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Assessing Gamma kernels and BSS/LSS processes

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Gamma kernels and BSS/LSS processes

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Abstract

This paper reviews the roles of gamma type kernels in the theory and modelling for Brownian and Lévy semistationary processes. Applications to financial econometrics and the physics of turbulence are pointed out.

Keywords: Ambit Stochastics; autocorrelation functions; Brownian semistationary processes; financial econometrics; fractional differentiation; identification; Levy semistationary processes; path properties; turbulence modelling; volatility/intermittency.

JEL: $C1$, $C5$.

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1 Introduction

The use of gamma kernels in modelling Brownian and Lévy semistationary (BSS and LSS) processes, at first introduced as a simple convenient choice, has turned out to be of a more significant nature than first envisaged. This paper reviews the roles the kernels have had in the study of these and related types of processes and their applications.[1]

BSS and LSS processes are prominent examples of the types of continuous time stationary processes on $\mathbb{R}$ studied in Ambit Stochastics, a concept introduced in [6]. Two main areas of applications of such processes are financial econometrics and the physics of turbulence, cf. for instance [21] respectively [43].

In its full generality Ambit Stochastics is a framework for modelling tempo-spatial dynamic fields. A main point of Ambit Stochastics is that it specifically incorporates terms modelling stochastic volatility. This is true in particular of BSS and LSS processes.

The papers [8], [49] and [11] review recent developments in the theory and applications of Ambit Stochastics.

Section 2 recalls the definitions of BSS and LSS processes and presents some instances of the role of the gamma kernels, including an illustration of the modelling capability. Section 3 points out that the gamma kernel has an interpretation as a Green's function corresponding to a certain fractional differential operator.

The asymptotic behaviour of the autocorrelation functions of BSS and LSS processes is of crucial importance for their applications, not least in regard to the modelling of turbulence, and this is reviewed in Section 4 under the gamma kernel assumption.

An outstanding issue is the establishment of an Ito type stochastic calculus for BSS and LSS processes; the point here is that these types of processes are not in general semimartingales. An important step in

[1] Proofs and technical details are, in most cases, not presented here. For these and related literature we refer to the papers cited.

this direction has been a detailed study of the path properties of such processes, as discussed in Section 5.

The questions of what can be deduced about the ingredients of a *BSS* or *LSS* process based on knowledge of its law and/or from high frequency observations of its sample path form the topic of Section 6. This involves both purely theoretical reasoning and central questions of inference.

2 *BSS* and *LSS* processes

The concept of Brownian semistationary processes, or *BSS* processes, was introduced in [16], cf. also [14], [15]. Such a process is of the form

$$Y_t = \mu + \int_{-\infty}^{t} g(t-s)\sigma_s \mathrm{d}B_s + \int_{-\infty}^{t} q(t-s)a_s \mathrm{d}s \tag{1}$$

where B is Brownian motion, σ and a are stochastic processes and g and q are deterministic kernels with $g(t) = h(t) = 0$ for $t \leq 0$. The process Y is stationary provided σ and a are stationary, as we shall henceforth assume. The intended role of the processes σ and a is to model volatility, or intermittency as it is called in turbulence.

For simplicity we assume from now on that σ and a are independent of the Brownian motion B. We note however that a very general treatment of stochastic integration theory for Ambit Stochastics is given in [28].

The specification (1) is a particular case of the general concept of *LSS* (Lévy semistationary) processes defined as

$$Y_t = \mu + \int_{-\infty}^{t} g(t-s)\sigma_s \mathrm{d}L_s + \int_{-\infty}^{t} q(t-s)a_s \mathrm{d}s. \tag{2}$$

where L denotes an arbitrary Lévy process on $\mathbb{R}$. This concept was introduced in [7]) and has been further studied for instance in [54], [21], [49], [48], and references given there. One of the roles of processes of type *LSS* is to model volatility/intermittency. General conditions for existence of the stochastic integrals in (1) and (2) are given in [18].

We refer to

$$G_t = \int_{-\infty}^{t} g(t-s)\,\mathrm{d}B_s \tag{3}$$

as the Gaussian base process.

The case where $a = \sigma^2$, i.e.

$$Y_t = \mu + \int_{-\infty}^{t} g(t-s)\sigma_s \mathrm{d}B_s + \int_{-\infty}^{t} q(t-s)\sigma_s^2 \mathrm{d}s, \tag{4}$$

can be seen as a stationary process analogue of the so-called *BNS* model, discussed extensively in financial econometrics, see for instance [22].

In this paper we discuss cases where g, and possibly also q, is of the gamma type

$$g(t;\nu,\lambda) = \frac{\lambda^\nu}{\Gamma(\nu)} t^{\nu-1} e^{-\lambda t}. \tag{5}$$

The form of the gamma kernel means that small and large lag behaviour of Y can be controlled separately. The more general form $g(t) = t^{\nu-1} f(t)$ with f continuous and slowly varying at 0 offers the same type of control, and many of the asymptotic results in the literature on *BSS*/*LSS* processes are derived under this latter asumption. However, the gamma kernel has some very particular properties of key relevance.

We note that with g as the gamma kernel the restriction $\nu > \frac{1}{2}$ is needed for the stochastic integral in (1) to be well defined and that (3) constitutes a semimartingale only if $\nu = 1$ or $\nu > \frac{3}{2}$. For $\nu \in \left(\frac{1}{2}, \frac{3}{2}\right)$ the process is Hölder continuous with index less than $\nu - \frac{1}{2}$. These aspects and some of their consequences are discussed in [16], [9], [10] and [19] and will be touched upon later in the paper.

As models for the timewise development of the main component of the velocity vector in a homogeneous turbulent field stochastic processes of *BSS* type have been extensively studied probabilistically and compared to empirical and simulated data, see [14], [15], [34], [43]. Of special interest in the context of turbulence are the cases where the roughness parameter ν of the gamma kernel satisfies either $\frac{1}{2} < \nu < 1$ or $1 < \nu < \frac{3}{2}$. Near 0 the gamma kernel behaves quite differently depending on whether $\frac{1}{2} < \nu < 1$ or $1 < \nu < \frac{3}{2}$, tending respectively to 0 and ∞. The dynamics of the process (1) is significantly different in the two cases and this has strong consequences with respect to path behaviour and to inference on volatility/intermittency, see Sections 5 and 6.

Considering the setting (4) we note that to accomodate the manifest observed skewness in the distribution of velocity differences in turbulence, cf. [3], the conditional mean $\mathrm{E}\{Y_{t+u} - Y_t|\sigma\}$ should be of the same order as the variance $\mathrm{V}\{Y_{t+u} - Y_t|\sigma\}^{\frac{1}{2}}$ for small values of u. This can be achieved by having

$$q^2(u) \sim \int_0^u g^2(s)\,\mathrm{d}s \tag{6}$$

for $u \downarrow 0$. With g as the gamma kernel a natural way of obtaining this is to take $q(t) = g\left(t; \nu - \frac{1}{2}, \lambda\right)$. For $\nu = \frac{5}{6}$ the resulting process may, for suitable choice of the volatility/intermittency process σ^2, be considered as a temporal stochastic model for fully developed turbulence; cf. the following Section 4.

It is natural to extend the concept of *LSS* processes to the specification

$$Y_t = \mu + \int_{-\infty}^t g(t-s)\sigma_s \mathrm{d}L_s^T + \int_{-\infty}^t q(t-s)a_s \mathrm{d}s \tag{7}$$

where T denotes a time change and $L_t^T = L(T(t))$. Here T and σ represent the two different aspects of the volatility: intensity and amplitude. For a discussion of time change in stochastic processes and its role in modelling volatility/intermittency see [17].

Note 1 *Convolution* Let

$$Y_t = \int_{-\infty}^t g(t-s)\sigma_s \mathrm{d}L_s^T$$

and let h be a shift kernel. Then under mild conditions (cf. the Fubini Theorem presented in [2]) we have

$$\int_{-\infty}^t h(t-s)Y_s \mathrm{d}s = \int_{-\infty}^t h * g(t-s)\sigma_s \mathrm{d}L_s^T$$

where $h * g$ is the convolution of h and g. The resulting process is again of *LSS* type, the left hand side constituting a natural operation on *LSS* processes while convolution of kernels, as on the right hand, is a useful way to flexible modelling. As a concrete example, in [43] the convolution of two gamma kernels is used succesfully as description of the spectral density function in well developed turbulence.

The paper [48] discusses the case where

$$Y_t = \int_{-\infty}^t g(t-s;\nu,\lambda)\mathrm{d}L_s \tag{8}$$

in great detail. Necessary and sufficient conditions for the existence of Y are given in terms of ν and the Lévy measure v of L, under the assumption that σ is predictable, strongly stationary and square integrable and satisfies $\mathrm{E}\left\{\sigma_0^{2(1-\nu)}\right\} < \infty$. Provided the Lévy measure has log moment, existence is guaranteed for all $\nu > \frac{1}{2}$ while for $0 < \nu \leq \frac{1}{2}$ an additional condition on the Lévy measure is needed.

Note 2 *Selfdecomposability* A striking example of the special character of the gamma kernel is the fact that whenever the process (8) is well-defined the one-dimensional law of Y is self-decomposable even if the driving Lévy process L does not have that property. In view of how the class of selfdecomposable distributions is defined this is both remarkable and difficult to explain. (The proof given in [48] is by direct analytical derivation and does not throw light on the probabilistic aspect.)

On the other hand, as shown in [13], the process Y as such is selfdecomposable if and only if L is selfdecomposable.

We conclude this Section by an illustration of the flexibility of modelling using gamma kernels. Here, as is often convenient, the volatility/intermittency process σ^2 is taken to be of *LSS* form.

Example 1 *BSS process with GH marginals* A stationary *BSS* processes with generalised hyperbolic marginals was used in [7] in connection with a study on modelling electricity spot prices by Lévy semistationary processes. As an illustration of the versatility of *BSS*/*LSS* processes we here represent the proof of the existence of such a GH related process.

We recall that the GH laws are analytically very tractable and have been found to fit empirical distributions in a wide range of applications (cf. for instance [3], [30], [31], [34]. Also, in a recent extensive development of the Kolmogorov-Obukhov statistical theory of turbulence, Björn Birnir ([23], [24], [25]) has formulated a stochastic version of the Navier-Stokes equations under which the velocity differences follow GH distributions and this theoretical study is backed by a detailed empirical and simulation based analysis showing excellent agreement to the GH form.

The existence of stationary BSS processes having generalised hyperbolic marginals is established on the basis of the form (4) by suitable choice of g and q as gamma kernels and by taking σ^2 as a particular LSS process, specifically as a generalised inverse Gaussian Ornstein-Uhlenbeck process (GIG-OU process).

Note first that, whatever g, q and σ^2, the conditional law of Y_t given σ is normal:

$$Y_t|\sigma \overset{law}{=} N\left(\mu+\beta\int_{-\infty}^{t} q\left(t-s\right)\sigma_s^2\mathrm{d}s, \int_{-\infty}^{t} g^2\left(t-s\right)\sigma_s^2\mathrm{d}s\right).$$

Now suppose that σ^2 follows an LSS process given by

$$\sigma_t^2=\int_{-\infty}^{t} h\left(t-s\right)\mathrm{d}L_s \tag{9}$$

where L is a subordinator. Then, by the stochastic Fubini theorem we find

$$\begin{aligned}\int_{-\infty}^{t} q\left(t-s\right)\sigma_s^2\mathrm{d}s &= \int_{-\infty}^{t}\int_{u}^{t} q\left(t-s\right)h\left(s-u\right)\mathrm{d}s\mathrm{d}L_u \\ &= \int_{-\infty}^{t} k\left(t-u\right)\mathrm{d}L_u\end{aligned}$$

where $k=q*h$, the convolution of q and h. Similarly,

$$\int_{-\infty}^{t} g^2\left(t-s\right)\sigma_s^2\mathrm{d}s=\int_{-\infty}^{t} m\left(t-u\right)\mathrm{d}L_u$$

with $m=g^2*h$.

Next, for $\frac{1}{2}<\nu<1$ define g by

$$g\left(t\right)=\left(\lambda\frac{\Gamma(2\nu-1)}{\Gamma(\nu)^2}\right)^{-\frac{1}{2}} g\left(t;\nu,\frac{\lambda}{2}\right).$$

Then we have

$$g^2(t)=g(t;2\nu-1,\lambda).$$

Hence, if

$$h(t)=g(t;2(1-\nu),\lambda)$$

and if, moreover,

$$q\left(t\right)=g\left(t;2\nu-1,\lambda\right)$$

we obtain

$$k\left(t\right)=m(t)=e^{-\lambda t}.$$

In other words,

$$Y_t|\sigma \overset{law}{=} N\left(\mu+\beta\vartheta_t^2,\vartheta_t^2\right).$$

where

$$\vartheta_t^2=\int_{-\infty}^{t} e^{-\lambda(t-u)}\mathrm{d}L_u. \tag{10}$$

It follows that if the subordinator L is such that ϑ_t^2 has the generalised inverse Gaussian law $GIG(\delta,\gamma)$ then the law of Y_t is the generalised hyperbolic $GH(\alpha,\beta,\mu,\delta)$ (where $\alpha=\sqrt{\beta^2+\gamma^2}$). The existence of such a subordinator follows from a theorem of Jurek and Verwaat, see [38], according to which a random variable X is representable in law on the form

$$X \overset{law}{=} \int_{0}^{\infty} e^{-\lambda t}\mathrm{d}L_t^T \tag{11}$$

if and only if the distribution of X is selfdecomposable; and selfdeconposability of GIG has been established in [33].

3 Gamma kernel as Green's function

For any $\gamma \in (0,1)$ and $n \in \mathbb{N}$ the Caputo fractional derivative $D^{n,\gamma}$ is, in its basic form, defined by

$$D^{n,\gamma} f(x) = \Gamma(1-\gamma)^{-1} \int_c^x (x-\xi)^{-\gamma} f^{(n)}(\xi)\,\mathrm{d}\xi$$

where f denotes any function on the interval $[c,\infty)$ which is n times differentiable there and such that $f^{(n)}$ is absolutely continuous on $[c,\infty)$. This concept was introduced by [27] and has since been much generalised and extensively applied in a great variety of scientific and technical areas. For a comprehensive exposition of this and other concepts of fractional differentiation, see [42], cf. also [44], [1] and [45].

For functions f on $\mathbb{R}$ let M_λ with $\lambda \geq 0$ be the operator $M_\lambda f(x) = e^{\lambda x} f(x)$ and, for $0 < \gamma < 1$ and $c \in \mathbb{R}$, define the operator $\mathbb{D}_\lambda^{n,\gamma}$ by

$$\mathbb{D}_\lambda^{n,\gamma} f(x) = M_\lambda^{-1} D D^{n,\gamma} M_\lambda f(x)$$

where D indicates ordinary differentiation and $D^{n,\gamma}$ is the Caputo fractional derivative.[2]

Now, suppose that $1 < \nu < \frac{3}{2}$ and consider the equation

$$\mathbb{D}_\lambda^{1,\nu-1} f(x) = \phi(x) \tag{12}$$

where ϕ is assumed known. We seek the solution f to this equation, stipulating that $f(c)$ should be equal to 0, and it turns out to be

$$f(x) = \Gamma(\nu)^{-1} \int_c^x (x-\xi)^{\nu-1} e^{-\lambda(x-\xi)} \phi(\xi)\mathrm{d}\xi. \tag{13}$$

In other words,

$$g(x) = g(x;\nu,\lambda) = \Gamma(\nu)^{-1} x^{\nu-1} e^{-\lambda x} \tag{14}$$

is the Green's function corresponding to the operator $\mathbb{D}_\lambda^{1,\nu-1}$ when $1 < \nu < \frac{3}{2}$.

The verification is by direct calculation. With f given by (13) we find

$$(M_\lambda f)(x) = \Gamma(\nu)^{-1} \int_c^x (x-\xi)^{\nu-1} e^{\lambda\xi} \phi(\xi)\mathrm{d}\xi$$

so

$$(M_\lambda f)'(x) = \Gamma(\nu-1)^{-1} \int_c^x (x-\xi)^{\nu-2} e^{\lambda\xi} \phi(\xi)\mathrm{d}\xi$$

and hence, for any $\gamma \in (0,1)$,

$$\begin{aligned}
\mathbb{D}_\lambda^{1,\gamma} f(x) &= \Gamma(\nu-1)^{-1}\Gamma(1-\gamma)^{-1} e^{-\lambda x} D \int_c^x (x-\xi)^{-\gamma} \int_c^\xi (\xi-\eta)^{\nu-2} e^{\lambda\eta}\phi(\eta)\mathrm{d}\eta\mathrm{d}\xi \\
&= \Gamma(\nu-1)^{-1}\Gamma(1-\gamma)^{-1} e^{-\lambda x} D \int_c^x e^{\lambda\eta}\phi(\eta) \int_\eta^x (x-\xi)^{-\gamma}(\xi-\eta)^{\nu-2}\mathrm{d}\xi\mathrm{d}\eta \\
&= \Gamma(\nu-1)^{-1}\Gamma(1-\gamma)^{-1} e^{-\lambda x} D \int_c^x (x-\eta)^{-\gamma+\nu-1} e^{\lambda\eta}\phi(\eta)\mathrm{d}\eta \\
&\quad \times \int_0^1 (1-w)^{(1-\gamma)-1} w^{(\nu-1)-1}\mathrm{d}w \\
&= \Gamma(\nu-\gamma)^{-1} e^{-\lambda x} D \int_c^x (x-\eta)^{-\gamma+\nu-1} e^{\lambda\eta}\phi(\eta)\mathrm{d}\eta.
\end{aligned}$$

Consequently, for $\gamma = \nu - 1$ we have

$$\mathbb{D}_\lambda^{1,\nu-1} f(x) = e^{-\lambda x} D \int_c^x e^{\lambda\eta}\phi(\eta)\mathrm{d}\eta = \phi(x).$$

[2] The differentiation term $DD^{n,\gamma}$ may be viewed as a special case of the more general definition

$$D^{m,n,\gamma} = D^m D^{n,\gamma}$$

where m, like n, is a nonnegative integer and $0 < \gamma < 1$. Then $D^{m,0,\gamma}$ equals the Riemann-Liouville fractional derivative while $D^{0,n,\gamma}$ is the Caputo fractional derivative.

On the other hand, in case $\nu \in \left(\frac{1}{2}, 1\right)$ the relevant equation is

$$\mathbb{D}_\lambda^{0,\nu} f(x) = \phi(x),$$

and the solution is again of the form (13). In fact,

$$\begin{aligned}
\mathbb{D}_\lambda^{0,\gamma} f(x) &= \Gamma(\nu)^{-1}\Gamma(1-\gamma)^{-1} e^{-\lambda x} D \int_c^x (x-\xi)^{-\gamma} \int_c^\xi (\xi-\eta)^{\nu-1} e^{\lambda\eta}\phi(\eta)\mathrm{d}\eta\mathrm{d}\xi \\
&= \Gamma(\nu)^{-1}\Gamma(1-\gamma)^{-1} e^{-\lambda x} D \int_c^x e^{\lambda\eta}\phi(\eta)\mathrm{d}\eta \int_\eta^x (x-\xi)^{-\gamma}(\xi-\eta)^{\nu-1}\mathrm{d}\xi \\
&= \Gamma(\nu)^{-1}\Gamma(1-\gamma)^{-1} e^{-\lambda x} D \int_c^x (x-\eta)^{-\gamma+\nu} e^{\lambda\eta}\phi(\eta)\mathrm{d}\eta \\
&\quad \times \int_0^1 (1-w)^{(1-\gamma)-1} w^{(\nu-1)-1}\mathrm{d}w \\
&= \Gamma(\nu)^{-1}\Gamma(1-\gamma)^{-1} e^{-\lambda x} D \int_c^x (x-\eta)^{-\gamma+\nu} e^{\lambda\eta}\phi(\eta)\mathrm{d}\eta \\
&\quad \times \int_0^1 (1-w)^{(1-\gamma)-1} w^{(\nu-1)-1}\mathrm{d}w \\
&= \Gamma(1-\gamma+\nu)^{-1} e^{-\lambda x} D \int_c^x (x-\eta)^{-\gamma+\nu} e^{\lambda\eta}\phi(\eta)\mathrm{d}\eta
\end{aligned}$$

and with $\gamma = \nu$ we have

$$\mathbb{D}_\lambda^{0,\nu} f(x) = \phi\left(x\right).$$

Thus, in both cases, $\frac{1}{2} < \nu < 1$ and $1 < \nu < \frac{3}{2}$, the gamma kernel (14) occurs as the Green's function. In the former case the differential operator is of Riemann-Liouville type and in the latter of Caputo type.

This suggests, in particular, that for suitable choice of g and q as gamma kernels there may exist an extension of the definition of Caputo derivatives (corresponding to taking the limit $c \to -\infty$) such that the process (4) may be viewed as the solution to a stochastic differential equation of the form $\mathbb{D}Y_t = \sigma_t \dot{B} + \beta\sigma_t^2$.

Note 3 Introducing the operator I_λ^ν by

$$I_\lambda^\nu \phi = M_\lambda^{-1} D^{0,\nu-1} M_\lambda \phi,$$

we may reexpress formula (13) as

$$f = I_\lambda^\nu \phi$$

and the calculation above shows that

$$\mathbb{D}_\lambda^{1,\nu-1} I_\lambda^\nu = I \tag{15}$$

where I denotes the identity operator. Thus $\mathbb{D}_\lambda^{1,\nu-1}$ is the left inverse of I_λ^ν.

The operator I_λ^ν also has a right inverse (where, again, $1 < \nu < \frac{3}{2}$). To determine that, let

$$J_\lambda^{1,\gamma} = M_\lambda^{-1} D^{1,\gamma} M_\lambda$$

(where $\gamma \in (0,1)$). Then

$$\begin{aligned}
I_\lambda^\nu J_\lambda^{1,\gamma} f\left(x\right) &= M_\lambda^{-1} D^{0,\nu-1} M_\lambda \Gamma\left(\gamma\right)^{-1} e^{-\lambda x} \int_c^x (x-\xi)^{-\gamma} \left(e^{\lambda\xi} f\left(\xi\right)\right)' \mathrm{d}\xi \\
&= \Gamma\left(\gamma\right)^{-1} e^{-\lambda x} D^{0,\nu-1} \int_c^x (x-\xi)^{-\gamma} \left(e^{\lambda\xi} f\left(\xi\right)\right)' \mathrm{d}\xi \\
&= \frac{1}{\Gamma\left(\gamma\right)\Gamma\left(\nu-1\right)} e^{-\lambda x} \int_c^x (x-\xi)^{1-\nu} \int_c^\xi (\xi-\eta)^{-\gamma} \left(e^{\lambda\eta} f\left(\eta\right)\right)' \mathrm{d}\eta\mathrm{d}\xi \\
&= \frac{1}{\Gamma\left(\gamma\right)\Gamma\left(\nu-1\right)} e^{-\lambda x} \int_c^x \left(e^{\lambda\eta} f\left(\eta\right)\right)' \mathrm{d}\eta \int_\eta^x (x-\xi)^{1-\nu} (\xi-\eta)^{-\gamma} \mathrm{d}\xi \\
&= \frac{1}{\Gamma\left(\gamma\right)\Gamma\left(\nu-1\right)} e^{-\lambda x} \int_c^x (x-\eta)^{2-\nu-\gamma} \left(e^{\lambda\eta} f\left(\eta\right)\right)' \mathrm{d}\eta \\
&\quad \times \int_0^1 (1-w)^{2-\nu-1} w^{1-\gamma-1}\mathrm{d}w \\
&= \frac{B\left(2-\nu, 1-\gamma\right)}{\Gamma\left(\gamma\right)\Gamma\left(1-\nu\right)} e^{-\lambda x} \int_c^x (x-\eta)^{2-\nu-\gamma} \left(e^{\lambda\eta} f\left(\eta\right)\right)' \mathrm{d}\eta.
\end{aligned}$$

So, for $\gamma = 2 - \nu$ we have

$$I_\lambda^\nu J_\lambda^{1,\gamma} f(x) = f(x),$$

i.e. $J_\lambda^{1,2-\nu}$ is the right inverse of I_λ^ν.

Remark 1 In view of the link to fractional differentiation established in the present Section it is pertinent briefly to refer to the broad range of studies of the relevance of (multi)fractional calculus to turbulence modelling existing in the literature. Some links between that literature and the ambit modelling approach are given in [52].

Another related line of study is that of space-time fractional diffusion equations and the possibility of interpreting the associated Green's functions as probability densities, see [46] and [55].

4 Autocorrelation

The autocorrelation function r of the Gaussian base process (3) is

$$r(u) = \frac{\int_0^\infty g(u+s)\, g(s)\, \mathrm{d}s}{\int_0^\infty g^2(s)\, \mathrm{d}s} \tag{16}$$

and

$$\mathrm{E}\left\{(G_{t+u} - G_t)^2\right\} = \int_0^\infty \psi_u(v)\, \mathrm{d}v = 2\int_0^\infty g^2(s)\, \mathrm{d}s \bar{r}(u) \tag{17}$$

where

$$\bar{r}(u) = 1 - r(u) \tag{18}$$

is the complementary autocorrelation function of G.

When g is the gamma kernel (5) the autocorrelation function is expressible in terms of the type K Bessel functions. Specifically

$$r(u) = 2^{-\nu+\frac{3}{2}} \Gamma\left(\nu - \frac{1}{2}\right)^{-1} \bar{K}_{\nu-\frac{1}{2}}(\lambda u) \tag{19}$$

where, for all real ν, $\bar{K}$ is defined as $\bar{K}_\nu(x) = x^\nu K_\nu(x)$. This function (19) equals the Whittle-Matérn autocorrelation function which is widely used in geostatistics and other areas of spatial statistics as the autocorrelation between two points a distance u apart in d-dimensional Euclidean space, see [37].

The asymptotic behaviour of r as u tends to 0 is of special interest and this is given by

$$\boxed{\bar{r}(u) \sim \begin{array}{lll} 2^{-2\nu+1}\frac{\Gamma\left(\frac{3}{2}-\nu\right)}{\Gamma\left(\frac{1}{2}+\nu\right)} u^{2\nu-1} + O(u^2) & \text{for} & \frac{1}{2} < \nu < \frac{3}{2} \\ \frac{1}{2} u |\ln \frac{u}{2}| + O(u^3 |\ln(u)|) & \text{for} & \nu = \frac{3}{2} \\ \frac{1}{2}\frac{\Gamma(\nu-\frac{5}{2})}{\Gamma\left(\nu-\frac{1}{2}\right)} u^2 + O(u^3|\ln(u)|) & \text{for} & \frac{3}{2} < \nu \end{array}} \tag{20}$$

In a paper from 1948 [40] von Karmann discussed the behaviour of the double correlation functions in three dimensional homogeneous and isotropic turbulence. These functions are defined by

$$\phi(r) = \frac{\overline{u(x_1, x_2, x_3)\, u(x_1 + r, x_2, x_3)}}{\overline{u^2}} \tag{21}$$

and

$$\psi(r) = \frac{\overline{u(x_1, x_2, x_3)\, u(x_1, x_2 + r, x_3)}}{\overline{u^2}} \tag{22}$$

where u denotes the main component of the three-dimensional velocity vector (i.e. the component in the mean wind direction) and the overbar indicates mean value. Due to the continuity equation for incompressible fluids the functions ψ and ϕ are related by

$$\psi(r) = \phi(r) + \frac{r}{2}\phi'(r), \tag{23}$$

see [41], cf. also Section 6.2.1 of [32]. Von Karmann sets up a series of physically based assumptions concerning this type of turbulence and supplementing these assumptions with some speculative reasoning he arrived at the following proposal for the functional form of ϕ

$$\phi(r) = \frac{2^{2/3}}{\Gamma(1/3)} r^{1/3} K_{1/3}(r). \tag{24}$$

A main point in von Karmann's argument was that the spectral density corresponding to this functional form interpolates smoothly between behaving as a fourth power near the origin and decaying at exponential rate $-5/3$ for large frequencies[3]. Both of these traits correspond to well documented empirical behaviour, and the 5/3 rate is the spectral counterpart to Kolmogorov's 2/3 law. In the same paper von Karmann compared this form, or rather that of the transversal correlation function ψ, to wind tunnel data obtained at California Institute of Technology and found a fair agreement between the observations and ψ, as determined from (23).

We note that if ϕ has the form (19) then ψ as determined from (23) is given by

$$\psi(r) = 2^{-\nu+\frac{3}{2}}\Gamma\left(\nu - \frac{1}{2}\right)^{-1}\left(\bar{K}_{\nu-\frac{1}{2}}(r) + \frac{1}{2} r \bar{K}'_{\nu-\frac{1}{2}}(r)\right). \tag{25}$$

It follows from elementary properties of the Bessel functions K that we have the simple relation

$$\bar{K}'_{\nu}(x) = -x\bar{K}_{\nu-1}(x). \tag{26}$$

Hence

$$\begin{aligned}
\psi(r) &= 2^{-\nu+\frac{3}{2}}\Gamma\left(\nu - \frac{1}{2}\right)^{-1}\left(\bar{K}_{\nu-\frac{1}{2}}(r) - \frac{1}{2} r^2 \bar{K}_{\nu-\frac{3}{2}}(r)\right) \\
&= 2^{-\nu+\frac{3}{2}}\Gamma\left(\nu - \frac{1}{2}\right)^{-1}\left(\bar{K}_{\nu-\frac{1}{2}}(r) - \frac{1}{2} r^{\nu+\frac{1}{2}} K_{\nu-\frac{3}{2}}(r)\right) \\
&= 2^{-\nu+\frac{3}{2}}\Gamma\left(\nu - \frac{1}{2}\right)^{-1}\left(\bar{K}_{\nu-\frac{1}{2}}(r) - \frac{1}{2} r^{\nu+\frac{1}{2}} K_{\frac{3}{2}-\nu}(r)\right)
\end{aligned}$$

i.e.

$$\psi(r) = 2^{-\nu+\frac{3}{2}}\Gamma\left(\nu - \frac{1}{2}\right)^{-1}\left(\bar{K}_{\nu-\frac{1}{2}}(r) - \frac{1}{2} r^{2\nu-1} \bar{K}_{\frac{3}{2}-\nu}(r)\right). \tag{27}$$

One sees that $r(u) = \bar{\phi}(u)$ where $\bar{\phi}(u) = 1 - \phi(u)$; thus the asymptotic behaviour of $\bar{\phi}(u)$ as $u \to 0$ is determined by (20). As regards the asymptotic properties of the complementary autocorrelation function $\bar{\psi}(u) = 1 - \psi(u)$ of the transversal velocities it follows immediately from the Table that, for $\frac{1}{2} < \nu < \frac{3}{2}$ and $u \to 0$, the leading terms of the expansions of $\bar{\phi}(u)$ and $\bar{\psi}(u)$ are both of order $u^{2\nu-1}$.

Formula (24) is a special case, obtained for $\nu = \frac{5}{6}$, of the general form of autocorrelation function (19). von Karmann's derivation was not based on any specified probability structure or model. The general form (19), obtained by a Fourier inversion, was proposed as correlation function by [53] (Russian Edition 1959). As mentioned above, that form is also known as the Whittle-Matérn correlation function. Note that the von Karmann-Tatarski specification refers to spatial correlations whereas that of (19) concerns timewise correlation. However, the Taylor Frozen Field Hypothesis[4] provides a direct physical link between the two results.

Note 4 Expressed in terms of $\phi(r)$ itself, rather than the spectrum of ϕ, the basis for von Karmann's proposal was that $\bar{\phi}(r)$ provides a good fit to the behaviour of the second order structure function both over the inertial subrange (the 2/3 law) and at large lags. It follows from the table (20) that the 2/3 behaviour in fact extends all the way down to 0. Turning this observation around, the indication is that, by the nature of the gamma kernel, the asymptotic behaviour of the third order structure function near 0, which is linear, extends to the inertial subrange.

Note 5 *Moving average processes with bi-gamma kernel* The results in (20) may also be used for describing the small scale behaviour of the following moving average process with a gamma type kernel.

Consider the stationary Gaussian moving average process given by

$$Y_t = \int_{-\infty}^{\infty} g(t - s; \nu, \lambda, \mu, \kappa)\, \mathrm{d}B_s \tag{28}$$

[3] The $-5/3$ behaviour is very manifest in the socalled inertial range but, as documented by later, extensive and accurate measurements, for the largest frequencies (the dissipation range) the spectral density decreases at a much faster rate. The total behaviour of the spectral density is accurately described by a formula due to Skharofsky, see for instance Figure 5 in [35].

[4] The Hypothesis states that spatial and temporal increments of the main component of the velocity vector are equivalent in law up to a proportional change of time. Cf. for instance [43].

where B is Brownian motion on $\mathbb{R}$ and

$$g(t;\nu,\lambda,\mu,\kappa) = \begin{cases} t^{\nu-1}e^{-\lambda t} & \text{for} \quad t>0 \\ |t|^{\mu-1}\,e^{-\kappa|t|} & \text{for} \quad t<0 \end{cases}$$

We refer to this as the bi-gamma kernel. In case $\mu=\nu$ and $\kappa=\lambda$ the kernel is symmetric around 0 and may be written as $g(|t|\,;\nu,\lambda)$. The process Y is well defined provided both ν and μ are greater than $\frac{1}{2}$, and it may be rewritten as

$$Y_t = \int_{-\infty}^{t}(t-s)^{\nu-1}\,e^{-\lambda(t-s)}\mathrm{d}B_s + \int_t^{\infty}(s-t)^{\mu-1}\,e^{-\kappa(s-t)}\mathrm{d}B_s.$$

Here

$$\begin{aligned} \mathrm{E}\{Y_0Y_u\} &= \int_0^\infty g(s;\nu,\lambda)\,g(u+s;\nu,\lambda)\,\mathrm{d}s \\ &\quad + \int_0^u g(s;\mu,\kappa)\,g(u-s;\nu,\lambda)\,\mathrm{d}s \\ &\quad + \int_u^\infty g(s;\mu,\kappa)\,g(s-u;\mu,\kappa)\,\mathrm{d}s. \end{aligned}$$

By formula 3.383.1 in [36] we find

$$\begin{aligned} \int_0^u g(s;\mu,\kappa)\,g(u-s;\nu,\lambda)\,\mathrm{d}s &= \int_0^u s^{\mu-1}e^{-\kappa s}\,(u-s)^{\nu-1}\,e^{-\lambda(u-s)}\mathrm{d}s \\ &= e^{-\lambda u}\int_0^u s^{\mu-1}\,(u-s)^{\nu-1}\,e^{-(\lambda+\kappa)s}\mathrm{d}s \\ &= e^{-\lambda u}B(\nu,\mu)\,{}_1F_1(\mu;\nu+\mu;\lambda+\kappa)\,u^{\lambda+\kappa-1} \end{aligned}$$

where ${}_1F_1$ is the general hypergeometric function, and we have ([36], formula 9.14.1)

$${}_1F_1(\mu;\nu+\mu;(\lambda+\kappa)) = \sum_{k=0}^{\infty}\frac{(\mu)_k}{(\nu+\mu)_k}\frac{(\lambda+\kappa)^k}{k!}.$$

All in all this implies that for $u\to 0$ the second order structure function of (28) is of the form

$$S_2(u) = cu^{2\nu-1} + c'u^{2\mu-1} - c''u^{\nu+\mu-1}$$

where c, c' and c'' are positive constants. Thus, in particular, if $\frac{1}{2}<\nu<\frac{3}{2}$ and $\frac{1}{2}<\kappa<\frac{3}{2}$ then

$$S_2(u) \sim \begin{cases} cu^{2\nu-1} & \text{for} \quad \nu<\mu; \\ (c+c'-c'')\,u^{2\nu-1} & \text{for} \quad \nu=\mu; \\ c'u^{2\mu-1} & \text{for} \quad \nu>\mu. \end{cases} \tag{29}$$

5 Pathwise behaviour

The fine structure of *BSS* and *LSS* processes is discussed in [47] and [49].

In [47] the authors establish a connection between the path behaviour of the *BSS* process

$$Y_t = \int_{-\infty}^{t}(t-s)^{\nu-1}\,e^{-\lambda(t-s)}\mathrm{d}B_s \tag{30}$$

and that of the fractional Ornstein-Uhlenbeck process Y^H with index H, that is

$$Y_t^H = \int_{-\infty}^{t} e^{-\lambda(t-s)}\mathrm{d}B_s^H \tag{31}$$

under the assumption that $H=\nu-\frac{1}{2}$ and $\nu\in\left(\frac{1}{2},1\right)\cup\left(1,\frac{3}{2}\right)$ so that we are in the nonsemimartingale case. While both of these are stationary Gaussian processes, the former is in many situations more realistic in regard to applications,.particularly for $\nu=5/6$ which corresponds to von Karmann's spectral density function. This is the case in particular for turbulence modelling. The key difference lies in the tail behaviour of the increments for large lags.

The following result is established in [47].

Theorem 1 For all $t > 0$

$$Y_t = Y_t^H - D_t \tag{32}$$

where $D \in C^1([0,\infty))$. A concrete representation of the process D is available:

$$D_t = \int_{-\infty}^{t} \int_{s}^{t} \left(e^{-\lambda(t-u)} - e^{-\lambda(t-s)} \right) \frac{\partial}{\partial u} K^H(u,r)\, \mathrm{d}u \mathrm{d}B_s$$

where K^H denotes the kernel for the fractional Brownian motion, i.e.

$$K^H(t,s) = (t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}.$$

Similarly Y may be represented as

$$Y_t = B_t^H - V_t$$

where V is an absolutely continous process.

The stochastic analysis for volatility modulated Lévy-driven Volterra processes, developed in [5], [4], [13], is an important tool in the derivation of these results. Together these three papers constitute a foundation for an Ito calculus for *BSS* and *LSS* processes.

6 Recovery and Inference

Once a *BSS* or *LSS* model has been formulated the question arises as to what can be learned about the components of the model, either in law or pathwise. In discussing this we will, unless otherwise mentioned, assume that the skewness terms are absent.

Thus let

$$Y_t = \int_{-\infty}^{t} g(t-s)\sigma_s \mathrm{d}L_s^T \tag{33}$$

where L^T denotes a Lévy process L time changed by a chronometer T (i.e. a cadlag increasing process such that $T(t) \to -\infty$ for $t \to -\infty$ and $T(t) \to \infty$ for $t \to \infty$). The question has several aspects: (i) In case the process has been observed continuously over an interval, can any of the model components be exactly determined (ii) Under what conditions does the law of Y uniquely determine the kernel g or the laws of σ or L or T (iii) If the data available consists of high frequency observations, what inference procedures for assessing some or all of the components might be available; in particular, what can be said about the volatility/intermittency process σ^2.

Below we exemplify these aspects. For additional results, proofs and references see [50], [51].

(i) The following example presents a case where concrete pathwise recovery of L^T over the interval $(-\infty, t)$ is possible if Y has been observed continuously over the same interval

Example 2 Suppose that g is the gamma kernel and that $\sigma \equiv 1$ and $T = t$, that is

$$Y_t = \int_{-\infty}^{t} g(t-s;\nu,\lambda)\, \mathrm{d}L_s.$$

By the stochastic Fubini theorem (cf. [2]) we find for $\nu \in \left(\frac{1}{2}, 1\right)$ and letting

$$Z_t = \int_{-\infty}^{t} g(t-u; 1-\nu, \lambda)\, Y_u \mathrm{d}u$$

that

$$\begin{aligned} Z_t &= \int_0^t \int_{-\infty}^{u} g(t-u;1-\nu,\lambda)\, g(u-s;\nu,\lambda)\, \mathrm{d}L_s \mathrm{d}u \\ &= \int_{-\infty}^{t} \int_{s}^{t} g(t-u;1-\nu,\lambda)\, g(u-s;\nu,\lambda)\, \mathrm{d}u \mathrm{d}L_s \\ &= \int_{-\infty}^{t} \int_{0}^{t-s} g(w;1-\nu,\lambda)\, g(t-s-w;\nu,\lambda)\, \mathrm{d}w \mathrm{d}L_s \\ &= \int_{-\infty}^{t} g(t-s;1,\lambda)\, \mathrm{d}L_s^T = c\int_{-\infty}^{t} e^{-\lambda(t-s)} \mathrm{d}L_s \end{aligned}$$

for a constant c. Hence

$$\begin{aligned}
Z_t^+ &= \int_0^t Z_s \mathrm{d}s = \int_0^t \int_{-\infty}^s e^{-\lambda(s-u)} \mathrm{d}L_u \mathrm{d}s \\
&= \int_{-\infty}^t \int_u^t e^{-\lambda(s-u)} \mathrm{d}s \mathrm{d}L_u \\
&= \lambda^{-1} \int_{-\infty}^t \left(1 - e^{-\lambda(t-u)}\right) \mathrm{d}L_u = \lambda^{-1} \left(L_t - Z_t\right)
\end{aligned}$$

or

$$L = \lambda Z^+ + Z.$$

It is noteworthy here that L is explicitly recoverable in spite of the fact that with $\nu \in \left(\frac{1}{2}, 1\right)$ the kernel $g(t; \nu, \lambda)$ tends to ∞ for $t \to 0$.

Note 6 Under a minor regularity condition on the time change T, the same argument goes through, giving that $L^T = \lambda Z^+ + Z$.

Note 7 For general moving average processes driven by Lévy noise

$$Y_t = \int_{-\infty}^{\infty} g(t-s) L(\mathrm{d}s)$$

recovery of L from complete knowledge of the realisation of Y on $\mathbb{R}$ is, subject to regularity restrictions on g and L, possible in terms of linear limit operations. This applies in particular for the gamma kernel and in that case $\{L_s : s \leq t\}$ is recoverable from $\{X_s : s \in \mathbb{R}\}$, see [50].

Consider the class $\mathcal{G}$ of kernels g such that g is integrable with non-vanishing Fourier transform. This is the case in particular for the gamma kernel who's Fourier transform is

$$\hat{g}(\zeta; \nu, \lambda) = \frac{\lambda^\nu}{\Gamma(\nu)} (\lambda - i\zeta)^{-\nu}.$$

It is shown in [51] that if $g \in \mathcal{G}$ and

$$Y_t = \int_{-\infty}^t g(t-s) \mathrm{d}L_s^T,$$

with T a subordinator independent of L, then the law of T is completely determined by the laws of L and Y. This is thus, in particular, the case when g is of the gamma type.

(ii) The paper [39], cf. also [26], introduces a powerful nonparametric procedure for estimation of the kernel function for *BSS* processes

$$Y_t = \int_{-\infty}^t g(t-s) \sigma_s \mathrm{d}B_s$$

with kernel function g in $\mathcal{G}$. This is done through determining g from the autocorrelation function

$$r(u) = \int_0^\infty g(s + |u|) g(s) \mathrm{d}s.$$

Numerically the gamma kernel is used as a test case. (Earlier approaches to the estimation of the kernel are presented in [26].)

(iii) A key question of inference for *BSS* and *LSS* models is how to assess the inherent volatility σ^2. More specifically, the wish will typically be to draw accurate inference on the accumulated volatility

$$\sigma_t^{2+} = \int_0^t \sigma_s^2 \mathrm{d}s.$$

The realised quadratic variation is a natural initial tool to this end and for *BSS* processes Y, as given by (1), that will under mild conditions yield a consistent estimator of σ_t^{2+} provided Y is a semimartingale, i.e. it will hold that

$$[Y_\delta]_t \xrightarrow{p} \sigma_t^{2+} \quad \text{as} \quad \delta \to 0.$$

However, suppose that

$$Y_t = \int_{-\infty}^{t} g\left(t-s;\nu,\lambda\right)\sigma_s \mathrm{d}B_s$$

with $\nu \in \left(\frac{1}{2},1\right) \cup \left(1,\frac{3}{2}\right]$. Then Y is not a semimartingale and the realised quadratic variation $\left[Y_\delta\right]_t$ converges to ∞ for $\nu \in \left(\frac{1}{2},1\right)$ and to 0 for $\nu \in \left(1,\frac{3}{2}\right]$. The rate of these convergences is determined by ν. For instance, when $\nu \in \left(\frac{1}{2},1\right)$ we have

$$c\delta^{2(1-\nu)}\left[Y_\delta\right]_t \xrightarrow{p} \sigma_t^{2+} \quad \text{as} \quad \delta \to 0$$

where $c = \lambda 2^{2(1-\nu)}\Gamma\left(2\nu-1\right)/\Gamma\left(\nu\right)^2$.

Obtaining estimates of ν as a way to inference on σ_t^{2+}, in particular through stable central limit theorems, requires advanced reasoning. The papers [9], [10] and [29] develop the theory of multipower variations for this and related purposes. In particular, the latter two papers discuss the use of COF (change-of-frequency) statistics. Similar points for *LSS* processes are discussed in [49].

So far we have, for simplicity of discussion, assumed that the skewness terms in the *BSS/LSS* processes are 0. When this is not the case it is still possible, under certain conditions, to establish stable central limit theorems, as shown in the above-mentioned papers. However, consider the case.where Y is of the form (4) and g and q are given respectively as $g\left(t;\nu,\lambda\right)$ and $g\left(t;\nu-\frac{1}{2},\lambda\right)$ for $\nu \in \left(\frac{1}{2},1\right)$, cf. (6). Then the skewness term contain information on the volatility/intermittency process. In fact, it can be shown that then

$$\delta^{2(1-\nu)}\left[Y_\delta\right]_t \xrightarrow{p} c\int_0^t \sigma_s^2 \mathrm{d}s + c'\int_0^t \sigma_s^4 \mathrm{d}s$$

for certain constants c and c' and where $\left[Y_\delta\right]_t$ is the quadratic variation of Y over the interval $(0,t)$ at lag δ.

Recently a powerful procedure for simulation of *BSS* processes is presented in [20], under the assumption that the kernel function is regularly varying at 0, as is the case for the gamma kernel with $\nu \in \left(\frac{1}{2},1\right)$, and the method is applied successfully for inference on ν (referred to as the roughness parameter). Approximation of *LSS* processes by Fourier methods is discussed in [21].

As long as the interest is solely in regard to relative volatility, inference on ν can be avoided by using the realised relative volatility, defined as $\left[Y_\delta\right]_t / \left[Y_\delta\right]_T$ for $0 < t < T$. This yields

$$\left[Y_\delta\right]_t / \left[Y_\delta\right]_T \xrightarrow{p} \sigma_t^{2+}/\sigma_T^{2+} \quad \text{as} \quad \delta \to 0.$$

Associated confidence intervals based on a stable central limit theorem have been developed in [12].

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