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Maximum Likelihood Estimation of Time-Varying Loadings in High-Dimensional Factor Models*

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Abstract

In this paper, we develop a maximum likelihood estimator of time-varying loadings in high-dimensional factor models. We specify the loadings to evolve as stationary vector autoregressions (VAR) and show that consistent estimates of the loadings parameters can be obtained by a two-step maximum likelihood estimation procedure. In the first step, principal components are extracted from the data to form factor estimates. In the second step, the parameters of the loadings VARs are estimated as a set of univariate regression models with time-varying coefficients. We document the finite-sample properties of the maximum likelihood estimator through an extensive simulation study and illustrate the empirical relevance of the time-varying loadings structure using a large quarterly dataset for the US economy.

Keywords: High-dimensional factor models, dynamic factor loadings, maximum likelihood, principal components.

JEL classification: C33, C55, C13.

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1 Introduction

In this paper, we develop a consistent maximum likelihood estimator of time-varying loadings in high-dimensional factor models where factors are estimated with principal components.

The problem of time-varying loadings in factor models is important because the assumption of constant loadings has been found to be implausible in a number of studies considering structural instability in factor models. In a large macroeconomic dataset for the U.S., Stock and Watson (2009) find considerable instability in factor loadings around 1984, and they improve factor-based forecast regressions of individual variables by allowing factor coefficients to change after the break point. Breitung and Eickmeier (2011) develop Chow-type tests for structural breaks in factor loadings and find similar evidence of structural instability around 1984. They also find evidence of structural breaks in the Euro area around 1992 and 1999. Del Negro and Otrok (2008), Liu et al. (2011), and Eickmeier et al. (2015) estimate factor models in which the factor loadings are modelled as random walks using large panels of data, but theoretical results for models with time-varying parameters in a high-dimensional setting are scant.

The econometric theory on factor models explicitly addresses the high dimensionality of these datasets by developing results in a large N and large T framework. The central results in the literature on consistent estimation of the factor space by principal components as $N, T \to \infty$ have been developed in Stock and Watson (1998, 2002), and Bai and Ng (2002). Forni et al. (2000) consider estimation in the frequency domain. Principal components have the advantage of being easy to compute and feasible even when the cross-sectional dimension N is larger than the sample size T. Bates et al. (2013) characterize the types and magnitudes of structural instability in factor loadings under which the principal components estimator of the factor space is consistent. Another strand of literature is concerned with estimation by maximum likelihood. Bai and Li (2012a,b) consider maximum likelihood estimation of factor loadings and idiosyncratic variances, while Doz et al. (2012) study functions of maximum likelihood estimators, also in a large N, T setting. Their analyses applies to factor models with constant loadings.

We consider a factor model of the form $X_{it} = \lambda_{it}' F_t + e_{it}$ for $i = 1, ...N$ and $t = 1, ..., T$, where the data X_{it} depend on a small number $r \ll N$ of unobserved common factors F_t. The $r \times 1$ vector of factor loadings λ_{it} evolves over time. We model λ_{it} for each i as a stationary vector autoregression, and our main contribution is to show that the parameters of these time-varying loadings can be consistently estimated by maximum likelihood. Our estimation procedure consists of two steps. In the first step, the common factors are estimated by principal components, and in the second step we estimate the loadings parameters by maximum likelihood, treating the principal components as observed data.

The principal components estimator is robust to stationary variations in the loadings. By averaging over the cross-section, the temporal instabilities in the loadings are smoothed out and the factor space is consistently estimated. Average consistency in t of the factor space is shown by Bates et al. (2013), and we extend the result to uniform consistency in t to analyse the maximum likelihood estimator.

In the second step, we estimate a panel of regression models with time-varying coefficients where the principal components are treated as the observed regressors, and the loadings are the time-varying coefficients. We allow for heteroskedasticity and serial correlation in the idiosyncratic errors e_{it}, but restrict our attention to cross-sectionally uncorrelated errors to avoid proliferation of parameters. Conditional on the factor estimates, the variables in the panel of regressions are therefore uncorrelated, and the loadings parameters can be estimated as a set of N univariate regression models with time-varying coefficients. Under the condition that $\frac{T}{N^2} \to 0$, the maximum likelihood estimator of the time-varying loadings is consistent as $N, T \to \infty$, and estimation error from the principal components can be ignored.

We point out that the computation of the maximum likelihood estimator is relatively simple. Principal components are simple to compute, and the set of N univariate regression models with time-varying parameters can be readily estimated by Kalman-filter procedures.

The rest of the paper is organized as follows. Section 2 introduces the model and the two-step estimation procedure. Section 3.1 states the assumptions and consistency results for the principal components estimator, and Section 3.2 discusses identification of the loadings parameters. Our main result on consistency of the maximum likelihood estimator of the time-varying loadings and the associated assumptions are stated in Section 3.3. In Section 4 we report the results of a Monte Carlo study, and in Section 5 we provide an empirical illustration. Section 6 concludes.

2 Model and Estimation

We consider the following model:

$$X_t = \Lambda_t F_t + e_t, \tag{1}$$

where $X_t = (X_{1t}, ..., X_{Nt})'$ is the N-dimensional vector of observed data at time t. The observations are generated by a small number $r \ll N$ of unobserved common factors $F_t = (F_{1t}, ..., F_{rt})'$, time-varying factor loadings $\Lambda_t = (\lambda_{1t}, ..., \lambda_{Nt})'$, and idiosyncratic errors $e_t = (e_{1t}, ..., e_{Nt})$ with covariance matrix $E(e_t e_t') = \Psi_0$. The $N \times r$ loadings matrix $\Lambda_t = (\lambda_{1t}, ..., \lambda_{Nt})'$ is time-varying and each $\lambda_{it} \in \mathbb{R}^{r \times 1}$ evolves as an r-dimensional vector autoregression:

$$B_i^0(L)(\lambda_{it} - \lambda_i^0) = \eta_{it}, \tag{2}$$

where $\lambda_i^0 = E(\lambda_{it})$ is the unconditional mean, and $B_i^0(L) = I - B_{i,1}^0 L - ... - B_{i,p}^0 L^p$ is a p^{th}-order lag polynomial with roots outside the unit circle. The autoregressive order p can be allowed to vary over i such that p_i differs over i. We suppress the subscript for notational convenience. The innovations η_{it} have covariance matrix $E(\eta_{it}\eta_{it}') = Q_i^0$.

Our goal is to estimate the parameters of each of the loadings processes (2) and the idiosyncratic variance matrix Ψ_0. To achieve a sufficiently parsimonious parametrization of the $N \times N$ matrix Ψ_0, we specify it to be diagonal, $\Psi_0 = diag(\psi_1^0, ..., \psi_N^0)$, such that the idiosyncratic errors are cross-sectionally uncorrelated. Conditional on the factors, X_i is therefore uncorrelated over i, and the model can be written as:

$$X_i = \mathbf{F}\Lambda_i + e_i, \tag{3}$$

where $X_i = (X_{i1}, ..., X_{iT})'$, $e_i = (e_{i1}, ..., e_{iT})'$, $\mathbf{F} = diag\,\{F_t'\}_{t=1,...,T}$ is a $T \times rT$ block-diagonal matrix, and $\Lambda_i = (\lambda_{i1}', ..., \lambda_{iT}')'$. The mean and variance of X_i are $E(X_i) = (F_1'\lambda_i^0, ..., F_T'\lambda_i^0)'$ and $\Sigma_i := Var(X_i) = \mathbf{F}\Phi_i\mathbf{F}' + \psi_i I_T$ where $\Phi_i = Var(\Lambda_i)$ is of dimension $rT \times rT$. We can thus specify a Gaussian likelihood function for X_i conditional on the factors $F = (F_1, ..., F_T)'$ as:

$$\mathcal{L}_T(X_i|F;\theta_i) = -\frac{1}{2}\log(2\pi) - \frac{1}{2T}\log|\Sigma_i| - \frac{1}{2T}(X_i - E(X_i))'\Sigma_i^{-1}(X_i - E(X_i)), \tag{4}$$

with parameter vector $\theta_i = \{B_i(L), \lambda_i, Q_i, \psi_i\}$. Equations (2) and (3) can be written as a linear state-space model and the likelihood can therefore be calculated with the Kalman filter.

It is not feasible to estimate θ_i with (4), however, as the likelihood depends on the unobservable factors F. We therefore replace the unobservable factors F in (4) with an estimate $\tilde{F}$ to form the feasible likelihood function $\tilde{\mathcal{L}}_T(X_i|\tilde{F};\theta_i)$. This gives us a set of N likelihood functions to estimate the parameters θ_i for each i. Define the estimator $\tilde{\theta}_i$ which maximizes the feasible likelihood function as:

$$\tilde{\theta}_i = \underset{\theta}{\operatorname{argmax}}\ \tilde{\mathcal{L}}_T(X_i|\tilde{F};\theta_i). \tag{5}$$

This is our object of interest and we show that the estimator $\tilde{\theta}_i \xrightarrow{p} \theta_i^0$ for each i, where $\theta_i^0 = \left\{B_i^0(L), \lambda_i^0, Q_i^0, \psi_i^0\right\}$ is the true value of the parameters.

We use the principal components estimator to estimate the factors. The principal components estimator treats the loadings as being constant over time, $\Lambda_t \equiv \Lambda$, and solves the minimization problem:

$$(\tilde{F}, \tilde{\Lambda}) = \min_{F,\Lambda}(NT)^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}(X_{it} - \lambda_i' F_t)^2, \tag{6}$$

where $\tilde{F}$ is $T \times r$ and $\tilde{\Lambda}$ is $N \times r$. To uniquely define the minimizers, it is necessary to impose identifying restrictions on the estimators, as only X_{it} is observed. By concentrating out Λ and using the normalization $F'F/T = I_r$, the problem is equivalent to maximizing $tr(F'(XX')F)$, where $X = (X_1, ..., X_T)'$ is the $T \times N$ matrix of observations. The resulting estimator $\tilde{F}$ is given by $\sqrt{T}$ times the eigenvectors corresponding to the r largest eigenvalues of the $T \times T$ matrix XX'. The solution is not unique: any orthogonal rotation of $\tilde{F}$ is also a solution. Bai and Ng (2008b) give an extensive treatment of the principal components estimator. We use $\tilde{F}$ to form the feasible likelihood function $\tilde{\mathcal{L}}_T(X_i|\tilde{F};\theta_i)$.

The estimation procedure thus consists of two steps. In the first step, we extract principal components from the observable data to estimate the factors F_t, under the assumption of constant loadings. In the second step, we use the factor estimates together with the observable data to maximize the likelihood function and estimate the parameters θ_i of the time-varying loadings. Our main result in Section 3.3 shows that this yields a consistent estimator for the parameters of the time-varying loadings.

3 Asymptotic Theory

In this section, we present the asymptotic theory for the two-step estimation method discussed in Section 2. The main result is Theorem 1 on consistent estimation of the loadings parameters by maximum likelihood; it is given in Section 3.3. Our result builds on the work by Bates et al. (2013), who show average consistency of the principal components estimator when loadings are subject to structural instability. We use a different rotation of the principal components estimator, and in Section 3.1 we therefore restate their result in Lemma 1. Furthermore, we provide a result on uniform consistency in t of the principal components estimator in Proposition 1. Section 3.2 discusses identification of the factors and loadings parameters. All results are for $N, T \to \infty$, and the factor rank r is assumed to be known.

We introduce the following notation. $\|A\| = [\mathrm{tr}(A'A)]^{1/2}$ denotes the Frobenius norm of the matrix A. The subscripts i, j are cross-sectional indices, t, s are time indices, and p, q are factor indices. The constant $M \in (0, \infty)$ is a constant common to all the assumptions below. Finally, define $C_{NT} = \min\{\sqrt{N}, \sqrt{T}\}$.

3.1 Principal Components Estimation

Let $\xi_{it} := \lambda_{it} - \lambda_i^0 = B_i^0(L)^{-1}\eta_{it}$ be the loadings innovations and write (1) as:

$$X_t = \Lambda^0 F_t + \xi_t F_t + e_t,$$

where $\Lambda^0 = (\lambda_1^0, ..., \lambda_N^0)'$ and $\xi_t = (\xi_{1t}, ..., \xi_{Nt})'$ are the $N \times r$ matrices of loadings means and innovations, respectively. The vector ξ_{it} is the moving average representation of the loadings. Assumptions A-C are standard for factor models and are the same as Assumptions A-C in Bai and Ng (2002):

Assumption A (Factors). $E\|F_t\|^k \leq M < \infty$ for some $k \geq 4$, and $T^{-1}\sum_{t=1}^T F_t F_t' \xrightarrow{p} \Sigma_F$ for some $r \times r$ positive definite matrix Σ_F.

Assumption B (Loadings). $\|\lambda_i^0\| \leq M < \infty$, and $\|\Lambda^{0\prime}\Lambda^0/N - \Sigma_\Lambda\| \to 0$ for some positive definite matrix Σ_Λ.

Assumption C (Idiosyncratic Errors). There exists a positive constant $M < \infty$ such that for all N and T:

1. $E(e_{it}) = 0$, $E|e_{it}|^8 \leq M$.

2. $E(e_s'e_t/N) = E(N^{-1}\sum_{i=1}^N e_{is}e_{it}) = \gamma_N(s,t)$, $|\gamma_N(s,s)| \leq M$ for all s, and $T^{-1}\sum_{s,t=1}^T |\gamma_N(s,t)| \leq M$.

3. $E(e_{it}e_{jt}) = \tau_{ij,t}$ with $|\tau_{ij,t}| \leq |\tau_{ij}|$ for some τ_{ij} and for all t. In addition $N^{-1}\sum_{i,j=1}^N |\tau_{ij}| \leq M$.

4. $E(e_{it}e_{js}) = \tau_{ij,ts}$, and $(NT)^{-1}\sum_{i,j=1}^N \sum_{t,s=1}^T |\tau_{ij,ts}| \leq M$.

5. For every (s,t), $E|N^{-1/2}\sum_{i=1}^N [e_{is}e_{it} - E(e_{is}e_{it})]|^4 \leq M$.

We leave the moment condition on the factors in Assumption A unspecified, as the result in Proposition 1 depends on k. Assumption B requires the columns of Λ^0 to be linearly independent, such that the matrix Σ_Λ is non-singular. Assumptions A and B together imply the existence of r common factors. Assumption C allows for heteroskedasticity and limited time-series and cross-section dependence in the idiosyncratic errors. It will later be strengthened to e_{it} being independent over i and t. Note that if e_{it} is independent for all i and t, Assumptions C.2-C.5 follow from C.1.

We impose the following assumption on the factor loadings innovations and the factors:

Assumption D (Factor Loadings Innovations). The following conditions hold for all N, T and factor indices $p_1, q_1, p_2, q_2 = 1, ..., r$:

1. $\sup_{s,t} \sum_{i,j=1}^N |E(\xi_{isp_1}\xi_{jtq_1}F_{sp_1}F_{tq_1})| = O(N)$.

2. $\sum_{s,t=1}^T \sum_{i,j=1}^N |E(\xi_{isp_1}\xi_{jsq_1}F_{sp_1}F_{sq_1}F_{tp_2}F_{tq_2})| = O(NT^2)$.

3. $\sup_{s,t} \sum_{s=1}^T \sum_{i,j=1}^N |E(\xi_{isp_1}\xi_{jsq_1}\xi_{itp_2}\xi_{jtq_2}F_{sp_1}F_{sq_1}F_{tp_2}F_{tq_2})| = O(N^2) + O(NT)$.

Assumption D is identical to Bates et al. (2013) except for D.3, which is stronger than their corresponding assumption. Assumption D.3 is needed for uniform consistency of the principal components and is still reasonable. We do not require independence between the factors and loadings, as the effect of the factors on the observable variables might reasonably be expected to change when the factors differ substantially from their mean levels. However, if the factors and loadings are assumed to be independent, and the loadings evolve as stationary vector autoregressions that are independent over i, Assumptions D.1-D.3 can easily be shown to hold: For simplicity, take $r = 1$. By Assumption A and cross-sectional independence of the loadings, the supremum in D.1 can be bounded by:

$$\sup_{s,t}\{|E(F_sF_t)| \sum_{i,j=1}^N |E(\xi_{is}\xi_{jt})|\} \leq M \sup_{s,t} \sum_{i,j=1}^N |E(\xi_{is}\xi_{jt})| = M \sum_{i=1}^N \sup_{s,t} |E(\xi_{is}\xi_{it})|.$$

The terms $E(\xi_{is}\xi_{it})$ are the autocovariances of the moving average representation of the loadings. As the loadings are stationary, these autocovariances are bounded, and the rate $O(N)$ follows. The rate $O(NT^2)$ in D.2 follows from D.1 when the factors and the loadings are independent. The sum in D.3 can be bounded by:

$$M \sup_{s,t} \sum_{s=1}^{T} \sum_{i,j=1}^{N} |E(\xi_{is}\xi_{js}\xi_{it}\xi_{jt})| = M \sup_{s,t} \sum_{s=1}^{T} \sum_{i=1}^{N} |E(\xi_{is}^2 \xi_{it}^2)| + M \sup_{s,t} \sum_{s=1}^{T} \sum_{i \neq j}^{N} |E(\xi_{is}\xi_{it})E(\xi_{js}\xi_{jt})|$$

$$\leq M \sum_{s=1}^{T} \sum_{i=1}^{N} \sup_{s,t} |E(\xi_{is}^4)| + M \sup_{s,t} \sum_{i \neq j}^{N} \left(\sum_{s=1}^{T} |E(\xi_{is}\xi_{it})|^2 \right)^{1/2} \left(\sum_{s=1}^{T} E|(\xi_{js}\xi_{jt})|^2 \right)^{1/2} .$$

The first term is $O(NT)$ if $E(\xi_{is}^4) < \infty$, and the second term is $O(N^2)$ if the autocovariances $E(\xi_{is}\xi_{it})$ are square-summable. Assumption D.3 is therefore satisfied when the loadings and the factors are independent. We assume the same rates to hold without imposing independence between the factors and the loadings.

Finally, we impose independence between the idiosyncratic errors and the factors and loadings innovations.

Assumption E (Independence). For all (i, j, s, t), e_{it} is independent of (F_s, ξ_{js}).

Assumptions A-E are sufficient to consistently estimate the space spanned by the factors. For this purpose, we use the result of Lemma 1 below, which is a modified version of Theorem 1 in Bates et al. (2013).[1] We use a rescaled estimator that is more convenient for the rest of the analysis and therefore restate their result:

Lemma 1. *Under Assumptions A-E there exists an $r \times r$ matrix H such that*

$$T^{-1} \sum_{t=1}^{T} \|\tilde{F}_t - H' F_t\|^2 = O_p(C_{NT}^{-2})$$

as $N, T \to \infty$.

Proof. See the Appendix.

Lemma 1 shows that the mean-squared deviation between the principal components and the common factors disappears as the sample size T and the cross-sectional dimension N tend to infinity.[2] The convergence rate C_{NT} is the same as in Bai and Ng (2002), and the principal components estimator is thus robust to stationary deviations in the loadings around a constant mean. Note that the common factors are only identified up to a rotation, so the principal components converge to a rotation of the common factors.

[1] Bates et al. (2013) use the estimator $\hat{F} = \tilde{F} V_{NT}$, where V_{NT} is the diagonal matrix of the r largest eigenvalues of $(NT)^{-1} X X'$.

[2] Lemma 1 also holds when the factor rank is unknown. By setting the number of estimated factors to any fixed $k \geq 1$, the Lemma can be stated as $T^{-1} \sum_{t=1}^{T} \|\tilde{F}_t^k - H^{k\prime} F_t\|^2 = O_p(C_{NT}^{-2})$, where $\tilde{F}_t^k$ is $k \times 1$ and H^k is a $r \times k$ matrix, and $\tilde{F}_t^k$ consistently estimates the space spanned by k of the true factors

Lemma 1 does not imply uniform convergence in t, but only average consistency of the principal components. In order to analyse the properties of the feasible likelihood function $\tilde{\mathcal{L}}_T(X_i, \tilde{F}|\theta_i)$, we need uniform consistency of the estimated factors, in addition to the average consistency of Lemma 1. To establish uniform convergence, we make additional assumptions, as in Bai and Ng (2006, 2008a):

Assumption F There exists a positive constant $M < \infty$ such that for all N and T:

1. $\sum_{s=1}^{T} |\gamma_N(s,t)| \leq M$ for all t.
2. $E\|(NT)^{-1/2} \sum_{s=1}^{T} \sum_{k=1}^{N} F_s[e_{ks}e_{kt} - E(e_{ks}e_{kt})]\|^2 \leq M$ for all t.
3. $E\|N^{-1/2} \sum_{i=1}^{N} \lambda_i^0 e_{it}\|^8 \leq M$ for all t.

Assumption F.1 is stronger than C.2, but still reasonable: If e_{it} is assumed to be stationary with absolutely summable autocovariances, Assumption F.1 holds. Assumptions F.2 and F.3 are reasonable as they involve zero-mean random variables. We can now present the uniform consistency result for the estimated factors.

Proposition 1. *Under Assumptions A-F and additionally if* $\max_t \|F_t\| = O_p(\alpha_T)$, *and* $T/N^2 \to 0$,

$$\max_t \|\tilde{F}_t - H'F_t\| = O_p\left(\frac{T^{1/8}}{N^{1/2}}\right) + O_p(\alpha_T N^{-1/2}) + O_p(\alpha_T T^{-1}) + O_p(C_{NT}^{-1}).$$

Proof. See the Appendix.

Proposition 1 shows that the maximum deviation between the factors and the principal components depends on α_T. The convergence rate thus depends on the assumption imposed on $\max_t \|F_t\|$. The factors can be modelled as a dynamic process with arbitrary dynamics to determine α_T. However, if the parameters governing these dynamics are not of direct interest, nothing is lost by assuming the factors to be a sequence of fixed and bounded constants. Thus $\max_t \|F_t\| \leq M$.[3] We can take $O_p(\alpha_T)$ to be $O(1)$ in our results, and $\max_t \|\tilde{F}_t - H'F_t\| = o_p(1)$. However, Proposition 1 is of independent interest, e.g. for deriving the limiting distribution for the maximum likelihood estimator, so we state Proposition 1 in its more general form. Bai (2003) and Bai and Ng (2008a) derive a similar result for factor models with constant loadings. Uniform convergence when loadings undergo small variations is also considered by Stock and Watson (1998), who obtain a much slower convergence rate and require $T = o(N^{1/2})$. Thus, Proposition 1 extends the uniform consistency result to the case of time-varying loadings.

[3]Bai and Li (2012a,b) treat the factors as a sequence of fixed constants when providing inferential theory for maximum likelihood estimation of factor models with constant loadings.

3.2 Identification

It is well known that without identifying restrictions, factors and loadings are not separately identified in (1). The common component $C_t = \Lambda_t F_t$ is identified, but normalizations are needed to separate factor and loadings from the common component. This has implications for the identification of the loadings parameters as well, which we now illustrate. The model defined by (1) and (2) is observationally equivalent to:

$$\begin{aligned}
&X_t = \Lambda_t H'^{-1} H' F_t + e_t, \\
&B_i(L) H^{-1}(\lambda_{it} - \lambda_i) = H^{-1}\eta_{it}, \qquad for \quad i = 1, ..., N.
\end{aligned}$$

Lemma 1 states that the principal components estimator $\tilde{F}_t$ is a consistent estimate of a rotation of the true factors, $H'F_t$. The two-step estimation procedure fixes the rotational indeterminacy by imposing the normalization in the principal components step. By replacing the unobserved factors F_t with $\tilde{F}_t$ for maximum likelihood estimation, we are thus estimating the parameters of $\lambda^*_{it} = H^{-1}\lambda_{it}$.

To clarify the issue, consider the following example. Using the same notation as previously, the elements of the $r \times 1$ vector $\lambda_{it} = (\lambda_{it,1}..., \lambda_{it,r})'$ refer to the loadings of variable i at time t on each of the r factors, and $\lambda_i = E(\lambda_{it}) = (\lambda_{i,1}, ..., \lambda_{i,r})'$ are the corresponding unconditional expectations of the factor loadings. Assume that the matrices Σ_F and Σ_Λ are diagonal. In this case it is not hard to show that the rotation matrix H converges to $\Sigma_F^{-1/2}$. Let the number of factors $r = 2$ with variance $\Sigma_F = diag(\sigma_1^2, \sigma_2^2)$ and let the data-generating parameters of the loadings be

$$\lambda_i^0 = \begin{pmatrix} \lambda_{i,1} \\ \lambda_{i,2} \end{pmatrix}, \quad Q_i^0 = \begin{pmatrix} q_{i,1} & 0 \\ 0 & q_{i,2} \end{pmatrix}, \quad B_i^0(L) = I_2 - \begin{pmatrix} b_{i,11} & 0 \\ 0 & b_{i,22} \end{pmatrix}.$$

We can now make precise what $\tilde{\theta}$ is estimating. With the normalization $\tilde{F}'\tilde{F}/T = I_2$, the principal components will be close to $\Sigma_F^{-1/2} F_t$ in large samples. Using the principal components in place of the unobserved factors means that we are estimating the following model:

$$\begin{aligned}
&X_t = \Lambda^*_t \tilde{F}_t + e_t, \\
&\lambda^*_{it} - \lambda^*_i = B^*_i(\lambda^*_{i,t-1} - \lambda^*_i) + v_{it}, \qquad for \quad i = 1, ..., N,
\end{aligned}$$

where $\lambda^*_{it} = \Sigma_F^{-1/2}\lambda_{it} = \begin{pmatrix} \sigma_1^{-1}\lambda_{it,1} \\ \sigma_2^{-1}\lambda_{it,2} \end{pmatrix}$ and $v_{it} = \Sigma_F^{-1/2}\eta_{it}$. The loadings λ_{it} are scaled by the standard deviations of the unobserved factors, and it is the parameters of the rotated loadings λ^*_{it} that can be estimated. In large samples the estimate of the loadings mean λ^*_i will be therefore close to

$$\Sigma_F^{-1/2}\lambda_i^0 = \begin{pmatrix} \sigma_1^{-1}\lambda_{i,1} \\ \sigma_2^{-1}\lambda_{i,2} \end{pmatrix},$$

and the variance estimate $Var(v_{it}) = Var(\Sigma_F^{-1/2}\eta_{it})$ will be close to

$$\Sigma_F^{-1/2} Q_i^0 \Sigma_F^{-1/2} = \begin{pmatrix} \sigma_1^{-2} q_{i,1} & 0 \\ 0 & \sigma_2^{-2} q_{i,2} \end{pmatrix}.$$

The mean and variance parameters are thus scaled by the standard deviation of the factors. The matrices $B_i(L)$ and Q_i^0 of the data-generating model are diagonal in this example, so the diagonal elements of B_i^* are the autocorrelations of $\lambda_{it,1}^*$ and $\lambda_{it,2}^*$. In large samples the first diagonal element of B_i^* will therefore be close to

$$\begin{aligned} b_{i,11}^* &= \frac{Cov(\lambda_{it,1}^*, \lambda_{i,t-1,1}^*)}{Var(\lambda_{it,1}^*)} = \frac{Cov(\sigma_1^{-1}\lambda_{it,1}, \sigma_1^{-1}\lambda_{i,t-1,1})}{Var(\sigma_1^{-1}\lambda_{it,1})} \\ &= \frac{\sigma_1^{-2} Cov(\lambda_{it,1}, \lambda_{i,t-1,1})}{\sigma_1^{-2} Var(\lambda_{it,1})} = \frac{Cov(\lambda_{it,1}, \lambda_{i,t-1,1})}{Var(\lambda_{it,1})} = b_{i,11}, \end{aligned}$$

and similarly for $b_{i,22}^*$. The estimates of the autoregressive matrix B_i^* are therefore unaffected by the normalization imposed on the principal components, and the estimate of B_i^* is consistent for the autoregressive parameters B_i of the data-generating process λ_{it}.

The arguments of this example apply to the general setting as well. The maximum likelihood estimator (5) of the loadings parameters is estimating $B_i(L)$, $H^{-1}\lambda_i$, and $H^{-1}Q_i H'^{-1}$. The mean and variance parameters of (2) are identified up to the unknown rotation matrix H, while the dynamic parameters $B_i(L)$ are not subject to any rotation. The rotation is determined by the restriction used to identify the principal components. Using another normalization in the first step will thus change the estimates of λ_i and Q_i, while the estimate of $B_i(L)$ is unaffected, except for small numerical differences owing to numerical optimization of the likelihood. The dynamic properties of the loadings are therefore uniquely identified. In the following, we assume for simplicity that $H = I_r$. This is just a normalization and can be achieved by imposing further assumptions on the matrices Σ_F and Σ_Λ.

3.3 Maximum Likelihood Estimation

Our method of proof relies on showing that the likelihood function (4) with principal components is asymptotically equivalent to the likelihood function with unobserved factors. To establish our result, we impose distributional assumptions on the loadings and idiosyncratic errors that enable maximum likelihood estimation of the parameters $\theta_i = \{B_i(L), \lambda_i, Q_i, \psi_i\}$. We make the following assumptions:

Assumption G (Distributions) For all $i = 1, ..., N$, the following statements hold:

1. The loadings λ_{it} follow a finite-order Gaussian VAR:

$$B_i(L)(\lambda_{it} - \lambda_i) = \eta_{it},$$

with the $r \times r$ filter $B_i(L) = I - B_{i,1}L - ... - B_{i,p}L^p$ having roots outside the unit circle, and η_{it} is an r-dimensional Gaussian white noise process, $\eta_{it} \sim$ i.i.d. $\mathcal{N}(0, Q_i)$, where Q_i is positive definite with all elements bounded.

2. The idiosyncratic errors e_t are cross-sectionally independent Gaussian white noise, $e_t \sim \mathcal{N}(0, \Psi)$, where Ψ is a diagonal matrix with elements $\psi_i > 0$ and bounded for all i.

G.1 assumes the loadings to evolve as stationary vector autoregressions. We rule out the possibility of $I(1)$ loadings as this would be in violation of Assumption D. With non-stationary loadings the principal components estimator cannot consistently estimate the factor space.[4] G.2 assumes the idiosyncratic errors to be i.i.d. over both t and i. This assumption can be relaxed to allow for serial correlation. The key part of Assumption G.2 is the cross-sectional independence. This enables us to analyse the likelihood separately for each i. This is a set of N independent univariate regressions with time-varying parameters. With observed regressors, consistency is known to hold, see e.g. Pagan (1980). Rather than proving consistency for the maximum likelihood estimator with observed factors we therefore assume consistency in the following assumption.

Assumption H (MLE with observed factors) For each i, the function $\mathcal{L}_T(X_i|F;\theta_i)$ satisfies:

1. There exists a function $\mathcal{L}_0(X_i|F;\theta_i)$ that is uniquely maximized at θ_0.

2. θ_i^0 is in the interior of a convex set Θ_i, and $\mathcal{L}_T(X_i|F;\theta_i)$ is concave.

3. $\mathcal{L}_T(X_i|F;\theta_i) \xrightarrow{p} \mathcal{L}_0(X_i|F;\theta_i)$ for all $\theta_i \in \Theta_i$.

Under Assumptions G and H, the maximum likelihood estimator with observed factors $\hat{\theta}_i = \underset{\theta}{\operatorname{argmax}}\, \mathcal{L}_T(X_i|F;\theta_i)$ exists with probability approaching 1 and is consistent for each i: $\hat{\theta}_i \xrightarrow{p} \theta_i^0$. This follows from standard arguments as in Newey and McFadden (1994).

Replacing the unobserved factors with the principal components estimates yields the feasible likelihood function $\tilde{\mathcal{L}}_T(X_i|\tilde{F};\theta_i)$ and the maximum likelihood estimator defined in (5). We now state our main result.

Theorem 1. *Let Assumption A-H hold. For each i, the estimator $\tilde{\theta}_i$ defined in (5) exists with probability approaching 1 and*

$$\tilde{\theta}_i \xrightarrow{p} \theta_i^0.$$

Proof. See the Appendix.

[4] Bates et al. (2013) consider random walk loadings of the form $\lambda_{it} = \lambda_{i,t-1} + T^{-3/4}\zeta_{it}$ and show that Assumption D is satisfied with this specification. However, the scaling of the loadings innovations by the factor $T^{-3/4}$ is crucial for Lemma 1 to hold. With a pure random walk of the form $\lambda_{it} = \lambda_{i,t-1} + \zeta_{it}$, principal components cannot estimate the factor space consistently.

Theorem 1 states that using the principal component estimates instead of the unobserved factors does not affect the consistency of the maximum likelihood estimator. The main argument in proving Theorem 1 is that the feasible likelihood function convergences uniformly to the infeasible likelihood function. Asymptotically, the feasible likelihood function therefore has the same properties as the infeasible likelihood function, for which consistency is known to hold. Assumption H thus holds for $\tilde{\mathcal{L}}_T(X_i|\tilde{F};\theta_i)$ and consistency follows. In the proof of Theorem 1 we use the following normalization that is convenient for the calculations: If $F'F/T = I_r$ and $\Lambda^{0\prime}\Lambda^0$ is a diagonal matrix with distinct elements, we show in the Appendix that the rotation matrix H converges to the identity I_r. Lemma 1 and Proposition 1 then holds with H replaced by the identity matrix, and θ_i can be estimated asymptotically without rotation. Such normalizations are inconsequential for the results as H is asymptotically bounded, and they are only imposed to avoid unnecessary complications. Without such normalizations the feasible likelihood converges to $\mathcal{L}_T(X_i|FH;\theta_i)$ and $\tilde{\theta}_i$ is consistent for the parameters of the process $\lambda^*_{it} = H^{-1}\lambda_{it}$ as discussed in Section 3.2.

We have assumed that the factors are estimated by the method of principal components. Note, however, that the proof of Theorem 1 does not rely on the principal components estimator. Theorem 1 holds for all estimators $\tilde{F}$ that satisfy the conditions for Lemma 1 and Proposition 1.

Our analysis does not make any formal statements about the limiting distribution of $\tilde{\theta}_i$. Simulation evidence in Section 4 does, however, suggest that an asymptotic normality result holds for $\tilde{\theta}_i$ as well. The simulations further indicate that the limiting distribution of $\tilde{\theta}_i$ is unaffected by the estimation error of F_t. Two-step estimators typically require adjusting the limiting distribution to account for estimation error from first-step estimation as in Newey (1984) and Pagan (1986), but that does not seem to be the case here. Bai and Ng (2006) show that the estimated-regressor problem can be ignored when using principal components in place of the unobserved factors in factor-augmented VARs. We expect that a similar result holds for our model, but leave a formal proof for future research.

In Assumption G.2 and the proof of Theorem 1 we assume that the model has an exact factor structure in the sense that idiosyncratic errors have no cross-sectional or temporal dependence. It is straightforward to relax the assumption of no temporal dependence. We could model the idiosyncratic errors as cross-sectionally uncorrelated autoregressions and estimate the parameters by including e_{it} in the state equation of the state space representation of the model and compute the likelihood with the Kalman filter. The proof of Theorem 1 applies with very minor changes. The assumption of no temporal dependence in e_{it} is thus only for expositional simplicity.

Relaxing the assumption of no cross-sectional correlation to allow for an approximate factor structure requires substantially more work. The essential contribution of Assumption G.2 is that it enables the likelihood of the full panel of data $X = (X_1, ..., X_N)$ to be analysed separately for each X_i. When the idiosyncratic elements of the model are cross-sectionally correlated, $E(e_t e_t') = \Psi_0$ is non-diagonal, and we cannot condition on the factors to make X_i independent over i. Analysis of the model with cross-sectionally correlated errors requires a different method of proof and is beyond the scope of this paper.[5] However, in the next section we provide simulation evidence showing that our results are robust to the assumption of no cross-sectional correlation in the idiosyncratic elements.

4 Monte Carlo Simulations

In this section, we conduct a simulation study to assess the finite-sample performance of the two-step estimator. We provide results for both principal components and maximum likelihood estimates. Section 4.1 describes the simulation design, and Section 4.2 reports and discusses the results.

4.1 Design

The simulation design broadly follows that of Stock and Watson (2002):

$$\begin{aligned}
&X_{it} = \lambda_{it}' F_t + e_{it}, && \\
&(I_r - B_i L)(\lambda_{it} - \lambda_i) = \eta_{it}, && \eta_{it} \sim \text{i.i.d. } \mathcal{N}(0, Q_i), \\
&F_{tp} = \rho F_{t-1,p} + u_{tp}, && u_{tp} \sim \text{i.i.d. } \mathcal{N}(0, 1-\rho^2), \\
&(1 - \alpha L) e_{it} = v_{it}, && v_t \sim \text{i.i.d. } \mathcal{N}(0, \Omega),
\end{aligned}$$

where $i = 1, ..., N,\ t = 1, ..., T,\ p = 1, ..., r$. The processes $\{\eta_{it}\}$,$\{u_{tp}\}$, and $\{v_t\}$ are mutually independent. The autoregressive matrix B_i determines the degree of persistence of the loadings and has eigenvalues inside the unit circle in all simulations. The unconditional mean of the loadings is $\lambda_i = (\lambda_{i1}, ..., \lambda_{ir})'$ and $\lambda_{ip} \sim$ i.i.d. $\mathcal{N}(0, 1)$ in all simulations. The matrix Q_i is the covariance matrix of the loadings innovations. The model allows for cross-sectional and temporal dependence in the errors e_{it}. The parameter α determines the degree of serial correlation in the idiosyncratic errors, and cross-sectional correlation is modelled by specifying the variance matrix of v_t as $\Omega = \left(\beta^{|i-j|}\sqrt{\psi_i \psi_j}\right)_{ij}$ for $i, j = 1, ..., N$. The matrix is thus a Toeplitz matrix and the cross-sectional correlation between the idiosyncratic elements is therefore limited and determined by the coefficient β. If $\beta = 0$, $\alpha = 0$, e_{it} is independent across i and t, and the model is an exact factor model and correctly specified according to Assumption G. We allow for factor persistence through the coefficient ρ.

We generate the model 2000 times for each of the different combinations of T and N. To avoid any dependence on initial values of the simulated processes we have a 'burn-in' period of 200 observations for each simulation. The principal components are calculated with the estimator $\tilde{F}_t$ defined in (6). The data X_{it} are standardized to have mean zero and variance equal to one prior to

[5] Bai and Li (2012a) analyse factor models with constant loadings in a maximum likelihood setting under weak cross-sectional correlation in e_{it}. They show that the estimates of ψ_i^0 are consistent for the diagonal elements of $T^{-1}\sum_{t=1}^{T} E(e_t e_t')$. However, they rely on a method of proof that differs substantially from ours and their results do not readily apply to our model.

extracting principal components. The principal components are identified only up to an orthogonal rotation. In order to directly compare the maximum likelihood estimates with data-generating parameters, we therefore rotate the principal components to resemble the simulated factors. More specifically, we solve for the orthogonal $r \times r$ matrix A^* that maximizes $\mathrm{tr}[\mathrm{corr}(F, \tilde{F}A)]$.[6] The estimates are then rescaled to have the same standard deviation as the true simulated factors:

$$\tilde{F}_p^* = \frac{\sigma(F_p)}{\sigma(\tilde{F}_p)}\tilde{F}_p, \quad p = 1, ..., r$$

where $\tilde{F}_p$ is the p^{th} column of the rotated principal components matrix $\tilde{F}A^*$. Such rotations are innocuous and allow us to directly compare the estimated parameter values with the data-generating parameters. The principal components are treated as data, and we maximize the likelihood $\tilde{\mathcal{L}}_T(X_i|\tilde{F}^*; \theta_i)$ to estimate θ_i.

The performance of the principal component estimator $\tilde{F}$ is measured by the trace statistic:

$$R^2_{\tilde{F},F} = \frac{\hat{E}[\mathrm{tr}(F'\tilde{F}(\tilde{F}'\tilde{F})^{-1}\tilde{F}'F)]}{\hat{E}[\mathrm{tr}(F'F)]},$$

where $\hat{E}$ denotes the average over the Monte Carlo simulations. The trace statistic $R^2_{\tilde{F},F}$ is a multivariate R^2 from a regression of the true data-generating factors on the principal components. It is smaller than 1 and tends to 1 as the canonical correlation between the factors and the principal components tends to 1.

For the maximum likelihood estimates $\tilde{\theta}_i$ we compute the mean estimates over the Monte Carlo repetitions for each parameter.[7] However, for the mean parameter λ_i we report the bias of the estimates $\tilde{\lambda}_i$ as the true value of λ_i changes for each combination of N, T. Furthermore, we calculate the root-mean-squared error of the estimates $\tilde{\theta}_i$ and also of the infeasible estimates $\hat{\theta}_i$ where the true data-generating factors are used in the maximum likelihood estimation. We report the relative root-mean-squared error between the estimates $\tilde{\theta}_i$ and $\hat{\theta}_i$. This gives us a measure of the estimation error in $\tilde{\theta}_i$ that is due to estimation error from the principal components estimates.

The parameters are identically chosen across the cross-section.[8] The properties of the estimated parameters $\tilde{\theta}_i$ are thus the same for all i and we only report the results for a single cross-section index.[9] In the baseline case, we set $B_i = \mathrm{diag}\{b_{ip}\}_{p=1,...,r}$, $Q_i = \mathrm{diag}\{q_{ip}\}_{p=1,...,r}$, and choose the

[6] The solution to this is $A^* = VU'$ where V and U are the orthogonal matrices of the singular value decomposition $\mathrm{corr}(F, \tilde{F}) = USV'$. When the number of principal components k is not equal to the true number of factors r, we only rotate the first $l = \min\{k, r\}$ principal components. Eickmeier et al. (2015) use the same rotation.

[7] Convergence is generally very good, with all 1-factor calibrations having over 99% convergence rate, and most calibrations with 2 and 3 factors have over 98% convergence rate. Exceptions are sample sizes of $T = 50$ for the 2- and 3-factor models where the lowest convergence rate is 92%. However, this is expected as we are estimating up to 10 parameters in a highly non-linear model with 50 observations. Convergence statistics using the true factors are similar, but with somewhat better convergence rates for calibrations with 2 and 3 factors and $T = 50$.

[8] The mean parameters λ_i are not the same for all i. This is necessary for Assumption B to be satisfied. With λ_i identical over i, the matrix Λ^0 does not have full rank and $\Lambda^{0\prime}\Lambda^0/N$ will not converge to a positive definite matrix.

[9] Simulations with loadings parameters calibrated with heterogeneous values across i show similar results as in Table 1. The results are available upon request.

loadings persistence and variance parameters to be $b_{ip} = 0.9$ and $q_{ip} = 0.2$. The idiosyncratic errors are cross-sectionally and temporally uncorrelated, i.e. $\alpha = 0$, $\beta = 0$, and the variance is set at $\psi_i = 1$. Finally, we set $\rho = 0$ such that the factors are white noise.

We introduce serial correlation and cross-sectional dependence separately in the idiosyncratic errors. We set $\alpha = 0.5$ and estimate this parameter by including e_{it} in the state equation. To consider the effect of cross-sectional correlation, i.e. misspecifying the model, we set $\beta = 0.5$. We also report results with persistent factors with the factor persistence set at both 0.9 and 0.5. Finally, we consider the consequences of estimating the wrong number of factors, i.e. extracting one fewer or one additional principal component than the true number of factors.

4.2 Results

Table 1 reports the results for one factor, $r = 1$. Panel I shows the results for the baseline model with no serial, no cross-sectional dependence in errors, and no factor dependence. The $R^2_{\tilde{F},F}$ statistics show that the factor estimates are close to the true factors even for small sample sizes. For the autoregressive parameter b_i, the estimates improve as the sample size T increases. Increasing the cross-sectional dimension N only gives minor improvements for fixed T. This is unsurprising as a larger N can only improve the parameter estimates through better factor estimates which are already quite good even for $N = 50$. The estimate of the loadings innovation variance q_i is closely related to the estimate of b_i. As b_i gets closer to its true value, so does q_i, and vice versa. For $T \geq 200$ the estimates are close to the true values. The small-sample bias of b_i is not a consequence of estimation error from principal components. Using the true factors instead of principal components to estimate the parameters of the latent process λ_{it} also shows that $T \geq 200$ is needed for the bias of b_i and q_i to be less than 10% of the true value. The loadings mean λ_i and the error variance ψ_i are very precisely estimated for all sample sizes.

In Panel II, the idiosyncratic errors are serially correlated, and the autoregressive parameters for the errors are estimated along with the other parameters. The $R^2_{\tilde{F},F}$ statistic is hardly affected by serially correlated errors. The results are very close to the corresponding values in the first panel. The results for the loadings parameters are also very similar and are not markedly affected. The autoregressive parameter for the errors α and the variance parameter ψ are very close to their true value for all sample sizes. The model with serially correlated errors can thus be estimated equally well as the model with i.i.d. errors.

Table 1 - Simulation results for 1-factor model								
	T	N	$R^2_{\tilde{F},F}$	b_i	λ_i	q_i	ψ_i	α
Panel	True values			0.9	0	0.2	1	
	$\alpha = 0, \quad \beta = 0, \quad \rho = 0$							
I	50	50	0.943	0.603	-0.022	0.302	0.970	-
	100	50	0.941	0.785	-0.005	0.258	1.005	-
	50	100	0.960	0.607	0.023	0.307	0.978	-
	100	100	0.969	0.792	-0.043	0.260	1.010	-
	100	200	0.976	0.797	-0.011	0.256	0.993	-
	200	200	0.984	0.864	0.003	0.224	1.003	-
	400	200	0.986	0.884	-0.019	0.213	1.009	-
	600	300	0.991	0.890	0.012	0.208	0.997	-
	$\alpha = 0.5, \quad \beta = 0, \quad \rho = 0$							
II	50	50	0.939	0.617	-0.005	0.296	0.934	0.500
	100	50	0.934	0.805	-0.009	0.246	0.992	0.494
	50	100	0.956	0.614	0.033	0.290	0.929	0.494
	100	100	0.966	0.805	-0.060	0.246	0.978	0.509
	100	200	0.974	0.812	0.015	0.242	0.969	0.493
	200	200	0.982	0.870	0.001	0.223	0.990	0.497
	400	200	0.985	0.885	-0.027	0.212	0.994	0.500
	600	300	0.989	0.890	0.011	0.206	0.995	0.499
	$\alpha = 0, \quad \beta = 0.5, \quad \rho = 0$							
III	50	50	0.944	0.540	-0.042	0.334	1.279	-
	100	50	0.942	0.762	-0.002	0.276	1.326	-
	50	100	0.959	0.556	0.029	0.335	1.295	-
	100	100	0.968	0.777	-0.029	0.275	1.324	-
	100	200	0.976	0.775	-0.004	0.273	1.328	-
	200	200	0.983	0.857	-0.005	0.235	1.326	-
	400	200	0.986	0.881	-0.019	0.215	1.353	-
	600	300	0.990	0.889	0.003	0.210	1.321	-
	$\alpha = 0, \quad \beta = 0, \quad \rho = 0.9$							
IV	50	50	0.652	0.587	0.000	0.355	1.177	-
	100	50	0.785	0.784	0.019	0.255	1.134	-
	50	100	0.651	0.606	-0.031	0.392	1.257	-
	100	100	0.809	0.828	0.010	0.291	1.385	-
	100	200	0.806	0.781	0.009	0.269	1.116	-
	200	200	0.896	0.865	-0.024	0.237	1.065	-
	400	200	0.941	0.894	-0.050	0.221	1.122	-
	600	300	0.961	0.895	0.026	0.211	1.040	-
	$\alpha = 0, \quad \beta = 0, \quad \rho = 0.5$							
V	50	50	0.905	0.625	-0.033	0.306	1.017	-
	100	50	0.920	0.803	-0.010	0.249	1.023	-
	50	100	0.916	0.606	0.044	0.307	1.045	-
	100	100	0.950	0.803	-0.060	0.254	1.076	-
	100	200	0.955	0.803	0.019	0.254	1.015	-
	200	200	0.973	0.865	-0.010	0.226	1.012	-
	400	200	0.981	0.885	-0.022	0.214	1.026	-
	600	300	0.987	0.889	0.012	0.209	1.006	-

Notes: The columns T and N report the sample sizes. The column $R^2_{\tilde{F},F}$ reports the convergence statistic for the principal components estimator. The remaining columns report the mean of the parameter estimates over the Monte Carlo simulations. For the parameter λ_i, the bias is reported.

Table 2 - Relative root-mean-squared error for 1-factor model

Panel	T	N	$R^2_{\tilde{F},F}$	b_i	λ_i	q_i	ψ_i	α
	True values			0.9	0	0.2	1	
			$\alpha = 0, \quad \beta = 0, \quad \rho = 0$					
I	50	50	0.943	1.043	1.056	0.983	1.024	-
	100	50	0.941	1.083	1.062	0.986	1.024	-
	50	100	0.960	1.032	1.042	1.039	1.030	-
	100	100	0.969	1.065	1.033	1.017	1.076	-
	100	200	0.976	1.009	1.033	1.032	1.020	-
	200	200	0.984	1.029	1.019	0.992	1.005	-
	400	200	0.986	0.995	1.013	1.008	1.039	-
	600	300	0.991	0.979	1.011	0.989	0.997	-
			$\alpha = 0.5, \quad \beta = 0, \quad \rho = 0$					
II	50	50	0.939	1.040	1.065	1.063	0.997	1.006
	100	50	0.934	1.022	1.072	0.985	1.025	1.014
	50	100	0.956	1.021	1.054	0.989	0.988	1.001
	100	100	0.966	1.105	1.051	1.032	1.015	1.017
	100	200	0.974	0.999	1.043	1.020	0.998	1.002
	200	200	0.982	0.977	1.021	1.011	0.998	0.999
	400	200	0.985	0.985	1.019	1.018	1.015	0.999
	600	300	0.989	0.980	1.012	1.007	1.000	1.007
			$\alpha = 0, \quad \beta = 0.5, \quad \rho = 0$					
III	50	50	0.944	1.013	1.055	0.998	1.053	-
	100	50	0.942	1.081	1.066	1.043	1.095	-
	50	100	0.959	1.016	1.041	0.989	1.068	-
	100	100	0.968	0.970	1.021	0.984	1.133	-
	100	200	0.976	1.022	1.033	1.011	1.049	-
	200	200	0.983	0.962	1.019	0.996	1.035	-
	400	200	0.986	1.001	1.013	0.995	1.076	-
	600	300	0.990	0.981	1.006	0.984	0.982	-
			$\alpha = 0, \quad \beta = 0, \quad \rho = 0.9$					
IV	50	50	0.652	1.060	1.147	1.339	1.926	-
	100	50	0.785	1.106	1.127	1.060	1.778	-
	50	100	0.651	1.038	1.147	1.451	2.415	-
	100	100	0.809	0.854	1.166	1.468	4.506	-
	100	200	0.806	1.047	1.085	1.083	1.665	-
	200	200	0.896	0.947	1.085	1.101	1.455	-
	400	200	0.941	0.851	1.099	1.130	2.743	-
	600	300	0.961	0.921	1.044	1.021	1.432	-
			$\alpha = 0, \quad \beta = 0, \quad \rho = 0.5$					
V	50	50	0.905	1.061	1.064	0.936	1.086	-
	100	50	0.920	1.043	1.067	1.049	1.039	-
	50	100	0.916	1.024	1.046	1.030	1.241	-
	100	100	0.950	0.956	1.043	1.060	1.415	-
	100	200	0.955	1.090	1.040	1.070	1.057	-
	200	200	0.973	1.012	1.028	1.023	1.038	-
	400	200	0.981	0.984	1.014	1.013	1.169	-
	600	300	0.987	0.971	1.011	0.990	1.040	-

Notes: The columns T and N report the sample sizes. The column $R^2_{\tilde{F},F}$ reports the convergence statistic for the principal components estimator. The remaining columns report the relative root-mean-squared error of the parameter estimates using principal components and the true simulated factors.

Table 3 - Simulation results for 2- and 3-factor model

Panel	T	N	$R^2_{\tilde{F},F}$	b_{i1}	λ_{i1}	q_{i1}	b_{i2}	λ_{i2}	q_{i2}	b_{i3}	λ_{i3}	q_{i3}	ψ_i
	True values			0.9	0	0.2	0.9	0	0.2	0.9	0	0.2	1
	Two factors − $\alpha = 0$, $\beta = 0$, $\rho = 0$												
I	50	50	0.920	0.615	0.009	0.277	0.447	-0.156	0.379	-	-	-	1.146
	100	50	0.934	0.825	0.018	0.236	0.731	0.033	0.287	-	-	-	1.017
	50	100	0.953	0.629	-0.017	0.271	0.490	-0.080	0.365	-	-	-	0.971
	100	100	0.953	0.819	0.047	0.234	0.735	-0.041	0.287	-	-	-	1.032
	100	200	0.973	0.813	-0.056	0.240	0.739	-0.051	0.291	-	-	-	1.088
	200	200	0.978	0.872	-0.006	0.216	0.854	0.002	0.239	-	-	-	1.006
	400	200	0.982	0.888	-0.020	0.205	0.883	0.036	0.217	-	-	-	1.029
	600	300	0.987	0.892	0.004	0.204	0.891	0.021	0.210	-	-	-	1.013
	Three factors − $\alpha = 0$, $\beta = 0$, $\rho = 0$												
II	50	50	0.883	0.584	-0.000	0.244	0.434	-0.143	0.334	0.340	0.006	0.516	1.396
	100	50	0.875	0.813	0.029	0.196	0.755	-0.038	0.245	0.587	-0.181	0.372	1.353
	50	100	0.906	0.605	-0.010	0.240	0.476	-0.051	0.333	0.351	-0.151	0.536	1.134
	100	100	0.938	0.831	-0.021	0.216	0.770	-0.103	0.262	0.639	0.051	0.361	1.132
	100	200	0.960	0.838	0.023	0.219	0.774	0.034	0.262	0.654	-0.120	0.365	1.040
	200	200	0.974	0.875	-0.014	0.210	0.862	0.014	0.226	0.823	0.062	0.276	1.015
	400	200	0.975	0.889	0.022	0.207	0.886	0.043	0.214	0.880	0.051	0.231	1.099
	600	300	0.985	0.893	-0.001	0.204	0.890	-0.003	0.210	0.887	-0.003	0.216	1.025

Notes: The columns T and N report the sample sizes. The column $R^2_{\tilde{F},F}$ reports the convergence statistic for the principal components estimator. The remaining columns report the mean of the parameter estimates over the Monte Carlo simulations. For the parameter λ_i, the bias is reported.

Table 4 - Relative root mean squared errors for 2- and 3-factor model

Panel	T	N	$R^2_{\tilde{F},F}$	b_{i1}	λ_{i1}	q_{i1}	b_{i2}	λ_{i2}	q_{i2}	b_{i3}	λ_{i3}	q_{i3}	ψ_i
	True values			0.9	0	0.2	0.9	0	0.2	0.9	0	0.2	1
	Two factors — $\alpha = 0$, $\beta = 0$, $\rho = 0$												
I	50	50	0.920	1.210	1.105	1.151	1.061	1.167	1.446	-	-	-	1.462
	100	50	0.934	0.989	1.063	1.079	1.048	1.090	1.027	-	-	-	1.040
	50	100	0.953	1.087	1.059	1.049	1.054	1.086	1.110	-	-	-	1.069
	100	100	0.953	1.250	1.055	1.030	1.097	1.093	1.062	-	-	-	1.118
	100	200	0.973	1.059	1.059	1.059	1.053	1.112	1.082	-	-	-	1.388
	200	200	0.978	0.978	1.027	1.007	0.929	1.039	1.019	-	-	-	1.027
	400	200	0.982	0.988	1.027	1.010	0.986	1.069	1.012	-	-	-	1.082
	600	300	0.987	0.963	1.037	1.000	0.970	1.022	1.002	-	-	-	1.042
	Three factors — $\alpha = 0$, $\beta = 0$, $\rho = 0$												
II	50	50	0.883	1.409	1.113	1.221	1.187	1.140	1.372	1.030	1.566	1.557	1.788
	100	50	0.875	1.604	1.082	0.999	1.236	1.119	1.164	1.199	1.877	1.466	1.821
	50	100	0.906	1.359	1.091	1.178	1.114	1.112	1.387	1.016	1.537	1.727	1.403
	100	100	0.938	1.204	1.079	1.021	1.150	1.104	1.121	1.016	1.312	1.286	1.285
	100	200	0.960	1.235	1.051	1.051	1.087	1.082	1.081	1.029	1.217	1.186	1.109
	200	200	0.974	1.073	1.030	1.024	1.062	1.053	1.045	1.029	1.100	1.156	1.047
	400	200	0.975	0.967	1.036	1.041	1.003	1.090	1.091	1.063	1.195	1.109	1.320
	600	300	0.985	0.970	1.014	1.021	0.987	1.021	1.037	0.893	1.033	1.061	1.051

Notes: The columns T and N report the sample sizes. The column $R^2_{\tilde{F},F}$ reports the convergence statistic for the principal components estimator. The remaining columns reports the relative root-mean-squared error of the parameter estimates using principal components and the true simulated factors.

Next, in Panel III we consider the effect of cross-sectional correlation in the errors. The factor estimates are again not affected. The misspecification of the variance matrix for the idiosyncratic errors deteriorates the estimates of b_i and q_i for the smaller sample sizes. Compared to the results for i.i.d. errors, estimates are worse, but do converge for the larger sample sizes. The mean parameter λ_i is not affected. One can show that the information matrix of the likelihood is block diagonal between λ_i and the other parameters, so misspecifying the variance does not lead to estimation error in the mean. Cross-sectionally correlated errors inflate the estimate of the idiosyncratic error variance ψ_i, however, but the loadings parameters remain consistent. This indicates that our result in Theorem 1 is robust to the assumption of no cross-sectional correlation in e_{it}.

High factor persistence has a larger impact on the $R^2_{\tilde{F},F}$ statistic. Panel IV shows much lower values of these statistics for all but the largest sample sizes. However, this estimation error does not seem to influence the estimate of the loadings parameters. The estimates for b_i, and accordingly q_i, are similar to the case of white noise factors. The most notable impact of the lower $R^2_{\tilde{F},F}$ is in the estimate of ψ_i. The increase in factor estimation error seems to inflate the error variance, which is larger for all sample sizes, but the results do show convergence for the largest sample size. Results for more moderate levels of factor persistence are shown in Panel V. The drop in the $R^2_{\tilde{F},F}$ is less severe in this case and the estimate of ψ_i thus less biased.

In Table 2, the relative root-mean-squared errors of the estimates using principal components and the true simulated factors are reported. Values close to 1 indicate that the asymptotic variance of the parameter estimates is unaffected by the estimation error from principal components estimation of the factors. In Panels I-III, all the statistics are close to 1 even for the smallest sample sizes. In Panel IV, the statistics for the loadings parameters are somewhat higher for the smaller sample sizes, but close to one for large sample sizes. The statistics for the idiosyncratic variances are much larger than 1. This is partly due to the bias of these estimates evident in Panel IV of Table 1, but also reflects higher variability of the estimates. High factor persistence thus mainly affects the idiosyncratic variance parameters. Unreported results show that the estimates improve for larger sample sizes. In Panel V, the factor persistence is more moderate and the relative root-mean-squared errors are much closer to 1.

Table 3 displays the simulations results for the model with 2 and 3 factors with i.i.d. errors and white noise factors. Compared to the 1-factor model, the $R^2_{\tilde{F},F}$ statistics are lower, reflecting the increasing difficulties in extracting additional factors. In Panel I, the estimates for the second set of loadings parameters are worse than for the first set and the same pattern is evident for the 3-factor model (Panel II). The results for the third set of loadings parameters are worse than for the second, which are worse than for the first. However, all the estimates are converging to their true values. Compared to the 1-factor model, larger sample sizes are generally needed to get precise estimates due to the increased number of parameters. Introducing serial and cross-sectional correlation in the errors or persistence in factors does not reveal any additional insights compared to the 1-factor model. The results generalize and are therefore omitted. Table 4 shows the relative root-mean-squared errors for the 2- and 3-factor model. The statistics are somewhat larger than 1 for the smaller sample sizes, but get increasingly closer to one as the sample sizes grow. This indicates that the estimation error of the principal components does not affect the asymptotic variance of the estimates.

Table 5 - Simulation results for incorrect number of principal components

Panel	T	N	$R^{2(1)}_{\tilde{F},F}$	$R^{2(2)}_{\tilde{F},F}$	b_{i1}	λ_{i1}	q_{i1}	b_{i2}	λ_{i2}	q_{i2}	ψ_i
	True values				0.9	0	0.2	0.9	0	0.2	1
	1 principal component, two factors — $\alpha = 0,\quad \beta = 0,\quad \rho = 0$										
I	50	50	0.954	0.797	0.497	0.011	0.387	-	-	-	6.703
	100	50	0.956	0.789	0.835	0.056	0.230	-	-	-	2.354
	50	100	0.967	0.807	0.649	-0.070	0.313	-	-	-	3.681
	100	100	0.971	0.808	0.831	0.040	0.238	-	-	-	2.587
	100	200	0.981	0.819	0.817	-0.070	0.248	-	-	-	3.566
	200	200	0.985	0.820	0.873	-0.031	0.218	-	-	-	2.078
	400	200	0.989	0.823	0.885	0.023	0.215	-	-	-	4.885
	600	300	0.992	0.826	0.889	0.023	0.212	-	-	-	6.630
	2 principal components, 1 factor — $\alpha = 0,\quad \beta = 0,\quad \rho = 0$										
II	50	50	0.475	0.951	0.315	-0.022	0.198	0.128	-0.008	0.100	0.886
	100	50	0.474	0.948	0.642	-0.003	0.196	0.241	0.016	0.075	0.928
	50	100	0.482	0.964	0.317	0.005	0.210	0.126	0.006	0.106	0.901
	100	100	0.486	0.972	0.605	-0.010	0.208	0.248	-0.003	0.079	0.961
	100	200	0.489	0.978	0.583	0.008	0.199	0.268	0.013	0.074	0.953
	200	200	0.492	0.984	0.826	-0.007	0.201	0.241	0.002	0.047	0.967
	400	200	0.493	0.986	0.884	-0.018	0.195	0.236	-0.007	0.025	0.994
	600	300	0.495	0.991	0.889	0.014	0.196	0.227	-0.004	0.017	0.991

Notes: The columns T and N report the sample sizes. The columns $R^{2(1)}_{\tilde{F},F}$ and $R^{2(1)}_{\tilde{F},F}$ are the two convergence statistics for the principal components estimator. The remaining columns report the mean of the parameter estimates over the Monte Carlo simulations. For the parameter λ_i, the bias is reported.

Table 5 shows the results of estimating the wrong number of factors. For these simulations, we report two convergence statistics for the principal components. The first is the R^2 from a regression of the principal components on the true factors, $R^{2(1)}_{\tilde{F},F} = \frac{\hat{E}[\tilde{F}'F(F'F)^{-1}F'\tilde{F}]}{\hat{E}[\tilde{F}'\tilde{F}]}$ and the second is the R^2 from a regression of the true factors on the principal components. In Panel I, the simulated data have two factors, but only 1 principal component is extracted. The first statistic $R^{2(1)}_{\tilde{F},F}$ is close to 1 for all sample sizes. Hence, the two factors explain all the variation in the single principal component. The second statistic $R^{2(2)}_{\tilde{F},F}$ does not tend to 1, as a single principal component cannot span the two-dimensional factor space. The results show that the loadings parameters for the first factor can still be estimated consistently. The consequence of excluding a factor is that the estimate of the error variance ψ_i gets larger, reflecting the variability in the data from the excluded factor and its loadings. Panel II displays results for the 1-factor model with two principal components extracted from the data. $R^{2(2)}_{\tilde{F},F}$ tends to 1, and the two principal components thus explain all the variation in the single factor. The other measure $R^{2(1)}_{\tilde{F},F}$ tends to 0.5 as the single factor can only span half of the two-dimensional space of the principal components. The loadings on the first factor are estimated consistently. For the second factor, the mean and variance of the loadings are being estimated as zero.[10] The estimated parameters thus show that the data do not load on the second factor and therefore correctly dismiss the second factor. The results are thus very encouraging even with the number of principal components different from the true number of factors.

The results can be summarized as follows:

- The loadings and idiosyncratic variance parameters are estimated consistently. The sample size T needs to be sufficiently large (≥ 200) for the bias in the autoregressive parameters to be less than 10%.

- The results are robust to the assumption of cross-sectionally uncorrelated errors. The loadings parameters are not affected by this misspecification, only the estimate of the error variance.

- Loadings parameters are consistently estimated even when an incorrect number of principal components are extracted. Too few principal components increase the error variance estimate, and loadings means and variances are correctly estimated as zero for principal components in excess of the true number of factors.

- The relative root-mean-squared errors indicate that the asymptotic variance is unaffected by replacing the factors with the principal components estimates and that the estimates have the same limiting distribution as if the factors are observed.

5 An Empirical Illustration

We provide an empirical illustration of the model using the data set of Stock and Watson (2009), who analyse a balanced panel of 144 quarterly time series for the United States, focusing on structural instability in factor loadings and its consequences for forecast regression. The data set consist of 144 quarterly time series for the United States, spanning 1959:I-2006:IV. The data series are transformed to be stationary, and the first two quarters are thus excluded because of differencing, resulting in $T = 190$ observations used for estimation. We exclude a number of series that are higher-level aggregates of the included series, which brings the number of series used for estimation to $N = 109$. For a complete data description and details on data transformations, see the appendix of Stock and Watson (2009).

Stock and Watson (2009) argue for 4 factors in the sample, and perform robustness checks of their results using different numbers of factors. We therefore extract 4 principal components from the standardized data and estimate the loadings parameters and the idiosyncratic variances for each of the 109 variables. The system matrices B_i and Q_i are specified to be diagonal, i.e., the loadings are estimated as univariate autoregressions uncorrelated over the factor indices. The lag polynomials $B_i(L)$ are of order one for all i.

[10] The results for b_{i2} are not indicative of any convergence. Histograms of the estimated values show that the parameter is not identified as the values are randomly estimated anywhere between -1 and 1.

Stock and Watson (2009) test for breaks in the factor model and find evidence of structural instability in a large number of the factor loadings. Using Chow statistics to test for breaks in loadings in 1984:I, they reject the null of no instability for 41% of the variables. We provide similar evidence of structural instability in the loadings. For each variable X_i, we test the null hypothesis of constant loadings by likelihood ratio statistics. In the restricted model, we thus set the diagonal elements of the matrix Q_i equal to zero, in which case the maximum likelihood estimator of the restricted model can be computed by ordinary least squares. The last column of Table 6 reports the rejection frequencies grouped by variable category as well as the rejection frequency for the entire panel. For 85% of the series, the likelihood ratio statistics rejects at the 5% significance level, and 76% of the series are rejected at the 1% level. When comparing the rejection frequencies across categories, no obvious pattern emerges: The rejection frequencies are high for all categories. The category with the lowest rejection frequency is consumption variables, where the null of constant loadings cannot be rejected for a third of the variables.

Table 6 - Rejection frequencies of likelihood ratio statistics

Category	F_1	F_2	F_3	F_4	All Factors
Output	0.63	0.75	0.38	0.25	0.88
Consumption	0.33	0.33	0.00	0.67	0.67
Labour market	0.70	0.61	0.30	0.17	0.78
Housing	1.00	0.60	0.60	0.20	1.00
Investment	0.63	0.88	0.50	0.38	1.00
Prices & Wages	0.50	0.68	0.21	0.29	0.79
Financial variables	0.68	0.74	0.68	0.53	0.95
Money & Credit	0.50	0.38	0.50	0.50	0.88
Other	0.71	0.71	0.29	0.00	0.86
All	0.62	0.66	0.39	0.31	0.85

Notes: Column F_1 reports the rejection frequencies across variable groups of the likelihood ratio statistics for testing the null hypothesis of constant loadings on the first factor and similarly for columns $F_2 - F_4$. The last column reports the rejection frequencies for the null hypothesis of constant loadings on all factors. All tests are evaluated at the 5% significance level.

We also test the null hypothesis of constant loadings on each individual factor. We reestimate the model for each variable 4 times and let the loadings on a single factor be constant each time, while the rest of the loadings follows first-order autoregressions. Columns $F_1 - F_4$ of Table 6 report the rejection frequencies for these likelihood ratio statistics. For the first two factors, approximately two thirds of the series show evidence of time-varying loadings. For the last two factors, the rejection frequencies are smaller, but time-variation in the loadings is still evident for a non-trivial number of the series. Comparing the rejection frequencies across variable categories does not reveal any general patterns: The rejection frequencies vary a lot across categories and factors. Closer inspection of the individual series indeed reveals that the way in which the data load on the factors is very heterogeneous across series.

Figure 1: Factor loadings for the one-year/three-month Treasury term spread

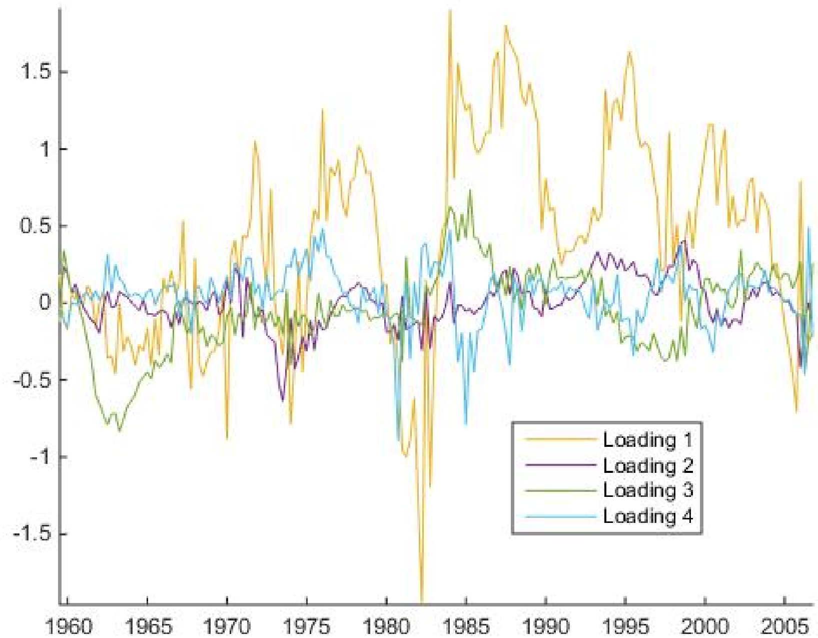
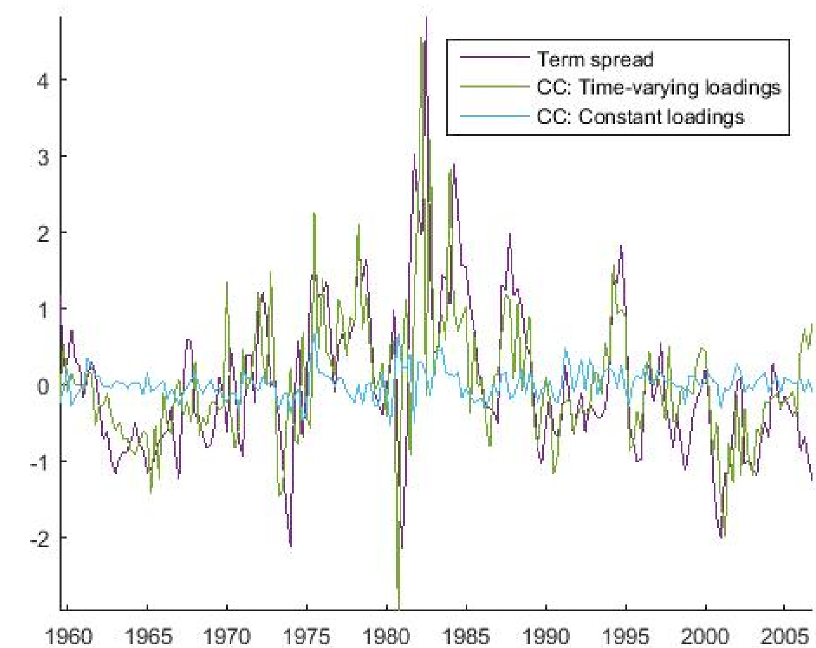


Figure 2: One-year/three-month Treasury term spread and two estimates of its common component (CC)



Comparing our results to those of Stock and Watson (2009), we find that a larger share of the variables have time-varying loadings. However, as our model considers stationary variations around a constant mean, and Stock and Watson (2009) consider a one-time break at a fixed point in time, it is possible for the likelihood ratio statistics to reject the null of constant loadings, even when the Chow tests of Stock and Watson (2009) cannot reject the null of no breaks.

Given the large number of series for which we reject the null of constant loadings, we can obtain a better in-sample fit of the common component by letting the loadings vary over the time dimension. Figure 1 displays the estimated loadings paths for the one-year/three-month Treasury term spread. This series exhibits very strong evidence of structural instability in the loadings. The p-values of the likelihood ratio statistics for joint and single restrictions are all zero to at last 4 decimal places. The loadings show a considerable amount of variation over the sample period with quite persistent dynamics. Figure 2 displays the standardized Treasury term spread and two estimates of its common component, computed with time-varying loadings and constant loadings, respectively. The common component based on time-varying loadings tracks the data much closer than the one with constant loadings. The correlations of the two estimates of the common component with the data are 0.56 and 0.19, respectively. This clearly illustrates the improvement in the in-sample fit by modelling the loadings as autoregressive processes.

6 Conclusion

We proposed a two-step maximum likelihood estimator for time-varying loadings in high-dimensional factor models. The loadings parameters are estimated by a set of N univariate regression models with time-varying coefficients, where the unobserved regressors are estimated by principal components. Replacing the unobservable factors with principal components gives a feasible likelihood function that is asymptotically equivalent to the infeasible one with unobservable factors and therefore gives consistent estimates of the loadings parameters as $N, T \to \infty$. The finite-sample properties of our estimator were assessed via an extensive simulation study. The results showed that the loadings means and idiosyncratic error variances are estimated precisely even for small sample sizes. A somewhat larger sample size is needed to get precise estimates of the loadings variance and dynamic parameters. Furthermore, the simulations showed very satisfactory results when the number of principal components is different from the number of factors in the data. We illustrated the empirical relevance of the time-varying loadings structure using the large quarterly dataset of Stock and Watson (2009) for the US economy. For the majority of the variables we found evidence of time-varying loadings, and we showed that a large increase in the in-sample fit of the common component can be obtained by modelling the loadings as time-varying.

Appendix

Let $X = (X_1, ..., X_T)'$ be the $T \times N$ matrix of observations, and let V_{NT} be the $r \times r$ diagonal matrix of the r largest eigenvalues of $(NT)^{-1}XX'$ in decreasing order. By the definition of eigenvalues and eigenvectors, we have $(NT)^{-1}XX'\tilde{F} = \tilde{F}V_{NT}$ or $(NT)^{-1}XX'\tilde{F}V_{NT}^{-1} = \tilde{F}$, where $\tilde{F}'\tilde{F}/T = I_r$. Let $H = (\Lambda^{0\prime}\Lambda^0/N)(F'\tilde{F}/T)V_{NT}^{-1}$ be the $r \times r$ rotation matrix. Assumption A and B together with Lemma A.1 below implies that $\|H\| = O_p(1)$. Let $w_t = \xi_t F_t$. We can write (1) as:

$$X_t = \Lambda^0 F_t + \xi_t F_t + e_t = \Lambda^0 F_t + w_t + e_t.$$

Define $e = (e_1, ..., e_T)'$ and $w = (w_1, ..., w_T)'$. We use the following expression from Bates et al. (2013):

$$XX' = F\Lambda^{0\prime}\Lambda^0 F' + F\Lambda^{0\prime}(e+w)' + (e+w)\Lambda^0 F' + (e+w)(e+w)'. \tag{7}$$

Let v_t denote a conforming unit vector with zeros in all entries except the t^{th}. We then have:

$$XX'v = F\Lambda^{0\prime}\Lambda^0 F_t + F\Lambda^{0\prime}(e_t + w_t) + (e+w)\Lambda^0 F_t + (e+w)(e_t + w_t).$$

Using the definition of $\tilde{F}_t$ and H, we can then write:

$$\begin{aligned}
\tilde{F}_t - H'F_t &= V_{NT}^{-1}(NT)^{-1}\tilde{F}'XX'v - V_{NT}^{-1}(\tilde{F}'F/T)(\Lambda^{0\prime}\Lambda^0/N)F_t \\
&= V_{NT}^{-1}(NT)^{-1}\left\{\tilde{F}'F\Lambda^{0\prime}e_t + \tilde{F}'e\Lambda^0 F_t + \tilde{F}'ee_t \right. \\
&\left. + \tilde{F}'F\Lambda^{0\prime}w_t + \tilde{F}'w\Lambda^0 F_t + \tilde{F}'ww_t + \tilde{F}'ew_t + \tilde{F}'we_t \right\}.
\end{aligned}$$

Denote each term on the right-hand as $A_{1t}, ..., A_{8t}$, respectively. We get:

$$\tilde{F}_t - H'F_t = V_{NT}^{-1}\sum_{n=1}^{8} A_{nt}. \tag{8}$$

The following is a generalization of Lemma A.3 in Bai (2003). They consider constant loadings; we generalize the proof to autoregressive loadings.

Lemma A.1. *Under Assumptions A-E, as $N, T \to \infty$:*

(i) $\left\|V_{NT} - \frac{\tilde{F}'F}{T}\frac{\Lambda^{0\prime}\Lambda^0}{N}\frac{F'\tilde{F}}{T}\right\|^2 = O_p(C_{NT}^{-2})$,

(ii) $\frac{\tilde{F}'F}{T}\frac{\Lambda^{0\prime}\Lambda^0}{N}\frac{F'\tilde{F}}{T} \xrightarrow{p} V$,

where V is the diagonal matrix consisting of the eigenvalues of $\Sigma_\Lambda \Sigma_F$.

Proof. From $V_{NT} = T^{-1}\tilde{F}'(NT)^{-1}XX'\tilde{F}$ we get using (7):

$$V_{NT} - \frac{\tilde{F}'F}{T}\frac{\Lambda^{0\prime}\Lambda^0}{N}\frac{F'\tilde{F}}{T} = T^{-1}\tilde{F}'(NT)^{-1}\left\{F\Lambda^{0\prime}(e+w)' + (e+w)\Lambda^0 F' + (e+w)(e+w)'\right\}\tilde{F}$$
$$= T^{-1}\sum_{t=1}^{T}\tilde{F}_t\sum_{n=1}^{8}A_{nt}'.$$

Hence,

$$\|T^{-1}\sum_{t=1}^{T}\tilde{F}_t\sum_{n=1}^{8}A_{nt}'\|^2 \leq \left(T^{-1}\sum_{t=1}^{T}\|\tilde{F}_t\|^2\right)\left(T^{-1}\sum_{t=1}^{T}\left\|\sum_{n=1}^{8}A_{nt}\right\|^2\right)$$
$$\leq 8rT^{-1}\sum_{t=1}^{T}\sum_{n=1}^{8}\|A_{nt}\|^2,$$

where the last inequality uses $tr(\tilde{F}'\tilde{F}/T) = tr(I_r) = r$ and Loève's inequality. The right-hand side is $O_p(C_{NT}^{-2})$ by Theorem 1 of Bates et al. (2013).[11] Statement (i) follows.

Statement (ii) is implicitly proven by Stock and Watson (1998). It should be noted that their paper considers the model $X_t = \Lambda^0 F_t + e_t$, i.e. a factor model with constant loadings. However, their proof only uses the asymptotic representation $V_{NT} = \frac{\tilde{F}'F}{T}\frac{\Lambda^{0\prime}\Lambda^0}{N}\frac{F'\tilde{F}}{T} + o_p(1)$ and the normalization $\tilde{F}'\tilde{F}/T = I_r$. Their proof is thus applicable for our model as well.

□

Proof of Lemma 1. From (8) we have:

$$T^{-1}\sum_{t=1}^{T}\|\tilde{F}_t - H'F_t\|^2 \leq \|V_{NT}^{-1}\|^2 8T^{-1}\sum_{t=1}^{T}\sum_{n=1}^{8}\|A_{nt}\|^2.$$

Since V_{NT} converges to a positive definite matrix, it follows that $\|V_{NT}^{-1}\|^2 = O_p(1)$. The right-hand side is thus $O_p(C_{NT}^{-2})$ by Theorem 1 in Bates et al. (2013).

□

Proof of Proposition 1. Using (8) we have:

$$\max_t\|\tilde{F}_t - H'F_t\| = \max_t\|V_{NT}^{-1}\sum_{n=1}^{8}A_{nt}\| \leq \|V_{NT}^{-1}\|\sum_{n=1}^{8}\max_t\|A_{nt}\|.$$

[11] Our Assumption D.3 differs from the corresponding assumption in Bates et al. (2013). They assume that $\sum_{s,t=1}^{T}\sum_{i,j=1}^{N}|E(\xi_{isp_1}\xi_{jsq_1}\xi_{itp_2}\xi_{jtq_2}F_{sp_1}F_{sq_1}F_{tp_2}F_{tq_2})|$ is bounded by the envelope function $Q_3(N,T)$, with $C_{NT}^2 Q_3(N,T) = O(N^2T^2)$. This is implied by Assumption D.3, and their Theorem 1 thus holds with Assumption A-E.

Lemma 1 implies that $\|V_{NT}^{-1}\| = O_p(1)$. We can write A_{1t} as:[12]

$$A_{1t} = (NT)^{-1}\sum_{s=1}^{T}(\tilde{F}_s - H'F_s)F_s'\Lambda^{0\prime}e_t + (NT)^{-1}\sum_{s=1}^{T}H'F_sF_s'\Lambda^{0\prime}e_t.$$

The first term is less than:

$$\left(T^{-1}\sum_{s=1}^{T}\|\tilde{F}_s - H'F_s\|^2\right)^{1/2}\left(N^{-2}T^{-1}\sum_{s=1}^{T}\|F_s'\Lambda^{0\prime}e_t\|^2\right)^{1/2}.$$

We have:

$$N^{-2}T^{-1}\sum_{s=1}^{T}\|F_s'\Lambda^{0\prime}e_t\|^2 \leq N^{-1}\|N^{-1/2}\Lambda^{0\prime}e_t\|^2T^{-1}\sum_{s=1}^{T}\|F_s\|^2.$$

By Assumption F.3, the maximum of $\|N^{-1/2}\Lambda^{0\prime}e_{it}\|^2$ over t is $O_p(T^{1/4})$, and Assumption A implies $\sum_{s=1}^{T}\|F_s\|^2 = O_p(1)$. By Lemma 2, we have $T^{-1}\sum_{s=1}^{T}\|\tilde{F}_s - H'F_s\|^2 = O_p(C_{NT}^{-2})$. Taking the square root then gives that the first term is $O_p\left(C_{NT}^{-1}\right)O_p\left(\frac{T^{1/8}}{N^{1/2}}\right)$. For the second term, we have:

$$(NT)^{-1}\|\sum_{s=1}^{T}H'F_sF_s'\Lambda^{0\prime}e_t\| \leq N^{-1/2}\|H\|\|N^{-1/2}\Lambda^{0\prime}e_t\|T^{-1}\sum_{s=1}^{T}\|F_s\|^2,$$

where $\|H\| = O_p(1)$ and $T^{-1}\sum_{s=1}^{T}\|F_s\|^2 = O_p(1)$ by Assumption A. The maximum of $\|N^{-1/2}\Lambda^{0\prime}e_t\|$ over t is $O_p(T^{1/8})$. The second term is thus equal to $O_p\left(\frac{T^{1/8}}{N^{1/2}}\right)$ and dominates the first.

Consider A_{2t}, which can be written as:

$$(NT)^{-1}\sum_{s=1}^{T}(\tilde{F}_s - H'F_s)e_s'\Lambda^0F_t + (NT)^{-1}\sum_{s=1}^{T}H'F_se_s'\Lambda^0F_t.$$

The first term is bounded by

$$\left(T^{-1}\sum_{s=1}^{T}\|\tilde{F} - H'F_s\|^2\right)^{1/2}\left(N^{-2}T^{-1}\sum_{s=1}^{T}\|e_s'\Lambda^0F_t\|^2\right)^{1/2}.$$

Now,

$$N^{-2}T^{-1}\sum_{s=1}^{T}\|e_s'\Lambda^0F_t\|^2 \leq \max_t\|F_t\|^2N^{-1}T^{-1}\sum_{s=1}^{T}\|N^{-1/2}e_s'\Lambda^0\|^2 = O_p(\alpha_T^2)N^{-1}$$

by Assumption F.3. The first term is thus equal to $O_p(C_{NT}^{-1}\alpha_TN^{-1/2})$. The second term is equal

[12]The terms A_{1t}, A_{2t}, A_{3t} have been shown to be $O_p(\alpha_TT^{-1}) + O_p(T^{1/8})N^{-1/2}$ by Bai and Ng (2008a). They do, however, rely on intermediate results, which we have not proved for the model with time-varying loadings. We therefore provide an alternative proof for A_{1t}, A_{2t}, A_{3t}.

to:

$$(NT)^{-1}\sum_{s=1}^{T}\sum_{i=1}^{N}H'F_s e_{is}\lambda_i^{0\prime}F_t,$$

which is bounded by:

$$N^{-1/2}M\max_t\|F_t\|\|H\|\left(T^{-1}\sum_{s=1}^{T}\|F_s\|^2\right)^{1/2}\left((NT)^{-1}\sum_{s=1}^{T}\sum_{i,j=1}^{N}e_{is}e_{js}\right)^{1/2}.$$

This is equal to $O_p(\alpha_T)N^{-1/2}$ by Assumption C.3 and dominates the first term.

We can write A_{3t} as:

$$(NT)^{-1}\sum_{s=1}^{T}(\tilde{F}_s - H'F_s)[e_s'e_t - E(e_s'e_t)] + (NT)^{-1}\sum_{s=1}^{T}H'F_s[e_s'e_t - E(e_s'e_t)]$$
$$+ (NT)^{-1}\sum_{s=1}^{T}(\tilde{F}_s - H'F_s)E(e_s'e_t) + (NT)^{-1}\sum_{s=1}^{T}H'F_sE(e_s'e_t).$$

The first term is bounded by:

$$\left(T^{-1}\sum_{s=1}^{T}\|\tilde{F}_s - H'F_s\|^2\right)^{1/2}\left(N^{-1}T^{-1}\sum_{s=1}^{T}\left|N^{-1/2}\sum_{i=1}^{N}[e_{is}'e_{it} - E(e_{is}'e_{it})]\right|^2\right)^{1/2}.$$

By Assumption C.5, $\max_t\left|N^{-1/2}\sum_{i=1}^{N}[e_{is}'e_{it} - E(e_{is}'e_{it})]\right|^2 = O_p(\sqrt{T})$, so the first term is equal to $O_p(C_{NT}^{-1})O_p(\frac{T^{1/4}}{N^{1/2}})$. The second term is bounded by:

$$(NT)^{-1/2}\|H\|\|(NT)^{-1/2}\sum_{s=1}^{T}\sum_{i=1}^{N}F_s[e_{is}'e_{it} - E(e_{is}'e_{it})]\|.$$

By Assumption F.2, the maximum of this expression over t is $O_p(N^{-1/2})$. The third term is bounded by:

$$T^{-1/2}\left(T^{-1}\sum_{s=1}^{T}\|\tilde{F}_s - H'F_s\|^2\right)^{1/2}\left(\sum_{s=1}^{T}\gamma_N(s,t)^2\right)^{1/2}.$$

By Assumption F.1 and Lemma 2, this is equal to $T^{-1/2}O_p(C_{NT}^{-1})$. The fourth term is bounded by:

$$T^{-1}\max_t\|F_t\|\|H\|\sum_{s=1}^{T}|\gamma_N(s,t)|,$$

which is $O_p(\alpha_T T^{-1})$.

For A_{4t}, we have:

$$(NT)^{-1}\|\tilde{F}'F\Lambda^{0\prime}w_t\| \leq \|T^{-1/2}\tilde{F}\|\|T^{-1/2}F\|\|N^{-1}\Lambda^{0\prime}w_t\|.$$

The first two terms are both $O_p(1)$. The last term can be bounded in expectation:

$$\begin{aligned}
E\left\|\frac{\Lambda^{0\prime}w_t}{N}\right\|^2 &\leq N^{-2}\sum_{i,j=1}^{N}|E(w_{it}w_{jt})\lambda_i^{0\prime}\lambda_j^0| \leq M^2N^{-2}\sum_{i,j=1}^{N}|E(\xi_{it}F_t\xi_{jt}F_t)| \\
&\leq M^2r^2N^{-2}\sup_{p.q}\sum_{i,j=1}^{N}|E(\xi_{itp}\xi_{jtq}F_{tp}F_{tq})| = O_p(N^{-1})
\end{aligned}$$

uniformly in t by Assumption D.1, so the maximum of the last term over t is $O_p(N^{-1/2})$.

Consider A_{5t}:

$$(NT)^{-1}\|\sum_{s=1}^{T}\tilde{F}_s w_s'\Lambda^0 F_t\| \leq \max_t\|F_t\|\left(T^{-1}\sum_{s=1}^{T}\|\tilde{F}_s\|^2\right)^{1/2}\left(T^{-1}\sum_{s=1}^{T}\|N^{-1}w_s'\Lambda^0\|^2\right)^{1/2}.$$

By Assumption D.1, this is equal to $O_p(\alpha_T N^{-1/2})$.

For A_{6t}, we have:

$$(NT)^{-1}\|\sum_{s=1}^{T}\tilde{F}_s w_s' w_t\| \leq N^{-1}T^{-1/2}\left(T^{-1}\sum_{s=1}^{T}\|\tilde{F}_s\|^2\right)^{1/2}\left(\sum_{s=1}^{T}\|w_s'w_t\|^2\right)^{1/2}.$$

By Assumption D.3, we have:

$$\begin{aligned}
&\sum_{s=1}^{T}E(w_s'w_t)^2 = \sum_{s=1}^{T}\sum_{i,j}^{N}E(w_{is}w_{it}w_{js}w_{jt}) \\
&\leq r^4\sup_{p_1,p_2,q_1,q_2}\sum_{s=1}^{T}\sum_{i,j}^{N}|E(\xi_{isp_1}\xi_{jsq_1}\xi_{itp_2}\xi_{jtq_2}F_{sp_1}F_{sq_1}F_{tp_2}F_{tq_2})| = O(N^2) + O(NT)
\end{aligned}$$

uniformly in t. We therefore have that $\max_t\|A_{6t}\| = N^{-1}T^{-1/2}[O_p(N) + O_p(N^{1/2}T^{1/2})] = O_p(C_{NT}^{-1})$.

The seventh term is bounded by:

$$\left(T^{-1}\sum_{s=1}^{T}\|\tilde{F}_s\|^2\right)^{1/2}\left(N^{-2}T^{-1}\sum_{s=1}^{T}\|e_s'w_t\|^2\right)^{1/2}.$$

The first term is $O(1)$. We can bound the second term in expectation:

$$
\begin{aligned}
N^{-2}T^{-1}\sum_{s=1}^{T} E\|e_s' w_t\|^2 &= N^{-2}T^{-1}\sum_{s=1}^{T}\sum_{i,j=1}^{N} E(e_{is}e_{js})E(w_{it}w_{jt}) \\
&\leq N^{-2}T^{-1}\sum_{s=1}^{T}\sum_{i,j=1}^{N} E(e_{is}^2)^{1/2}E(e_{js}^2)^{1/2}|E(w_{it}w_{jt})| \\
&\leq Mr^2N^{-2}T^{-1}\sup_{p,q}\sum_{s=1}^{T}\sum_{i,j=1}^{N}|E(\xi_{ipt}\xi_{jqt}F_{pt}F_{tq})|,
\end{aligned}
$$

which is $O_p(N^{-1})$ uniformly in t by Assumption D.1. Taking the square root then gives $O_p(N^{-1/2})$.

Finally, A_{8t} is bounded by:

$$
(NT)^{-1}\|\tilde{F}' w e_t\| = \left(T^{-1}\sum_{s=1}^{T}\|\tilde{F}_s\|^2\right)^{1/2}\left(N^{-2}T^{-1}\sum_{s=1}^{T}\|w_s' e_t\|^2\right)^{1/2}.
$$

The first term is again $O(1)$, and the last term can be bounded in expectation:

$$
\begin{aligned}
N^{-2}T^{-1}\sum_{s=1}^{T} E\|w_s' e_t\|^2 &= N^{-2}T^{-1}\sum_{s=1}^{T}\sum_{i,j=1}^{N} E(e_{it}e_{jt})E(w_{is}w_{js}) \\
&\leq N^{-2}T^{-1}\sum_{s=1}^{T}\sum_{i,j=1}^{N} E(e_{it}^2)^{1/2}E(e_{jt}^2)^{1/2}|E(w_{is}w_{js})| \\
&\leq Mr^2N^{-2}T^{-1}\sup_{p,q}\sum_{s=1}^{T}\sum_{i,j=1}^{N}|E(\xi_{ips}\xi_{jqs}F_{ps}F_{sq})|,
\end{aligned}
$$

which is $O_p(N^{-1})$ uniformly in t by Assumption D.1. The last term is thus $O_p(N^{-1/2})$. All terms are dominated by $O_p(\frac{T^{1/8}}{N^{1/2}}) + O_p(\alpha_T N^{-1/2}) + O_p(\alpha_T T^{-1}) + O_p(C_{NT}^{-1/2})$, and Proposition 1 follows.

□

Lemma A.2. *Let Assumption A-E hold. If $F'F/T = I_r$ and $\Lambda^{0\prime}\Lambda^0$ is a diagonal matrix with distinct entries,*

$$
H = I_r + O_p(C_{NT}^{-2}).
$$

Proof. First we need to show that $(\tilde{F} - FH)'F/T$ and $(\tilde{F} - FH)'\tilde{F}/T$ are both $O_p(C_{NT}^{-2})$. We have:

$$
\begin{aligned}
&\|(\tilde{F} - FH)'F/T\|^2 = \|T^{-1}\sum_{t=1}^{T}(\tilde{F}_t - H'F_t)F_t'\|^2 \\
&\leq \left(T^{-1}\sum_{t=1}^{T}\|\tilde{F}_t - H'F_t\|^2\right)\left(T^{-1}\sum_{t=1}^{T}\|F_t\|^2\right) = O_p(C_{NT}^{-2}),
\end{aligned}
$$

where the last equality follows from Lemma 1 and Assumption A. By similar arguments $(\tilde{F} - FH)'\tilde{F}/T = O_p(C_{NT}^{-2})$. The rest of the proof is identical to the proof of equation (2) in Bai and Ng (2013).

□

Lemma A.2 shows that if the imposed normalization holds for the process generating the data, the factors can be estimated without rotation. This implies that θ_i can be estimated without rotation as well. In the proof of Theorem 1 below, we assume that $H = I_r$, and note that in general, the feasible likelihood converges to $\mathcal{L}_T(X_i|FH;\theta_i)$, and $\tilde{\theta}_i$ is consistent for a rotation of θ_i^0 as discussed in Section 3.2.

Proof of Theorem 1. It suffices to show that the feasible likelihood function $\tilde{\mathcal{L}}_T(X_i|\tilde{F};\theta_i)$ converges uniformly to the infeasible one $\mathcal{L}_T(X_i|F;\theta_i)$.[13] This will imply that $\tilde{\mathcal{L}}_T(X_i|\tilde{F};\theta_i)$ satisfies the conditions of Assumption H and $\tilde{\theta}_i \xrightarrow{p} \theta_i^0$. We thus need:

$$\sup_{\theta\in\Theta}\left|\tilde{\mathcal{L}}_T(X_i|\tilde{F};\theta_i) - \mathcal{L}_T(X_i|F;\theta_i)\right| \xrightarrow{p} 0.$$

By the mean value expansion, we can write:

$$\tilde{\mathcal{L}}_T(X_i|\tilde{F};\theta_i) = \mathcal{L}_T(X_i|F;\theta_i) + \sum_{t=1}^{T}\nabla_{F_t}\mathcal{L}_T(X_i|F^*;\theta_i)(\tilde{F}_t - F_t),$$

where $\nabla_{F_t}\mathcal{L}_T(X_i|F^*;\theta_i) = \frac{\partial\mathcal{L}_T(X_i|F;\theta_i)}{\partial F_t}\Big|_{F=F^*}$, and F^* is between F and $\tilde{F}$. For uniform convergence the last term needs to be $o_p(1)$ uniformly in Θ, when F_t^* is in a neighbourhood of F_t, such that $\max_t\|F_t^* - F_t\| = o_p(1)$.

Let $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the largest and smallest eigenvalue of a matrix A, and let $(A)_{(s,t)}$ denote entry (s,t) of a $T\times T$ matrix A. Furthermore, let ϕ_i be the $r\times r$ block matrix on the diagonal of Φ_i, i.e. $\phi_i = Var(\lambda_{it})$. The derivative of $\mathcal{L}_T(X_i|F;\theta_i)$ takes the form:[14,15]

$$\begin{aligned}\nabla_{F_t}\mathcal{L}_T(X_i|F;\theta_i)' =& -T^{-1}\phi_i F_t\Sigma^{-1}_{i,(t,t)} + T^{-1}\lambda_i\sum_{s=1}^{T}(X_{is} - F_s'\lambda_i)\Sigma^{-1}_{i,(s,t)}\\ &+ T^{-1}\phi_i F_t\left(\Sigma_i^{-1}(X_i - E(X_i))(X_i - E(X_i))'\Sigma_i^{-1}\right)_{t,t},\end{aligned}$$

where F_t is to be evaluated at F_t^*. Denote the three terms above by B_{nt}, for $n = 1,...,3$. We can

[13] Pointwise convergence would suffice as Assumption H(iii) requires convergence for all $\theta_i\in\Theta_i$. However, there are no additional difficulties in showing uniform convergence, and we therefore prove convergence uniformly in Θ_i.

[14] The calculations of the derivative are omitted for brevity. They are available upon request.

[15] With autocorrelated errors, the derivative takes the same form, but the variance matrix is $\Sigma_i = F\Phi_i F' + \Psi_i$, where $\Psi_i = E(e_ie_i')$ is non-diagonal.

then write:

$$\begin{aligned}\sup_{\theta\in\Theta}\left|\tilde{\mathcal{L}}_T(X_i|\tilde{F};\theta_i)-\mathcal{L}_T(X_i|F;\theta_i)\right| &= \sup_{\theta\in\Theta}\left|\sum_{t=1}^{T}\sum_{n=1}^{3}(\tilde{F}_t-F_t)'B_{nt}\right| \\ \leq sup_{\theta\in\Theta}\left|\sum_{t=1}^{T}(\tilde{F}_t-F_t)'B_{1t}\right| &+ \sup_{\theta\in\Theta}\left|\sum_{t=1}^{T}(\tilde{F}_t-F_t)'B_{2t}\right| + \sup_{\theta\in\Theta}\left|\sum_{t=1}^{T}(\tilde{F}_t-F_t)'B_{3t}\right|.\end{aligned} \tag{9}$$

For the term involving B_{1t}, we have:

$$\left|T^{-1}\sum_{t=1}^{T}(\tilde{F}_t-F_t)'\phi_iF_t^*\Sigma^{-1}_{i,(t,t)}\right| \leq \lambda_{\max}(\Sigma^{-1})T^{-1}\sum_{t=1}^{T}\|\tilde{F}_t-F_t\|\|\phi_i\|\|F_t^*\|, \tag{10}$$

since each entry in Σ_i^{-1} is bounded by the largest eigenvalue. For the largest eigenvalue of Σ_i^{-1}, we have $\lambda_{\max}(\Sigma_i^{-1})=[\lambda_{\min}(\Sigma_i)]^{-1}$, and it therefore follows from the Weyl inequality that $\lambda_{\max}(\Sigma_i^{-1})\leq M$ as:[16]

$$\lambda_{\min}(\Sigma_i)\geq\lambda_{\min}(\mathbf{F}\Phi_i\mathbf{F}')+\lambda_{\min}(\psi_iI_T)\geq\psi_i>0$$

uniformly in Θ_i. The term $\|\phi_i\|$ is also uniformly bounded, as the parameters of $B_i(L)$ are in the stationary region, and the elements of Q_i are bounded. We can therefore bound (10) by:

$$O(1)T^{-1}\sum_{t=1}^{T}\|\tilde{F}_t-F_t\|\|F_t^*-F_t\|+O(1)T^{-1}\sum_{t=1}^{T}\|\tilde{F}_t-F_t\|\|F_t\|.$$

Since F_t^* is between F_t and $\tilde{F}_t$, the first term is less than $T^{-1}\sum_t\|\tilde{F}_t-F_t\|^2$ and is $O_p(C_{NT}^{-2})$ by Lemma 1. Note that $T^{-1}\sum_t\|\tilde{F}_t-F_t\|^2$ does not depend on θ_i, and the result is thus uniform in Θ_i. For the second term, we can write:

$$T^{-1}\sum_{t=1}^{T}\|\tilde{F}_t-F_t\|\|F_t\|\leq\left(T^{-1}\sum_{t=1}^{T}\|\tilde{F}_t-F_t\|^2\right)^{1/2}\left(T^{-1}\sum_{t=1}^{T}\|F_t\|^2\right)^{1/2},$$

which is $O_p(C_{NT}^{-1})$ by Lemma 1 and Assumption A, also uniformly in Θ_i.

For the term involving B_{3t} in (9), we can write:

$$\left|T^{-1}\sum_{t=1}^{T}(\tilde{F}_t-F_t)'\phi_iF_t^*\left(\Sigma_i^{-1}(X_i-E(X_i))(X_i-E(X_i))'\Sigma_i^{-1}\right)_{t,t}\right|\leq$$
$$\max_t\left|(\tilde{F}_t-F_t)'\phi_iF_t^*\right|T^{-1}\left|\sum_{t=1}^{T}\left(\Sigma_i^{-1}(X_i-E(X_i))(X_i-E(X_i))'\Sigma_i^{-1}\right)_{t,t}\right|.$$

[16]This also holds with $\Psi_i=E(e_ie_i')$ non-diagonal, as we can bound the smallest eigenvalue of Σ_i uniformly i Θ_i.

For the term outside the sum, we have:

$$\begin{aligned}\max_t \left|(\tilde{F}_t - F_t)'\phi_i F_t^*\right| &\le \|\phi_i\| \max_t \|\tilde{F}_t - F_t\| \|F_t^*\| \\ &\le O(1)\max_t \|\tilde{F}_t - F_t\|^2 + O(1)\max_t \|\tilde{F}_t - F_t\| \|F_t\|.\end{aligned}$$

If we take F_t to be a sequence of fixed and bounded constants, $\max_t \|F_t\| \le M$, and the second term is then $o_p(1)$ by Proposition 1, which is uniform in Θ_i as the proof of Proposition 1 does not depend on θ_i. The first term is bounded by the second.

The term involving the sum can be written as

$$\begin{aligned} & T^{-1}\left|\sum_{t=1}^{T}\left(\Sigma_i^{-1}(X_i - E(X_i))(X_i - E(X_i))'\Sigma_i^{-1}\right)_{t,t}\right| \\ &= T^{-1}\left|\mathrm{tr}\left(\Sigma_i^{-1}(X_i - E(X_i))(X_i - E(X_i))'\Sigma_i^{-1}\right)\right|, \end{aligned} \tag{11}$$

which is bounded by

$$\begin{aligned}&\lambda_{\max}(\Sigma^{-2})T^{-1}|\mathrm{tr}(X_i - E(X_i))(X_i - E(X_i))'| \le M^2 T^{-1}\sum_{t=1}^{T}\|X_{it} - F_t^{*\prime}\lambda_i\|^2 \\ &\le 4M^2T^{-1}\sum_{t=1}^{T}\left(\|F_t'\lambda_i^0\|^2 + \|F_t'(\lambda_{it} - \lambda_i^0)\|^2 + \|e_{it}\|^2 + \|F_t^{*\prime}\lambda_i\|^2\right).\end{aligned}$$

The first term in the sum is bounded by $T^{-1}M^2\sum_{t=1}^{T}\|F_t\|^2 = O_p(1)$. For the second term in the sum, we can write:

$$T^{-1}\sum_{t=1}^{T}\|F_t'(\lambda_{i,t} - \lambda_i^0)\|^2 \le \left(T^{-1}\sum_{t=1}^{T}\|F_t\|^4\right)^{1/2}\left(T^{-1}\sum_{t=1}^{T}\|\lambda_{i,t} - \lambda_i^0\|^4\right)^{1/2}.$$

This is $O_p(1)$ by Assumption A and G. By Assumption C we have $T^{-1}\sum_{t=1}^{T}e_{it}^2 = O_p(1)$, and for the last term, we can write:

$$T^{-1}\sum_{t=1}^{T}\|F_t^{*\prime}\lambda_i\|^2 \le M^2T^{-1}\sum_{t=1}^{T}\|F_t^{*\prime} - F_t\|^2 + M^2T^{-1}\sum_{t=1}^{T}\|F_t\|^2 = O_p(C_{NT}^{-2}) + O_p(1),$$

as λ_i is estimated in a bounded parameter space. The second term in (9) is thus $\max_t \|\tilde{F}_t - F_t\| O_p(1) = o_p(1)$ uniformly in Θ_i.

For the term involving B_{2t} in (9), we can write:

$$\left| T^{-1} \sum_{t=1}^{T} (\tilde{F}_t - F_t)' \lambda_i \sum_{s=1}^{T} (X_{is} - F_s^{*\prime} \lambda_i) \Sigma_{i,(s,t)}^{-1} \right|$$
$$\leq \left(T^{-1} \sum_{t=1}^{T} |(\tilde{F}_t - F_t)' \lambda_i|^2 \right)^{1/2} \left(T^{-1} \sum_{t=1}^{T} \left| \sum_{s=1}^{T} (X_{is} - F_s^{*\prime} \lambda_i) \Sigma_{s,t}^{-1} \right|^2 \right)^{1/2}.$$

The first term in parentheses is less than $M^2 T^{-1} \sum_{t=1}^{T} \|\tilde{F}_t - F_t\|^2 = O_p(C_{NT}^{-2})$ uniformly in Θ_i. The second term in parentheses is equal to

$$T^{-1} \left| \mathrm{tr} \left(\Sigma_i^{-1} (X_i - E(X_i))(X_i - E(X_i))' \Sigma_i^{-1} \right) \right|,$$

which is $O_p(1)$ uniformly in Θ_i from the arguments above, see (11). By taking the square root, the second term is thus $O_p(C_{NT}^{-1})$ and dominated by the third. Collecting the results gives:

$$\sup_{\theta \in \Theta} \left| \tilde{\mathcal{L}}_T(X_i | \tilde{F}; \theta_i) - \mathcal{L}_T(X_i | F; \theta_i) \right| = O_p \left(\max_t \|\tilde{F}_t - F_t\| \right) = o_p(1).$$

□

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