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Component shares in continuous time

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Abstract: We formulate a continuous-time price discovery model and investigate how the standard price discovery measures vary with respect to the sampling frequency. We find that the component share measure is invariant to the sampling frequency, and hence a continuous-time price discovery measure can be identified from discrete sampled prices. We also contribute by proposing a novel estimation strategy for the continuous-time component share. We establish consistency and asymptotic normality of a kernel-based estimator that compares favourably to the standard daily VECM regression. Finally, we compute daily estimates of price discovery for 30 stocks in the U.S. from 2007 to 2013.

JEL classification numbers: C13, C32, C51, G14

Keywords: high-frequency data, price discovery, continuous-time model, sampling frequency, time-varying coefficients

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1 Introduction

Equity markets in the U.S. and Europe have experienced an ongoing process of market fragmentation following regulatory policy changes that aimed to increase competition (see Menkveld, 2014; O'Hara, 2015; Menkveld, 2016; among others).[1] The immediate consequence of this process is that the market share of the listing exchanges have decreased dramatically, while the new entrants have captured significant order flow. High-frequency trading has contributed to scatter quotes across the different exchanges and has also made markets much faster with time scales of microseconds or even nanoseconds (O'Hara, 2015; Hasbrouck, 2018). In turn, quotes, trades, and information are now dispersed across a variety of exchanges and markets that are populated by players possessing different strategic behaviours. It is thus important to investigate how exchanges and markets impound information to the efficient price of securities in an environment of highly competitive fragmented markets that operate in extremely fast time frames. This paper addresses this issue by formulating a price discovery measure in a continuous-time setting and showing how to identify it from discrete sampled prices.

There are essentially two standard price discovery measures, both in a discrete-time setting. The first comprises any variant of Hasbrouck's (1995) information share that gauges the contribution of each market/venue to the total variation of the efficient price innovation (see, for instance, Grammig, Melvin and Schlag, 2005; Lien and Shrestha, 2009; Fernandes and Scherrer, 2018). The second measure is the component share, which results from the application of the permanent-transitory decomposition of Gonzalo and Granger (1995) and Gonzalo and Ng (2001) to price discovery analysis (see, among others Booth, So and Tseh, 1999; Chu, Hsieh and Tse, 1999; Figuerola-Ferretti and Gonzalo, 2010). Both information share and component share rely on the estimation of a vector error-correction model (VECM) for price changes. The speed-of-adjustment parameter from the VECM is particularly important. Baillie, Booth, Tse and Zabotina (2002), de Jong (2002) and Yan and Zivot (2010) provide a formal comparison between component and information shares in a discrete-time setting. They show that these measures could render similar results (up to a different normalization) if market innovations are contemporaneously uncorrelated and market variances

[1] Specifically, Regulation ATS (alternative trading systems; RegATS) in 2000, and Regulation National Market System (Reg NMS) in 2007 in the U.S., and Markets in Financial Instruments Directive (MiFiDin) in 2007 in Europe set the foundation to the existence of multiple trading venues linked together and competing for liquidity and trades.

are similar in value. However, these conditions are highly unlikely to hold in practice as time aggregation usually leads to stronger contemporaneous correlation across markets. In addition, market-specific variances often differ markedly for sufficiently small sampling intervals.

This paper makes two contributions to the price discovery literature. First, we examine both component share and information share in a continuous-time setting. In particular, we put forward a k-dimensional reduced-rank multivariate Ornstein-Uhlenbeck process as the price process for a homogeneous asset traded at different venues. Using the exact discretization from the continuous-time process, we investigate how the component and information shares vary with respect to the sampling frequency. We find that the component share is invariant, implying that one may learn about the continuous-time price discovery at lower frequencies. This is in contrast with the information share measure, which converges to the value of $1/k$ as the sampling frequency decreases, where k denotes the number of markets in the analysis. This essentially means that price discovery analyses based on the information share measure should take the sampling frequency into consideration as well as the continuous-time contemporaneous correlation across markets. In fact, this is the main motivation for Hasbrouck (2018) to examine price discovery in higher resolutions. Next, we extend the early comparison between the discrete-time component and information share measures (Baillie et al. (2002), de Jong (2002) and Yan and Zivot (2010)) to the continuous-time setting. We show that the conditions under which these measures yield similar results in discrete time become less likely to hold as the sampling frequency decreases.

The second contribution concerns the estimation of time-varying component shares in a continuous-time setting. The importance of allowing for time variation in the price discovery process has already been highlighted in the literature. The standard approach to capture this time-varying nature of price discovery is to estimate daily VECMs (Hasbrouck, 2003; Chakravarty, Gulen and Mayhew, 2004; Hansen and Lunde, 2006; Mizrach and Neely, 2008). Estimating individual daily VECMs essentially boils down to treating the VECM parameters as if they were independent across days. Differently, we assume that the VECM parameters are persistent over time and then exploit the inter-daily information to obtain better finite-sample performance in the estimation of daily price discovery measures. In particular, we estimate the daily speed-of-adjustment parameters in the VECM using Giraitis, Kapetanios and Yates' (2013) kernel least squares estimator. This allows for either deterministic or stochastic variation of unknown form in the VECM parameters, as opposed to the parametric nature of Ozturk, van der Welv and van Dijk's (2017) state-space approach for price discovery analysis, for instance. We first establish the consistency and asymptotic distribution of the kernel least squares estimator and then show through an extensive Monte Carlo exercise that it compares favourably to price discovery measures based on daily VECMs. The results indicate that our estimation strategy is able to alleviate most of the noise in the daily VECM estimation and hence offer a more precise picture of the relative informativeness of each market.

Our empirical application examines the daily price informativeness of Arca, Nasdaq and New York Stock Exchange (NYSE) for both Nasdaq- and NYSE-listed stocks from 2007 to 2013. By entertaining such a long time span, we deviate from the current practice of considering at most one year of data in price discovery analyses at the high-frequency level (see, for instance, de Jong and Schotman, 2010; Benos and Sagade, 2016; Hasbrouck, 2018). Using midquotes and transaction prices of 30 stocks that differ in terms of listing venues and trading activity, we find statistical evidence that there is indeed significant daily variation in the component share measures with relative market informativeness alternating between the listing venue and the competing exchange. In particular, Elliott and Müller's (2006) test strongly rejects the null hypothesis of time-invariant continuous-time component share measures against the alternative of persistent time variation for virtually every stock.

Next, we show that our daily estimates of the continuous-time component share measures are more precise than the least squares estimates from daily VECMs in that the standard deviation of the former estimator is about three times smaller. When jointly assessing the informational content of midquotes and transaction prices, we find that the former strongly dominates the price discovery process, reinforcing our main set of results. Finally, to better understand the daily variation in the price discovery mechanism, we study the long-run relationship between price discovery and trading volume. The results reveal that the listing venue contribution to the price discovery increases with its relative volume. We find that for virtually every stock, the trading volume seems to respond more significantly to deviations from the long-run equilibrium than the price discovery measure.

The remainder of this paper is organized as follows. Section 2 describes the continuous-time setting for price discovery and Section 3 discusses how the information share and component share measures are affected by sampling frequency. Section 4 shows how to estimate price discovery measures in a consistent manner accounting for daily stochastic changes in the speed-of-adjustment parameter. Section 5 investigates the finite sample performance of the proposed kernel least squares estimator vis-à-vis the daily VECM estimation. Section 6 investigates how the price informativeness of NYSE- and Nasdaq-listed stocks changes over time and the long-run relationship between price discovery and volume. Finally, we offer some concluding remarks in Section 7.

2 A continuous-time setting for price discovery

In this section, we propose a continuous-time model for price discovery. Let prices for a given asset that trades on multiple venues follow the process

$$\mathrm{d}P_t = \Pi P_t \,\mathrm{d}t + C\,\mathrm{d}W_t, \qquad \text{with} \quad P_0 = p_0, \tag{1}$$

where $P_t = (p_{1,t}, \ldots, p_{k,t})'$ is a $k \times 1$ vector of log-prices with k denoting the number of trading venues, $\Pi = \alpha\beta'$ is a $k \times k$ reduced-rank matrix with rank equal to $r = k - 1$, α and β are $k \times r$ full-rank matrices, W is a $k \times 1$ vector of Brownian motions, and C is a $k \times k$ matrix such that the covariance matrix $\Sigma = CC'$ is positive definite. Prices at the different markets should not drift much apart, oscillating around the (latent) efficient price, as they refer to the same asset. Accordingly, there is $k - 1$ cointegrating relationships ($r = k - 1$), with log-prices sharing the asset's efficient price as the single common stochastic trend. We assume without loss of generality that β is known and takes the form of $\beta = (I_r, -\iota_r)$, where ι_r denotes a $r \times 1$ unit vector. In turn, α determines how quickly each market reacts to deviations from the long-run equilibria $\beta' P_t$.

The solution to (1) is a homogenous Gaussian Markov process given by

$$P_t = \exp(t\Pi)\left[P_0 + \int_0^t \exp(-u\Pi)C\,\mathrm{d}W_u\right], \tag{2}$$

giving way to a homoskedastic Gaussian VAR(1) process in discrete time. Due to the reduced rank, it will also admit a homoskedastic Gaussian VECM(0) representation as in Hasbrouck (1995). Despite the restricted lag structure in the discrete-time VECM, the reduced-rank Ornstein-Uhlenbeck (OU) process in (1) provides a useful framework to study the dynamics of a single asset traded at multiple venues given that it ensures that the stochastic trend is a martingale and that returns follow an infinitive VMA process. Nevertheless, it is possible to contemplate reduced-rank continuous-time processes that yield more general lag structures to their discrete-time counterparts by applying the Laplace transform function to the lag operator as in Nguenang (2016).[2] See also Cochrane (2012) for more details on ARMA processes in continuous time.

We assume prices are observed regularly and equidistantly over the unit interval $[0,1]$ that characterizes, say, one trading day (calendar-time sampling, as discussed in Hansen and Lunde, 2006). Denote each interval in $[0,1]$ as $[t_{i-1}, t_i]$, where $i = 1, 2, \ldots, n$ and n is the total number of intervals such that $0 = t_0 < t_1 < \ldots < t_n = 1$. The length of each interval is $\delta = t_i - t_{i-1} = 1/n$ in $[0,1]$. For instance, the usual trading day in the U.S. market lasts for 6.5 hours (23,400 seconds), and thus, sampling one observation per minute yields $n = 390$ intraday observations, with $\delta = 1/390$. Denoting by $\exp(A)$ the matrix exponential of a $k \times k$ matrix A such that $\exp(A) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} A^{\ell}$, the exact discretization of (1) at interval length δ reads

$$\Delta P_{t_i} = \Pi_\delta P_{t_{i-1}} + \varepsilon_{t_i}, \tag{3}$$

where $\Pi_\delta = \alpha_\delta \beta'$ and $\alpha_\delta = \alpha(\beta'\alpha)^{-1}\left[\exp(\delta\beta'\alpha) - I_r\right]$, with I_r denoting a r-dimensional identity matrix, and P_{t_i} is a $k \times 1$ vector of log-prices observed at discrete time. The innovation ε_{t_i} is iid Gaussian with zero mean and covariance matrix given by $\Sigma_\delta = \int_0^\delta \exp(u\Pi)\Sigma\exp(u\Pi')\,\mathrm{d}u$.

Kessler and Rahbek (2004) provide the conditions under which the mapping given $\theta = (\Pi, \Sigma) \overset{\psi}{\mapsto} \psi(\theta) = (\Pi_\delta, \Sigma_\delta)$ is unique, θ is identifiable, and the space spanned by the columns of α is equal to the one spanned by the columns of α_δ. Specifically, if all eigenvalues of Π are real and no elementary divisor of Π occurs more than once, Proposition 1 in Kessler and Rahbek (2004) shows that the mapping ψ is injective and θ is identifiable. It is important to note that temporal aggregation preserves the cointegration rank, i.e., $\operatorname{rank}\Pi_\delta = \operatorname{rank}\Pi$, and that the definition of (co)integration for OU processes in continuous time is consistent with the definition in discrete time (Kessler and Rahbek, 2004). This means that one may conduct inference about rank and cointegrating space using discrete-time procedures and then interpret the results in the continuous-time setting.

[2] Nguenang (2016) assumes a baseline process similar to (1), but with a stochastic covariance matrix, to introduce a version of the information share measure based on Pesaran and Shin's (1998) generalized impulse response function in both continuous and discrete times. However, he does not examine how it changes once we move from continuous to discrete time.

To compute price discovery measures, one must decompose the price vector into a permanent $I(1)$ component and a transitory covariance-stationary $I(0)$ component. There are essentially two alternative decompositions in the price discovery literature: Gonzalo and Granger's (1995) permanent-transitory decomposition (PTD) and the Granger representation theorem (GRT). Although they have different implications for the permanent component (de Jong, 2002), both PTD and GRT build on the orthogonal complements of α (or α_δ if in discrete time) and β. Let A be a $k \times r$ matrix with full column rank $r \leq k$. We define the orthogonal complement of A as any matrix $A_\perp$ with dimension $k \times (k-r)$ and rank $k-r$ such that $AA_\perp = 0$.[3] In particular, let the $k \times 1$ vectors $\alpha_\perp$ and $\alpha_{\delta,\perp}$ respectively denote the orthogonal complements of the vector of speed-of-adjustment parameters in continuous and discrete time, whereas we fix the orthogonal complement of $\beta = (I_r, -\iota_r)$ to $\beta_\perp = \iota_r$ (any multiple would do). These orthogonal complements relate to the nonstationary directions of the processes and hence to the permanent component.

The PTD posits that $P_{t_i} = a_f f_{t_i} + a_z z_{t_i}$, where the common factor f_{t_i} is a linear combination of the elements of P_{t_i}. Identification requires the absence of long-run Granger causality from the transitory component z_{t_i} to f_{t_i}, implying $a_f = \alpha_{\delta,\perp}$. However, changes in the PTD common factor display serial correlation and hence f_{t_i} is not a martingale. In contrast, the GRT extends the Beveridge-Nelson decomposition to a multivariate setting (Stock and Watson, 1988; Johansen, 1991; Hansen, 2005), resulting in a permanent component that follows a random walk process with serially uncorrelated increments (and, hence, a martingale). In turn, the transitory term admits a covariance-stationary VMA(∞) representation, thus inheriting all the remaining serial correlation. Given that the price discovery measures based on the PTD and GRT coincide up to a scale factor (de Jong, 2002), we will henceforth adopt the GRT decomposition in order to ensure that the martingale property holds for the efficient price.[4]

To formally establish a bridge between the continuous- and discrete-time price discovery measures, the GRT must hold in both settings. Kessler and Rahbek (2001) show that the GRT follows in continuous time directly by assuming that α and β have full column ranks r, $\beta'\alpha$ has full rank

[3] Alternatively, denoting by $sp(A)$ the subspace in $\mathbb{R}^k$ spanned by the columns of A, $A_\perp$ is any matrix with dimension $k \times (k-r)$ such that $sp(A_\perp) = sp(A)_\perp$.

[4] This essentially boils down to saying that the instantaneous risk premium is zero, which is in practice a very good approximation at the high frequency.

r, and all eigenvalues of $\beta'\alpha$ have negative real parts. Under these assumptions,

$$I_k = \beta_\perp(\alpha'_\perp\beta_\perp)^{-1}\alpha'_\perp + \alpha(\beta'\alpha)^{-1}\beta', \tag{4}$$

and hence a straightforward application of (4) to the solution of (1) yields

$$P_t = \Xi(CW_t + P_0) + \eta_t, \tag{5}$$

where $\Xi = \beta_\perp\left(\alpha'_\perp\beta_\perp\right)^{-1}\alpha'_\perp$, P_0 contains initial values, and $\eta_t = \alpha\left(\beta'\alpha\right)^{-1}Z_t$, with $Z_t = \beta' P_t$ denoting a stationary OU process given by $\mathrm{d}Z_t = \beta'\alpha Z_t\,\mathrm{d}t + \beta' C\,\mathrm{d}W_t$.

The exact discretization in (3) is such that

- (i) prices are not explosive in that the roots of the characteristic polynomial $|I_k - (\Pi_\delta + I_k)z| = 0$ are either outside the unit circle or equal to one;
- (ii) $\Pi_\delta = \alpha_\delta\beta'$ has reduced rank $r < k$, with both α_δ and β having full column rank r; and
- (iii) the number of unit roots equals $k - r$.

It then follows from Johansen's (1995) Theorem 4.2 that the GRT holds in discrete time irrespective of the distribution of the innovations in (3). Moreover, conditions (i) to (iii) also ensure that the projection identity in (4) holds in discrete time, namely,

$$I_k = \beta_\perp(\alpha'_{\delta,\perp}\beta_\perp)^{-1}\alpha'_{\delta,\perp} + \alpha_\delta(\beta'\alpha_\delta)^{-1}\beta'. \tag{6}$$

Pre-multiplying (3) by β' yields the r-dimensional stationary process $\beta' P_{t_i} = \sum_{h=0}^{\infty}(I_r + \beta'\alpha_\delta)^h\beta'\varepsilon_{t_{i-h}}$, whereas pre-multiplying by $\alpha'_{\delta,\perp}$ entails the nonstationary component $\alpha'_{\delta,\perp}P_{t_i} = \alpha'_{\delta,\perp}P_{t_0} + \sum_{i=1}^{n}\alpha'_{\delta,\perp}\varepsilon_{t_i}$.

Using the projection identity in (6), the GRT in discrete time reads

$$P_{t_i} = \Xi_\delta\sum_{h=1}^{i}\varepsilon_{t_h} + \sum_{h=0}^{\infty}\Upsilon_{\delta,h}\varepsilon_{t_{i-h}} + \Xi_\delta P_{t_0}, \tag{7}$$

where $\Xi_\delta = \beta_\perp(\alpha'_{\delta,\perp}\beta_\perp)^{-1}\alpha'_{\delta,\perp}$, $\Upsilon_{\delta,h} = (I_k - \Xi_\delta)(I_k + \alpha_\delta\beta')^h$ such that $\sum_{h=0}^{\infty}\Upsilon_{\delta,h}\varepsilon_{t_{i-h}}$ is a stationary process, and P_{t_0} is a vector of initial values.[5] The stochastic common trend given by the first term on the right-hand side of (7) reflects the efficient price of the asset. In view that $\beta_\perp = \iota_r$, not only does Ξ_δ have common rows, but also the efficient price relates to a weighted-average of the prices

[5] Apart from closed-form expressions for the MA component, Hansen (2005) also provides an alternative proof for the GRT (see Theorem 1 and Corollary 2 herein).

at the different trading venues. Finally, it is also reassuring to observe that the stochastic trend, $\Xi_\delta \sum_{h=1}^{i} \varepsilon_{t_h}$, is a martingale and so consistent with non-arbitrage requirements (see discussion in Hansen and Lunde, 2006).

In view of the GRT, a key parameter in any measure of price discovery is $\alpha_{\delta,\perp}$. Alternatively, the speed-of-adjustment matrix α_δ also contains information about the price discovery mechanism. The matrix α_δ reflects the adjustment that each market implements such that their prices do not deviate from the efficient latent price. This means that the closer the α_δ of a given market is to zero, the less it adjusts to the efficient price. In the limit, $\alpha_{\delta,m} = 0$ means that the price at market m coincides with the efficient price, therefore leading the price discovery.

3 The effect of the sampling frequency

A natural step forward is to investigate how the sampling frequency affects the component share (CS) and information share (IS) measures. In this section, we first establish how both price discovery measures behave under temporal aggregation starting from the continuous-time setting. We then show how to identify the continuous-time CS measure from prices at different sampling frequencies. Next, we compare the information content of the CS and IS measures in continuous time and wrap up the discussion with two simple examples. For simplicity of exposition, we henceforth consider a single asset traded on two trading venues ($k = 2$ and $r = 1$).

3.1 Component share

The component share relies on the orthogonal complement of α_δ, namely, $\alpha_{\delta,\perp}$ such that $\alpha'_{\delta,\perp}\alpha_\delta = 0$ (see, among others, Booth et al., 1999; Chu et al., 1999; Harris, McInish and Wood, 2002; Hansen and Lunde, 2006). Because $\alpha_{\delta,\perp}$ is not unique, one typically imposes $\alpha_{\delta,\perp,1} + \alpha_{\delta,\perp,2} = 1$. While α_δ corresponds to the stationary direction of the process in (3), $\alpha_{\delta,\perp}$ relates to the nonstationary direction. This makes $\alpha_{\delta,\perp}$ a natural quantity to assess how the efficient price relates to each market innovation. The market with the highest $\alpha_{\delta,\perp}$ has the least need of adjustment towards the latent efficient price and hence it is the one that leads the price discovery process.

Using the normalization $\alpha_{\delta,\perp,1} + \alpha_{\delta,\perp,2} = 1$, it follows from the exact discretization of the reduced-rank OU process in (3) that

$$\alpha_{\delta,\perp} = \left(\frac{\alpha_{\delta,2}}{\alpha_{\delta,2}-\alpha_{\delta,1}}, -\frac{\alpha_{\delta,1}}{\alpha_{\delta,2}-\alpha_{\delta,1}}\right)' = \left(\frac{\alpha_2}{\alpha_2-\alpha_1}, -\frac{\alpha_1}{\alpha_2-\alpha_1}\right)', \tag{8}$$

given that $(\beta'\alpha)^{-1}[\exp(\delta\beta'\alpha) - I_r]$ cancels out for appearing in both numerators and denominators. It is now clear that $\alpha_{\delta,\perp}$ is invariant to the sampling frequency in that $\alpha_{\delta,\perp} = \alpha_\perp$ for any $0 < \delta < 1$. This means that identification and inference of the continuous-time price discovery measure arises directly from estimating $\alpha_{\delta,\perp}$ at any sampling frequency. From an empirical perspective, (8) allows for us to learn about the continuous-time price discovery mechanism even if using data at a lower frequency (and hence less prone to market microstructure noise).

3.2 Information share

Perhaps the most popular price discovery measure in the literature is Hasbrouck's (1995) information share (see, among others, Baillie et al., 2002; de Jong, 2002; Grammig, Melvin and Schlag, 2005; Yan and Zivot, 2010). In short, the IS measure gives the share of each market contribution to the total variance of the efficient price ($IS_{\delta,1} + IS_{\delta,2} = 1$). Using the exact discretization of (1), the IS measure of a given market $m \in \{1,2\}$ for $0 < \delta < 1$ is

$$IS_{\delta,m} = \frac{[\xi_\delta C_\delta]_m^2}{\xi_\delta \Sigma_\delta \xi_\delta'}, \tag{9}$$

where $\Sigma_\delta = C_\delta C_\delta' = \int_0^\delta \exp(u\Pi)\Sigma\exp(u\Pi')\,\mathrm{d}u$, ξ_δ is the common row of Ξ_δ in (7) that follows from $\beta_\perp = (1,1)'$, and $[.]_m$ denotes the mth element of a vector.

We now investigate the effect of δ on the IS measure. Considering the two-market case, we can re-write (9) as function of the market-specific variances and correlations. Specifically, let

$$\Sigma_\delta = \begin{pmatrix} \sigma_{\delta,1}^2 & \sigma_{\delta,1}\sigma_{\delta,2}\rho_\delta \\ \sigma_{\delta,1}\sigma_{\delta,2}\rho_\delta & \sigma_{\delta,2}^2 \end{pmatrix} \quad \underline{C}_\delta = \begin{pmatrix} \sigma_{\delta,1} & 0 \\ \sigma_{\delta,2}\rho_\delta & \sigma_{\delta,2}\sqrt{1-\rho_\delta^2} \end{pmatrix} \quad \bar{C}_\delta = \begin{pmatrix} \sigma_{\delta,1}\sqrt{1-\rho_\delta^2} & \sigma_{\delta,1}\rho_\delta \\ 0 & \sigma_{\delta,2} \end{pmatrix},$$

where $\underline{C}_\delta$ and $\bar{C}_\delta$ are the Cholesky decompositions of Σ_δ resulting from the different orderings of the variables such that $\Sigma_\delta = \underline{C}_\delta \underline{C}_\delta' = \bar{C}_\delta \bar{C}_\delta$. We denote the variance of market $m \in \{1,2\}$ by $\sigma_{\delta,m}^2$ and the contemporaneous correlation between the two markets by ρ_δ for any δ. The Cholesky decomposition depends on the ordering of the variables. Most studies average the maximum and minimum IS measures, computed over all possible orderings. Using the fact that $\alpha_{\delta,\perp,m} = \alpha_{\perp,m}$

for any $0 < \delta < 1$, the average IS measure in a given market $m \in \{1, 2\}$ for $0 < \delta < 1$ then reads

$$\overline{IS}_{\delta,m} = \frac{1}{2}\left(\frac{[\xi_\delta \underline{C}_\delta]_m^2}{\xi_\delta \Sigma_\delta \xi_\delta'} + \frac{\left[\xi_\delta \bar{C}_\delta\right]_m^2}{\xi_\delta \Sigma_\delta \xi_\delta'}\right) = \begin{cases} \frac{(\alpha_{\perp,1}\sigma_{\delta,1}+\alpha_{\perp,2}\sigma_{\delta,2}\rho_\delta)^2+\alpha_{\perp,1}^2\sigma_{\delta,1}^2(1-\rho_\delta^2)}{2\left(\alpha_{\perp,1}^2\sigma_{\delta,1}^2+\alpha_{\perp,2}^2\sigma_{\delta,2}^2+2\alpha_{\perp,1}\alpha_{\perp,2}\sigma_{\delta,1}\sigma_{\delta,2}\rho_\delta\right)}, & \text{if } m = 1, \\ \frac{(\alpha_{\perp,2}\sigma_{\delta,2}+\alpha_{\perp,1}\sigma_{\delta,1}\rho_\delta)^2+\alpha_{\perp,2}^2\sigma_{\delta,2}^2(1-\rho_\delta^2)}{2\left(\alpha_{\perp,1}^2\sigma_{\delta,1}^2+\alpha_{\perp,2}^2\sigma_{\delta,2}^2+2\alpha_{\perp,1}\alpha_{\perp,2}\sigma_{\delta,1}\sigma_{\delta,2}\rho_\delta\right)}, & \text{if } m = 2. \end{cases} \tag{10}$$

As opposed to the component share, $\overline{IS}_{\delta,m}$ is not invariant to the sampling frequency because the market-specific variances and correlation across markets in (10) depend on δ. In particular, the contemporaneous correlation absorbs most of the lead-lag patterns as δ increases because both markets have now sufficient time to impound the news. In fact, exact discretization yields $|\rho_\delta| \to 1$ as $\delta \to 1$, and thus, $\lim_{\delta\to1} \overline{IS}_{\delta,1} = \lim_{\delta\to1} \overline{IS}_{\delta,2} = 1/2$. This means that a fair comparison of IS measures must take into consideration not only the sampling frequency but also the contemporaneous correlation across markets in continuous time. However, this is not straightforward. Teasing out the continuous-time covariance matrix from estimates of $\Sigma_\delta = \int_0^\delta \exp(u\Pi)\Sigma\exp(u\Pi')\,du$ tends to produce poor results in finite samples, typically resulting in negative semi-definite estimates of Σ for prices sampled at frequencies lower than 10 seconds. Moreover, markets are currently very fast and interconnected given the rise of high-frequency trading and statistical arbitrage across and within markets (see, among others, Menkveld, 2014, 2016; O'Hara, 2015), implying higher contemporaneous correlation across markets even at the very high frequency and, in turn, IS measures that converge to 1/2.

3.3 A continuous-time comparison of CS and IS

Using the continuous-time version of the GRT in (5), the continuous-time IS measure of a given market $m \in (1, 2)$ reads

$$\overline{IS}_m = \begin{cases} \frac{(\alpha_{\perp,1}\sigma_1+\alpha_{\perp,2}\sigma_2\rho)^2+\alpha_{\perp,1}^2\sigma_1^2(1-\rho^2)}{2\left(\alpha_{\perp,1}^2\sigma_1^2+\alpha_{\perp,2}^2\sigma_2^2+2\alpha_{\perp,1}\alpha_{\perp,2}\sigma_1\sigma_2\rho\right)}, & \text{if } m = 1, \\ \frac{(\alpha_{\perp,2}\sigma_2+\alpha_{\perp,1}\sigma_1\rho)^2+\alpha_{\perp,2}^2\sigma_2^2(1-\rho^2)}{2\left(\alpha_{\perp,1}^2\sigma_1^2+\alpha_{\perp,2}^2\sigma_2^2+2\alpha_{\perp,1}\alpha_{\perp,2}\sigma_1\sigma_2\rho\right)}, & \text{if } m = 2. \end{cases} \tag{11}$$

where σ_1^2 and σ_2^2 are the market-specific variances and ρ denotes the correlation between the two markets in continuous time. There is a direct link between CS and IS only if $\rho = 0$ and $\sigma_1 = \sigma_2$. In this case, the information share in continuous time reads

$$\overline{IS}_m = \frac{\alpha_{\perp,m}^2\sigma_m^2}{\alpha_{\perp,1}^2\sigma_1^2 + \alpha_{\perp,2}^2\sigma_2^2}, \qquad m = 1, 2. \tag{12}$$

This essentially reproduces in continuous time the results that Baillie et al. (2002) and de Jong (2002) obtain in discrete time by imposing $\rho_\delta = 0$ and $\sigma_{\delta,1} = \sigma_{\delta,2}$. It should be stressed, however, that these conditions are highly unlikely to hold either in continuous or discrete time. Given that $\lim_{\delta \to 1} |\rho_\delta| = 1$, it becomes increasingly more unrealistic to assume that $\rho_\delta = 0$ as the frequency decreases. On the other hand, the market-specific variances should diverge as frequency increases due to the different market microstructure features and trading clientele (Dias, Scherrer and Papailias, 2016). We thus conclude that the continuous-time CS and IS measures will most often deliver different empirical results.

3.4 A simple illustration

In this section, we illustrate our theoretical findings using a continuous-time version of Hasbrouck's (1995) example. Suppose that a homogeneous asset trades on two markets. Market 1 is the leading trading venue, with prices fully reflecting the efficient price, whereas the price on market 2 reacts to deviations with respect to the (efficient) price on market 1. Prices cointegrate with $\alpha = (0, \alpha_2)'$ and $\beta' = (1, -1)$:

$$\mathrm{d}\begin{pmatrix} p_{1,t} \\ p_{2,t} \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha_2 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} p_{1,t} \\ p_{2,t} \end{pmatrix} \mathrm{d}t + C \, \mathrm{d}W_t, \tag{13}$$

where C and W are defined as in (1). The absence of adjustment in market 1 ($\alpha_1 = 0$) implies that $p_{1,t}$ coincides with the stochastic trend and hence a coherent price discovery measure should identify market 1 as the sole contributor to the price discovery process, regardless of the sampling frequency. This setup is general enough to nest the case in which the CS and IS measures are equal in continuous time, and it is also informative to show how the two price discovery measures diverge when the sampling frequency decreases. In particular, we entertain the exact discretization of (13) for $\sigma_1^2 = \sigma_2^2$ and $\rho \in \{0, 0.3, 0.5, 0.7, 0.9\}$, with δ ranging from $1/23{,}400$ to $1/13$ (implying frequencies of one observation per second to one observation per 30 minutes in a trading day of 6.5 hours).

The left panel of Figure 1 displays the CS and IS measures for market 1, whereas the right panel plots the contemporaneous correlation at each frequency. As expected, discretization affects the price discovery measures in different manners. While CS is completely immune to the sampling frequency in that $\alpha_\perp = \alpha_{\delta,\perp}$ for any $0 < \delta < 1$, the IS measure converges to $1/2$ as δ increases because $|\rho_\delta|$ converges to one. At lower frequencies, information hits both markets within the same time interval, preventing the IS measure from identifying market 1 in (13) as the unique contributor to the price discovery process. As expected, given that $\sigma_1^2 = \sigma_2^2$, CS and IS yield the same result in continuous time only in the absence of contemporaneous correlation between markets.

3.5 Allowing for a stochastic covariance matrix

We next assess what happens if we assume that prices follow

$$\mathrm{d}P_t = \Pi P_t \,\mathrm{d}t + C_t \,\mathrm{d}W_t, \qquad \text{with} \quad P_0 = p_0, \tag{14}$$

where C_t is a cadlag stochastic volatility process that evolves intradaily such that the covariance matrix $\Sigma_t = C_t C_t'$ is positive definite. Allowing for C_t to evolve over time implies that the exact discretization in Section 2 no longer holds. Proposition 1 in Nguenang (2016) documents nonetheless that allowing for a stochastic covariance matrix does not affect how the speed-of-adjustment parameters change with the sampling interval. As such, the CS measure at any sampling frequency still coincides with the CS measure in continuous time, reflecting as well the consistency of the LS estimator even in the presence of stochastic volatility. In contrast, the exact discretization of the IS measure will depend on the specification of the stochastic process that governs the dynamics of the instantaneous covariance matrix, even if one could approximate the integrated covariance matrix arbitrarily well using realized measures (Barndorff-Nielsen and Shephard, 2004).

We next illustrate how the price discovery measures change with the sampling frequency by simulating the price process in (14), with $\alpha = (0, \alpha_2)'$ and $\beta' = (1, -1)$ as in (13). As for C_t, we assume that the instantaneous correlation is constant, whereas the market-specific variances follow a single-factor stochastic volatility (SV1F) process as in, e.g., Huang and Tauchen (2005), Gonçalves and Meddahi (2009) and Barndorff-Nielsen, Hansen, Lunde and Shephard (2008). We let $\sigma_{m,t}^2 = \exp(\varsigma_0 + \varsigma_1 V_{m,t})$, with $\mathrm{d}V_{m,t} = \gamma V_{m,t}\,\mathrm{d}t + \mathrm{d}B_{m,t}$ for $m = 1, 2$. In addition, we consider that $\mathrm{Corr}(\mathrm{d}W_t, \mathrm{d}B_{m,t}) = \nu_1$ for $m = 1, 2$ and that $\mathrm{Corr}(\mathrm{d}B_{1,t}, \mathrm{d}B_{2,t}) = \nu_2$, where ν_1 and ν_2 are (possibly nonzero) constants. We further impose that the parameter vector $(\varsigma_0, \varsigma_1, \gamma, \nu_1)$ takes the same values for both markets, so that their variance processes exhibit the same time series properties and unconditional distributions. We set $\varsigma_1 = 0.125$, $\gamma = -0.025$, and $\nu_1 = -0.30$ as in Barndorff-Nielsen et al. (2008), whereas we impose $\varsigma_0 = \varsigma_1^2/(2\gamma)$, so that the instantaneous variances integrate to one in both markets, and $\nu_2 = 0.95$. Altogether, this specification is convenient because it makes the results directly comparable to the ones in Figure 1 given that the IS is essentially driven by the amount of correlation if the market-specific variances do not differ considerably.

We run simulations using an iterative method based on an Euler scheme that accounts for the exact discretization of the stochastic volatility process to obtain prices at the 1-second frequency. We set $n = 23,400$ to match the number of seconds in a trading day of 6.5 hours. Next, we estimate the speed-of-adjustment parameters by least squares (LS) and then compute the corresponding CS measures at the 1-, 5-, 10-, 30-, 60-, 300-, 600-, and 1,800-second frequencies, corresponding respectively to samples of 23,400, 4,680, 2,340, 1,170, 780, 390, 78, 39, and 13 observations. Finally, we also estimate the traditional IS measures based on the sample covariance matrix as if it were not changing over time.

Figure 2 displays the box plots of the CS and IS estimates, as well as of the contemporaneous correlation between markets, at the different sampling intervals across 50,000 replications. We find that the median IS estimates are very close to the IS measures that the exact discretization of (13) would imply. This is not surprising given that we force the market-specific instantaneous variances to have the same unconditional distributions. More generally, allowing for stochastic volatility does not seem to alter the main conclusion from Figure 1 in that the IS measures still converge to 1/2 as the sampling frequency increases due to the increase in the contemporaneous correlation.

As for the CS estimates in the lower panel of Figure 2, it is apparent that they remain invariant to the sampling frequency, with their median values coinciding with the continuous-time CS regardless of the contemporaneous correlation between markets. The CS signature plots also shed some light on the behavior of the noise-to-signal ratio as the frequency decreases. The variance of the LS estimates increases markedly with the sampling frequency for two reasons. First, as the sampling frequency decreases, identification of α_δ weakens and, as a result, the LS estimates become noisier. Second, the convergence rate of the LS estimator is of order $\sqrt{n}$, and thus, precision deteriorates as n declines. In the next section, we propose an alternative estimator based on kernel least squares, whose root mean squared error (RMSE) is much smaller than the RMSE of the least squares estimator (namely, half at the 30-second frequency and a quarter at the 5-minute frequency).

4 Estimation of the time-varying continuous-time price discovery

This section discusses a consistent and asymptotically normal estimator of the daily speed-of-adjustment parameters and hence of the daily component shares. In particular, we allow for the parameters in (7) to change over time at the daily frequency. This flexible parametrization is consistent with previous studies that identify time variation in the price discovery mechanism (see, for instance, Hasbrouck, 1995, 2003; Mizrach and Neely, 2008). The ability of a trading venue to impound new information mostly depends on market features (e.g., cost structure, market design, technological infrastructure, and relative presence of high-frequency traders) and market characteristics (e.g., trading intensity, trading volume and volatility). It turns out that both change over time, but neither in a continuous nor in a brusk fashion. This is well in line with Eun and Sabherwal (2003) and Frijns, Gilbert and Tourani-Rad (2015), who show that some of the main price discovery drivers are highly persistent over time (e.g., volume and volatility). Accordingly, we assume that the speed-of-adjustment parameters follow a bounded local stable stochastic process at the daily frequency in order to cope with persistence.

Consistent estimation of the daily component shares requires only a consistent estimator of the speed-of-adjustment parameters for any frequency given that $\beta = (1, -1)'$ is known. Because our sample consists of prices observed intra-daily over different days and our estimation method uses the entire sample, it is convenient to adapt our notation as follows. Denote by $\alpha_\delta^{(d)}$ the speed-of-adjustment parameter at interval length δ for day d and $T = nD$ the total number of observations, where n is the number of intraday observations and D is the number of trading days. As standard in the daily VECM approach, we augment the lag structure in (3) and allow the corresponding autoregressive matrices $\Gamma_{\delta,1}^{(d)}, \ldots, \Gamma_{\delta,\ell}^{(d)}$ also to vary over time:

$$\Delta P_\tau = \alpha_\delta^{(d)} \beta' P_{\tau-1} + \sum_{j=1}^{\ell} \Gamma_{\delta,j}^{(d)} \Delta P_{\tau-j} + \varepsilon_\tau, \qquad d = 1, \ldots, D \quad \tau = 1, \ldots, T \tag{15}$$

or, in a more compact notation,

$$\Delta P_\tau = B^{(d)'} X_\tau + \varepsilon_\tau, \tag{16}$$

where $X_\tau = (P_{1,\tau-1} - P_{2,\tau-1}, \Delta P'_{\tau-1}, \ldots, \Delta P'_{\tau-\ell})'$ is a $(2\ell + 1) \times 1$ vector of covariance stationary regressors and $B^{(d)}$ is a $(2\ell + 1) \times 2$ random coefficient matrix that collects the free parameters in (15), namely, $B^{(d)} = (\alpha_\delta^{(d)}, \Gamma_{\delta,1}^{(d)}, \ldots, \Gamma_{\delta,\ell}^{(d)})'$.

The standard practice is to capture the daily variation in price discovery by estimating a daily VECM (see, for instance, Hasbrouck, 2003; Chakravarty et al., 2004; Mizrach and Neely, 2008). In contrast, we employ Giraitis, Kapetanios and Yates' (2013) kernel-based estimator to retrieve daily estimates of the VECM parameters. As opposed to the daily VECM least-squares estimates, our CS estimates are not independent across days. The kernel least squares (KLS) estimator exploits the assumption that the daily variations in the alpha and Gamma matrices are persistent processes (either deterministic or stochastic) in order to obtain more efficient estimators. This is convenient because as CS is invariant to the sampling frequency, one may prefer to use prices at a lower frequency to mitigate the effects of market microstructure noise (Hupperets and Menkveld, 2002).

The VECM in (15) remains a valid representation of cointegrated prices as long as the roots of $\mathcal{C}(z) = \left|(1-z)I_2 - \alpha_\delta^{(d)}\beta' z - \sum_{j=1}^{\ell}\Gamma_{\delta,j}^{(d)}(1-z)z^j\right| = 0$ are outside the unit circle or equal to one for all $d = 1, 2, \ldots, D$ (Hansen, 2003). In what follows, we assume that $B^{(d)}$ is a bounded stochastic process that meets this condition.

Assumption RM *$B^{(d)}$ forms a sequence of random matrices satisfying $\mathcal{C}(z) = 0$ for $|z| \geq 1$, $\sup_{d\leq D}\left\|B^{(d)}\right\| = O_p(1)$, and $\sup_{\imath:|\imath-d|\leq h}\left\|B^{(d)} - B^{(\imath)}\right\|_{sp}^2 = O_p(h/d)$ for $h = o(d) \to \infty$ as $d \to \infty$.*

The local stability conditions in Assumption RM are very mild, holding for the bounded random walk process we consider in our Monte Carlo study, for instance. Note that we implicitly assume that the elements of $B^{(d)}$ vary over time in a stochastic fashion as in Giraitis, Kapetanios and Yates (2018). Alternatively, in the case of deterministic variation in $B^{(d)}$, asymptotic normality of the KLS estimator would require the parameters to satisfy a Lipschitz condition (Robinson, 1989). Regardless of whether the variation is stochastic or deterministic, the matrix of regressors X_τ is the same for all equations of (16), and thus, it is possible to estimate the free parameters in each of the two equations separately. Define the $(2\ell+1)\times 1$ vector $B_m^{(d)}$ as the mth column of $B^{(d)}$, containing all the parameters of the mth equation of (16). The KLS estimator then reads

$$\widehat{B}_m^{(d)} = \left(\sum_{\tau=1}^{T} K\left(\frac{nd-\tau}{H}\right) X_\tau X_\tau'\right)^{-1} \sum_{\tau=1}^{T} K\left(\frac{nd-\tau}{H}\right) X_\tau \Delta P_{m,\tau}, \quad m = 1, 2, \tag{17}$$

where $K(\cdot)$ and H respectively denote a kernel function and the corresponding bandwidth, and $\Delta P_{m,\tau}$ is the mth row of ΔP_τ. The next assumption regulates the kernel properties and the rate at which the bandwidth grows.

Assumption K *The kernel function $K(v)$ is nonnegative for any $v \in \mathbb{R}$. It is continuous and bounded, with a bounded first derivative, such that $\int K(v)\,\mathrm{d}v = 1$ and $\int [K(v)]^2\,\mathrm{d}v = c_K < \infty$. There is also a constant $c > 0$ such that $K(v) = O\left(e^{-cv^2}\right)$. As for the bandwidth, for fixed n, $H = o(nD) \to \infty$ as $D \to \infty$.*

Most standard kernels in the literature (e.g., the flat, Epanechnikov and Gaussian kernels) meet the conditions in Assumption K. It now remains to regulate the higher-order moments of errors and regressors, so as to ensure that the limiting distribution of the KLS estimator is well defined.

Assumption COV *The error ε_τ forms a covariance stationary martingale difference sequence with finite fourth moment uniformly over τ, and orthogonal to the regressors X_τ. The latter are covariance stationary such that $\mathbb{E}(X_\tau X_\tau') = Q_X < \infty$ uniformly over τ and T and that $\frac{1}{\kappa}\sum_{\kappa=0}^{\infty} \sup_{1\le i,\imath,j,\jmath\le 2\ell+1} \sup_{1\le\tau\le T} |Cov(X_{i,\tau}X_{\imath,\tau}, X_{j,\tau+\kappa}X_{\jmath,\tau+\kappa})| < \infty$. Finally, ε_τ and X_τ are such that $\zeta_\tau = (X_\tau\varepsilon_{1,\tau}, X_\tau\varepsilon_{2,\tau})'$ has a finite covariance matrix $Q_{\zeta,\tau} = \mathbb{E}[\zeta_\tau\zeta_\tau']$ uniformly over τ, with $\frac{1}{T}\sum_{\tau=1}^{T} Q_{\zeta,\tau} \xrightarrow{p} Q_\zeta$, $\frac{1}{H}\sum_{|\tau-nd|<H}\left[K\left(\frac{nd-\tau}{H}\right)\right]^2 \zeta_\tau\zeta_\tau' \xrightarrow{p} c_K\, Q_\zeta$, and $\mathbb{E}(\zeta_{i,\tau}\zeta_{\imath,\tau}\zeta_{j,\tau}\zeta_{\jmath,\tau}) < \infty$ for all $i,\imath,j,\jmath = 1,\ldots,4\ell+2$.*

Assumptions COV, K and RM coincide with Giraitis, Kapetanios and Marcellino's (2017) Assumptions 1 to 3 in the context of exogenous regressors, and hence asymptotic normality of the KLS estimator readily follows as $D \to \infty$ from their Theorem 3. We then employ the delta method to back out the asymptotic behavior of the continuous-time component share measures. In particular, it follows from (8) that

$$\Lambda_\perp^{(d)} \equiv \frac{\partial \alpha_\perp^{(d)}}{\partial \alpha_\delta^{(d)\prime}} = \begin{pmatrix} \frac{\alpha_{\delta,2}^{(d)}}{\left(-\alpha_{\delta,1}^{(d)}+\alpha_{\delta,2}^{(d)}\right)^2} & -\frac{\alpha_{\delta,1}^{(d)}}{\left(-\alpha_{\delta,1}^{(d)}+\alpha_{\delta,2}^{(d)}\right)^2} \\ -\frac{\alpha_{\delta,2}^{(d)}}{\left(-\alpha_{\delta,1}^{(d)}+\alpha_{\delta,2}^{(d)}\right)^2} & \frac{\alpha_{\delta,1}^{(d)}}{\left(-\alpha_{\delta,1}^{(d)}+\alpha_{\delta,2}^{(d)}\right)^2} \end{pmatrix}. \tag{18}$$

The next result documents the asymptotic normality of the KLS estimators of the daily VECM parameters and of the daily component shares, providing analytical expressions for their asymptotic variances.

Theorem *Let Assumptions COV, K and RM hold, and let $b^{(d)} = vecB^{(d)}$ and $\widehat{b}^{(d)} = vec\widehat{B}^{(d)}$. If*

$H = o(\sqrt{T})$,

$$\sqrt{H}(\widehat{b}^{(d)} - b^{(d)}) \xrightarrow{d} N\left(0, c_K\,[I_2 \otimes Q_X^{-1}]\,Q_\zeta\,[I_2 \otimes Q_X^{-1}]\right), \tag{19}$$

where $Q_X = \mathbb{E}[X_\tau X_\tau']$ and $\otimes$ denotes the Kronecker product. In turn,

$$\sqrt{H}(\widehat{\alpha}_\perp^{(d)} - \alpha_\perp^{(d)}) \xrightarrow{d} N\left(0, c_K\,\Lambda_\perp^{(d)} R_\alpha\,[I_2 \otimes Q_X^{-1}]\,Q_\zeta\,[I_2 \otimes Q_X^{-1}]\,R_\alpha' \Lambda_\perp^{(d)\prime}\right), \tag{20}$$

where $R_\alpha = (e_1, 0_{2\times 2\ell}, e_2, 0_{2\times 2\ell})$ is a deterministic matrix that selects the elements of $\alpha_\delta^{(d)}$ from $\widehat{b}^{(d)}$ with $e_1 = (1,0)'$ and $e_2 = (0,1)'$.

To consistently estimate the asymptotic covariance matrix, it suffices to substitute $\widehat{Q}_X = \frac{1}{H}\sum_{|\tau - nd| < H} K\left(\frac{nd-\tau}{H}\right) X_\tau X_\tau'$ and $\widehat{Q}_\zeta = \frac{1}{H}\sum_{|\tau - nd| < H} \left[K\left(\frac{nd-\tau}{H}\right)\right]^2 \widehat{\zeta}_\tau \widehat{\zeta}_\tau'$ into (19) and (20), with $\widehat{\zeta}_\tau = (X_\tau \widehat{\varepsilon}_{1,\tau}, X_\tau \widehat{\varepsilon}_{2,\tau})'$, given that $\widehat{Q}_\zeta \xrightarrow{p} c_K\,Q_\zeta$ due to the consistency of $\widehat{\varepsilon}_\tau$ and to Assumptions K and COV.

5 Monte Carlo study

We assess the performance of the KLS estimator relative to the standard approach of estimating time-varying measures of price discovery using daily VECMs. As before, we contemplate one asset traded at $k = 2$ trading venues, with prices following either (1) or (14). In the first setting, we simulate prices within a given day from the exact discretization of the continuous-time process in (1) at the 1-second frequency (i.e., $\delta = 1/23{,}400$ for a trading day of 6.5 hours) for $D = 500$ trading days (about 2 years), with contemporaneous correlation $\rho \in \{0, 0.3, 0.5, 0.7, 0.9\}$. We then sample prices at fixed intervals of 1/2, 1, 3, and 5 minutes, corresponding to δ values of 1/780, 1/390, 1/195, 1/130, and 1/78, respectively. As a result, sample size ranges from 39,000 to 390,000 intraday observations.

The data-generation process we entertain requires that the elements of $\alpha^{(d)}$ follow independent bounded random walk processes at the daily frequency. The bounds are such that $\alpha_{1,1/78}^{(d)} \in [-0.49, -0.01]$ and $\alpha_{2,1/78}^{(d)} \in [0.01, 0.49]$ at the 5-minute frequency ($\delta = 1/78$). Note that restricting the speed-of-adjustment parameters in their lowest frequency ensures that the eigenvalues of $\alpha_\delta^{(d)}\beta'$ are strictly smaller than one in magnitude for any $0 < \delta \leq 1/78$. Similarly to Giraitis, Kapetanios and Yates (2013), we assume that

$$\alpha^{(d)}_{1,1/78} = \bar{\alpha}\left(\frac{a^{(d)}_1}{\max_{0\le d^*\le d}\left|a^{(d^*)}_1\right|} - 1\right) - 0.01 \qquad \alpha^{(d)}_{2,1/78} = \bar{\alpha}\left(\frac{a^{(d)}_2}{\max_{0\le d^*\le d}\left|a^{(d^*)}_2\right|} + 1\right) + 0.01,$$

with $\bar{\alpha} = 0.24$, so as to satisfy the upper and lower bounds, and $(a^{(d)}_1, a^{(d)}_2)'$ following independent driftless random walk processes driven by white noise innovations. We then back out $\Pi^{(d)} = \alpha^{(d)}\beta'$ by imposing $\beta = (1, -1)'$ and inverting the matrix exponential operator in the exact discretization of the reduced-rank OU process:

$$\Pi^{(d)} = \alpha^{(d)}\beta' = \left(\frac{1}{78}\right)^{-1} \log\left(\alpha^{(d)}_{\delta=1/78}\,\beta' + I_k\right), \tag{21}$$

where $Z = \log(A)$ if Z is such that $\exp(Z) = A$ for any square matrix A.

As for the daily covariance matrix $\Sigma^{(d)}$, we assume that

$$\ln\sigma^2_{m,d} = \phi_0 + \phi_1 \ln\sigma^2_{m,d-1} + \varsigma\, \upsilon_{m,d}, \qquad \text{for } \ m = 1, 2 \ \text{ and } \ d = 1, 2, \ldots, D \tag{22}$$

with $\sigma^2_{1,d}$ and $\sigma^2_{2,d}$ denoting the diagonal elements of $\Sigma^{(d)}$. The volatility innovations $\upsilon_{1,d}$ and $\upsilon_{2,d}$ are Gaussian white noises with a constant correlation of 0.95 and unit variances. As in Jacquier, Polson and Rossi (1994), we fix the autoregressive parameter to 0.98 and calibrate ϕ_0 and ς in (22) such that the expected annual volatility is 20% and the coefficient of variation given by $\mathbb{V}(\sigma^2_{m,d})/\mathbb{E}(\sigma^2_{m,d}) = \exp\left(\varsigma/(1-\phi^2_1)\right) - 1$ is equal to 1/2.

After generating prices at the different frequencies using the exact discretization in (3) with the above continuous-time parameters, we estimate $\alpha^{(d)}_{\delta}$ by LS as in the daily VECM approach and by KLS using a Epanechnikov kernel, with bandwidth $H \in \{n^{8/10}\sqrt{D}, n^{9/10}\sqrt{D}, n\sqrt{D}\}$. Table 1 documents that the root mean squared errors of the KLS estimates of $\alpha^{(d)}_{\delta,\perp,1}$ relative to their LS counterparts over 1,000 replications as well as their bias. It should be noted that given the normalization of the orthogonal complements, the bias magnitude and the relative root mean squared errors (RRMSE) of the $\alpha^{(d)}_{\delta,\perp,1}$ and $\alpha^{(d)}_{\delta,\perp,2}$ estimates are identical by construction.

Both estimators are clearly unbiased in that sample biases are very close to zero in magnitude. There is a clear decreasing pattern with the sampling interval and bandwidth value, whereas the magnitude of the bias does not seem to vary with the amount of correlation across markets. As for the RRMSE figures, they show that the KLS-based component shares are much more precise than the LS estimates almost regardless of the amount of correlation between markets, sampling frequency, and bandwidth choice. The difference in performance increases with the contemporaneous correlation but declines with the sampling frequency. This happens because estimating independent VECM models for each day becomes more difficult not only at lower frequencies due to the smaller sample sizes but also as correlation increases due to the lesser amount of information.

Finally, we entertain the continuous-time price process in (14) with C_t evolving intradaily, but with drift parameters varying on a daily basis through $\Pi^{(d)} = \alpha^{(d)}\beta'$. In particular, the drift parameters are as in the previous set of simulations, whereas we assume that the market-specific variances follow correlated SV1F processes as in Section 3.4. To ensure that the results are comparable with the previous simulations, we let the instantaneous market-specific variances integrate to the corresponding diagonal element of the daily integrated covariance matrix $\Sigma^{(d)}$. We simulate prices at the 1-second frequency using the same procedure as in Section 3.5 and then sample them over different intervals. Table 2 reports the relative performance of the KLS estimator over the LS estimator of $\alpha^{(d)}_{\delta,\perp,1}$. As before, the KLS estimator outclasses the LS estimator in virtually every instance, with the difference in performance increasing with the correlation across markets and declining with the sampling frequency.

All in all, the Monte Carlo results favor the KLS estimator over the LS estimator for any contemporaneous correlation, bandwidth choice, and covariance matrix specification.

6 Price informativeness

In this section, we estimate the continuous-time CS measures a broad set of NYSE- or Nasdaq-listed stocks using high-frequency data from January 2007 to December 2013. Such a long time span is unusual for price discovery analyses in that most studies typically consider much shorter periods of up to 12 months (see, for instance, de Jong and Schotman, 2010; Riordan and Storkenmaier, 2012; Benos and Sagade, 2016; Ozturk et al., 2017; Hasbrouck, 2018). In what follows, we first carry out Elliott and Müller's (2006) test to assess the time-varying nature of the continuous-time CS measures in a formal manner. We then estimate the continuous-time CS measures by both LS and KLS and investigate how they relate to liquidity.

6.1 Data

Our data set consists of 30 stocks that markedly differ in terms of industry, listing venue, and trading activity. We group them into three subsamples: 10 Nasdaq-listed stocks, 10 actively-traded NYSE-listed stocks and 10 less-actively-traded NYSE-listed stocks. The first group consists of a random sample from the Nasdaq-100 stock market index constituents that have been trading since January 2007: Adobe Systems (ADBE), Align Technology (ALGN), Amazon.com (AMZN), CA Technologies (CA), Expedia (EXPE), Alphabet (GOOG), Micron Technology (MU), Starbucks Corporation (SBUX), Vodafone Group (VOD), and Wendy's (WEN).

As for the actively-traded NYSE-listed stocks, we select at random from the S&P 500 index constituents: Bank of America (BAC), General Electric (GE), Hewlett-Packard (HPQ), International Business Machines (IBM), J.C. Penney Company (JCP), JP Morgan Chase (JPM), Coca-Cola Company (KO), Altria Group (MO), Verizon Communications (VZ), and ExxonMobil (XOM). Finally, we randomly select 10 less-liquid NYSE-listed stocks from the Russell 1000 index constituents: Canon (CAJ), Cooper Companies (COO), Dolby Laboratories (DLB), Kirby Corporation (KEX), Lazard (LAZ), Corporate Office Properties Trust (OFC), Everest Re Group (RE), Regal Beloit Corp (RBC), RPC Inc (RES), Rollins (ROL) and Thor Industries (THO). These stocks exhibit, on average, 70% less trading intensity, as measured by the number of trades, than the actively-traded stocks in the previous group. See also Figure 6 for the trading volume of each stock.

We extract quotes data from TAQ and implement the same cleaning filters as in Barndorff-Nielsen, Hansen, Lunde and Shephard (2009), discarding any observation with a zero quote, negative bid-ask spread, or outside the main trading hours (9:30 to 16:00). We also discard any data point either with a bid-ask spread higher than 50 times the median spread on that day or with a midquote deviating by more than 10 mean absolute deviations from a rolling centered median of 50 observations. Finally, we take the median bid and ask quotes at each second in the presence of multiple ticks. We then synchronize the NYSE and Nasdaq midquotes by sampling at regularly spaced intervals of 1 minute. This not only alleviates concerns with market microstructure noise (see, among others, Hupperets and Menkveld, 2002; Grammig et al., 2005) but also helps with the convergence rate of the KLS estimator given that the $n = 390$ intraday observations at this frequency allows for a reasonably large bandwidth H. Table 3 details the cleaning process and provides the final number of time-series observations for each stock.

Although using midquotes is standard in the price discovery literature (see, among others, Hasbrouck, 1995; Baillie et al., 2002; Menkveld, Koopman and Lucas, 2007), we redo the analysis in Section 6.3.1 using transaction prices. Accordingly, we also collect transactions data from TAQ, including prices, number of trades, and trading volume. We handle transactions data using the same cleaning filters as in Barndorff-Nielsen et al. (2009). In particular, we discard any observation with a zero trade, outside the main trading hours, or with a flag for trade correction or abnormal sale condition. In addition, we take the median over prices with the same time stamp and delete any price above/below the ask/bid quote plus/minus the bid-ask spread.

6.2 Time variation in the continuous-time component shares

There is seemingly a consensus in the literature that the price discovery processes change over time, with many studies running daily VECM specifications to address this issue (see, among others, Hasbrouck, 2003; Chakravarty et al., 2004; Hansen and Lunde, 2006; Mizrach and Neely, 2008). Additionally, empirical evidence suggests that the price discovery changes with some highly persistent market indicators such as trading volume and volatility. For instance, Figuerola-Ferretti and Gonzalo (2010) posit an equilibrium model of commodity spot and future prices in which the speed-of-adjustment parameters of a discrete-time VECM depend on the relative number of market participants. As a result, they establish a direct link between component shares and market activity indicators, such as relative volume or trade intensity.

We start with a formal test of whether component shares change over time. In particular, we employ Elliott and Müller's (2006) test for the null hypothesis of constant speed-of-adjustment parameters against the alternative hypothesis that they display persistent variation in time. In the context of one asset trading at two markets, time-varying speed-of-adjustment parameters automatically imply that the CS measures also change over time. The Elliott-Müller test is convenient because it accommodates well enough the sort of variation we describe in Section 4. In addition, the test entails good size and power properties even under conditional heteroskedasticity. The asymptotic distribution of the test statistic is nonstandard, with small values indicating rejection of the null hypothesis of parameter stability. The critical values for a bivariate VECM are respectively -12.80, -14.32, and -17.57 at the 10%, 5%, and 1% significance levels, regardless of the lag structure.

The second column in Table 4 displays the Elliott-Müller test results, which are overwhelmingly in favor of time-varying speed-of-adjustment parameters. Specifically, we cannot reject at the usual significance levels the stability of the speed-of-adjustment parameters over time only for MU, JPM and XOM. We reject the null at the 1% significance level for every less-liquid NYSE-listed stock as well as for most Nasdaq-listed and actively-traded NYSE-listed stocks. This provides strong evidence that the price discovery indeed changes over time, corroborating previous findings in the empirical and theoretical literature on price discovery.

The second and third panels of Table 4 report the median and standard deviation of the LS and KLS estimates of the speed-of-adjustment parameters and of the CS measures over the sample period for each stock. The subscripts 1 and 2 in the parameter estimates denote Nasdaq and NYSE/Arca markets, respectively.[6] The LS and KLS median estimates of the component share are similar enough to lead to the same conclusion about overall market leadership. However, the standard deviations of the daily VECM least-squares estimates are much larger than the corresponding figures for the KLS estimates (threefold for the CS measures, for instance). This is not surprising given the better finite-sample performance of the KLS estimator (see Section 5), and accordingly, we restrict attention in the next section to the KLS-based price discovery analysis.

6.3 Daily evolution of the continuous-time CS measures

Before discussing the CS estimates, it is important to provide a couple of details about model selection and estimation. First, we set the bandwidth to $H = n\sqrt{D}$ as in Giraitis et al. (2013) given that the Monte Carlo results confirm that this bandwidth choice performs very well. Second, we determine the lag structure by minimizing the Bayesian information criterion (BIC), though selecting the most parsimonious specification in which we cannot reject the absence of residual autocorrelation at the 5% significance level as in Hansen and Lunde (2006) does not change the qualitative results. As expected, we find only one cointegrating vector for every pair of stock prices using Johansen's maximum eigenvalue and trace tests at the 1% significance level.[7]

Figures 3 to 5 plot the KLS estimates of the daily component shares in continuous time and

[6] For the Nasdaq-listed stocks, we use Arca as the competing trading venue, as NYSE does not trade equities listed on other exchanges.

[7] The results for the most parsimonious congruent specification and of the cointegration analyses are available upon request.

their respective 95% confidence intervals for each group of stocks we consider. Figure 3 focuses on the Nasdaq-listed stocks trading on Nasdaq and Arca. The price discovery changes markedly over time, with relative market informativeness alternating between trading venues. Even though results vary substantially across stocks, Nasdaq seems more important than Arca in general. This is not very surprising given that Nasdaq not only is the listing venue for these stocks but also offers more liquidity. See the trading volume dynamics in Figure 6. Finally, the standard errors are generally small enough to distinguish between the price discovery measures in the two markets for most of the days in our sample. This is mainly due to the use of 1-minute data, which alleviates the slow convergence rates of the KLS estimator while retaining the signal that identifies the CS measures from the data.

Figure 4 unveils the price discovery for the actively-traded NYSE-listed stocks trading on Nasdaq and NYSE. As before, market leadership alternates between the two exchanges, with mixed results especially in the beginning of the sample period. The Nasdaq contribution to the price discovery picks up for the majority of the stocks as of 2008, likely due to a severe decline in the NYSE market share (see Figure 6). This coincides with the period of increase in market fragmentation in the U.S. market and significant gain in market share of the new entrant markets (see, for instance, Menkveld, 2014, 2016; O'Hara, 2015). The NYSE gradually recovers its relevance in the price discovery process for most of the stocks as from mid 2010 and, by mid 2011, it leads the price discovery for 7 of the 10 stocks in this group. The price discovery contribution typically moves in tandem with relative liquidity and NYSE experiences much more volume than Nasdaq on average.

Figure 5 exhibits the daily component share estimates for the less-actively-traded NYSE-listed stocks. In contrast to the previous results, NYSE clearly dominates Nasdaq in terms of market leadership for every single stock in this group. This corroborates the importance of market activity and trade intensity to the price discovery in that the trading volume at NYSE is much larger than at the Nasdaq for the less liquid stocks. Furthermore, it is also in line with Frijns and Schotman's (2009) evidence that market makers lead the price discovery process for less liquid stocks.

In the next section, we check whether our results are sensitive to the use of midquotes rather than transaction prices. We not only redo the empirical analysis using transaction prices for a subset of five actively-traded NYSE-listed stocks, but also consider a price system with both midquotes and transaction prices to assess whether the latter brings about any additional information on the price discovery. Finally, we also exploit transactions data to link the daily variation in the price discovery measures to relative volume across markets.

6.3.1 Taking advantage of transactions data

Most price discovery analyses employ midquotes not only because they usually have a higher information content than transactions data, but also because they do not suffer from the bid-ask bounce that plagues transaction prices (Menkveld et al., 2007). Transaction prices indeed exhibit different statistical properties, e.g., negative autocorrelation (see Hasbrouck, 2007 for a comprehensive analysis of the Roll model of trade prices). We thus run two robustness checks for five actively-traded NYSE-listed stocks.[8] First, we estimate the continuous-time CS measures using their transactions data. As before, we choose the VECM lag structure that minimizes the BIC criterion, which turns out to entail no residual autocorrelation up to the 10th lag at the 5% significance level. The plots in the first column of Figure 7 indicate a very similar CS pattern than we observe using midquotes in that time variation in the continuous-time CS measures is evident with NYSE and Nasdaq alternating over time as the most informative market.

Second, we consider a price system with both midquotes and transaction prices to understand which contributes most to the price discovery. The plots in the second column of Figure 7 reveal a very clear dominance of midquotes, indicating that they are more informative than transaction prices. As before, there is strong evidence that the continuous-time CS measures change over time. In short, these results support our choice of using midquotes instead of transactions data as standard in the price discovery literature.

6.3.2 Long-run relationship between CS measures and trading volume

Finally, to better understand the daily variation in the price discovery, we estimate a VECM model for the daily CS estimates and the logarithm of the relative volume at the listing venue.[9] Both time series are very persistent, indicating the presence of unit roots. This is not surprising given that we allow for the component shares to follow a bounded random walk. In addition, standard

[8] The results for the Nasdaq-listed stocks are qualitatively very similar, and hence we do not report them. They are of course available upon request.

[9] Depending on the stock, we need between 5 and 9 lags in the VAR specification to cope with persistence. To simplify matters, we carry out the cointegration analysis using a VAR(9) specification for every stock, which leads to VECM representations with 8 lags.

cointegration tests strongly suggest a long-run positive relationship between component shares and relative volume. This is in line with Figuerola-Ferretti and Gonzalo's (2010) model as well as with the empirical findings of Eun and Sabherwal (2003) and Frijns et al. (2015).[10]

Table 5 shows that every cointegrating vector is such that the contribution to the price discovery of a given trading venue increases with liquidity. The relative volume coefficient estimates in the cointegrating vector are on average about -1/2 for the Nasdaq-listed stocks and twice that for the NYSE-listed stocks (on average, -1.1437 for the very liquid stocks and -0.9427 for the less liquid stocks). The listing venue also matters for the response of the CS estimates to their long-run relationship with the relative volume. In particular, the corresponding speed-of-adjustment parameters are always significant at the 5% level for the Nasdaq-listed stocks, whereas they are significant for less than half of the NYSE-listed stocks. Finally, the relative volume seems to react in a significant manner to the long-run relationship for every stock apart from ADBE.

The VECM specifications linking price discovery to liquidity have a quite good fit. This is especially so for the Nasdaq-listed stocks, whose average adjusted R^2 is 0.7675, with a median of 0.8330. Altogether, it seems that regardless of liquidity and listing venue issues, relative volume and price discovery adjust to maintain their long-run relationship.

7 Conclusion

This paper entertains price discovery in a continuous-time setting. We first show that the component share measure of price discovery is invariant to the discretization frequency, allowing us to make inference on the continuous-time price discovery mechanism from discrete sampled prices. This is in contrast with Hasbrouck's (1995) information share, which depends on the contemporaneous correlation across markets, which naturally increases in magnitude as the sampling frequency decreases.

We then make use of Giraitis et al.'s (2013) KLS method to estimate daily component shares. By exploiting the inter-dependence across days, the KLS approach yields more efficient estimates than we would otherwise obtain by treating the daily variation in the VECM parameters as independent over time. Monte Carlo simulations confirm under different scenarios that the KLS estimator easily

[10] As before, the unit root and cointegration test results are available from the authors upon request.

outperforms the standard practice of estimating component shares for each day using individual VECM specifications.

Empirically, we assess the informativeness of Nasdaq and NYSE/Arca for a set of 30 stocks. We find statistical evidence that the component shares indeed change over time for virtually every stock in our sample. Our estimates indicate that market leadership alternates over time, depending on the relative liquidity of each trading venue.

Appendix: Proof

It follows from the orthogonality condition that $[\mathbb{E}(X_\tau X_\tau')]^{-1}\mathbb{E}(X_\tau \Delta P_{m,\tau})$ minimizes the LS objective function for the mth equation of (16). Giraitis et al. (2013, 2018) propose the use of sample kernel-based counterparts of the expectation terms, leading to the KLS estimator in (17). The only difference is that they allow for the parameters to change at each instant of time, whereas we restrict variation by assuming that parameters vary only at a lower (daily) frequency.

This modification yields

$$\widehat{B}_m^{(d)} = \left(\sum_{\tau=1}^{T} K\left(\frac{nd-\tau}{H}\right) X_\tau X_\tau'\right)^{-1} \sum_{\tau=1}^{T} K\left(\frac{nd-\tau}{H}\right) X_\tau \Delta P_{m,\tau} \tag{23}$$

$$= B_m^{(d)} + \left(\sum_{\tau=1}^{T} K\left(\frac{nd-\tau}{H}\right) X_\tau X_\tau'\right)^{-1} \sum_{\tau=1}^{T} K\left(\frac{nd-\tau}{H}\right) X_\tau \varepsilon_{m,\tau}, \qquad m = 1, 2, \tag{24}$$

with $H = o(D) \to \infty$ as $D \to \infty$.[11] It then follows that

$$\sqrt{H}(\widehat{B}_m^{(d)} - B_m^{(d)}) = \left(\frac{1}{H}\sum_{\tau=1}^{T} K\left(\frac{nd-\tau}{H}\right) X_\tau X_\tau'\right)^{-1} \frac{1}{\sqrt{H}} \sum_{\tau=1}^{T} K\left(\frac{nd-\tau}{H}\right) X_\tau \varepsilon_{m,\tau}, \quad m = 1, 2.$$

It is straightforward to show along the same lines as in Giraitis et al.'s (2017) Lemma 4 that

$$\frac{1}{H}\sum_{\tau=1}^{T} K\left(\frac{nd-\tau}{H}\right) X_\tau X_\tau' = \frac{1}{H}\sum_{|\tau-nd|<H} K\left(\frac{nd-\tau}{H}\right) X_\tau X_\tau' + o_p(1/H),$$

$$\frac{1}{\sqrt{H}}\sum_{\tau=1}^{T} K\left(\frac{nd-\tau}{H}\right) X_\tau \varepsilon_{m,\tau} = \frac{1}{\sqrt{H}}\sum_{|\tau-nd|<H} K\left(\frac{nd-\tau}{H}\right) X_\tau \varepsilon_{m,\tau} + o_p(H^{-1/2}).$$

This means that

$$\sqrt{H}(\widehat{B}_m^{(d)} - B_m^{(d)}) = \widehat{Q}_X^{-1} \frac{1}{\sqrt{H}} \sum_{|\tau-nd|<H} K\left(\frac{nd-\tau}{H}\right) X_\tau \varepsilon_{m,\tau} + o_p(1), \tag{25}$$

[11] Note that it suffices to assume that $n = 1$ and $D = T$ to recover the standard KLS setting in which the random matrix changes at every point in time.

where $\widehat{Q}_X = \frac{1}{H}\sum_{|\tau-nd|<H} K\left(\frac{nd-\tau}{H}\right) X_\tau X_\tau'$ is common to both markets. We may then stack the VECM parameters and their estimates into

$$\sqrt{H}(\widehat{b}^{(d)} - b^{(d)}) = \left(I_2 \otimes \widehat{Q}_X^{-1}\right) \frac{1}{\sqrt{H}} \sum_{|\tau-nd|<H} K\left(\frac{nd-\tau}{H}\right) \zeta_\tau + o_p(1). \tag{26}$$

Along similar lines to Giraitis et al.'s (2018) Lemma 5, it follows from Assumptions K and COV that $\lim_{H\to\infty} \mathbb{E}(\widehat{Q}_X) = Q_X$ for $\int K(v)\,\mathrm{d}v = 1$ and that $\lim_{H\to\infty} \mathbb{V}[\widehat{Q}_X(\imath,\jmath)] = \lim_{H\to\infty} \frac{c_K}{H}\,\mathbb{V}(X_{\imath,\tau}X_{\jmath,\tau}) = 0$ for all $\imath,\jmath = 1,2,...,2\ell+1$ with $\widehat{Q}_X(\imath,\jmath)$ denoting the $(\imath,\jmath)$th element of $\widehat{Q}_X$. Altogether, this implies that $\widehat{Q}_X \stackrel{p}{\longrightarrow} Q_X$, so that $\widehat{Q}_X^{-1} \stackrel{p}{\longrightarrow} Q_X^{-1}$. Assumption COV ensures that ζ_τ is a vector of martingale difference sequences such that $\frac{1}{H}\sum_{|\tau-nd|<H}\left[K\left(\frac{nd-\tau}{H}\right)\right]^2 \zeta_\tau\zeta_\tau' \stackrel{p}{\longrightarrow} c_K\, Q_\zeta$. It then follows from a standard central limit theorem for martingale arrays and Davidson's (1994) Theorem 19.2 that $\sqrt{H}(\widehat{b}^{(d)} - b^{(d)}) \stackrel{d}{\longrightarrow} \mathcal{N}\left(0, c_K\,[I_2\otimes Q_X^{-1}]\,Q_\zeta\,[I_2\otimes Q_X^{-1}]\right)$. Asymptotic normality of the CS estimates readily ensues from a straightforward delta method application, completing the proof.■

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Table 1: Relative performance of the daily component share estimators

We document the bias of the KLS and LS estimators of $\alpha^{(d)}_{\delta,\perp,1}$ as well as their relative root mean squared error (RRMSE) for sample sizes of $D = 500$ days over 1,000 replications. RRMSE figures below unit imply better performance of the KLS estimator. The instantaneous correlation between markets ρ ranges from 0 to 0.90, whereas the sampling frequency ranges from one observation per 30 seconds to one observation per 5 minutes: $\delta \in \{1/780, 1/390, 1/195, 1/130, 1/78\}$. We compute the KLS estimator using a bandwidth $H = n^{b/10}\sqrt{D}$, with $b \in \{8, 9, 10\}$.

		$100 \times \text{bias}(\widehat{\alpha}^{(d)}_{\delta,\perp,1})$					$\text{RRMSE}(\widehat{\alpha}^{(d)}_{\delta,\perp,1})$				
	ρ	$\delta = 1/780$	$\delta = 1/390$	$\delta = 1/195$	$\delta = 1/130$	$\delta = 1/78$	$\delta = 1/780$	$\delta = 1/390$	$\delta = 1/195$	$\delta = 1/130$	$\delta = 1/78$
least squares	0.00	-0.01	-0.03	0.06	0.01	0.12					
	0.30	-0.25	-0.05	-0.02	0.06	0.05					
	0.50	0.03	-0.04	0.04	0.02	0.06					
	0.70	-0.03	-0.08	0.06	-0.04	0.01					
	0.90	-0.05	-0.32	0.13	0.14	0.17					
$H = n^{8/10}\sqrt{D}$	0.00	-0.02	-0.01	0.00	0.00	0.03	0.68	0.56	0.42	0.37	0.29
	0.30	-0.02	-0.01	-0.01	0.04	0.05	0.40	0.45	0.36	0.33	0.27
	0.50	-0.03	-0.02	0.02	0.02	0.06	0.45	0.40	0.33	0.31	0.26
	0.70	-0.04	-0.03	0.01	-0.02	0.00	0.42	0.35	0.30	0.28	0.25
	0.90	-0.07	-0.03	0.08	0.09	0.16	0.34	0.27	0.26	0.26	0.23
$H = n^{9/10}\sqrt{D}$	0.00	-0.02	-0.01	0.00	0.00	0.03	0.82	0.63	0.44	0.37	0.27
	0.30	-0.03	-0.02	-0.02	0.02	0.05	0.48	0.49	0.36	0.31	0.25
	0.50	-0.03	-0.01	0.02	0.03	0.07	0.50	0.42	0.32	0.28	0.23
	0.70	-0.05	-0.03	0.00	-0.01	-0.03	0.43	0.34	0.28	0.25	0.21
	0.90	-0.07	-0.03	0.06	0.08	0.13	0.30	0.24	0.22	0.21	0.19
$H = n\sqrt{D}$	0.00	-0.03	-0.02	-0.03	0.00	0.01	1.02	0.74	0.49	0.39	0.27
	0.30	-0.03	-0.03	-0.02	-0.02	-0.02	0.57	0.59	0.40	0.32	0.23
	0.50	-0.04	-0.02	0.00	0.00	0.02	0.60	0.48	0.34	0.28	0.21
	0.70	-0.05	-0.04	-0.01	-0.02	-0.03	0.50	0.37	0.28	0.23	0.19
	0.90	-0.08	-0.04	0.05	0.05	0.08	0.31	0.23	0.19	0.18	0.16

Table 2: Relative performance of the daily component share estimators with stochastic covariance matrix

We document the bias of the KLS and LS estimators of $\alpha^{(d)}_{\delta,\perp,1}$ as well as their relative root mean squared error (RRMSE) for sample sizes of $D = 500$ days over 1,000 replications. RRMSE figures below unit imply better performance of the KLS estimator. The instantaneous correlation between markets ρ ranges from 0 to 0.90, whereas the sampling frequency ranges from one observation per 30 seconds to one observation per 5 minutes: $\delta \in \{1/780, 1/390, 1/195, 1/130, 1/78\}$. We compute the KLS estimator using a bandwidth $H = n^{b/10}\sqrt{D}$, with $b \in \{8, 9, 10\}$.

		$100 \times \text{bias}(\widehat{\alpha}^{(d)}_{\delta,\perp,1})$					$\text{RRMSE}(\widehat{\alpha}^{(d)}_{\delta,\perp,1})$				
	ρ	$\delta = 1/780$	$\delta = 1/390$	$\delta = 1/195$	$\delta = 1/130$	$\delta = 1/78$	$\delta = 1/780$	$\delta = 1/390$	$\delta = 1/195$	$\delta = 1/130$	$\delta = 1/78$
least squares	0.00	0.19	0.07	0.04	0.02	0.19					
	0.30	0.00	-0.07	-0.02	-0.08	-0.03					
	0.50	0.06	0.03	-0.12	0.04	0.02					
	0.70	-0.02	0.09	0.10	0.08	0.11					
	0.90	0.03	0.19	0.22	-0.22	-0.01					
$H = n^{8/10}\sqrt{D}$	0.00	0.05	0.01	0.00	0.01	0.05	0.76	0.64	0.50	0.42	0.33
	0.30	-0.01	-0.01	-0.01	-0.01	-0.04	0.62	0.53	0.42	0.36	0.29
	0.50	0.04	0.05	0.01	0.06	0.04	0.54	0.46	0.38	0.33	0.28
	0.70	0.01	0.03	0.08	0.03	0.07	0.45	0.39	0.33	0.30	0.26
	0.90	0.00	0.01	0.12	-0.04	-0.11	0.36	0.33	0.29	0.27	0.25
$H = n^{9/10}\sqrt{D}$	0.00	0.00	0.01	0.00	0.01	0.05	0.94	0.76	0.55	0.44	0.33
	0.30	-0.01	-0.01	-0.01	-0.01	-0.03	0.75	0.61	0.45	0.36	0.28
	0.50	0.05	0.05	0.03	0.07	0.06	0.62	0.51	0.38	0.32	0.26
	0.70	0.01	0.03	0.09	0.04	0.11	0.48	0.40	0.32	0.28	0.23
	0.90	-0.02	0.00	0.06	-0.07	-0.20	0.33	0.29	0.25	0.23	0.21
$H = n\sqrt{D}$	0.00	0.03	0.04	0.04	0.06	-0.01	1.22	0.95	0.65	0.50	0.35
	0.30	-0.02	-0.01	-0.01	0.00	-0.03	0.95	0.74	0.52	0.40	0.29
	0.50	0.02	0.00	0.00	0.04	0.06	0.77	0.60	0.43	0.34	0.25
	0.70	-0.02	0.02	0.02	-0.03	-0.09	0.57	0.46	0.33	0.28	0.22
	0.90	-0.06	-0.08	-0.12	-0.04	-0.09	0.36	0.30	0.24	0.21	0.18

Table 3: Data description

We report summary statistics for raw and cleaned data for Nasdaq and NYSE/Arca. For the Nasdaq-listed stocks in the top panel, we report summary statistics of Nasdaq and Arca, whereas we consider Nasdaq and NYSE for the actively- and less-actively-traded stocks listed at NYSE respectively in the middle and bottom panels. In particular, we display the number of quotes (in millions) for each stock on the two trading venues before any cleaning filter (raw data) as well as after the implementation of the cleaning procedure (clean data). We also report the daily average number of quotes (in thousands) for both trading venues, and the total number of days we have for each stock in the sample period (January 2007 to December 2013).

	raw ('000,000)		clean ('000,000)		obs per day ('000)		number of days
	Nasdaq	NYSE/Arca	Nasdaq	NYSE/Arca	Nasdaq	NYSE/Arca	
ADBE	37	19	9	7	5.20	3.83	1,734
ALGN	188	136	23	22	13.09	12.56	1,734
AMZN	202	80	21	17	12.37	9.57	1,734
CA	176	76	19	16	11.19	8.98	1,734
EXPE	604	212	32	27	18.74	15.81	1,734
GOOG	230	94	22	19	12.90	10.88	1,734
MU	218	140	20	19	11.46	10.93	1,734
SBUX	50	28	11	9	6.39	5.44	1,734
VOD	524	204	31	27	18.05	15.60	1,734
WEN	262	114	23	20	13.44	11.65	1,734
BAC	523	503	31	34	17.85	19.05	1,734
GE	363	427	29	31	16.53	17.78	1,734
HPQ	326	277	26	28	14.84	15.86	1,734
IBM	122	149	21	25	11.96	14.13	1,734
JCP	175	149	20	22	11.55	12.26	1,734
JPM	696	542	32	33	18.43	18.64	1,734
KO	244	205	23	25	13.27	14.44	1,734
MO	178	204	22	26	12.40	14.90	1,734
VZ	264	257	25	29	14.44	16.19	1,734
XOM	503	417	31	33	18.10	18.84	1,734
CAJ	27	30	8	9	4.58	4.99	1,734
COO	21	29	7	9	4.09	5.42	1,734
DLB	22	37	7	10	4.26	6.05	1,734
DNB	24	30	8	10	4.67	5.49	1,734
OFC	29	42	9	11	5.13	6.36	1,734
RBC	25	30	8	10	4.76	5.67	1,734
RE	21	26	7	8	4.13	4.88	1,734
RES	21	28	7	9	3.82	5.24	1,734
ROL	11	28	5	8	2.74	4.51	1,734
THO	21	29	7	9	4.22	5.19	1,734

Table 4: Elliott-Müller test for daily variation in the component share estimates

The column "EM test" reports the test statistics of the Elliott-Müller test for the null hypothesis of time-invariant VECM parameters. The asymptotic critical values for a bivariate VECM are respectively -17.57, -14.32, and -12.80 at the 1%, 5%, and 10% significance levels, with ***, **, and * denoting the corresponding rejections. The remaining columns report the median and standard deviation (within parentheses) over the entire sample period of the LS and KLS daily estimates of the speed-of-adjustment parameters and of the Nasdaq component share $\alpha^{(d)}_{\delta,\perp,1}$. The subscript 2 refers to Arca for the Nasdq-listed stocks in the top panel, and to NYSE for the actively- and less-actively-traded stocks listed at NYSE in the middle and bottom panels, respectively.

	EM test	least squares			kernel least squares		
		$\alpha^{(d)}_{\delta,1}$	$\alpha^{(d)}_{\delta,2}$	$\alpha^{(d)}_{\delta,\perp,1}$	$\alpha^{(d)}_{\delta,1}$	$\alpha^{(d)}_{\delta,2}$	$\alpha^{(d)}_{\delta,\perp,1}$
ADBE	-30.37***	-0.42 (0.71)	0.49 (0.72)	0.54 (0.83)	-0.13 (0.19)	0.29 (0.30)	0.67 (0.28)
ALGN	-203.75***	-0.21 (0.26)	0.52 (0.27)	0.71 (0.37)	-0.16 (0.10)	0.40 (0.17)	0.69 (0.12)
AMZN	-28.80***	-0.39 (0.64)	0.51 (0.64)	0.57 (0.68)	-0.18 (0.18)	0.24 (0.22)	0.61 (0.19)
CA	-57.48***	-0.41 (0.65)	0.47 (0.66)	0.53 (0.82)	-0.16 (0.16)	0.27 (0.23)	0.64 (0.18)
EXPE	-15.74**	-0.35 (0.65)	0.52 (0.66)	0.59 (0.71)	-0.22 (0.19)	0.39 (0.24)	0.63 (0.30)
GOOG	-21.21***	-0.39 (0.39)	0.49 (0.36)	0.56 (0.40)	-0.26 (0.21)	0.34 (0.18)	0.56 (0.25)
MU	-7.69	-0.54 (0.81)	0.34 (0.80)	0.39 (0.93)	-0.18 (0.23)	0.11 (0.19)	0.57 (0.25)
SBUX	-58.77***	-0.40 (0.72)	0.53 (0.72)	0.57 (0.76)	-0.14 (0.26)	0.22 (0.28)	0.62 (0.38)
VOD	-30.61***	-0.54 (0.54)	0.34 (0.54)	0.39 (1.09)	-0.08 (0.33)	0.08 (0.31)	0.44 (0.52)
WEN	-35.01***	-0.44 (0.56)	0.31 (0.56)	0.43 (0.70)	-0.28 (0.24)	0.20 (0.16)	0.45 (0.30)
BAC	-18.67***	-0.59 (0.74)	0.36 (0.74)	0.38 (0.76)	-0.36 (0.29)	0.19 (0.23)	0.47 (0.30)
GE	-28.21***	-0.61 (0.72)	0.33 (0.72)	0.35 (0.77)	-0.29 (0.25)	0.20 (0.16)	0.45 (0.24)
HPQ	-56.10***	-0.48 (0.80)	0.46 (0.80)	0.48 (0.89)	-0.19 (0.35)	0.24 (0.37)	0.49 (0.45)
IBM	-21.94***	-0.55 (0.44)	0.35 (0.44)	0.39 (0.50)	-0.36 (0.22)	0.23 (0.17)	0.39 (0.19)
JCP	-57.18***	-0.53 (0.66)	0.36 (0.65)	0.39 (0.72)	-0.35 (0.24)	0.19 (0.16)	0.37 (0.24)
JPM	-3.45	-0.42 (0.90)	0.52 (0.90)	0.55 (0.93)	-0.26 (0.30)	0.27 (0.33)	0.54 (0.31)
KO	-14.74**	-0.52 (0.70)	0.39 (0.70)	0.43 (1.12)	-0.26 (0.26)	0.12 (0.17)	0.39 (0.19)
MO	-42.00***	-0.55 (0.73)	0.38 (0.73)	0.40 (0.89)	-0.27 (0.45)	0.12 (0.35)	0.38 (0.40)
VZ	-18.63***	-0.52 (0.74)	0.41 (0.73)	0.44 (1.23)	-0.22 (0.27)	0.17 (0.22)	0.44 (0.28)
XOM	-3.51	-0.40 (0.73)	0.54 (0.74)	0.57 (0.99)	-0.20 (0.26)	0.27 (0.32)	0.56 (0.28)
CAJ	-287.48***	-0.43 (0.22)	0.21 (0.19)	0.32 (0.28)	-0.42 (0.14)	0.15 (0.11)	0.30 (0.13)
COO	-132.32***	-0.50 (0.24)	0.18 (0.20)	0.26 (0.86)	-0.39 (0.19)	0.08 (0.10)	0.19 (0.13)
DLB	-143.27***	-0.51 (0.23)	0.17 (0.22)	0.23 (0.64)	-0.46 (0.21)	0.09 (0.11)	0.24 (0.18)
DNB	-55.93***	-0.49 (0.23)	0.21 (0.20)	0.29 (0.44)	-0.43 (0.21)	0.10 (0.08)	0.24 (0.23)
OFC	-469.83***	-0.56 (0.23)	0.18 (0.21)	0.24 (1.15)	-0.45 (0.21)	0.12 (0.08)	0.21 (0.11)
RBC	-1,790.14***	-0.47 (0.20)	0.17 (0.20)	0.25 (0.34)	-0.43 (0.15)	0.11 (0.12)	0.23 (0.16)
RE	-37.61***	-0.49 (0.22)	0.20 (0.20)	0.29 (0.33)	-0.46 (0.22)	0.09 (0.13)	0.23 (0.14)
RES	-68.25***	-0.49 (0.21)	0.18 (0.20)	0.26 (0.35)	-0.49 (0.20)	0.11 (0.11)	0.25 (0.17)
ROL	-201.00***	-0.46 (0.20)	0.12 (0.16)	0.20 (0.85)	-0.45 (0.19)	0.07 (0.08)	0.16 (0.12)
THO	-134.02***	-0.52 (0.22)	0.19 (0.19)	0.26 (0.53)	-0.51 (0.21)	0.12 (0.08)	0.25 (0.10)

Table 5: Long-run relationship between price discovery and liquidity measures

We report the results of a VECM(8) for the daily price discovery measure and relative volume at the listing venue. We report the cointegrating vector estimates and their standard errors (note that we omit the one related to the price discovery measure as we force its loading to one) as well as the corresponding speeds of adjustment for the price discovery measure and for the relative volume. Finally, the last column displays the adjusted R^2 to gauge how much the VECM explains of the overall variation of the daily changes in the price discovery measure.

	cointegrating vector		speed of adjustment		R^2
	intercept	relative volume	component share	relative volume	
ADBE		−0.2107 (0.0253)	−0.0020 (0.0004)	0.0443 (0.0471)	0.8339
ALGN		−0.3196 (0.0269)	−0.0015 (0.0004)	0.1971 (0.0505)	0.8321
AMZN		−0.5301 (0.0488)	−0.0014 (0.0004)	0.0765 (0.0274)	0.8530
CA		−0.5557 (0.0352)	−0.0011 (0.0003)	0.1430 (0.0342)	0.8448
EXPE		−0.2456 (0.0263)	−0.0024 (0.0005)	0.1232 (0.0477)	0.6496
GOOG		−0.4878 (0.0354)	−0.0048 (0.0013)	0.1264 (0.0328)	0.4437
MU		−0.7771 (0.0713)	−0.0014 (0.0004)	0.0552 (0.0203)	0.8273
SBUX		−0.4962 (0.0274)	−0.0009 (0.0003)	0.1644 (0.0388)	0.8770
VOD		−0.5485 (0.1147)	−0.0008 (0.0003)	0.0797 (0.0257)	0.8638
WEN		−0.7331 (0.1962)	−0.0004 (0.0002)	0.0611 (0.0214)	0.6495
BAC	−0.0749 (0.1732)	−1.6909 (0.3827)	-7.33×10^{-5} (0.0002)	0.0417 (0.0097)	0.4772
GE		−1.0747 (0.0825)	−0.0011 (0.0006)	0.1242 (0.0228)	0.4387
HPQ		−1.9721 (0.2823)	-7.16×10^{-5} (0.0003)	0.0482 (0.0102)	0.2347
IBM		−0.5962 (0.0617)	−0.0049 (0.0014)	0.1211 (0.0282)	0.2520
JCP		−1.8361 (0.2824)	-5.33×10^{-5} (0.0005)	0.0403 (0.0091)	0.2648
JPM	−0.3339 (0.0926)	−1.1122 (0.2241)	−0.0010 (0.0004)	0.0678 (0.0140)	0.3263
KO		−0.6748 (0.0501)	−0.0054 (0.0013)	0.1499 (0.0323)	0.2405
MO		−0.4463 (0.0552)	−0.0040 (0.0014)	0.1324 (0.0362)	0.2416
VZ		−0.9991 (0.0864)	−0.0032 (0.0012)	0.1224 (0.0238)	0.1291
XOM		−1.0343 (0.0897)	−0.0011 (0.0007)	0.1150 (0.0214)	0.3262
CAJ	1.8425 (0.3354)	−1.9411 (0.2674)	−0.0013 (0.0010)	0.1048 (0.0163)	0.2618
COO		−0.8027 (0.0782)	−0.0029 (0.0015)	0.1240 (0.0272)	0.2571
DLB		−0.9201 (0.0507)	−0.0022 (0.0013)	0.1812 (0.0284)	0.4880
DNB		−0.9477 (0.0768)	−0.0032 (0.0015)	0.1193 (0.0229)	0.4829
OFC		−0.5100 (0.0221)	−0.0043 (0.0015)	0.2536 (0.0477)	0.2745
RBC	0.4429 (0.1501)	−1.1042 (0.1401)	−0.0012 (0.0008)	0.1637 (0.0256)	0.2699
RE	0.4619 (0.1594)	−1.1997 (0.1619)	−0.0012 (0.0012)	0.1379 (0.0223)	0.2810
RES		−0.8941 (0.0446)	−0.0009 (0.0008)	0.2108 (0.0320)	0.2652
ROL		−0.4790 (0.0343)	−0.0012 (0.0008)	0.1832 (0.0404)	0.2576
THO		−0.6285 (0.0357)	−0.0032 (0.0011)	0.2305 (0.0395)	0.2421

Figure 1: Information share and contemporaneous correlation as sampling frequency decreases

The first plot displays the information and component shares of market 1 in continuous time and for δ ranging from $1/23,400$ (frequency of one observation per second) to $1/13$ (frequency of one observation per 30 minutes). The second plot depicts the exact correlation across markets at each sampling frequency. We consider a range of values for the correlation across markets in continuous time, and then compute their discrete-time counterparts using the exact discretization of the reduced-rank OU process as in (3), with $\alpha = (0, 522)'$ so that $\alpha_{1/390} = (0, 0.74)'$ at the 1-minute frequency.

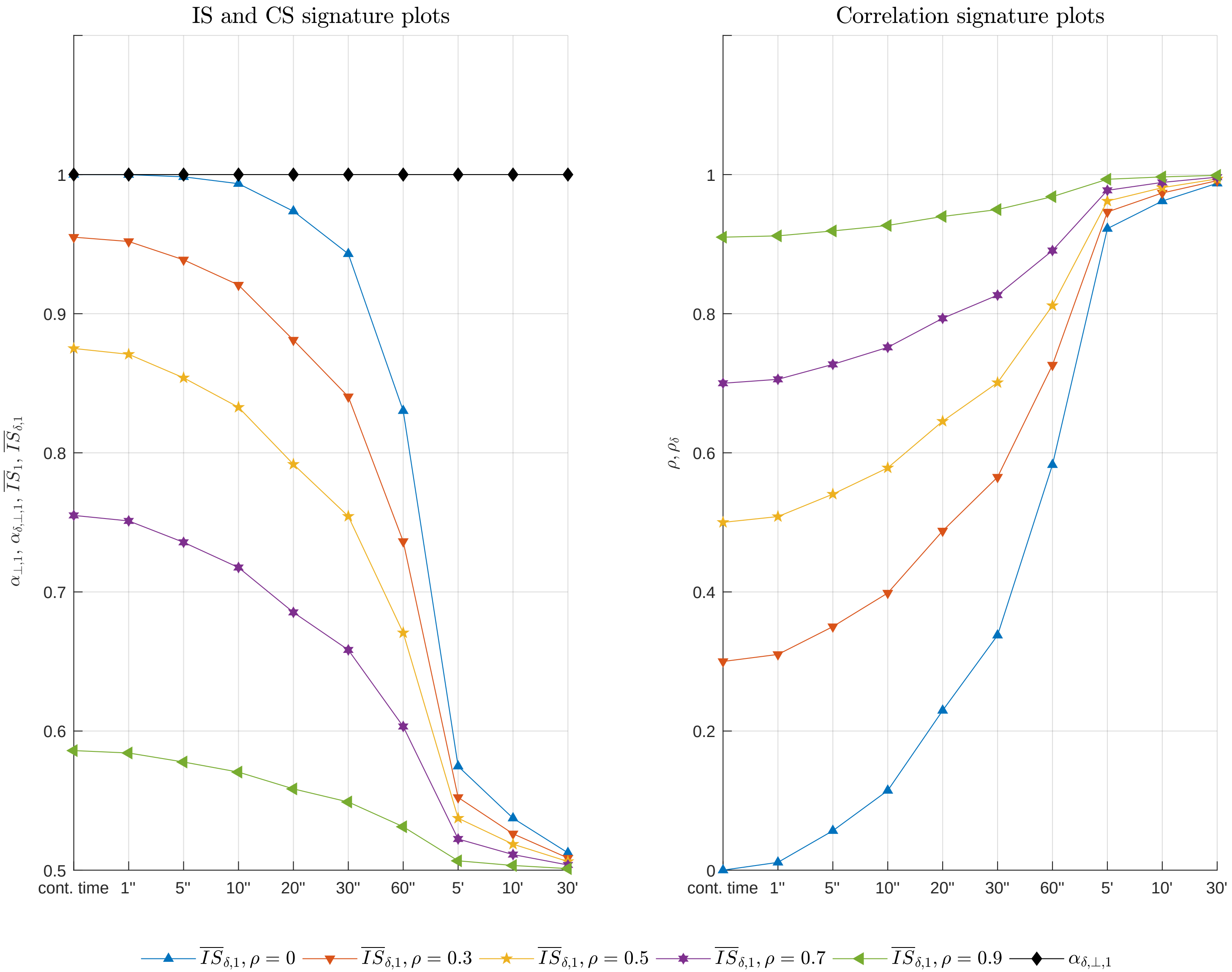


Figure 2: Information share and contemporaneous correlation as sampling frequency decreases with stochastic covariance matrix

The first plot in the upper panel displays the theoretical information share based on the exact discretization of (1) and the box plots of the estimates of the IS measures at the different frequencies based on the simulations of (14). The second plot on the upper panel portrays the exact correlation across markets in continuous time and the box plots of the corresponding estimates at the different sampling frequencies. Finally, the bottom panel plots depict the theoretical continuous-time CS measures (solid markers in black) and the box plot of the estimates obtained at the different sampling frequencies from the simulations. As for the continuous-time parameters, we set $\alpha = (0, 522)'$ so that $\alpha_{1/390} = (0, 0.74)'$ at the 1-minute frequency. The edges of boxes in the box plots refer to the 25% and 75% percentiles, whereas the black dot inside a white circle represents the median.

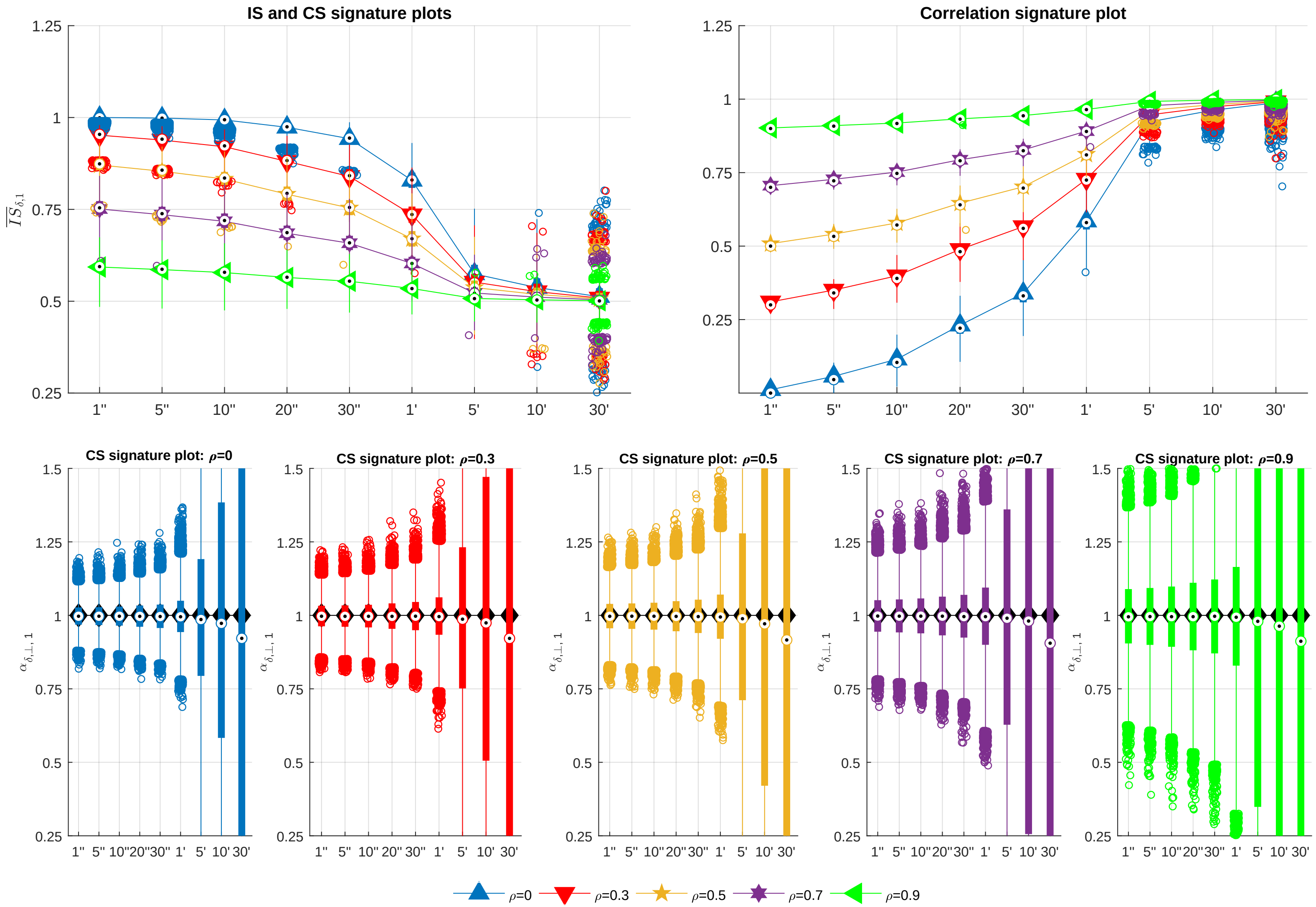


Figure 3: KLS estimates of the daily component shares for the Nasdaq-listed stocks

The plots depict the KLS estimates of $\alpha_{\perp}^{(d)}$ with their 95% confidence bands (in shades) for the 10 actively-traded Nasdaq listed stocks. We fix the bandwidth at $n\sqrt{D}$, where n is the number of intraday observations (average of 390 observations per day) and $D = 1,735$ is the number of trading days. We normalize the orthogonal complements such that the elementwise estimates sum up to one, i.e., $\widehat{\alpha}_{1,\perp}^{(d)} + \widehat{\alpha}_{2,\perp}^{(d)} = 1$, with subscripts 1 and 2 denoting Nasdaq and Arca, respectively.

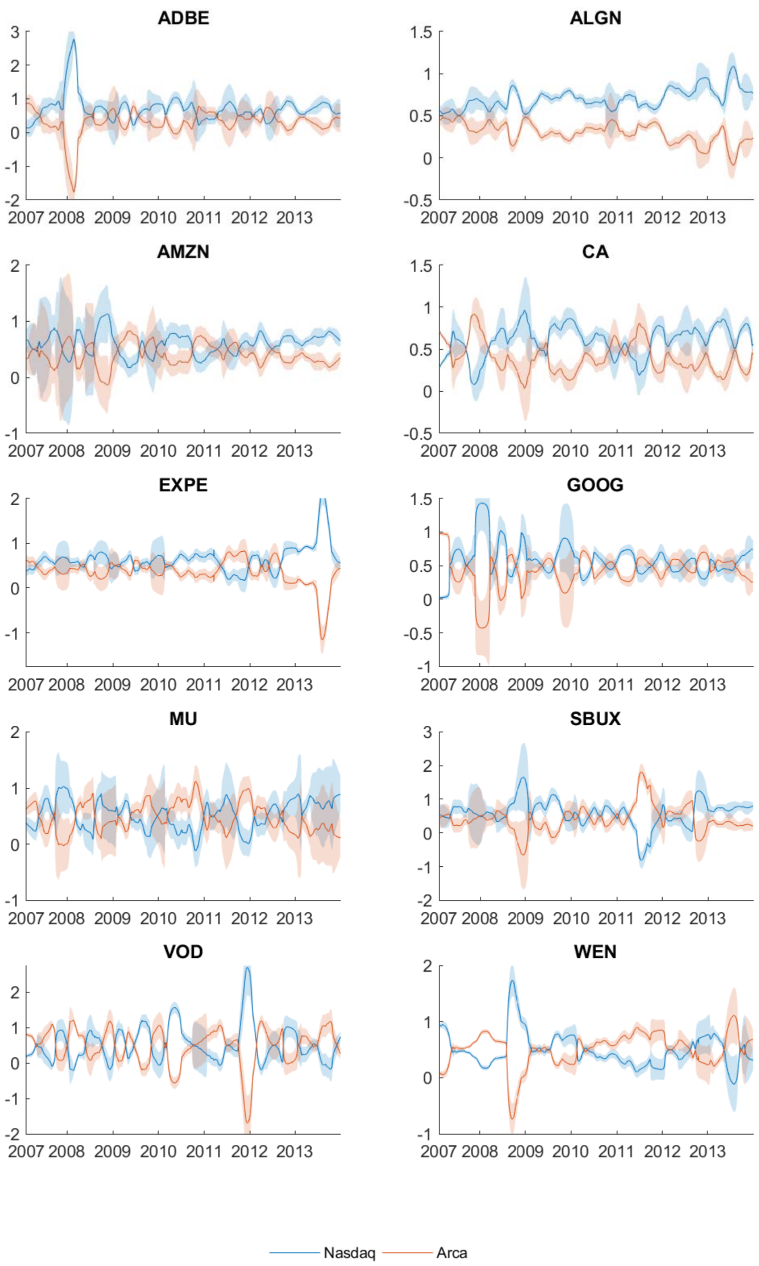


Figure 4: KLS estimates of the daily component shares for the actively-traded NYSE-listed stocks

The plots depict the KLS estimates of $\alpha_{\perp}^{(d)}$ with their 95% confidence bands (in shades) for the 10 actively-traded NYSE-listed stocks. We fix the bandwidth at $n\sqrt{D}$, where n is the number of intraday observations (average of 390 observations per day) and $D = 1,735$ is the number of trading days. We normalize the orthogonal complements such that the elementwise estimates sum up to one, i.e., $\widehat{\alpha}_{1,\perp}^{(d)} + \widehat{\alpha}_{2,\perp}^{(d)} = 1$, with subscripts 1 and 2 denoting Nasdaq and NYSE, respectively.

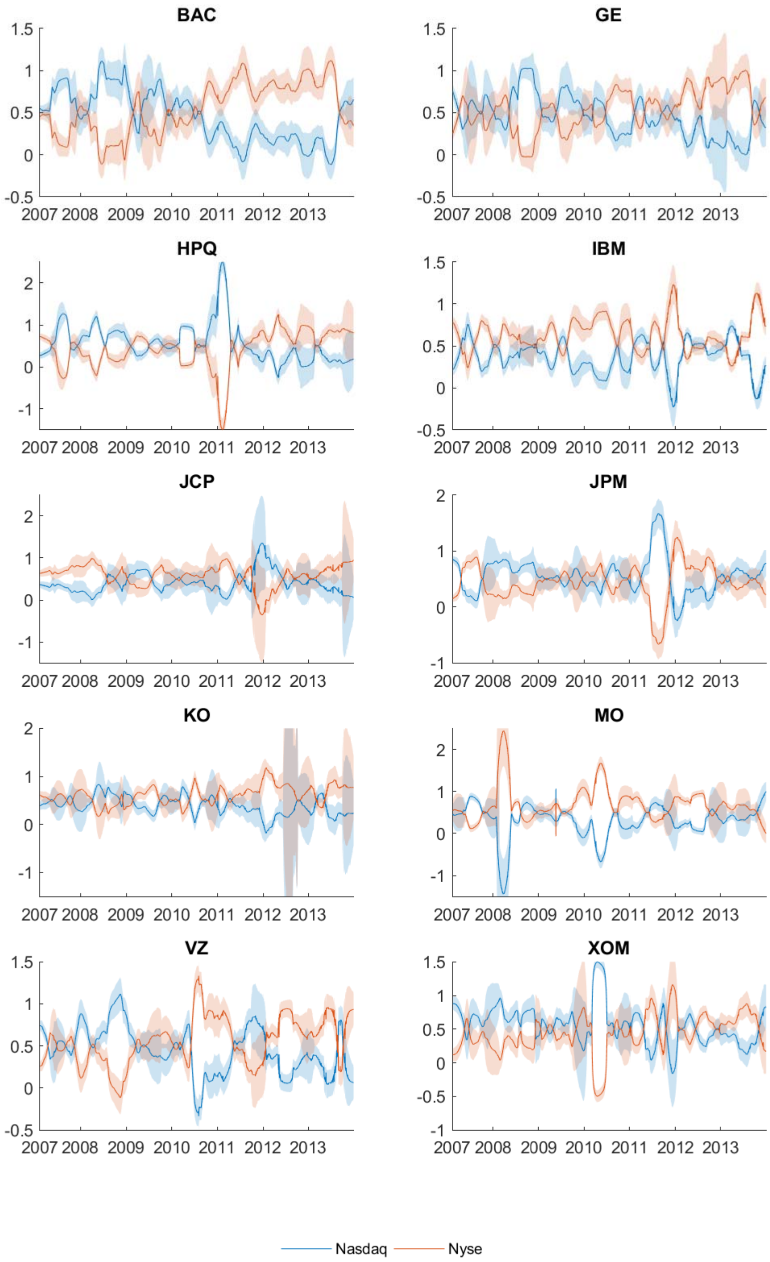


Figure 5: KLS estimates of the daily component shares for the less-liquid NYSE-listed stocks

The plots depict the KLS estimates of $\alpha_{\perp}^{(d)}$ with their 95% confidence bands (in shades) for the 10 less-actively-traded NYSE-listed stocks. We fix the bandwidth at $n\sqrt{D}$, where n is the number of intraday observations (average of 390 observations per day) and $D = 1,735$ is the number of trading days. We normalize the orthogonal complements such that the elementwise estimates sum up to one, i.e., $\widehat{\alpha}_{1,\perp}^{(d)} + \widehat{\alpha}_{2,\perp}^{(d)} = 1$, with subscripts 1 and 2 denoting Nasdaq and NYSE, respectively.

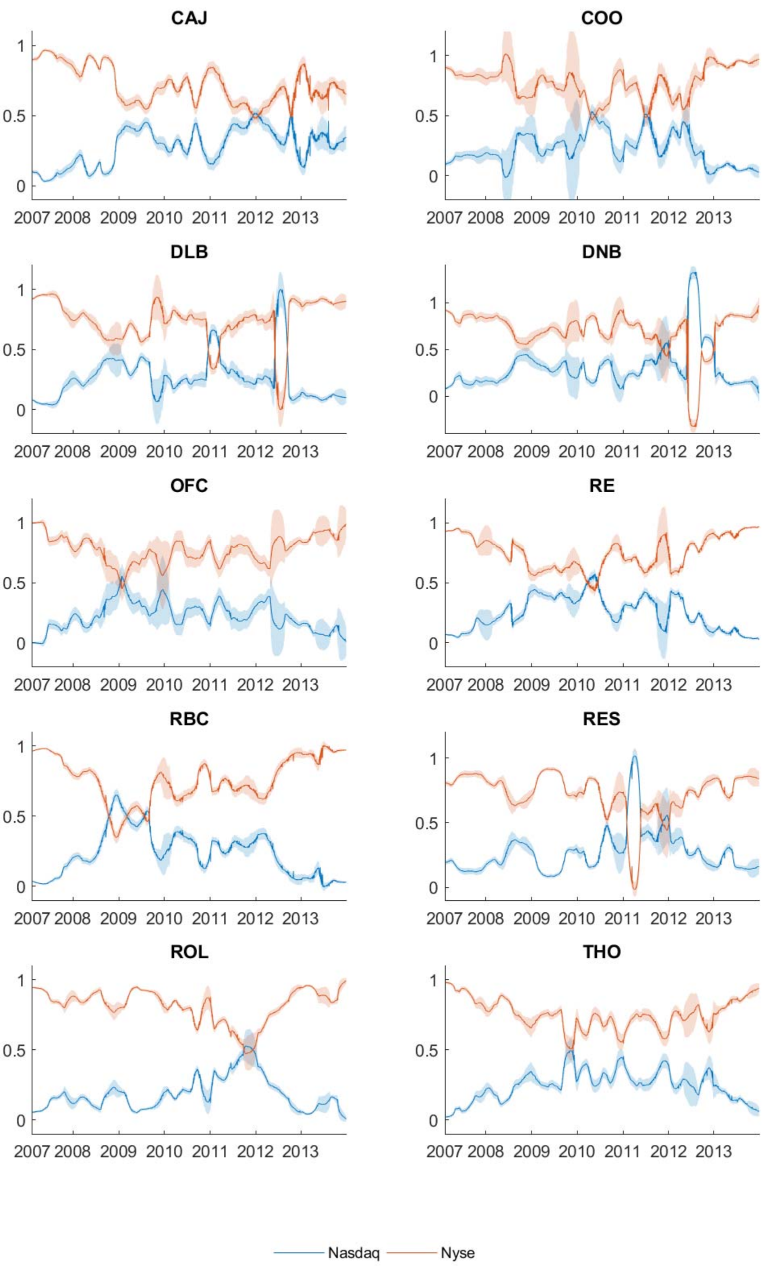


Figure 6: Relative trading volume

The plots portray the volume shares of Nasdaq and NYSE/Arca for each stock. For example, the volume share of Nasdaq for a NYSE-listed stock corresponds to the trading volume at Nasdaq over the sum of the trading volume at Nasdaq and NYSE. Recall that, for Nasdaq-listed stocks, we report volume shares of Nasdaq and Arca.

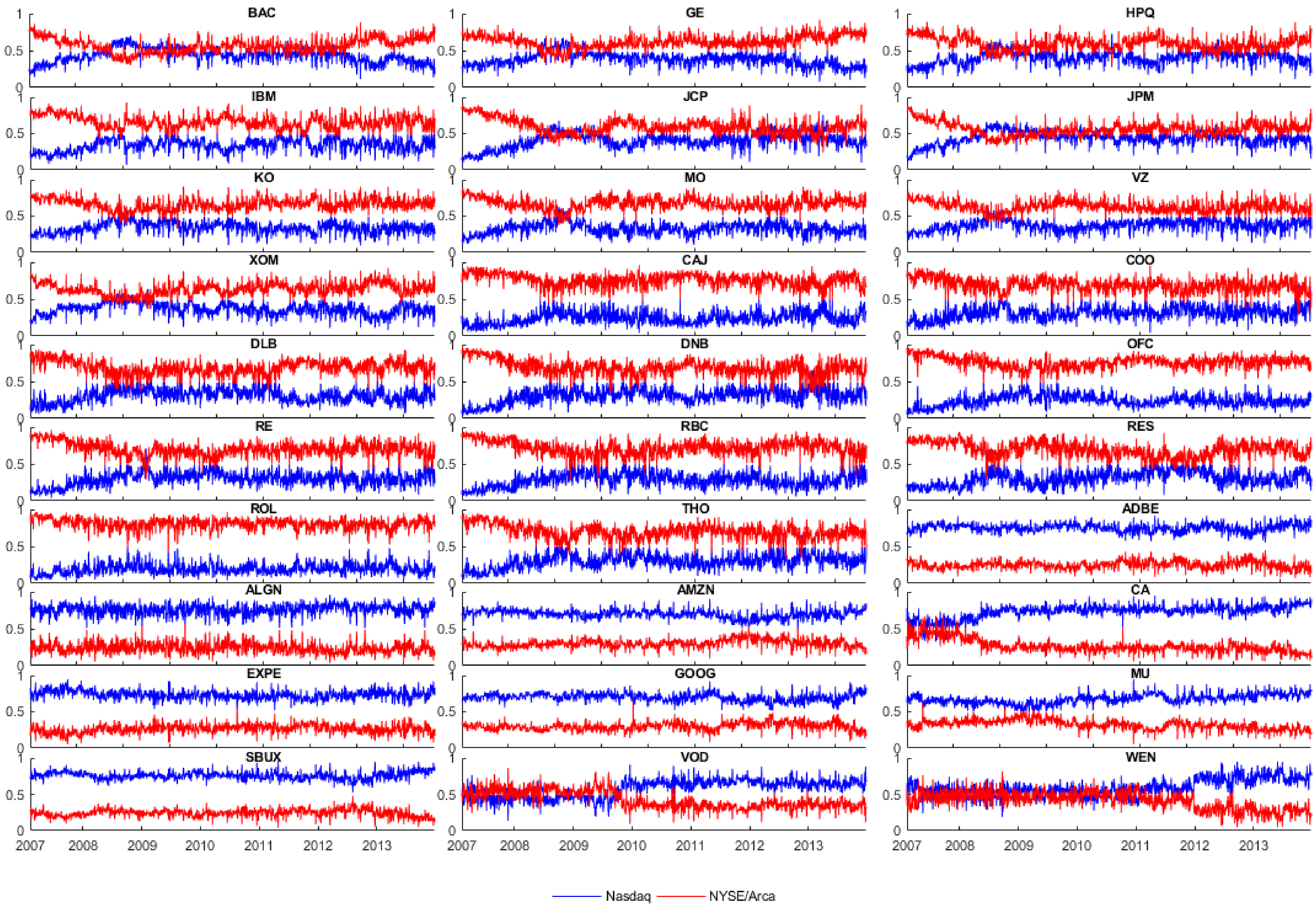
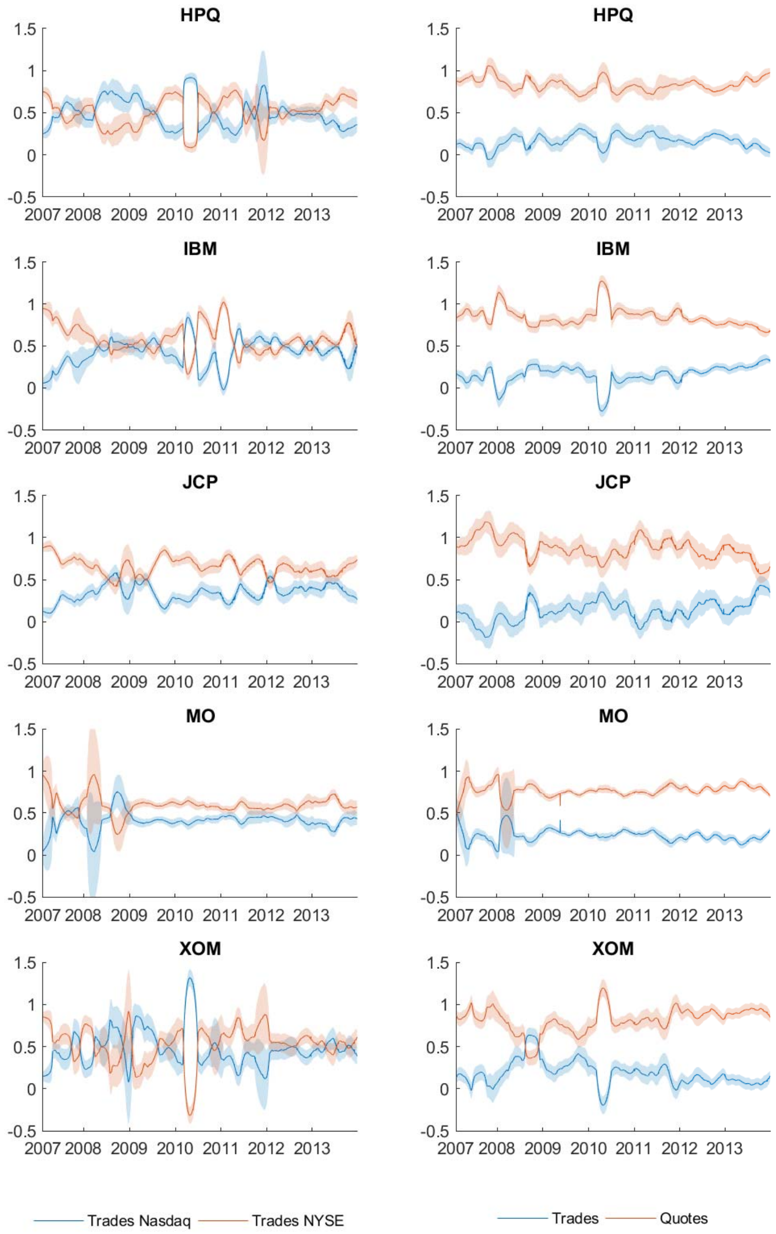


Figure 7: Trades and Quotes: KLS estimates of the daily component shares, with 95% confidence intervals

The plots portray the KLS daily estimates of $\alpha_{\perp}^{(d)}$ with their 95% confidence bands (in shades) for 5 actively-traded NYSE-listed stocks. We fix the bandwidth at $n\sqrt{D}$, where n is the number of intraday observations (average of 390 observations per day) and $D = 1,735$ is the number of trading days. We normalize the orthogonal complements such that the element wise estimates sum up to one, i.e., $\widehat{\alpha}_{1,\perp}^{(d)} + \widehat{\alpha}_{2,\perp}^{(d)} = 1$, with subscripts 1 and 2 denoting Nasdaq and NYSE, respectively. The first column presents the estimates using transactions prices, whereas the second column displays the estimates considering both quotes and trades at NYSE.



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