$$

with $J_W = \partial J(W)/\partial W$ and $J_{WW} = \partial^2 J(W)/\partial W^2$, and where we have suppressed time indexes to reflect the recursive nature of the corresponding dynamic programming problem. The next proposition summarizes the closed form solution to the first stage problem.

Proposition 1. *In the absence of commodity income, i.e., $Y_t = 0$, and constant investment opportunities, the manager's optimal value function for any $t \geq \hat{T}$ is given by*

$$J(W) = \frac{\beta^\theta}{1-\gamma} \mathbb{G}_\infty^{-\frac{\theta}{\psi}} W^{1-\gamma}, \tag{3.10}$$

where the constant $\mathbb{G}_\infty$ is given by

$$\mathbb{G}_\infty = \beta\psi + (1-\psi) r + \frac{(1-\psi)}{2\gamma} \left(\frac{\mu - r}{\sigma_S} \right)^2. \tag{3.11}$$

The optimal share of financial wealth invested in the risky asset is given by

$$\alpha_t = \frac{1}{\gamma} \frac{\mu - r}{\sigma_S^2}, \tag{3.12}$$

while the optimal consumption-to-financial wealth ratio is

$$\frac{C_t}{W_t} = \mathbb{G}_\infty. \tag{3.13}$$

Proof. See Appendix A. □

Equation (3.12) suggests that the optimal demand for the risky asset for $t \geq \hat{T}$ is constant and completely characterized by the market price of risk (the ratio of the expected risk premium to the asset's volatility) rescaled by the asset's volatility, and the manager's coefficient of RRA. It is also independent of the EIS and the subjective discount rate. The higher the coefficient of RRA, the lower the investment in stocks and the higher the investment in the risk-free asset (see Merton, 1969).

On the other hand, the optimal consumption is given by the modified Keynes-Ramsey rule in (3.13). It suggests that optimal consumption is a linear function of the fund's financial wealth. The marginal propensity to consume is constant and its value is determined by the coefficient of RRA, the manager's EIS, and the subjective discount rate. We can rewrite the optimal consumption-to-financial wealth ratio as

$$\frac{C_t}{W_t} = r_P + \psi (\beta - r_P),$$

where $r_P = \mu_P - \frac{\gamma}{2}\sigma_P^2$ is the (risk-adjusted) expected return on the composite portfolio (see van den Bremer et al., 2016 and Wang et al., 2016).

Second stage. The problem faced by the fund's manager from the perspective of time

$t < \hat{T}$ corresponds to a finite horizon allocation problem with stochastic income similar to that in Munk and Sørensen (2010) and Wang et al. (2016) with terminal utility given by the optimal value function at time of depletion. As shown in Appendix B, the HJB equation for this problem is

$$0 = \max_{C,\alpha} \Bigg\{ f(C,J) + J_t + J_W \left[rW + \alpha(\mu - r) W + Y - C \right] \\ + \frac{1}{2}\sigma_S^2 (\alpha W)^2 J_{WW} + \kappa Y J_Y + \frac{1}{2}\sigma_P^2 Y^2 J_{YY} + \sigma_S \sigma_P \rho_{PS} (\alpha W) Y J_{WY} \Bigg\}, \quad (3.14)$$

where $J_t = \partial J(t,W,Y)/\partial t$, $J_Y = \partial J(t,W,Y)/\partial Y$, $J_{YY} = \partial^2 J(t,W,Y)/\partial Y^2$, and $J_{WY} = \partial^2 J(t,W,Y)/\partial W \partial Y$. The terminal condition to this problem is given by the power utility of financial wealth in (3.10) evaluated at $t = \hat{T}$

$$J\left(\hat{T}, W, Y = 0\right) = \frac{\beta^\theta}{1-\gamma} \mathbb{G}_\infty^{-\frac{\theta}{\psi}} W_{\hat{T}}^{1-\gamma}. \quad (3.15)$$

The first order conditions for an interior solution for any $t < \hat{T}$ are given by

$$\frac{C_t}{W_t} = \left(\frac{\beta}{J_W}\right)^\psi \frac{\left[(1-\gamma) J\right]^{\frac{1-\psi\gamma}{1-\gamma}}}{W_t} \quad (3.16)$$

$$\alpha_t = \underbrace{\frac{1}{-\frac{W_t J_{WW}}{J_W}} \frac{\mu - r}{\sigma_S^2}}_{\text{Myopic demand}} + \underbrace{\frac{Y_t J_{WY}}{J_W} \frac{1}{-\frac{W_t J_{WW}}{J_W}} \frac{\sigma_P \rho_{PS}}{\sigma_S}}_{\text{Hedging demand}}. \quad (3.17)$$

Equation (3.16) results from the standard envelope condition $f_C = J_W$. Accordingly, the optimal consumption-to-financial wealth ratio is such that the marginal benefit of an additional unit of consumption is equal to the marginal utility of an additional unit of financial wealth. Equation (3.17) determines the optimal share of financial wealth allocated to the risky asset as the sum of two components. The first term, usually referred to as the *myopic* or *speculative demand,* corresponds to the investment strategy that a manager with a short investment horizon will follow, i.e. an investor that ignores what happens beyond the immediate next instant. It is defined by the standard mean-variance analysis of Markowitz (1952) that suggest that the demand for risky assets should be proportional to the asset's risk premium over the risk free asset, $(\mu - r)$, and inversely proportional to the asset's volatility, σ_S, and the investors's risk aversion captured by the curvature of the value function, $-W J_{WW}/J_W$.

The second term, usually referred to as *intertemporal hedging demand* or *excess risky demand*, represents the additional demand required by an investor with a long investment horizon in order to hedge against the risk of unexpected changes in the commodity income that cannot be fully eliminated (see Merton, 1969, 1971, 1973). It is determined by the volatility of income, σ_P, and its correlation with the stock returns, ρ_{PS}, the manager's coefficient of RRA, and his aversion to income risk as measured by YJ_{WY}/J_W. Importantly, this component suggests that the investor should increase her holding of the risky asset for increased levels of aversion to income risk and whenever its returns covary negatively with changes in the commodity income.

As seen from (3.16) and (3.17), the solution to the optimal consumption and investment share depend on the unknown time-dependent value function $J(t, W, Y)$. When financial markets are incomplete, i.e. for $|\rho_{PS}| < 1$, no closed-form solution is available and we need to resort to numerical methods in order to approximate the optimal behavior of the fund's manager. However, under the simplifying assumption of complete markets, i.e. $|\rho_{PS}| = 1$, the oil income can be valued as a stream of dividends which allows us to derive an analytical solution. Although this assumption is challenged by the empirically observed correlation between stock and oil prices, we now make use of the complete market solution to build the economic intuition on the main determinants of the optimal consumption and investment decisions when oil income is not perfectly spanned by the stock market. However, our main results are computed under the assumption of incomplete markets.

Complete market solution. Let us first define $\mathcal{O}_t \equiv \mathcal{O}\left(Y_t, t; \hat{T}\right)$ as the time t value of the SWF's oil wealth, i.e., the present discounted value of all future oil income streams.

Lemma 1 (Oil wealth under complete financial markets)**.** *Assume a complete financial market, i.e., $|\rho_{PS}| = 1$. Then, the SWF's oil wealth at time t is given by*

$$\mathcal{O}\left(Y_t, t; \hat{T}\right) = Y_t \mathcal{M}\left(t; \hat{T}\right), \tag{3.18}$$

for all $t < \hat{T}$. The time-dependent function $\mathcal{M}\left(t; \hat{T}\right)$ denotes the commodity income multiplier

$$\mathcal{M}\left(t; \hat{T}\right) = \frac{1}{r - \kappa \pm \sigma_P \lambda}\left[1 - \exp\left\{-\left(r - \kappa \pm \sigma_P \lambda\right)\left(\hat{T} - t\right)\right\}\right], \tag{3.19}$$

for $(r - \kappa \pm \sigma_P \lambda) \neq 0$, and where $\lambda = (\mu - r)/\sigma_S$ is the market price of risk. For $t \geq \hat{T}$ the oil income is zero, $Y_t = 0$, and thus $\mathcal{O}\left(Y_t, t; \hat{T}\right) = 0$.

Proof. See Appendix B.2. □

Lemma 1 shows that when markets are complete it is possible to decompose the level of oil wealth as the product between the current level of oil income, Y_t, and the the time-dependent income multiplier, $\mathcal{M}\left(t; \hat{T}\right)$. For a fixed time to depletion, the oil wealth will decrease as the natural resource is depleted at a rate that depends on the financial market returns, the expected growth rate of oil income, and the volatility of oil income.

Furthermore, the oil wealth will be higher for an income stream that is negatively correlated with the stock market, than for a similar income stream, but positively correlated with the stock market. Given the fixed time to depletion, a positive (negative) correlation implies that the future expected income will be discounted at a rate higher (lower) than the return on the risk-free asset.

Proposition 2. *Assume a complete financial market, i.e., $|\rho_{PS}| = 1$. Let $Y_t \mathcal{M}\left(t; \hat{T}\right)$ denote the market value of the fund's oil wealth at time t. Then, the optimal consumption-to-financial wealth ratio for all $t < \hat{T}$ is given by*

$$\frac{C_t}{W_t} = \mathbb{G}_\infty \left(1 + \frac{Y_t}{W_t}\mathcal{M}\left(t; \hat{T}\right)\right), \tag{3.20}$$

with $\mathbb{G}_\infty$ is given in (3.11).

The optimal share of financial wealth invested in the risky asset for all $t < \hat{T}$ is

$$\alpha_t = \frac{1}{\gamma}\left(\frac{\mu - r}{\sigma_S^2}\right)\left(1 + \frac{Y_t}{W_t}\mathcal{M}\left(t; \hat{T}\right)\right) - \frac{Y_t}{W_t}\mathcal{M}\left(t; \hat{T}\right)\frac{\sigma_P \rho_{PS}}{\sigma_S}. \tag{3.21}$$

Proof. See Appendix B.2. □

The optimal consumption is given by a modified Keynes-Ramsey rule in (3.20). In the presence of stochastic oil income, consumption at any point in time is linear in the fund's total wealth, $(W_t + \mathcal{O}_t)$. The marginal propensity to consume out of total wealth is constant, and its determinants are the same as in the case of no oil wealth: the coefficient of RRA, the manager's EIS, the market price or risk, and subjective discount rate. However, using Lemma 1 it is straightforward to show that the propensity to consume out of income is increasing in the financial wealth-to-oil income ratio, W_t/Y_t, and the expected income growth rate (oil income multiplier), and decreasing in the current income rate. As opposed to the case $Y_t = 0$, the optimal consumption-to-financial wealth ratio is no longer constant. Instead, the marginal propensity to consume out of financial wealth will fall over time in line with the decrease in the oil wealth-to-financial wealth ratio. As the fund's oil wealth runs out over time, the consumption-to-financial wealth ratio converges to the constant level given in (3.13).

The first term on the right hand side of (3.21) is the myopic demand for the risky asset, while the second term is the intertemporal hedging demand. Similar to the case without income, the optimal investment share is independent of the EIS. However, the presence of stochastic oil income ($\sigma_P > 0$) will have a magnifying effect on the investment share through the intertemporal hedging component, as long as the commodity income is correlated with the returns of the risky asset, $\rho_{PS} \neq 0$. The direction of this effect will depend on the sign of the instantaneous correlation. A negative (positive) correlation implies a positive (negative) hedging demand. Importantly, the optimal investment share is no longer constant over time. In particular, its trajectory will depend on the path of the oil wealth-to-financial wealth ratio, $\mathcal{O}_t/W_t$, and as the natural resource is depleted, it will converge to the constant level in (3.12). This result also holds for the case of a deterministic stream of income ($\sigma_P = 0$), or a stochastic stream of income that is uncorrelated with asset returns ($\rho_{PS} = 0$). The optimal demand for the risky asset in (3.21) can be alternatively written as

$$\alpha_t = \frac{1}{\gamma}\left(\frac{\mu - r}{\sigma_S^2}\right) + \left(\frac{1}{\gamma}\left(\frac{\mu - r}{\sigma_S^2}\right) - \frac{\sigma_P \rho_{PS}}{\sigma_S}\right)\frac{Y_t}{W_t}\mathcal{M}\left(t;T\right),$$

where the first term is identical to the optimal investment without income and the second term represents the effect of commodity income on the optimal investment strategy. Consequently, as long as the oil wealth-to-financial wealth ratio is positive, the convergence of the optimal investment strategy to its long-run value will depend on the coefficient of RRA, the expected excess return, and the correlation coefficient. For $\left(\mu - r\right)/\sigma_S > \gamma\sigma_P\rho_{PS}$, the convergence will be from above: if the expected excess return exceeds the covariance between the stock and oil prices, the fund's manager should decrease the fraction of financial wealth invested in the stock market as the oil reserves are depleted. Furthermore, low values of the coefficient of RRA are associated with an intertemporal demand for the risky asset that exhibits larger deviations from the long-run value along its transition.

Incomplete market solution. Whenever $|\rho_{PS}| < 1$, it is no longer possible to value the oil income as a traded asset and thus the portfolio strategy in Proposition 2 turns out to be suboptimal. In particular, it will overestimate the hedging demand component by incorrectly assuming that the income risk can be fully replicated in the financial markets. This will lead to higher investment shares and lower consumption rates. For the case of an investor with time-additive preferences ($\theta = 1$) and stochastic labor income, Munk and Sørensen (2010) show that the complete market solution can be used to approximate the optimal allocations in incomplete markets when the investor does not face liquidity nor investment constraints. The reason is that although the allocation policies are suboptimal, they imply small utility losses even for small correlations between income and asset prices. However, their recommendation does not carry to the case of a commodity SWF like the one studied in this paper. Even though the correlation coefficient between the price of oil and the price of equity is low, the high volatility of the oil price will result in a portfolio weight on the risky asset that is unreasonable high relative that suggested by the optimal strategy. This in turn can lead to large utility losses from implementing suboptimal allocations due to an excessive exposure to risk.

To characterize the model's optimal policies when oil income is not perfectly hedgeable we use a finite difference approach based on the work in Gomez (2019). In particular, the solution to the HJB equation in (3.14) is approximated backwards in time starting from the terminal value in (3.15). At each point in time, we maximize the HJB equation over all possible consumption and investment choices along a pre-defined grid for the state variables. To obtain a stable and more efficient approximation of the unknown value function we use the state space reduction of Duffie et al. (1997) to reduce the number of state variables from two to one by exploiting the homogeneity properties of the HJB equation. A complete description of the state reduction problem as well as of the finite difference solution method can be found in Appendix B.1.

4 Quantitative model under incomplete markets

This section explores the quantitative predictions of the model. We begin our analysis by calibrating the model parameters. Using these values we can compute the optimal value function, as well as the consumption and investment policies, that determine the long-run behavior of the fund in the absence of oil income due to the depletion of the nonrenewable natural resource. Using the resulting indirect utility, we then approximate the solution to finite horizon problem in order to obtain the optimal consumption and investment profiles in the presence of a continuous stream of oil income. We also analyze the effects of different values for the correlation coefficient and the coefficient of RRA on the optimal portfolio strategy.

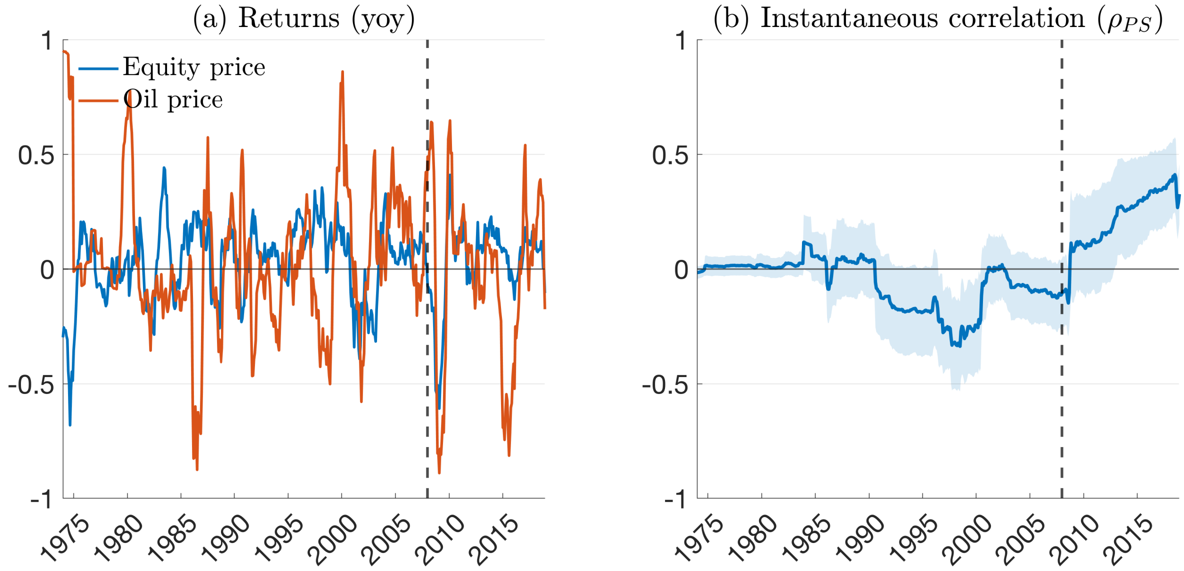
4.1 Calibration

We split the parameters of the model into two groups, $\mathbf{\Theta} = \{\boldsymbol{\theta}_1, \boldsymbol{\theta}_2\}^\intercal$. The first group consists of all the parameters associated with the exogenous processes that drive the dynamics of the stock and oil prices, $\boldsymbol{\theta}_1 = \{\mu, \sigma_S, \rho_{PS}, \kappa_P, \sigma_P\}^\intercal$. The second group includes those parameters related to the fund's preferences, the return on the market's risk-free asset, and some additional parameters associated to the generation of income, $\boldsymbol{\theta}_2 = \left\{\beta, \gamma, \psi, r, \kappa_Q, \hat{T}\right\}^\intercal$. Table 1 summarizes our benchmark calibration. Time is measured in years and parameters should be interpreted accordingly.

Estimated parameters ($\boldsymbol{\theta}_1$). In what follows we assume that the risky asset is represented by the U.S. stock market, and the oil price is gauged by the West Texas Intermediate (WTI) price of crude oil. To measure stock prices, we use monthly nominal returns on the value-weighted index excluding distribution from CRSP, while the monthly WTI price of oil is retrieved from the FRED database. All prices in the model are real. We use the monthly value of the Consumer Price Index from the U.S. Bureau of Labor Statistics as deflator. Panel (a) in Figure 4.1 illustrates the monthly year-over-year (yoy) returns for each of the variables for the period 1974:1-2018:12.

Let $\mathbf{X}_t = (S_t, P_t)^\intercal$. According to (3.1) and (3.3), the dynamic of prices can be repre-

Figure 4.1. Crude oil and equity prices (1974:1 - 2018:12). Panel (a) plots the monthly year-over-year (yoy) change in the real price of stocks (VWI CRSP) and crude oil (WTI). Panel (b) plots the estimated instantaneous correlation between the real price of stocks and crude oil using a 10 years estimation rolling window. Dashed areas represent 95% confidence intervals. The dashed vertical line delimits the end of the sample used in the maximum likelihood estimation (December 2017).



sented by the following system of Markovian stochastic differential equations (SDE)

$$\mathrm{d}\mathbf{X}_t = \mu\left(\mathbf{X}_t;\boldsymbol{\theta}_1\right)\mathrm{d}t + \sigma\left(\mathbf{X}_t;\boldsymbol{\theta}_1\right)\mathrm{d}\mathbf{Z}_t, \tag{4.22}$$

where $\mathbf{Z}_t = \left(Z_{S,t}, Z_{P,t}\right)^\intercal$ is a vector of independent standard Brownian motions, and where

$$\mu\left(\mathbf{X}_t;\boldsymbol{\theta}_1\right) = \begin{bmatrix} \mu S_t \\ \kappa_P P_t \end{bmatrix}, \quad \sigma\left(\mathbf{X}_t;\boldsymbol{\theta}_1\right) = \begin{bmatrix} \sigma_S S_t & 0 \\ \sigma_P \rho_{PS} P_t & \sigma_P \sqrt{1-\rho_{PS}^2} P_t \end{bmatrix}.$$

Let $\mathbb{P}\left(\mathbf{X}_0, \mathbf{X}_\Delta, \ldots, \mathbf{X}_{n\Delta}\right)$ denote the joint density of a sample of n discrete measurements $\mathbf{X} = \{\mathbf{X}_{i\Delta}\}_{i=0}^n$, where $\Delta = 1/12$ denotes the fixed (monthly) observation frequency. Using the properties of joint densities, and the Markovian nature of the process in (4.22), it is possible to decompose $\mathbb{P}\left(\mathbf{X}_0, \mathbf{X}_\Delta, \ldots, \mathbf{X}_{n\Delta}\right)$ as the product of a conditional and a marginal density

$$\mathbb{P}\left(\mathbf{X}_0, \mathbf{X}_\Delta, \ldots, \mathbf{X}_{n\Delta};\boldsymbol{\theta}_1\right) = \mathbb{P}\left(\mathbf{X}_0;\boldsymbol{\theta}_1\right)\prod_{i=1}^{n}\mathbb{P}\left(\mathbf{X}_{i\Delta}|\mathbf{X}_{(i-1)\Delta}\right).$$

Ignoring the dependence on the initial observation, $\mathbb{P}\left(\mathbf{X}_0;\boldsymbol{\theta}_1\right)$, and taking logarithms, the log-likelihood of the data reads

$$\mathcal{L}_n\left(\mathbf{X};\boldsymbol{\theta}_1\right) = \sum_{i=1}^{n}\log\mathbb{P}\left(\mathbf{X}_{i\Delta}|\mathbf{X}_{(i-1)\Delta};\boldsymbol{\theta}_1\right), \tag{4.23}$$

whilst the maximum likelihood estimator (MLE) of $\boldsymbol{\theta}_1$ is defined as

$$\hat{\boldsymbol{\theta}}_1 = \arg\max_{\boldsymbol{\theta}_1} \ \mathcal{L}_n\left(\boldsymbol{\theta}_1; \mathbf{X}\right).$$

In general, the conditional probability density $\mathbb{P}\left(\mathbf{X}_{i\Delta}|\mathbf{X}_{(i-1)\Delta}; \boldsymbol{\theta}_1\right)$, and hence the log-likelihood function, is not available in closed form. We follow Ait-Sahalia (2002, 2008) and approximate the log-likelihood function in (4.23) according to

$$\mathcal{L}_n\left(\mathbf{X}; \boldsymbol{\theta}_1\right) \approx -\frac{1}{2}\log\left|\boldsymbol{\Sigma}\left(\mathbf{X}_t; \boldsymbol{\theta}_1\right)\right| + \mathcal{L}_n\left(\mathbf{Y}; \boldsymbol{\theta}_1\right), \tag{4.24}$$

where $\boldsymbol{\Sigma} = \sigma\sigma^{\intercal}$ is the infinitesimal variance-covariance of the stochastic process $\mathbf{X}$, and $\mathcal{L}_n\left(\mathbf{Y}; \boldsymbol{\theta}_1\right)$ is the first order closed-form approximation to the log-likelihood function of the transformed unitary diffusion process $\mathbf{Y} = \{\mathbf{Y}_{i\Delta}\}_{i=0}^{n}$ given by[3]

$$\mathcal{L}_n\left(\mathbf{Y}; \boldsymbol{\theta}_1\right) = -\log\left(2\pi\Delta\right) + \frac{C_{\mathbf{Y}}^{(-1)}\left(\mathbf{Y} \mid \mathbf{Y}_0; \boldsymbol{\theta}_1\right)}{\Delta} + C_{\mathbf{Y}}^{(0)}\left(\mathbf{Y} \mid \mathbf{Y}_0; \boldsymbol{\theta}_1\right) + C_{\mathbf{Y}}^{(1)}\left(\mathbf{Y} \mid \mathbf{Y}_0; \boldsymbol{\theta}_1\right)\Delta.$$

The approximation constants $C_{\mathbf{Y}}^{(-1)}$, $C_{\mathbf{Y}}^{(0)}$ and $C_{\mathbf{Y}}^{(1)}$ are provided in Theorem 1 of Ait-Sahalia (2008). The approximated log-likelihood function in (4.24) converges to the true log-likelihood function of the data as $\Delta \to 0$, and thus all the standard statistical properties of the (quasi-)MLE, including classical inference, carry over.

Panel (a) in Table 1 reports the estimation results for the model in (4.22) using monthly data on stock and oil prices that span the period from 1973:1 - 2007:12. It also presents standard errors robust to autocorrelation and heteroskedasticity. The estimated (annual) volatilities of the stock and oil price changes are 15.92% and 35.22%, respectively, while the instantaneous drift parameters, although not statistically significant at conventional confidence levels, imply an annual expected stock return of 4.13%, and an annual expected growth of oil price of 9.73%. The estimated instantaneous correlation between stock and oil prices indicates a negative and statistically significant association equal to -6.76% (per annum) during these sample period. Our estimate is consistent with the evidence provided in Jones and Kaul (1996), Sadorsky (1999), Gorton and Rouwenhorst (2006), Lee and Chiou (2011) and Bhardwaj et al. (2015) for the period prior to the Great Recession.

Although the negative correlation between oil prices and equity is in line with conventional wisdom, recent evidence suggest that this correlation has increased considerable during the last decade. Buyuksahin and Robe (2014) attribute this increase to the observed growth in commodity-market activity led by hedge funds but also to macroeconomic fundamentals and the TED spread. Lombardi and Ravazzolo (2016) provide further

[3] The multivariate diffusion in (4.22) is reducible in the sense that it is possible to transform the diffusion process $\mathbf{X}$ into a diffusion process $\mathbf{Y}$ with diffusion matrix equal to the identity matrix (see Ait-Sahalia, 2008).

Table 1. Parameter values. Panel (a) reports the maximum likelihood estimates and associated standard errors for the parameters that describe the dynamics of the exogenous driving forces in the economy. The estimation uses monthly data on U.S. real stock prices and real WTI oil prices for the period 1973-2007. Effective number of observations is 420. Panel (b) report calibrated parameter values that describe the investor's preferences and some additional parameters that replicate salient features of the investment problem faced by the Norwegian SWF.

Panel (a): Estimated parameters, $\boldsymbol{\theta}_1$			
Description	Parameter	MLE	Standard error
Drift of oil price growth	κ_P	0.0973	0.0606
Drift stock price growth	μ	0.0413	0.0264
Diffusion oil price growth	σ_P	0.3522	0.0908
Diffusion stock price growth	σ_S	0.1592	0.0087
Corr. stock price and oil price growth	ρ_{PS}	-0.0676	0.0403
Panel (b): Calibrated parameters, $\boldsymbol{\theta}_2$			
Description	Parameter	Value	
Risk-free rate	r	0.011	
Discount rate	β	0.020	
Risk aversion coefficient	γ	3.000	
Elasticity of intertemporal substitution	ψ	2.000	
Extraction rate	κ_Q	0.078	
Time to depletion (years)	$\hat{T}$	60	

evidence of the higher correlation observed at the onset of the financial crisis of 2008. Additionally, Datta et al. (2018) argue that the increase in the oil-stock price correlation that started in 2008 can be explained by the nominal interest rates being constrained by the zero lower bound. To verify this claim, we extend our sample period until the end of 2018 and, using the maximum likelihood procedure described above, produce rolling estimates using a fixed window of 10 years. Panel (b) in Figure 4.1 reports the rolling estimates together with 95% confidence bands. The results suggest that the long-term co-movement between oil prices and stock prices is not constant. Instead, the estimates suggest three different phases over the last 45 years: (i) a period of zero correlation between 1973 and 1989; (ii) a period of negative correlation between 1990 and 2007; and (iii) a period of positive correlation from 2008 until today. We use the variation in these correlations to perform a sensitivity analysis.

Fixed parameters ($\boldsymbol{\theta}_2$). The calibration of the second group of parameters is summarized in Panel (b) of Table 1. We set the subjective discount rate to $\beta = 2\%$ per year, the coefficient of RRA to $\gamma = 3.0$ and the EIS to $\psi = 2.0$. These values are standard in the asset pricing and asset allocation literature and, as it is shown below, imply a relatively smooth path for the consumption-to-financial wealth ratio that is consistent with the spending rule mandate of the Norwegian SWF.

Using data from BP (2019) we calibrate the extraction rate of crude oil and the time-to-depletion (see Figure 2.2). At the end of 2018, Norway had 8,600 million of barrels of proved reserves. During the same year, it produced 1,844 thousand barrels per day. The production-to-reserves ratio implies an extraction rate for 2018 of $\kappa_Q = 7.8\%$ per year. Together with the estimated average growth rate of oil price, they imply and expected oil income growth rate of $\kappa = 1.9\%$ per year. On the other hand, we set the time-to-depletion equal to $\hat{T} = 60$ years. This corresponds to the number of years that it would take to exhaust 99% of the oil reserves available in 2018 under the assumption of zero exploration (and discoveries) of new reserves, and a constant extraction rate equal to κ_Q.

Finally, the return on the risk-free asset is measured as the sample average of the annualized real return on the 90 day U.S. Treasury bill. For the U.S. postwar period, this corresponds to $r = 1.11\%$ per year. Together with the estimated values for the equity's expected return, μ, and volatility, σ_S, our calibration implies that in the absence of oil income, i.e. starting from year $\hat{T} = 60$ and onwards, the fund's manager should investment 40% of its financial wealth on the risky asset and consume 2.3 percent of its financial wealth each period in perpetuity.

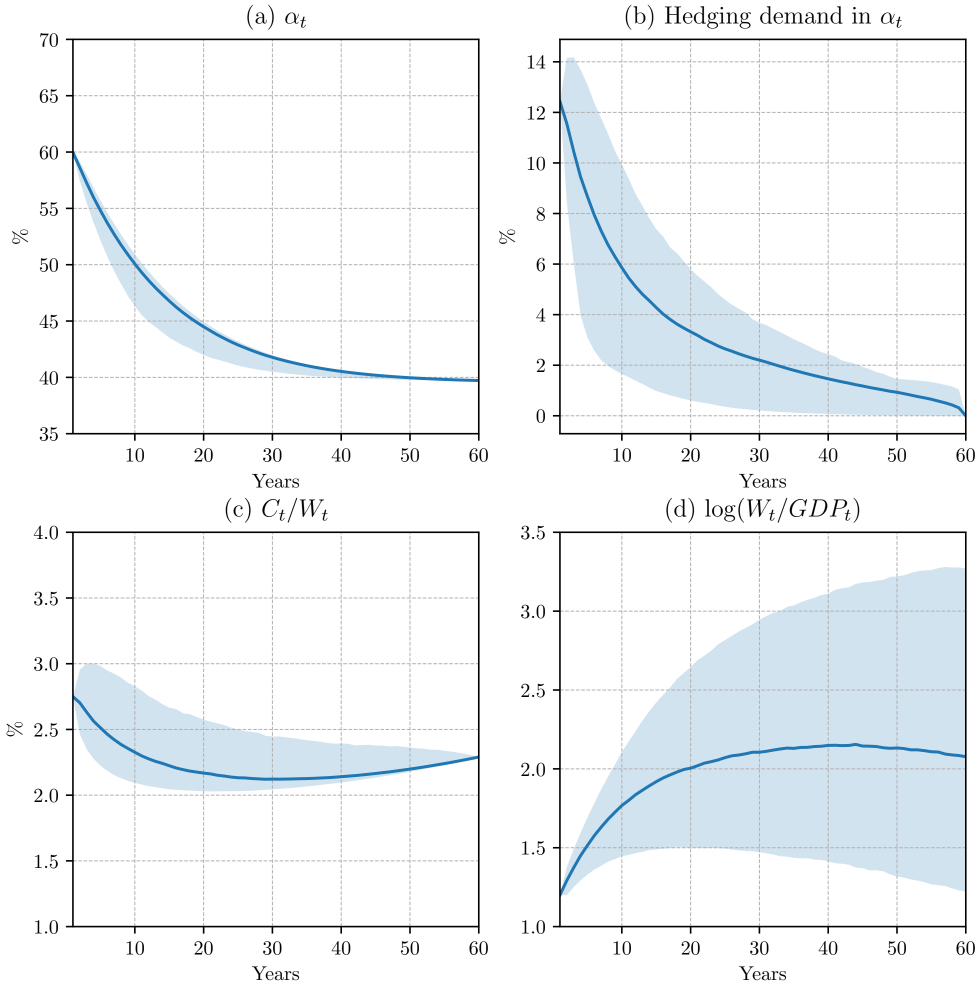
4.2 Optimal consumption and investment policies

In the following we use the benchmark calibration in Table 1 to illustrate the solution to the model when the oil income is not fully spanned by the financial markets. Due to market incompleteness, we approximate the solution numerically at each point in time until the natural resource has been depleted, i.e. for all $t \in [0, \hat{T}]$. We then simulate each of the variables in the model and report the median value over 10,000 samples. In the simulations, we assume an initial financial wealth-to-oil income ratio equal to $W_0/Y_0 = 9.4$, a number that is consistent with that reported by the Norwegian GPFG in 2018.

Figure 4.2 plots the optimal path for selected variables for all $t \in [0, \hat{T}]$, together with intervals around the median that represent the 15th and 85th percentiles of the simulated distributions. Panel (a) illustrates the optimal portfolio share of financial wealth invested in equity. The results indicate that the fund's manager should initially invest 60 percent of her financial wealth into stocks, a figure that is 20 percent above the optimal share that would prevail in the long-run in the absence of oil income, i.e. for $t \geq \hat{T}$. As the reserves of oil are depleted, the fund's manager should thereafter decrease gradually its position in the risky asset until she reaches a long-run allocation of 40 percent in about 60 years.

The excess demand for the risky asset, relative to the long-run position, is the result of two complementary effects. First, a wealth or leverage effect that results from the capitalized value of all future oil income transferred to the fund, i.e., the non-tradable "underground" oil wealth available at the beginning of the investment horizon. Second, a positive hedging demand that, given our benchmark calibration, is primarily driven by the high volatility of oil prices, and not by their correlation with the risky asset. Panel (b) plots this hedging component using the decomposition in (3.17). We see that the intertemporal hedging demand accounts for nearly 12 percent of the additional financial wealth invested

Figure 4.2. Optimal Asset Allocation under Incomplete Markets. Panels (a)–(d) plot, respectively, the optimal share of financial wealth invested in equity, the hedging demand as a fraction of financial wealth, the optimal consumption-to-financial wealth ratio, and the evolution of financial wealth-to-mainland GDP ratio using the parameters in Table 1. The solid lines represent the median value over M =10,000 simulated paths, each of them of $T = 60$ sample points. The shaded areas represents the 15 and 85 percentiles from the sampling distribution of the simulated series.



in equity markets at the beginning of the investment horizon. This additional demand creates a hedge against fluctuations in oil prices and thus in the fund's revenues. Similar to the overall investment share in equity, the hedging demand decreases over time hand-in-hand with the oil wealth-to-financial wealth ratio. In the long run, once the commodity is fully depleted and the fund stops receiving oil revenues, the hedging demand becomes zero.

Our results suggest the optimal portfolio profile is in sharp contrast with the effective investment strategy executed by the Norwegian GPFG between 1998 and 2007 shown in Figure 2.1. In a period characterized by a low and negative correlation between stock and oil prices, the fund's investment strategy was rather conservative. Although in line with the mandate given by the Executive Board to the NBIM, the share of financial wealth invested in equity remained relatively constant at around 40 percent.

The median consumption-to-financial wealth ratio is illustrated in Panel (c). Our benchmark calibration produces a relatively stable optimal spending rule over time as a function of the fund's financial wealth. At the beginning of the investment horizon, when the total wealth is large, the optimal consumption is 2.7 percent of the fund's financial wealth. After 10 years, the spending path stabilizes at around 2.2 percent of the financial wealth for the remaining time horizon, until it reaches its long-run value of 2.3 percent after 60 years. Our results, although somewhat lower, are consistent with the constant transfer rule described in the investment mandate of the Norwegian GPFG.

Finally, Panel (d) shows the path of the financial wealth-to-mainland GDP ratio over time. In the simulations we use an initial value of $W_0/GDP_0 = 3.32$, a value that is consistent with the real GDP of mainland Norway for the year 2019. Assuming a constant growth rate for real GDP of mainland Norway of 1.07 percent per year (the average growth rate observed over the period 2000-2019), our results suggest that following the optimal consumption-investment policies allows the fund to reach, over the course of 60 years, a level of financial wealth-to-GDP ratio that doubles its initial endowment. This in turn implies that the fund's financial wealth will grow at an average annual rate of 2.55 percent from 9.4 in year zero to 42.36 in year 60 once the commodity has been depleted.

Our results can also be used to argue against the popular advice of using the complete market solution to approximate the optimal consumption and investment strategies of an unconstrained investor with stochastic income when markets are incomplete (see for example Bick et al., 2009; Munk and Sørensen, 2010; van den Bremer et al., 2016). Suppose that the oil income risk is perfectly spanned by the risky asset and use Lemma 1 to compute the value of the underground oil wealth at each point in time. Then, using Proposition 2, and the observed correlation coefficient between oil and stock price changes, we can approximate the investment and consumption policies[4]. In general, we find that the use of the complete market solution would command the SWF to invest 535 percent of its financial wealth at the beginning of the planning horizon, a number that far exceeds the true portfolio share of 60 percent that prevails under incomplete markets. The associated intertemporal heding demand component amounts to nearly 140 percent, when the true excess demand is 12 percent. Under this scenario, the SWF needs to borrow an unreasonable large amount of funds in order to achieve the optimal portfolio. The reason behind this large demand for the risky asset is that, given the covariance structure between oil and stock price changes, the SWF will overestimate the value of its underground oil wealth when erroneously assuming that markets are complete. This will, in turn, imply that the investor will rely on a wrong measure of its true leverage possibilities, over invest in the risky asset, and expose the fund to unnecessary large amounts of risk. Therefore, our results suggest not to approximate the optimal investment strategy for oil-based SWFs using the assumption of complete markets, since the resulting portfolio weights on the

[4]The corresponding median trajectories from 10,000 simulations of the model together with their 15 and 85 percentiles are reported in Appendix C.

risky asset will be unreasonable high in the absence of liquidity or borrowing constraints. However, as shown in Munk and Sørensen (2010), the accuracy of the approximation remains valid in this case if the volatility of equity returns exceeds that of income[5].

4.3 Parameter sensitivity

In this subsection we examine the sensitivity of the optimal investment strategy to changes in the correlation between the stock price and the oil price, and the SWF's coefficient of RRA.

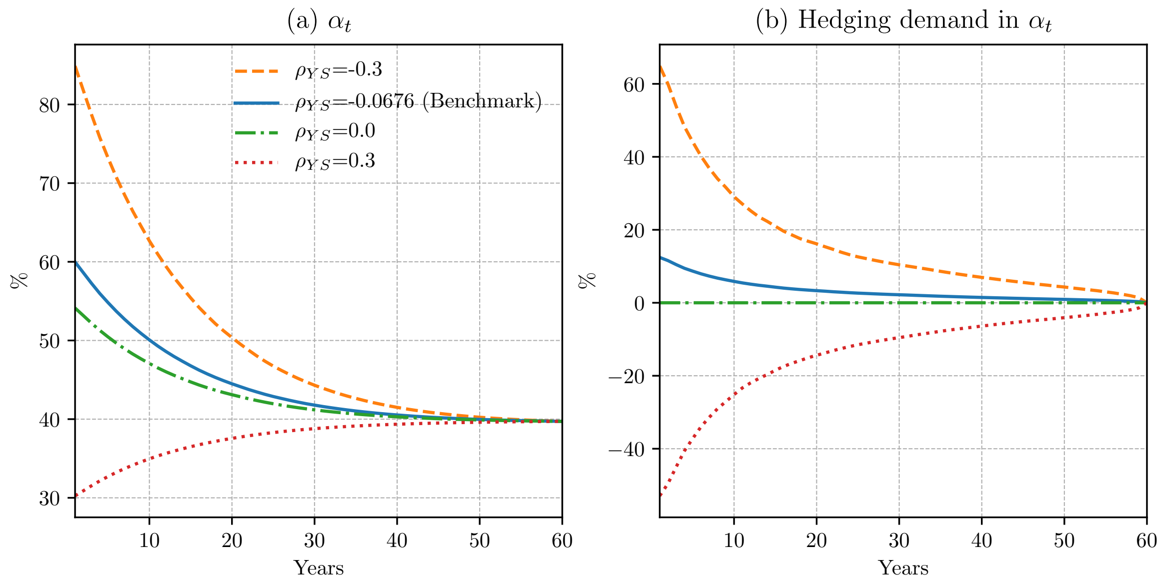
Correlation between asset returns and oil price changes. Motivated by time-varying estimates reported in Figure 4.1, we ask what are the consequences of different values of the correlation between the shocks to the oil price and the stock prices for the optimal portfolio allocation. The results are illustrated in Figure 4.3 where we consider different values of ρ_{PS}, all of them consistent with the three different episodes observed between 1974 and 2018. These include periods of high negative and positive correlation, as well as periods of zero correlation.

Panel (a) plots the median share of financial wealth invested in equity across 10,000 simulated paths. It shows that the optimal investment strategy is sensitive to changes in the correlation coefficient. In particular, the portfolio weight on the risky asset is a nonlinear and decreasing function of ρ_{PS}. As discussed previously, a negative correlation between the price of oil and the price of equity commands an initial position in stocks that exceeds its long-run value. Using (3.21) under complete markets as an approximation, we observe that the larger is the negative association between stock and oil prices, the larger are the gains from hedging, and thus the higher is the optimal demand for equity. In particular, while a correlation of nearly -7 percent implies an initial allocation of 60 percent, a correlation of -30 percent increases this position to over 80 percent. As shown in Panel (b), the hedging demand in these two examples account, respectively, for 12 percent and 65 percent of the initial financial wealth allocated to stocks.

If the price of oil and the price of equity are uncorrelated, then the optimal portfolio share is below 60 percent. This allocation is completely determined by the speculative demand component since there is no room for hedging. The decreasing path in this scenario is exclusively explained by a falling oil wealth-to-financial wealth ratio.

[5]Using quarterly data on U.S. aggregate income from the National Income and Product Accounts (NIPA) for the period 1951-2003, Munk and Sørensen (2010) estimate a volatility of disposable labor income of 2.08%, and a correlation coefficient between income and equity price changes of 16.73%. Their estimate of the volatility of the S&P500 index is 16.13%. When using aggregate income data from the Panel Study of Income Dynamics (PSID) survey, their estimate of the labor income volatility is 1.64%. On the other hand, and consistent with the evidence reported in Carroll and Samwick (1997) and Chamberlain and Hirano (1999) using PSID data, Viceira (2001) uses a volatility of labor income of 10% in his benchmark calibration. A similar value is estimated in Cocco et al. (2005), who additionally estimate a correlation coefficient with equity returns between 0% and -1.75%.

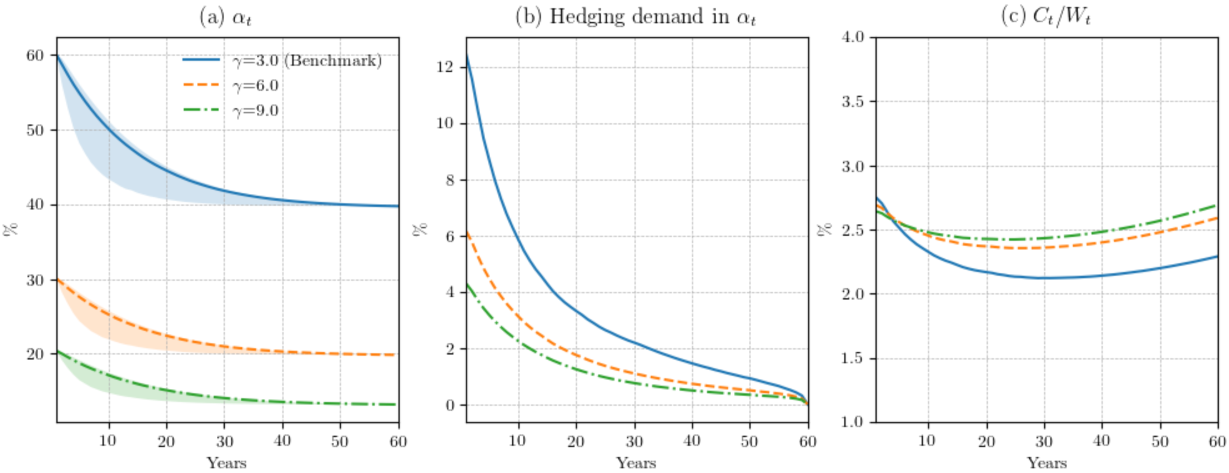
Figure 4.3. Sensitivity of the optimal investment share to the correlation coefficient, ρ_{PS}. Panel (a) plots the median value of the fraction of financial wealth invested into equity. The median is computed from M =10,000 simulated paths of the model for different values of the instantaneous correlation between real asset returns and changes in the real price of oil, ρ_{PS}. Panel (b) plots the associated median hedging demand.



Finally, a positive correlation of 30 percent results in a portfolio that invest a large fraction of wealth into the risk-free asset. In particular, the initial fraction of wealth invested in equity is reduced to 30 percent, a value that represents half of the share under our benchmark calibration. This low but positive fraction invested in the risky asset is the result of a long position that is simultaneously counterbalanced by a short position that aims to minimize the oil income risk via the hedging component. Moreover, the optimal investment rule should increase monotonically over time to reach its long-run value of 40 percent at time of depletion. Hence, if the correlation stays positive, as suggested by the recent empirical evidence, the current stock/bond mix in the portfolio of the Norwegian GPFG implies an unnecessary large exposure to risk.

Risk Aversion. As discussed previously, the coefficient of RRA, γ, plays an important role in shaping the intertemporal demand for the risky asset. Consequently, panel (a) in Figure 4.4 illustrates the median value of the optimal demand for equity for different values of this parameter across 10,000 simulated paths. In general, larger risk aversion leads the investor to take less risk. While our benchmark calibration with $\gamma = 3$ commands the fund to invest nearly 60 percent of its financial wealth into the risky asset at the beginning of the planning horizon, this fraction drops to 30 percent and 21 percent for coefficients of RRA equal to $\gamma = 6$ and $\gamma = 9$, respectively. Moreover, the uncertainty around the optimal investment strategy, as indicated by the shaded areas, also becomes smaller the lower is the coefficient of RRA. Similar conclusions are reported in Campbell and Viceira (1999) in a dynamic asset allocation model of an infinitely lived investor

Figure 4.4. Sensitivity of optimal allocations to the risk aversion coefficient. Panels (a), (b) and (c) plot, respectively, the median value of the investment share on equity, of the hedging demand, and of the consumption-to-financial wealth ratio over M =10,000 simulated paths of the model for different values in γ.



without stochastic income but with a time-varying equity premium.

As the commodity gets depleted, the allocation on the risky asset converges to an investment share that remains constant through time. As shown in (3.12), this stationary level decreases with the value of the coefficient of RRA. In particular, the long-run optimal allocation on the risky asset is nearly 20 percent for $\gamma = 6$, and around 13 percent for $\gamma = 9$. These values represent a sizable correction relative to the 40 percent share implied by our benchmark scenario.

Panel (b) plots the median value of the hedging demand component associated with the total demand in Panel (a). In general, we find that the larger is the investor's risk aversion, the more conservative she is to hedge against the oil income risk, thus the less is the hedging component. This is in line with the intuition obtained in the complete market case. As shown in equation (3.21), when the oil income and stock price are negative correlated, then the larger is the coefficient of RRA, the smaller is the hedging component.

Finally, in Panel (c) we report the effects of different values of the coefficient of RRA on the dynamics of the median optimal consumption-to-financial wealth ratio. In general, we find that the impact of different levels of risk aversion on the consumption ratio is less pronounced than for equity holdings. In other words, changes in the investor's risk aversion have a larger effect on the manager's asset allocation than on the consumption choice over time. For $\gamma \in (3.0, 6.0, 9.0)$, the optimal ratio is very smooth over time, and it fluctuates between 2.0 and 3.0 percent. Given the value of the EIS in our benchmark calibration ($\psi = 2.0$), we find that the optimal consumption ratio declines with γ. As an example consider year $\hat{T} = 60$. The median optimal consumption is 2.3 percent of the financial wealth for an investor with $\gamma = 3.0$, whereas it increases to 2.6 percent for an investor with $\gamma = 9.0$. As shown in Campbell and Viceira (1999) this is not necessarily always the case as one should expect to see an opposite relation between the optimal consumption ratio and the coefficient of RRA for investors that are extremely reluctant to substitute consumption across periods and hence have low values of the EIS.

5 Welfare costs of suboptimal investment rules

So far, we have shown how a SWF should optimally allocate its wealth among securities, and how to use the equity markets in order to hedge against fluctuations in the commodity income. In particular, the optimal investment policy implies a portfolio mix that is time dependent. Its evolution over time is determined by the path followed by the oil wealth-to-financial wealth ratio, the coefficient of RRA, and the correlation between the commodity income and the asset returns. However, the investment strategy followed by most commodity-based SWFs is exogenously given by the fund's owner. More specifically, the fund's exposure to risk is determined by a long-term investment mandate that usually recommends to hold a relatively constant position in equity without necessarily timing the market, as opposed to the strategy that would otherwise maximize welfare. To shed light about the potential costs of following policies that are suboptimal, this section calculates the welfare cost of following alternative policy rules. To this end, we ask what is the amount of additional initial financial wealth that needs to be transferred to the fund at the beginning of the planning horizon so that the implementation of a suboptimal policy provides the same level of utility that could be achieved otherwise with the optimal rule (see Cochrane, 1989). In other words, we compute the wealth compensating variation τ_0 that yields

$$J(t_0, W_0, Y_0) = \widetilde{J}(t_0, (1 + \tau_0)W_0, Y_0),$$

where J is the indirect utility that solves (3.7), and $\widetilde{J}$ is the value function that results from following an investment strategy different from that implied by (3.17), but where consumption is allowed to adjust optimally. Note that τ_0 can alternatively be interpreted as a wealth-equivalent *loss* for the owner of the SWF since in the absence of compensation the fund's (indirect) utility from following a suboptimal policy will be lower. If the wealth compensation τ_0 is small, then the welfare gains of implementing the maximizing investment rule can be probably outweighed by some of the features our model extracts from, e.g. transaction costs related to portfolio rebalancing, stochastic investment opportunities, etc. Thus, our stylized framework provides a lower bound on the welfare costs of following suboptimal investment policies.

In what follows we consider two different suboptimal investment strategies. The first policy fixes the fraction of wealth invested in the risky asset to $\widetilde{\alpha}_t = 70\%$ for all $t \leq \hat{T}$. This fixed rule echoes the investment mandate given by the Norwegian GPFG to the NBIM according to which the allocation on equity should amount to 60-80 percent of the total portfolio. The second rule assumes instead that every period the fund invests a fraction of its financial wealth into equity that is equal to the median investment share recommended by the optimal asset allocation model across M =10,000 simulations, i.e. $\widetilde{\alpha}_t = \text{Median}\left(\alpha_t^1, \ldots, \alpha_t^M\right)$. In this case, the investment rule is no longer constant but time varying. We regard this rule as a near-optimal policy in the sense that it corresponds to a perturbed version of the optimal strategy and therefore can be used to study the welfare costs of small mis-specifications in the dynamic asset allocation model or the use of inaccurate parameter values.

Table 2 summarizes our findings, where we report the wealth-equivalent compensation required at the beginning of the planning horizon for different values of coefficient of RRA, γ, and different values of the correlation coefficient between stock and oil price changes, ρ_{PS}. All the values are measured as a percentage of the fund's initial endowment of financial wealth.

Panel (a) shows the results when the SWF follows the constant investment rule of 70%, regardless of the value of the coefficient of RRA and of the correlation coefficient. This strategy implies that the SWF does not time the market, and instead assumes that holding a constant position in equity will imply a reduced level of risk for the overall portfolio over the long-run (see Siegel, 2014). However, our results suggest that such an strategy can lead to large wealth losses. As an example, for our benchmark calibration with $\gamma = 3$ and $\rho_{PS} = -0.07$, the wealth-equivalent loss is equivalent to 12.5 percent of the initial financial wealth. In other words, for the suboptimal policy to deliver the same level of welfare that can be obtained using the optimal investment policy, the fund will require an injection of capital equivalent to 12.5% of the initial endowment. To understand why the magnitude of the loss, recall that under the optimal strategy the fund should initially investment 60 percent of its financial wealth and thereafter decrease this fraction over time to reach a long-run value of around 40 percent. On the contrary, when following the suboptimal policy the fund invests instead 70 percent period-by-period, a value that exceeds the optimal allocation at every point in time. Hence, the wealth-equivalent loss in the benchmark scenario reflects the excessive exposure to risk implied by the suboptimal policy, an exposition that becomes larger the closer the natural resource is to depletion.

Our results also suggest that the welfare losses tend to be substantial for moderate to highly risk averse investors, particularly when the absolute value of the correlation between oil and stock price changes is large. For example, for a correlation of 30 percent which resembles that observed in during the last decade, the losses that result from implementing a constant investment rule of 70 percent in equity over time fluctuate between 16 and 82 percent of the initial endowment of the fund. Although smaller in magnitude, large losses are also incurred for large and negative correlation coefficients.

In Panel (b) we report the wealth-equivalent losses incurred by the fund when following a suboptimal, but time-dependent, investment rule corresponding to the median of the op-

Table 2. Wealth-equivalent compensation under suboptimal portfolio rules Panel (a) reports the wealth compensation required (% of initial financial wealth) when following a suboptimal portfolio rule that is constant and equal to 70% over time, while consumption adjusts optimally. Panel (b) reports the wealth compensation required (% of initial financial wealth) when following an ad-hoc suboptimal portfolio rule that in each period fixes the investment share equal to the median value of the optimal portfolio rule, while consumption adjusts optimally.

	(a) Constant rule				(b) Median rule			
	ρ_{PS}				ρ_{PS}			
γ	-0.3	-0.07	0	0.3	-0.3	-0.07	0	0.3
2	23.82	6.42	4.14	1.41	23.38	5.82	3.51	0.04
3	17.85	12.49	12.33	16.29	9.77	2.01	1.02	0.44
4	26.17	26.64	27.76	36.33	5.72	1.00	0.43	0.64
5	38.22	43.32	45.58	58.47	3.94	0.60	0.22	0.72
6	51.83	61.32	64.71	81.98	2.98	0.41	0.13	0.74

timal share obtained from repeated simulations of the model. In contrast to the previous rule, this alternative fixed rule is now a function of the investor's coefficient of RRA and the correlation coefficient between oil and stock price changes. Our results indicate that using a time-dependent rule, whose path is allowed to adjust to the investor preferences and the market interdependencies, leads to considerably smaller losses. For our benchmark calibration, the loss from not implementing the optimal strategy is equivalent to 2 percent of the fund's initial endowment. Interestingly, if $\rho_{PS} \leq 0$ the loss becomes smaller the more risk averse is the investor, a pattern that is in contrast to that documented in Panel (a). This inverse relation is partly explained by the reduced variability in the optimal investment share (see Figure 4.4) that accompanies the lower median equity holdings of highly risk averse agents. However, in the case of a positive correlation we find that the losses, although small, increase with coefficient of RRA. For $\rho_{PS} = 30\%$ the losses never exceed 1 percent, which makes the use of a time-dependent fixed rule an attractive alternative in case that the optimal rule is not readily available to the fund. A similar conclusion can be drawn for small negative correlation coefficients and relatively high coefficients of RRA.

6 Conclusions

In this paper, we have extended the standard dynamic asset allocation problem for long-term investors with stochastic income to accommodate the portfolio problem faced by commodity-based SWFs. In particular, we study the optimal consumption-investment decision of a SWF whose primary source of income comes from oil-related activities. Fluctuations in the income stream are assumed to be primarily driven by variations in the exogenous and volatile price of oil. The model features Epstein-Zin-Weil recursive preferences that conveniently separates risk aversion from the elasticity of intertemporal substitution. Since most SWFs are set up by countries interested in sustaining a standard of living for all future generations, we assume that the fund's planning horizon is infinite.

However, the oil revenues are received only for a fixed number of periods until the time of depletion of the nonrenewable natural resource.

Using data on the S&P500 index and the WTI price for crude oil for the period 1973-2019 we find statistical evidence of a time-varying, but imperfect, correlation between the risky asset and the commodity price. This suggests that the income risk faced by the SWF cannot be perfectly replicated by a trading strategy in the financial markets. More specifically, we find that the average correlation for the period prior to 2007 was -7 percent. Interestingly, the direction and magnitude of this correlation changed considerably with the start of the Great Recession, and by the end of 2018 it had reached a positive value of 30 percent. Consistent with this evidence, we solve the SWF's manager allocation problem under the assumption of incomplete markets. The remaining parameters are chosen to replicate some salient features of the Norwegian SWF, the Government Pension Fund Global (GPFG). We find that the fund should initially allocate around 60 percent of its financial wealth into risky assets and thereafter decrease its position gradually until it reaches a long-run share of 40 percent. This gradual adjustment is estimated to last approximately 60 years, equivalent to the time it will take to deplete the reserves of oil. The implied intertemporal hedging demand component is found to be large at the beginning of the planning horizon accounting for 20 percent of the total demand for risky assets.

In contrast with our findings, the fraction of financial wealth invested by the GFPG in equity has exhibited an upward trend since its inception: 40 percent in 1998 to 70 percent in 2019. Given the imperfect, but negative, correlation between stock prices and oil income, our results suggest that the GPFG has followed a suboptimal investment strategy that has not exploited all the available hedging opportunities, and thus has taken larger and unnecessary amounts of risk. If instead we consider a positive correlation, similar to that observed after the Great Recession, we find an initial allocation of financial wealth into the risky asset of around 30 percent that should gradually increase towards its long-run share of 40 percent. Although in this case the model implies an increasing investment profile, the optimal portfolio share in equity is substantially lower than any of the allocations reported by the Norwegian SWF during the last 20 years, and thus, also suggests a suboptimal use of the hedging possibilities. In a world where the positive correlation is becoming stronger, SWFs should therefore reduce their exposure to risk by moving away from stocks in the near future.

We also studied the welfare costs of not following an investment strategy that optimally exploits the intertemporal hedging opportunities available to the SWF. To do so, we use a measure of wealth-equivalent welfare compensation. This is defined as the percentage of additional initial financial wealth that the government would need to transfer to the portfolio administrator in order to achieve the same welfare that could be otherwise obtained by following the optimal investment strategy. We found significant welfare costs for a SWF that follows a constant investment policy with a fixed equity/bond mix equal to 70/30%. We also found that these losses are considerably reduced if instead the SWF implements a time-dependent, but ad-hoc, investment policy that although suboptimal resembles the decreasing path of the oil-wealth-to-financial wealth ratio. For practical purposes, this alternative policy may be considered as a second-best policy in an environment with institutional constraints that prevent the SWF's manager to hedge oil price fluctuations periodically.

Our study offers an unified framework that can be used by commodity-based SWFs to design optimal investment policies that are consistent with the long-term objective of ensuring a smooth intergenerational consumption, and the short-run objective of shielding the economy from fluctuations in the volatile oil revenues. Given the importance of the covariance structure between oil prices and asset returns, future work should further investigate the time-varying nature of the correlation between tradable and non-tradable assets, and the inclusion of more asset classes with time-varying risk premiums.

References

Achdou, Y., J. Han, J.-M. Lasry, P.-L. Lions, and B. Moll (2017): "Income and Wealth Distribution in Macroeconomics: A Continuous-Time Approach," Working Paper 23732, National Bureau of Economic Research.

Ait-Sahalia, Y. (2002): "Maximum Likelihood Estimation of Discretely Sampled Diffusions: A Closed-form Approximation Approach," *Econometrica*, 70, 223–262.

——— (2008): "Closed-Form Likelihood Expansions for Multivariate Diffusions," *The Annals of Statistics*, 36, 906–937.

Bellman, R. (1957): *Dynamic Programming*, Princeton University Press.

Bernanke, B. (2016): "The Relationship between Stocks and Oil Prices," The Brookings Institution, Retrieved from https://www.brookings.edu/blog/ben-bernanke.

Bhardwaj, G., G. Gorton, and G. Rouwenhorst (2015): "Facts and Fantasies about Commodity Futures Ten Years Later," NBER Working Papers, National Bureau of Economic Research, Inc.

Bick, B., H. Kraft, and C. Munk (2009): "Investment, Income, and Incompleteness," Working Paper Series: Finance and Accounting Nr. 202.

Bodie, Z., R. C. Merton, and W. F. Samuelson (1992): "Labor Supply Flexibility and Portfolio Choice in a Life Cycle Model," *Journal of Economic Dynamics and Control*, 16, 427–449.

BP (2019): "BP Statistical Review of World Energy 2019," Retrieved from https://www.bp.com/en/global/corporate/energy-economics/statistical-review-of-world-energy.html.

Buyuksahin, B. and M. Robe (2014): "Speculators, Commodities and Cross-Market Linkages," *Journal of International Money and Finance*, 42, 38–70.

Campbell, J. Y. (1996): "Understanding Risk and Return," *Journal of Political Economy*, 104, 298–345.

Campbell, J. Y. and L. M. Viceira (1999): "Consumption and Portfolio Decisions when Expected Returns are Time Varying," *The Quarterly Journal of Economics*, 114, 433–495.

——— (2002): *Strategic Asset Allocation: Portfolio Choice for Long-Term Investors*, Oxford University Press.

Candler, G. (1999): *Finite Difference Methods for Continuous-Time Dynamic Programming,*, in R. Marimon and A. Scott (Eds.): Computational Methods for the Study of Dynamic Economies, Oxford University Press.

Carroll, C. and A. Samwick (1997): "The Nature of Precautionary Wealth," *Journal of Monetary Economics*, 40, 41–71.

Chamberlain, G. and K. Hirano (1999): "Predictive Distributions based on Longitudinal Earnings Data," *Annals of Economics and Statistics*, 211–242.

Cocco, J. F., F. J. Gomes, and P. J. Maenhout (2005): "Consumption and Portfolio Choice over the Life Cycle," *Review of Financial Studies*, 18, 491–533.

Cochrane, J. H. (1989): "The Sensitivity of Tests of the Intertemporal Allocation of Consumption to Near-Rational Alternatives," *American Economic Review*, 79, 319–337.

Datta, D. D., B. K. Johannsen, H. Kwon, and R. J. Vigfusson (2018): "Oil, Equities, and the Zero Lower Bound," Finance and Economics Discussion Series 2018-058, Board of Governors of the Federal Reserve System (U.S.).

Davis, S. and P. Willen (2000): "Using Financial Assets to Hedge Labor Income Risks: Estimating the Benefits," Working paper.

Deaton, A. and G. Laroque (1992): "On the Behaviour of Commodity Prices," *Review of Economic Studies*, 59, 1–23.

Døskeland, T. (2007): "Strategic Asset Allocation for a Country: the Norwegian Case," *Financial Markets and Portfolio Management*, 21, 167–201.

Duffie, D. and L. G. Epstein (1992a): "Asset Pricing with Stochastic Differential Utility," *The Review of Financial Studies*, 5, 411–436.

——— (1992b): "Stochastic Differential Utility," *Econometrica*, 60, 353–394.

Duffie, D., W. Fleming, H. M. Soner, and T. Zariphopoulou (1997): "Hedging in Incomplete Markets with HARA Utility," *Journal of Economic Dynamics and Control*, 21, 753–782.

Epstein, L. and S. Zin (1989): "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework," *Econometrica*, 57, 937–69.

Fages, R., L. Heredia, J. Carrubba, O. Eyal, D. Frankle, K. Kurihara, B. Mace, E. Palmisani, N. Pardasani, T. Schulte, B. Sheridan, and Q. Xu (2019): "Global Asset Management 2019: Will These '20s Roar?" Retrieved from https://www.bcg.com/publications/2019.

Filis, G., S. Degiannakis, and C. Floros (2011): "Dynamic Correlation between Stock Market and Oil Prices: The Case of Oil-Importing and Oil-Exporting Countries," *International Review of Financial Analysis*, 20, 152 – 164.

Gomes, F. J. (2007): "Exploiting Short-Run Predictability," *Journal of Banking & Finance*, 31, 1427 – 1440.

Gomez, M. (2019): "Solving PDEs Associated with Economic Models," Tech. rep., available at https://github.com/matthieugomez/EconPDEs.jl.

Gorton, G. and K. G. Rouwenhorst (2006): "Facts and Fantasies about Commodity Futures," *Financial Analysts Journal*, 62, 47–68.

Heaton, J. and D. Lucas (1997): "Market Frictions, Savings Behavior, and Portfolio Choice," *Macroeconomic Dynamics*, 1, 76–101.

Jones, C. M. and G. Kaul (1996): "Oil and the Stock Markets," *The Journal of Finance*, 51, 463–491.

Kreps, D. and E. L. Porteus (1978): "Temporal Resolution of Uncertainty and Dynamic Choice Theory," *Econometrica*, 46, 185–200.

Larsen, L. S. and C. Munk (2012): "The Costs of Suboptimal Dynamic Asset Allocation: General Results and Applications to Interest Rate Risk, Stock Volatility Risk, and Growth/Value Tilts," *Journal of Economic Dynamics and Control*, 36, 266–293.

Lee, Y.-H. and J.-S. Chiou (2011): "Oil Sensitivity and its Asymmetric Impact on the Stock Market," *Energy*, 36, 168 – 174.

Lindset, S. and K. A. Mork (2019): "Risk Taking and Fiscal Smoothing with Sovereign Wealth Funds in Advanced Economies," *International Journal of Financial Studies*, 7, 1–24.

Lombardi, M. and F. Ravazzolo (2016): "On the Correlation between Commodity and Equity Returns: Implications for Portfolio Allocation," *Journal of Commodity Markets*, 2, 45–57.

Markowitz, H. (1952): "Portfolio Selection," *Journal of Finance*, 7, 77–91.

Merton, R. (1969): "Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case," *The Review of Economics and Statistics*, 51, 247–57.

——— (1973): "An Intertemporal Capital Asset Pricing Model," *Econometrica*, 41, 867–87.

Merton, R. C. (1971): "Optimum Consumption and Portfolio Rules in a Continuous-Time Model," *Journal of Economic Theory*, 3, 373–413.

Munk, C. and C. Sørensen (2010): "Dynamic Asset Allocation with Stochastic Income and Interest Rates," *Journal of Financial Economics*, 433–462.

NBIM (2016): "Discussion Note 04/16: Asset Allocation with Government Revenues and Spending Commitments," Retrieved from https://www.nbim.no.

Sadorsky, P. (1999): "Oil Price Shocks and Stock Market Activity," *Energy Economics*, 21, 449–469.

Scherer, B. (2011): "Portfolio Choice for Oil-Based Sovereign Wealth Funds," *The Journal of Alternative Investments*, 13.

Siegel, J. (2014): *Stocks for the Long Run*, McGraw-Hill Education, 5 ed.

Statista (2019): "Statista Dossier on Hedge Funds," Retrieved from https://www.statista.com/topics/5064/hedge-funds/.

Svensson, L. (1989): "Portfolio Choice with Non-Expected Utility in Continuous Time," *Economics Letters*, 30, 313–317.

SWF Institute (2020): "Sovereign Wealth Fund Rankings," Retrieved from https://www.swfinstitute.org.

van den Bremer, T., v. d. P. Frederick, and S. Wills (2016): "The Elephant In The Ground: Managing Oil And Sovereign Wealth," *European Economic Review*, 82, 113–131.

van den Bremer, T. S. and F. van der Ploeg (2013): "Managing and Harnessing Volatile Oil Windfalls," *IMF Economic Review*, 61, 130–167.

Viceira, L. M. (2001): "Optimal Portfolio Choice for Long-Horizon Investors with Non-tradable Labor Income," *The Journal of Finance*, 56, 433–470.

Wang, C., N. Wang, and J. Yang (2016): "Optimal Consumption and Savings with Stochastic Income and Recursive Utility," *Journal of Economic Theory*, 165, 292–331.

Weil, P. (1990): "Nonexpected Utility in Macroeconomics," *The Quarterly Journal of Economics*, 105, 29–42.

Appendix

A Optimal allocation: First stage solution

Proof of Proposition 1. Following Campbell and Viceira (2002), the optimal allocation problem faced by the SWF when there is no revenues from the commodity exploitation is given by

$$J\left(W_{\hat{T}}\right)=\max_{\{C_t,\alpha_t\}_{t=\hat{T}}^{\infty}}\mathbb{E}_{\hat{T}}\left[\int_{\hat{T}}^{\infty}f(C_t,J\left(W_t\right))\mathrm{d}t\right]$$

subject to the evolution of wealth

$$\mathrm{d}W_t=\left(rW_t+\left(\mu-r\right)\alpha_tW_t-C_t\right)\mathrm{d}t+\sigma_S\alpha_tW_t\mathrm{d}Z_{S,t},$$

and where the aggregator $f\left(C,J\right)$ is given by (3.6). A necessary condition for optimality for any $t\in\left[\hat{T},\infty\right)$ is given by the Hamilton-Jacobi-Bellman (HJB) equation

$$0=\max_{\{C,\alpha\}}\left\{f\left(C,J\left(W\right)\right)+\frac{1}{\mathrm{d}t}\mathbb{E}_t\left[\mathrm{d}J\left(W\right)\right]\right\}.$$

An application of Itô's lemma implies

$$\mathrm{d}J\left(W\right)=J_W\mathrm{d}W+\frac{1}{2}J_{WW}\left(\sigma_S\alpha W\right)^2\mathrm{d}t,$$

where $J_W\equiv\partial J\left(W\right)/\partial W$ and $J_{WW}=\partial^2J\left(W\right)/\partial W^2$. Using the martingale difference properties of stochastic integrals, we arrive at

$$0=\max_{\{C,\alpha\}}\left\{f\left(C,J\right)+J_W\left[rW+\left(\mu-r\right)\alpha W-C\right]+\frac{1}{2}\sigma_S^2J_{WW}\left(\alpha W\right)^2\right\}. \tag{A.25}$$

The first order conditions for an interior solution read

$$C^\star=\left(\frac{\beta}{J_W}\right)^{\psi}\left[\left(1-\gamma\right)J\right]^{\frac{1-\psi\gamma}{1-\gamma}} \tag{A.26}$$

$$\alpha^\star=\frac{1}{-\frac{W_tJ_{WW}}{J_W}}\frac{\mu-r}{\sigma_S^2}. \tag{A.27}$$

By substituting (A.26) and (A.27) in (A.25) we arrive to the maximized HJB equation

$$0=f\left(C^\star,J\right)+J_W\left[rW+\left(\mu-r\right)\alpha^\star W-C^\star\right]+\frac{1}{2}\sigma_S^2J_{WW}\left(\alpha^\star W\right)^2, \tag{A.28}$$

which corresponds to a non-linear partial difference equation in $J(W)$. We conjecture that a solution to (A.28) is given by

$$J(W) = \frac{\beta^{\theta}}{1-\gamma} \mathbb{G}_{\infty}^{-\frac{\theta}{\psi}} W^{1-\gamma}, \tag{A.29}$$

where $\mathbb{G}_{\infty}$ is an unknown constant to be determined. Our conjecture implies that

$$J_W = \beta^{\theta} \mathbb{G}_{\infty}^{-\frac{\theta}{\psi}} W^{-\gamma}, \tag{A.30}$$

$$J_{WW} = -\gamma \beta^{\theta} \mathbb{G}_{\infty}^{-\frac{\theta}{\psi}} W^{-\gamma-1}. \tag{A.31}$$

Substituting our guess together with its partial derivatives into the maximized HJB (A.28) yields

$$\begin{aligned} 0 = \frac{\beta\psi}{\psi-1} \beta^{\theta} \mathbb{G}_{\infty}^{-\frac{\theta}{\psi}} W^{1-\gamma} \left\{\beta^{-1}\mathbb{G}_{\infty} - 1\right\} \\ + \beta^{\theta} \mathbb{G}_{\infty}^{-\frac{\theta}{\psi}} W^{1-\gamma} \left[r + \alpha^{\star}(\mu - r) - \frac{C^{\star}}{W}\right] - \frac{1}{2}\gamma\beta^{\theta}\mathbb{G}_{\infty}^{-\frac{\theta}{\psi}} W^{1-\gamma} (\alpha^{\star})^2 \end{aligned}$$

where

$$\frac{C^{\star}}{W} = \mathbb{G}_{\infty}, \tag{A.32}$$

$$\alpha^{\star} = \frac{1}{\gamma}\frac{\mu - r}{\sigma_S^2}. \tag{A.33}$$

Dividing both sides by $\beta^{\theta}\mathbb{G}_{\infty}^{-\frac{\theta}{\psi}} W^{1-\gamma}$ and solving for $\mathbb{G}_{\infty}$ yields

$$\mathbb{G}_{\infty} = \beta\psi + (1-\psi) r + \frac{(1-\psi)}{2\gamma}\left(\frac{\mu - r}{\sigma_S}\right)^2, \tag{A.34}$$

which confirms our conjecture. Equation (A.32) and (A.33) show that both the consumption-wealth ratio and the share of wealth invested in the risky asset are constant for all $t \geq \hat{T}$. $\square$

B Optimal allocation: Second stage solution

The optimal allocation problem faced by the fund's manager for all $t < \hat{T}$ is given by

$$J(t, W_t, Y_t) = \max_{\{C_s, \alpha_s\}_{s=t}^{\hat{T}}} \mathbb{E}_t \left[\int_t^{\hat{T}} f(C_s, J(s, W_s, Y_s)) \mathrm{d}s + \frac{\varepsilon}{1-\gamma} W_{\hat{T}}^{1-\gamma} \right]$$

subject to

$$\begin{aligned}
\mathrm{d}W_t &= (r_{P,t} W_t - C_t + Y_t)\, \mathrm{d}t + \sigma_S \alpha_t W_t \mathrm{d}Z_{S,t}. \\
\mathrm{d}Y_t &= \kappa Y_t \mathrm{d}t + \sigma_P Y_t \left(\rho_{PS} \mathrm{d}Z_{S,t} + \sqrt{1-\rho_{PS}^2} \mathrm{d}Z_{P,t} \right),
\end{aligned}$$

where $\varepsilon \equiv \beta^{\theta} \mathbb{G}_{\infty}^{-\frac{\theta}{\psi}}$. The terminal condition fixes the value of the value function at time $\hat{T}$ to that in (3.15), i.e., $J\left(\hat{T}, W, Y\right) = J(W)$. It implies that at time of depletion the value function should equal the stationary value function that solves the fund's allocation problem in the absence of oil revenues.

A necessary condition for optimality for any $t \in \left[0, \hat{T}\right)$ is given by the Hamilton-Jacobi-Bellman (HJB) equation

$$0 = \max_{\{C, \alpha\}} \left\{ f(C, J(t, W, Y)) + \frac{1}{\mathrm{d}t} \mathbb{E}_t [\mathrm{d}J(t, W, Y)] \right\}.$$

An application of Itô's lemma implies

$$\begin{aligned}
\mathrm{d}J(t, W, Y) = {}& \left[J_t + [rW + \alpha(\mu - r) W + Y - C] J_W \right. \\
& \left. + \frac{1}{2} \sigma_S^2 (\alpha W)^2 J_{WW} + \kappa Y J_Y + \frac{1}{2} \sigma_P^2 Y^2 J_{YY} + \sigma_S \sigma_P \rho_{PS} (\alpha W) Y J_{WY} \right] \mathrm{d}t \\
& + (\sigma_S \alpha_t W_t J_W + \sigma_P Y_t \rho_{PS} J_Y)\, \mathrm{d}Z_{S,t} + \sigma_P Y_t \sqrt{1-\rho_{PS}^2} J_Y \mathrm{d}Z_{P,t},
\end{aligned}$$

where $J_t \equiv \partial J(t, W, Y) / \partial t$, $J_W \equiv \partial J(t, W, Y) / \partial W$, $J_Y \equiv \partial J(t, W, Y) / \partial Y$, $J_{WW} \equiv \partial^2 J(t, W, Y) / \partial W^2$, $J_{YY} \equiv \partial^2 J(t, W, Y) / \partial Y^2$, and $J_{WY} \equiv \partial^2 J(t, W, Y) / \partial W \partial Y$. Using the martingale difference properties of stochastic integrals, we arrive at

$$\begin{aligned}
0 = \max_{\{C, \alpha\}} & \left\{ f(C, J) + J_t + [rW + \alpha(\mu - r) W + Y - C] J_W \right. \\
& \left. + \frac{1}{2} \sigma_S^2 (\alpha W)^2 J_{WW} + \kappa Y J_Y + \frac{1}{2} \sigma_P^2 Y^2 J_{YY} + \sigma_S \sigma_P \rho_{PS} (\alpha W) Y J_{WY} \right\}.
\end{aligned}$$

Under the assumptions of incomplete markets, the allocation problem for $t < \hat{T}$, has no closed form solution. Section B.1 shows how to obtain a numerical approximation to

$J(W,Y,t)$ for all $t < \hat{T}$ that solves the maximized HJB equation in (3.14)

$$0 = \beta\theta J\left\{\left[\frac{C}{\left[(1-\gamma)J\right]^{\frac{1}{1-\gamma}}}\right]^{1-\frac{1}{\psi}} - 1\right\}$$
$$+ J_t + \left[rW + \alpha(\mu - r)W + Y - C\right]J_W + \frac{1}{2}\sigma_S^2(\alpha W)^2 J_{WW}$$
$$+ \kappa Y J_Y + \frac{1}{2}\sigma_P^2 Y^2 J_{YY} + \sigma_S\sigma_P\rho_{PS}(\alpha W)YJ_{WY}, \quad \text{(B.35)}$$

where C and α are given by the first order conditions in (3.16) and (3.17). The solution must satisfy the terminal condition $J\left(\hat{T}, W, Y = 0\right)$. On the contrary, if financial markets are complete it is possible to obtain an analytical solution to the allocation problem. This solution is derived in Section B.2.

B.1 Incomplete markets solution

B.1.1 The transformed problem

The problem in (B.35) consists of solving a nonlinear partial differential equation (PDE) in three state variables: time, financial wealth and income. To simplify the implementation of the numerical approximation, we exploit the homogeneity of the value function with respect to financial wealth and income to reduce the number of state variables from three to two.

As discussed in Wang et al. (2016), the value function $J(W,Y,t)$ is homogeneous of degree $(1-\gamma)$ in W and Y. Hence, for any given function $k(t)$ it holds that

$$J(k(t)W, k(t)Y, t) = k(t)^{1-\gamma} J(W,Y,t).$$

In line with Munk and Sørensen (2010), we set $k(t) = e^{-\delta t}/Y$, implying that

$$J(W,Y,t) = Y^{1-\gamma}e^{-\delta(\gamma-1)t}J\left(x, e^{-\delta t}, t\right) \equiv Y^{1-\gamma}\frac{F(x,t)^{1-\gamma}}{1-\gamma}, \quad \text{(B.36)}$$

where we have defined $x \equiv e^{-\delta t}W/Y$ to be the scaled-adjusted financial wealth-to-income ratio, with $\partial x/\partial Y = -x/Y$, $\partial x/\partial t = -\delta x$ and $\partial x/\partial W = e^{-\delta t}/Y$. The parameter $\delta \geq 0$ prevents the financial wealth-to-income ratio from taking very large values as $t \to \hat{T}$ which will prevent the numerical algorithm to converge on a fixed grid for x. The value for δ is found by trial-and-error conditional on the calibration of the structural parameters.

The introduction of new state variable x allows us to simplify the original problem. In fact, substituting (B.36) into (B.35) yields

$$0=\frac{\beta\psi}{\psi-1}\left(FY\right)^{1-\gamma}\left\{\left(\frac{C}{YF}\right)^{1-\frac{1}{\psi}}-1\right\}+\underbrace{F^{-\gamma}Y^{1-\gamma}\left(F_t-\delta F_x x\right)}_{\equiv J_t}$$
$$+\left[r\frac{W}{Y}+\alpha\left(\mu-r\right)\frac{W}{Y}+1-\frac{C}{Y}\right]Y\underbrace{e^{-\delta t}F^{-\gamma}Y^{-\gamma}F_x}_{\equiv J_W}+\frac{1}{2}\sigma_S^2\alpha^2\frac{W^2}{Y^2}Y^2\underbrace{e^{-2\delta t}F^{-\gamma-1}Y^{-\gamma-1}\left[FF_{xx}-\gamma F_x^2\right]}_{\equiv J_{WW}}$$
$$+\kappa Y\underbrace{\left(FY\right)^{-\gamma}\left(F-xF_x\right)}_{\equiv J_Y}+\frac{1}{2}Y^2\sigma_Y^2\underbrace{\left(FY\right)^{-1-\gamma}\left(x^2FF_{xx}-\gamma\left(F-xF_x\right)^2\right)}_{\equiv J_{YY}}$$
$$+\sigma_S\sigma_Y\rho_{YS}\alpha\frac{W}{Y}Y^2\underbrace{e^{-\delta t}\left(FY\right)^{-\gamma-1}\left[-xFF_{xx}-\left(\gamma F_x\right)\left(F-xF_x\right)\right]}_{\equiv J_{WY}}.$$

After some algebra, the transformed PDE reads

$$\begin{aligned}0=\frac{\beta\psi}{\psi-1}\hat{c}^{1-\frac{1}{\psi}}F^{\frac{1}{\psi}}+\left(\kappa-\frac{\beta\psi}{\psi-1}-\frac{\gamma\sigma_P^2}{2}\right)F+F_t\\+\left[\left(r-\delta-\kappa+\sigma_P^2\gamma\right)x+\left(\mu-r-\gamma\sigma_S\sigma_P\rho_{PS}\right)\alpha x+e^{-\delta t}-e^{-\delta t}\hat{c}\right]F_x\\+\frac{1}{2}x^2\left(\sigma_S^2\alpha^2+\sigma_P^2-2\sigma_S\sigma_P\rho_{PS}\alpha\right)\left(F_{xx}-\gamma F^{-1}F_x^2\right),\end{aligned}\tag{B.37}$$

where the optimal consumption-to-income ratio, $\hat{c}\equiv C/Y$, and the optimal investment share, α, are given by

$$\hat{c}=e^{\psi\delta t}\beta^{\psi}F_x^{-\psi}F\tag{B.38}$$
$$\alpha=\frac{FF_x}{x\left[\gamma F_x^2-FF_{xx}\right]}\left(\frac{\mu-r}{\sigma_S^2}-\frac{\gamma\sigma_P\rho_{PS}}{\sigma_S}\right)+\frac{\sigma_P\rho_{PS}}{\sigma_S}.\tag{B.39}$$

The transformed problem is now that of finding the two state variable function $F\left(x,t\right)$ for all $t<\hat{T}$ that solves the PDE in (B.37). The terminal condition for the transformed problem, $F\left(x,\hat{T}\right)$, is related to the original terminal condition through (B.36). In particular, note that at $t=\hat{T}$, the optimal value function can be written as

$$\begin{aligned}J\left(W,Y,\hat{T}\right)&=\frac{\beta^{\theta}}{1-\gamma}\mathbb{G}_{\infty}^{-\frac{\theta}{\psi}}W^{1-\gamma}\\&=\frac{\beta^{\theta}}{1-\gamma}\mathbb{G}_{\infty}^{-\frac{\theta}{\psi}}\left[Y\left(e^{-\delta\hat{T}}\frac{W}{Y}e^{\delta\hat{T}}\right)\right]^{1-\gamma}\\&=\frac{\beta^{\theta}}{1-\gamma}\mathbb{G}_{\infty}^{-\frac{\theta}{\psi}}\left(e^{\delta\hat{T}}Yx\right)^{1-\gamma}\end{aligned}$$

where we have used the fact that $x\left(\hat{T}\right)=e^{-\delta\hat{T}}\frac{W}{Y}$. Then, by the homogeneity property of the value function it follows that

$$Y^{1-\gamma}\frac{F\left(x,\hat{T}\right)^{1-\gamma}}{1-\gamma}=\frac{\beta^{\theta}}{1-\gamma}\mathbb{G}_{\infty}^{-\frac{\theta}{\psi}}\left(e^{\delta\hat{T}}Yx\right)^{1-\gamma}$$

which implies the following value for the terminal condition

$$F\left(x,\hat{T}\right)=\left(\beta^{\theta}\mathbb{G}_{\infty}^{-\frac{\theta}{\psi}}\right)^{\frac{1}{1-\gamma}}e^{\delta\hat{T}}x, \tag{B.40}$$

where $\mathbb{G}_{\infty}$ is given in (A.34).

B.1.2 Finite difference approximation

We approximate the solution to the PDE in (B.37) for $t\leq\hat{T}$ using the finite difference algorithm for nonlinear PDEs introduced in Gomez (2019). In particular, the finite difference method approximates $F\left(x,t\right)$ on a $(J+1)\times(N+1)$ rectangular grid of equally spaced points on the $\left(x,t\right)-$space with values $\{(x_j,t_n)\,|\,j=0,1,...,J,n=0,1,\ldots,N\}$, where $x_j=x_0+j\triangle x$ and $t_n=n\Delta t$ for some fixed spacing parameters Δx and Δt.

Let $F_{j,n}\equiv F\left(x_j,t_n\right)$ denote the approximated value function at grid point (x_j,t_n). At depletion time, $\hat{T}=t_N=N\Delta t$, the approximated value function is set equal to

$$F_{j,\hat{T}}=\left(\beta^{\theta}\mathbb{G}_{\infty}^{-\frac{\theta}{\psi}}\right)^{\frac{1}{1-\gamma}}e^{\delta\hat{T}}x_j, \tag{B.41}$$

for all $j=0,1.\ldots,J$. Given (B.41), the optimal investment share and consumption-to-income ratio at depletion time, $\hat{c}_{j,\hat{T}}$ and $\alpha_{j,\hat{T}}$, are computed from (3.12) and (3.13), respectively.

Now, for each $j=0,1\ldots,J$ and $t=0,1,\ldots,N-1$ in the interior of the grid we compute the time derivative of the value function using the forward difference approximation

$$F_t^{+}\approx D_t^{+}F_{j,n}=\frac{F_{j,n+1}-F_{j,n}}{\Delta t},$$

whereas the first order derivative with respect to the scale-adjusted wealth-to-income ratio is computed with either a forward or a backward difference operator

$$F_x^{+}\approx D_x^{+}F_{j,n}=\frac{F_{j+1,n}-F_{j,n}}{\triangle x},$$
$$F_x^{-}\approx D_x^{-}F_{j,n}=\frac{F_{j,n}-F_{j-1,n}}{\triangle x}.$$

Finally, the second order derivatives are approximated using the central difference operator

$$F_{xx} \approx D_x^2 F_{j,n} = \frac{F_{j+1,n} - 2F_{j,n} + F_{j-1,n}}{(\Delta x)^2}.$$

Following Candler (1999) and Achdou et al. (2017), the choice of difference operator for F_x is based on an upwind differentiation scheme according to which the correct approximation, $D_x F_{j,n}$, is determined by the direction of state variable. In what follows, the direction will be determined by the sign of

$$z_{j,n} \equiv \left(r - \delta - \kappa + \sigma_P^2 \gamma\right) x_{j,n} + \left(\mu - r - \gamma \sigma_S \sigma_P \rho_{PS}\right) \alpha_{j,n} x_{j,n} + e^{-\delta n \Delta t} - e^{-\delta n \Delta t} \hat{c}_{j,n},$$

where

$$\begin{aligned}
\hat{c}_{j,n} &= e^{\psi \delta n \Delta t} \beta^{\psi} \left(D_x F_{j,n}\right)^{-\psi} F_{j,n}, \\
\alpha_{j,n} &= \frac{F_{j,n}\left(D_x F_{j,n}\right)}{x\left[\gamma\left(D_x F_{j,n}\right)^2 - F_{j,n}\left(D_x^2 F_{j,n}\right)\right]} \left(\frac{\mu - r}{\sigma_S^2} - \frac{\gamma \sigma_P \rho_{PS}}{\sigma_S}\right) + \frac{\sigma_P \rho_{PS}}{\sigma_S}
\end{aligned}$$

are the optimal consumption-to-income ratio and investment rate at grid point (j, n). Thus, if the "drift" variable $z_{j,n}$ is positive we use the forward operator and if it is negative we use the backward operator. This gives rise to the following upwind operator

$$D_x F_{j,n} = \left(D_x^+ F_{j,n}\right) \mathbb{1}_{\{z^+ \geq 0\}} + \left(D_x^- F_{j,n}\right) \mathbb{1}_{\{z^- < 0\}},$$

where $\mathbb{1}$ denotes the indicator function, and z^+ and z^- the "drift" variables computed with the forward and backward operators respectively.

Then, the finite difference approximation to the HJB equation at grid point (j, n) is given by

$$\begin{aligned}
-\frac{F_{j,n+1} - F_{j,n}}{\Delta t} &= \frac{\beta \psi}{\psi - 1} \left(\hat{c}_{j,n}\right)^{1 - \frac{1}{\psi}} \left(F_{j,n}\right)^{\frac{1}{\psi}} \\
&+ \left(\kappa - \frac{\beta \psi}{\psi - 1} - \frac{\gamma \sigma_P^2}{2}\right) F_{j,n} + z_{j,n}^+ \left(D_x^+ F_{j,n}\right) \mathbb{1}_{\{z^+ \geq 0\}} + z_{j,n}^- \left(D_x^- F_{j,n}\right) \mathbb{1}_{\{z^- < 0\}} \\
&+ \frac{1}{2} x_{j,n}^2 \left(\sigma_S^2 \alpha_{j,n}^2 + \sigma_P^2 - 2 \sigma_S \sigma_P \rho_{PS} \alpha_{j,n}\right) \Big[\left(D_x^2 F_{j,n}\right) \\
&- \gamma \left(F_{j,n}\right)^{-1} \left(\left(D_x^+ F_{j,n}\right) \mathbb{1}_{\{z^+ \geq 0\}} + \left(D_x^- F_{j,n}\right) \mathbb{1}_{\{z^- < 0\}}\right)^2 \Big].
\end{aligned} \tag{B.42}$$

Given a value $F_{j,n+1}$ for all j, the approximation in (B.42) can be compactly written as a system of $(J + 1)$ nonlinear equations: one for each $n = 0, 1, \ldots, N - 1$. An approximation to the value function at time t_n is therefore given by the vector $\mathbf{F}_n = [F_{0,n}, F_{1,n}, \ldots, F_{J,n}]'$ that solves $\mathbf{G}\left(\mathbf{F}_n\right) = \mathbf{0}$, where $\mathbf{F}_n$ denotes the unknown value function at all the grid points in the $x-$lattice at time step n. To compute the approximation

$\mathbf{F}_n$ for all $n = 0, 1, \ldots, N-1$, we iterate backwards on time starting from the terminal condition $\mathbf{F}_N$ in (B.41). This recursion can be written as

$$\mathbf{0} = \mathbf{G}(\mathbf{F}_n) + \frac{1}{\triangle t}(\mathbf{F}_{n+1} - \mathbf{F}_n). \tag{B.43}$$

B.2 Complete markets solution

Proof of Lemma 1. Under the assumption of complete markets it follows that $|\rho_{PS}| = 1$. Hence, the dynamics of the SFW's income under the physical probability measure $\mathbb{P}$ is given by the Geometric Brownian motion

$$\frac{\mathrm{d}Y_t}{Y_t} = \kappa \mathrm{d}t + \xi \mathrm{d}Z_{S,t}, \tag{B.44}$$

where $\xi = \sigma_P \times \rho_{PS}$. Let the market price of risk be given by $\lambda = (\mu - r)/\sigma_S$. Then, by Girsanov's theorem, the fund's oil income has the following equivalent Geometric Brownian motion representation under the risk-neutral probability measure $\mathbb{Q}$

$$\frac{\mathrm{d}Y_t}{Y_t} = (\kappa - \xi\lambda)\,\mathrm{d}t + \xi \mathrm{d}Z^{\mathbb{Q}}_{S,t}, \tag{B.45}$$

with solution

$$Y_u = Y_t e^{\left(\left(\kappa - \xi\lambda - \frac{1}{2}\xi^2\right)(u-t) + \xi\left(Z^{\mathbb{Q}}_{S,u} - Z^{\mathbb{Q}}_{S,t}\right)\right)}, \quad \text{for } u \geq t. \tag{B.46}$$

Let $\mathcal{O}_t \equiv \mathcal{O}\left(Y_t, t; \hat{T}\right)$ denote the expected present discounted value at time t of all future oil income, $\{Y_u\}_{u=t}^{\hat{T}}$, i.e.,

$$\mathcal{O}_t = \mathbb{E}^{\mathbb{Q}}_t\left[\int_t^{\hat{T}} e^{-r(u-t)} Y_u \mathrm{d}u\right]. \tag{B.47}$$

Multiplying both sides of (B.46) by $e^{-r(u-t)}$ and integrating from t to $\hat{T}$ we arrive at

$$\int_t^{\hat{T}} e^{(-r(u-t))} Y_u \mathrm{d}u = Y_t \int_t^{\hat{T}} e^{\left(\left(-r + \kappa - \xi\lambda - \frac{1}{2}\xi^2\right)(u-t) + \xi\left(Z^{\mathbb{Q}}_{S,u} - Z^{\mathbb{Q}}_{S,t}\right)\right)} \mathrm{d}u. \tag{B.48}$$

Since $Z^{\mathbb{Q}}_{S,u} - Z^{\mathbb{Q}}_{S,t}$ is normally distributed the term inside the integral on the right hand side of (B.48) is log-normally distributed. Hence, the expected value conditional on the information at time t is

$$\mathbb{E}^{\mathbb{Q}}_t\left[\int_t^{\hat{T}} e^{-r(u-t)} Y_u \mathrm{d}u\right] = Y_t \int_t^{\hat{T}} e^{-(r - \kappa + \xi\lambda)(u-t)} \mathrm{d}u. \tag{B.49}$$

Hence, it follows that

$$\mathcal{O}_t = Y_t \mathbb{1}_{\{t<\hat{T}\}} \underbrace{\begin{cases} \frac{1}{r-\kappa+\xi\lambda}\left(1-e^{-(r-\kappa+\xi\lambda)(\hat{T}-t)}\right) & \text{if } r-\kappa+\xi\lambda \neq 0 \\ (T-t) & \text{if } r-\kappa+\xi\lambda = 0. \end{cases}}_{\equiv \mathcal{M}(t)\text{: Income multiplier}} \tag{B.50}$$

Note that under the risk-neutral probability measure $\mathbb{Q}$, Feynman-Kac Theorem suggests that $\mathcal{O}\left(Y_t, t; \hat{T}\right)$ satisfies the following partial differential equation (PDE)

$$\frac{\partial \mathcal{O}_t}{\partial t} + (\kappa - \xi\lambda) Y_t \frac{\partial \mathcal{O}_t}{\partial Y_t} + \frac{1}{2}\xi^2 Y_t^2 \frac{\partial^2 \mathcal{O}_t}{\partial Y_t^2} - r\mathcal{O}_t + Y_t = 0, \tag{B.51}$$

with terminal condition $\mathcal{O}\left(Y_{\hat{T}}, \hat{T}; \hat{T}\right) = 0$. □

Proof of Proposition 2. Conjecture that the value function takes the form

$$J(t, W, Y) = \frac{\beta^\theta}{1-\gamma} \mathbb{G}(t)^{\frac{\theta}{\psi}} (W + \mathcal{O})^{1-\gamma} \tag{B.52}$$

where $W + \mathcal{O}$ is the fund's total wealth at a given point in time, and $\mathbb{G}(t)$ is an unknown deterministic function to be determined. Our conjecture uses the idea in Bodie et al. (1992) according to which it is possible to think of the fund's manager as having an initial financial wealth of and no oil income, instead of having an initial financial wealth of and an inflow of income. Our conjecture imply that

$$\begin{aligned} J_t &= \frac{\beta^\theta}{(\psi-1)} \mathbb{G}(t)^{\frac{\theta}{\psi}-1} \frac{\partial \mathbb{G}(t)}{\partial t} (W+\mathcal{O})^{1-\gamma} + \beta^\theta \mathbb{G}(t)^{\frac{\theta}{\psi}} (W+\mathcal{O})^{-\gamma} \frac{\partial \mathcal{O}}{\partial t} \\ J_W &= \beta^\theta \mathbb{G}(t)^{\frac{\theta}{\psi}} (W+\mathcal{O})^{-\gamma} \\ J_{WW} &= -\gamma\beta^\theta \mathbb{G}(t)^{\frac{\theta}{\psi}} (W+\mathcal{O})^{-\gamma-1} \\ J_Y &= \beta^\theta \mathbb{G}(t)^{\frac{\theta}{\psi}} (W+\mathcal{O})^{-\gamma} \mathcal{O}_Y \\ J_{YY} &= -\gamma\beta^\theta \mathbb{G}(t)^{\frac{\theta}{\psi}} (W+\mathcal{O})^{-\gamma-1} (\mathcal{O}_Y)^2 \\ J_{WY} &= -\gamma\beta^\theta \mathbb{G}(t)^{\frac{\theta}{\psi}} (W+\mathcal{O})^{-\gamma-1} \mathcal{O}_Y, \end{aligned}$$

where we have used the fact that according to Lemma 1 $\mathcal{O}_{YY} = 0$. Subscripts on the value function J denote partial derivatives with respect to the respective state variables. Substituting in (3.16) and (3.17) yields

$$C_t = \mathbb{G}(t)^{-1} (W_t + \mathcal{O}_t) \tag{B.53}$$

$$\alpha_t = \frac{1}{\gamma}\left(\frac{\mu - r}{\sigma_S^2}\right)\left(1 + \frac{\mathcal{O}_t}{W_t}\right) - \frac{\mathcal{O}_t}{W_t}\frac{\sigma_P \rho_{PS}}{\sigma_S}, \tag{B.54}$$

where we have used (B.50) to conclude that $\mathcal{O}_t = Y_t \mathcal{O}_Y$.

Substituting into the maximized HJB equation (B.35) and using (B.51) together with the fact $rW = r(W + \mathcal{O}) - r\mathcal{O}$, we arrive to the linear ODE

$$\frac{\partial \mathbb{G}(t)}{\partial t} - \left[\beta\psi + (1-\psi)r + (1-\psi)\frac{1}{2\gamma}\left(\frac{\mu - r}{\sigma_S}\right)^2\right]\mathbb{G}(t) + 1 = 0. \tag{B.55}$$

Using the terminal condition for the HJB equation

$$J\left(\hat{T}, W, Y\right) = \frac{1}{1-\gamma}\beta^{\theta}\mathbb{G}_{\infty}^{-\frac{\theta}{\psi}} W_{\hat{T}}^{1-\gamma}.$$

together with our conjecture, we obtain derive the terminal condition the ODE in (B.55) has to satisfy. In particular

$$\begin{aligned} \frac{\beta^{\theta}}{1-\gamma}\mathbb{G}\left(\hat{T}\right)^{\frac{\theta}{\psi}} W_{\hat{T}}^{1-\gamma} &= \frac{1}{1-\gamma}\beta^{\theta}\mathbb{G}_{\infty}^{-\frac{\theta}{\psi}} W_{\hat{T}}^{1-\gamma} \\ &\Downarrow \\ \mathbb{G}\left(\hat{T}\right) &= \mathbb{G}_{\infty}^{-1}, \end{aligned}$$

which implies that

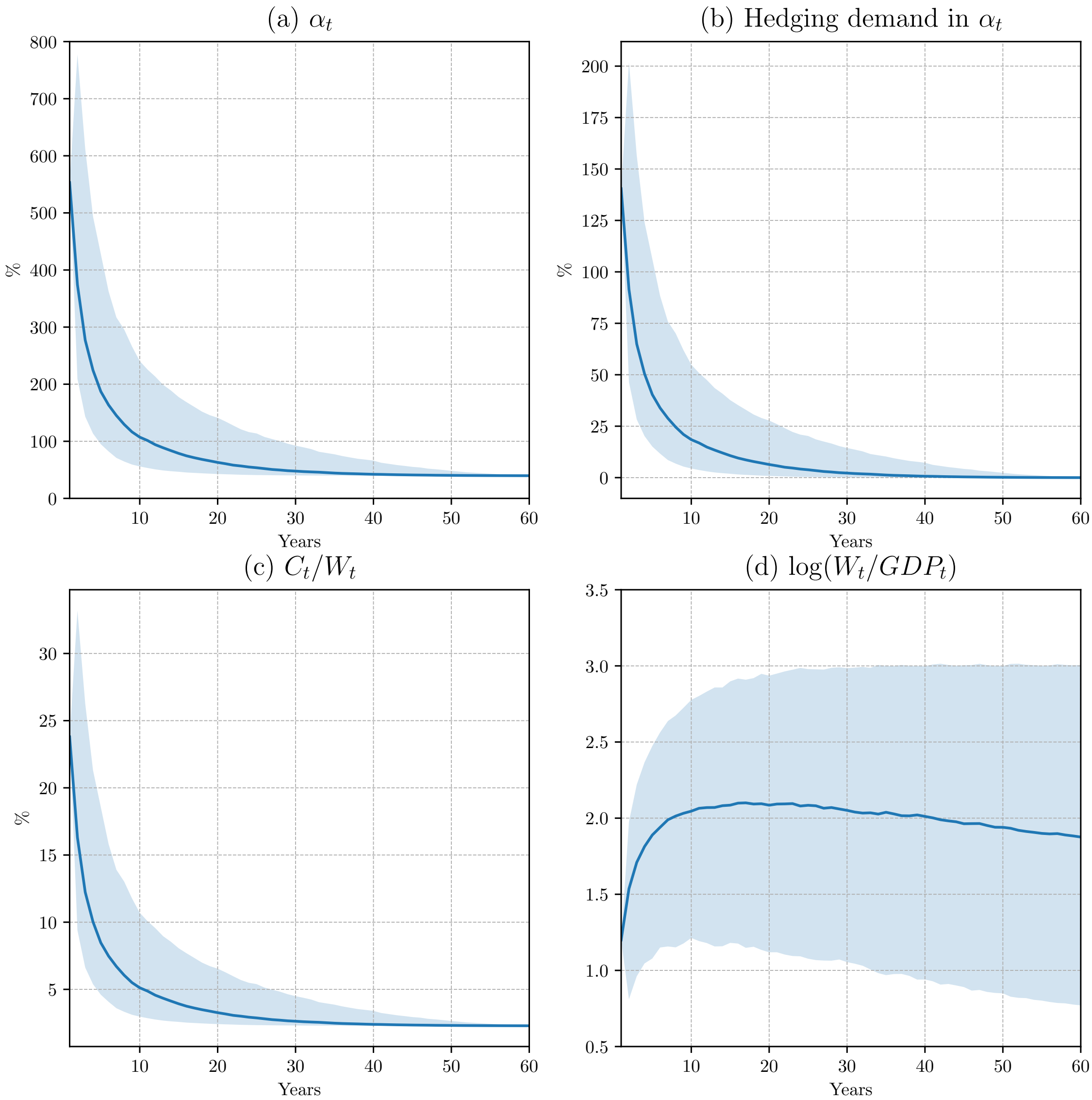
$$\mathbb{G}(t) = \mathbb{G}_{\infty}^{-1} \quad \forall t < \hat{T}, \tag{B.56}$$

and our conjecture has been verified.

C Optimal trajectories under complete markets

This appendix illustrates the optimal path for selected variables for all $t \in \left[0, \hat{T}\right]$, together with intervals around the median that represent the 15th and 85th percentiles of their distribution generated from 10,000 simulations of the model. We assume that markets are complete and therefore use Lemma 1 to compute the value of the underground oil wealth at each point in time. The optimal consumption-to-financial wealth ratio, and the optimal demand for the risky asset follow from Proposition 2 when replacing ρ_{PS} by the estimated correlation coefficient reported in Table 1.

Figure C.1. Optimal Asset Allocation under Complete Markets. Panels (a)–(d) plot, respectively, the optimal share of financial wealth invested in equity, the hedging demand as a fraction of financial wealth, the optimal consumption-to-financial wealth ratio, and the evolution of financial wealth-to-mainland GDP ratio. The optimal consumption-to-financial wealth ratio and the optimal demand for equity are given by (3.20) and (3.21) in Proposition 2, respectively. The solid lines represent the median value over M =10,000 simulated paths using the parameters in Table 1, each of them of $T = 60$ sample points. The shaded areas represents the 15 and 85 percentiles from the sampling distribution of the simulated series.



Research Papers 2020

CREATES
Center for Research in Econometric Analysis of Time Series

2019-18: Changli He, Jian Kang, Timo Teräsvirta and Shuhua Zhang: Long monthly temperature series and the Vector Seasonal Shifting Mean and Covariance Autoregressive model

2019-19: Changli He, Jian Kang, Timo Teräsvirta and Shuhua Zhang: Comparing long monthly Chinese and selected European temperature series using the Vector Seasonal Shifting Mean and Covariance Autoregressive model

2019-20: Malene Kallestrup-Lamb, Søren Kjærgaard and Carsten P. T. Rosenskjold: Insight into Stagnating Life Expectancy: Analysing Cause of Death Patterns across Socio-economic Groups

2019-21: Mikkel Bennedsen, Eric Hillebrand and Siem Jan Koopman: Modeling, Forecasting, and Nowcasting U.S. CO2 Emissions Using Many Macroeconomic Predictors

2019-22: Anne G. Balter, Malene Kallestrup-Lamb and Jesper Rangvid: The move towards riskier pensions: The importance of mortality

2019-23: Duván Humberto Cataño, Carlos Vladimir Rodríguez-Caballero and Daniel Peña: Wavelet Estimation for Dynamic Factor Models with Time-Varying Loadings

2020-01: Mikkel Bennedsen: Designing a sequential testing procedure for verifying global CO2 emissions

2020-02: Juan Carlos Parra-Alvarez, Hamza Polattimur and Olaf Posch: Risk Matters: Breaking Certainty Equivalence

2020-03: Daniel Borup, Bent Jesper Christensen, Nicolaj N. Mühlbach and Mikkel S. Nielsen: Targeting predictors in random forest regression

2020-04: Nicolaj N. Mühlbach: Tree-based Synthetic Control Methods: Consequences of moving the US Embassy

2020-05: Juan Carlos Parra-Alvarez, Olaf Posch and Mu-Chun Wang: Estimation of heterogeneous agent models: A likelihood approach

2020-06: James G. MacKinnon, Morten Ørregaard Nielsen and Matthew D. Webb: Wild Bootstrap and Asymptotic Inference with Multiway Clustering

2020-07 Javier Hualde and Morten Ørregaard Nielsen: Truncated sum of squares estimation of fractional time series models with deterministic trends

2020-08 Giuseppe Cavaliere, Morten Ørregaard Nielsen and Robert Taylor: Adaptive Inference in Heteroskedastic Fractional Time Series Models

2020-09 Daniel Borup, Jonas N. Eriksen, Mads M. Kjær and Martin Thyrsgaard: Predicting bond return predictability

2020-10 Alfonso A. Irarrazabal, Lin Ma and Juan Carlos Parra-Alvarez: Optimal Asset Allocation for Commodity Sovereign Wealth Funds