DEPARTMENT OF ECONOMICS AND BUSINESS ECONOMICS
AARHUS UNIVERSITY

CREATES
Center for Research in Econometric Analysis of Time Series

Bond Risk Premiums at the Zero Lower Bound

Martin Møller Andreasen, Kasper Jørgensen
and Andrew Meldrum

CREATES Research Paper 2019-10

Department of Economics and Business Economics
Aarhus University
Fuglesangs Allé 4
DK-8210 Aarhus V
Denmark

Email: oekonomi@au.dk
Tel: +45 8716 5515

Bond Risk Premiums at the Zero Lower Bound

Martin M. Andreasen, Kasper Jørgensen, and Andrew Meldrum*

May 14, 2019

Abstract

This paper documents a significantly stronger relationship between the slope of the yield curve and future excess bond returns on Treasuries from 2008-2015 than before 2008. This new predictability result is not matched by the standard shadow rate model with Gaussian factor dynamics, but extending the model with regime-switching in the (physical) dynamics of the factors at the lower bound resolves this shortcoming. The model is also consistent with the downwards trend in surveys on short rate expectations at long horizons, but requires a break in the level of its factors to closely fit the low level of these surveys since 2015.

Keywords: Dynamic term structure model, bond return predictability, shadow rate model, structural break, regime-switching.

JEL: E43, E44, G12.

*Andreasen, mandreasen@econ.au.dk, Aarhus University, CREATES, and Danish Finance Institute; Jørgensen, kasper.joergensen@frb.gov, Board of Governors of the Federal Reserve System; Meldrum, andrew.c.meldrum@frb.gov, Board of Governors of the Federal Reserve System. We thank Eric Engstrom, Joachim Grammig, Thomas B. King, and Cynthia Wu for useful comments and discussions. The analysis and conclusions are those of the authors and do not indicate concurrence by the Board of Governors of the Federal Reserve System or other members of the research staff of the Board.

1 Introduction

One of the most widely used dynamic term structure models (DTSMs) in recent years is the shadow rate model (SRM) proposed by Black (1995) (see Kim and Singleton (2012), Christensen and Rudebusch (2015), Bauer and Rudebusch (2016), Wu and Xia (2016), Andreasen and Meldrum (2018), among others). The popularity of the SRM stems from its ability to enforce the zero lower bound (ZLB) on nominal bond yields, while preserving a (near-)linear relationship between bond yields and the pricing factors away from the ZLB. However, in this paper we show that this standard version of the SRM has two important shortcomings when applied to U.S. We then address each of these shortcomings by proposing a new SRM.

The first shortcoming of the standard SRM that we document is its inability to match a shift in bond return predictability at the ZLB. Beginning with the seminal studies of Fama and Bliss (1987) and Campbell and Shiller (1991), regressions of excess bond returns on the slope of the yield curve have received substantial attention in the finance literature. Dai and Singleton (2002) use such return regressions as a diagnostic test for DTSMs by requiring that a well-specified model should imply population regression coefficients that match those in the data. We document a shift in these bond return regressions at the ZLB and therefore propose a modified diagnostic test for DTSMs. Specifically, we argue that a well-specified ZLB-consistent DTSM should match the slope coefficients in these predictability regressions when the short rate is close to the ZLB and when it is away from the lower bound. We show that the standard SRM matches these slope coefficients away from the lower bound (as is well known), but that the model is unable to match the shift in these slope coefficients at the ZLB. This suggests that simply enforcing the ZLB by truncating the short rate at zero in a Gaussian model is not sufficient to properly capture the change in bond yield dynamics that occurred at the ZLB.

We then consider whether two extensions of the standard SRM can match the shift in bond return predictability at the ZLB. The first extension introduces regime-switching in the SRM by allowing the (physical) dynamics of the pricing factors to change at the lower bound. This extension is motivated by a desire to accommodate the effects of various unconventional monetary policy measures that are largely expected to have a temporary effect on bond yields. The second extension we consider allows for a permanent break in the (physical) dynamics of the pricing factors in 2008 to accommodate long-lasting effects of recent developments, such as a possible "secular stagnation" as proposed by Summers (2015). The SRM with regime-switching is able to match the shift in bond return predictability at the ZLB, meaning that this shift can be attributed to a re-pricing of risk among bond investors at the lower bound. On the other hand, the SRM with a permanent break does not improve on the performance of the standard SRM.

The second shortcoming of the standard SRM that we document is its inability to match the low level of long-horizon short rate expectations in surveys since 2015. If we include surveys when estimating DTSMs, as suggested by Kim and Orphanides (2012), we find that the standard SRM broadly matches the downward trend in short rate expectations at long horizons since 1990, but that the model cannot explain the low level of these surveys from 2015 and onwards. This could indicate that the standard SRM fails to match a recent decline in the natural rate of interest, as documented by Laubach and Williams (2016), Del Negro, Giannone, Giannoni and Tambalotti (2017), Christensen and Rudebusch (2019), among others. Unfortunately, neither of the two extensions that we consider are able to improve on the performance of the standard SRM in terms of matching these low short rate expectations. However, a further extension to the regime-switching model that incorporates a permanent break in the level of the (physical) pricing factors in 2015 is able to match the recent low level of short rate expecations in surveys. This finding therefore provides tentative evidence of long-lasting effects from the recent financial crisis, as argued in Summers (2015).

The remainder of this paper proceeds as follows. Section 2 documents the shift in bond return predictability during the recent ZLB period. Section 3 shows that the standard SRM cannot match this new empirical finding. The two modifications of the standard SRM are presented in 4. Section 5 studies the ability of the considered SRMs to match long-horizon short rate expectations from surveys. Section 6 discusses the timing of a permanent break in the SRM and introduces a permanent break in the level of the pricing factors for the SRM with regime-switching. Concluding comments are provided in Section 7.

2 Shift in Bond Return Predictability at the ZLB

This section documents a significant shift in the ability of the yield spread to predict future excess bond returns during the recent ZLB period. We proceed by showing instability in the classic bond return predictability regressions in Section 2.1, while Section 2.2 presents our new empirical finding about the nature of this instability. Various robustness checks are provided in Section 2.3.

2.1 Instability in the Classic Return Predictability Regression

The risk premium implied by a long-maturity bond is often measured by its expected return in excess of the return on a short-maturity bond. These expected returns are not directly observed and therefore are typically estimated by regressing realized excess bond returns $rx_{t,t+h}^{(n)}$ on a set of predictors $\mathbf{z}_t$. That is, by running the regression

$$rx_{t,t+h}^{(n)} = \beta_0^{(n)} + \boldsymbol{\beta}^{(n)\prime}\mathbf{z}_t + \varepsilon_{t,t+h}^{(n)}, \tag{1}$$

where $\varepsilon_{t,t+h}^{(n)}$ is a residual. Here, $rx_{t,t+h}^{(n)} \equiv p_{t+h}^{(n-h)} - p_t^{(n)} + p_t^{(h)}$ denotes the excess return on an n-month bond relative to an h-period bond between time t and $t+h$, with $p_t^{(n)}$ being the log price at time t of a zero-coupon bond with n periods to maturity. A natural benchmark is the expectations hypothesis, which implies $\boldsymbol{\beta}^{(n)} = 0$ and hence no bond return predictability.

One of the most prominent and robust predictors of excess bond returns is the slope of the yield curve, as used in the seminal work of Campbell and Shiller (1991).[1] Letting $y_t^{(n)}$ denote the yield at time t on an n period zero-coupon bond, the specification in Campbell and Shiller (1991) is equivalent to

$$rx_{t,t+h}^{(n)} = \beta_0^{(n)} + \beta_1^{(n)} \left(y_t^{(n)} - y_t^{(h)} \right) + \varepsilon_{t,t+h}^{(n)}, \tag{2}$$

where the yield spread $y_t^{(n)} - y_t^{(h)}$ measures the slope of the yield curve.

However, the recent financial crisis has generated several unusual developments in the bond market that are likely to have affected the relationship between excess bond returns and the yield spread. The most obvious is probably that U.S. short-term interest rates were constrained by the ZLB from late 2008 to late 2015. This introduced an obvious nonlinearity in the yield curve that most likely caused the yield spread $y_t^{(n)} - y_t^{(h)}$ to be smaller than it would otherwise have been, because the short-term yield $y_t^{(h)}$ was constrained from below. Such a slope compression is likely to have increased $\beta_1^{(n)}$ because a given yield spread during the ZLB period carried a stronger signal about future bond returns than before the financial crisis.

A second important development was the Large-Scale Asset Purchase (LSAP) programs that the Federal Reserve used during the ZLB period to stimulate the U.S. economy. These programs were mainly designed to temporarily lower long-term yields by reducing bond risk premiums, as discussed in D'Amico, English, Lopez-Salido and Nelson (2012), Li and Wei (2013), and Bonis, Ihrig and Wei (2017). Thus, the LSAP programs are likely to have affected both the level of excess bond returns and the yield spread, implying that the coefficients in equation (2) may be affected.

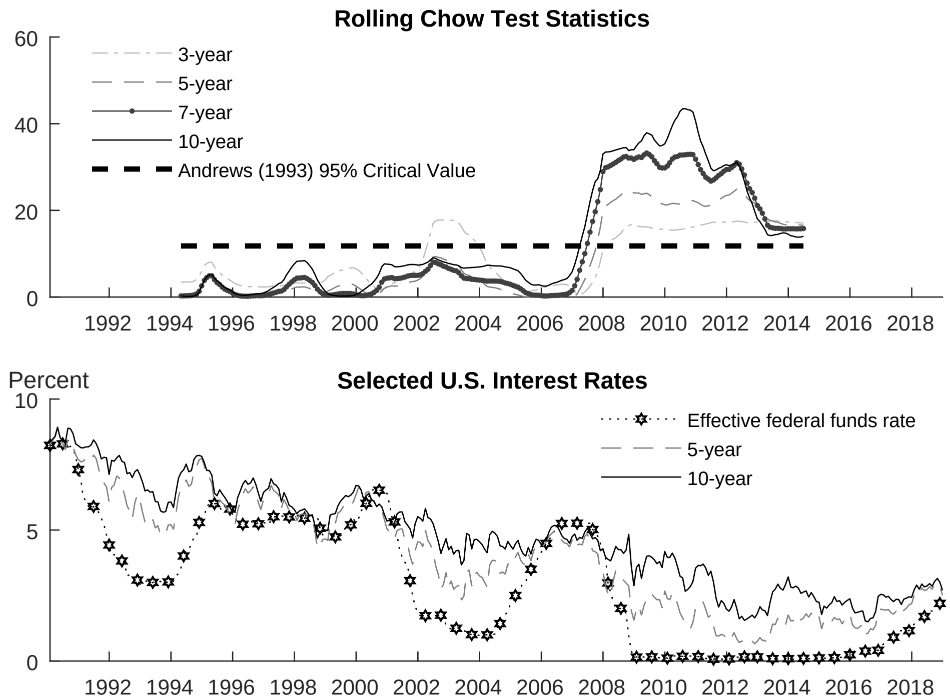
Finally the large financial shock in 2008 may permanently have reduced expectations about the long-term level of the short rate due to concerns about a secular stagnation in the U.S. (see, for instance, Summers (2015)). Such a shift in expectations seems also capable of

[1] A number of other yield curve and macroeconomic variables have recently been shown to predict excess returns (see, for example, Cochrane and Piazzesi (2005), Ludvigson and Ng (2009), Joslin, Priebsch and Singleton (2014), Cieslak and Povala (2015), and Bauer and Rudebusch (2017)).

affecting the coefficients in equation (2).

Figure 1: Chow Tests and Historical Bond Yields

The top chart reports the Chow test statistic for breaks in equation (2) using at least 15 percent of the observations for the pre- and post-break sample. The reported 95 percent critical value is for the maximum Chow test of Andrews (1993). The bottom chart shows selected historical bond yields. The 5- and 10-year yields are end-month from Gürkaynak, Sack and Wright (2007) starting in January 1990 and ending in December 2018.



Given these considerations, we formally test whether the coefficients in equation (2) are constant across time using a monthly sample of U.S. bond yields from Gürkaynak et al. (2007) between January 1990 and December 2018, adopting a 1-year holding period ($h = 12$). The top chart in Figure 1 shows the Chow test statistics for the 3-, 5-, 7-, and 10-year bond yield for all possible break points after January 1990, reserving at least 15 percent of the observations for the pre- and post-break sample. The reported 95 percent critical value is for the maximum Chow test of Andrews (1993). We find that the Chow test statistics increase strongly around 2008 for all bond yields and generally reach their highest level between 2008 and 2009. The bottom chart in Figure 1 shows that this break in equation (2) coincides with the short rate approaching the ZLB during 2008.

Thus, the classic predictability regressions in equation (2) display evidence of a break around 2008 when the short rate in the U.S. approached the ZLB.

2.2 Formalizing the Shift in Bond Return Predictability

The next step is to examine the nature of this break in the classic return predictability regression. The first two of the unusual bond market developments mentioned above (that is, the ZLB episode and the LSAP programs) were specific to the period between 2008 and late 2015. These developments seem therefore likely to only have had an effect on bond yields that is largely contained to that period. Hence, a natural extension of equation (2) is to allow for a threshold effect depending on the level of the short rate. That is, to consider a regression of the form

$$rx_{t,t+h}^{(n)} = \beta_{0,1}^{(n)}\mathcal{I}_{\{r_t \geq c\}} + \beta_{0,2}^{(n)}\mathcal{I}_{\{r_t < c\}} + \left(\beta_{1,1}^{(n)}\mathcal{I}_{\{r_t \geq c\}} + \beta_{1,2}^{(n)}\mathcal{I}_{\{r_t < c\}}\right)\left(y_t^{(n)} - y_t^{(h)}\right) + \varepsilon_{t,t+h}^{(n)}, \quad (3)$$

where $\mathcal{I}_{\{r_t \geq c\}}$ is an indicator function that takes a value of 1 at time t if the short rate $r_t \geq c$ and 0 otherwise. As a result, equation (3) allows the regression coefficients in equation (2) to shift when r_t becomes sufficiently low.

On the other hand, concerns about secular stagnation and a lower long-term level for the short rate are not specific to the ZLB period and may therefore affect bond yields for several years after the ZLB period. Hence, an alternative extension of equation (2) is to allow for a permanent break at time τ. That is, to consider a regression of the form

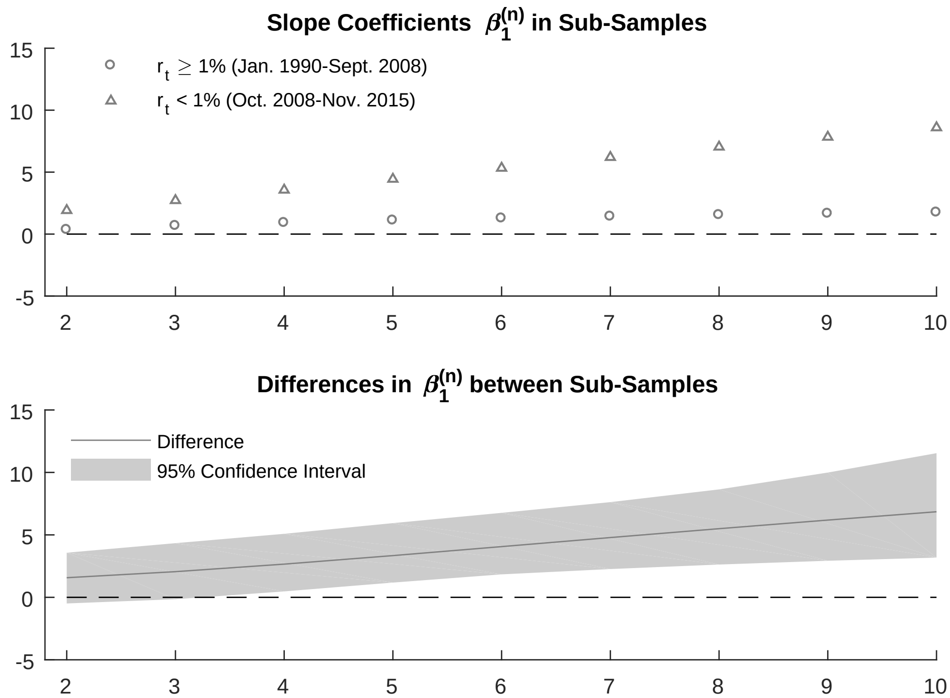
$$rx_{t,t+h}^{(n)} = \beta_{0,1}^{(n)}\mathcal{I}_{\{t<\tau\}} + \beta_{0,2}^{(n)}\mathcal{I}_{\{t\geq\tau\}} + \left(\beta_{1,1}^{(n)}\mathcal{I}_{\{t<\tau\}} + \beta_{1,2}^{(n)}\mathcal{I}_{\{t\geq\tau\}}\right)\left(y_t^{(n)} - y_t^{(h)}\right) + \varepsilon_{t,t+h}^{(n)}, \quad (4)$$

where $\mathcal{I}_{\{t<\tau\}}$ is an indicator function that takes a value of 1 at time t if $t < \tau$ and 0 otherwise.

Given that there is only one ZLB period for the postwar U.S. economy in our sample, it is hard to tell which of the nonlinear specifications in equations (3) and (4) we should

Figure 2: Bond Return Predictability

The top chart reports the slope coefficient in $rx^{(n)}_{t,t+h} = \beta^{(n)}_0 + \beta^{(n)}_1 \left(y^{(n)}_t - y^{(h)}_t\right) + \varepsilon^{(n)}_{t,t+h}$ with $h = 12$ when estimated from January 1990 to September 2008 (excluding December 2003) where $r_t \geq 0.01$, and when estimated from October 2008 to November 2015 where $r_t < 0.01$. The bottom chart reports the difference in the slope estimates at a given maturity. The 95 percent confidence interval for these differences are computed using a block bootstrap with 5,000 repetitions and a block window of 24 months. In each bootstrap sample it is required that there are at least 50 observations where the short rate is above and below 1 percent, respectively. The x-axes reports maturity in years, and all yields are end-month from Gürkaynak et al. (2007) with r_t measured by the effective federal funds rate.



prefer. To avoid taking a view on the best specification at this stage, we simply estimate the standard predictability regression in equation (2) using two separate samples. The first sample ("Regime 1") runs from January 1990 to September 2008, excluding December 2003 when the short rate was slightly below 1 percent. The second sample ("Regime 2") runs from October 2008 to November 2015, which is the period where the short rate was close to the ZLB. Because the first sample includes only observations before October 2008 with a short rate above 1 percent, and the second sample includes only observations after October 2008 with a short rate below 1 percent, the estimates are consistent with either equation

(3) with $c = 0.01$ or equation (4) with $\tau = 225$ (October 2008 is the 225th month of our sample).

We estimate the regressions for both regimes using monthly observations of yields and excess returns on bonds with maturities from 2 to 10 years using a 1-year holding period as above. The gray circles in the top chart of Figure 2 for the pre-ZLB period reveal the usual empirical pattern that the slope coefficients in equation (2) are positive and increase with maturity. Our new empirical finding is that these slope coefficients are larger and increase faster with maturity during the recent ZLB period (the gray triangles) when compared to the pre-ZLB period. The bottom chart in Figure 2 shows that these differences in the slope coefficients are significant at the 5 percent level for maturities beyond 3 years. Thus, recent developments in the bond market imply a shift in bond return predictability with a much stronger relationship between the yield spread and excess bond returns than observed in the pre-ZLB period.

2.3 Robustness

We examine the robustness of this shift in bond return predictability in Table 1, where $\Delta\beta_1^{(n)} \equiv \beta_{1,2}^{(n)} - \beta_{1,1}^{(n)}$ refers to the change in the slope coefficient in equation (2). The first column considers biannual bond returns ($h = 6$) instead of the annual horizon used so far. The shift in the slope coefficients is again positive, and it is significant beyond the 4-year maturity. Our new finding is also robust to measuring the slope of the yield curve by the second principal component (PCA2) of bond yields, as shown by the second column in Table 1. The next column shows that the shift in bond return predictability is also robust to replacing the yield spread by the forward spread $f_t^{(n,h)} - y_t^{(h)}$, as in Fama and Bliss (1987), where the forward rate is given by $f_t^{(n,h)} = \log\left(P_t^{(n+h)}/P_t^{(n)}\right)$. The fourth robustness check reported in Table 1 adds the Chicago Fed National Activity Index (CFNAI) as in Joslin et al. (2014) and the inflation trend factor of Cieslak and Povala (2015) as additional regressors in equation (2) to control for the effect of macro variables. Table 1 shows that we again find a positive and significant shift in the ability of the yield spread to predict future bond returns during the ZLB period.

Table 1: Bond Return Predictability: Robustness

Column (1) estimates equation (2) with $h = 6$. Column (2) replaces $y_t^{(n)} - y_t^{(h)}$ in equation (2) by the second principal component of bond yields. Column (3) replaces $y_t^{(n)} - y_t^{(h)}$ in equation (2) by the forward spread $f_t^{(n,h)} - y_t^{(h)}$. Column (4) adds CFNAI and the inflation trend factor of Cieslak and Povala (2015) as two additional macro control variables to equation (2). Column (5) adds $\mathcal{I}_{\{r_t<c\}} i_t^*$ to equation (2) to control for the effect of the natural nominal short rate i_t^*, when measured by the expected 3-month Treasury bill yield from 5 to 10 years ahead in the Blue Chip Economic Indicators survey. These alternative specifications are estimated from January 1990 to September 2008 (excluding December 2003) where $r_t \geq 0.01$ and from October 2008 to November 2015 where $r_t < 0.01$. Standard errors are provided in parenthesis using a block bootstrap with 5,000 repetitions and a block window of 24 months. In each bootstrap sample it is required that there are at least 50 observations where the short rate is above and below 1 percent, respectively. Significance at the 10, 5, and 1 percent level is denoted by *, **, and ***. All yields are end-month from Gürkaynak et al. (2007) with r_t measured by the effective federal funds rate.

	(1)		(2)		(3)		(4)		(5)	
	$h = 6$		PCA2		$f_t^{(n,h)} - y_t^{(h)}$		Macro		Trend	
n	$\beta_1^{(n)}$	$\Delta\beta_1^{(n)}$	$\beta_1^{(n)}$	$\Delta\beta_1^{(n)}$	$\beta_1^{(n)}$	$\Delta\beta_1^{(n)}$	$\beta_1^{(n)}$	$\Delta\beta_1^{(n)}$	$\beta_1^{(n)}$	$\Delta\beta_1^{(n)}$
2	0.64	0.79	0.03	0.46*	0.16	0.78	0.04	1.81*	0.32	1.18
	(0.57)	(0.69)	(0.25)	(0.28)	(0.50)	(0.52)	(1.00)	(1.06)	(0.97)	(0.98)
3	0.83	1.13	0.17	1.12**	0.33	1.05*	0.41	2.24*	0.64	1.63
	(0.63)	(0.82)	(0.46)	(0.52)	(0.57)	(0.61)	(1.08)	(1.21)	(1.05)	(1.10)
4	0.95	1.53	0.38	1.97***	0.46	1.50**	0.68	2.77**	0.87	2.22*
	(0.64)	(0.98)	(0.62)	(0.69)	(0.59)	(0.68)	(1.08)	(1.28)	(1.06)	(1.21)
5	1.02	1.93*	0.62	2.90***	0.56	2.06***	0.92	3.39***	1.07	3.00**
	(0.63)	(1.12)	(0.74)	(0.82)	(0.60)	(0.69)	(1.06)	(1.30)	(1.03)	(1.32)
6	1.08*	2.33*	0.89	3.83***	0.64	2.67***	1.12	4.08***	1.24	4.00***
	(0.61)	(1.20)	(0.83)	(0.98)	(0.61)	(0.70)	(1.04)	(1.33)	(1.00)	(1.45)
7	1.12*	2.70**	1.17	4.72***	0.71	3.31***	1.30	4.80***	1.39	5.21***
	(0.60)	(1.25)	(0.89)	(1.19)	(0.61)	(0.86)	(1.02)	(1.41)	(0.98)	(1.57)
8	1.16**	3.04**	1.45	5.52***	0.77	3.99***	1.46	5.54***	1.52	6.63***
	(0.59)	(1.32)	(0.94)	(1.45)	(0.63)	(1.13)	(1.00)	(1.59)	(0.96)	(1.66)
9	1.19**	3.35**	1.72*	6.22***	0.83	4.72***	1.59	6.26***	1.63*	8.22***
	(0.59)	(1.41)	(0.99)	(1.75)	(0.64)	(1.46)	(1.00)	(1.85)	(0.95)	(1.71)
10	1.22**	3.64**	1.98*	6.81***	0.88	5.54***	1.71*	6.97***	1.73*	9.93***
	(0.59)	(1.54)	(1.03)	(2.07)	(0.67)	(1.83)	(1.00)	(2.17)	(0.95)	(1.73)

The fifth column in Table 1 controls for the trend in the natural (or equilibrium) nominal rate i_t^*, which Bauer and Rudebusch (2017) show has a strong effect on bond yields. We use the approach in Christensen and Rudebusch (2019) among others and measure the natural rate by its long-horizon expectations from 5 to 10 years into the future. Specially, we measure i_t^* by the average expected 3-month Treasury bill yield from 5 to 10 years ahead, as reported biannually in the Blue Chip Economic Indicators survey.[2] Given that i_t^* operates as a level

[2]The forecast horizons are not the same in each edition of the survey, so we linearly interpolate the mean

factor affecting all yields equally, it should cancel out in the yield spread away from the ZLB but not close to the bound, where short-term interest rates cannot respond one-to-one with i_t^*. This suggests that i_t^* should only affect excess bond returns close to the ZLB, and we therefore add the regressor $\mathcal{I}_{\{r_t<c\}}i_t^*$ to equation (2). The last column in Table 1 shows that controlling for i_t^* cannot explain our new empirical result, as we also in this case find a positive and significant shift in the ability of the yield spread to predict bond returns.[3]

Thus, the shift in bond return predictability documented in Section 2.2 is robust to several modifications and extensions of the classic predictability regression in (2).

3 A Shortcoming of the Standard Shadow Rate Model

This section shows that the standard SRM with Gaussian factor dynamics cannot explain the shift in bond return predictability documented in Section 2. We proceed by describing the model in Section 3.1, our estimation method in Section 3.2, and the inability of the standard SRM to generate more return predictability from the yield spread at the ZLB in Section 3.3.

3.1 A Shadow Rate Model

Suppose that the U.S. economy can be described by a set of factors collected in $\mathbf{x}_t$ of dimension $n_x \times 1$ that evolves as

$$\mathbf{x}_{t+1} = \mathbf{h}\left(\mathbf{x}_t\right) + \boldsymbol{\Sigma}\boldsymbol{\varepsilon}_{t+1}^{\mathbb{P}}. \tag{5}$$

expectations across respondents to get short rate expectations at the desired horizon. This biannual series is then extended to a monthly frequency by linear interpolation. The data source is Wolters Kluwer Legal and Regulatory Solutions U.S., Blue Chip Economic Indicators.

[3] Unreported results show that we also find a positive and significant shift in $\beta_1^{(n)}$ if we simply add our measure of i_t^* as an extra regressor in equation (2) to account for the effect of i_t^* away from the ZLB and close to the lower bound.

The function $\mathbf{h}(\mathbf{x}_t)$ is potentially nonlinear and $\boldsymbol{\varepsilon}_{t+1}^{\mathbb{P}}$ is an $n_x \times 1$ vector of independent standard Gaussian shocks under the physical probability measure $\mathbb{P}$, denoted $\boldsymbol{\varepsilon}_{t+1}^{\mathbb{P}} \sim \mathcal{NID}(\mathbf{0}, \mathbf{I})$. Throughout, we consider models with the standard 3 pricing factors (that is, $n_x = 3$). A stochastic discount factor M_{t+1} is assumed to have the flexible form

$$M_{t+1} = \exp\left\{-r_t - \boldsymbol{\lambda}(\mathbf{x}_t)' \boldsymbol{\lambda}(\mathbf{x}_t) - \boldsymbol{\lambda}(\mathbf{x}_t)\boldsymbol{\varepsilon}_{t+1}^{\mathbb{P}}\right\},$$

where r_t is the one-period nominal short rate and $\boldsymbol{\lambda}(\mathbf{x}_t)$ is a potentially nonlinear function for the market prices of risk. Many macroeconomic models relate M_{t+1} to consumption and inflation. The precise structural underpinnings of M_{t+1} are, however, not provided in reduced-form DTSMs to reduce the risk of model misspecification, although the factors in $\mathbf{x}_t$ at an overall level are linked to the U.S. economy. The short rate is given by

$$r_t = \max\{0, s_t\}, \tag{6}$$

where the use of the maximum function following Black (1995) constrains r_t from below at zero and $s_t \equiv \alpha + \boldsymbol{\beta}'\mathbf{x}_t$ is an unconstrained "shadow rate." To make the bond pricing tractable, it is assumed that the market prices of risk are given by

$$\boldsymbol{\lambda}(\mathbf{x}_t) = \boldsymbol{\Sigma}^{-1}\left(\mathbf{h}(\mathbf{x}_t) - \boldsymbol{\Phi}\boldsymbol{\mu} - (\mathbf{I} - \boldsymbol{\Phi})\mathbf{x}_t\right), \tag{7}$$

because it implies a first-order vector auto-regression (VAR) under the risk-neutral probability measure $\mathbb{Q}$. That is,

$$\mathbf{x}_{t+1} = \boldsymbol{\Phi}\boldsymbol{\mu} + (\mathbf{I} - \boldsymbol{\Phi})\mathbf{x}_t + \boldsymbol{\Sigma}\boldsymbol{\varepsilon}_{t+1}^{\mathbb{Q}}, \tag{8}$$

where $\boldsymbol{\varepsilon}_{t+1}^{\mathbb{Q}} \sim \mathcal{NID}(\mathbf{0}, \mathbf{I})$.

The standard SRM with Gaussian factor dynamics is obtained by letting $\mathbf{h}(\mathbf{x}_t) = \mathbf{h}_0 +$

$\mathbf{H}_x \mathbf{x}_t$, implying that equation (5) reduces to

$$\mathbf{x}_{t+1} = \mathbf{h}_0 + \mathbf{H}_x \mathbf{x}_t + \mathbf{\Sigma} \boldsymbol{\varepsilon}_{t+1}^{\mathbb{P}}. \tag{9}$$

Yields do not have closed-form expressions in this version of the SRM, and we therefore use the second-order approximation of Priebsch (2013). For identification, we impose the standard restrictions that $\boldsymbol{\beta}' = \mathbf{1}$, $\boldsymbol{\mu} = \mathbf{0}$, $\mathbf{\Sigma}$ is lower triangular, and $\mathbf{\Phi}$ is in Jordan form with increasing diagonal elements (see Joslin, Singleton and Zhu (2011)). A preliminary analysis reveals that the first eigenvalue of $\mathbf{\Phi}$ is often indistinguishable from zero, meaning that the pricing factors have a near unit root under $\mathbb{Q}$. This implies that the unconditional mean of the shadow rate under $\mathbb{Q}$ is badly identified, and we therefore impose $\mathbf{\Phi}(1,1) = \alpha = 0$ to ensure identification (see Hamilton and Wu (2012)).[4] Finally, the $\mathbb{P}$ transition parameters $\mathbf{h}_0$ and $\mathbf{H}_x$ are unrestricted.

3.2 Estimation Method and Data

Most previous studies that estimate DTSMs use quasi-maximum likelihood (QML) methods. However, the joint optimization of all parameters required by standard QML is computationally challenging and the asymptotic properties of these QML estimators are unknown in nonlinear models such as the SRM. We overcome these limitations by using the sequential regression (SR) approach of Andreasen and Christensen (2015), which is computationally simpler and provides asymptotically Gaussian parameter estimates.

To apply the SR approach to the standard SRM, it is convenient to define two vectors with partly overlapping sub-sets of parameters. First, $\boldsymbol{\theta}_1$ collects the "$\mathbb{Q}$ parameters" in equation (8) that determine the cross-sectional relationship between the pricing factors and bond yields. Second, $\boldsymbol{\theta}_2$ collects the "$\mathbb{P}$ parameters" in equation (9) that determine the time-series dynamics of the pricing factors. Because $\mathbf{\Sigma}$ appears in both $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$, it is convenient

[4]Previous studies that have imposed these restrictions on $\mathbf{\Phi}(1,1)$ and α include Christensen and Rudebusch (2015).

to partition these vectors further as $\boldsymbol{\theta}_1 = \begin{bmatrix} \boldsymbol{\theta}_{11}' & vech\left(\boldsymbol{\Sigma}\right)' \end{bmatrix}'$ and $\boldsymbol{\theta}_2 = \begin{bmatrix} \boldsymbol{\theta}_{22}' & vech\left(\boldsymbol{\Sigma}\right)' \end{bmatrix}'$, where $\boldsymbol{\theta}_{11} = \begin{bmatrix} \boldsymbol{\Phi}\left(2,2\right) & \boldsymbol{\Phi}\left(3,3\right) \end{bmatrix}'$ and $\boldsymbol{\theta}_{22} = \begin{bmatrix} \mathbf{h}_0' & vec\left(\mathbf{H}_x\right)' \end{bmatrix}'$.

The SR approach proceeds in three steps. At step 1, $\boldsymbol{\theta}_1$ and the pricing factors $\mathbf{x}_t$ are jointly estimated using non-linear regressions from $n_{y,t}$ yields with maturities $m_1, m_2, \ldots, m_{n_{y,t}}$ in period t. The observed yield with maturity m_j at time t is denoted $y_t^{(m_j)} = g_{m_j}\left(\mathbf{x}_t; \boldsymbol{\theta}_1\right) + v_{m_j,t}$, where $g_{m_j}\left(\mathbf{x}_t; \boldsymbol{\theta}_1\right)$ is the model-specific function that relates the pricing factors to the cross section of yields and $v_{m_j,t}$ is a measurement error. We assume that these measurement errors have zero means and finite, positive-definite covariance matrices. For a given value of $\boldsymbol{\theta}_1$, we obtain the pricing factors in each period by

$$\widehat{\mathbf{x}}_t\left(\boldsymbol{\theta}_1\right) = \underset{\mathbf{x}_t \in \mathbb{R}^{n_x}}{\arg\min}\ \frac{1}{2n_{y,t}} \sum_{j=1}^{n_{y,t}} \left(y_t^{(m_j)} - g_{m_j}\left(\mathbf{x}_t; \boldsymbol{\theta}_1\right) \right)^2. \tag{10}$$

The estimate of $\boldsymbol{\theta}_1$ is then given by minimizing the sum of squared residuals from equation (10). That is,

$$\widehat{\boldsymbol{\theta}}_1^{step1} = \underset{\boldsymbol{\theta}_1 \in \boldsymbol{\Theta}_1}{\arg\min} \frac{1}{2N} \sum_{t=1}^{T} \sum_{j=1}^{n_{y,t}} \left(y_t^{(m_j)} - g_{m_j}\left(\widehat{\mathbf{x}}_t\left(\boldsymbol{\theta}_1\right); \boldsymbol{\theta}_1\right) \right)^2, \tag{11}$$

where $\widehat{\boldsymbol{\theta}}_1^{step1}$ denotes the step 1 estimate of $\boldsymbol{\theta}_1$, $N = \sum_{t=1}^{T} n_{y,t}$, and $\boldsymbol{\Theta}_1$ is the feasible domain of $\boldsymbol{\theta}_1$. Andreasen and Christensen (2015) show that $\widehat{\boldsymbol{\theta}}_1^{step1}$ is consistent and asymptotically Gaussian for $n_{y,t} \to \infty$ for all t given standard regularity conditions.

At step 2 of the SR approach, the time-series parameters $\boldsymbol{\theta}_2$ are estimated using the estimated factors $\widehat{\mathbf{x}}_t\left(\widehat{\boldsymbol{\theta}}_1^{step1}\right)$ from step 1 with a correction for estimation uncertainty in these factors. As shown in Andreasen and Christensen (2015), this procedure corresponds to running a modified VAR, where all second moments are corrected for estimation uncertainty in $\widehat{\mathbf{x}}_t\left(\widehat{\boldsymbol{\theta}}_1^{step1}\right)$. The details are provided in Appendix A.

At step 3 of the SR approach, the estimates of $\boldsymbol{\Sigma}$ from step 1 and 2 can be combined optimally and, conditional on the optimal estimate of $\boldsymbol{\Sigma}$, the remaining $\mathbb{Q}$ parameters $\boldsymbol{\theta}_{11}$

and the $\mathbb{P}$ parameters $\boldsymbol{\theta}_2$ are re-estimated. Preliminary results for our applications reveal that $\boldsymbol{\Sigma}$ is estimated very imprecisely at step 1 compared with step 2.[5] So, for simplicity, we settle by conditioning on the estimate $\widehat{\Sigma}^{step2}$ at step 2 and re-estimate $\boldsymbol{\theta}_{11}$ as

$$\widehat{\boldsymbol{\theta}}_{11}^{step3} = \underset{\boldsymbol{\theta}_{11}\in\boldsymbol{\Theta}_{11}}{\arg\min} \frac{1}{2N}\sum_{t=1}^{T}\sum_{j=1}^{n_{y,t}}\left(y_t^{(m_j)} - g_{m_j}\left(\widehat{\mathbf{x}}_t\left(\boldsymbol{\theta}_{11}, \widehat{\Sigma}^{step2}\right); \boldsymbol{\theta}_{11}, \widehat{\Sigma}^{step2}\right)\right)^2, \tag{12}$$

where $\widehat{\boldsymbol{\theta}}_{11}^{step3}$ denotes the step 3 estimate of $\boldsymbol{\theta}_{11}$ and $\boldsymbol{\Theta}_{11}$ is the feasible domain of $\boldsymbol{\theta}_{11}$. Finally, we update the estimate of $\boldsymbol{\theta}_2$ by re-running step 2 using the estimated factors from step 3, that is $\widehat{\mathbf{x}}_t\left(\widehat{\boldsymbol{\theta}}_{11}^{step3}, \widehat{\Sigma}^{step2}\right)$.

The SR approach is designed for a setting with a large cross section, and we therefore include more yields than typically used when estimating DTSMs. That is, we represent the yield curve by 25 points, using the 3-month yield, the 6-month yield, yields in the 1-year to 3-year range at 3-month intervals, and yields in the 3- to 10-year range at 6-month intervals. For the 3- and 6-month yields, we use Treasury bill yields, while yields with longer maturities are obtained from the Gürkaynak et al. (2007) data set. Our monthly sample covers the period from January 1990 through December 2018, where the starting point is chosen to reduce the possibility of a structural break associated with the shift in U.S. monetary policy during the 1980s (see Rudebusch and Wu (2007)).

3.3 Matching the Shift in Bond Return Predictability

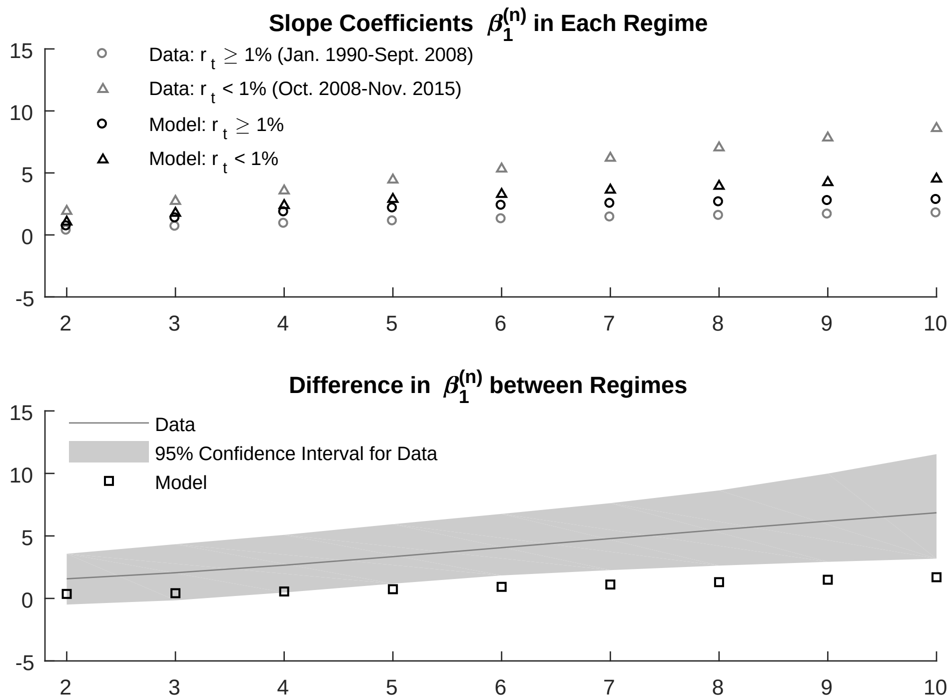
We next examine whether the standard SRM can explain the shift in bond return predictability when the short rate approaches the ZLB. Given that the standard SRM has constant parameters, it is obvious that it cannot match this shift by generating a permanent break in the regression coefficients, as implied by equation (4). However, this model may be able to match the shift in bond return predictability through a threshold effect due to

[5] This finding is similar to the result of Joslin et al. (2011) for affine DTSMs, because their estimates of $\boldsymbol{\Sigma}$ from the time-series dynamics of their (observed) factors hardly change when taking account of the cross section of bond yields.

its nonlinear mapping between the pricing factors and bond yields. We therefore simulate a sample of 1 million observations from the standard SRM and estimate equation (3) with a threshold of 1 percent.

Figure 3: Bond Return Predictability: Standard SRM

The top chart shows the ability of the standard SRM to match the shift in the slope coefficients in equation (3) with $h = 12$ and $c = 0.01$ using a simulated sample of 1 million observations. The corresponding data moments are from equation (2) with $h = 12$ when estimated from January 1990 to September 2008 (excluding December 2003) where $r_t \geq 0.01$, and when estimated from October 2008 to November 2015 where $r_t < 0.01$. The bottom chart shows the difference in the slope coefficients in equation (3) at a given maturity as implied by the standard SRM and the data. All x-axes show the maturity in years.



When the short rate is above 1 percent, the top chart in Figure 3 shows that the standard SRM (the black circles) matches closely the corresponding data moment (the gray circles). This finding is not too surprising, given that the standard SRM away from the ZLB reduces to the Gaussian affine model, which is able to match these moments (see, for instance, Dai and Singleton (2002)). Turning to the more interesting case when the short rate is below 1 percent, we find that the standard SRM (the black triangles) is indeed able to generate larger slope coefficients when compared to the loadings away from the ZLB. But this increase in the slope coefficients is insufficient to match the shift in the data. This is shown by the bottom chart in Figure 3, as the differences between the model-implied slope coefficients lie outside the 95 percent confidence interval for the corresponding data moment for maturities exceeding 3 years.[6]

Thus, it appears that simply enforcing the ZLB by the standard SRM is insufficient to capture the shift in bond return predictability that occurred during the recent ZLB period.

4 Two Extended Shadow Rate Models

This section proposes two extensions of the standard SRM by introducing nonlinearities in the $\mathbb{P}$ dynamics of the pricing factors. Section 4.1 describes the two extensions, which allow for i) regime-switching in the pricing factors or ii) a permanent break in the factor dynamics. Section 4.2 shows that it is impossible to distinguish between these two models (or the standard SRM) from the in-sample fit to the cross-section of yields, but that the models have substantially different implications for the time-series dynamics of bond yields. Section 4.3 shows that the SRM with regime-switching goes a long way in explaining the shift in bond return predictability documented in Section 2, whereas the extension with a permanent break performs even worse than the standard SRM.

4.1 Nonlinear Factor Dynamics in the SRM

As discussed in Section 2, there are at least two interpretations of the recent shift in bond return predictability. If we believe that the shift is temporary, as implied by the threshold regression in equation (3), it seems natural to accommodate regime-switching in

[6]This result supports the findings of Andreasen and Meldrum (2018), who show that once the ZLB period is included in the estimation sample, the standard SRM cannot match the desired slope coefficients in equation (2) conditional on yields being *away* from the ZLB. This is because the estimated factor dynamics change once the ZLB period is included in the estimation sample, and this distorts the time-series dynamics of yields when they are *away* from the ZLB.

the $\mathbb{P}$ dynamics of the pricing factors. That is, to replace equation (9) with

$$\mathbf{x}_{t+1} = \mathbf{h}_0^{(1)}\mathcal{I}_{\{r_t \geq c\}} + \mathbf{h}_0^{(2)}\mathcal{I}_{\{r_t < c\}} + \left(\mathbf{H}_x^{(1)}\mathcal{I}_{\{r_t \geq c\}} + \mathbf{H}_x^{(2)}\mathcal{I}_{\{r_t < c\}}\right)\mathbf{x}_t + \boldsymbol{\Sigma}\boldsymbol{\varepsilon}_{t+1}^{\mathbb{P}}, \tag{13}$$

where both the level and the persistence of the factors may change when r_t falls below the threshold c. In our application, we consider a threshold of 1 percent, which means that the two regimes implied by the model are consistent with the two regimes for the predictability regression in equation (2). The short rate and the $\mathbb{Q}$ dynamics remain given by equations (6) and (8), respectively, meaning that bond prices also for this extended model can be computed using the second-order approximation of Priebsch (2013). We refer to this modified shadow rate model with regime-switching dynamics as the R-SRM.

Another possibility is to interpret the shift in bond return predictability as being permanent, as implied by equation (4). In this case, it seems more natural to allow for a permanent break in the $\mathbb{P}$ dynamics of the pricing factors. That is, to replace equation (9) with

$$\mathbf{x}_{t+1} = \mathbf{h}_0^{(1)}\mathcal{I}_{\{t < \tau\}} + \mathbf{h}_0^{(2)}\mathcal{I}_{\{t \geq \tau\}} + \left(\mathbf{H}_x^{(1)}\mathcal{I}_{\{t < \tau\}} + \mathbf{H}_x^{(2)}\mathcal{I}_{\{t \geq \tau\}}\right)\mathbf{x}_t + \boldsymbol{\Sigma}\boldsymbol{\varepsilon}_{t+1}^{\mathbb{P}}, \tag{14}$$

where both the level and the persistence of the factors are allowed to change after the τth month in the sample. In our application, we set $\tau = 225$ to obtain a break in October 2008, which implies that the two regimes in the model are consistent with the two regimes for the predictability regression in equation (2). We again compute bond yields using the second-order approximation of Priebsch (2013), as the short rate and the $\mathbb{Q}$ dynamics remain given by equations (6) and (8), respectively. We refer to this modified shadow rate model with a permanent break as the B-SRM.

These two modifications of the standard SRM do not change the cross-sectional relationship between the pricing factors and yields, but correspond to introducing nonlinearities in the market prices of risk given by equation (7). To guide intuition for why such an extension of the standard SRM has the potential to explain the shift in equation (2), consider a the well-known affine DTSM where $r_t = s_t$ with closed-form expressions for bond yields and returns. The slope coefficient in equation (2) with $h = 1$ is then

$$\beta_1^{(n)} = \frac{(n-1)\,\mathbf{b}_{n-1}\boldsymbol{\lambda}_x \mathbb{V}\left[\mathbf{x}_t\right]\left(\frac{1}{n}\mathbf{b}_n + \boldsymbol{\beta}'\right)'}{\left(\frac{1}{n}\mathbf{b}_n + \boldsymbol{\beta}'\right)\mathbb{V}\left[\mathbf{x}_t\right]\left(\frac{1}{n}\mathbf{b}_n + \boldsymbol{\beta}'\right)'}, \tag{15}$$

where $\boldsymbol{\lambda}_x = \mathbf{H}_x - (\mathbf{I} - \boldsymbol{\Phi})$ are the loadings of the market prices of risk on $\mathbf{x}_t$ and yields are given by $y_t^{(n)} = -\frac{1}{n}\left(a_n + \mathbf{b}_n'\mathbf{x}_t\right)$ (see Duffee (2002)). Thus, allowing for a shift in $\mathbf{H}_x$ without changing $\boldsymbol{\Phi}$ induces a shift in $\boldsymbol{\lambda}_x$ that may explain the observed shift in the predictability regression in equation (2).

An even more flexible extension of the standard SRM than implied by the R-SRM and the B-SRM might also allow for regime-switching in i) the short rate in equation (6), ii) the $\mathbb{Q}$ dynamics in equation (8), and iii) the conditional covariance matrix $\boldsymbol{\Sigma}$, as considered in the regime-switching model of Dai, Singleton and Yang (2007). Such a model would display even greater flexibility in fitting the cross section of bond yields than implied by the standard SRM and the two extensions we propose. However, previous studies have shown that the standard SRM with fixed parameters is able to fit the cross-section of bond yields closely, both when yields are away from and close to the ZLB (see, for example, Christensen and Rudebusch (2015) and Andreasen and Meldrum (2018)). Hence, any improvements to the cross-sectional fit in a more flexible SRM seems likely to be economically marginal and would increase the risk of over-fitting the data. We therefore prefer to consider more parsimonious models that only allow for structural changes in the $\mathbb{P}$ dynamics.

Estimation of the R-SRM and B-SRM proceeds as described in Section 3.2, except that step 2 of the SR approach is extended by running either a threshold vector autoregression (for the R-SRM) or an autoregression with a break (for the B-SRM). This implies that the additional parameters introduced in the R-SRM and the B-SRM in comparison to the standard SRM are estimated in closed form and without any additional computational cost. The details are described in Appendix B.

4.2 In-Sample Results

We start our comparison of the standard SRM, the R-SRM, and the B-SRM by examining their in-sample fit to bonds yields. The results reported in Table 2 show that the root mean squared errors (RMSEs) between observed and model-implied yields are almost identical for the three models and below 10 basis points at all maturities. This result is fairly unsurprising because all the models have the same short rate specification and $\mathbb{Q}$ dynamics, meaning that the cross-sectional mapping between the pricing factors and bond yields only differs slightly due to different estimates of $\mathbf{\Sigma}$.

Table 2: In-Sample Fit to Bond Yields

This table reports the root mean squared errors (RMSEs) in annualized basis points between actual and model-implied yields at selected maturities at step 3 of the SR approach.

	Maturity (in months)				
	6	12	24	60	120
SRM	8.28	9.93	2.60	5.37	9.81
R-SRM	8.28	9.93	2.59	5.37	9.78
B-SRM	8.28	9.93	2.59	5.37	9.75

However, the models have different implications for the time-series dynamics of bond yields due to the nonlinearities introduced in the $\mathbb{P}$ dynamics. For the R-SRM, we find a significant shift in the persistence of the pricing factors, as we reject the null hypothesis of $\mathbf{H}_x^{(1)} = \mathbf{H}_x^{(2)}$ in a Wald test (p-value = 0.000). On the other hand, the intercepts in (13) do not appear to shift when we approach the ZLB (p-value = 0.34). For the B-SRM, we get qualitatively the same results, as we find a significant shift in the persistence of the pricing factors but not in their intercepts after the break in October 2008.[7]

One way to illustrate these differences in $\mathbb{P}$ dynamics is to compare the model-implied term premiums $TP_t^{(n)}$, which are defined as

$$TP_t^{(n)} = y_t^{(n)} - \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{E}_t \left[r_{t+i}\right]. \tag{16}$$

[7] A table with the full set of estimates is provided in the online appendix.

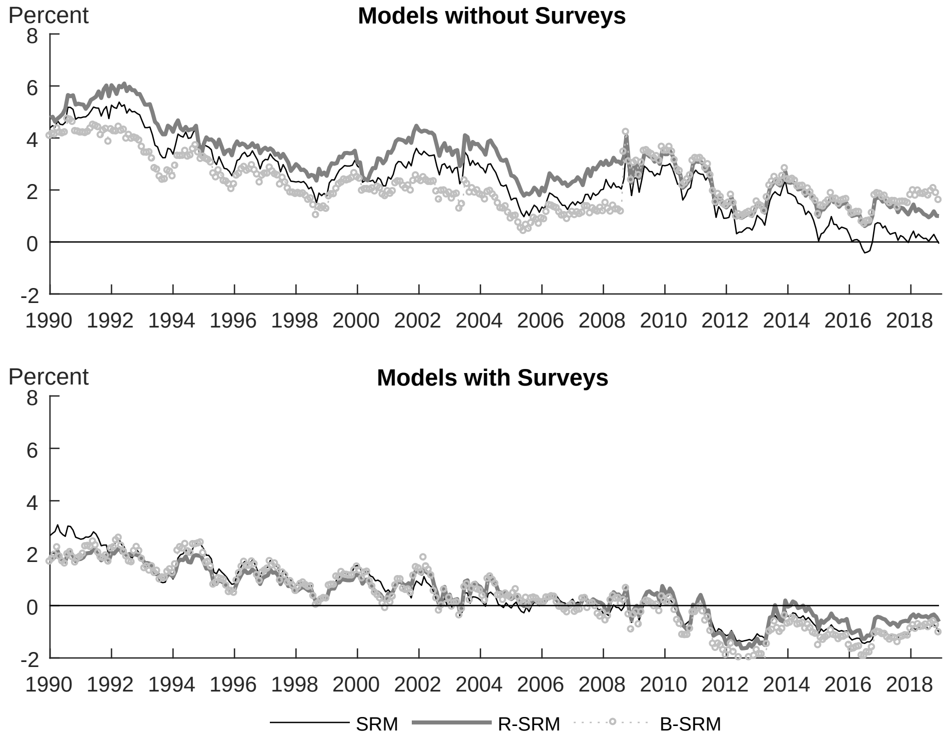
Given that the three models provide the same close fit to bond yields, any differences in $TP_t^{(n)}$ must primarily reflect different model-implied projections of the short rate as determined by the $\mathbb{P}$ dynamics of the pricing factors. The top chart in Figure 4 shows the 10-year term premiums implied by the three models. Over much of the sample, the three term premium estimates are highly correlated. However, there are also important differences. For instance, the term premium implied by the R-SRM before 2008 is somewhat higher when compared to the term premium implied by the B-SRM. This is because the R-SRM assigns the last three years of the sample (that is, December 2015 to December 2018) with low interest rates to the regime away from the ZLB, whereas these observations in the B-SRM are in the post-break regime. This implies that the R-SRM for the pre-ZLB regime has slightly lower expected short rate paths than the B-SRM, which then generates higher term premiums in the R-SRM than in the B-SRM. Another interesting difference appears toward the end of the sample, where the B-SRM implies higher term premiums than either the standard SRM or the R-SRM. This result arises because the model with the permanent break implies a permanent downward shift in the long-run mean of the short rate, which generates a lower expected short rate path in the B-SRM and hence higher term premiums than in the two other models.

4.3 Matching the Shift in Bond Return Predictability

We next consider whether the two extensions of the standard SRM can match the shift in bond return predictability documented in Section 2. For the R-SRM, it is natural to view this shift as being generated by the proximity of the short rate to the ZLB. Thus, we adopt the same procedure as for the standard SRM in Section 3.3 and estimate equation (3) with a threshold of 1 percent using a simulated sample of 1 million observations from the R-SRM. The upper left chart in Figure 5 shows that the R-SRM preserves the satisfying ability of the standard SRM to generate the desired degree of return predictability in the pre-ZLB period when $r_t \geq 0.01$. Even more encouraging is the ability of the R-SRM to generate a

Figure 4: Model-Implied 10-Year Term Premiums

This figure reports model-implied 10-year term premiums. For the R-SRM, we compute the expected average short rate over the next 10 years in a given period using Monte Carlo integration with 1 million draws.

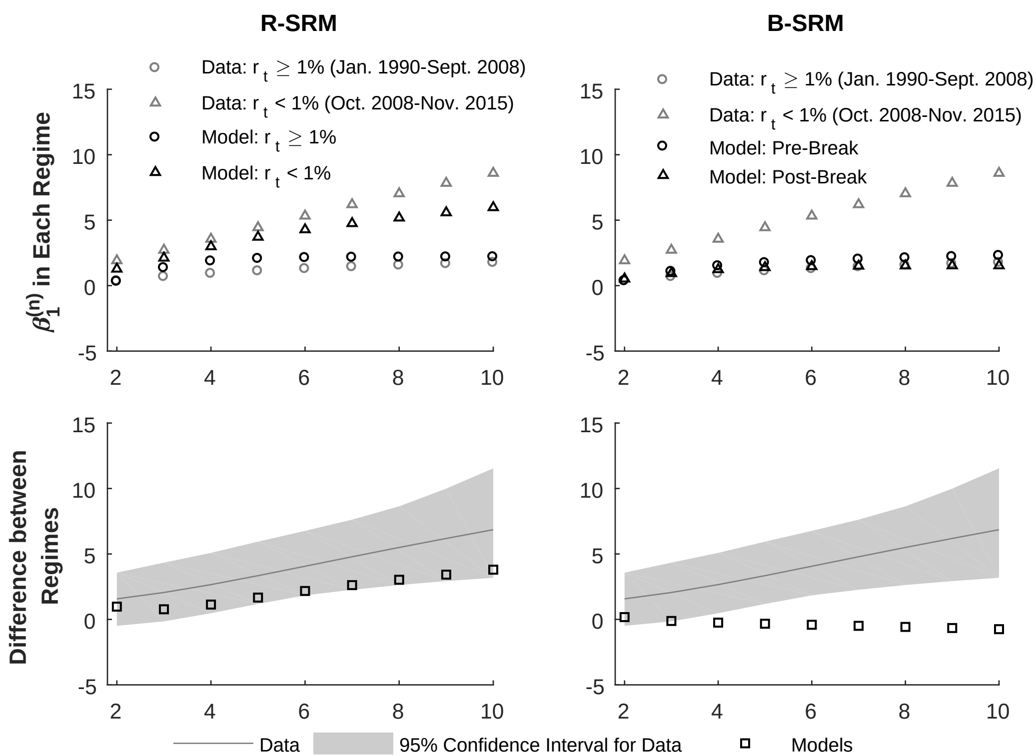


much stronger relationship between the yield spread and excess bond returns close to the ZLB with $r_t < 0.01$ than in the pre-ZLB period. This shift in return predictability is not as strong as seen in the data, but the lower left chart in Figure 5 shows that the increase in the slope coefficients lie within the 95 percent confidence interval for the sample moment at all maturities.

For the B-SRM, it seems natural to view the shift in bond return predictability as the result of a permanent break. Thus, we obtain the model-implied predictability coefficients for the B-SRM by estimating equation (4) on a simulated sample of 1 million observations with a break at $\tau = 500,001$ to generate two regimes of $500,000$ observations. The top right chart in Figure 5 shows that the B-SRM matches the degree of bond predictability in the pre-ZLB sample, but the model is simply unable to explain the shift in bond predictability after the break in 2008.

Figure 5: Bond Return Predictability: Extended SRMs

The left charts show the ability of the R-SRM to match the shift in the slope coefficients in equation (3) with $h = 12$ and $c = 0.01$ using a simulated sample of 1 million observations. The right charts show the ability of the B-SRM to match the shift in the slope coefficients in equation (4) with $h = 12$ using a simulated sample of 1 million observations with $\tau = 500,001$. The corresponding data moments are from equation (2) with $h = 12$ when estimated from January 1990 to September 2008 (excluding December 2003) where $r_t \geq 0.01$, and when estimated from October 2008 to November 2015 where $r_t < 0.01$. All x-axes show the maturity in years.



In summary, the R-SRM goes a long way in explaining the stronger link between the yield spread and bond returns during the recent ZLB period, meaning that this shift in return predictability can be attributed to a temporary re-pricing of risk. The results are less encouraging for the B-SRM with a permanent break, as this model is unable to generate stronger bond predictability after the break in 2008.

5 Matching Survey Expectations

This section examines the ability of the three SRMs to match surveys on short rate expectations at long horizons. In Section 5.1 we show that none of the three models in Sections 3 and 4 can match these surveys when the models are estimated using only data on bond yields as done above. Section 5.2 explains how survey expectations can be included in the data set for estimating the three SRMs using the SR approach. Section 5.3 shows that this modification allows the three models to match survey expectations reasonably well and that our key conclusion from Section 4 is unaffected, that is, only the R-SRM is able to match the documented shift in bond return predictability.

5.1 Out-of-Sample Fit to Surveys

We explore the ability of the three SRMs to match the expected short rate in 5 years and the average expected short rate from 5 to 10 years ahead. That is, we focus on long-horizon survey expectations and disregard more short-term expectations, because it is unclear whether surveys represent the mean or the mode of respondents' probability distributions and the model-implied mean and mode are likely to be materially different near the ZLB.[8] The distinction between mean and mode matters less at long horizons, where the ZLB is likely to have a much smaller effect on the conditional model-implied short rate distributions.

[8] The SRM and extensions of it are only able to address differences between the mean and mode of the conditional short rate distribution to a limited extent because the model implies that the mean must always lie above the mode.

Another benefit of including long-term short rate expectations is that they serve as observable proxies for i_t^*, and hence allow us to explore whether the models match the evolution in the natural nominal short rate.

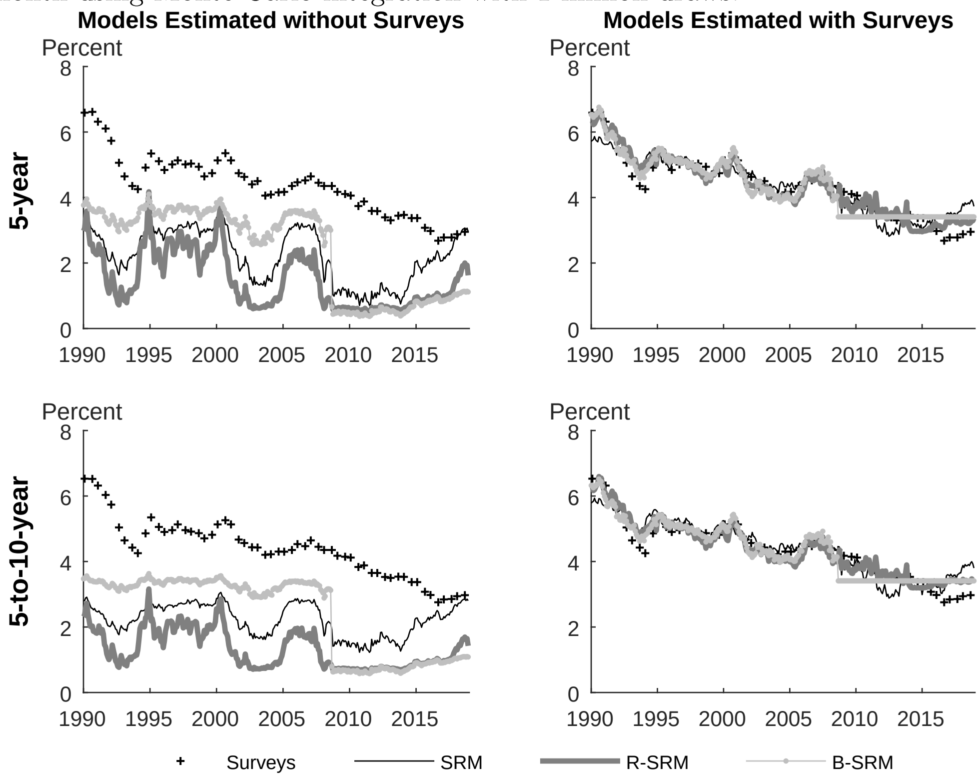
Our empirical source for these long-horizon expectations is the Blue Chip Economic Indicators survey, where repondents every 6 months report a point expectation for the 3-month Treasury bill yield at horizons up to 11 years ahead. The forecast horizons are not the same in each edition of the survey, so we linearly interpolate the mean expectations across respondents at different horizons to get a measure of the average expected 3-month Treasury bill yield at a constant horizon of 5 years and from 5 to 10 years ahead - that is, $\mathbb{E}_t\left[y_{t+59}^{(3)}\right]$ and $\mathbb{E}_t\left[\frac{1}{60}\sum_{i=60}^{119} y_{t+i}^{(3)}\right]$, respectively. These expectations are evaluated exactly in the SRM and B-SRM and by Monte Carlo integration in the R-SRMs. We exploit the close correlation between $\mathbb{E}_t\left[y_{t+i}^{(3)}\right]$ and $\mathbb{E}_t\left[r_{t+i}\right]$ at long forecast horizons to approximate $\mathbb{E}_t\left[y_{t+59}^{(3)}\right]$ by $\mathbb{E}_t\left[r_{t+59}\right]$ and $\mathbb{E}_t\left[\frac{1}{60}\sum_{i=60}^{119} y_{t+i}^{(3)}\right]$ by $\mathbb{E}_t\left[\frac{1}{60}\sum_{i=60}^{119} r_{t+i}\right]$. This substantially reduces the computational burden, because we avoid to repeatedly solve for the three-month yield in the SRMs.

The left charts in Figure 6 show the survey-based measure at the 5 year horizon (top left) and at the 5-to-10 year ahead (bottom left) along with the corresponding expectations in the three SRMs. The survey-based measures (the black plus signs) generally decline at both horizons throughout the sample, from about 6.5 percent to about 3 percent. In contrast, the short rate expectations implied by the standard SRM (the black line) are typically lower than 3 percent, have no clear downwards trend, and display only a weak correlation with the two surveys. The short expectations from the R-SRM (the dark gray line) are generally even further below the survey-based measure but are otherwise very similar to those from the standard SRM. Finally, the B-SRM (the light gray line) implies long-horizon expectations that are relatively stable (with the obvious exception of the drop at the time of the break in October 2008), and these expectations differ therefore also substantially from the surveys.

In summary, we conclude that none of the considered models are able to produce plausible short rate expectations at long horizons, and hence match the downward trend in the survey-

Figure 6: Matching Long-Horizon Short Rate Surveys

This figure shows the ability of the SRMs to match the expected short rate in 5 years and the average expected short rate from 5 to 10 years ahead as implied by responses to Blue Chip Economic Indicators surveys. The left charts show the results for models estimated using only a panel of bond yields, while the right charts show the results for models estimated using a data set that includes a panel of bond yields and surveys. For the R-SRM, we compute the expected average short rate at each month using Monte Carlo integration with 1 million draws.



based measure of the natural nominal short rate, when these models are estimated solely based on a panel of bond yields.

5.2 Incorporating Surveys into the SR Approach

The reason that the SRMs do not produce short rate expectations at long horizons that match those from surveys is most likely explained by the fairly short span of our sample (from 1990 to 2018). Given the high persistence in bond yields, it is well-known that short samples imply estimates of the $\mathbb{P}$ dynamics that may be biased and subject to substantial estimation uncertainty (see Bauer, Rudebusch and Wu (2012), Kim and Orphanides (2012), and Wright (2014)). One way to mitigate these problems is to incorporate survey data when estimating DTSMs, as proposed by Kim and Orphanides (2012). We incorporate surveys at step 2 of the SR approach when estimating the $\mathbb{P}$ parameters in $\boldsymbol{\theta}_2$, but not at step 1 and 3 when determining the $\mathbb{Q}$ parameters and the pricing factors, because they are already accurately identified from the considered panel of bond yields. Although this assumption implies a degree of simplification, it means that we can continue to separate the $\mathbb{Q}$ and $\mathbb{P}$ parameters, which is the key computational advantage of the SR approach.

More formally, at step 2 of the SR approach, we allow surveys to be measured with errors, which we denote by $\eta_t^{(5y)}$ and $\eta_t^{(5-10y)}$ for the short rate expectations in 5 years and from 5 to 10 years ahead, respectively. Each of these errors are assumed to be independent and identically distributed with zero mean and standard deviation σ_η. For the standard SRM and the R-SRM, we augment the moment conditions for the $\mathbb{P}$ parameters by the first and second moments of $\eta_t^{(5y)}$ and $\eta_t^{(5-10y)}$ when $r_t \geq 0.01$ and when $r_t < 0.01$. That is, a total of 8 additional moment conditions. For the B-SRM, we include the first and second moments of $\eta_t^{(5y)}$ and $\eta_t^{(5-10y)}$ both before and after the break point, which also imply 8 additional moment conditions. Further details are provided in Appendix C.

For our application, we assume that the measurement errors in surveys have a standard deviation of 10 basis points, i.e. $\sigma_\eta = 0.1$, which seems reasonable given the fairly stable evolution in the considered surveys.[9]

5.3 Estimation Results with Surveys

The in-sample fit to bond yields is basically unaffected by the inclusion of surveys, and the three SRMs therefore display the same satisfying fit as reported in Table 2. For the R-SRM we once again find a significant shift in the persistence of the pricing factors but not in the intercepts. The shift in the persistence of the pricing factors is also significant in the

[9] A preliminary analysis also considered $\sigma_\eta = 0.75$ as in Kim and Orphanides (2012), but this implied a very low weight to surveys in our estimation, and hence fairly small effects of including surveys at the second step of the SR approach. The intention of this paper is not to suggest that $\sigma_\eta = 0.1$ is the optimum calibration but to show that the models are capable of fitting the broad movements in the surveys without changing our key conclusions about return predictability.

B-SRM (p-value = 0.000), but now we also find a significant shift in the intercepts (p-value = 0.047) whereas it was insignificant without surveys.[10]

The right charts in Figure 6 show that the models track the overall evolution in surveys reasonably well once they are included in the data set for estimation. That is, the standard SRM now matches the downward trend in long-term short rate expectations, and hence this empirical proxy for the equilibrium short rate. But it is also evident from Figure 6 that this model generates too high short rate expectations at the end of our sample, up to 1 percentage point higher than in the data. Figure 6 further shows that the R-SRM and the B-SRM are also able to match the downward trend in surveys, and that these models produce less elevated short rate expectations at the end of our sample when compared to the standard SRM.

The bottom panel of Figure 4 reveals that the inclusion of surveys in the estimation reduces the level of the 10-year term premium substantially in all three SRMs. The main differences between the three models now only appear at the end of our sample, where the R-SRM implies a slightly higher term premium than seen in the B-SRM.

Figure 7 finally explores the ability of the SRMs to broadly match the shift in bond return predictability once surveys are included in the estimation. The encouraging finding is that the R-SRM remains able to match the shift, whereas both the standard SRM and the B-SRM fail to generate a sufficiently large shift in bond return predictability.

In summary, we conclude that the models are capable of matching long-term short rate expectations reasonably closely without changing our fundamental conclusion that only the R-SRM is able to match the documented shift in bond return predictability at the ZLB.

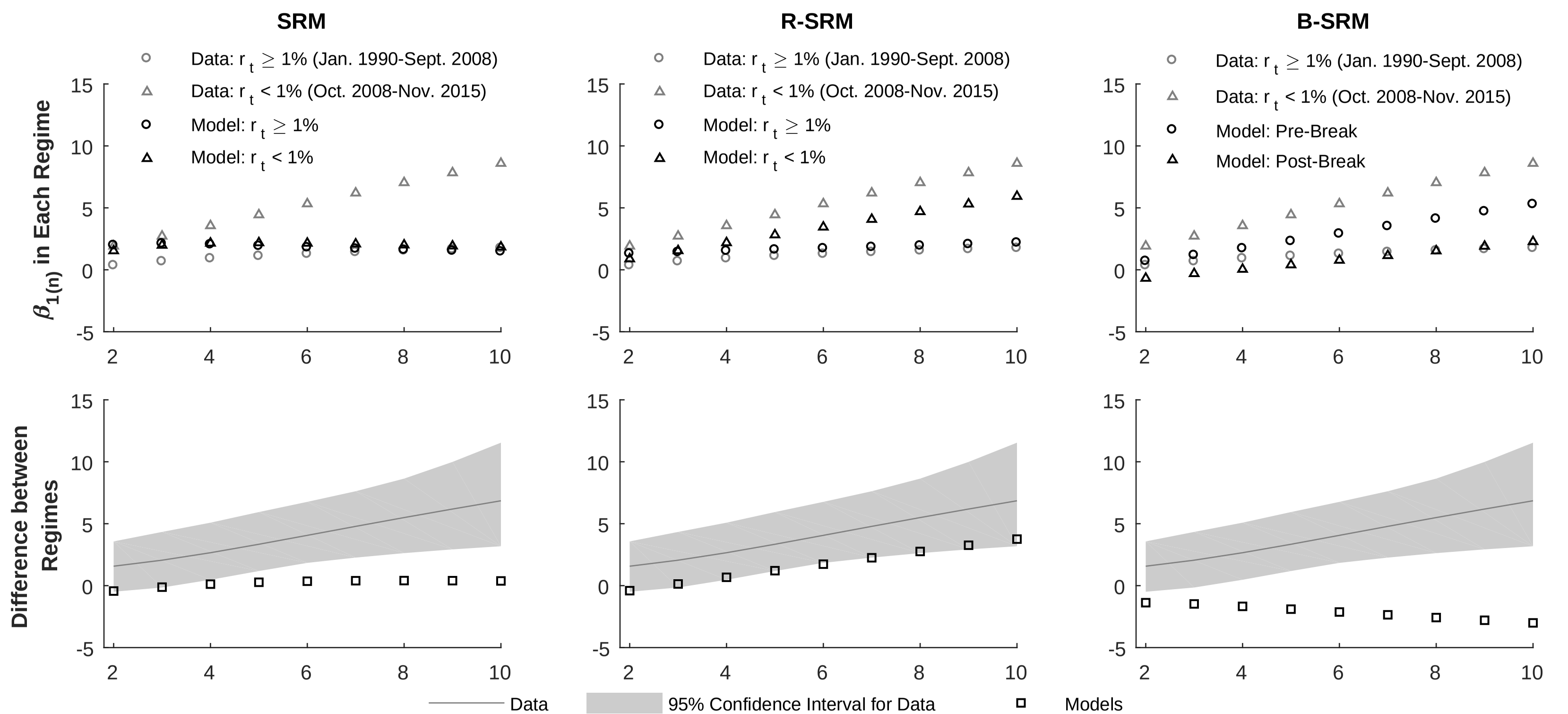
6 Alternative Specifications

This section considers two alternative specifications of the models considered above. We first show in Section 6.1 that the performance of the B-SRM does not improve by changing

[10] A table with the full set of estimates is provided in the online appendix.

Figure 7: Bond Return Predictability: SRMs with Surveys

The left and middle charts show the ability of the standard SRM and R-SRM, respectively, to match the shift in the slope coefficients in equation (3) with $h = 12$ and $c = 0.01$ using a simulated sample of 1 million observations. The right charts show the ability of the B-SRM to match the shift in the slope coefficients in equation (4) with $h = 12$ using a simulated sample of 1 million observations with $\tau = 500,001$. The corresponding data moments are from equation (2) with $h = 12$ when estimated from January 1990 to September 2008 (excluding December 2003) where $r_t \geq 0.01$, and when estimated from October 2008 to November 2015 where $r_t < 0.01$. All x-axes show the maturity in years.



the timing of the break point. Section 6.2 improves the ability of the R-SRM to match the recent low level of long-term short rate expectations by allowing for a break in the intercepts of the pricing factors after the ZLB period.

6.1 Alternative Timing of a Permanent Break

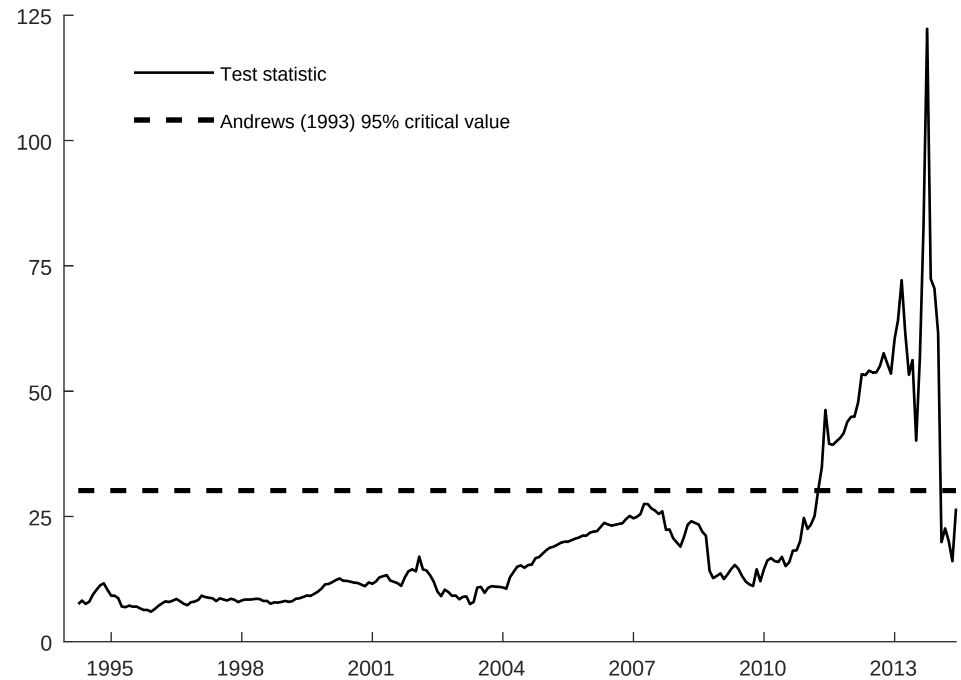
We have so far assumed that the break in the B-SRM coincides with the start of the ZLB period based on the instability of the predictability regression in equation (2). But, a different timing of the break point may be more suitable for the $\mathbb{P}$ dynamics in the B-SRM and this may improve its performance. We therefore briefly explore whether the results for the B-SRM are robust to determining the break point directly from the historical evolution of the estimated pricing factors. Here, we exploit a convenient property of the SR approach that the pricing factors are estimated non-parametrically without any assumption about the $\mathbb{P}$ dynamics, unlike typical QML estimators based on Kalman filtering.[11] This property of the SR approach implies that we can directly test for break in the $\mathbb{P}$ dynamics by using Chow tests applied to the estimated pricing factors in step 1 of the SR approach. Given that the timing of the break is unknown, we compute Chow tests using the same range of potential break points as considered in Section 2.1. The solid line in Figure 8 shows the Chow test statistic when estimating equation (9) using the pricing factors from the standard SRM. The test statistic is generally higher during the ZLB period than before, but it does not peak until November 2013, when it is well above the 95 percent critical value of Andrews (1993). This result suggests that a break in the $\mathbb{P}$ dynamics during the fall of 2013 may be more appropriate than during the fall of 2008.[12]

Given this finding, we briefly consider whether a version of the B-SRM with a break in November 2013 ($\tau = 287$) is able to match the documented shift in bond return predictability. To give the model the best possible chance of matching this shift, we also re-estimate the

[11] Here, we refer to the step 1 estimates of the pricing factors in the SR approach.

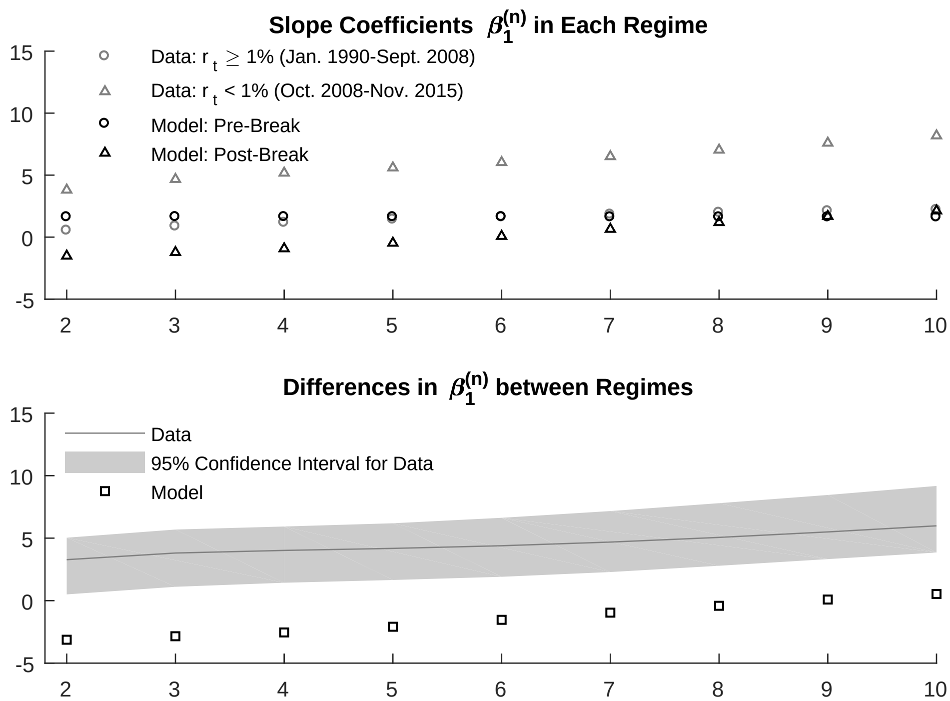
[12] One possible explanation for this result could be that several bond yields did not appear to be constrained by the ZLB until after 2010, as shown in Swanson and Williams (2014).

Figure 8: Chow Tests for a Break in the Pricing Factors
This chart reports the Chow test statistic for breaks in the estimated pricing factors in the standard SRM (first step estimates) using at least 15 percent of the observations for the pre- and post-break sample. The reported 95 percent critical value is for the maximum Chow test in Andrews (1993).



regression coefficients in the data using the same sample split. Hence, the gray markers on the top chart in Figure 9 show the slope estimates in equation (2) for the sub-sample from January 1990 through October 2013 ("Regime 1") and the sub-sample from November 2013 through December 2018 ("Regime 2"). The black markets are for the B-SRM estimated using surveys on short rate expectations as described in Section 5. The clear message from Figure 9 is that this alternative timing for the break point does not change our conclusion from above, as the B-SRM also in this case is unable to match the recent shift in bond return predictability.

Figure 9: Bond Return Predictability: Different Break Point in the B-SRM
The top chart shows the slope coefficient in equation (4) with $h = 12$ using a simulated sample of 1 million observations with $\tau = 500,001$ using an estimated version of the B-SRM with a break in November 2013. The corresponding data moments are from equation (2) with $h = 12$ when estimated from January 1990 to October 2013, and when estimated from November 2008 to December 2018. The bottom chart shows the difference in the slope estimates at a given maturity. The 95 percent confidence interval for these differences are computed using a block bootstrap with 5,000 repetitions and a block window of 24 months. In each bootstrap sample it is required that there are at least 50 observations in each regime. The x-axes reports maturity in years, and all yields are end-month from Gürkaynak et al. (2007)



6.2 The Recent Low Short Rate Expectations in Surveys

The empirical evidence presented so far clearly favors the R-SRM compared to the standard SRM and the B-SRM. Hence, considering the recent financial crisis as having a temporary impact on U.S. Treasury yields goes a long way in explaining the observed shift in bond return predictability documented in Section 2. We have also seen that the R-SRM is able to match short rate expectations at long horizons in surveys reasonably well, but that all the considered models (including the R-SRM) struggle to explain the low short rate expectations since 2015.[13] Although the magnitude of these fitting errors is not larger than seen prior to the financial crisis, they could indicate more long-lasting effects of the financial crisis, as argued in Summers (2015). We explore this possibility by considering an extension of the R-SRM that allows for a permanent break in the level of the $\mathbb{P}$ dynamics as follows

$$\begin{aligned}
\mathbf{x}_{t+1} &= \underbrace{\mathbf{h}_0^{(1,1)}\mathcal{I}_{\{r_t \geq c\}}\mathcal{I}_{\{t<\tau\}}}_{\text{pre ZLB period}} + \underbrace{\mathbf{h}_0^{(2)}\mathcal{I}_{\{r_t < c\}}\mathcal{I}_{\{t<\tau\}}}_{\text{ZLB period}} + \underbrace{\mathbf{h}_0^{(1,2)}\mathcal{I}_{\{t\geq\tau\}}}_{\text{post ZLB period}} \\
&\quad + \left(\mathbf{H}_x^{(1)}\mathcal{I}_{\{r_t \geq c\}} + \mathbf{H}_x^{(2)}\mathcal{I}_{\{r_t < c\}}\right)\mathbf{x}_t + \boldsymbol{\Sigma}\boldsymbol{\varepsilon}_{t+1}^{\mathbb{P}}. \qquad (17)
\end{aligned}$$

with $c = 0.01$. That is, we allow for a separate intercepts for the post ZLB-period to accommodate a permanent effect on the level of the pricing factors from the recent financial crisis. As for the timing of the break, we let $\tau = 302$ to introduce a break in March 2015, when the R-SRM starts to generate large fitting errors for the short rate expectations in Section 5.3. The short rate and the $\mathbb{Q}$ dynamics remain given by equations (6) and (8), respectively, meaning that bond prices also for this extended R-SRM can be computed using the second-order approximation of Priebsch (2013).

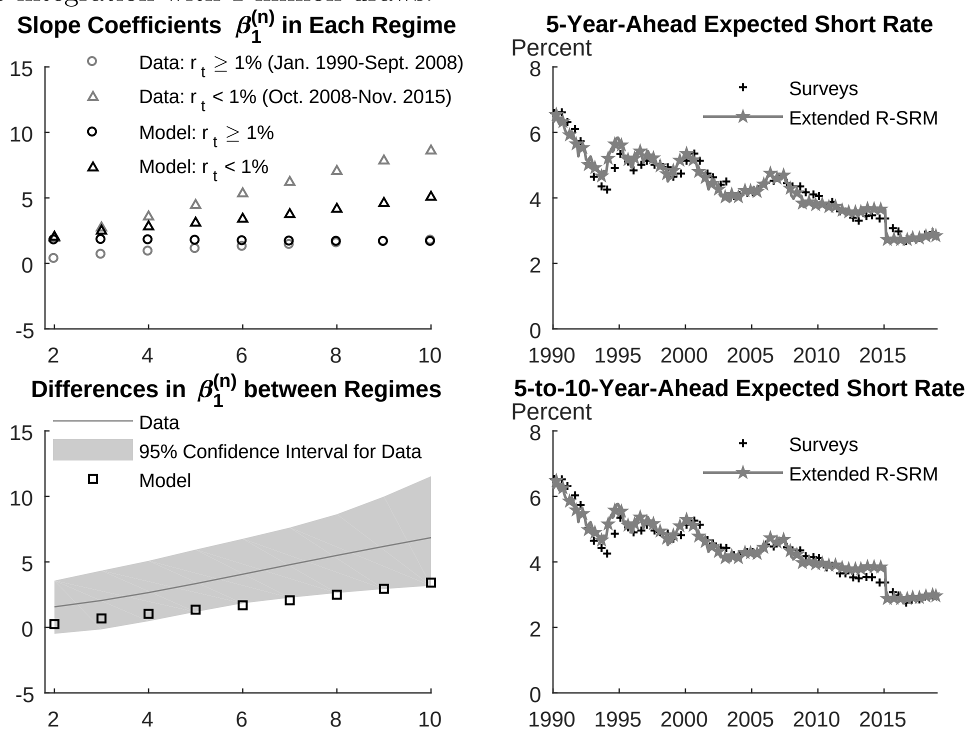
We estimate this extended R-SRM using surveys as described in Section 5.2. The shift in the persistence of the pricing factors remains significant as in Section 5.3. We also find a shift in the intercepts away from the ZLB period, as we reject the null hypothesis of $\mathbf{h}_0^{(1,1)} = \mathbf{h}_0^{(1,2)}$ in a Wald test (p-value = 0.000). That is, the level of the pricing factors appears to change from $\mathbf{h}_0^{(1,1)}$ to $\mathbf{h}_0^{(1,2)}$ when exiting the ZLB period, which may be interpreted as evidence of non-stationarity in the $\mathbb{P}$ dynamics. Our results are therefore in line with those obtained in Bauer and Rudebusch (2017), which also find evidence of non-stationarity in the $\mathbb{P}$ dynamics, although they use a different formulation of non-stationarity within a DTSM.

To consider whether this extended R-SRM can match the shift in bond return predictability, we compute the model-implied slope coefficients in equation (3) using only the pre-break

[13] Unreported results show that reducing the value of σ_η below does not materially improve the fit to surveys since 2015.

Figure 10: Bond Return Predictability: The Extended R-SRM

The left charts show the ability of the extended R-SRM to match the shift in the slope coefficients in equation (3) with $h = 12$ and $c = 0.01$ using a simulated sample of 1 million observations. The corresponding data moments are from equation (2) with $h = 12$ when estimated from January 1990 to September 2008 (excluding December 2003) where $r_t \geq 0.01$, and when estimated from October 2008 to November 2015 where $r_t < 0.01$. The x-axes show the maturity in years. The right charts show the ability of the extended R-SRMs to match the expected short rate in 5 years and the average expected short rate from 5 to 10 years ahead as implied by responses to Blue Chip Economic Indicators surveys. The model-implied short rate expectations are computed using Monte Carlo integration with 1 million draws.



intercepts in the simulation.[14] The left charts in Figure 10 show that the extended R-SRM preserves the satisfying ability of the R-SRM to match bond predictability away from the ZLB, and that it also goes a long way in matching the documented shift in bond predictability at the lower bound. The right charts in Figure 10 show that the extended R-SRM generates a close fit to long-term short rate expectations from 2015 and hence improves upon the performance of the R-SRM.

[14]This exercise is therefore only able to explore whether the extended R-SRM can match the ability of the yield spread to predict excess bond returns before the break in March 2015. The period after the break is likely too short to draw any firm conclusions.

In summary, we conclude that a new level for the pricing factors after 2015 allows the R-SRM to explain the recent low level of survey-based measures of the natural nominal short rate, while at the same time generating a notable shift in bond return predictability during the ZLB period.

7 Conclusion

This paper documents a shift in the predictability of excess bond returns during the recent ZLB period that a standard SRM cannot replicate. This shows that simply enforcing the ZLB by truncating the short rate at zero in a Gaussian model is insufficient to properly capture the change in bond yield dynamics that occurred at the ZLB. We find that this new predictability result is consistent with a SRM that allows the $\mathbb{P}$ dynamics of the pricing factors to change at the lower bound. In contrast, a SRM that introduces a permanent break in the pricing factors in 2008 is not able to explain this shift in bond return predictability at the ZLB. None of the considered models are able to match the low level of long-term short rate expectations in surveys since 2015. However, a further extension to the regime-switching model that incorporates a permanent break in the level of the pricing factors in 2015 is able to address this shortcoming of the R-SRM.

References

Andreasen, M. M. and Christensen, B. J. (2015), 'The SR Approach: A New Estimation Procedure for Non-Linear and Non-Gaussian Dynamic Term Structure Models', *Journal of Econometrics* **184**(2), 420–451.

Andreasen, M. M., Engsted, T., Möller, S. V. and Sander, M. (2016), 'Bond Market Asymmetries across Recessions and Expansions: New Evidence on Risk Premia', *Working paper* .

Andreasen, M. M. and Meldrum, A. C. (2018), 'A Shadow Rate or a Quadratic Policy Rule? the Best Way to Enforce the Zero Lower Bound in the United States', *Journal of Financial and Quantitative Analysis* **Forthcoming**.

Andrews, D. W. K. (1993), 'Tests for Parameter Instability and Structural Change With Unkown Change Point', *Econometrica* **61**(4), 821–856.

Bauer, M. D. and Rudebusch, G. D. (2016), 'Monetary Policy Expectations at the Zero Lower Bound', *Journal of Money, Credit, and Banking* **48**(7), 1439–1465.

Bauer, M. D. and Rudebusch, G. D. (2017), 'Interest Rates Under Falling Stars', *Working Paper* .

Bauer, M. D., Rudebusch, G. D. and Wu, J. C. (2012), 'Correcting Estimation Bias in Dynamic Term Structure Models', *Journal of Business and Economic Statistics* **30**(3), 454–467.

Black, F. (1995), 'Interest Rates as Options', *Journal of Finance* **50**(5), 1371–1376.

Bonis, B., Ihrig, J. and Wei, M. (2017), 'Projected Evolution of the SOMA Portfolio and the 10-year Treasury Term Premium Effect', *FEDS Notes, Washington: Board of Governors of the Federal Reserve System,* .

Campbell, J. Y. and Shiller, R. J. (1991), 'Yield Spreads and Interest Rate Movements: A Bird's Eye View', *Review of Economic Studies* **58**(3), 495–514.

Christensen, J. H. E. and Rudebusch, G. D. (2015), 'Estimating Shadow-Rate Term Structure Models with Near-Zero Yields', *Journal of Financial Econometrics* **13**(2), 226–259.

Christensen, J. H. E. and Rudebusch, G. D. (2019), 'A New Normal for Interest Rates? Evidence from Inflation-Indexed Debt', *Federal Reserve Bank of San Francisco Working Paper Series* .

Cieslak, A. and Povala, P. (2015), 'Expected Returns in Treasury Bonds', *Review of Financial Studies* **28**(10), 2859–2901.

Cochrane, J. H. and Piazzesi, M. (2005), 'Bond Risk Premia', *American Economic Review* **95**(1), 138–160.

Dai, Q. and Singleton, K. J. (2002), 'Expectation Puzzles, Time-Varying Risk Premia, and Affine Models of the Term Structure', *Journal of Financial Economics* **63**(3), 415–441.

Dai, Q., Singleton, K. J. and Yang, W. (2007), 'Regime Shifts in a Dynamic Term Structure Model of U.S. Treasury Bond Yields', *Review of Financial Studies* **20**(5), 1669–1706.

D'Amico, S., English, W., Lopez-Salido, D. and Nelson, E. (2012), 'The Federal Reserve's Large-Scale Asset Purchase Programmes: Rationale and Effects', *Economic Journal* **122**, F415–F446.

Del Negro, M., Giannone, D., Giannoni, M. P. and Tambalotti, A. (2017), 'Safety, Liquidity, and the National Rate of Interest', *Federal Reserve Bank of New York, Staff Reports No. /812* pp. 1–52.

Duffee, G. (2002), 'Term Premia and Interest Rate Forecast in Affine Models', *Journal of Finance* **57**(1), 405–443.

Fama, E. F. and Bliss, R. R. (1987), 'The Information in Long-Maturity Forward Rates', *American Economic Review* **77**(4), 680–692.

Gürkaynak, R. S., Sack, B. and Wright, J. H. (2007), 'The U.S. Treasury Yield Curve: 1961 to the present', *Journal of Monetary Economics* **54**(8), 2291–2304.

Hamilton, J. D. and Wu, J. C. (2012), 'Identification and Estimation of Gaussian Affine Term Structure Models', *Journal of Econometrics* **168**(2), 315–331.

Joslin, S., Priebsch, M. and Singleton, K. J. (2014), 'Risk Premiums in Dynamic Term Structure Models with Unspanned Macro Risk', *Journal of Finance* **69**(3), 453–468.

Joslin, S., Singleton, K. J. and Zhu, H. (2011), 'A New Perspective on Gaussian Dynamic Term Structure Models', *Review of Financial Studies* **24**(3), 926–970.

Kim, D. H. and Orphanides, A. (2012), 'Term Structure Estimation with Survey Data on Interest Rate Forecasts', *Journal of Financial and Quantitative Analysis* **47**(1), 241–272.

Kim, D. H. and Singleton, K. J. (2012), 'Term Structure Models and the Zero Bound: An Empirical Investigation of Japanese Yields', *Journal of Econometrics* **170**(1), 32–49.

Laubach, T. and Williams, J. C. (2016), 'Measuring the National Rate of Interest Redux', *Business Economics* **51**(2), 57–67.

Li, C. and Wei, M. (2013), 'Term Structure Modeling with Supply Factors', *International Journal of Central Banking* **9**(1), 3–39.

Ludvigson, S. C. and Ng, S. (2009), 'Macro Factors in Bond Risk Premia', *Review of Financial Studies* **22**(12), 5027–5067.

Priebsch, M. A. (2013), 'Computing Arbitrage-Free Yields in Multi-Factor Gaussian Shadow-Rate Term Structure Models', *Board of Governors of the Federal Reserve System (U.S.) Finance and Economics Discussion Series 2013-63* .

Rudebusch, G. D. and Wu, T. (2007), 'Accounting for a Shift in Term Structure Behavior with No-Arbitrage and Macro-Finance Models', *Journal of Money, Credit, and Banking* **39**(2-3), 395–422.

Summers, L. H. (2015), 'Demand Side Secular Stagnation', *American Economic Review: Papers and Proceedings* **105**(5), 60–65.

Swanson, E. T. and Williams, J. C. (2014), 'Measuring the Effect of the Zero Lower Bound on Medium- and Longer-Term Interest Rates', *American Economic Review* **104**(10), 3154–3185.

Wright, J. H. (2014), 'Term Premia and Iinflation Uncertainty: Empirical Evidence from an International Panel Dataset: Reply', *American Economic Review* **104**(1), 338–341.

Wu, J. C. and Xia, F. D. (2016), 'Measuring the Macroeconomic Impact of Monetary Policy at the Zero Lower Bound', *Journal of Money, Credit, and Banking* **48**(2-3), 253–291.

Appendix A: Step 2 in the SR Approach

To estimate $\boldsymbol{\theta}_2$ in Step 2 of the SR approach we follow Andreasen and Christensen (2015) and use the moment conditions

$$\begin{bmatrix} \mathbb{E}_t\left[\widehat{\boldsymbol{\varepsilon}}^{\mathbb{P}}_{t+1}\right] \\ vec\left(\mathbb{E}_t\left[\widehat{\boldsymbol{\varepsilon}}^{\mathbb{P}}_{t+1}\widehat{\mathbf{x}}_t\left(\boldsymbol{\theta}_1\right)'\right]\right) \\ vech\left(\mathbb{V}_t\left[\widehat{\boldsymbol{\varepsilon}}^{\mathbb{P}}_{t+1}\right]\right) \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ vec\left(\mathbb{C}_t\left[\mathbf{u}_{t+1},\mathbf{u}_t\right] - \mathbf{H}_x\mathbb{V}_t\left[\mathbf{u}_t\right]\right) \\ vech\begin{pmatrix} \mathbb{V}_t\left[\boldsymbol{\varepsilon}^{\mathbb{P}}_{t+1}\right] + \mathbb{V}_t\left[\mathbf{u}_t\right] + \mathbf{H}_x\mathbb{V}_t\left[\mathbf{u}_t\right]\mathbf{H}_x' \\ -\mathbb{C}_t\left[\mathbf{u}_{t+1},\mathbf{u}_t\right]\mathbf{H}_x' - \mathbf{H}_x\mathbb{C}_t\left[\mathbf{u}_t,\mathbf{u}_{t+1}\right] \end{pmatrix} \end{bmatrix}, \quad (18)$$

where $\widehat{\boldsymbol{\varepsilon}}^{\mathbb{P}}_t$ denotes the estimated values of $\boldsymbol{\varepsilon}^{\mathbb{P}}_t$ and $\mathbf{u}_t \equiv \widehat{\mathbf{x}}_t\left(\widehat{\boldsymbol{\theta}}_1^{step1}\right) - \mathbf{x}_t$. Consistent estimates of the second moments $\mathbb{V}_t\left[\mathbf{u}_{t+1}\right]$, $\mathbb{C}_t\left[\mathbf{u}_{t+1},\mathbf{u}_t\right]$, and $\mathbb{C}_t\left[\mathbf{u}_t,\mathbf{u}_{t+1}\right]$ follow from step 1 of the SR approach. Thus, $\boldsymbol{\theta}_2$ can be estimated using the generalized method of moments (GMM), which has a closed-form solution as shown in Andreasen and Christensen (2015)

Appendix B: Estimating the Extended SRMs

The parameters to be estimated at step 2 can again be written as $\boldsymbol{\theta}_2 = \begin{bmatrix} \boldsymbol{\theta}_{22}' & vech\left(\boldsymbol{\Sigma}\right)' \end{bmatrix}'$, but now we have

$$\boldsymbol{\theta}_{22} = \begin{bmatrix} \mathbf{h}_0^{(1)\prime} & vec\left(\mathbf{H}_x^{(1)}\right)' & \mathbf{h}_0^{(2)\prime} & vec\left(\mathbf{H}_x^{(2)}\right)' \end{bmatrix}'.$$

For the R-SRM, consider the revised set of moment conditions are

$$\begin{bmatrix} \mathbb{E}_t\left[\widehat{\boldsymbol{\varepsilon}}^{\mathbb{P}}_{t+1}\mathcal{I}_{\{r_t\geq c\}}\right] \\ \mathbb{E}_t\left[\widehat{\boldsymbol{\varepsilon}}^{\mathbb{P}}_{t+1}\mathcal{I}_{\{r_t< c\}}\right] \\ vec\left(\mathbb{E}_t\left[\widehat{\boldsymbol{\varepsilon}}^{\mathbb{P}}_{t+1}\widehat{\mathbf{x}}^{(1)}_t\left(\boldsymbol{\theta}_1\right)'\right]\right) \\ vec\left(\mathbb{E}_t\left[\widehat{\boldsymbol{\varepsilon}}^{\mathbb{P}}_{t+1}\widehat{\mathbf{x}}^{(2)}_t\left(\boldsymbol{\theta}_1\right)'\right]\right) \\ vech\left(\mathbb{V}_t\left[\widehat{\boldsymbol{\varepsilon}}^{\mathbb{P}}_{t+1}\right]\right) \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ vec\left(\mathbb{C}_t\left[\mathbf{u}^{(1)}_{t+1},\mathbf{u}^{(1)}_t\right]-\mathbf{H}^{(1)}_x\mathbb{V}_t\left[\mathbf{u}^{(1)}_t\right]\right) \\ vec\left(\mathbb{C}_t\left[\mathbf{u}^{(2)}_{t+1},\mathbf{u}^{(2)}_t\right]-\mathbf{H}^{(2)}_x\mathbb{V}_t\left[\mathbf{u}^{(2)}_t\right]\right) \\ vech\left(\mathbb{V}_t\left[\boldsymbol{\varepsilon}^{\mathbb{P}}_{t+1}\right]+\boldsymbol{\Omega}_{t+1}\right) \end{bmatrix}, \tag{19}$$

where $\widehat{\mathbf{x}}^{(1)}_t\left(\boldsymbol{\theta}_1\right)=\widehat{\mathbf{x}}_t\left(\widehat{\boldsymbol{\theta}}_1\right)\mathcal{I}_{\{r_t\geq c\}}$, $\widehat{\mathbf{x}}^{(2)}_t\left(\boldsymbol{\theta}_1\right)=\widehat{\mathbf{x}}_t\left(\widehat{\boldsymbol{\theta}}_1\right)\mathcal{I}_{\{r_t< c\}}$, $\mathbf{u}^{(1)}_t=\mathbf{u}_t\mathcal{I}_{\{r_t\geq c\}}$, $\mathbf{u}^{(2)}_t=\mathbf{u}_t\mathcal{I}_{\{r_t> c\}}$, and

$$\begin{aligned} \boldsymbol{\Omega}_{t+1} \;=\; & \mathbb{V}_t\left[\mathbf{u}_{t+1}\right]+\mathbf{H}^{(1)}_x\mathbb{V}_t\left[\mathbf{u}^{(1)}_t\right]\mathbf{H}^{(1)\prime}_x+\mathbf{H}^{(2)}_x\mathbb{V}_t\left[\mathbf{u}^{(2)}_t\right]\mathbf{H}^{(2)\prime}_x \\ & -\mathbb{C}_t\left[\mathbf{u}^{(1)}_{t+1},\mathbf{u}^{(1)}_t\right]\mathbf{H}^{(1)\prime}_x-\mathbf{H}^{(1)}_x\mathbb{C}_t\left[\mathbf{u}^{(1)}_t,\mathbf{u}^{(1)}_{t+1}\right] \\ & -\mathbb{C}_t\left[\mathbf{u}^{(2)}_{t+1},\mathbf{u}^{(2)}_t\right]\mathbf{H}^{(2)\prime}_x-\mathbf{H}^{(2)}_x\mathbb{C}_t\left[\mathbf{u}^{(2)}_t,\mathbf{u}^{(2)}_{t+1}\right]. \end{aligned}$$

Andreasen, Engsted, Möller and Sander (2016) provide the closed-form solution for $\boldsymbol{\theta}_2$ using these moment conditions. It is easy to verify that a similar result applies for the B-SRM, where regimes are defined based on the time point in the sample instead of the value of r_t.

Appendix C: The SR Approach with Surveys

The model parameters to be estimated at step 2 are the same as when surveys are not included. For the standard SRM and the R-SRM, we consider the following revised set of moment conditions

$$\mathbf{R}_t = \begin{bmatrix} \mathbb{E}_t\left[\widehat{\boldsymbol{\varepsilon}}^{\mathbb{P}}_{t+1}\mathcal{I}_{\{r_t\geq c\}}\right] \\ \mathbb{E}_t\left[\widehat{\boldsymbol{\varepsilon}}^{\mathbb{P}}_{t+1}\mathcal{I}_{\{r_t< c\}}\right] \\ vec\left(\mathbb{E}_t\left[\widehat{\boldsymbol{\varepsilon}}^{\mathbb{P}}_{t+1}\widehat{\mathbf{x}}^{(1)}_t\left(\boldsymbol{\theta}_1\right)'\right]\right) \\ vec\left(\mathbb{E}_t\left[\widehat{\boldsymbol{\varepsilon}}^{\mathbb{P}}_{t+1}\widehat{\mathbf{x}}^{(2)}_t\left(\boldsymbol{\theta}_1\right)'\right]\right) \\ vech\left(\mathbb{V}_t\left[\widehat{\boldsymbol{\varepsilon}}^{\mathbb{P}}_{t+1}\right]\right) \\ \mathbb{E}_t\left[\eta^{(5y)}_t\mathcal{I}_{\{r_t\geq c\}}\right] \\ \mathbb{E}_t\left[\eta^{(5y)}_t\mathcal{I}_{\{r_t< c\}}\right] \\ \mathbb{E}_t\left[\eta^{(5-10y)}_t\mathcal{I}_{\{r_t\geq c\}}\right] \\ \mathbb{E}_t\left[\eta^{(5-10y)}_t\mathcal{I}_{\{r_t< c\}}\right] \\ \mathbb{E}_t\left[\left(\eta^{(5y)}_t\right)^2\mathcal{I}_{\{r_t\geq c\}}\right] \\ \mathbb{E}_t\left[\left(\eta^{(5y)}_t\right)^2\mathcal{I}_{\{r_t< c\}}\right] \\ \mathbb{E}_t\left[\left(\eta^{(5-10y)}_t\right)^2\mathcal{I}_{\{r_t\geq c\}}\right] \\ \mathbb{E}_t\left[\left(\eta^{(5-10y)}_t\right)^2\mathcal{I}_{\{r_t< c\}}\right] \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ vech\left(\mathbb{V}_t\left[\boldsymbol{\varepsilon}^{\mathbb{P}}_{t+1}\right]\right) \\ 0 \\ 0 \\ 0 \\ 0 \\ \sigma^2_\eta \\ \sigma^2_\eta \\ \sigma^2_\eta \\ \sigma^2_\eta \end{bmatrix}. \tag{20}$$

Here, we ignore the estimation uncertainty in the factors as accommodating this feature would make the estimation computationally infeasible.[15] With more moment conditions than parameters, we obtain the GMM estimates for $\boldsymbol{\theta}_2$ using numerical optimization of the squared residuals in equation (20), where we upscale the weight assigned to the survey-based moments by 6 to account for the fact that surveys are only observed once every 6 months. For the B-SRM, we use a similar estimation procedure with regimes defined based on the break point in the sample instead of the value of r_t. In the standard SRM and the B-SRM, all short rate expectations are computed in closed form. In the R-SRM, the short rate expectations are computed by Monte Carlo simulation.[16]

[15] Our experience with other SRMs suggests that the effects of allowing for estimation uncertainty in the factors are generally very small when representing the yield curve by 25 points each period.

[16] We find that a simulation using 500 simulated short rate paths generated using antithetic shocks is sufficiently accurate to allow estimation of the model.

Research Papers
2019

2018-31: Stan Hurn, Nicholas Johnson, Annastiina Silvennoinen and Timo Teräsvirta: Transition from the Taylor rule to the zero lower bound

2018-32: Sebastian Ankargren, Måns Unosson and Yukai Yang: A mixed-frequency Bayesian vector autoregression with a steady-state prior

2018-33: Carlos Vladimir Rodríguez-Caballero and Massimiliano Caporin: A multilevel factor approach for the analysis of CDS commonality and risk contribution

2018-34: James G. MacKinnon, Morten Ørregaard Nielsen, David Roodman and Matthew D. Webb: Fast and Wild: Bootstrap Inference in Stata Using boottest

2018-35: Sepideh Dolatabadim, Paresh Kumar Narayan, Morten Ørregaard Nielsen and Ke Xu: Economic significance of commodity return forecasts from the fractionally cointegrated VAR model

2018-36: Charlotte Christiansen, Niels S. Grønborg and Ole L. Nielsen: Mutual Fund Selection for Realistically Short Samples

2018-37: Niels S. Grønborg, Asger Lunde, Kasper V. Olesen and Harry Vander Elst: Realizing Correlations Across Asset Classes

2018-38: Riccardo Borghi, Eric Hillebrand, Jakob Mikkelsen and Giovanni Urga: The dynamics of factor loadings in the cross-section of returns

2019-01: Andrea Gatto and Francesco Busato: Defining, measuring and ranking energy vulnerability

2019-02: Federico Carlini and Paolo Santucci de Magistris: Resuscitating the co-fractional model of Granger (1986)

2019-03: Martin M. Andreasen and Mads Dang: Estimating the Price Markup in the New Keynesian Model

2019-04: Daniel Borup, Bent Jesper Christensen and Yunus Emre Ergemen: Assessing predictive accuracy in panel data models with long-range dependence

2019-05: Antoine A. Djogbenou, James G. MacKinnon and Morten Ørregaard Nielsen: Asymptotic Theory and Wild Bootstrap Inference with Clustered Errors

2019-06: Vanessa Berenguer-Rico, Søren Johansen and Bent Nielsen: The analysis of marked and weighted empirical processes of estimated residuals

2019-07: Søren Kjærgaard, Yunus Emre Ergemen, Kallestrup-Lamb, Jim Oeppen and Rune Lindahl-Jacobsen: Forecasting Causes of Death using Compositional Data Analysis: the Case of Cancer Deaths

2019-08: Søren Kjærgaard, Yunus Emre Ergemen, Marie-Pier Bergeron Boucher, Jim Oeppen and Malene Kallestrup-Lamb: Longevity forecasting by socio-economic groups using compositional data analysis

2019-09: Debopam Bhattacharya, Pascaline Dupas and Shin Kanaya: Demand and Welfare Analysis in Discrete Choice Models with Social Interactions

2019-10: Martin Møller Andreasen, Kasper Jørgensen and Andrew Meldrum: Bond Risk Premiums at the Zero Lower Bound