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CREATES Research Paper 2015-32

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Which pricing approach for options under GARCH with non-normal innovations?*

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July 2015

Abstract

Two different pricing frameworks are typically used in the literature when pricing options under GARCH with non-normal innovations: the equilibrium approach and the no-arbitrage approach. Each framework can accommodate various forms of GARCH and innovation distributions, but empirical implementation and tests are typically done in one framework or the other because of the computational challenges that are involved in obtaining the relevant pricing parameters. We contribute to the literature by comparing and documenting the empirical performance of a GARCH specification which can be readily implemented in both pricing frameworks. The model uses a parsimonious GARCH specification with skewed and leptokurtic Johnson s_u innovations together with either the equilibrium based framework or the no-arbitrage based framework. Using a large sample of options on the S&P 500 index, we find that the two approaches give rise to very similar pricing errors when implemented with time-varying pricing parameters. However, when implemented with constant pricing parameters, the performance of the no-arbitrage approach deteriorates in periods of high volatility relative to the equilibrium approach whose performance remains stable and at par with the models with time-varying pricing parameters.

JEL Classification: C22, C53, G13

Keywords: Option pricing, Equilibrium approach, No-arbitrage approach.

*The authors thank Jeroen Rombouts as well as participants and various discussants at the 7th CSDA International Conference on Computational and Financial Econometrics in London, the 5th Annual CIRPÉE Applied Financial Time Series Workshop in Montreal, the 2014 Workshop on Modelling and Forecasting Moment Risk Premia in Paris, and the 2015 Midwest Finance Association meeting in Chicago for valuable comments. Lars Stentoft acknowledges support from CREATES - Center for Research in Econometric Analysis of Time Series (DNRF78), funded by the Danish National Research Foundation.

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1 Introduction

Motivated by the empirical success of GARCH models with non-normal innovations to fit asset returns, an important literature on discrete-time option pricing based on these processes has emerged (see for example the recent survey in Christoffersen, Jacobs and Ornthanalai (2013)). Because GARCH models are discrete-time processes with continuously distributed innovations for the underlying asset returns, the market is incomplete and there is no unique pricing measure. Hence, this literature has mainly relied on two different pricing approaches. A first approach is based on an equilibrium pricing framework and was used in e.g. Duan (1999). This framework relies on a representative agent and strong assumptions about preferences to derive the restrictions that must be satisfied in equilibrium to obtain a coherent pricing model. A second approach is based on the no-arbitrage assumption as proposed in e.g. Christoffersen, Elkamhi, Feunou, and Jacobs (2010) (hereafter CEFJ). This framework is less restrictive and essentially requires an assumption about the form of the Radon-Nikodym derivative and the absence of arbitrage.[1]

The two methods lead to equivalent pricing restrictions in the Gaussian case, but such a result is not available when the distribution is non-normal. In this empirically more relevant case the methods differ in the way the risk premium is taken into account when deriving the risk neutral dynamics. This may consequentially result in different estimated option prices. Although both approaches allow coherent computations of option prices, the literature does not provide any guidance about which one should be used by someone wishing to implement GARCH models with non-normal innovations. Furthermore, while both approaches allow for time-varying pricing parameters, most empirical implementations rely on a constant pricing parameter and the literature is silent about the relevance of such an assumption. In this paper, we attempt to fill these gaps by comparing and documenting the empirical performance of the equilibrium and no-arbitrage pricing approaches and by assessing if the

[1] An alternative method to that of CEFJ which would provide a similar set of conditions is to specify a candidate stochastic discount factor directly as is done in e.g. Gourieroux and Monfort (2007).

use of a constant or time-varying pricing parameter produces significant pricing differences.

Our main findings can be summarized as follows. First, when implemented with time-varying pricing parameters, both approaches obtain very similar pricing errors for a large sample of option prices on the S&P 500 index from 2006 to 2011. This indicates that the choice of a pricing approach in the GARCH framework should be mostly based upon convenience of implementation. However, when implemented with a constant pricing parameter as is done most often in the existing empirical literature, the approaches are not equivalent. With the equilibrium pricing approach, a constant pricing parameter obtains option prices highly correlated with those from the time-varying parameter case, an indication that this simpler setting provides a pricing precision equivalent to the more intricate time-varying case. For the no-arbitrage approach, when implemented with a constant pricing parameter, a deterioration of the performance is observed in periods of high volatility. Our results show that these larger pricing errors are caused, in part, by the pricing parameter of the no-arbitrage approach which is proportional to the variance, while the equilibrium approach has a pricing parameter proportional to the standard deviation. This apparent deficiency of the no-arbitrage approach can be fixed by scaling the pricing parameter with the GARCH volatility.

The equilibrium approach in option pricing takes its roots in the work of Rubinstein (1976) and Brennan (1979). Duan (1999) generalizes the local Risk Neutral Valuation Relationship introduced in Duan (1995), to the case of non-normal innovations. In his study, the implementation of the pricing framework is done by assuming a symmetric Generalized Error distribution and relies on nested Monte Carlo simulations for risk premia determination and option price computations. The numerical complexity of the nested simulations prevents using this specification for large scale empirical testing. This motivates Christoffersen, Dorion, Jacobs and Wang (2010) to propose a practical specification of this equilibrium based pricing model. They rely on an approximation based on the symmetry of the error distribution allowing for a convenient computation of the risk premium, making the specification amenable to an empirical implementation. In Lehnert (2003), the Generalized Error distribution used in Duan (1999) and Christoffersen, Dorion, Jacobs and Wang (2010) is extended to include skewness, an important feature of stock index returns. However, the proposed model does not include any pricing restrictions preventing the absence of arbitrage. Finally, in Stentoft (2008), a feasible approach including both skewness and kurtosis different from the normal distribution is proposed with the Normal Inverse Gaussian distribution. This approach is used in Stentoft (2015) to price a large sample of individual stock options in an empirical exercise that documents the importance of allowing for GARCH features and non-normal innovations.

The second category of option pricing papers trying to address volatility clustering with non-normal innovations uses the approach proposed in CEFJ. In this framework, using an assumption about the form of the Radon-Nikodym derivative and the absence of arbitrage, an equivalent martingale measure is characterized. This differs from the equilibrium approach since no explicit assumptions are made about a representative investor or their preferences. As discussed in CEFJ the no-arbitrage framework nests several interesting recent option pricing models. For example, the model used in Christoffersen, Heston, and Jacobs (2006) which uses an Inverse Gaussian distribution allowing for both conditional skewness and kurtosis is a special case, and so is the heteroskedastic model with Poisson-normal innovations developed in Duan, Ritchken, and Sun (2005). Other recent contributions that uses the no-arbitrage approach include the use of Gaussian mixtures as the underlying conditional distribution in Rombouts and Stentoft (2014) and Rombouts and Stentoft (2015). These distributions capture well the skewness and kurtosis in stock return data and help producing theoretical option prices that are closer to observed values.

Although the two frameworks are based on fundamentally different premises, they are in fact related since specific assumptions about investor preferences are implicit in the specification of the risk premium in the return dynamics of the no-arbitrage framework. To our knowledge however, none of the existing papers compare the performance of these pricing approaches in the general setting where the distribution is non-normal and the risk is priced. The current paper is thus the first to document the potential equivalence of these frameworks and the high correlation between the predicted price errors from the two different approaches to option pricing. The main reason for this gap in the existing literature is most likely related to the computational challenges involved in obtaining the pricing parameters, especially for the equilibrium based framework, either through direct estimation on historical return data alone or together with calibrating the models to existing option prices.

To fill this gap in the existing literature, we introduce a particular specification of the GARCH option pricing model with Johnson s_u innovations. This specification is more parsimonious than Gaussian mixtures but fits well historical time series of index or individual stock return data. Besides the empirical relevance of this distribution, an important reason motivating its use is the ease with which it can be implemented in both pricing frameworks. Unlike other non-Gaussian choices such as the Generalized Error, Normal Inverse Gaussian or the asymmetric t, the Johnson s_u is a natural candidate to implement the equilibrium approach because it is built upon a monotone and continuous transformation of a Gaussian distribution. This in turn provides a simple solution of the risk-neutral distribution and pricing parameter identification avoiding the computational burden involved in, for example, Duan (1999). Moreover, the tools required to compute the moments, density, distribution functions and random numbers are straightforward to use and only involves functions associated with normal distributions allowing Monte Carlo simulation to be used for option price computation.

A valid concern that one might have is that our findings are specific to using the Johnson s_u distribution as the conditional distribution. Thus, in the last section of the paper, robustness checks are performed to assess if the findings of our analysis are in fact restricted to the distributional assumption adopted above. First, a distribution-free version of the no-arbitrage approach is implemented and is shown to result in pricing errors similar to the Johnson based model. Second, we examine if an alternative specification of our GARCH model where both the skewness and kurtosis are time-varying obtains different results. Such a specification thus uses a conditional time-varying non-normal distribution which is implemented and tested with market data. Although these specifications show slightly different pricing precisions, the results reveal similar qualitative findings about the two pricing frameworks: i.e. they obtain highly correlated pricing errors when implemented with time-varying pricing parameters.

The rest of this paper is organized as follows. Section 2 presents the physical return process with Johnson s_u error terms used in this study and some empirical evidence regarding its fit. Section 3 and 4 explain how this specification can be implemented in the equilibrium and no-arbitrage pricing approaches. Section 5 provides empirical results regarding how each pricing approach performs with a sample of option prices. Finally, Section 6 presents the robustness checks while Section 7 concludes.

2 The return process

With $\mathcal{F}_t$ denoting the information set up to time t, the Johnson NGARCH process for the continuously compounded stock price return under the physical measure is given by:

$$\ln \frac{S_t}{S_{t-1}} + \delta_t = \alpha - \gamma_t + \sigma_t \varepsilon_t, \tag{1}$$

$$\sigma_t^2 = \beta_0 + \beta_1 \sigma_{t-1}^2 + \beta_2 \sigma_{t-1}^2 \left(\varepsilon_{t-1} - \theta\right)^2, \tag{2}$$

$$\varepsilon_t | \, \mathcal{F}_{t-1} \sim J_{s_u}\left(a, b\right) \tag{3}$$

where S_t is the stock price at t, δ_t is the continuously compounded deterministic dividend yield, and σ_t^2 is the conditional variance of the continuously compounded return. Here, α is a constant parameter interpreted as the expected return over the next period. This interpretation is warranted because γ_t is determined according to $e^{\gamma_t} = E_{t-1}\left(e^{\sigma_t \varepsilon_t}\right)$ which implies $E_{t-1}\left[S_t\right] = S_{t-1} e^{\alpha - \delta_t}$. The conditional variance follows a standard NGARCH(1,1) process with the usual parameter interpretation. We choose this GARCH specification since it is usually found to provide an adequate fit to stock return data. Other GARCH specifications could however be used here since our option pricing approach will rely on Monte Carlo simulation, a numerical approach capable of handling most specifications.

The unit variance innovations $\varepsilon_t | \mathcal{F}_{t-1}$ are assumed to be distributed according to a Johnson s_u distribution with parameters a and b, denoted by $J_{s_u}(a, b)$. Such a distribution, proposed in Johnson (1949), allows the skewness and kurtosis to be different from the standard normal distribution with $-\infty < a < \infty$ controlling the skewness and $b > 0$ controlling the kurtosis. A positive (negative) value of a induces a negative (positive) skewness while smaller (higher) values of b are associated with larger (smaller) kurtosis values. As mentioned in the introduction, such random variables are simple to use, have finite higher moments and produce a large region of admissible skewness and kurtosis pairs which is very similar to the asymmetric t, a popular alternative to introduce non-normalities in GARCH models (see for example Hansen (1994)). In the context of the present study, an important advantage of Johnson s_u distribution is its link with the standard normal distribution. A Johnson random variable is a monotone transformation of a standard normal random variable. As shown in the next section, such a structure makes this distribution a natural candidate which greatly facilitate the computation of the risk-premium parameter in the context of the equilibrium pricing approach. A disadvantage associated with this distribution is the absence of a closed form expression for the moment generating function that would facilitate the computation of option prices in the no-arbitrage framework. The absence of such a function can however be bypassed by using a precise analytical approximation based on the first four moments which are available in closed form. Appendix A describes how a standard normal random variable can be used to build a Johnson random variable, and provides the likelihood function for the above model with such random shocks.

Table 1 reports estimation results with a time series of daily S&P 500 index returns obtained from the Center for Research in Security Prices (CRSP) for the period from January 1, 1990 to December 31, 2011. As a first set of result, under the heading "Gaussian physical", this table shows the parameter estimates obtained by maximising the likelihood of the system (1) to (3) as if the innovations where Gaussian. These numbers thus represent quasi-maximum likelihood (QML) estimates. As shown by the standard errors in parenthesis, all parameters are statistically significant. However, the Jarque Bera test on the standardized residuals reveals that the normality assumption is strongly rejected. As a second set of results, the maximum likelihood (ML) estimation of the above system is reported under the header "Johnson s_u physical". Again all parameters are statistically significant, including the skewness and kurtosis parameters a and b. As expected, the estimated GARCH parameter values for the QML and ML approaches are very close to each others. Although the specifications are non-nested (the normal distribution is not a special case of the Johnson s_u), it is interesting to notice that the likelihood value obtained with the Johnson assumption (18138) is much higher than the one obtained with the Gaussian distribution assumption (18042), with only two additional parameters. The validity of the Johnson error term assumption can be tested by computing the Jarque Bera test on the standard normal residuals implied from the model and computed with the procedure explained in Appendix A. As shown by the results of this test, the normality assumption cannot be rejected at the 5% confidence level. Figure 1 shows the quantile to quantile plot of the implied normal residuals and offers a visual validation of the distributional assumption. For both the QML and ML estimation, as indicated by the Ljung-Box test, the error terms (normalized by the estimated standard deviations) still show some significant autocorrelations. A detailed look at the residuals reveals that these are caused by small but significant autocorrelation coefficients in the first lag.

3 Equilibrium pricing approaches

As shown in Duan (1999), the generalized local risk-neutral valuation relationship (GLRNVR) implies the following risk-neutralized dynamics for the above physical system:

$$R_t^* = \alpha - \delta_t - \gamma_t + \sigma_t^* \varepsilon_t^*, \tag{4}$$

$$\sigma_t^{*2} = \beta_0 + \beta_1 \sigma_{t-1}^{*2} + \beta_2 \sigma_{t-1}^{*2} \left(\varepsilon_{t-1}^* - \theta\right)^2, \tag{5}$$

$$\varepsilon_t^* | \mathcal{F}_{t-1} = F_{s_u}^{-1} \left[\Phi\left(z_t^* - \lambda_t\right)\right] \tag{6}$$

where z_t^* is a standard Gaussian random variable under the risk-neutral measure Q, $F_{s_u}^{-1}$ is the inverse of the Johnson s_u distribution function, Φ is the standard normal distribution function, and where λ_t is a sequence that must satisfy the following restriction:

$$\alpha - \gamma_t + \ln E_{t-1}^Q \left[e^{\sigma_t^* F_{s_u}^{-1}\left[\Phi\left(z_t^* - \lambda_t\right)\right]}\right] = r_t \tag{7}$$

with r_t the continuously compounded periodic risk-free rate. To get a feasible pricing model, the computation of the expected value appearing in the above expression is required in order to determine λ_t. We tackle this problem in two steps.

In a first step, we develop an explicit expression for $\varepsilon_t^* = F_{s_u}^{-1} \left[\Phi\left(z_t^* - \lambda_t\right)\right]$ appearing in the exponential function. As shown in Appendix B, such terms are characterized by a four parameter Johnson s_u random variable written as

$$\varepsilon_t^* = c + d \times \sinh\left(\frac{z_t^* - a_t^*}{b}\right) \tag{8}$$

where $a_t^* = a + \lambda_t$, $c = -M\left(a, b\right) / \sqrt{V(a, b)}$, $d = 1/\sqrt{V(a, b)}$, and where expressions for $M\left(\cdot\right)$ and $V\left(\cdot\right)$ are available in Appendix A. Parameters c and d are location and scale parameters. Hence, the risk neutralization does not change the distribution of the random shock which remains a Johnson s_u random variable. Only the skewness parameter is affected by a shift induced by λ_t. The fact that we can characterize the risk neutral distribution explicitly sets this model apart from previous applications of the equilibrium framework in which numerical approximations have to be used (see for example, Duan (1999), Stentoft (2008), and Christoffersen, Dorion, Jacobs and Wang (2010)).

In a second step, rewriting $E_{t-1}^Q \left[e^{\sigma_t^* F_{s_u}^{-1}\left[\Phi\left(z_t^* - \lambda_t\right)\right]}\right]$ as $E_{t-1}^Q \left[e^{\sigma_t^* \varepsilon_t^*}\right]$, we use a fourth order Taylor series to obtain an analytical approximation for computing such expected values from the first four moments of the random quantity ε_t^*. This analytical approximation can also be used to compute $\gamma_t = \ln E_{t-1}\left(e^{\sigma_t \varepsilon_t}\right)$, which appears in the restriction allowing the determination of λ_t. The analytical approximation is written as:

$$E_{t-1}^{Q}\left[e^{\sigma_t^* \varepsilon_t^*}\right] \simeq 1 + \sigma_t^* \mu_1^* + \frac{1}{2}\sigma_t^{*2}\mu_2^* + \frac{1}{6}\sigma_t^{*3}\mu_3^* + \frac{1}{24}\sigma_t^{*4}\mu_4^* \tag{9}$$

where $\mu_i^* = E_{t-1}^{Q}\left[\varepsilon_t^{*i}\right]$. For Johnson s_u random error terms, these expected values are available in closed form and are described in Appendix C. Table 2 reports some results about the performance of the approximation to compute $\ln E_{t-1}^{Q}\left[e^{\sigma_t^* \varepsilon_t^*}\right]$. The approximation is compared to a precise Monte Carlo simulation with 15 million sample paths which generates Johnson s_u random errors ε_t^* according to equation (8) for various values of a and b. In these simulations, $\sigma_t^* = 0.2 \times \sqrt{1/252}$ while $\lambda_t = 0.05$. As shown in Table 2, this approximation is precise enough for all practical purposes. Other computations with different values of σ_t^* and λ_t yield similar results.

With the above specification, a key step is the computation of the values of the pricing parameter λ_t that allow implementing the risk-neutral system for pricing options. In the Gaussian case λ_t is readily interpreted as the risk premium. Though a similar overall interpretation can be given to the parameter in the non-Gaussian case, this interpretation is no longer exact (see also the discussion in Stentoft (2015)). The λ_t parameter nevertheless remains connected to the risk premium irrespectively of the assumed underlying distribution and Duan (1999) refers to it as the risk premium parameter. To avoid any kind of uncertainty, we refer to λ_t in the following as the pricing parameter in the equilibrium pricing framework. In the next subsections, two avenues are examined to obtain this parameter.

3.1 Equilibrium approach #1: constant λ

As in Stentoft (2008), a possible approach to estimating the pricing parameter λ_t is to assume a constant value for it i.e. $\lambda_t = \lambda$. Substituting the pricing restriction (7) and the analytical approximation (9) in equation (1) results in the following system for the return process under the physical measure:

$$R_t = r_t - \delta_t - \ln\left[1 + \sigma_t^* \mu_1^* + \frac{1}{2}\sigma_t^{*2}\mu_2^* + \frac{1}{6}\sigma_t^{*3}\mu_3^* + \frac{1}{24}\sigma_t^{*4}\mu_4^*\right] + \sigma_t \varepsilon_t, \tag{10}$$

$$\sigma_t^2 = \beta_0 + \beta_1 \sigma_{t-1}^2 + \beta_2 \sigma_{t-1}^2 \left(\varepsilon_{t-1} - \theta\right)^2, \tag{11}$$

$$\varepsilon_t \sim J_{s_u}\left(a, b\right). \tag{12}$$

The quantities in square brackets are functions of the constant pricing parameter λ, which can therefore be obtained straightforwardly with maximum likelihood (ML) estimation of the above system. In the estimation procedure, the term between the square brackets can be computed with the physical variance σ_t^2 which is locally equal to the risk-neutral variance, a property of the GLRNVR pricing framework.

With the estimated value of the pricing parameter λ, option pricing using this approach can be done with Monte Carlo simulation, with the following steps to simulate a risk-neutral price on a sample path, given estimated values for the parameters β_0, β_1, β_2, θ, a, b, λ and a starting value for σ_t:

- Simulate ε_t^*, a risk-neutral Johnson s_u innovation, with equation (8).

- Use the risk-neutral system (4) and (5) with the simulated risk-neutral Johnson s_u shock to obtain the stock price at time t, with $\alpha - \delta - \gamma_t$ set to $r_t - \delta_t - \ln\left[1 + \sigma_t^* \mu_1^* + \frac{1}{2}\sigma_t^{*2}\mu_2^* + \frac{1}{6}\sigma_t^{*3}\mu_3^* + \frac{1}{24}\sigma_t^{*4}\mu_4^*\right]$ in the return equation.[2]

Using the simulated risk-neutral sample paths for the stock prices, Monte Carlo option price estimates are then obtained the usual way. The fourth column of Table 1 reports the ML estimation results for the above time series model with a constant pricing parameter. The results are similar to those reported in the third column for the physical process. Imposing the equilibrium pricing restriction does not change the likelihood and yields a statistically significant value for λ. We also notice that the P-value of the Q-stat goes from 3.8% to 5.1%, giving some indication that having a risk-premium which is a function of the time-varying volatility improves the fit.

[2]Without loss of generality, for the parameter estimation and option prices computations, the risk-free rate and the dividend yield are considered constant through time and are set to the values observed at the date at which the option prices are computed. In unreported trials, specifications with time-varying dividend and interest rates were implemented without noticeable differences in results.

3.2 Equilibrium approach #2: time-varying λ_t

As an alternative to the above approach, the pricing parameter could be determined directly when computing the option price with the risk-neutral system. In a Monte Carlo simulation context, at every simulation step and for every path, a nonlinear equation must be solved for the unknown pricing parameter. This system is obtained from substituting the approximation (9) for γ_t and $E^Q\left[e^{\sigma_t^* \varepsilon_t^*}\right]$ in the restriction (7). With other distributions, for which numerical approximations are required for the risk-neutral expectation, this approach is clearly too time consuming. However, with our proposed Johnson s_u distribution it remains feasible, and obtains the following equation:

$$\alpha - r_t - \ln\left[\frac{1+\frac{1}{2}\sigma_t^2+\frac{1}{6}\sigma_t^3\mu_3+\frac{1}{24}\sigma_t^4\mu_4}{1+\sigma_t^*\mu_1^*+\frac{1}{2}\sigma_t^{*2}\mu_2^*+\frac{1}{6}\sigma_t^{*3}\mu_3^*+\frac{1}{24}\sigma_t^{*4}\mu_4^*}\right]=0 \tag{13}$$

where $\mu_i = E_{t-1}\left[\varepsilon_t^i\right]$ with $\mu_1 = 0$ and $\mu_2 = 1$ because ε_t is a zero-mean unit-variance innovation. Recall that μ_i^* are functions of λ_t. The pricing parameter can thus be solved for numerically by finding the value of λ_t yielding the above equality. As shown in Figure 2, this restriction, when expressed as a function of λ_t, is linear for all practical purposes. Moreover, this function is equal to $\alpha - r_t$ when $\lambda_t = 0$, since the numerator and denominator in the square brackets of restriction (13) become equal. This suggests that λ_t can be solved for with a simple linear interpolation as described in Appendix D. Table 3 compares the values of λ_t computed with a bisection algorithm and with the linear interpolation. As seen in this table, the linear interpolation obtains a performance which is precise enough for all practical purposes. Using this approach, computing the solution takes a fraction of a second and can be readily used in a Monte Carlo simulation to quickly compute the pricing parameter values.

Option pricing using this approach can be done with Monte Carlo simulation, with the following steps to simulate a risk-neutral price on a sample path, given estimated values for the parameters α, β_0, β_1, β_2, θ, a, b and a starting value for σ_t:

- Compute the pricing parameter λ_t with the linear interpolation method in Appendix

D.

- Simulate ε_t^*, a risk-neutral Johnson s_u innovation, with equation (8).
- Use the risk-neutral system (4) and (5) with the simulated risk-neutral Johnson s_u shock to obtain the stock price at time t, with γ_t computed with $\ln(1+\frac{1}{2}\sigma_t^{*2}+\frac{1}{6}\sigma_t^{*3}\mu_3+\frac{1}{24}\sigma_t^{*4}\mu_4)$.

Using the simulated risk-neutral sample paths for the stock prices, Monte Carlo option price estimates are obtained the usual way. Notice here that γ_t is computed with the risk-neutral variance σ_t^{*2} which is locally equal to the physical variance, a property of the GLRNVR pricing framework. It should also be noticed that, instead of the linear interpolation, the exact solution to the restriction (13) could be obtained with a bisection algorithm, using as a starting point the linearly interpolated λ_t. However, this would be significantly more time consuming and as the results above show it would not affect the performance in any noticeable way.

4 No-arbitrage pricing approaches

With GARCH models, markets are incomplete and hence there is no unique pricing measure needed for option pricing. In this section, we examine the general pricing approach proposed in CEFJ for deriving one such measure, which does not require strong assumptions about preferences. In this framework, assuming the absence of arbitrage, a candidate equivalent martingale measure is specified from the following likelihood ratio (or Radon Nikodym derivative):

$$L_t = \exp\left(-\sum_{i=1}^{t}\left(\nu_i\varepsilon_i\sigma_i + \Psi_i\left(\nu_i\right)\right)\right)$$

where $\Psi\left(u\right) = \ln E\left[e^{-u\sqrt{\sigma}\varepsilon}\right]$ is the logarithm of the conditional moment generating function (MGF), and where ν_t is a sequence that must satisfy the following restriction:

$$\alpha - \Psi_t\left(-1\right) + \Psi_t\left(\nu_t - 1\right) - \Psi_t\left(\nu_t\right) = r_t \tag{14}$$

where $\Psi_t(-1) = \gamma_t$. In the general case, the risk neutral dynamics are characterized through the log conditional MGF which can then be used to derive the specific parametric risk-neutral distribution using the Inversion Theorem (see for example Billingsley (1995, Theorem 26.2) or Davidson (1997, Theorem 11.12)). Thus, given a set of values ν_t agreeing with the above restriction, the risk-neutral dynamics for a given GARCH process can be obtained, and Monte Carlo option prices can be computed the usual way with these risk-neutral dynamics.

The risk neutral dynamics for the volatility with non-normal innovations typically involves time-varying GARCH parameters. Alternatively, a computationally simpler approach which uses the physical constant GARCH parameters can be used even if the risk neutral distribution is unavailable explicitly. This approach simulates the stock prices under the physical process, in conjunction with the likelihood ratio L_t, to obtain a Monte Carlo option price from

$$\frac{e^{-rT}}{n}\sum_{j=1}^{n} pay\left(S_{T,j}\right) \times L_{T,j} \tag{15}$$

where $pay(\cdot)$ is the payoff function of the option, n is the number of sample paths, and $L_{T,j}$ is the likelihood ratio computed along the jth path. Here, the likelihood ratio is a factor adjusting the simulated option payoff under the measure P, into an option payoff under the required risk-neutral measure Q. In order to apply this pricing method, a closed form expression for the function $\Psi_t(\cdot)$ must be available to implement the pricing restriction. For this purpose, we again rely here on the fourth order Taylor series approximation to obtain:

$$\Psi_t(u) \approx \ln\left[1 + \frac{1}{2}u^2\sigma_t^2 + \frac{1}{6}u^3\sigma_t^3\mu_3 + \frac{1}{24}u^4\sigma_t^4\mu_4\right]. \tag{16}$$

As in the case of the GLRNVR, the computation of the values of the pricing parameter ν_t in the pricing restriction is a key step that allows implementing the above pricing system. In CEFJ it is noted that in the Gaussian case ν_t is related to the price of risk, though it is not exactly equal to the risk premium in this case. More generally, ν_t is related to the risk premium even in the non-Gaussian case. To avoid any kind of uncertainty related to the exact interpretation one can give to this parameter, we refer to ν_t in the following as the pricing parameter in the no-arbitrage pricing framework. In the next subsections, two avenues are examined to obtain values for this parameter.

4.1 No-arbitrage approach #1: constant ν

As in Rombouts and Stentoft (2015), a possible approach to estimating the pricing parameter ν_t is to assume a constant value for it i.e. $\nu_t = \nu$. Substituting the pricing restriction (14) in equation (1) results in the following system for the return process under the physical measure:

$$R_t = [r_t - \delta_t - \Psi_t(\nu - 1) + \Psi_t(\nu)] + \sigma_t \varepsilon_t, \tag{17}$$

$$\sigma_t^2 = \beta_0 + \beta_1 \sigma_{t-1}^2 + \beta_2 \sigma_{t-1}^2 (\varepsilon_{t-1} - \theta)^2, \tag{18}$$

$$\varepsilon_t \sim J_{s_u}(a, b) \tag{19}$$

which can be estimated with a ML approach using the approximation for the logarithm of the MGF in (16).

With the estimated value of the pricing parameter ν, option pricing using this approach can be done with Monte Carlo simulation, with the following steps to simulate a sample path of stock returns, given estimated values for the parameters β_0, β_1, β_2, θ, a, b, ν and a starting value for σ_t:

- Simulate ε_t, a physical Johnson s_u innovation, with the procedure outlined in Appendix A.
- Use the system (17) and (18) with the simulated Johnson s_u shock to obtain the stock price at time t and the likelihood ratio L_t with the estimated ν used to calculate $\Psi_t(\nu)$ and $\Psi_t(\nu - 1)$ with equation (16).

Using the simulated physical sample paths and likelihood ratios, Monte Carlo option price estimates for European options are obtained with the approach outlined above in equation (15). The fifth column of Table 1 reports the ML estimation results for the above time series model with a constant value for ν. Again, the results are almost identical to those reported for the physical process and the physical process with the equilibrium pricing restriction. Imposing the no-arbitrage pricing restriction does not change the likelihood and yields a statistically significant value for the pricing parameter ν and results in a higher P-value of the Q-stat which indicates that having a risk-premium which is a function of the time-varying variance improves the fit.

4.2 No-arbitrage approach #2: time-varying ν_t

Instead of assuming a constant value, ν_t can be computed for every time point and every sample path. Given numerical values for the parameters a and b of the Johnson s_u distribution, which allows the computations of the moments μ_3 and μ_4, the pricing restriction (14) can be solved for ν_t using a bisection algorithm. Alternatively, one can use the approximation proposed in CEFJ which is written as:

$$\nu_t = \frac{\alpha - r_t - \gamma_t}{\sigma_t^2} + \frac{1}{2}. \tag{20}$$

As with the equilibrium approach, the pricing restriction (14) is approximately linear in ν_t as shown in Figure 3. This suggests that a linear interpolation similar to the one presented in Appendix D could be used to solve for the required values of ν_t. Table 4 provides a comparison of these three different approaches which can be used to solve for the pricing restrictions. Except for small volatility values, the three methods give very similar answers.

Option pricing using this approach can be done with Monte Carlo simulation, with the following steps to simulate a price on a sample path, given estimated values for the parameters α, β_0, β_1, β_2, θ, a, b, and a starting value for σ_t:

- Compute the pricing parameter ν_t with the approximation given by equation (20).
- Simulate ε_t, a physical Johnson s_u innovation, with the procedure outlined in Appendix A.

- Use the system (1) and (2) with the simulated Johnson s_u shock to obtain the stock price at time t and the likelihood ratio L_t with the computed value of ν_t used to calculate $\Psi_t(\nu_t)$ with equation (16).

Using the simulated physical sample paths and likelihood ratios, Monte Carlo option price estimates for European options are then obtained with the approach outlined above in equation (15). It should be noticed that, instead of using approximation (20), the exact solution to the restriction (14) could be solved with a bisection algorithm, using as a starting point the approximated ν_t given by equation (20). However, this would again be significantly more time consuming and as the results above show would not affect the performance in any noticeable way.

5 Results

The previous two sections demonstrate how our proposed GARCH specification with Johnson s_u innovations can be implemented in the equilibrium based framework of Duan (1999) and the no-arbitrage framework of CEFJ with constant as well as with time-varying pricing parameters. In this section we examine how each of the four pricing models presented above perform in relative terms when taken to actual data. For convenience, we will refer to the pricing approaches as follows: EquCst is for the equilibrium approach with a constant pricing parameter λ; EquTva is for the equilibrium approach with a time-varying pricing parameter λ_t; NoaCst is for the no-arbitrage approach with a constant pricing parameter ν; NoaTva is for the no-arbitrage approach with a time-varying parameter ν_t.

5.1 Call option prices from the MLE estimates

As a first test, Table 5 presents the Black-Scholes implied volatilities (IV) for a sample of artificial option prices computed with the parameter values obtained from the time series estimates reported in Table 1. The numbers are for European call options with maturities of 30, 90 and 270 days, a stock price of 50, and strike prices ranging from 47 to 53. The variance is initialized with the variance estimated for the last data point in the time series estimation. The prices are computed using Monte Carlo simulations with paths generated as described in the earlier sections. Each option price is computed with 1,000,000 sample paths and common random numbers for all approaches. A Black-Scholes control variate and the Empirical Martingale simulation approach from Duan and Simonato (1998) are also used as variance reduction techniques. For the equilibrium approach, computing the option prices with the time-varying parameter involves more computations but can still be done quickly. For example, computing the option prices in the table with 100,000 paths requires around 7.5 second with the constant λ case, while it takes around 8.5 seconds for the time-varying case. For the no-arbitrage approach, the additional computations required by the time-varying ν_t are negligible.

As shown in this table, the four different specifications result in values differing only by small amounts. For the time-varying cases, such results can be explained in part by the behavior of the pricing parameters λ_t and ν_t since both approaches share the exact same set of GARCH and Johnson parameters obtained from the ML estimation of the physical process. As shown by equations (13) and (20), when computing options prices, the pricing parameters λ_t and ν_t are fluctuating in response to changes in the volatility level. Figure 4 shows a plot of λ_t and ν_t as a function of the volatility level (on a two scale graph). From this graph we see that both parameters exhibit similar responses to changes in volatility levels, although they have a different scale (an issue discussed later). For the constant pricing parameter case, the absence of sensitivity to changes in the variance does not seem to affect the computed prices which are similar to those of the time-varying cases. The results presented here are however dependant on the GARCH and Johnson parameters estimated in Table 1 and may not translate to other parameter values and variance levels.

5.2 Pricing errors from a sample of observed option prices

To examine results with different sets of GARCH and Johnson parameter values, we use a large sample of call and put options on the S&P 500 index from OptionMetrics covering the years 2006 to 2011. This sample thus includes the turbulent periods associated with the financial crisis of 2008. We impose the following standard restrictions on our sample: firstly, we consider weekly data only and choose the options traded on Wednesdays. If Wednesday is not a trading day we pick the date closest to it. This choice is made to balance the tradeoff between having a long time period against the computational complexity from model estimation and option pricing. We choose Wednesdays as these options are the least affected by weekend effects. Secondly, we choose to work only with those contracts which had a daily traded volume of at least 100 contracts. Thirdly, we exclude options which have a price, defined as $(bid + ask)/2$, below 50 cents. Fourthly, we exclude options with less than 7 or more than 252 calendar days to maturity. Finally, we eliminate options in the LEAPS series as the contract specifications for these options do not correspond to that of the standard options.

Each week, to compute the theoretical option prices, the parameters of the GARCH models are estimated with a ML estimation approach on a rolling window using the most recent historical time series of 5,000 daily index returns with dividends. Using these parameter estimates, a constant risk-free rate and dividend yield taken from the OptionMetrics database for the day at which the option prices are observed, we compute the theoretical option prices using the Monte Carlo procedures outlined earlier for each of the four pricing approaches. We use 100,000 paths, common random numbers for all pricing approaches, and the same variance reduction technique that are used in the previous table. The theoretical prices are then converted into Black Scholes implied volatilities and compared to the implied volatility of the observed option prices.

Table 6 reports the results for the pricing errors computed as the differences between the IV of the option prices computed with the GARCH models and the IV of the observed prices.

The first and second panel report the correlations between the pricing errors and the absolute pricing errors. The EquCst, EquTva and NoaTva pricing errors show correlations over 0.94 between each other. The NoaCst case stands out from the group with smaller correlations, indicating that it generates results different from the other approaches. The third panel of the table also shows that this last approach stands out with different mean errors (ME), mean absolute errors (MAE) and root-mean-squared errors (RMSE), unlike the other three approaches who have similar performances. The fourth and fifth panels report the MAE examined with respect to maturities and moneyness of the options. A similar phenomena emerge from these panels which show that the NoaCst approach stands-out, and performs particularly poorly for options that have long maturities and are in the money. Finally, panels six and seven examine the MAE by volatility and year. With these, a clearer picture emerge and we see that the NoaCst shows differences with the other methods for higher implied volatility levels and for years where volatility was much higher i.e. the financial crisis in 2008. For example, when $\sigma > 40\%$ the MAE is 25% larger for the NoaCst and during 2008 the errors were 54% larger. A closer examination of the results for 2008 does not reveal any systematic pattern across moneyness or maturities though.

5.3 Alternative specifications in the no-arbitrage framework

In order to examine further the source of the discrepancies for the constant pricing parameter case, we notice that an important difference between the equilibrium and no-arbitrage approach is how the pricing parameter enters the return process. This can be seen by looking at the case of normally distributed errors with a constant pricing parameter. As shown in Appendix E, in such a context, the return process can be rewritten as

$$R_t = r - \delta_t + \lambda \sigma_t - \frac{1}{2}\sigma_t^2 + \sigma_t \varepsilon_t$$

for the equilibrium approach, while it can be rewritten as

$$R_t = r - \delta_t + \nu \sigma_t^2 - \frac{1}{2}\sigma_t^2 + \sigma_t \varepsilon_t$$

for the no-arbitrage approach. Thus with the equilibrium approach, the pricing parameter λ loads on the volatility, while ν loads on the variance for the no-arbitrage approach. This explains the scale differences between these parameters: because the daily volatility level is smaller than one, the squared volatility is much smaller in magnitude than the volatility, and thus commands a higher value of the pricing parameter. In periods of high volatility, the combination of a constant pricing parameter loading on the level of the squared volatility appears to be an important issue explaining the differences. Although the above analysis is for the Gaussian case, a similar relationship holds for the non-Gaussian case but with additional effects brought on by higher moments.

To examine more closely this issue, we look at two additional specifications of the no-arbitrage case with a constant pricing parameter: a first specification where the parameter is scaled by the volatility i.e. $\nu_t = \nu/\sigma_t$ and a second version where the parameter is scaled by the the variance i.e. $\nu_t = \nu/\sigma_t^2$. As for the NoaCst case, using these specifications and the pricing restriction (14), we obtain physical return specifications for which the pricing parameter appears as a constant which may be estimated by a ML estimation approach. Table 7 presents the pricing error statistics obtained with these alternative approaches. The second column of this table reports, for convenience, the results for the NoaCst approach ($\nu_t = \nu$) and the fifth column reports the results for the EquCst approach ($\lambda_t = \lambda$), both of which also appear in Table 6. The third and fourth columns present the $\nu_t = \nu/\sigma_t$ and $\nu_t = \nu/\sigma_t^2$ cases, respectively. As shown by these results, when the pricing parameter is scaled by the volatility or the variance, the pricing error become similar to those of the EquCst, and therefore also to those of the EquTva and NoaTva cases.

From the analysis above, we see that the equilibrium and no-arbitrage pricing approaches provide similar option prices with time-varying pricing parameters and with the constant pricing parameters (when properly scaled). Hence the choice of which pricing approach to use should be largely dictated by convenience issues associated with the implementation. While the equilibrium approach is simple to implement with the Johnson distribution, it is difficult to do so with other distribution choices, as shown by the earlier literature. This is not the case for the no-arbitrage approach which can be easily implemented as long as the first four moments of the error distributions are known in closed form, or more generally by deriving the risk neutral dynamics which exist and are generally from the same distribution, provided the physical conditional distribution is an infinitely devisable distribution with finite second moment (see CEFJ, proposition 1.3)). Furthermore, our results show that the simple case of a constant pricing parameter, properly scaled by the standard deviation, gives results equivalent to those obtained with the more intricate case of a time-varying pricing parameter. Thus, for actual applications one can use the more convenient constant pricing method instead of the significantly more time-consuming and complicated time-varying method without sacrificing precision.

6 Robustness checks

As mentioned previously the two pricing frameworks yield equivalent pricing restrictions in the Gaussian case and when risk premia are zero. In reality however, returns are neither Gaussian nor are risk premia zero. Thus, there is no theoretical reason that the two frameworks should produce pricing results that are as close to each other as what we have found in the previous section. A valid concern that one might have is that our findings are driven by the particular choice of conditional distribution. In this section several robustness checks are performed to assess if the findings of our analysis are restricted to the distributional assumption adopted above.

6.1 A distribution-free approach

As implemented in the previous sections, the no-arbitrage approach with a time-varying parameter uses the distribution assumption in the parameter estimation step (to obtain the likelihood function), and in the simulation step (to generate random error terms with the appropriate skewness and kurtosis). However, in both steps the Johnson distribution assumption can be avoided, and we now examine an approach that does not assume any specific distribution for the innovations in the no-arbitrage framework. In the estimation step, the QML estimator of Table 1 can be used to compute non-parametric estimates of the skewness and kurtosis. These can then be used in the simulation step which can be adapted by replacing the simulation of the Johnson random numbers with a Cornish-Fisher approximation that can generate random error terms with the target skewness and kurtosis. The procedure to obtain Cornish-Fisher random numbers is taken from Maillard (2012).

Table 8 reports the results obtained for the artificial sample of call options when using the QML parameter estimates. The first panel reports the case of a constant pricing parameter ν while the second reports the time-varying ν_t case. In each panel, for convenience, we first show the results obtained using the Johnson distribution in the estimation and pricing steps that have already been reported in a previous table (the NoaTva and NoaCst cases). For the constant ν case (first panel), the parameter estimates are taken from the last column in Table 1. For this case, (unlike the time-varying ν_t case), a genuine QML estimator is not possible. Instead, the parameter estimates are obtained using a two-step approximate QML procedure. In a first step, the system given by equations (17) and (18) is estimated with a Gaussian likelihood and μ_3 and μ_4 set to 0 and 3. Using the residuals from this step, estimated values of μ_3 and μ_4 are obtained and used as inputs in the second step where the Gaussian likelihood associated to the system is optimized once more. As shown by the results in Table 1, the GARCH parameter estimates are very close to the genuine QML estimator with a likelihood only sightly higher. The pricing parameter ν is estimated precisely and is very close in value to the one reported in the Johnson s_u case.

Looking at the computed IV, the differences in the option implied volatilities are small, showing that a distribution free approach gives results similar to the NoaTva approach, which itself obtains results very similar to the other approaches examined in this study. The third set of option prices reported in this panel shows the implied volatilities of the options computed with the approximate QML estimates using Johnson random numbers instead of

Cornish-Fisher. The parameters for the Johnson distribution have been obtained by finding the a and b parameters that allow matching the estimated skewness and kurtosis obtained from the QML residuals. This panel shows IVs that agree in most cases to the third digit after the decimal point. This indicates that the differences in the NoaTva and QML Cornish-Fisher option prices are mainly due to the changes in the parameters used as inputs, and are not caused by the Johnson assumption for the random shocks used to compute the the option prices. Looking at the time-varying ν_t case in the second panel of Table 8, results similar to those from the first panel are obtained. Small differences between the ML Johnson s_u and QML Cornish-Fisher case are observed, while the IV from these option prices are almost indistinguishable from those reported for the QML Johnson s_u case.

To get a clearer picture of the possible discrepancies, Table 9 shows the pricing differences for our sample of option prices. For convenience, the second column reports the NoaTva case already reported in an earlier table. The third column reports the case of QML estimates with time-varying ν_t, but with option prices computed with Johnson random numbers with moment matched a and b parameters, as in the previous table. The fourth column shows the QML case with time-varying ν_t, and Cornish-Fisher random numbers. As with the results from the previous table, the option prices obtained with the QML-Johnson and QML-Cornish-Fisher approaches are very close, with a pricing error correlation of 99% and very small discrepancies between the different statistics about the errors. The fifth column reports the pricing errors obtained with the approximate QML estimates and a fixed ν. The results obtained with this approach are similar to those of the NoaCst case, showing a marked deterioration of the precision when the variance is high. Finally, unreported in this table, the performance of a QML no-arbitrage model with a constant pricing parameter scaled by the volatility was also examined. As in Table 7, such a model restores the error to the level of those associated to the QML time-varying ν_t model, showing again that such a result is not solely associated to our earlier assumption of Johnson s_u shocks.

6.2 Time-varying non-normal distributions

Differences between the two pricing frameworks are linked to the non-normality of the innovations and could thus be more pronounced in periods with more skewness or excess kurtosis. To examine this issue we now consider an alternative specification which relies on time-varying distributions where the skewness and kurtosis are modeled as conditional mean reverting processes. We introduce the time-varying skewness and kurtosis through a dynamic specification of the parameters of the random shock distribution. The a and b parameters, which govern the third and fourth moments of $e_t = \varepsilon_t \sigma_t$ are time-varying and allowed to revert to long-run values of $\overline{a}$ and $\overline{b}$. Such a specification is in the spirit of Jondeau and Rockinger (2003), who also use a dynamic specification for the parameters of the distribution, which evolves according to the shocks affecting the returns.[3]

More specifically, we assume that the stock return with dividends under the physical measure is given by:

$$R_t + \delta_t = \alpha - \gamma_t + e_t, \tag{21}$$

with

$$\sigma_t^2 = \beta_0 + \beta_1 \sigma_{t-1}^2 + \beta_2 \sigma_{t-1}^2 \left(e_{t-1}/\sigma_t - \theta\right)^2, \tag{22}$$

and

$$a_t = \overline{a} + \phi_{a,1}\left(a_{t-1} - \overline{a}\right) + \phi_{a,2} \times \left(e_{t-1}^3 - \mu_{3,e}\left(\overline{a}, \overline{b}\right)\right), \tag{23}$$

$$b_t = \overline{b} + \phi_{b,1}\left(b_{t-1} - \overline{b}\right) + \phi_{b,2} \times \left(e_{t-1}^4 - \mu_{4,e}\left(\overline{a}, \overline{b}\right)\right). \tag{24}$$

In this specification, $\alpha, \beta_0, \beta_1, \beta_2, \theta, \overline{a}, \phi_{a,1}, \phi_{a,2}, \overline{b}, \phi_{b,1}, \phi_{b,2}$ are constant parameters. Here, $\mu_{3,e}\left(\overline{a}, \overline{b}\right)$ and $\mu_{4,e}\left(\overline{a}, \overline{b}\right)$ are the long-run third and fourth central moments of e_t, which are functions of $\overline{a}$ and $\overline{b}$. The values of these functions are the third and fourth moments of a Johnson s_u random variable with parameter $\overline{a}$ and $\overline{b}$ with a variance equal to the

[3]We note here that the return shock e_t is also a Johnson random variable with skewness and kurtosis determined by a_t and b_t, but with a variance given by σ_t^2 which is a known quantity at t. More technically, e_t is a Johnson s_u random variable written as $e_t = \widetilde{c} + \widetilde{d} \times x_t$ where $\widetilde{c}_t = \sigma_t c_t$ and $\widetilde{d}_t = \sigma_t d_t$ where the quantities x_t, c_t and d_t are described in Appendix A.

unconditional variance of the daily return i.e.

$$\mu_{3,e}\left(\overline{a},\overline{b}\right)=E\left(e^{3}\right) \quad \text{and} \quad \mu_{4,e}\left(\overline{a},\overline{b}\right)=E\left(e^{4}\right)$$

where e is a $J_{s_u}\left(\overline{a},\overline{b}\right)$ with a mean of zero and a variance set equal to the sample variance of the returns. Using this specification for the returns, it is possible to examine the four pricing approaches examined earlier i.e. the EquCst, EquTva, NoaCst, and NoaTva specifications.

Table 10 presents the ML estimation for the parameters of the physical process without pricing restrictions and the physical processes with the pricing restrictions associated with the constant pricing parameter approaches. As shown in this table, all parameters are statistically significant with a large increase in the likelihood when compared with the constant a and b parameter case examined earlier. The mean reversion parameters $\phi_{a,1}$ and $\phi_{b,1}$ suggest that the a and b parameters strongly revert towards their long run estimates, which correspond to long run skewness and kurtosis estimates of -0.15 and 3.82 respectively. The reverting behavior exhibited by the conditional parameters a_t and b_t is accompanied by skewness and kurtosis shock effects with both coefficients $\phi_{a,2}$ and $\phi_{b,2}$ significant at the 5% level. Imposing the pricing restriction improves the likelihood slightly and yield a statistically significant value for λ and ν, respectively. When compared with the physical case, we also notice an improvement in the Q-stat when imposing the pricing restrictions, giving some indications that a pricing parameter which is a function of the time-varying volatility or variance improves the fit.

Table 11 shows the statistics about the pricing errors computed with our sample of options with the time-varying non-normal distribution model. The same rolling window estimation procedure is adopted to compute the parameter estimates for the four pricing models. The results presented in this table show a pattern which is qualitatively very close to that presented earlier. In particular, the NoaCst approach stands out from the other methods in general and with much larger pricing errors for options that have long maturities and are in the money. Moreover, across volatility and through time a clear pattern arises and this model performs particularly bad in periods of high volatility like the financial crisis in 2008. Thus, the conclusions reached earlier translate to this model showing the robustness of the results to a more general framework with conditional non-normalities.

7 Conclusion

This paper examines and compares the performance of the two main pricing approaches that have been used in the GARCH option pricing literature with non-normal innovations. Using Johnson s_u innovations for our GARCH process, we show how the equilibrium approach and the no-arbitrage approach can be used to compute option prices with constant and time-varying pricing parameters. The numerical and empirical results show that both pricing frameworks obtain very similar option prices when implemented with time-varying pricing parameters. Hence, the choice of a pricing framework is more a matter of convenience than a matter of pricing precision. When implemented with a constant pricing parameter, we find that the equilibrium approach gives similar results to the time-varying parameter case. This indicates that such a simplifying assumption can obtain results equivalent to the more computationally complex time-varying case and represents a reasonable choice to someone wishing to use the equilibrium approach. However, unlike the equilibrium case, the no-arbitrage approach with a constant pricing parameter obtains pricing errors that are larger than those obtained with the time-varying case. As shown by numerical experiments, these differences are mainly caused by the specification of the no-arbitrage approach which has a pricing parameter proportional to the variance, unlike the equilibrium approach parameter which loads on the volatility. The differences disappear when a constant pricing parameter specification scaled by the volatility is used. Finally, we examine the robustness of our results to changes in our main distributional assumption, which reveal that similar results are obtained with alternative assumptions.

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A Johnson s_u innovations

This appendix describes the Johnson s_u random variables that have been introduced in Johnson (1949).

A zero-mean, unit variance random variable following a two parameter Johnson s_u distribution with parameters a and b can be written as

$$\varepsilon_t = c + d \times x_t \tag{25}$$

where $x_t = \sinh\left(\frac{z_t - a}{b}\right)$, with z_t is a standard normal random variable and

$$c = -M(a,b)/\sqrt{V(a,b)}, \quad d = 1/\sqrt{V(a,b)},$$

$$M(a,b) = -w^{\frac{1}{2}} \sinh(\Omega),$$

and

$$V(a,b) = \frac{1}{2}(w-1)(w\cosh(2\Omega)+1).$$

Parameters a and b control the skewness and kurtosis, $\cosh(u) = (e^u + e^{-u})/2$ and $\sinh(u) = (e^u - e^{-u})/2$ are the hyperbolic cosine and sine functions while $w = e^{\frac{1}{b^2}}$ and $\Omega = \frac{a}{b}$. Here $M(\cdot)$ and $V(\cdot)$ are the mean and variance of x_t, the unstandardized two parameter Johnson s_u random variable. The values c and d, which are here functions of a and b, are used to change the location and scale of x in such a way that it becomes a standardized Johnson s_u random variable.

From Simonato (2012), the log likelihood function of the time series model described by equation (1) and (2), for a time series of T observations, can be written as:

$$\ln L = -\frac{1}{2}\sum_{t=1}^{T}\ln\sigma_t^2 - \frac{T}{2}\ln 2\pi - \frac{1}{2}\sum_{t=1}^{T}\left(a + b\cdot\sinh^{-1}\left(M\left(a,b\right) + \varepsilon_t\sqrt{V\left(a,b\right)}\right)\right)^2 +$$
$$\frac{T}{2}\ln V\left(a,b\right) + T\ln b - \frac{1}{2}\sum_{t=1}^{T}\ln\left(\left(M\left(a,b\right) + \varepsilon_t\sqrt{V\left(a,b\right)}\right)^2 + 1\right).$$

The standard normal residual at the source of the Johnson errors can be recovered with

$$z_t = a + b\cdot\sinh^{-1}\left(M\left(a,b\right) + \varepsilon_t\sqrt{V\left(a,b\right)}\right) \tag{26}$$

where $\sinh^{-1}\left(u\right) = \ln\left(u + \sqrt{u^2+1}\right)$. The validity of the Johnson distribution assumption can be verified by testing if the z's from a ML estimation are normally distributed.

B Risk-neutral Johnson s_u innovations

We want to find an explicit expression for the risk-neutral innovations $\varepsilon_t^* = F_{s_u}^{-1}\left(\Phi\left(z_t^* - \lambda_t\right)\right)$. For the standardized two parameter Johnson s_u random variable defined in Appendix A, the inverse of the distribution function is

$$F_{s_u}^{-1}\left(u\right) = c + d\times\sinh\left(\frac{\Phi^{-1}\left(u\right) - a}{b}\right)$$

where u is a probability. In the present context, $u = \Phi\left(z_t^* - \lambda_t\right)$ and

$$F_{s_u}^{-1}\left[\Phi\left(z_t^* - \lambda_t\right)\right] = c + d\times\sinh\left(\frac{\Phi^{-1}\left(\Phi\left(z_t^* - \lambda_t\right)\right) - a}{b}\right)$$

which yields

$$\varepsilon_t^* = c + d\times\sinh\left(\frac{z_t^* - a_t^*}{b}\right)$$

since $\varepsilon_t^* = F_{s_u}^{-1}\left[\Phi\left(z_t^* - \lambda_t\right)\right]$ and where $a_t^* = a + \lambda_t$. It should be emphasized here that the c and d quantities (described in Appendix A) are not the location and scale parameter making $\sinh\left(\frac{z_t^* - a_t^*}{b}\right)$ a zero-mean unit-variance random variable. Hence, ε_t^* can be seen here as a four parameter Johson s_u random variable, with parameters a_t^*, b, c and d.

C Moments of a Johnson s_u random variable

Let z denote a standard normal random variable and consider x, a two parameter Johnson s_u random variable defined as $x = \sinh\left(\frac{z-a}{b}\right)$ where $-\infty < a < \infty$, $b > 0$, $\sinh(u) = (e^u - e^{-u})/2$. Using the moment generating function of a normal random variable, the first four moments of x can be derived analytically. The formulas for these moments are:

$$\begin{aligned} E[x] &= -w^{\frac{1}{2}} \sinh(\Omega), \\ E\left[x^2\right] &= \frac{1}{2}\left[w^2 \cosh(2\Omega) - 1\right], \\ E\left[x^3\right] &= \frac{1}{4}\left(3w^{\frac{1}{2}} \sinh(\Omega) - w^{\frac{9}{2}} \sinh(3\Omega)\right), \\ E\left[x^4\right] &= \frac{1}{8}\left[w^8 \cosh(4\Omega) - 4w^2 \cosh(2\Omega) + 3\right], \end{aligned}$$

with $\cosh(u) = (e^u + e^{-u})/2$, $w = e^{\frac{1}{b^2}}$ and $\Omega = \frac{a}{b}$.

For a four parameter Johnson s_u random variable written as $\varepsilon = c + d \times x$, the moments can be computed from those above with:

$$\begin{aligned} E[\varepsilon] &= c + d \times E[x], \\ E\left[\varepsilon^2\right] &= c^2 + 2cdE[x] + d^2 E\left[x^2\right], \\ E\left[\varepsilon^3\right] &= c^3 + 3c^2 dE[x] + 3cd^2 E\left[x^2\right] + d^3 E\left[x^3\right], \\ E\left[\varepsilon^4\right] &= c^4 + 4c^3 dE[x] + 6c^2 d^2 E\left[x^2\right] + 4cd^3 E\left[x^3\right] + d^4 E\left[x^4\right]. \end{aligned}$$

D Solving for λ_t with a linear interpolation

Denote the pricing restriction as a function of the unknown value for λ_t to be:

$$f(\lambda_t) = \alpha - r - \ln\left[\frac{1 + \frac{1}{2}\sigma_t^2 + \frac{1}{6}\sigma_t^3 \mu_3 + \frac{1}{24}\sigma_t^4 \mu_4}{1 + \sigma^* \mu_1^* + \frac{1}{2}\sigma_t^{*2} \mu_2^* + \frac{1}{6}\sigma_t^{*3} \mu_3^* + \frac{1}{24}\sigma_t^{*4} \mu_4^*}\right]$$

where the μ_i^* are functions of λ_t. We want to find a value λ_t^* such that $f(\lambda_t^*) = 0$. Given the approximate linearity of the function and a first point on the function given by $f(0) = \alpha - r$, we use a second point $\widetilde{\lambda}_t$ to obtain:

$$\lambda_t^* = \frac{\alpha - r}{\alpha - r - f\left(\widetilde{\lambda}_t\right)} \times \widetilde{\lambda}_t.$$

E Equilibrium and no-arbitrage under normality and a constant pricing parameter

The pricing restriction for the equilibrium approach, given by equation (7), is

$$\alpha = r + \gamma_t - \ln E^{Q}_{t-1}\left[e^{\sigma^*_t F^{-1}_{s_u}[\Phi(z^*_t - \lambda_t)]}\right].$$

With a normal distribution for the shocks, $\gamma_t = \frac{1}{2}\sigma_t^2$ and $F^{-1}_{s_u} = \Phi^{-1}$, which leads, under the assumption of a constant pricing parameter, to

$$\alpha = r + \sigma_t \lambda$$

since $\sigma^*_t = \sigma_t$. This expression can be substituted in the physical return dynamics to obtain

$$R_t = r - \delta_t + \lambda\sigma_t - \frac{1}{2}\sigma_t^2 + \sigma_t \varepsilon_t.$$

For the no-arbitrage case, the pricing restriction is given by equation (14) and is written as:

$$\alpha = r - \Psi_t(\nu_t - 1) + \Psi_t(\nu_t) + \Psi_t(-1).$$

Using a constant value for the pricing parameter, the definition for $\Psi_t(\cdot)$ and the moment generating function of a normal random variable, we can write

$$\Psi_t(-1) = \gamma_t,$$

$$\Psi_t(\nu) = \ln E\left[e^{-\nu\varepsilon_t}\right] = \frac{1}{2}\sigma_t^2\nu^2,$$

and

$$\Psi_t(\nu - 1) = \ln E\left[e^{-(\nu-1)\varepsilon_t}\right] = \ln e^{\frac{1}{2}\sigma_t^2\nu^2 - \sigma_t^2\nu + \frac{1}{2}\sigma_t^2}.$$

Substituting in the physical return expression gives

$$R_t = r - \delta_t + \nu\sigma_t^2 - \frac{1}{2}\sigma_t^2 + \sigma_t\varepsilon_t.$$

Table 1: Time series estimation results

	Gaussian physical	Johnson s_u physical	Johnson s_u equilibrium	Johnson s_u no-arbitrage	QML no-arbitrage
β_0	1.4e-06	1.1e-06	1.2e-06	1.4e-06	1.7e-06
	(1.3e-07)	(1.7e-07)	(1.6e-07)	(1.8e-07)	(1.5e-07)
β_1	0.8653	0.8664	0.8638	0.8600	0.8600
	(0.0067)	(0.0098)	(0.0099)	(0.0102)	(0.0070)
β_2	0.0638	0.0631	0.0631	0.0642	0.0648
	(0.0046)	(0.0064)	(0.0064)	(0.0065)	(0.0047)
θ	0.9937	1.0316	1.0308	1.0413	0.9959
	(0.0811)	(0.1099)	(0.1100)	(0.1091)	(0.0809)
a	–	0.3478	0.3410	0.3604	–
	–	(0.0869)	(0.0877)	(0.0867)	–
b	–	2.1610	2.1621	2.1622	–
	–	(0.1307)	(0.1307)	(0.1308)	–
α	3.1e-04	3.3e-04	–	–	–
	(1.1e-04)	(1.1e-04)	–	–	–
λ	–	–	0.0311	–	–
	–	–	(0.0135)	–	–
ν	–	–	–	1.7772	1.9879
	–	–	–	(0.9755)	(1.0025)
Loglik	18042	18138	18138	18138	18043
JB	649.7317	0.3909	0.4148	0.3063	637.4257
	[0.0000]	[0.8187]	[0.8112]	[0.8691]	[0.0000]
$Q(20)$	32.2506	32.5304	31.3165	30.3431	30.3369
	[0.0407]	[0.0380]	[0.0511]	[0.0645]	[0.0646]
$Q^2(20)$	23.4912	21.6906	22.1540	22.8584	24.6091
	[0.2653]	[0.3575]	[0.3322]	[0.2958]	[0.2168]

This table reports the QML ("Gaussian physical") and ML estimates ("Johnson s_u physical") for the physical process described by equations (1), (2) and (3), the ML estimates of the physical process with a constant pricing parameter for the equilibrium pricing approach ("Johnson s_u equilibrium") given by equation (10), (11) and (12), and the ML and QML estimates of the physical process with a constant pricing parameter for the no-arbitrage approach ("Johnson s_u no-arbitrage" and "QML no-arbitrage") given by equations (17), (18) and (19). Standard errors are reported in parenthesis below the parameter estimates. "Loglik" is the log-likelihood value and "JB" is the Jarque-Bera normality test for the standard normal residuals computed with equation (26) in Appendix A. $Q(20)$ is the Ljung-Box portmanteau test for up to 20th-order serial correlation in the standardized residuals, whereas $Q^2(20)$ is the same test for the squared standardized residuals. The p-values for these tests are reported below in square brackets.

Table 2: Analytical approximation performance for $\ln E_{t-1}^{Q}[e^{\sigma_t^* \varepsilon_t^*}]$

a	b	simu.	appr.	diff.
0.0	1.5	-0.000546	-0.000540	0.000006
0.0	2.0	-0.000543	-0.000547	0.000004
0.0	2.5	-0.000545	-0.000549	0.000004
0.0	3.0	-0.000545	-0.000550	0.000005
1.0	1.5	-0.000522	-0.000521	0.000001
1.0	2.0	-0.000541	-0.000541	0.000000
1.0	2.5	-0.000546	-0.000547	0.000001
1.0	3.0	-0.000551	-0.000549	0.000002
2.0	1.5	-0.000504	-0.000499	0.000005
2.0	2.0	-0.000532	-0.000529	0.000003
2.0	2.5	-0.000541	-0.000541	0.000000
2.0	3.0	-0.000547	-0.000546	0.000001
3.0	1.5	-0.000492	-0.000490	0.000002
3.0	2.0	-0.000524	-0.000521	0.000003
3.0	2.5	-0.000532	-0.000535	0.000003
3.0	3.0	-0.000547	-0.000542	0.000005

This table reports results on the performance of the approximation in (9): "simu." is the expected value computed with a Monte Carlo simulation using 15 million sample paths and "appr." is the analytical value computed with the approximation formula given by equation (9). In these computations $\sigma_t^* = 0.2 \times \sqrt{1/252}$ and $\lambda_t = 0.05$.

Table 3: Solving for λ_t with a linear interpolation

a	b	bisec.	lin. int.	diff.
0.0	1.0	0.023895	0.023879	0.000016
0.0	2.0	0.022162	0.022157	0.000005
0.0	3.0	0.022076	0.022068	0.000008
0.0	4.0	0.022052	0.022054	0.000002
1.0	1.0	0.026996	0.026250	0.000746
1.0	2.0	0.022467	0.022265	0.000202
1.0	3.0	0.022137	0.022045	0.000093
1.0	4.0	0.022076	0.022024	0.000052
2.0	1.0	0.028632	0.027670	0.000961
2.0	2.0	0.022968	0.022631	0.000336
2.0	3.0	0.022284	0.022117	0.000167
2.0	4.0	0.022137	0.022033	0.000105
3.0	1.0	0.028912	0.027909	0.001003
3.0	2.0	0.023285	0.022883	0.000402
3.0	3.0	0.022430	0.022213	0.000217
3.0	4.0	0.022198	0.022062	0.000136

This table reports results for two methods which can be used to solve the pricing restriction in (13): "bisec." is the value of λ_t computed with a bisection algorithm and "lin. int." is the value of λ_t computed with the linear interpolation approach. In these computations $\sigma_t^* = 0.2 \times \sqrt{1/252}$ and $\alpha = 0.1/252$, $r = 0.03/252$.

Table 4: Solving for ν_t

a	b	σ	bisec.	lin. int.	approx.
1.0	1.0	0.10	6.315247	7.104352	7.005481
1.0	2.0	0.10	6.884111	7.017292	7.000914
1.0	3.0	0.10	6.952161	7.007081	7.000375
1.0	4.0	0.10	6.973826	7.003864	7.000204
1.0	1.0	0.20	1.703993	1.801053	1.760663
1.0	2.0	0.20	1.742703	1.758614	1.751819
1.0	3.0	0.20	1.747009	1.753530	1.750747
1.0	4.0	0.20	1.748368	1.751926	1.750408
1.0	1.0	0.30	0.784489	0.811006	0.793322
1.0	2.0	0.30	0.779146	0.783499	0.780493
1.0	3.0	0.30	0.778345	0.780124	0.778894
1.0	4.0	0.30	0.778087	0.779058	0.778388
1.0	1.0	0.60	0.207850	0.209674	0.222830
1.0	2.0	0.60	0.196964	0.197271	0.199795
1.0	3.0	0.60	0.195483	0.195606	0.196656
1.0	4.0	0.60	0.195011	0.195079	0.195654

This table reports results for the three approaches which can be used to solve the pricing restriction in (14): "bisec." is the value of ν_t computed with a bisection algorithm, "lin. int." is the the value of ν_t computed with the linear interpolation approach, and "approx." is the value of ν_t computed with the approximation suggested in CEFJ. In these computations $\alpha = 0.1/252$, $r = 0.03/252$ and σ is multiplied by $\sqrt{1/252}$.

Table 5: Black-Scholes implied volatilities of European call option prices on the S&P 500

	Strike prices						
	47	48	49	50	51	52	53
				Equilibrium: constant λ			
$T = 30$	0.2745	0.2577	0.2415	0.2261	0.2114	0.1978	0.1855
$T = 90$	0.2669	0.2540	0.2415	0.2294	0.2178	0.2068	0.1963
$T = 270$	0.2668	0.2589	0.2512	0.2437	0.2366	0.2297	0.2230
				Equilibrium: time-varying λ_t			
$T = 30$	0.2745	0.2577	0.2414	0.2258	0.2111	0.1974	0.1850
$T = 90$	0.2671	0.2541	0.2415	0.2293	0.2177	0.2066	0.1961
$T = 270$	0.2674	0.2593	0.2516	0.2441	0.2369	0.2299	0.2233
				No-arbitrage: constant ν			
$T = 30$	0.2742	0.2573	0.2410	0.2255	0.2108	0.1972	0.1848
$T = 90$	0.2664	0.2534	0.2408	0.2287	0.2170	0.2060	0.1956
$T = 270$	0.2667	0.2586	0.2508	0.2433	0.2360	0.2291	0.2224
				No-arbitrage: time-varying ν_t			
$T = 30$	0.2742	0.2574	0.2412	0.2256	0.2109	0.1972	0.1849
$T = 90$	0.2671	0.2541	0.2415	0.2293	0.2176	0.2065	0.1960
$T = 270$	0.2677	0.2596	0.2518	0.2443	0.2370	0.2301	0.2234

This table reports the Black-Scholes implied volatilities obtained from theoretical call option prices computed with a Monte-Carlo simulation with one million sample paths. The NGARCH-Johnson parameter values are taken from the time series estimations reported in Table 1. In each case, the initial variance is set to the estimated variance for the last data point in the time series estimation. The initial stock price is set to 50, the interest rate to 0.03/252 and the dividend yield to 0.01/252.

Table 6: Statistics about the pricing errors computed with a weekly sample of call and put options on the S&P 500 index from 2006 to 2011

	EquCst	EquTva	NoaCst	NoaTva
	Correlations of pricing errors			
EquCst	–	0.9987	0.8780	0.9679
EquTva	–	–	0.8721	0.9689
NoaCst	–	–	–	0.8800
	Correlations of absolute pricing errors			
EquCst	–	0.9960	0.8225	0.9434
EquTva	–	–	0.8042	0.9465
NoaCst	–	–	–	0.8174
	Pricing errors			
me	-0.0119	-0.0146	-0.0062	-0.0142
mae	0.0386	0.0392	0.0431	0.0391
$rmse$	0.0576	0.0580	0.0705	0.0578
	mae by maturity			
$T \leq 30$	0.0413	0.0414	0.0455	0.0412
$30 < T \leq 90$	0.0375	0.0383	0.0417	0.0382
$90 < T \leq 180$	0.0345	0.0364	0.0394	0.0361
$T > 180$	0.0328	0.0353	0.0423	0.0346
	mae by moneyness			
out	0.0509	0.0526	0.0566	0.0523
near	0.0312	0.0315	0.0347	0.0313
in	0.1187	0.1184	0.1506	0.1210
	mae by implied volatility			
$\sigma \leq 0.20$	0.0224	0.0225	0.0226	0.0224
$0.20 < \sigma \leq 0.30$	0.0384	0.0395	0.0401	0.0393
$0.30 < \sigma \leq 0.40$	0.0559	0.0575	0.0635	0.0573
$\sigma > 0.40$	0.0816	0.0817	0.1086	0.0816
	mae by years			
2006	0.0171	0.0165	0.0174	0.0166
2007	0.0235	0.0238	0.0234	0.0237
2008	0.0474	0.0461	0.0714	0.0463
2009	0.0541	0.0559	0.0558	0.0556
2010	0.0408	0.0428	0.0400	0.0427
2011	0.0388	0.0402	0.0392	0.0396

This table reports statistics about the pricing errors, expressed as differences between Black-Scholes implied volatilities, computed with a weekly sample of call and put options on the S&P 500 index from 2006 to 2011. "EquCst" is for the equilibrium approach with a constant pricing parameter; "EquTva" is for the equilibrium approach with a time-varying pricing parameter; "NoaCst" is for the no-arbitrage approach with a constant pricing parameter; "NoaTva" is for the no-arbitrage approach with a time-varying pricing parameter. "*me*" are mean pricing errors; "*mae*" are mean absolute pricing errors; "*rmse*" are root-mean-squared errors. "out" are out-of-the-money options defined as $S/K < 0.9$ for calls and $S/K > 1.1$ for puts. "in" are in-the-money options defined as $S/K > 1.1$ for calls and $S/K < 0.9$ for puts. "near" are near-the-money options with S/K between 0.9 and 1.1.

Table 7: Statistics about the pricing errors computed with a weekly sample of call and put options on the S&P 500 index from 2006 to 2011 for alternative specifications of the no-arbitrage approach with constant pricing parameters

	$\nu_t = \nu$	$\nu_t = \nu/\sigma_t$	$\nu_t = \nu/\sigma_t^2$	$\lambda_t = \lambda$
	Correlations of pricing errors			
$\nu_t = \nu$	–	0.9045	0.8842	0.8780
$\nu_t = \nu/\sigma_t$	–	–	0.9678	0.9301
$\nu_t = \nu/\sigma_t^2$	–	–	–	0.9409
	Correlations of absolute pricing errors			
$\nu_t = \nu$	–	0.8668	0.8302	0.8225
$\nu_t = \nu/\sigma_t$	–	–	0.9441	0.8817
$\nu_t = \nu/\sigma_t^2$	–	–	–	0.8971
	Pricing errors			
me	-0.0062	-0.0109	-0.0123	-0.0119
mae	0.0431	0.0383	0.0382	0.0386
rmse	0.0705	0.0581	0.0561	0.0576
	mae by maturity			
$T \leq 30$	0.0455	0.0413	0.0409	0.0413
$30 < T \leq 90$	0.0417	0.0373	0.0372	0.0375
$90 < T \leq 180$	0.0394	0.0338	0.0344	0.0345
$T > 180$	0.0423	0.0315	0.0324	0.0328
	mae by moneyness			
out	0.0566	0.0503	0.0510	0.0509
near	0.0347	0.0310	0.0307	0.0312
in	0.1506	0.1222	0.1137	0.1187
	mae by implied volatility			
$\sigma \leq 0.20$	0.0226	0.0223	0.0221	0.0224
$0.20 < \sigma \leq 0.30$	0.0401	0.0379	0.0379	0.0384
$0.30 < \sigma \leq 0.40$	0.0635	0.0553	0.0560	0.0559
$\sigma > 0.40$	0.1086	0.0816	0.0801	0.0816
	mae by years			
2006	0.0174	0.0173	0.0169	0.0171
2007	0.0234	0.0230	0.0232	0.0235
2008	0.0714	0.0482	0.0464	0.0474
2009	0.0558	0.0533	0.0538	0.0541
2010	0.0400	0.0402	0.0409	0.0408
2011	0.0392	0.0382	0.0383	0.0388

This table reports statistics about the pricing errors, expressed as differences between Black-Scholes implied volatilities, computed with a weekly sample of call and put options on the S&P 500 index from 2006 to 2011. "$\nu_t = \nu$" is for the no-arbitrage approach with a constant pricing parameter; "$\nu_t = \nu/\sigma_t$" is for the no-arbitrage approach with a constant pricing parameter scaled by the volatility; "$\nu_t = \nu/\sigma_t^2$" is for the no-arbitrage approach with a constant pricing parameter scaled by the variance; "*me*" are mean pricing errors; "*mae*" are mean absolute pricing errors; "*rmse*" are root-mean-squared errors. "out" are out-of-the-money options defined as $S/K < 0.9$ for calls and $S/K > 1.1$ for puts. "in" are in-the-money options defined as $S/K > 1.1$ for calls and $S/K < 0.9$ for puts. "near" are near-the-money options with S/K between 0.9 and 1.1.

Table 8: Black-Scholes implied volatilities of European call option prices on the S&P 500 in the no-arbitrage pricing framework

	Strike prices						
	47	48	49	50	51	52	53
	constant ν						
	ML Johnson s_u						
$T = 30$	0.2803	0.2648	0.2503	0.2370	0.2249	0.2142	0.2052
$T = 90$	0.2743	0.2623	0.2508	0.2399	0.2296	0.2201	0.2114
$T = 270$	0.2867	0.2781	0.2699	0.2621	0.2548	0.2478	0.2414
	QML Cornish-Fisher						
$T = 30$	0.2711	0.2555	0.2409	0.2276	0.2157	0.2053	0.1967
$T = 90$	0.2598	0.2483	0.2373	0.2271	0.2176	0.2087	0.2007
$T = 270$	0.2618	0.2543	0.2471	0.2404	0.2341	0.2281	0.2225
	QML Johnson s_u						
$T = 30$	0.2710	0.2555	0.2410	0.2277	0.2158	0.2054	0.1967
$T = 90$	0.2598	0.2483	0.2374	0.2272	0.2177	0.2088	0.2008
$T = 270$	0.2618	0.2542	0.2471	0.2404	0.2341	0.2281	0.2225
	time-varying ν_t						
	ML Johnson s_u						
$T = 30$	0.2800	0.2654	0.2516	0.2390	0.2274	0.2172	0.2084
$T = 90$	0.2737	0.2624	0.2516	0.2413	0.2315	0.2224	0.2139
$T = 270$	0.2713	0.2641	0.2572	0.2506	0.2442	0.2381	0.2323
	QML Cornish-Fisher						
$T = 30$	0.2700	0.2553	0.2414	0.2287	0.2173	0.2072	0.1988
$T = 90$	0.2585	0.2477	0.2374	0.2277	0.2185	0.2101	0.2023
$T = 270$	0.2514	0.2450	0.2389	0.2331	0.2275	0.2222	0.2172
	QML Johnson s_u						
$T = 30$	0.2700	0.2553	0.2415	0.2288	0.2174	0.2073	0.1988
$T = 90$	0.2586	0.2478	0.2375	0.2278	0.2187	0.2102	0.2023
$T = 270$	0.2515	0.2451	0.2390	0.2331	0.2276	0.2223	0.2172

This table reports Black-Scholes implied volatilities of call option prices computed with a Monte-Carlo simulation with one million sample paths for the no-arbitrage models with constant and time-varying ν_t. "ML Johnson s_u" are for option prices obtained from the no-arbitrage pricing framework with Johnson s_u shocks, "QML Cornish-Fisher" are for option prices obtained with a quasi-maximum likelihood estimator and Cornish-Fisher random numbers, and "QML Johnson s_u" are for option prices obtained with a quasi-maximum likelihood estimator and Johnson s_u random numbers. In each case, the initial variance is set to the estimated variance for the last data point in the time series estimation. The initial stock price is set to 50, the constant interest rate to 0.03/252 and the dividend yield to 0.01/252.

Table 9: Statistics about the pricing errors computed with a weekly sample of call and put options on the S&P 500 index from 2006 to 2011

	NoaTva	QML-John Tva	QML-CF Tva	QML-CF Cst
		Correlations of pricing errors		
NoaTva	–	0.9478	0.9452	0.8796
QML-John	–	–	0.9950	0.8859
QML-CF	–	–	–	0.8867
		Correlations of absolute pricing errors		
NoaTva	–	0.9089	0.9036	0.8362
QML-John	–	–	0.9914	0.8211
QML-CF	–	–	–	0.8213
		Pricing errors		
me	-0.0142	-0.0191	-0.0196	-0.0146
mae	0.0391	0.0382	0.0382	0.0415
rmse	0.0578	0.0545	0.0545	0.0630
		mae by maturity		
$T \leq 30$	0.0412	0.0374	0.0374	0.0410
$30 < T \leq 90$	0.0382	0.0374	0.0375	0.0398
$90 < T \leq 180$	0.0361	0.0409	0.0410	0.0448
$T > 180$	0.0346	0.0425	0.0426	0.0484
		mae by moneyness		
out	0.0523	0.0504	0.0506	0.0537
near	0.0313	0.0310	0.0310	0.0338
in	0.1210	0.1118	0.1111	0.1407
		mae by implied volatility		
$\sigma \leq 0.20$	0.0224	0.0222	0.0220	0.0225
$0.20 < \sigma \leq 0.30$	0.0393	0.0402	0.0404	0.0418
$0.30 < \sigma \leq 0.40$	0.0573	0.0568	0.0570	0.0611
$\sigma > 0.40$	0.0816	0.0716	0.0717	0.0911
		mae by years		
2006	0.0166	0.0186	0.0181	0.0179
2007	0.0237	0.0233	0.0232	0.0227
2008	0.0463	0.0378	0.0378	0.0566
2009	0.0556	0.0590	0.0593	0.0590
2010	0.0427	0.0421	0.0423	0.0417
2011	0.0396	0.0395	0.0396	0.0403

This table reports statistics about the pricing errors, expressed as differences between Black-Scholes implied volatilities, computed with a weekly sample of call and put options on the S&P 500 index from 2006 to 2011 with the no-arbitrage approach. "NoaTva" are for option prices computed with a maximum likelihood approach and a time-varying ν; "QML-John-Tva" are for option prices computed with a quasi-maximum likelihood approach, Johnson s_u random numbers and a time-varying ν_t; "QML-CF-Tva" are for option prices computed with a quasi-maximum likelihood approach, Cornish-Fisher random numbers and a time-varying ν_t; "QML-CF-Cst" are for option prices computed with a quasi-maximum likelihood approach, Cornish-Fisher random numbers and a constant ν. "*me*" are mean pricing errors; "*mae*" are mean absolute pricing errors; "*rmse*" are root-mean-squared errors. "out" are out-of-the-money options defined as $S/K < 0.9$ for calls and $S/K > 1.1$ for puts. "in" are in-the-money options defined as $S/K > 1.1$ for calls and $S/K < 0.9$ for puts. "near" are near-the-money options with S/K between 0.9 and 1.1.

Table 10: Time series estimation results for time-varying conditional non-normal distributions

	physical	equilibrium	no-arbitrage
β_0	1.8e-06	1.9e-06	2.2e-06
	(2.7e-07)	(2.6e-07)	(2.9e-07)
β_1	0.8446	0.8429	0.8371
	(0.0108)	(0.0109)	(0.0113)
β_2	0.0662	0.0660	0.0669
	(0.0067)	(0.0066)	(0.0067)
θ	1.1097	1.1031	1.1162
	(0.1138)	(0.1144)	(0.1147)
$\bar{a}$	0.2808	0.2691	0.2905
	(0.0796)	(0.0776)	(0.0731)
$\phi_{a,1}$	0.5277	0.5408	0.5670
	(0.2634)	(0.2557)	(0.2482)
$\phi_{a,2}$	-47968	-48464	-47753
	(26640)	(26632)	(26377)
$\bar{b}$	2.5434	2.5455	2.5634
	(0.2298)	(0.2294)	(0.2293)
$\phi_{b,1}$	0.9855	0.9857	0.9863
	(0.0033)	(0.0032)	(0.0031)
$\phi_{b,2}$	222798	221687	222092
	(91444)	(90414)	(88809)
α	2.9e-04	–	–
	(1.1e-04)	–	–
λ	–	0.0297	–
	–	(0.0138)	–
ν	–	–	2.0797
	–	–	(1.0111)
loglik	18154	18156	18156
JB	2.0823	2.1726	1.9204
	[0.3427]	[0.3421]	[0.3793]
$Q(20)$	31.3145	30.4335	29.4389
	[0.0512]	[0.0631]	[0.0795]
$Q^2(20)$	27.1567	27.5678	30.0118
	[0.1309]	[0.1200]	[0.0697]

This table reports the maximum likelihood estimates for the time series model with conditional time-varying non-normal distribution described by equations (21) to (24). Standard errors are reported in parenthesis below the parameter estimates. "Loglik" is the log-likelihood value and "JB" is the Jarque-Bera normality test for the standard normal residuals computed with equation (26). $Q(20)$ is the Ljung-Box portmanteau test for up to 20th-order serial correlation in the standardized residuals, whereas $Q^2(20)$ is the same test for the squared standardized residuals. The p-values for these tests are reported below in square brackets.

Table 11: Statistics about the pricing errors computed with a weekly sample of call and put options on the S&P 500 index from 2006 to 2011 with the time-varying non-normal distribution model

	EquCst	EquTva	NoaCst	NoaTva
	Correlations of pricing errors			
EquCst	–	0.9932	0.8258	0.9672
EquTva	–	–	0.8181	0.9625
NoaCst	–	–	–	0.8285
	Correlations of absolute pricing errors			
EquCst	–	0.9878	0.7586	0.9453
EquTva	–	–	0.7410	0.9390
NoaCst	–	–	–	0.7577
	Pricing errors			
me	-0.0170	-0.0194	-0.0118	-0.0192
mae	0.0375	0.0385	0.0417	0.0384
$rmse$	0.0559	0.0571	0.0678	0.0566
	mae by maturity			
$T \leq 30$	0.0398	0.0401	0.0431	0.0400
$30 < T \leq 90$	0.0362	0.0373	0.0401	0.0373
$90 < T \leq 180$	0.0348	0.0370	0.0402	0.0369
$T > 180$	0.0345	0.0387	0.0441	0.0366
	mae by moneyness			
out	0.0507	0.0528	0.0556	0.0525
near	0.0298	0.0304	0.0330	0.0302
in	0.1159	0.1162	0.1500	0.1200
	mae by implied volatility			
$\sigma \leq 0.20$	0.0204	0.0206	0.0204	0.0205
$0.20 < \sigma \leq 0.30$	0.0381	0.0395	0.0392	0.0394
$0.30 < \sigma \leq 0.40$	0.0567	0.0588	0.0630	0.0585
$\sigma > 0.40$	0.0794	0.0812	0.1072	0.0809
	mae by years			
2006	0.0172	0.0166	0.0182	0.0165
2007	0.0232	0.0236	0.0231	0.0235
2008	0.0398	0.0397	0.0639	0.0397
2009	0.0616	0.0635	0.0614	0.0636
2010	0.0384	0.0403	0.0374	0.0405
2011	0.0365	0.0384	0.0363	0.0376

This table reports statistics about the pricing errors, expressed as differences between Black-Scholes implied volatilities, computed with a weekly sample of call and put options on the S&P 500 index from 2006 to 2011. "EquCst" is for the equilibrium approach with a constant pricing parameter; "EquTva" is for the equilibrium approach with a time-varying pricing parameter; "NoaCst" is for the no-arbitrage approach with a constant pricing parameter; "NoaTva" is for the no-arbitrage approach with a time-varying pricing parameter. "me" are mean pricing errors; "mae" are mean absolute pricing errors; "$rmse$" are root-mean-squared errors. "out" are out-of-the-money options defined as $S/K < 0.9$ for calls and $S/K > 1.1$ for puts. "in" are in-the-money options defined as $S/K > 1.1$ for calls and $S/K < 0.9$ for puts. "near" are near-the-money options with S/K between 0.9 and 1.1.

Figure 1: Quantile to quantile plot of standardized normal residuals implied by the Johnson s_u distribution

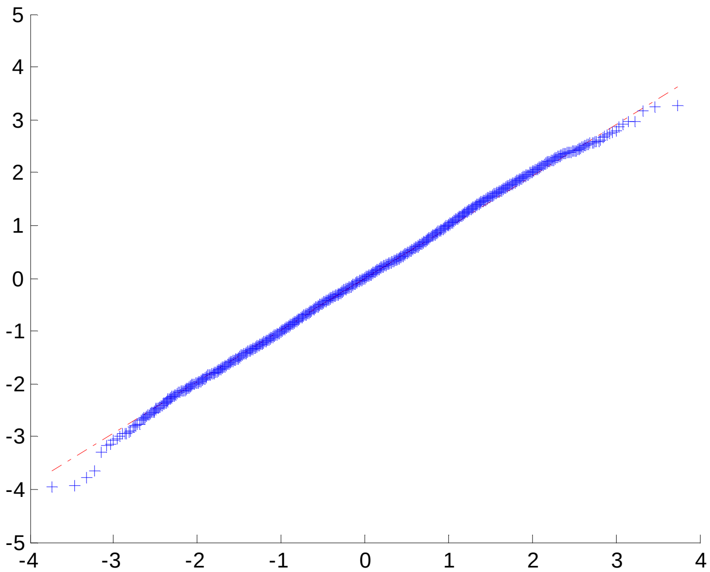


Figure 2: Pricing restriction as a function of λ in the equilibrium pricing framework

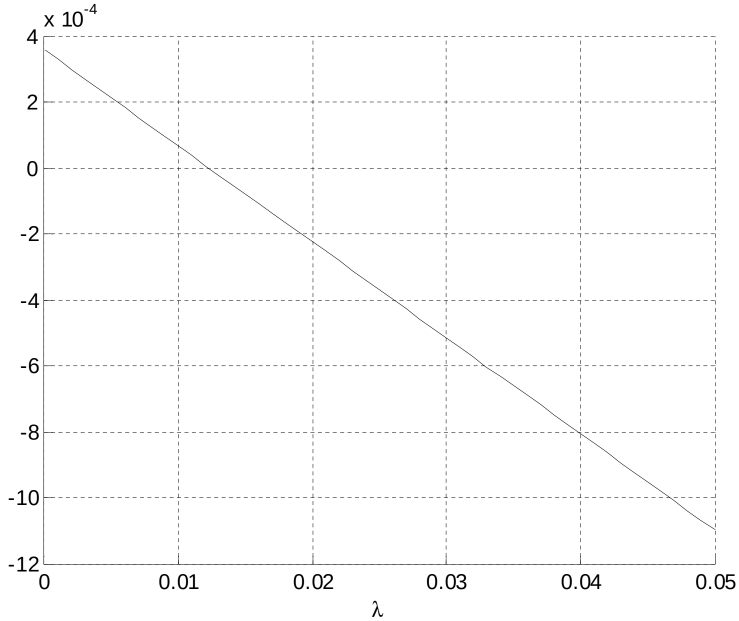


Figure 3: Pricing restriction as a function of ν in the no-arbitrage pricing framework

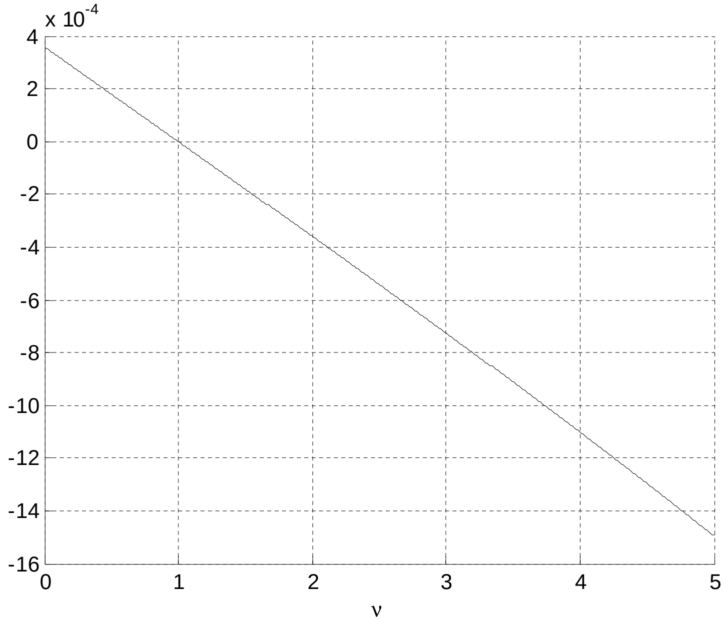
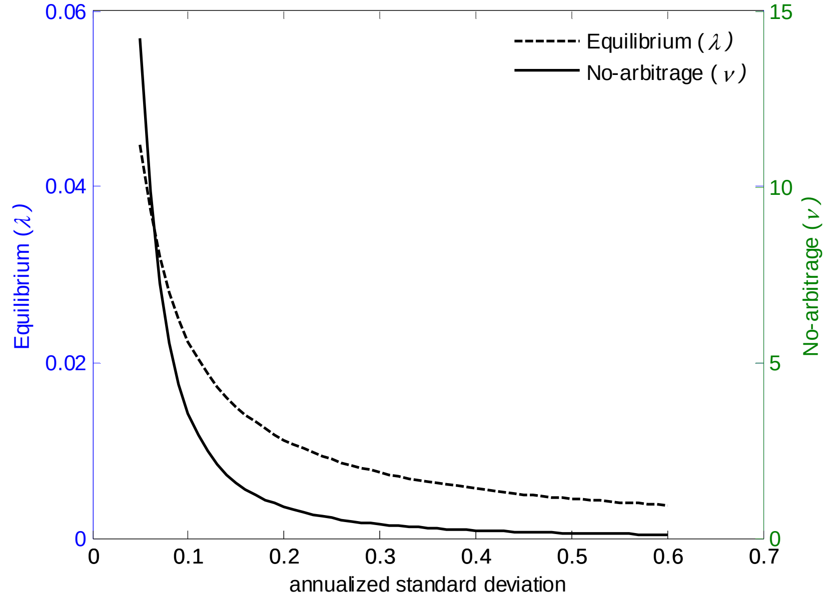


Figure 4: Equilibrium and no-arbitrage pricing parameter as a function of the volatility level



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