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Bootstrapping integrated covariance matrix estimators in noisy jump-diffusion models with non-synchronous trading *

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Abstract

We propose a bootstrap method for estimating the distribution (and functionals of it such as the variance) of various integrated covariance matrix estimators. In particular, we first adapt the wild blocks of blocks bootstrap method suggested for the pre-averaged realized volatility estimator to a general class of estimators of integrated covolatility. We then show the first-order asymptotic validity of this method in the multivariate context with a potential presence of jumps, dependent microstructure noise, irregularly spaced and non-synchronous data. Due to our focus on non-studentized statistics, our results justify using the bootstrap to estimate the covariance matrix of a broad class of covolatility estimators. The bootstrap variance estimator is positive semi-definite by construction, an appealing feature that is not always shared by existing variance estimators of the integrated covariance estimator. As an application of our results, we also consider the bootstrap for regression coefficients. We show that the wild blocks of blocks bootstrap, appropriately centered, is able to mimic both the dependence and heterogeneity of the scores, thus justifying the construction of bootstrap percentile intervals as well as variance estimates in this context. This contrasts with the traditional pairs bootstrap which is not able to mimic the score heterogeneity even in the simple case where no microstructure noise is present. Our Monte Carlo simulations show that the wild blocks of blocks bootstrap improves the finite sample properties of the existing first-order asymptotic theory. We illustrate its practical use on high-frequency equity data.

JEL Classification: C15, C22, C58
Keywords: High-frequency data, market microstructure noise, non-synchronous data, jumps, realized measures, integrated covariance, wild bootstrap, block bootstrap.

1 Introduction

The covariation between asset returns is indispensable for risk management, portfolio selection, hedging and pricing of derivatives, etc. Presently, the availability of high-frequency financial intraday data such as stock prices or currencies allows us to accurately estimate the integrated covariance. An early popular estimator is realized covariance matrix, computed as the sum of outer product of vectors of high-frequency returns. The underlying idea is to use quadratic covariation as an ex-post covariance

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measure, whose increments can be studied to learn about the dependence of asset returns over a given period (see e.g., Andersen et al. (2003) and Barndorff-Nielsen et al. (2004a)). An important characteristic of high frequency financial data is the presence of market microstructure effects: prices are observed with contamination errors (the so-called noise) due to the presence of bid-ask bounce effects, rounding errors, etc., which contribute to a discrepancy between the latent efficient price process and the price observed by the econometrician (e.g. Hasbrouck (2007)). In a univariate setting, market microstructure noise makes the standard realized volatility estimator biased and inconsistent. This has motivated the development of alternative estimators. Currently, there are four main univariate approaches to restore the consistency of realized volatility estimator, namely linear combination of realized volatilities obtained by subsampling (Zhang et al. (2005), and Zhang (2006)), kernel-based autocovariance adjustments (Barndorff-Nielsen et al. (2008)), the pre-averaging method (Podolskij and Vetter (2009), and Jacod et al. (2009)), and the maximum likelihood-based approach (Xiu (2010)).

In a multivariate setting, matters are further complicated with the distinctive feature of multivariate financial data: the phenomenon of non-synchronous trading, i.e. the prices of two assets are often not observed at the same time, leading to the well-known Epps effect, highlighted by Epps (1979). These factors create a further level of challenge to the problem of integrated covariance matrix estimation. The most prominent estimators of integrated covolatility that are consistent under non-synchronous observed data and contaminated by market microstructure noise include but are not limited to, the pre-averaged Hayashi-Yoshida estimator studied by Christensen et al. (2010), the multivariate realized kernel estimator of Barndorff-Nielsen et al. (2011), the flat-top realized kernel by Varneskov (2014), the two-scales covariance estimator of Zhang (2011), the generalized multi-scale covariance estimator of Bibinger (2011), the maximum likelihood based-estimator of Ait-Sahalia, Fan and Xiu (2010), Corsi, Peluso and Audrino (2014), Liu and Tang (2014), Shephard and Xiu (2014), the Fourier based estimator of covariances of Park and Linton (2012), and the local method of moments estimator of Bibinger et al. (2014).

Despite the fact that these statistics are measured over large samples, their finite sample distributions are not necessarily well approximated by their asymptotic mixed normal distribution. Indeed, Zhang et al. (2011) showed in the univariate case that the asymptotic normal approximation is often inaccurate for the subsampling realized volatility estimator of Zhang et al. (2005), whose finite sample distribution is skewed and heavy tailed. They proposed Edgeworth corrections for this estimator as a way to improve upon the standard normal approximation. Similarly, Bandi and Russell (2011) discussed the limitations of asymptotic approximations in the context of realized kernels and proposed a finite sample procedure. As an alternative tool of inference in this context, Gonçalves and Meddahi (2009) introduced bootstrap methods for the realized volatility under no market microstructure noise, whereas Hounyo et al. (2013) and Gonçalves et al. (2014) extend the work of Gonçalves and Meddahi (2009) by allowing market microstructure effects.

In this paper, we focus on the class of estimators of integrated covolatility that can be written as the sum of miniature realized covolatility measure. Examples of potential estimators of integrated covolatility in this class include the realized covariance matrix, the cumulative covariance estimator developed in Hayashi and Yoshida (2005), the truncation-based estimators of integrated covariance of Mancini and Gobbi (2012), and some noise-robust estimators listed above (pre-averaging, realized kernel, two and multi-scale based covariance estimators), among others.

The main contribution of this paper is to propose a general bootstrap method for estimating the distribution as well as the variance of integrated covariance matrix estimators. The bootstrap technique employed here is related to previous work in the univariate case, in particular, the wild blocks of blocks bootstrap suggested in Hounyo et al. (2013) for the pre-averaging estimator. To handle both the dependence and heterogeneity of pre-averaged returns (most often in the form of heteroskedasticity), Hounyo et al. (2013) propose to combine the wild bootstrap with the blocks of blocks bootstrap. This procedure relies on the fact that the heteroskedasticity can be handled elegantly by use of the wild bootstrap, and a block-based bootstrap can be used to treat the serial correlation in the data. The current article draws ideas from this paper, but here we are faced with two additional challenges at the same time. We have to extend their univariate wild blocks of blocks bootstrap method to the multivariate case, but we also need to adapt this method for a broad class of covolatility estimators (not only for the pre-averaging based-estimator). The univariate method cannot be applied directly in this general context. We provide intuition of this in Section 4.3. This generalization faces the additional complexity of possibly having to deal with jumps, various types of noise, irregularly spaced and non-synchronous data. In particular, in a multivariate setting we first adapt the wild blocks of blocks bootstrap method studied by Hounyo et al. (2013) to a general class of statistics. Next, we give a set of high level conditions such that any bootstrap method is asymptotically valid when estimating the distribution as well as the variance of integrated covariance matrix estimator. We then verify these high-level conditions for various estimators of integrated covolatility in different settings which allow for a potential presence of jumps, dependent microstructure noise, irregularly spaced and non-synchronous data. The bootstrap variance estimator is positive semi-definite by construction, an appealing feature that is not always shared by existing variance estimators of the integrated covariance estimator.

Our findings have many implications and improve existing results in different settings. Firstly, in the idealized world where the mechanics of trading is perfect such that there is no market microstructure effects and prices are observed synchronously, apart from border terms which are $O_P\left(\frac{1}{n}\right)$ (where n denotes the sample size), our bootstrap variance estimator of the variance of the realized covariance matrix coincides with the sophisticated consistent variance estimator proposed by Barndorff-Nielsen and Shephard (2004a). This is in contrast with the pairs bootstrap studied by Dovonon et al. (2013), which is not able to estimate the long run variance of the realized covariance matrix, except when the volatility is constant. Secondly, in a more interesting setting where data are non-synchronous, however, ruling out the presence of noise, our bootstrap variance estimator of the variance of the Hayashi and Yoshida (2005) covariance estimator is an alternative to the consistent variance estimator proposed recently by Mykland (2012), which is not guaranteed to be positive semi-definite. Thirdly, in a framework where we allow the presence of market microstructure noise, but we rule out asynchronicity, the bootstrap variance estimator is an alternative to the variance estimator of the bias-corrected multivariate pre-averaged estimator proposed by Christensen et al. (2010), which is also not guaranteed to be positive semi-definite. Fourthly, and more realistically, we investigate the combination of asynchronicity, irregularly spaced and microstructure noise. We find that our bootstrap method consistently estimates the variance and the entire distribution of the pre-averaged Hayashi-Yoshida estimator of Christensen et al. (2013). We also explore how and to what extent the wild blocks of blocks bootstrap can be applied to the multivariate realized kernel estimator of Barndorff-Nielsen et al. (2011). Lastly, in the context where the covariance between the risk factors of asset prices is due to both Brownian and jump components, but we rule out asynchronicity and microstructure effects, the bootstrap variance estimator is an alternative to the asymptotic variance estimator for the truncation-based estimators of integrated covariance recently proposed by Mancini and Gobbi (2012). This result extends the work of Hounyo (2013), where a local Gaussian bootstrap method has been proposed for inference on integrated volatility under no jumps by allowing for the latter. It also provides an alternative to the general local Gaussian bootstrap method recently introduced by Dovonon et al. (2014) for jump tests.

As an application of our results, we also consider the bootstrap for realized regression coefficients. We show that the wild blocks of blocks bootstrap, appropriately centered, is able to mimic both the dependence and heterogeneity of the scores, thus justifying the construction of bootstrap percentile intervals as well as asymptotic variance estimates in this context. This contrasts with the traditional pairs bootstrap analysed in Dovonon et al. (2013), which is not able to mimic the score heterogeneity even in the simple case where microstructure noise is absent and prices are regularly spaced and synchronous. Our Monte Carlo simulations suggest that the wild blocks of blocks bootstrap method improves upon the first-order asymptotic theory in finite samples. Although the wild blocks of blocks bootstrap that we propose here requires the choice of an additional tuning parameter (the block size), we follow Hounyo et al. (2013) and use an empirical procedure to select the block size that performs well in our simulations.

The remainder of this paper is organized as follows. In the next section, we provide the framework and introduce the general class of statistics of interest. In Section 3, after introducing the bootstrap method, we give a set of high level conditions such that any bootstrap method is asymptotically valid when estimating the distribution as well as the asymptotic variance matrix of integrated covariance matrix estimator. Section 4 illustrates the bootstrap method and verifies these high level conditions for various estimators of integrated covolatility. In Section 5, we present the Monte Carlo results, while an empirical illustration is conducted in Section 6. Section 7 concludes. Two appendices are provided. Appendix A contains the tables with simulation and empirical results whereas Appendix B is a mathematical appendix providing the proofs.

2 General framework

2.1 Setup

It is well-known in finance that, under the no-arbitrage assumption, price processes must follow a semimartingale (see, e.g., Delbaen and Schachermayer (1994)). We consider a d-dimensional latent efficient log-price process $X_t = \left(X_t^{(1)}, \cdots, X_t^{(d)}\right)'$ defined on a probability space $\left(\Omega^{(0)}, \mathcal{F}^{(0)}, P^{(0)}\right)$ equipped with a filtration $\left(\mathcal{F}_t^{(0)}\right)_{t\geq 0}$. We model X as an Itô semimartingale process defined by the equation

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t \sigma_s dW_s + \int_0^t \int \kappa\left(\delta\left(s,z\right)\right)\left(\mu - \nu\right)\left(ds,dz\right) + \int_0^t \int \kappa'\left(\delta\left(s,z\right)\right)\mu\left(ds,dz\right), \quad (1)$$

where $a = (a_t)_{t\geq 0}$ is a d-dimensional predictable locally bounded drift vector, $W = (W_t)_{t\geq 0}$ is d-dimensional Brownian motion and $\sigma = (\sigma_t)_{t\geq 0}$ is an adapted càdlàg $d \times d$ locally bounded process such that $\Sigma_t = \sigma_t \sigma_t'$ is the spot covariance matrix of X at time t. Whereas μ is a a d-dimensional Poisson random measure on $\mathbb{R}_+ \times E$, with $(E, \mathcal{E})$ an auxiliary measurable space, on the space $\left(\Omega^{(0)}, \mathcal{F}^{(0)}, \left(\mathcal{F}_t^{(0)}\right)_{t\geq 0}, P^{(0)}\right)$ and the predictable compensator (or intensity measure) of μ is $\nu\left(ds, dz\right) = ds \otimes \lambda\left(dz\right)$ for some given finite or σ-finite measure λ on $(E, \mathcal{E})$, δ is a d-dimensional predictable function on $\Omega^{(0)} \times \mathbb{R}_+ \times E$. Moreover, κ is a continuous truncation function on $\mathbb{R}^d$, that is a function from $\mathbb{R}^d$ into itself with compact support and $\kappa(x) = x$ on a neighbourhood of zero, and we set $\kappa'(x) = x - \kappa(x)$ to separate the martingale part of small jumps and the large jumps. Note that a, σ and δ should be such that the integrals in (1) make sense (see, e.g., Jacod and Shiryaev for a precise definition of the last two integrals).

In the special case where X is continuous, it has the form

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t \sigma_s dW_s. \quad (2)$$

Under (1), the quadratic (co)variation of X is given by

$$\begin{aligned} [X]_t &= \int_0^t \Sigma_s ds + \sum_{s\leq t} \left(\Delta X_s\right)\left(\Delta X_s\right)' \\ &\equiv \Gamma_t + JC_t, \end{aligned}$$

where $\Delta X_s = X_s - X_{s-}$, $X_{s-} = \lim_{t\to s,\, t<s} X_t$. Thus $[X]_t$ is the sum of Γ_t (the integrated covolatility) and JC_t (the sum of products of simultaneous jumps (called co-jumps)). For empirical applications, one may be concerned with the behavior of Γ_t and JC_t in isolation making interesting to decompose the two sources of covariability in the price process. In this paper, our parameter of interest is integrated covariance matrix Γ_t. Without loss of generality, we let $t = 1$ (which we think of as a given day), omit the index t and define $\Gamma \equiv \Gamma_1 = \int_0^1 \Sigma_s ds$.

The presence of market frictions such as price discreteness, rounding errors, bid-ask spreads, gradual response of prices to block trades, etc, prevent us from observing the efficient price process X. Instead, we observe a noisy price process $Y = \left(Y^{(1)}, \cdots, Y^{(d)}\right)'$, given by

$$Y_t = X_t + \epsilon_t,$$

where ϵ_t represents the noise term that collects all the market microstructure effects. These prices are observed irregularly and non-synchronously over the interval $[0,1]$. In particular, for all $k = 1, \ldots, d$, we observed the component process $\left(Y^{(k)}\right)$ at time points t_i^k for $i = 0, \ldots, n_k$, given by

$$Y^k_{t_i^k} = X^k_{t_i^k} + \epsilon^k_{t_i^k},$$

from which we compute n_k intraday returns defined as,

$$\Delta Y^k_{t_i^k} \equiv Y^k_{t_i^k} - Y^k_{t_{i-1}^k}, \; i = 1, \ldots, n_k, \tag{3}$$

with $0 = t_0^k < \ldots < t_{n_k}^k = 1$ being partitions of the interval $[0,1]$, which satisfies $\max_{1 \leq i \leq n_k} \left|t_i^k - t_{i-1}^k\right| \to 0$ as $n_k \to \infty$ for all $1 \leq k \leq d$.

In order to make both X and Y measurable with respect to the same kind of filtration, we define a new probability space $\left(\Omega, (\mathcal{F}_t)_{t \geq 0}, P\right)$, which accommodates both processes. To this end, we follow Jacod et al. (2009) and assume one has a second space $\left(\Omega^{(1)}, \left(\mathcal{F}_t^{(1)}\right)_{t \geq 0}, P^{(1)}\right)$, where $\Omega^{(1)}$ denotes $\mathbb{R}^{[0,1]}$ and $\mathcal{F}^{(1)}$ the product Borel-σ-field on $\Omega^{(1)}$. Next, for any $t \in [0,1]$, we define $Q_t\left(\omega^{(0)}, dy\right)$ to be the probability measure on $\mathbb{R}$, which corresponds to the transition from $X_t\left(\omega^{(0)}\right)$ to the observed process Y_t. In the case of i.i.d. noise, this transition kernel is rather simple, but it becomes more pronounced in a general framework. $P^1\left(\omega^{(0)}, d\omega^{(1)}\right)$ denotes the product measure $\otimes_{t \in [0,1]} Q_t\left(\omega^{(0)}, \cdot\right)$. The filtered probability space $\left(\Omega, (\mathcal{F}_t)_{t \in [0,1]}, P\right)$ on which the process Y lives is then defined with $\Omega = \Omega^{(0)} \times \Omega^{(1)}$, $\mathcal{F} = \mathcal{F}^{(0)} \times \mathcal{F}^{(1)}$, $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s^{(0)} \times \mathcal{F}_s^{(1)}$, and $P\left(d\omega^{(0)}, d\omega^{(1)}\right) = P^0\left(\omega^{(0)}\right) P^1\left(\omega^{(0)}, d\omega^{(1)}\right)$.

2.2 Statistics of interest

The statistics of interest in this paper can be written as smooth functions of $\widehat{\Gamma}^n \equiv \left(\widehat{\Gamma}^n_{kl}\right)_{1 \leq k,l \leq d}$ where $\widehat{\Gamma}^n$ is a consistent estimator of the integrated covariance matrix Γ, such that a central limit theorem holds. We have, as $n \to \infty$,

$$\tau_n\left(\widehat{\Gamma}^n - \Gamma\right) \to^{st} MN(0, V), \tag{4}$$

where n denotes the sample size, $\tau_n = n^{\delta_1}$ with $\delta_1 \in (0,1)$ is a known rate of convergence, $\to^{st} MN$ denotes stable convergence to a mixed Gaussian distribution (see Jacod and Shiryaev (2003, Ch. 8, Sect. 5c) for the definition and properties of stable convergence) and $V = \left(V_{kl,k'l'}\right)_{1 \leq k,k'l,'l' \leq d}$ is a $d \times d \times d \times d$ array, whose generic element $V_{kl,k'l'}$ corresponding to the asymptotic covariance between $\tau_n \widehat{\Gamma}^n_{kl}$ and $\tau_n \widehat{\Gamma}^n_{k'l'}$. In particular, we focus on the class of estimators of Γ which can be written as

$$\widehat{\Gamma}^n = \sum_{\alpha=1}^{J_n} \mathcal{Z}^n(\alpha) - \widetilde{b}^n,$$

or equivalently using the individual entries of $\widehat{\Gamma}^n$, $\mathcal{Z}^n(\alpha) \equiv (\mathcal{Z}^n_{kl}(\alpha))_{1\leq k,l\leq d}$ and $\widetilde{b}^n \equiv \left(\widetilde{b}^n_{kl}\right)_{1\leq k,l\leq d}$, we have

$$\widehat{\Gamma}^n_{kl} = \sum_{\alpha=1}^{J_n} \mathcal{Z}^n_{kl}(\alpha) - \widetilde{b}^n_{kl}, \tag{5}$$

where $J_n = \left\lfloor \frac{n}{b_n} \right\rfloor$, with $\lfloor\cdot\rfloor$ the integer part function and b_n is a sequence of integers such that

$$b_n \propto n^{\delta_2}, \tag{6}$$

where $\delta_2 \in (0,1)$. $\widetilde{b}^n$ can be interpreted as a bias-corrected estimator, which does not contribute to the asymptotic variance of the statistic of interest. This means that $\tau_n\widehat{\Gamma}^n$ and $\tau_n \sum_{\alpha=1}^{J_n} \mathcal{Z}^n(\alpha)$ have the same asymptotic variance. Usually, the following results also holds, as $n \to \infty$,

$$\tau_n\left(\widetilde{b}^n - \widetilde{b}\right) \to^P 0 \text{ and } \tau_n\left(\sum_{\alpha=1}^{J_n} \mathcal{Z}^n(\alpha) - \Gamma - \widetilde{b}\right) \to^{st} MN(0,V), \tag{7}$$

where $\widetilde{b} = p\lim_{n\to\infty} \widetilde{b}^n$. In the simple case where no bias-correction is needed (i.e. $\widetilde{b}^n_{kl} = 0$), for each $\alpha = 1,\ldots,J_n$, the statistic $\mathcal{Z}^n_{kl}(\alpha)$ is essentially the same quantity as $\widehat{\Gamma}^n_{kl}$, with the difference that it is computed only over time points t^k_i from the smaller interval $B_n(\alpha) = \left[\frac{(\alpha-1)b_n}{n}, \frac{\alpha b_n}{n}\right)$, whereas $\widehat{\Gamma}^n_{kl}$ is computed over the whole interval $[0,1]$. Thus in this case, $\mathcal{Z}^n(\alpha)$ is a miniature realized measure, which can help to get information about $\int_{\frac{(\alpha-1)b_n}{n}}^{\frac{\alpha b_n}{n}} \Sigma_s ds$. Similarly, when $\widetilde{b}^n_{kl} \neq 0$, $\mathcal{Z}^n_{kl}(\alpha)$ is the analogue of $\sum_{\alpha=1}^{J_n} \mathcal{Z}^n_{kl}(\alpha)$, but computed over time points t^k_i from $B_n(\alpha)$. The main advantage of writing $\widehat{\Gamma}^n$ as in (5) is that it provides a unified bootstrap theory to dealing with a broad class of estimators of Γ. As we show in the next section, as long as this is possible and under some other regularity conditions, the wild blocks of blocks bootstrap method studied by Hounyo et al. (2013) applies now to the statistics $\mathcal{Z}^n_{kl}(\alpha)$ is first-order valid. Examples of potential estimators of integrated covolatility that can be written as (5) are listed in the introduction.

The exact expression of the conditional asymptotic variance V may be rather complicated and can involve substantially more complex quantities than the original parameter of interest Γ. One of our contributions is to justify the use of the bootstrap to estimate V. Let $V^n = \left(V^n_{kl,k'l'}\right)_{1\leq k,k'l,'l'\leq d}$ denote a consistent estimator of V, then together with the CLT result (4) we have that

$$\left(\tilde{V}^n\right)^{-1/2} \tau_n\left(vec\left(\widehat{\Gamma}^n\right) - vec(\Gamma)\right) \to^{st} N(0, I_{d^2}),$$

where vec is the vectorization operator that stacks columns of a matrix below one another, I_{d^2} is a d^2-dimensional identity matrix and $\tilde{V}^n = \left(\tilde{V}^n_{kl}\right)_{1\leq k,l\leq d^2}$ is a $d^2\times d^2$ matrix, whose generic element $\tilde{V}^n_{kl}$ is given by

$$\tilde{V}^n_{kl} = V^n_{k-d\lfloor(k-1)/d\rfloor,\lfloor(k-1)/d\rfloor+1,l-d\lfloor(l-1)/d\rfloor,\lfloor(l-1)/d\rfloor+1}, \quad 1\leq k,l\leq d^2.$$

This result can be applied in order to compute confidence region for some functionals of Γ that are important in practice, such as covariance, regression coefficient and correlation estimates. In particular, the asymptotic variance estimates for standard measures of dependence between two asset returns such as the realized covariance, the realized regression and the realized correlation coefficients are obtained by the delta method, whose finite sample properties are often poor. This motivates the bootstrap as an alternative method of inference in these contexts. The next section details how the bootstrap methodology can be used for these purposes in our general setup, which accommodates the potential presence of jumps, microstructure noise, irregularly spaced and non-synchronous trading.

3 The wild blocks of blocks bootstrap

3.1 Main results

Our aim in this section is to extend the wild blocks of blocks bootstrap method proposed by Hounyo et al. (2013) to the multivariate context allowing for the presence of jumps, noise, irregularly spaced and non-synchronous data. In particular, we propose a bootstrap method that can be used to consistently estimate the distribution of $\tau_n \left(h \left(vec \left(\widehat{\Gamma}^n \right) \right) - h \left(vec \left(\Gamma \right) \right) \right)$, where $h : \mathbb{R}^{d^2} \to \mathbb{R}$ denotes a real valued function with continuous derivatives. This justifies for instance, the construction of bootstrap percentile (bootstrap unstudentized statistic) confidence intervals for covariance, regression and correlation. The bootstrap percentile intervals are easier to implement as they do not require an explicit estimator of the variance which is hard to compute in our context.

Gonçalves and Meddahi (2009) proposed the wild bootstrap method for the realized volatility in the absence of market microstructure noise and Gonçalves et al. (2014) extend their work by allowing for the latter. In particular, they focus on the pre-averaged realized volatitity estimator proposed by Podolskij and Vetter (2009). In their ideal setting, pre-averaged returns are non-overlapping, implying that they are asymptotically uncorrelated as $n \to \infty$, but possibly heteroskedastic due to stochastic volatility, thus motivating the use of a wild bootstrap method.

When pre-averaged returns are overlapping, they are strongly dependent. This implies that the wild bootstrap is no longer valid when applied to pre-averaged returns. Instead, a block bootstrap method applied to the pre-averaged returns would seem appropriate. This amounts to a "blocks of blocks" bootstrap, as proposed by Politis and Romano (1992) and further studied by Bühlmann and Künsch (1995) (see also Künsch (1989)). Nevertheless, as Hounyo et al. (2013) show in the univariate case, such a bootstrap scheme is only consistent when volatility is constant. They argue that squared pre-averaged returns are heterogenously distributed (in particular, their mean and variance are time-varying) and this creates a bias term in the blocks of blocks bootstrap variance estimator when volatility is stochastic. To avoid this problem, Hounyo et al. (2013) propose to combine the wild bootstrap with the blocks of blocks bootstrap. Here, we generalize their bootstrap method to the class of estimators of integrated covolatility, which can be written as in (5).

The general multivariate wild blocks of blocks bootstrap pseudo-data is given by

$$\mathcal{Z}_{kl}^{n*}(\alpha) = \begin{cases} \mathcal{Z}_{kl}^{n}(\alpha+1) + \left(\mathcal{Z}_{kl}^{n}(\alpha) - \mathcal{Z}_{kl}^{n}(\alpha+1) \right) \eta_\alpha, & \text{if } \alpha = 1, \ldots, J_n - 1 \\ \mathcal{Z}_{kl}^{n}(\alpha), & \text{if } \alpha = J_n, \end{cases} \tag{8}$$

where the external random variable η_α is an i.i.d. random variable independent of the data and whose moments are given by $\mu_q^* \equiv E^*\left(|\eta_\alpha|^q\right)$. As usual in the bootstrap literature, P^* (E^* and Var^*) denotes the probability measure (expected value and variance) induced by the bootstrap resampling, conditional on a realization of the original time series. In addition, for a sequence of bootstrap statistics Z_n^*, we write $Z_n^* = o_{P^*}(1)$ in probability, or $Z_n^* \to^{P^*} 0$, as $n \to \infty$, in probability, if for any $\varepsilon > 0$, $\delta > 0$, $\lim_{n\to\infty} P\left[P^*\left(|Z_n^*| > \delta\right) > \varepsilon\right] = 0$. Similarly, we write $Z_n^* = O_{P^*}(1)$ as $n \to \infty$, in probability if for all $\varepsilon > 0$ there exists a $M_\varepsilon < \infty$ such that $\lim_{n\to\infty} P\left[P^*\left(|Z_n^*| > M_\varepsilon\right) > \varepsilon\right] = 0$. Finally, we write $Z_n^* \to^{d_*} Z$ as $n \to \infty$, in probability, if conditional on the sample, Z_n^* weakly converges to Z under P^*, for all samples contained in a set with probability P converging to one.

The bootstrap analogue of (5) is defined by

$$\widehat{\Gamma}_{kl}^{n*} = \sum_{\alpha=1}^{J_n} \mathcal{Z}_{kl}^{n*}(\alpha), \tag{9}$$

and $\widehat{\Gamma}^{n*} \equiv \left(\widehat{\Gamma}_{kl}^{n*}\right)_{1\le k,l\le d}$. Note that although $\widehat{\Gamma}_{kl}^{n}$ contains a bias correction term (when $\widetilde{b}_{kl}^{n} \neq 0$), we do not consider bias correction in the bootstrap world, even in the case where $\widetilde{b}_{kl}^{n} \neq 0$. This is because the bias correction term $\widetilde{b}^{n}$ by definition does not affect the asymptotic variance of $\widehat{\Gamma}^{n}$. As long as the bootstrap method is able to consistently estimate this variance, no bias correction is needed in the bootstrap world. Since we can always center the bootstrap statistic $\widehat{\Gamma}_{kl}^{n*}$ at its own theoretical mean $E^*\left(\widehat{\Gamma}_{kl}^{n*}\right)$ without affecting the bootstrap variance. For example, the bias correction term $\widetilde{b}_{kl}^{n}$ for pre-averaged realized covolatility estimator (which we will introduce in Section 4.3) is crucially dependent of the noise assumption whereas the bootstrap estimator is robust regardless.

Our bootstrap method can be seen as a generalization of the wild blocks of blocks bootstrap method of Hounyo et al. (2013) to the general context described by (5). In particular, here we resample the statistics $\mathcal{Z}_{kl}^{n}(\alpha)$, which may be a block sum of functions of $\left(\Delta Y_{t_i^k}^k, \Delta Y_{t_j^l}^l\right)$ (see Section 4 for examples of statistics $\mathcal{Z}_{kl}^{n}(\alpha)$). As in the univariate case, to preserve the weak dependence, we divide the interval $[0,1]$ into J_n non-overlapping sub-interval of length $\frac{b_n}{n}$ and generate the bootstrap observations within a given sub-interval $B_n(\alpha) = \left[\frac{(\alpha-1)b_n}{n}, \frac{\alpha b_n}{n}\right)$ using the same external random variable η_α. This preserves the dependence within each sub-interval. Also, as mentioned in Hounyo et al. (2013), we show that by centering around $\mathcal{Z}_{kl}^{n}(\alpha+1)$ instead of $J_n^{-1}\sum_{\alpha=1}^{J_n} \mathcal{Z}_{kl}^{n}(\alpha)$ (as in the plain wild boostrap method of Wu (1986) and Liu (1988)) yields an asymptotically valid bootstrap method for $\widehat{\Gamma}_{kl}^{n}$. This is not necessary the case for the naive application of the original wild bootstrap of Liu (1988), which generates bootstrap observations $\mathcal{Z}_{kl}^{n*}(\alpha)$ as

$$\mathcal{Z}_{kl}^{n*}(\alpha) = J_n^{-1}\sum_{\alpha=1}^{J_n} \mathcal{Z}_{kl}^{n}(\alpha) - \left(\mathcal{Z}_{kl}^{n}(\alpha) - J_n^{-1}\sum_{\alpha=1}^{J_n} \mathcal{Z}_{kl}^{n}(\alpha)\right)\eta_\alpha,\ \alpha = 1,\ldots,J_n, \tag{10}$$

where η_α is i.i.d. $(0,1)$. As we show in this paper, the new wild blocks of blocks bootstrap preserves the mean heterogeneity property of the statistics $\mathcal{Z}_{kl}^{n}(\alpha)$ even when volatility is stochastic, in our multivariate setting that allows for jumps, noise, irregularly spaced and non-synchronous data. The following result gives the bootstrap moments of $\left(\widehat{\Gamma}^{n*}_{kl}, \widehat{\Gamma}^{n*}_{k'l'}\right)'$. In order to state our results, let $V^{n*}_{kl,k'l'} \equiv Cov^*\left(\tau_n \widehat{\Gamma}^{n*}_{kl}, \tau_n \widehat{\Gamma}^{n*}_{k'l'}\right)$ denote the wild blocks of blocks bootstrap covariance between $\tau_n \widehat{\Gamma}^{n*}_{kl}$ and $\tau_n \widehat{\Gamma}^{n*}_{k'l'}$ based on an external random variables $\eta_\alpha \sim$ i.i.d. with mean $E^*(\eta_\alpha)$ and variance $Var^*(\eta_\alpha)$, and V^{n*} a $d \times d \times d \times d$ array, whose generic element is $V^{n*}_{kl,k'l'}$ such that (8) holds.

Lemma 3.1. *Given (8) and (9), we have*

a)

$$
\begin{aligned}
E^*\left(\widehat{\Gamma}^{n*}_{kl}\right) &= \sum_{\alpha=1}^{J_n-1} \mathcal{Z}^n_{kl}(\alpha+1) + \mathcal{Z}^n_{kl}(J_n) \\
&\quad + \sum_{\alpha=1}^{J_n-1} \left(\mathcal{Z}^n_{kl}(\alpha) - \mathcal{Z}^n_{kl}(\alpha+1)\right) E^*(\eta_\alpha),
\end{aligned}
$$

in particular, if $E^*(\eta_\alpha) = 1$, we have that $E^*\left(\mathcal{Z}^{n*}_{kl}\right) = \sum_{\alpha=1}^{J_n} \mathcal{Z}^n_{kl}(\alpha) = \widehat{\Gamma}^n_{kl} + \widetilde{b}^n_{kl}$.

b)

$$
V^{n*}_{kl,k'l'} = 2Var^*(\eta) \underbrace{\frac{\tau_n^2}{2} \sum_{\alpha=1}^{J_n-1} \left(\mathcal{Z}^n_{kl}(\alpha) - \mathcal{Z}^n_{kl}(\alpha+1)\right)\left(\mathcal{Z}^n_{k'l'}(\alpha) - \mathcal{Z}^n_{k'l'}(\alpha+1)\right).}_{\equiv V^n_{kl,k'l'}}
$$

Part a) of Lemma 3.1 states that in the case where $\widetilde{b}^n_{kl} = 0$, if we let $E^*(\eta_\alpha) = 1$ then $\widehat{\Gamma}^{n*}_{kl}$ is an unbiased estimator of the integrated covariance Γ_{kl}. Part b) shows that the bootstrap covariance of $\tau_n \widehat{\Gamma}^{n*}_{kl}$ and $\tau_n \widehat{\Gamma}^{n*}_{k'l'}$ depends on the variance of the external random variable η, as well as the statistic $V^n_{kl,k'l'}$ which is based on a "local estimation" of the covariance of $\mathcal{Z}^n_{kl}$ and $\mathcal{Z}^n_{k'l'}$. It follows then that a sufficient condition for the bootstrap to provide a consistent estimator of the conditional asymptotic variance V is that $Var^*(\eta) = \frac{1}{2}$, and the sequence of $\mathcal{Z}^n_{kl}(\alpha)$, $\alpha = 1, \ldots, J_n$, is such that $V^n_{kl,k'l'} \to^P V_{kl,k'l'}$, as $n \to \infty$. Next, we provide a set of high level conditions that allow us to derive the first-order asymptotic validity of the bootstrap method. Note that this is a high level condition that does not depend on specifying whether the process X is a continous martingale or observed with error or not. However, for some estimators, it might hold only with some restrictions.

Condition A

A.1. The choice of the external random variable η is such that $Var^*(\eta) = \frac{1}{2}$, and as $n \to \infty$

$$
V^n_{kl,k'l'} = \frac{\tau_n^2}{2} \sum_{\alpha=1}^{J_n-1} \left(\mathcal{Z}^n_{kl}(\alpha) - \mathcal{Z}^n_{kl}(\alpha+1)\right)\left(\mathcal{Z}^n_{k'l'}(\alpha) - \mathcal{Z}^n_{k'l'}(\alpha+1)\right) \to^P V_{kl,k'l'}.
$$

A.2. $\left(\frac{n}{b_n}\right)^{1+\varepsilon} \sum_{\alpha=1}^{J_n} \left|\mathcal{Z}^n_{kl}(\alpha)\right|^{2+\varepsilon} = O_P(1)$, for some $\varepsilon > 0$, as $n \to \infty$.

A.3. For the same $\varepsilon > 0$, as in A.2., it holds that, as $n \to \infty$

$$\frac{b_n}{n} = o\left(\tau_n^{-\frac{2+\varepsilon}{1+\varepsilon}}\right).$$

A.1. requires that the choice of the external random variable η as well as the statistic $\mathcal{Z}_{kl}^n(\alpha)$ are such that the bootstrap variance V^{n*} yields a consistent estimator of the asymptotic variance V. This condition is very general and do not impose any structure on $\mathcal{Z}_{kl}^n(\alpha)$. We could replace A.1. by a condition on the sequence of $\mathcal{Z}_{kl}^n(\alpha)$, $\alpha = 1, \ldots, J_n$ such that they are conditionally independent, a moment condition on $\mathcal{Z}_{kl}^n(\alpha)$ ($E\left(|\mathcal{Z}_{kl}^n(\alpha)|^{2+\varepsilon} |\mathcal{F}_{\frac{(\alpha-1)b_n}{n}}^n\right) < \infty$, for some $\varepsilon > 0$) and more importantly the following homogeneity condition on the means

$$M_n \equiv \tau_n^2 \sum_{\alpha=1}^{J_n-1} \left(\mu_{kl}^n(\alpha) - \mu_{kl}^n(\alpha+1)\right)\left(\mu_{k'l'}^n(\alpha) - \mu_{k'l'}^n(\alpha+1)\right) \to^P 0, \tag{11}$$

where $\mu_{kl}^n(\alpha) = E\left(\mathcal{Z}_{kl}^n(\alpha) |\mathcal{F}_{\frac{(\alpha-1)b_n}{n}}^n\right)$. This mean homogeneity condition is suitable for financial high frequency data, in particular for estimators of integrated covolatility. This is not necessary the case of a naive application of the original wild bootstrap of Liu (1988), which will require in our context to verify the following condition

$$M_n^L \equiv \tau_n^2 \sum_{\alpha=1}^{J_n}\left(\mu_{kl}^n(\alpha) - J_n^{-1}\sum_{\alpha=1}^{J_n}\mu_{kl}^n(\alpha)\right)\left(\mu_{k'l'}^n(\alpha) - J_n^{-1}\sum_{\alpha=1}^{J_n}\mu_{k'l'}^n(\alpha)\right) \to^P 0. \tag{12}$$

In the context of time series, see e.g. Liu (1988) and Gonçalves and White (2002) (cf. Assumption 2.2) for similar restriction of the heterogeneity on the means. It is easy to see that in our setting, the homogeneity condition defined in (12) does not hold even in the very simple univariate stochastic volatility model without noise, where we also rule out drift, leverage effect, jumps and we suppose that prices are observed at equidistant date. In particular, in this case (for simplicity) we can let $J_n = n$ and consider as statistic of interest the realized volatility estimator defined by $\widehat{\Gamma}^n = \sum_{\alpha=1}^{J_n} \mathcal{Z}^n(\alpha) = \sum_{\alpha=1}^{n}\left(\Delta Y_{\frac{\alpha}{n}}\right)^2$. We can show that

$$\begin{aligned} M_n^L &= n\sum_{\alpha=1}^{n}\left(\int_{\frac{\alpha-1}{n}}^{\frac{\alpha}{n}}\sigma_s^2 ds - n^{-1}\sum_{\alpha=1}^{n}\int_{\frac{\alpha-1}{n}}^{\frac{\alpha}{n}}\sigma_s^2 ds\right)^2 = n\sum_{\alpha=1}^{n}\left(\int_{\frac{\alpha-1}{n}}^{\frac{\alpha}{n}}\sigma_s^2 ds\right)^2 - \left(\int_0^1 \sigma_s^2 ds\right)^2 \\ &\to^P \int_0^1 \sigma_s^4 ds - \left(\int_0^1 \sigma_s^2 ds\right)^2, \end{aligned}$$

which is not equal to zero (one exception is when the volatility is constant). Whereas for the new bootstrap method, the mean homogeneity condition requires that

$$M_n = n\sum_{i=1}^{n-1}\left(\int_{\frac{i-1}{n}}^{\frac{i}{n}}\sigma_s^2 ds - \int_{\frac{i}{n}}^{\frac{i+1}{n}}\sigma_s^2 ds\right)^2 \to^P 0.$$

In contrast to Liu's condition, we can show that under some regularity conditions (Riemann integrability of σ), we always have $M_n \to^P 0$, even if the volatility is stochastic. This explains the new centering suggested in (8). See Section 4, for more general stochastic volatility model.

Condition A.2. and A.3. are conditions used to show that a central limit theorem holds for $\tau_n\left(\widehat{\Gamma}^{n*} - E^*\left(\widehat{\Gamma}^{n*}\right)\right)$ in the bootstrap world. Part A.2. is a Lyapounov type condition that drives the asymptotic normality of $\sum_{\alpha=1}^{J_n} \mathcal{Z}_{kl}^n(\alpha)$, whereas part A.3. restricts the choice of the block size b_n, such that the CLT holds. Note that, when the sequence of $\mathcal{Z}_{kl}^n(\alpha)$, $\alpha = 1, \ldots, J_n$ can be shown to be conditionally independent by letting $b_n = 1$, in this case we will simply use $b_n = 1$, i.e. $J_n = n$.

Under this high level condition, we can prove the following results. Theorem 3.1 is the main result of our paper, and its proof is postponed to the Appendix.

Theorem 3.1. *Under Condition A, as $n \to \infty$*

a)

$$V_{kl,k'l'}^{n*} \to^P V_{kl,k'l'}, \text{ so that } V^{n*} \to^P V.$$

b) *Let $S^n = \tau_n\left(vec\left(\widehat{\Gamma}^n\right) - vec\left(\int_0^1 \Sigma_s ds\right)\right)$ and $S^{n*} = \tau_n\left(vec\left(\widehat{\Gamma}^{n*}\right) - E^*\left(vec\left(\widehat{\Gamma}^{n*}\right)\right)\right)$, if for some $\varepsilon > 0$, $E^*\left|\eta_\alpha\right|^{2+\varepsilon} \leq \Delta < \infty$, then*

$$\sup_{x \in \mathbb{R}^{d^2}} \left|P^*\left(S^{n*} \leq x\right) - P\left(S^n \leq x\right)\right| \to^P 0.$$

Part a) of Theorem 3.1 shows that the bootstrap variance estimator is consistent for the asymptotic variance V according to Condition A. Part b) provides a theoretical justification for using the wild blocks of blocks bootstrap to consistently estimate the entire distribution of $\widehat{\Gamma}^n$.

The statistics of interest in this paper can be written as smooth functions of $\widehat{\Gamma}^n$. The following theorem proves that the wild blocks of blocks bootstrap is first-order asymptotically valid when applied to smooth functions of the vectorized of $\widehat{\Gamma}^n$. Let $h : \mathbb{R}^{d^2} \to \mathbb{R}$ denote a real valued function with continuous derivatives, and let the $d \times 1$ vector-valued function ∇h denote its gradient. We suppose that $\nabla h\left(vec\left(\Gamma\right)\right)$ is non-zero for any sample path of Γ. The statistic of interest is defined as

$$S_h^n = \tau_n\left(h\left(vec\left(\widehat{\Gamma}^n\right)\right) - h\left(vec\left(\Gamma\right)\right)\right), \tag{13}$$

the wild blocks of blocks bootstrap version of S_h^n is

$$S_h^{n*} = \tau_n\left(h\left(vec\left(\widehat{\Gamma}^{n*}\right)\right) - h\left(E^*\left(vec\left(\widehat{\Gamma}^{n*}\right)\right)\right)\right). \tag{14}$$

Let $V_h^{n*} \equiv \nabla' h\left(E^*\left(vec\left(\widehat{\Gamma}^{n*}\right)\right)\right) \tilde{V}^{n*} \nabla h\left(E^*\left(vec\left(\widehat{\Gamma}^{n*}\right)\right)\right)$ denote the wild blocks of blocks bootstrap variance of $\tau_n h\left(vec\left(\widehat{\Gamma}^{n*}\right)\right)$, where $\tilde{V}^{n*} = \left(\tilde{V}_{kl}^{n*}\right)_{1 \leq k,l \leq d^2}$ is a $d^2 \times d^2$ matrix, whose generic element $\tilde{V}_{kl}^{n*}$ is given by

$$\tilde{V}_{kl}^{n*} = V_{k-d\lfloor(k-1)/d\rfloor, \lfloor(k-1)/d\rfloor+1, l-d\lfloor(l-1)/d\rfloor, \lfloor(l-1)/d\rfloor+1}^{n*},$$

with $1 \leq k, l \leq d^2$. The next theorem establishes the first-order asymptotic validity of the bootstrap for some smooth functions of the vectorized of $\widehat{\Gamma}^n$.

Theorem 3.2. *Under the same conditions of Theorem 3.1, as* $n \to \infty$,

a) $V_h^{n*} \to^P V_h \equiv \lim_{n\to\infty} Var\left(\tau_n h\left(vec\left(\widehat{\Gamma}^n\right)\right)\right)$.

b) *If for some* $\varepsilon > 0$, $E^* |\eta_\alpha|^{2+\varepsilon} \le \Delta < \infty$, *then*

$$\sup_{x\in\mathbb{R}} |P^*(S_h^{n*} \le x) - P(S_h^n \le x)| \to^P 0.$$

3.2 The bootstrap for realized covariation measures

In this section we show how we can apply Theorem 3.2 in order to prove first-order asymptotic validity of the bootstrap for some functionals of the matrix $\widehat{\Gamma}_{kl}^n$ that are important in practice. The focus will be on realized covariance, realized regression and realized correlation coefficients. For the kth and lth asset, these quantities are given by

$$\widehat{\Gamma}_{kl}^n,\ \hat{\beta}_{lk}^n = \frac{\widehat{\Gamma}_{kl}^n}{\widehat{\Gamma}_{kk}^n} \text{ and } \hat{\rho}_{lk}^n = \frac{\widehat{\Gamma}_{kl}^n}{\sqrt{\widehat{\Gamma}_{kk}^n \widehat{\Gamma}_{ll}^n}},$$

which under certain conditions consistently estimate

$$\Gamma_{kl} = \int_0^1 \Sigma_{kl}(s)\, ds,\ \beta_{lk} = \frac{\Gamma_{kl}}{\Gamma_{kk}} \text{ and } \rho_{lk} = \frac{\Gamma_{kl}}{\sqrt{\Gamma_{kk}\Gamma_{ll}}},$$

respectively. For each of these measures, the non-studentized statistics analogue of (13) are given by

$$S_{\Gamma_{kl}}^n \equiv \tau_n\left(\widehat{\Gamma}_{kl}^n - \Gamma_{kl}\right),\ S_{\beta_{lk}}^n \equiv \tau_n(\hat{\beta}_{lk}^n - \beta_{lk}), \text{ and } S_{\rho_{lk}}^n \equiv \tau_n\left(\hat{\rho}_{lk}^n - \rho_{lk}\right),$$

respectively. Similarly, the corresponding bootstrap percentile statistics (analogue of (14) for $\widehat{\Gamma}_{kl}$, $\hat{\beta}_{lk}^n$ and $\hat{\rho}_{lk}^n$ are given by

$$S_{\Gamma_{kl}}^{n*} \equiv \tau_n\left(\widehat{\Gamma}_{kl}^{n*} - E^*\left(\widehat{\Gamma}_{kl}^{n*}\right)\right),\ S_{\beta_{lk}}^{n*} \equiv \tau_n(\hat{\beta}_{lk}^{n*} - \frac{E^*\left(\widehat{\Gamma}_{kl}^*\right)}{E^*\left(\widehat{\Gamma}_{kk}^*\right)}), \text{ and } S_{\rho_{lk}}^{n*} \equiv \tau_n\left(\hat{\rho}_{lk}^{n*} - \frac{E^*\left(\widehat{\Gamma}_{kl}^*\right)}{\sqrt{E^*\left(\widehat{\Gamma}_{kk}^*\right)}\sqrt{E^*\left(\widehat{\Gamma}_{ll}^*\right)}}\right),$$

respectively, where $\widehat{\Gamma}_{kl}^{n*}$ is defined in (9), $\hat{\beta}_{lk}^{n*} = \frac{\widehat{\Gamma}_{kl}^*}{\widehat{\Gamma}_{kk}^*}$ and $\hat{\rho}_{lk}^{n*} = \frac{\widehat{\Gamma}_{kl}^*}{\sqrt{\widehat{\Gamma}_k^*}\sqrt{\widehat{\Gamma}_l^*}}$. According to part b) of Theorem 3.2, we can use the wild blocks of blocks bootstrap variance of $S_{\Gamma_{kl}}^{n*}$, $S_{\beta_{lk}}^{n*}$ and $S_{\rho_{lk}}^{n*}$ to consistently estimate the variance of $S_{\Gamma_{kl}}^n$, $S_{\beta_{lk}}^n$ and $S_{\rho_{lk}}^n$, respectively. In particular, for the realized covariance measure, a consistent estimator of $V_{\Gamma_{kl}} = \lim_{n\to\infty} Var\left(\tau_n \widehat{\Gamma}_{kl}^n\right)$ based on the bootstrap method is given by

$$V_{\Gamma_{kl}}^{n*} = V_{kl,kl}^{n*} = \frac{\tau_n^2}{2}\sum_{\alpha=1}^{J_n-1}\left(\mathcal{Z}_{kl}^n(\alpha) - \mathcal{Z}_{kl}^n(\alpha+1)\right)^2. \qquad (15)$$

Similarly, for the realized regression,

$$V_{\beta_{lk}}^{n*} = \left(\widehat{\Gamma}_{kk}^n\right)^{-2}\hat{g}_{\beta_{lk}} \to^P V_{\beta_{lk}} \equiv \lim_{n\to\infty} Var\left(\tau_n \hat{\beta}_{lk}\right), \qquad (16)$$

where $\hat{g}_{\beta_{lk}} = \begin{pmatrix} 1, & -\frac{E^*\left(\widehat{\Gamma}^*_{kl}\right)}{E^*\left(\widehat{\Gamma}^*_{kk}\right)} \end{pmatrix} B^{n*}_{\beta_{lk}} \begin{pmatrix} 1, & -\frac{E^*\left(\widehat{\Gamma}^*_{kl}\right)}{E^*\left(\widehat{\Gamma}^*_{kk}\right)} \end{pmatrix}'$ with $B^{n*}_{\beta_{lk}} = \begin{pmatrix} V^{n*}_{kl,kl} & V^{n*}_{kl,kk} \\ \bullet & V^{n*}_{kk,kk} \end{pmatrix}$.

For the realized correlation, the bootstrap estimator of $V_{\rho_{lk}} = \lim_{n\to\infty} Var\left(\tau_n \hat{\rho}^n_{lk}\right)$ is given by

$$V^{n*}_{\rho_{lk}} = \left(\widehat{\Gamma}^n_{kk}\widehat{\Gamma}^n_{ll}\right)^{-1} \hat{g}_{\rho_{lk}}, \quad (17)$$

where $\hat{g}_{\rho_{lk}}$ is defined by $\hat{g}_{\rho_{lk}} = \begin{pmatrix} -\frac{1}{2}\frac{E^*\left(\widehat{\Gamma}^*_{kl}\right)}{E^*\left(\widehat{\Gamma}^*_{kk}\right)}, & 1, & -\frac{1}{2}\frac{E^*\left(\widehat{\Gamma}^*_{kl}\right)}{E^*\left(\widehat{\Gamma}^*_{ll}\right)} \end{pmatrix} B^{n*}_{\rho_{lk}} \begin{pmatrix} -\frac{1}{2}\frac{E^*\left(\widehat{\Gamma}^*_{kl}\right)}{E^*\left(\widehat{\Gamma}^*_{kk}\right)}, & 1, & -\frac{1}{2}\frac{E^*\left(\widehat{\Gamma}^*_{kl}\right)}{E^*\left(\widehat{\Gamma}^*_{ll}\right)} \end{pmatrix}'$ with $B^{n*}_{\rho_{lk}} = \begin{pmatrix} V^{n*}_{kk,kk} & V^{n*}_{kk,kl} & V^{n*}_{kk,ll} \\ \bullet & V^{n*}_{kl,kl} & V^{n*}_{kl,ll} \\ \bullet & \bullet & V^{n*}_{ll,ll} \end{pmatrix}$.

Note that all the required terms are easy to compute (see Lemma 3.1), so it is rather simple to implement the bootstrap variance estimator of the variance of $V_{\Gamma_{kl}}$, $V_{\beta_{lk}}$ and $V_{\rho_{lk}}$.

4 Illustration of the bootstrap scheme

The general results presented so far for a multivariate diffusion model with a potential presence of jumps, noise, irregularly spaced and non-synchronous data are stated quite compactly. Hence, it is helpful to focus on some particular cases in order to enhance intuition. In this section, we provide a list of possible multivariate noisy semimartingale models, showing in details how our bootstrap scheme can be applied. We then verify the high level Condition A for various estimators of integrated covolatility. First, we look at a benchmark multivariate model where no market microstructure noise is present and prices are observed synchronously at equidistant time stamps. Secondly, we show how those results change when the observed data are non-synchronous. Thirdly, we discuss the case of multivariate model with noisy prices observed synchronously at equidistant time points. Fourthly, we deal with asynchronicity in noisy irregularly spaced diffusion model. Lastly, we study the case of presence of jumps, but we rule out asynchronicity and microstructure noise. In order to discuss these results, let us first introduce the assumptions on the sampling scheme. The assumptions made here are specific for the pre-averaging estimator, and others may be considered when using a different estimator. We follow Christensen et al. (2013) and assume that the observation times $t^k_i, i = 0, \ldots, n_k$, $k = 1, \ldots, d$ satisfy the following conditions:

Assumption 1 - Sampling scheme

(a) (Time transformation) t^k_i's are transformations of an equidistant grid, i.e. there exist strictly monotonic (deterministic) functions $f_k : [0,1] \to [0,1]$ in $C^1([0,1])$ with non-zero right and left derivative in 0 and 1, respectively, and with $f_k(0) = 0$, $f_k(1) = 1$ such that

$$t^k_i = f^{-1}_k(i/n_k), \quad i = 0, ..., n_k,\ k = 1, ..., d.$$

(b) (Boundedness of f_k') There exists a natural number $M > 0$ such that

$$M^{-1} < \sup_{x \in [0,1]} \left| f_k'(x) \right| < M, \;\; k = 1, ..., d.$$

(c) (Comparable number of observations) Set $n = \sum_{k=1}^{d} n_k$. It holds that

$$\frac{n_k}{n} \to m_k \in (0, 1], \qquad k = 1, ..., d.$$

(d) (Joint grid points) The grids $\left(t_i^k\right), \left(t_j^l\right)$ $(1 \leq k, l \leq d)$ have n_{kl} common points which are denoted by $\left(t_p^{kl}\right)_{1 \leq p \leq n_{kl}}$. They have the representation $t_p^{kl} = f_{kl}^{-1}\left(p/n_{kl}\right) \to m_{kl} \in [0, 1]$, where the functions f_{kl} satisfy the same assumptions as f_k in (a) and (b).

Assumption 1 amounts to Assumption T in Christensen et al. (2013). As they explain, condition (a) makes the explicit computation of the asymptotic covariance matrix of the pre-averaged Hayashi-Yoshida estimator (which we will introduce in Section 4.4) possible. Condition (c) implies that the observation numbers n_k have the same order. Condition (b) means that the points of the lth grid do not lie dense between any two successive points of the kth grid, i.e. the number of points t_j^l that lie in the interval $[t_{i-1}^k, t_i^k]$ is uniformly bounded by a constant for all $1 \leq k, l \leq d$. When these last two conditions (similar number of observations and uniform boundedness of the number of points t_j^l that belong to $[t_{i-1}^k, t_i^k]$) are fulfilled we say that the sampling schemes are comparable. See for instance Lemma 6.1 of Christensen et al. (2013) where conditions (b) and (c) imply that the amount of time points t_i^k contained in all sub-interval $[a, b]$ of $[0, 1]$ is of the same order as in the equidistant case for all k. Finally, condition (d) means that the number of common points can be negligible compared to n (if $m_{kl} = 0$) or it can be of order n (if $m_{kl} > 0$).

We assume that ϵ_t is m-dependent in tick time and that ϵ_t is independent of X_t. Assumption 2 below collects these assumptions.

Assumption 2 - Noise component.

(a) The noise component ϵ_t is m-dependent in tick time, which means that for $t_i^k \leq t_j^l$ the random variables $\epsilon_{t_i^k}^k$ and $\epsilon_{t_j^l}^l$ are independent if $\left\| t_i^k - t_j^l \right\| > m$ with

$$\left\| t_i^k - t_j^l \right\| = \min\left(j - \max\left\{ z | \; t_z^l \leq t_i^k \right\}, \min\left\{ z | \; t_z^k \geq t_j^l \right\} - i \right)$$

and similarly for $t_j^l < t_i^k$.

(b) $E\left(\epsilon_t\right) = 0$, and $E\left(\epsilon_t \epsilon_t'\right) = \Psi \in \mathbb{R}^{d \times d}$, and the marginal law Q of ϵ has finite eight moments.

(c) ϵ_t is independent from the latent log-price X_t.

Note that this assumption is specific for the pre-averaging estimator, and can be called to question at very high frequencies. See, e.g. Hansen and Lunde (2006), Voev and Lunde (2007) and Diebold and

Strasser (2012) for further discussion of this assumption. For instance, for the pre-averaged covolatility estimator, we could allow for dependence between X and ϵ, at the cost of slowing down the speed at which this estimator converges to the true integrated covariation (see Christensen et al. (2010), Section 3.4 for details). We could also consider a general noise model, allowing for both exogenous and endogenous components with polynomially decaying autocovariances as in Varneskov (2014) for realized kernel-based estimators.

In some of our results we rule out jumps in σ_t, formally, we make the following assumption.

Assumption 3 - Volatility

σ_t is locally bounded away from zero and is a continuous semimartingale.

This assumption is common in the realized volatility literature (e.g. equation (3) of Barndorff-Nielsen et al. (2008); Assumption 2 of Mykland and Zhang (2009) or equation (3) of Gonçalves and Meddahi (2009)). Assumption 3 can be relaxed (see Assumption H1 of Barndorff-Nielsen et al. (2006) for a weaker assumption on σ).

4.1 Noise-free, synchronous data and no jumps

In the simple case where no market microstructure noise is present and prices are observed synchronously at equidistant time points with no jumps. It follows that $Y = X$, where X follows (2), in addition $f_k(u) = f_{kl}(u) = u$, then $\Delta Y^k_{t^k_i} = \Delta Y^k_{\frac{i}{n}} = \Delta X^k_{\frac{i}{n}}$ for $i = 1, \dots, n$, $k = 1, \dots, d$. In applied work, this refers to a situation where the sampling frequencies are low enough for the effects of market microstructure to be negligible, e.g., 5, 15, or 30 minutes. In this relatively simple scenario, a popular consistent estimator of integrated covariance is the realized covariance matrix. Here, we can simply take $b_n = 1$, since with this the summands are conditionally asymptotically independent, it follows that $J_n = n$. There is no bias-corrected estimator term, $\widetilde{b}^n_{kl} = 0$. We have that $\tau_n = \sqrt{n}$ and

$$\widehat{\Gamma}^n = \sum_{i=1}^{n} \left(\Delta Y_{\frac{i}{n}}\right)\left(\Delta Y_{\frac{i}{n}}\right)' = \left(\widehat{\Gamma}^n_{kl}\right)_{1 \le k,l \le d}, \tag{18}$$

where

$$\widehat{\Gamma}^n_{kl} = \sum_{\alpha=1}^{J_n} \underbrace{\Delta Y^k_{\frac{\alpha}{n}} \Delta Y^l_{\frac{\alpha}{n}}}_{=\mathcal{Z}^n_{kl}(\alpha)}.$$

The bootstrap scheme decribed in (8) becomes

$$\mathcal{Z}^{n*}_{kl}(\alpha) = \begin{cases} \Delta Y^k_{\frac{\alpha+1}{n}} \Delta Y^l_{\frac{\alpha+1}{n}} + \left(\Delta Y^k_{\frac{\alpha}{n}} \Delta Y^l_{\frac{\alpha}{n}} - \Delta Y^k_{\frac{\alpha+1}{n}} \Delta Y^l_{\frac{\alpha+1}{n}}\right)\eta_\alpha, & \text{for } 1 \le \alpha \le n-1, \\ \Delta Y^k_{\frac{\alpha}{n}} \Delta Y^l_{\frac{\alpha}{n}}, & \text{for } \alpha = n. \end{cases} \tag{19}$$

Then, in this simple case, the bootstrap resample the cross product returns instead of returns as in Gonçalves and Meddahi (2009). It follows from Theorem 3.1 that the wild blocks of blocks bootstrap covariance between $\sqrt{n}\widehat{\Gamma}_{kl}^{n*}$ and $\sqrt{n}\widehat{\Gamma}_{k'l'}^{n*}$ is given by

$$V_{kl,k'l'}^{n*} = \frac{n}{2}\sum_{\alpha=1}^{n-1}\left(\Delta Y_{\frac{\alpha}{n}}^{k}\Delta Y_{\frac{\alpha}{n}}^{l} - \Delta Y_{\frac{\alpha+1}{n}}^{k}\Delta Y_{\frac{\alpha+1}{n}}^{l}\right)\left(\Delta Y_{\frac{\alpha}{n}}^{k'}\Delta Y_{\frac{\alpha}{n}}^{l'} - \Delta Y_{\frac{\alpha+1}{n}}^{k'}\Delta Y_{\frac{\alpha+1}{n}}^{l'}\right). \tag{20}$$

Next we verify Condition A. It is easy to see that Condition A.3. holds by replacing b_n by 1. To check Condition A.2., apply Theorem 2.1 of Barndorff-Nielsen et al. (2006). A.1. follows since we have let $\mathcal{Z}_{kl}^{n}(\alpha) = \Delta Y_{\frac{\alpha}{n}}^{k}\Delta Y_{\frac{\alpha}{n}}^{l}$, and $J_n = n$, then we can write

$$\begin{aligned}
V_{kl,k'l'}^{n} &= \frac{n}{2}\sum_{\alpha=1}^{n-1}\left(\Delta Y_{\frac{\alpha}{n}}^{k}\Delta Y_{\frac{\alpha}{n}}^{l} - \Delta Y_{\frac{\alpha+1}{n}}^{k}\Delta Y_{\frac{\alpha+1}{n}}^{l}\right)\left(\Delta Y_{\frac{\alpha}{n}}^{k'}\Delta Y_{\frac{\alpha}{n}}^{l'} - \Delta Y_{\frac{\alpha+1}{n}}^{k'}\Delta Y_{\frac{\alpha+1}{n}}^{l'}\right) \\
&= n\left(\sum_{\alpha=1}^{n}\Delta Y_{\frac{\alpha}{n}}^{k}\Delta Y_{\frac{\alpha}{n}}^{l}\Delta Y_{\frac{\alpha}{n}}^{k'}\Delta Y_{\frac{\alpha}{n}}^{l'} - \frac{1}{2}\sum_{\alpha=1}^{n-1}\left(\Delta Y_{\frac{\alpha}{n}}^{k}\Delta Y_{\frac{\alpha}{n}}^{l}\Delta Y_{\frac{\alpha+1}{n}}^{k'}\Delta Y_{\frac{\alpha+1}{n}}^{l'} + \Delta Y_{\frac{\alpha+1}{n}}^{k}\Delta Y_{\frac{\alpha+1}{n}}^{l}\Delta Y_{\frac{\alpha}{n}}^{k'}\Delta Y_{\frac{\alpha}{n}}^{l'}\right)\right) \\
&\quad \underbrace{-\frac{n}{2}\left(\Delta Y_{\frac{1}{n}}^{k}\Delta Y_{\frac{1}{n}}^{l}\Delta Y_{\frac{1}{n}}^{k'}\Delta Y_{\frac{1}{n}}^{l'} + \Delta Y_{1}^{k}\Delta Y_{1}^{l}\Delta Y_{1}^{k'}\Delta Y_{1}^{l'}\right)}_{=O_P\left(\frac{1}{n}\right)} \\
&\to {}^{P} V_{kl,k'l'},
\end{aligned}$$

where the last step uses Theorem 2 of Barndorff-Nielsen and Shephard (2004a). More specifically, we may let $y_i = vec\left(\left(\Delta Y_{\frac{i}{n}}\right)\left(\Delta Y_{\frac{i}{n}}\right)'\right)$ for $i = 1, \ldots, n$, then we can write

$$V^{n} = \underbrace{n\left(\sum_{i=1}^{n} y_i y_i' - \frac{1}{2}\sum_{i=1}^{n-1}\left(y_i y_{i+1}' + y_{i+1} y_i'\right)\right)}_{=\widehat{V}_{\text{BN-S}}^{n}} - \underbrace{\frac{n}{2}\left(y_1 y_1' + y_n y_n'\right)}_{=O_P\left(\frac{1}{n}\right)}, \tag{21}$$

where $V^n = \left(V_{kl,k'l'}^{n}\right)_{1\leq k,k'l,'l'\leq d}$ and $\widehat{V}_{\text{BN-S}}^{n}$ is the consistent estimator of the asymptotic variance of $\sqrt{n}\sum_{i=1}^{n}\left(\Delta Y_{\frac{i}{n}}\right)\left(\Delta Y_{\frac{i}{n}}\right)'$ proposed by Barndorff-Nielsen and Shephard (2004a). Thus, apart from border terms, which are $O_P\left(\frac{1}{n}\right)$, our bootstrap variance estimator of the variance of the realized covariance matrix coincides with the sophisticated consistent variance estimator proposed by Barndorff-Nielsen and Shephard (2004a). This is in contrast with the pairs bootstrap studied by Dovonon et al. (2013), which is not able to estimate the long run variance of the realized covariance matrix, except when the volatility is constant. Note that in the univariate case ($d = 1$), the wild blocks of blocks bootstrap variance $V_{kk,kk}^{n*}$ becomes

$$Var^{*}\left(\sqrt{n}\sum_{i=1}^{n}\left(\Delta Y_{\frac{i}{n}}^{k*}\right)^{2}\right) = \frac{n}{2}\sum_{i=1}^{n-1}\left(\left(\Delta Y_{\frac{i}{n}}^{k}\right)^{2} - \left(\Delta Y_{\frac{i+1}{n}}^{k}\right)^{2}\right)^{2} \to^{P} 2\int_{0}^{1}\sigma_s^4 ds,$$

which is a consistent estimator of the asymptotic variance of $\sqrt{n}\sum_{i=1}^{n}\left(\Delta Y_{\frac{i}{n}}^{k}\right)^{2}$. This is not the case of the bootstrap methods studied by Gonçalves and Meddahi (2009). In particular, the i.i.d. bootstrap variance estimator for the asymptotic variance of the realized volatility is given by

$$n\sum_{i=1}^{n}\left(\Delta Y^k_{\frac{i}{n}}\right)^4-\left(\sum_{i=1}^{n}\left(\Delta Y^k_{\frac{i}{n}}\right)^2\right)^2\to^P 3\int_0^1\sigma_s^4ds-\left(\int_0^1\sigma_s^2ds\right)^2,$$

which is equal to $2\int_0^1\sigma_s^4ds$ only when the volatility is constant.

4.2 Noise-free, asynchronous data and no jumps

We now turn to the case of non-synchronously observed data, but we do not allow jumps and market microstructure noise. In this particular case, it follows that $Y=X$, where X follows (2), and consequently we have $\Delta Y^k_{t^k_i}=\Delta X^k_{t^k_i}$ for $i=1,\ldots,n_k$, $k=1,\ldots,d$. The "standard" estimator of integrated covolatility, given in (18) is not robust to asynchronous data. An alternative to the realized covariance estimator that solves the non-synchronicity problem using tick-by-tick data is for example the cumulative covariance estimator developped in Hayashi and Yoshida (2005). This is defined as

$$\widehat{\Gamma}^n_{kl}=\sum_{i=0}^{n_k}\sum_{j=0}^{n_l}\Delta Y^k_{t^k_i}\Delta Y^l_{t^l_j}1_{C^{kl}_{ij}}=\sum_{\alpha=1}^{J_n}\mathcal{Z}^n_{kl}(\alpha)+\widetilde{b}^n_{kl},\qquad(22)$$

where $C^{kl}_{ij}=\left\{(i,j):\left(t^k_{i-1},t^k_i\right]\cap\left(t^l_{i-1},t^l_i\right]\neq\emptyset\right\}$. The idea of Hayashi and Yoshida (2005) is to select only some of the cross variations $\Delta Y^k_{t^k_i}\Delta Y^l_{t^l_j}$ in order to estimate $\int_0^1\Sigma^{kl}_s ds$, and precisely the ones for which there is an intersection between the time intervals $\left(t^k_{i-1},t^k_i\right]$ and $\left(t^l_{i-1},t^l_i\right]$. Here, we can also take $b_n=1$, then $J_n=n$ and $B_n(\alpha)=\left[\frac{\alpha-1}{n},\frac{\alpha}{n}\right)$. There is also no bias-corrected estimator term, i.e. $\widetilde{b}^n_{kl}=0$. Thus we have set

$$\mathcal{Z}^n_{kl}(\alpha)=\sum_{t^k_i\in B_n(\alpha)}\sum_{j=0}^{n_l}\Delta Y^k_{t^k_i}\Delta Y^l_{t^l_j}1_{C^{kl}_{ij}}.\qquad(23)$$

It is easy to verify that Condition A holds, then we can apply all results in Theorems 3.1 and 3.2 to $\widehat{\Gamma}^n_{kl}$ defined by (22). The proof of this result is achieved by using arguments alike the ones presented in the more general case in Section 4.4, where in addition to asynchronicity we allow noise. In particular, $\Delta Y^k_{t^k_i}$ plays the role of $\bar{Y}^k_{t^k_i}$; see Christensen et al. (2013), for further details. Thus, our bootstrap variance estimator of the variance of the Hayashi and Yoshida (2005) integrated covariance estimator is an alternative to the consistent variance estimator proposed recently by Mykland (2012).

4.3 Noisy, synchronous data and no jumps

Let us study the case where we allow for the presence of market microstructure noise, but we rule out asynchronicity, jumps and we suppose that prices are observed at equidistant time stamps. Specifically, we consider the multivariate model given by (2), then we have $\Delta Y^k_{\frac{i}{n}}=\Delta X^k_{\frac{i}{n}}+\Delta\epsilon^k_{\frac{i}{n}}$, for $i=1,\ldots,n$, $k=1,\ldots,d$. There exists many estimators alternative to the realized covariance estimator that are robust to the presence of market microstructure noise. Let us consider the bias-corrected pre-averaging estimator of Christensen et al. (2010), which yields the optimal rate of convergence. The pre-averaging approach proposed by Podolskij and Vetter (2009), studied by Jacod et al. (2009) and further extended to the multivariate context by Christensen et al. (2010) and Christensen et al. (2013) is one way to lessen the influence of the noise and help us to get information about Γ.

To describe this technique, let k_n be a sequence of integers, which defines the window length over which the pre-averaging of returns is performed. In particular, suppose

$$\frac{k_n}{\sqrt{n}} = \theta + o\left(n^{-1/4}\right), \tag{24}$$

for some $\theta > 0$. Similarly, let g be a weighting function on $[0,1]$ such that $g(0) = g(1) = 0$, $\int_0^1 g(s)^2 ds > 0$, and assume g is continuous and piecewise continuously differentiable with a piecewise Lipschitz derivative g'. An example of a function that satisfies these restrictions is $g(x) = \min(x, 1-x)$.

For all $k = 1, \ldots, d$, $i = 0, \ldots, n_k - k_n + 1$, the pre-averaged returns in tick time $\bar{Y}^k_{t^k_i}$ are obtained by computing the weighted sum of all consecutive returns performed in (3) over each block of size k_n

$$\bar{Y}^k_{t^k_i} = \sum_{j=1}^{k_n} g\left(\frac{j}{k_n}\right) \Delta Y^k_{t^k_{i+j}}. \tag{25}$$

Based on the pre-averaged returns $\bar{Y}^k_{t^k_i}$, Christensen et al. (2010) defined $\widehat{\Gamma}^n$ as:

$$\widehat{\Gamma}^n = \frac{1}{\psi_2 k_n} \sum_{i=0}^{n-k_n+1} \bar{Y}_{\frac{i}{n}} \left(\bar{Y}_{\frac{i}{n}}\right)' - \underbrace{\frac{\psi_1}{2n\theta^2\psi_2} \sum_{i=1}^{n} \Delta Y_{\frac{i}{n}} \left(\Delta Y_{\frac{i}{n}}\right)'}_{\text{bias correction term}}, \tag{26}$$

where $\psi_1 = \int_0^1 g'(u)^2 du$ and $\psi_2 = \int_0^1 g(u)^2 du$. The pre-averaging estimator is then simply the analogue of the realized covariance but based on pre-averaged returns and an additional term to remove bias due to noise. As discussed in Jacod et al. (2009), this bias term does not contribute to the asymptotic variance of $\widehat{\Gamma}^n$. Note that in (26), the bias correction term $\widetilde{b}^n = \frac{\psi_1}{2n\theta^2\psi_2} \sum_{i=1}^{n} \Delta Y_{\frac{i}{n}} \left(\Delta Y_{\frac{i}{n}}\right)'$ works only for i.i.d. noise. In the univariate case, e.g., Hautsch and Podolskij (2013) for the corrected estimator of the bias $\widetilde{b}$ under m-dependent noise. In order to apply the wild blocks of blocks bootstrap method, we can let

$$\mathcal{Z}^n_{kl}(\alpha) = \frac{1}{\psi_2 k_n} \sum_{i=1}^{b_n} \bar{Y}^k_{\frac{i-1+(\alpha-1)b_n}{n}} \bar{Y}^l_{\frac{i-1+(\alpha-1)b_n}{n}}. \tag{27}$$

Note that since the pre-averaged returns are strongly dependent, we cannot use $b_n = 1$ as before, instead we will let b_n tend to infinity as $n \to \infty$; since in this way we will asymptotically be able to mimic the dependence in the pre-averaged returns nonparametrically. In particular, b_n follows (6) but additionally we require that $1/2 < \delta_2 < 2/3$. In this case and under Assumptions 2 (with i.i.d noise), it is easy to verify that Condition A holds, then we can apply all results in Theorems 3.1 and 3.2 to the pre-averaging estimator $\widehat{\Gamma}^n_{kl}$ defined by (26). In particular the validity of A.1. is detailed in the proof of Lemma 7.1 in Appendix B. Condition A.2. also follows since under our assumptions we have that $\bar{Y}^k_{\frac{i}{n}} = O_P\left(\frac{1}{n^{1/4}}\right)$ uniformly in i and similarly $\mathcal{Z}^n_{kl}(\alpha) = O_P\left(\frac{b_n}{n}\right)$ uniformly in α (see for instance Lemma 6.2 of Christensen et al. (2013)). Finally A.3. follows since for any $\varepsilon > 0$ and $1/2 < \delta_2 < 2/3$ we have that $-2 - 3\varepsilon + 4\delta_2(1+\varepsilon) < 0$.

Note that when $d = 1$, (26) amounts to the pre-averaging estimator proposed by Jacod et al. (2009) on which Hounyo et al. (2013) first introduced the univariate wild blocks of blocks bootstrap method. Our new general multivariate wild blocks of blocks bootstrap method given in (8), differs from the univariate bootstrap method of Hounyo et al. (2013) in important ways. The later resamples the squared pre-averaged returns $\bar{Y}^2_{\frac{i}{n}}$. Here, in the present paper, we resample the block sum of the squared pre-averaged returns that belong to $B_n(\alpha) = \left[\frac{(\alpha-1)b_n}{n}, \frac{\alpha b_n}{n}\right)$, i.e. $\mathcal{Z}^n_{kk}(\alpha) = \frac{1}{\psi_2 k_n}\sum_{i=1}^{b_n}\left(\bar{Y}^k_{\frac{i-1+(\alpha-1)b_n}{n}}\right)^2$. In addition, in Hounyo et al. (2013) the choice of the bootstrap block size b_n is such that $b_n = (p+1)k_n$, where k_n is the block length of the interval over which the pre-averaging is done given in (24) and p is either fixed such that $p \geq 1$, or $p \to \infty$. This choice of b_n is more specific for the pre-averaging estimator. In this paper, $b_n \propto n^{\delta_2}$ where $\delta_2 \in (0,1)$. These modifications are important in order to generalize the wild blocks of blocks bootstrap method to a broad class of statistics.

It follows that, the bootstrap covariance between $\tau_n\widehat{\Gamma}^{n*}_{kl}$ and $\tau_n\widehat{\Gamma}^{n*}_{k'l'}$ with $\tau_n = n^{1/4}$ is given by $V^{n*}_{kl,k'l'} = \frac{\sqrt{n}}{2}\sum_{\alpha=1}^{J_n-1}\left(\mathcal{Z}^n_{kl}(\alpha) - \mathcal{Z}^n_{k'l'}(\alpha+1)\right)\left(\mathcal{Z}^n_{kl}(\alpha) - \mathcal{Z}^n_{k'l'}(\alpha+1)\right)$, where $\mathcal{Z}^n_{kl}(\alpha)$ is given by (27). Given Theorem 3.1, we have that as $n \to \infty$, $V^{n*}_{kl,k'l'} \to^P V_{kl,k'l'}$. Also, notice that the bootstrap variance estimator is positive semi-definite by construction, this is an appealing feature not shared by the existing variance estimator of $V_{kl,k'l'}$ proposed by Christensen et al. (2010).

4.4 Noisy, asynchronous data and no jumps

In this subsection, we allow for asynchronicity and as in Section 4.3, we consider a setup where we do not observe the true efficient prices X, but instead a process Y. These prices are observed irregularly and non-synchronous over the interval $[0,1]$. In this pratical situation, we study two different integrated covolatility estimators. First, we verify the validity of the high level Condition A for the pre-averaged Hayashi-Yoshida estimator studied by Christensen et al. (2013). Second, we show that the multivariate realized kernel estimator of Barndorff-Nielsen et al. (2011) as well as the flat-top realized kernel by Varneskov (2014) can also be written as an example of estimators of Γ given in (5). Then, we outline what a simple bootstrap variance estimator of the asymptotic variance V of the multivariate realized kernel estimator would look like, if our high level conditions hold.

4.4.1 The pre-averaged Hayashi-Yoshida estimator

Based on the pre-averaged returns $\bar{Y}^k_{t^k_i}$ (given by (25)), Christensen et al. (2010) defined a Hayashi-Yoshida-type estimator for the integrated covariance Γ_{kl} between assets k and l as follows

$$\widehat{\Gamma}^n_{kl} = \frac{1}{(\psi k_n)^2}\sum_{i=0}^{n_k-k_n+1}\sum_{j=0}^{n_l-k_n+1}\bar{Y}^k_{t^k_i}\bar{Y}^l_{t^l_j}1_{A^{kl}_{ij}}, \tag{28}$$

where k_n is given by (24), $\psi = \int_0^1 g(s)\,ds$, $A_{ij}^{kl} = \left\{(i,j) : \left(t_i^k, t_{i+k_n}^k\right] \cap \left(t_j^l, t_{j+k_n}^l\right] \neq \emptyset\right\}$, $1_{\{\cdot\}}$ is the indicator function discarding pre-averaged returns that do not overlap in time. For the simple function $g(x) = \min(x, 1-x)$, $\psi = 1/4$. This estimator has the profound advantage that it does not throw away information that is typically lost using a synchronization procedure. Note that under Assumption 1, n, n_k and n_l are of the same order and that n controls the universal pre-averaging window k_n. In order to apply the boostrap method given in (8), we can let

$$\mathcal{Z}_{kl}^n(\alpha) = \sum_{t_i^k \in B_n(\alpha)} \sum_{j=0}^{n_l - k_n + 1} \bar{Y}_{t_i^k}^k \bar{Y}_{t_j^l}^l 1_{A_{ij}^{kl}}. \tag{29}$$

Thus, under Assumptions 1-3, (k_n, θ) satisfying (24) and b_n follows (6) such that $1/2 < \delta_2 < 2/3$, we can show that Condition A holds for the pre-averaged Hayashi-Yoshida estimator $\widehat{\Gamma}_{kl}^n$ defined by (28). In particular, the validity of A.1. is detailed in the proof of Lemma 7.2 in Appendix B. Condition A.2. also holds because under our assumptions we have that $\bar{Y}_{\frac{i}{n}}^k = O_P\left(\frac{1}{n^{1/4}}\right)$ uniformly in i and similarly $\mathcal{Z}_{kl}^n(\alpha) = O_P\left(\frac{b_n}{n}\right)$ uniformly in α (see for instance Lemma 6.2 of Christensen et al. (2013)). Finally, A.3. follows since for any $\varepsilon > 0$ and $1/2 < \delta_2 < 2/3$ we have that $-2 - 3\varepsilon + 4\delta_2(1+\varepsilon) < 0$.

4.4.2 Multivariate realized kernels estimator

In the univariate setting, Jacod et al (2009) show that apart from border terms, i.e. terms close to 0 and 1, the pre-averaging estimator given by (26) coincides with the one-lag "flat top" realized kernel estimator in Barndorff-Nielsen et al. (2008) using kernel weights

$$k(s) = \psi_2^{-1} \int_s^1 g(u)\, g(u-s)\, du, \tag{30}$$

where $g(u)$ is defined as in Section 4.3. In particular, when we choose the bandwidth of the realized kernel estimator equal to the size of the pre-averaging window k_n, the realized kernel and pre-averaging based-estimators have the same asymptotic distribution. Consequently, for the bootstrap we can resample the same statistics as we did for the pre-averaging estimator to estimate the distribution as well as the variance of realized kernel based-estimator, provided that we use the weight function as given by (30). Some of our arguments here are heuristic. To fix ideas, let consider synchronous data in the following. According to equation (1) of Barndorff-Nielsen et al. (2011) (see also equation (5) of Varneskov (2014)), the multivariate realized kernel can be rewritten as

$$\widehat{\Gamma}^n = \sum_{i=1}^n k(0)\left(\Delta Y_{\frac{i}{n}}\right)\left(\Delta Y_{\frac{i}{n}}\right)' + \sum_{i=1}^{n-1}\sum_{h=1}^{n-i} k\left(\frac{h}{H}\right)\left(\left(\Delta Y_{\frac{i}{n}}\right)\left(\Delta Y_{\frac{i+h}{n}}\right)' + \left(\Delta Y_{\frac{i+h}{n}}\right)\left(\Delta Y_{\frac{i}{n}}\right)'\right), \tag{31}$$

where $\Delta Y_{\frac{i}{n}} = Y_{\frac{i}{n}} - Y_{\frac{i-1}{n}}$ and $k : \mathbb{R} \to \mathbb{R}$ is a non-stochastic weight function. That is characterised by:

Assumption K. (i) $k(0) = 1, k'(0) = 0$; (ii) k is twice differentiable with continous derivatives; (iii)

$\int_0^\infty k(x)^2 dx < \infty$, $\int_0^\infty k'(x)^2 dx < \infty$, $k''(x)^2 dx dx < \infty$; (iv) $\int_{-\infty}^{\infty} k(x) \exp(ix\lambda) dx \geq 0$ for all $\lambda \in \mathbb{R}$.

We follow Barndorff-Nielsen et al. (2011) and we average m prices at the very beginning and end of the day. More specifically, we set

$$Y_0 = \frac{1}{m}\sum_{i=1}^{m} Y_{\frac{i}{n}}, \text{ and } Y_1 = \frac{1}{m}\sum_{i=1}^{m} Y_{\frac{n-m+i}{n}}.$$

Note that, (31) can be written as

$$\widehat{\Gamma}^n = \sum_{\alpha=1}^{J_n} \mathcal{Z}^n(\alpha),$$

where for $1 \leq \alpha \leq J_n$,

$$\mathcal{Z}^n(\alpha) = \sum_{i=(\alpha-1)b_n+1}^{\alpha b_n} \left(\left(\Delta Y_{\frac{i}{n}}\right)\left(\Delta Y_{\frac{i}{n}}\right)' + \sum_{h=1}^{n-i} k\left(\frac{h}{H}\right)\left(\left(\Delta Y_{\frac{i}{n}}\right)\left(\Delta Y_{\frac{i+h}{n}}\right)' + \left(\Delta Y_{\frac{i+h}{n}}\right)\left(\Delta Y_{\frac{i}{n}}\right)'\right)\right), \tag{32}$$

given that $k(0) = 1$, and we suppose by simplicity that J_n is an integer such that $n = J_n \cdot b_n$. The statistics $\mathcal{Z}^n(\alpha)$ involve many increments of Y, that are not in the sub-interval $B_n(\alpha) = \left[\frac{(\alpha-1)b_n}{n}, \frac{\alpha b_n}{n}\right)$. Thus $\mathcal{Z}^n(\alpha)$ may be strongly dependent even if we let b_n tend to infinity as $n \to \infty$ because they rely on many common observations $\Delta Y_{\frac{i}{n}}$. However, when we use as weight function the Parzen kernel (which is advocated by Barndorff-Nielsen et al. (2011)), we show that we can remove substantially many common observations $\Delta Y_{\frac{i}{n}}$ in $\mathcal{Z}^n(\alpha)$. In particular, all observations in $\mathcal{Z}^n(\alpha)$ such that $\frac{h}{H} > 1$ (since by definition, for the Parzen kernel $k(x) = 0$ for $x > 1$). Thus, according that $k(x)$ is the Parzen kernel or any others kernel such that Assumption K holds and $k(x) = 0$ for $x > 1$, we can write (31) as follows, for $1 \leq \alpha \leq J_n - 1$

$$\mathcal{Z}^n(\alpha) = \sum_{i=(\alpha-1)b_n+1}^{\alpha b_n} \left(\left(\Delta Y_{\frac{i}{n}}\right)\left(\Delta Y_{\frac{i}{n}}\right)' + \sum_{h=1}^{H} k\left(\frac{h}{H}\right)\left(\left(\Delta Y_{\frac{i}{n}}\right)\left(\Delta Y_{\frac{i+h}{n}}\right)' + \left(\Delta Y_{\frac{i+h}{n}}\right)\left(\Delta Y_{\frac{i}{n}}\right)'\right)\right), \tag{33}$$

whereas for $\alpha = J_n$

$$\mathcal{Z}^n(\alpha) = \sum_{i=n-b_n+1}^{n} \left(\left(\Delta Y_{\frac{i}{n}}\right)\left(\Delta Y_{\frac{i}{n}}\right)' + \sum_{h=1}^{\min(H,n-i)} k\left(\frac{h}{H}\right)\left(\left(\Delta Y_{\frac{i}{n}}\right)\left(\Delta Y_{\frac{i+h}{n}}\right)' + \left(\Delta Y_{\frac{i+h}{n}}\right)\left(\Delta Y_{\frac{i}{n}}\right)'\right)\right), \tag{34}$$

where $H \leq b_n$. It is conjecture that the statistics $\mathcal{Z}^n(\alpha)$, as defined by (33) and (34) will verify our high level Condition A. If this is the case, then a positive semi-definite consistent estimator of the asymptotic variance V of the multivariate realized kernel estimator will be $V^n = \left(V^n_{kl,k'l'}\right)_{1 \leq k,k'l,'l' \leq d}$, where

$$V^n_{kl,k'l'} = \frac{n^{2/5}}{2}\sum_{\alpha=1}^{J_n-1}\left(\mathcal{Z}^n_{kl}(\alpha) - \mathcal{Z}^n_{kl}(\alpha+1)\right)\left(\mathcal{Z}^n_{k'l'}(\alpha) - \mathcal{Z}^n_{k'l'}(\alpha+1)\right).$$

It would cleary be desirable to have a formal proof of this, but this is beyond the scope of this paper.

We emphasize that the paper by Barndorff-Nielsen et al. (2011) goes much futher in developing the multivariate realized kernel estimation technology, including non-synchronous trading and allowing certain types of measurement error (such as endogenous noise). Furthermore, their results are extended in Varneskov (2014), who also suggests a class of kernels that are $n^{1/4}$-consistent and efficient.

In the univariate context, given that we can fit the subsampling-based estimator of Zhang et al. (2005) and Zhang (2006) into the realized kernel setting (e.g. Barndorff-Nielsen et al. (2008)), we conjecture that similar analysis as for kernel-based estimators holds for the subsampling-based estimators, but a full exploration of this is left for future research.

4.5 Jumps, noise-free and synchronous data

It has long been recognized that asset prices do not always evolve continuously over a given time interval (e.g. Huang and Tauchen (2005), Barndorff-Nielsen and Shephard (2006)). So far we have focused on the case where X is continuous In this subsection, we allow for jumps in X_t and suppose that no market microstructure noise is present and prices are observed synchronously at equidistant date. In particular, we observe $Y = X + Z$, where X is given by (2) at regular time points $t_i = \frac{i}{n}$, for $i = 0, \ldots, n$, where Z^k is any finite activity jump process. This means that they have the following representation, for all $k = 1, \ldots, d$,

$$Z_t^k = \int_0^t C_s^k dN_s^k ds = \sum_{r=1}^{N_t^k} C_{\pi_r^k}^k,$$

where $N^k = \left(N_t^k\right)_{t\in[0,1]}$ is a counting process with $E\left(N_1^k\right) < \infty$, $\left\{\pi_r^k,\ r = 1, \ldots, N_1^k\right\}$ denote the instants of jump of Z^k and $C_{\pi_r^k}^k$ denote the sizes ΔZ_t^k of jumps at π_r^k.

In this context, the covariance between risk factors of asset prices is due to both Brownian and jump components. To separate the two terms of the quadratic covariation given by the sum of Γ (integrated covariance) with the sum of co-jumps, we can for instance used the threshold estimator of Mancini and Gobbi (2012) (see also Barndorff-Nielsen and Shephard (2004b), Jacod and Todorov (2009), Bollerslev and Todorov (2010) and Boudt, Croux and Laurent (2011), among others). Following Mancini and Gobbi (2012), we have that

$$\widehat{\Gamma}^n = \sum_{i=1}^n \left(\widetilde{\Delta Y}_{\frac{i}{n}}\right)\left(\widetilde{\Delta Y}_{\frac{i}{n}}\right)' \equiv \left(\widehat{\Gamma}_{kl}^n\right)_{1\le k,l\le d}, \tag{35}$$

where $\widetilde{\Delta Y}_{\frac{i}{n}} \equiv \left(\widetilde{\Delta Y}^1_{\frac{i}{n}}, \ldots, \widetilde{\Delta Y}^d_{\frac{i}{n}}\right)' = \left(\Delta Y^1_{\frac{i}{n}} 1_{\left\{\left|\Delta Y^1_{\frac{i}{n}}\right| \le \widetilde{\alpha} n^{-\widetilde{\lambda}}\right\}}, \ldots, \Delta Y^d_{\frac{i}{n}} 1_{\left\{\left|\Delta Y^d_{\frac{i}{n}}\right| \le \widetilde{\alpha} n^{-\widetilde{\lambda}}\right\}}\right)'$, $\widetilde{\alpha} \ge 0$, and $\widetilde{\lambda} \in \left(0, \frac{1}{2}\right)$, and $\widehat{\Gamma}^n_{kl} = \sum_{i=1}^n \Delta Y^k_{\frac{i}{n}} 1_{\left\{\left|\Delta Y^k_{\frac{i}{n}}\right| \le \widetilde{\alpha} n^{-\widetilde{\lambda}}\right\}} \Delta Y^l_{\frac{i}{n}} 1_{\left\{\left|\Delta Y^l_{\frac{i}{n}}\right| \le \widetilde{\alpha} n^{-\widetilde{\lambda}}\right\}}$. As in Section 4.1, here we can take

$b_n = 1$. There is no bias-corrected estimator term, i.e. $\widetilde{b}_{kl}^n = 0$. It follows that $J_n = n$, and

$$\mathcal{Z}_{kl}^n(\alpha) = \widetilde{\Delta Y}^k_{\frac{\alpha}{n}} \widetilde{\Delta Y}^l_{\frac{\alpha}{n}}, \text{ for } \alpha = 1, \ldots, n. \tag{36}$$

Next we verify Condition A. It is easy to see that Condition A.3. holds by replacing b_n by 1. To check Condition A.2., apply Theorem 2.1 of Barndorff-Nielsen et al. (2006). The proof of validity of A.1. is achieved by using arguments alike the ones presented, in detail, in the proof of Lemma 7.1, where now $\widetilde{\Delta Y}^k_{\frac{\alpha}{n}}$ plays the role of $\bar{Y}^k_{t_i^k}$. It follows from Theorem 3.1 that the bootstrap covariance $V^{n*}_{kl,k'l'}$ is given by

$$V^{n*}_{kl,k'l'} = \frac{n}{2}\sum_{i=1}^{n-1}\left(\widetilde{\Delta Y}^k_{\frac{i}{n}}\widetilde{\Delta Y}^l_{\frac{i}{n}} - \widetilde{\Delta Y}^k_{\frac{i+1}{n}}\widetilde{\Delta Y}^l_{\frac{i+1}{n}}\right)\left(\widetilde{\Delta Y}^{k'}_{\frac{i}{n}}\widetilde{\Delta Y}^{l'}_{\frac{i}{n}} - \widetilde{\Delta Y}^{k'}_{\frac{i+1}{n}}\widetilde{\Delta Y}^{l'}_{\frac{i+1}{n}}\right).$$

In particular, when $(k,l) = \left(k', l'\right)$, we have that

$$\begin{aligned} V^{n*}_{kl,kl} &= \frac{n}{2}\sum_{i=1}^{n-1}\left(\widetilde{\Delta Y}^k_{\frac{i}{n}}\widetilde{\Delta Y}^l_{\frac{i}{n}} - \widetilde{\Delta Y}^k_{\frac{i+1}{n}}\widetilde{\Delta Y}^l_{\frac{i+1}{n}}\right)^2 \\ &= \underbrace{n\sum_{i=1}^{n}\left(\widetilde{\Delta Y}^k_{\frac{i}{n}}\widetilde{\Delta Y}^l_{\frac{i}{n}}\right)^2 - n\sum_{i=1}^{n-1}\left(\widetilde{\Delta Y}^k_{\frac{i}{n}}\widetilde{\Delta Y}^l_{\frac{i}{n}}\right)\left(\widetilde{\Delta Y}^k_{\frac{i+1}{n}}\widetilde{\Delta Y}^l_{\frac{i+1}{n}}\right)}_{=\widehat{V}^n_{\text{M-G}}} - \underbrace{\frac{n}{2}\left(\left(\widetilde{\Delta Y}^k_1\widetilde{\Delta Y}^l_1\right)^2 + \left(\widetilde{\Delta Y}^k_n\widetilde{\Delta Y}^l_n\right)^2\right)}_{=O_P\left(\frac{1}{n}\right)}, \end{aligned}$$

where $\widehat{V}^n_{\text{M-G}}$ is the consistent estimator of the asymptotic variance of $\sqrt{n}\sum_{i=1}^{n}\widetilde{\Delta Y}^k_{\frac{i}{n}}\widetilde{\Delta Y}^l_{\frac{i}{n}}$ proposed by Mancini and Gobbi (2012) (cf. Proposition 3.7). Thus, apart from border terms which are $O_P\left(\frac{1}{n}\right)$, we have $V^{n*}_{kl,kl} = \widehat{V}^n_{\text{M-G}} \to^P V_{kl,kl}$, as $n \to \infty$. This result extends the work of Hounyo (2013), where a local Gaussian bootstrap method have been proposed for inference on integrated covolatility under no jumps by allowing for the latter. It also provides an alternative to the recent general local Gaussian bootstrap method introduced by Dovonon et al. (2014) for jump tests.

Note that in the univariate context, the jump robust estimators of integrated volatility called bipower variation introduced by Barndorff-Nielsen and Shephard (2004) and its multipower version, analysed among others by Barndorff-Nielsen et al. (2006), can also be written as an example of estimators of Γ given in (5). Following Barndorff-Nielsen et al. (2006), we have that

$$\widehat{\Gamma}^n = \frac{1}{\prod_{l=1}^{L} m_{p_l}}\sum_{i=1}^{n-L+1}\prod_{l=1}^{L}\left|\Delta Y^k_{\frac{i+l-1}{n}}\right|^{p_l}, \tag{37}$$

such that $\sum_{l=1}^{L} p_l = 2$, where $p_l \geq 0$ and $m_p = E\left|N(0,1)\right|^p$. In particular, under some regularity conditions, we can apply the wild blocks of blocks bootstrap method by resampling as in (8) the statistics $\mathcal{Z}_{kk}^n(\alpha)$ given by

$$\mathcal{Z}_{kk}^n(\alpha) = \frac{1}{\prod_{l=1}^{L} m_{p_l}}\sum_{i=1}^{b_n}\prod_{l=1}^{L}\left|\Delta Y^k_{\frac{i+l-1+(\alpha-1)b_n}{n}}\right|^{p_l}, \text{ for } \alpha = 1, \ldots, J_n, \tag{38}$$

where here $J_n = \left\lfloor \frac{n-L+1}{b_n} \right\rfloor$. The full exploration of the multipower variation-based bootstrap is left for future research.

5 Monte Carlo results

In this section, we assess by Monte Carlo simulation the accuracy of the feasible asymptotic theory approach of Christensen et al. (2013). We find that this approach leads to important coverage probability distortions when returns are not sampled too frequently. We also compare the finite sample performance of this approach with the wild blocks of blocks bootstrap method. The design of our Monte Carlo study is roughly identical to that used by Christensen et al. (2010) and Barndorff-Nielsen et al. (2011) with some minor differences. In particular, in addition to the case of i.i.d. noise, we look at the case of autocorrelated noise. Here we briefly describe the Monte Carlo design we use.

To simulate log-prices we consider the following bivariate stochastic volatility model

$$dX_t^{(i)} = a^{(i)}dt + \rho^{(i)}\sigma_t^{(i)}dB_t^{(i)} + \sqrt{1-\left[\rho^{(i)}\right]^2}\sigma_t^{(i)}dW_t, \text{ for } i = 1, 2,$$

where $B^{(i)}$ and W are independent Brownian motions. In this model, the term $\rho^{(i)}\sigma_t^{(i)}dB_t^{(i)}$ is an idiosyncratic component, while $\sqrt{1-\left[\rho^{(i)}\right]^2}\sigma_t^{(i)}dW_t$ is a common factor.

The spot volatility is modeled as $\sigma_t^{(i)} = \exp\left(\beta_0^{(i)} + \beta_1^{(i)}\varrho_t^{(i)}\right)$ with an Ornstein-Uhlenbeck specification for $\varrho_t^{(i)} : d\varrho_t^{(i)} = \alpha^{(i)}\varrho_t^{(i)}dt + dB_t^{(i)}$. This implies that there is perfect correlation between the innovations of $\rho^{(i)}\sigma_t^{(i)}dB_t^{(i)}$ and $\sigma_t^{(i)}$, while it is $\rho^{(i)}$ between the increments of $X_t^{(i)}$ and $\varrho_t^{(i)}$. Finally, the magnitude of correlation between the two underlying price processes $X_t^{(1)}$ and $X_t^{(2)}$ is $\sqrt{1-\left[\rho^{(1)}\right]^2}\sqrt{1-\left[\rho^{(2)}\right]^2}$. The reported results are based on the following configuration of parameters for both processes: $\left(a^{(i)}, \beta_0^{(i)}, \beta_1^{(i)}, \rho^{(i)}, \alpha^{(i)}\right) = (0.03, -5/16, 1/8, -1/40, -0.3)$, so that $\beta_0^{(i)} = \left[\beta_1^{(i)}\right]^2 / \left[2\alpha^{(i)}\right]$. We note that this particular choice of parameters also means that the volatility process has been normalized, in the sense that $E\left(\int_0^1 \left[\sigma_s^{(i)}\right]^2\right) = 1$.

We simulate data for the unit interval $[0,1]$, and normalize one second to be $1/23400$, so that $[0,1]$ represent 6.5 hours worth of trading, which is then further decomposed into $N = 23,400$ subintervals of equal length $1/N$. In constructing noisy prices $Y^{(i)}$, we first generate a complete high frequency record of N equidistant observations of the efficient price $X^{(i)}$ using a standard Euler scheme. We initialize the spot volatility $\sigma_t^{(i)}$ at the start of each interval by drawing the initial values for the $\varrho_t^{(i)}$ processes from its stationary distribution, i.e. $\varrho_0^{(i)} \sim N\left(0, \left[2\alpha^{(i)}\right]^{-1}\right)$. The size of the market microstucture noise is an important parameter. We follow Barndorff-Nielsen et al. (2011) and model the noise magnitude as $\xi^2 = \omega^2 / \sqrt{\int_0^1 \sigma_s^4 ds}$. We fix ξ^2 equal to 0, 0.001 and 0.01 (which covers scenarios with no noise through low-to-high levels of noise) and let $\omega^2 = \xi^2\sqrt{\int_0^1 \sigma_s^4 ds}$. This means that the variance of the noise process increases with the level of volatility of the efficient price $X^{(i)}$, as documented by Bandi and Russell (2006). These values are motivated by the empirical study of Hansen and Lunde

(2006), who investigate 30 stocks of the Dow Jones Industrial Average. We follow Kalnina (2011) and add autocorrelated microstructure noise simulated as an $MA(1)$ process (for a given frequency of the observations):

$$\epsilon^{(i)}_{\frac{j}{n}} = u^{(i)}_{\frac{j-1}{n}} + \gamma u^{(i)}_{\frac{j}{n}}, \text{ where } u^{(i)} | \{\sigma, X\} \overset{i.i.d.}{\sim} N\left(0, \frac{\omega^2}{1+\gamma^2}\right),$$

so that $Var\left(\epsilon^{(i)}\right) = \omega^2$. The observed process is then given by $Y^{(i)} = X^{(i)} + \epsilon^{(i)}$. Three different values of γ are considered, $\gamma = 0$, $\gamma = -0.5$ and $\gamma = -0.9$ (which covers scenarios of i.i.d. noise, moderate and high level of correlation of noise). We follow Christensen et al. (2010) and use the conservative choice of k_n ($\theta = 1$, implying that $k_n = \sqrt{n}$). We also follow the literature and use the weight function $g(x) = \min(x, 1-x)$ to compute the pre-averaged returns. In order to reduce finite sample biases associated with Riemann integrals, we replace in (28), $\psi = \int_0^1 g(s)\, ds$ by its Riemann approximation given by $\psi_n = \frac{1}{k_n} \sum_{i=0}^{k_n} g\left(\frac{i}{k_n}\right)$.

Finally, we extract irregular, non-synchronous data from the complete high-frequency record using Poisson process sampling to generate actual observation times, $\left\{t^{(i)}_j\right\}$. In particular, we consider two independent Poisson processes with intensity parameter $\lambda = (\lambda_1, \lambda_2)$. Here λ_i denotes the average waiting time (in seconds) for new data from process $Y^{(i)}$, so that an average day will have N/λ_i observations of $Y^{(i)}$, $i = 1, 2$. We vary λ_1 through $(3, 10, 60)$ to capture the influence of liquidity on the performance of the pre-averaged multivariate volatility estimator and we set $\lambda_2 = 2\lambda_1$ such that on average $Y^{(2)}$ refreshes at half the pace of $Y^{(1)}$.

Table 1 gives the actual coverage probability rates of 95% confidence intervals of the three covariation measures (integrated covariance, integrated correlation and integrated regression coefficients) as well as the average lengths of the confidence intervals, computed over 10,000 replications. Results based on the asymptotic normal distribution and the wild blocks of blocks bootstrap method are included under the label CLT and WBBB, respectively.

In our simulations, bootstrap intervals use 999 bootstrap replications for each of the 10,000 Monte Carlo replications. We consider the bootstrap percentile method computed at the 95% level. To generate the bootstrap data we use the following external random variables $\eta \sim$ i.i.d. $N(1, 1/2)$. The choice of the bootstrap block size is critical. We follow Politis, Romano and Wolf (1999) and Hounyo et al. (2013) and use the Minimum Volatility Method to choose the bootstrap block (for further details see Hounyo et al. (2013)).

For the three covariation measures, all intervals tend to undercover. The degree of undercoverage is especially large, when the average arrival times of trades is not too frequent. Results are not very sensitive to the noise magnitude nor to the level of correlation. The gains associated with the wild blocks of blocks bootstrap method can be quite substantial, especially for larger values of λ_1 and λ_2 (long average waiting time for new data from process $Y^{(1)}$ and $Y^{(2)}$), when distortions of the CLT-based intervals are larger. For instance, when $\gamma = -0.5$ (moderate level of correlation of noise), $\xi^2 = 0.01$

(high level of noise), and $\lambda = (60, 120)$ (illiquid assets), for the regression coefficient, the coverage rate for a symmetric bootstrap percentile interval is equal to 87.52%, whereas it is equal to 70.20% for the feasible asymptotic theory of Christensen et al. (2013). The gains are especially important for the correlation coefficient, when the asymptotic theory-based intervals does worst. The bootstrap interval has a rate of 90.82%, whereas the Christensen et al. (2013) interval has a rate of 69.32%. For the covariance, these numbers are equal to 87.52% and 70.15%, for the bootstrap and the Christensen et al. (2013) interval, respectively. When the average arrival times of trades become frequent, the bootstrap intervals have coverage rates closer to the desired level, whereas the undercoverage problem persists for the CLT-based intervals. For instance, for the CLT-based intervals, when $\gamma = -0.9$ (high level of correlation of noise), $\xi^2 = 0.001$ (low level of noise), and $\lambda = (3, 6)$ (liquid assets), a two-sided 95% confidence interval for the covariance measure between the two assets has coverage rate equal to 89.19%, whereas it is equal to 88.70% for the regression coefficient. These numbers increase to 94.91% and 94.78% for the bootstrap-based intervals. The bootstrap performance is quite remarkable for the correlation coefficient where it essentially removes all finite sample bias associated with the first-order asymptotic theory of Christensen et al. (2013).

In summary, the results in Table 1 show that the performance of the asymptotic theory-based intervals and the bootstrap percentile intervals in terms of coverage rate crucially depends on the average arrival times of trades. In fact for non-frequent arrival times of trade, the asymptotic normal approximation is often inaccurate and leads to important coverage distortions. In all cases, the bootstrap outperforms the existing first order asymptotic theory.

6 Empirical application

To illustrate some empirical features of the wild blocks of blocks bootstrap theory developed above, we analyse high-frequency assets prices for four assets. In the analysis we focus on the realized beta estimator based on pre-averaged returns. In particular, we compare the empirical properties of the bootstrap to the existing feasible asymptotic procedure of Christensen et al. (2013). The data is the collection of trades recorded on the NYSE in July 2013, taken from the TAQ database through the Wharton Research Data Services (WRDS) system. This results in 22 distinct trading days. We picked 3 equities at random from the S&P 500 constituents list as of July 1, 2014. They are Microsoft Co. (listed under the ticker symbol (MSFT)), Boeing Co. (BA) and WPX Energy Inc. (CPWR). We then added a 4th element, namely the S&P 500 Depository Receipt (ticker symbol SPY). The SPY is an exchange-traded fund that tracks the large-cap segment of the US stock market. As such, it can be viewed as generating market-wide index returns. For each day, we consider data from the regular exchange opening hours from time stamped between 9:30 a.m. until 4 p.m. Eastern Standard Time. Our procedure for cleaning the data is identical to that used by Barndorff-Nielsen et al. (2011) (for further details see this paper). Table 2 reports some summary statistics of the data (before and after cleaning). As can be seen, these equities display varying degrees of liquidity with MSFT and SPY being the most liquid, while CPWR is the least liquid.

To implement the pre-averaged returns in tick time as given in (25), we select the tuning parameter θ by following the conservative rule ($\theta = 1$, implying that $k_n = \sqrt{n}$). For the bootstrap, to choose the block size b_n, we follow Politis, Romano and Wolf (1999) and use the minimum volatility method (see Appendix A of Hounyo et al. (2013) for details).

We start by analysing the high frequency data. Figure 1 shows time series, autocorrelation and histogram of raw returns as well as of pre-averaged returns for SPY. We observe a pronounced serial correlation in raw returns and in pre-averaged returns. In particular, for raw returns the first autocorrelation is large and negative. This is typical of noisy data and unlikely to arise from a Brownian semimartingale. Note that, the strong autocorrelation observed for pre-averaged returns in Panel D of Figure 1 is due to the fact that we have considered overlapping pre-averaged returns, which rely on many common raw returns. This has nothing to do with the fact that raw returns are possibly noisy. In fact, the correlogram (not reported here) of non-overlapping pre-averaged returns shows that the latter are almost uncorrelated (even for the first lag). The effect of pre-averaging is nicely illustrated by comparing Panel E and F of Figure 1. It appears that pre-averaging helps to reduce price discreteness effect observed in raw returns. At the same time, return distribution is now much closer to being Gaussian. These results are not surprising, it confirms theoretical properties of pre-averaged returns. In particular, under mild conditions on the dynamics of the price process we have that $n^{1/4}\bar{Y}_{\frac{\alpha}{n}}|\mathcal{F}^n_{\frac{(\alpha-1)b_n}{n}} \overset{a}{\sim} N\left(0, \theta\psi_2\sigma^2_{\frac{\alpha}{n}} + \frac{\psi_1}{\theta}\omega^2\right)$. Similar patterns (not reported here) are observed for MSFT, BA and CPWR.

We now turn to the realized beta for MSFT, BA and CPWR. We consider bootstrap percentile intervals, computed at the 95% level. The results are displayed in Figure 2 in terms of daily 95% symmetric confidence intervals for the latent realized beta. Two types of intervals are presented: our proposed wild blocks of blocks bootstrap method and the feasible asymptotic theory-based of Christensen et al. (2013). The pre-averaged Hayashi-Yoshida estimator-based beta estimate is in the center of both confidence intervals by construction. In fact, similar series of confidence intervals for beta was also graphed by Dovonon et al. (2013) in their Figures 1 and 2, except that they used daily log-returns to calculate estimated betas (based on realized covariance) over intervals of one quarter. The emphasis of their paper was to illustrate the usefulness of the bootstrap as a method of inference on beta in a context, where the mechanics of trading is perfect so that there is no market microstructure effects and prices are observed synchronously. In Figure 2, beta is estimated using full record transaction prices. For all stocks considered in the present study, the width of confidence intervals (the bootstrap and the asymptotic theory-based) varies through time. Also, there are a lot of variability in the daily estimate of beta, but all of them lie in the positive region. This means that, these stocks move in the same direction as the market.

As illustrated below, a closer analysis of Figure 2 show that these common patterns observed for

MSFT, BA and CPWR hide different empirical features which allow us to gain valuable insights into the empirical performance of the wild blocks of blocks bootstrap method. For MSFT: the most liquid stock after SPY considered in our analysis, a comparison of the bootstrap intervals with the intervals based on the feasible asymptotic approach of Christensen et al. (2013) suggests that the two types of interval tend to be quite similar. In contrast to MSFT, for the less liquid stock considered here, i.e. CPWR, in most of the cases the confidence intervals for daily beta based on the bootstrap method are usually wider than the confidence intervals using the feasible asymptotic theory. For BA, there is no evidence about the relative empirical performance of the bootstrap and the asymptotic theory-based. These observations lead us to conclude that the degree of liquidity of assets, specifically the non-trading of MSFT, BA or CPWR versus SPY influences the width of confidence intervals, although the conclusion might change for other data sets. Note that, as our Monte Carlo simulations showed, the asymptotic theory-based approach typically have undercoverage problems whereas the bootstrap intervals have coverage rates closer to the desired level. Therefore, if the goal is to control the coverage probability, shorter intervals are not necessarily better.

7 Conclusion

This paper proposes the bootstrap as a method of inference for integrated covariance matrix. We show that the wild blocks of blocks bootstrap studied by Hounyo et al. (2013) can be used to simultaneously handle the presence of dependence, jumps, heterogeneity, irregularly spaced and non-synchronous trading properties of high-frequency data. This combination of properties is unique in the bootstrap literature, so it is worthwhile exploring this bootstrap method in some detail. The bootstrap method is particularly useful because it circumvents the need for an explicit estimator of the asymptotic variance, which has proved difficult in our context.

We provide a set of conditions under which this method is asymptotically valid to first order. We then verify these conditions for various estimators of integrated covolatility. Our Monte Carlo simulations show that the wild blocks of blocks bootstrap improves the finite sample properties of the existing (pre-averaging-based estimator) first order asymptotic theory. Furthermore, an empirical illustration highlights the usefulness of our approach as an alternative method of inference for realized covariation measures and its applicability to real high-frequency data. In future work, we plan to study the higher-order accuracies of this bootstrap method. Another important extension is to provide a theoretical optimal choice of the block size b_n for confidence interval construction.

Appendix A

Tables 1 reports the actual coverage rates for the feasible asymptotic theory approach of Christensen et al. (2013) and for our bootstrap methods, as well as the average lengths of the confidence intervals using the optimal block size by minimizing confidence interval volatility.

Table 1. Summary results for the asymptotic theory and the bootstrap

	$\gamma = 0$ (i.i.d. noise)				$\gamma = -0.5$				$\gamma = -0.9$			
	Coverage rate 95%		Avg. CI length		Coverage rate 95%		Avg. CI length		Coverage rate 95%		Avg. CI length	
	CLT	WBBB	CLT	WBBB	CLT	WBBB	CLT	WBBB	CLT	WBBB	CLT	WBBB
Covariance												
$\xi^2 = 0$												
$\lambda = (3, 6)$	89.19	95.03	0.678	0.872	89.23	94.94	0.664	0.866	89.18	94.91	0.665	0.871
$\lambda = (10, 20)$	83.82	93.82	1.021	2.042	83.82	93.80	1.142	2.043	83.94	93.81	1.144	2.039
$\lambda = (60, 120)$	70.14	87.61	1.482	2.996	70.14	87.52	1.477	2.998	70.10	87.50	1.483	2.992
$\xi^2 = 0.001$												
$\lambda = (3, 6)$	89.21	95.04	0.679	0.874	89.24	94.96	0.666	0.871	89.19	94.91	0.666	0.873
$\lambda = (10, 20)$	83.83	93.84	1.024	2.042	83.84	93.80	1.145	2.043	83.95	93.85	1.145	2.040
$\lambda = (60, 120)$	70.15	87.62	1.484	2.997	70.15	87.52	1.481	2.998	70.12	87.51	1.484	2.995
$\xi^2 = 0.01$												
$\lambda = (3, 6)$	89.22	95.05	0.681	0.874	89.25	94.96	0.667	0.871	89.20	94.92	0.667	0.875
$\lambda = (10, 20)$	83.85	93.85	1.025	2.043	83.85	93.81	1.146	2.043	83.95	93.85	1.146	2.041
$\lambda = (60, 120)$	70.15	87.62	1.485	2.997	70.15	87.52	1.482	2.998	70.13	87.51	1.484	3.001
Regression												
$\xi^2 = 0$												
$\lambda = (3, 6)$	89.09	94.76	0.689	0.972	89.02	94.76	0.699	0.978	88.67	94.78	0.689	0.976
$\lambda = (10, 20)$	83.85	93.78	1.127	2.106	83.83	93.72	1.135	2.111	83.56	93.66	1.132	2.108
$\lambda = (60, 120)$	70.18	87.32	1.485	3.104	70.19	87.50	1.496	3.109	69.38	87.38	1.491	3.107
$\xi^2 = 0.001$												
$\lambda = (3, 6)$	89.12	94.77	0.693	0.977	89.03	94.78	0.701	0.979	88.70	94.78	0.692	0.981
$\lambda = (10, 20)$	83.87	93.79	1.134	2.111	83.84	93.73	1.136	2.114	83.58	93.67	1.134	2.112
$\lambda = (60, 120)$	70.18	87.34	1.491	3.108	70.20	87.51	1.497	3.111	69.41	87.39	1.494	3.108
$\xi^2 = 0.01$												
$\lambda = (3, 6)$	89.12	94.78	0.699	0.981	89.03	94.79	0.702	0.982	88.71	94.79	0.699	0.983
$\lambda = (10, 20)$	83.89	93.79	1.138	2.115	83.85	93.74	1.136	2.115	83.60	93.69	1.137	2.114
$\lambda = (60, 120)$	70.21	87.34	1.496	3.111	70.20	87.52	1.498	3.113	69.42	87.42	1.496	3.109
Correlation												
$\xi^2 = 0$												
$\lambda = (3, 6)$	89.17	94.83	0.762	0.987	88.95	94.86	0.769	0.988	89.04	94.79	0.769	0.991
$\lambda = (10, 20)$	82.93	93.87	1.204	2.148	82.85	93.82	1.212	2.154	82.77	93.82	1.207	2.161
$\lambda = (60, 120)$	69.31	90.87	1.521	3.121	69.29	90.81	1.508	3.121	69.25	90.71	1.506	3.135
$\xi^2 = 0.001$												
$\lambda = (3, 6)$	89.19	94.84	0.765	0.991	88.99	94.86	0.772	0.989	89.06	94.80	0.771	0.992
$\lambda = (10, 20)$	82.94	93.88	1.209	2.150	82.86	93.83	1.215	2.154	82.79	93.83	1.211	2.161
$\lambda = (60, 120)$	69.34	90.88	1.523	3.124	69.31	90.82	1.511	3.123	69.26	90.72	1.507	3.135
$\xi^2 = 0.01$												
$\lambda = (3, 6)$	89.21	94.85	0.771	0.993	89.01	94.87	0.772	0.989	89.06	94.80	0.771	0.992
$\lambda = (10, 20)$	82.95	93.89	1.212	2.152	82.88	93.83	1.215	2.155	82.80	93.83	1.211	2.161
$\lambda = (60, 120)$	69.37	90.88	1.529	3.125	69.32	90.82	1.511	3.123	69.27	90.72	1.508	3.137

Notes: CLT-intervals based on the Normal; BBB-intervals based on the bloks of blocks bootstrap. 10,000 Monte Carlo trials with 999 bootstrap replications each.

Table 2. Descriptive statistics and number of data before and after filtering.

Stock	BA	CPWR	MSFT	SPY
Raw trades	783,150	155,413	3,160,226	5,557,249
Corrected/Abnormal/Zeros	10	26	36	12
Time aggregation	645,249	125,242	2,889,825	5,191,067
# Trades	137,891	30,145	270,365	366,170
Intensity	6,268	1,370	12,289	16,644

Note. This table reports some descriptive statistics and liquidity measures for the selection of stocks included in our empirical application. Raw trades is the total number of data available from these exchanges during the trading session, while # trades is the total sample remaining after filtering the data. Intensity is the average number of data per day.

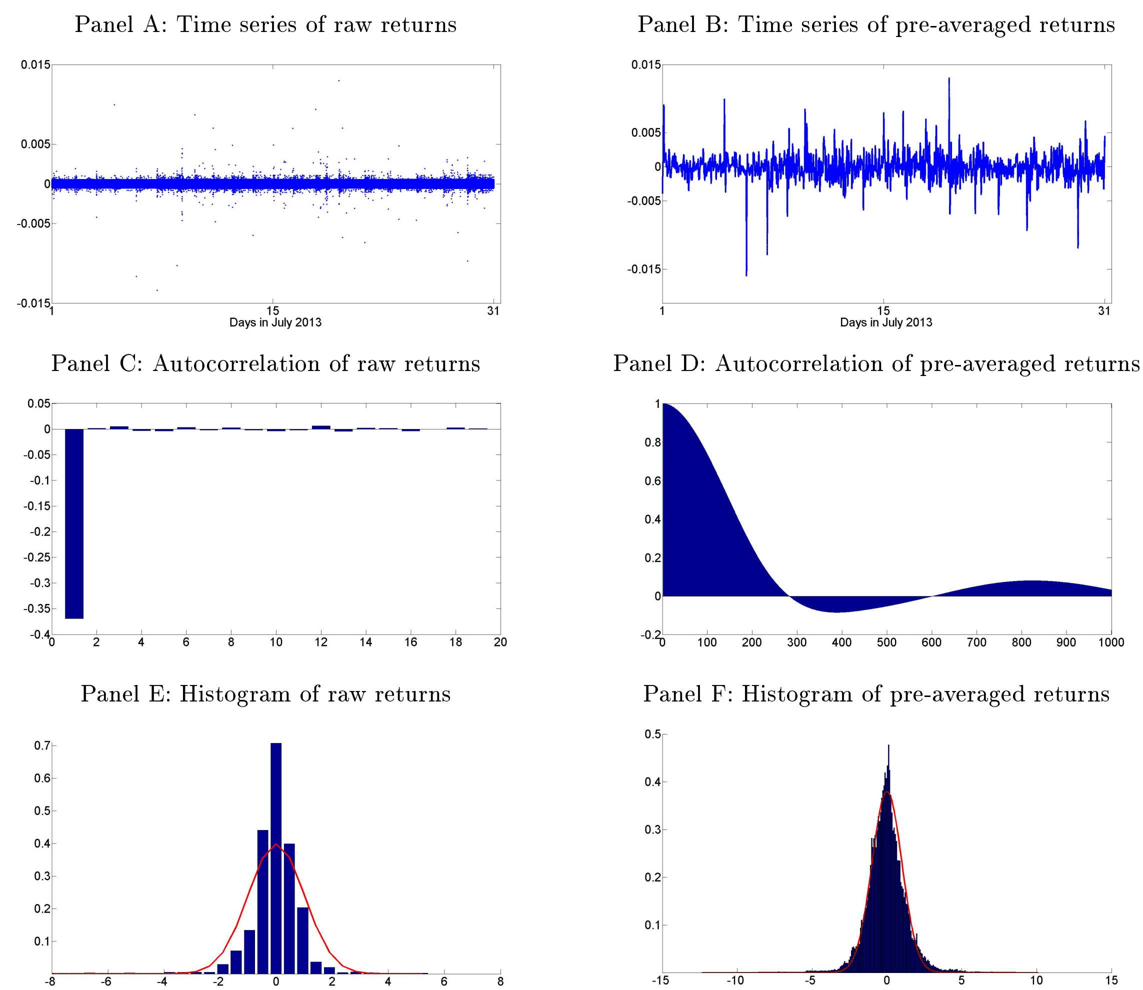

Figure 1: Summary statistics of raw and pre-averaged SPY trade data over regular exchange opening days in July 2013.

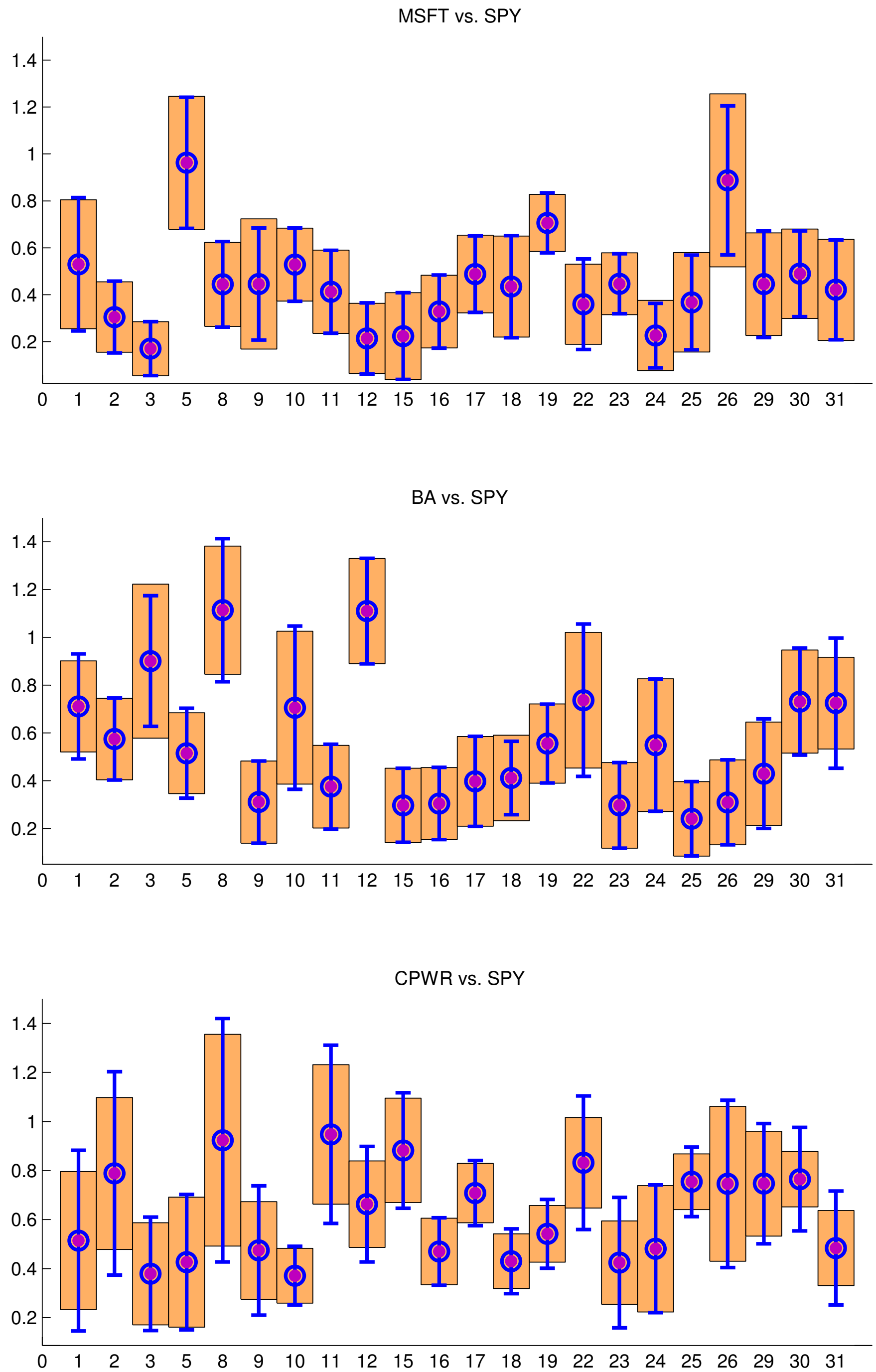

Figure 2: 95% Confidence Intervals (CI's) for the daily pre-averaged Hayashi-Yoshida estimator -based beta estimates, for each regular exchange opening days for BA, CPWR and MSFT in July 2013, calculated using the asymptotic theory of Christensen et al. (2013) (CI's with bars), and the wild blocks of blocks bootstrap method (CI's with lines). The pre-averaged Hayashi-Yoshida estimator -based beta estimate is the middle of all CI's by construction. Days on the x-axis.

Appendix B

Proof of Lemma 3.1 Part a). Given (8) and (9), result follows directly since we can write

$$\begin{aligned} E^*\left(\widehat{\Gamma}^{n*}_{kl}\right) &= \sum_{\alpha=1}^{J_n} E^*\left(\mathcal{Z}^{n*}_{kl}(\alpha)\right) \\ &= \sum_{\alpha=1}^{J_n-1} E^*\left(\mathcal{Z}^{n*}_{kl}(\alpha)\right) + E^*\left(\mathcal{Z}^{n*}_{kl}(J_n)\right) \\ &= \sum_{\alpha=1}^{J_n-1} \left[\mathcal{Z}^{n}_{kl}(\alpha+1) + \left(\mathcal{Z}^{n}_{kl}(\alpha) - \mathcal{Z}^{n}_{kl}(\alpha+1)\right) E^*(\eta_\alpha)\right] + \mathcal{Z}^{n}_{kl}(J_n). \end{aligned}$$

Then, under the condition $E^*(\eta_\alpha) = 1$, we have that

$$\begin{aligned} E^*\left(\widehat{\Gamma}^{n*}_{kl}\right) &= \sum_{\alpha=1}^{J_n} \mathcal{Z}^{n}_{kl}(\alpha) \\ &= \widehat{\Gamma}^{n}_{kl} + \widetilde{b}^{n}_{kl},. \end{aligned}$$

Proof of Lemma 3.1 Part b). Given the definition of $V^{n*}_{kl,k'l'}$, equations (8) and (9) we have that

$$\begin{aligned} V^{n*}_{kl,k'l'} &= \tau_n^2 E^*\left(\left(\mathcal{Z}^{n*}_{kl} - E^*\left(\mathcal{Z}^{n*}_{kl}\right)\right)\left(\mathcal{Z}^{n*}_{k'l'} - E^*\left(\mathcal{Z}^{n*}_{k'l'}\right)\right)\right) \\ &= \tau_n^2 \sum_{\alpha=1}^{J_n-1}\sum_{\alpha'=1}^{J_n-1} E^*\left(\left(\mathcal{Z}^{n*}_{kl}(\alpha) - E^*\left(\mathcal{Z}^{n*}_{kl}(\alpha)\right)\right)\left(\mathcal{Z}^{n*}_{k'l'}\left(\alpha'\right) - E^*\left(\mathcal{Z}^{n*}_{k'l'}\left(\alpha'\right)\right)\right)\right) \\ &= \tau_n^2 \sum_{\alpha=1}^{J_n-1}\sum_{\alpha'=1}^{J_n-1} \left(\left(\mathcal{Z}^{n}_{kl}(\alpha) - \mathcal{Z}^{n}_{kl}(\alpha+1)\right)\right)\left(\mathcal{Z}^{n}_{k'l'}\left(\alpha'\right) - \mathcal{Z}^{n}_{k'l'}\left(\alpha'+1\right)\right) Cov^*\left(\eta_\alpha, \eta_{\alpha'}\right). \end{aligned}$$

Using the fact that $\eta_\alpha \sim$ i.i.d., result follows, then we get

$$\begin{aligned} V^{n*}_{kl,k'l'} &= 2Var^*(\eta)\frac{\tau_n^2}{2}\sum_{\alpha=1}^{J_n-1}\left(\mathcal{Z}^{n}_{kl}(\alpha) - \mathcal{Z}^{n}_{kl}(\alpha+1)\right)\left(\mathcal{Z}^{n}_{k'l'}(\alpha) - \mathcal{Z}^{n}_{k'l'}(\alpha+1)\right) \\ &= 2Var^*(\eta)\, V^{n}_{kl,k'l'}. \end{aligned}$$

Proof of Theorem 3.1 Part a). Result follows directly given part b) of Lemma 3.1 and Condition A1.

Proof of Theorem 3.1 Part b). Let $\widehat{\Gamma}^{n*}_{kl}(\alpha) \equiv \left(\mathcal{Z}^{n*}_{kl}(\alpha)\right)_{1\le k,l\le d}$, where $\mathcal{Z}^{n*}_{kl}(\alpha)$ is defined in (8), and let $x^*_\alpha \equiv vec\left(\widehat{\Gamma}^{n*}_{kl}(\alpha)\right)$. We have that $S^{n*} \equiv n^{1/4}\left(vec\left(\widehat{\Gamma}^{n*}\right) - E^*\left(vec\left(\widehat{\Gamma}^{n*}\right)\right)\right) = \tau_n \sum_{\alpha=1}^{J_n}\left(x^*_\alpha - E^*\left(x^*_\alpha\right)\right)$. The proof follows from showing that for any $\lambda \in \mathbb{R}^{d^2}$ such that $\lambda'\lambda = 1$, $\sup_{x\in\mathbb{R}}|P^*(\sum_{\alpha=1}^{J_n}\tilde{x}^*_\alpha \le x) - \Phi(x/\left(\lambda'\tilde{V}\lambda\right))| \xrightarrow{P} 0$, where $\tilde{x}^*_\alpha = \tau_n\lambda'(x^*_\alpha - E^*(x^*_\alpha))$, and $\tilde{V} = \left(\tilde{V}_{kl}\right)_{1\le k,l\le d^2}$ is a $d^2 \times d^2$ matrix, whose generic element $\tilde{V}_{kl}$ is given by

$$\tilde{V}_{kl} = V_{k-d\lfloor (k-1)/d\rfloor,\lfloor (k-1)/d\rfloor+1,l-d\lfloor (l-1)/d\rfloor,\lfloor (l-1)/d\rfloor+1},$$

with $1 \le k,l \le d^2$. Clearly, $E^*\left(\sum_{\alpha=1}^{J_n}\tilde{x}^*_\alpha\right) = 0$ and $Var^*\left(\sum_{\alpha=1}^{J_n}\tilde{x}^*_\alpha\right) = \lambda\tilde{V}^{n*}\lambda \xrightarrow{P} \lambda'\tilde{V}\lambda$ by part a). Thus, by Katz's (1963) Berry-Essen Bound, for some small $\varepsilon > 0$ and some constant $K > 0$ which changes from line to line, $\sup_{x\in\mathbb{R}}\left|P^*\left(\sum_{\alpha=1}^{J_n}\tilde{x}_\alpha^*\leq x\right)-\Phi(x/\left(\lambda' LVL'\lambda\right))\right|\leq K\sum_{\alpha=1}^{J_n}E^*|\tilde{x}_\alpha^*|^{2+\epsilon}$. Next, we show that $\sum_{\alpha=1}^{J_n}E^*|\tilde{x}_\alpha^*|^{2+\varepsilon}=o_p(1)$. We have that

$$\begin{aligned}\sum_{\alpha=1}^{J_n}E^*|\tilde{x}_\alpha^*|^{2+\varepsilon} &= \sum_{\alpha=1}^{J_n}E^*\left|\tau_n\lambda'(x_\alpha^*-E^*\left(x_\alpha^*\right))\right|^{2+\varepsilon}\\ &\leq 2^{2+\epsilon}\tau_n^{2+\varepsilon}\sum_{\alpha=1}^{J_n}E^*\left|\lambda' x_\alpha^*\right|^{2+\varepsilon}\\ &\leq 2^{2+\epsilon}\tau_n^{2+\varepsilon}\sum_{\alpha=1}^{J_n}E^*\left|x_\alpha^*\right|^{2+\varepsilon}\\ &\leq K\tau_n^{2+\varepsilon}E^*\left|\eta_1\right|^{2+\varepsilon}\sum_{\alpha=1}^{J_n}\left|x_\alpha\right|^{2+\varepsilon},\end{aligned}$$

where the first inequality follows from the C_r and the Jensen inequalities; the second inequality uses the Cauchy-Schwarz inequality and the fact that $\lambda'\lambda=1$; and the third inequality follows from the C_r and the Jensen inequalities. We let $|z|^2=(z'z)$ for any vector z. It follows that

$$\begin{aligned}\sum_{\alpha=1}^{J_n}E^*|\tilde{x}_\alpha^*|^{2+\varepsilon} &\leq K\tau_n^{2+\epsilon}E^*\left|\eta_1\right|^{2+\varepsilon}\sum_{\alpha=1}^{J_n}\left|x_\alpha\right|^{2(1+\varepsilon/2)}\\ &\leq K\tau_n^{2+\varepsilon}E^*\left|\eta_1\right|^{2+\varepsilon}\sum_{\alpha=1}^{J_n}\left(\sum_{k=1}^{d}\sum_{l=1}^{d}\left(\mathcal{Z}_{kl}^n\left(\alpha\right)\right)^2\right)^{1+\varepsilon/2}\\ &\leq \underbrace{KE^*\left|\eta_1\right|^{2+\varepsilon}}_{=O(1)}\underbrace{\tau_n^{2+\varepsilon}\left(\frac{b_n}{n}\right)^{1+\varepsilon}}_{=o(1)}\underbrace{\left(\frac{n}{b_n}\right)^{1+\varepsilon}\sum_{\alpha=1}^{J_n}\left|\mathcal{Z}_{kl}^n\left(\alpha\right)\right|^{2+\varepsilon}}_{=O_P(1)}=o_P\left(1\right).\end{aligned}$$

where consistency follows since for any $\varepsilon>0$, $E^*\left|\eta_\alpha\right|^{2+\varepsilon}\leq\Delta<\infty$, and by using Conditions A.2. and A.3.

Proof of Theorem 3.2. **Parts a) and b)**. Since S^n converges stably in distribution to $N(0,\tilde{V})$, by an application of the delta method (see Podolskij and Vetter (2010, Proposition 2.5(iii))),

$$S_h^n\to^{st}N\left(0,\nabla' h\left(vec\left(\int_0^1\Sigma_s ds\right)\right)\tilde{V}\nabla h\left(vec\left(\int_0^1\Sigma_s ds\right)\right)\right).$$

Similarly, by a mean value expansion, and conditionally on the original sample,

$$S_h^{n*}=\tau_n\nabla' h\left(vec(\widehat{\Gamma}^{n*})\right)\left(vec(\widehat{\Gamma}^{n*})-vec\left(\widehat{\Gamma}^n\right)\right)+o_{P^*}(1),$$

since $\widehat{\Gamma}_{kl}^{n*}-\widehat{\Gamma}_{kl}^n\to^{P^*}0$ in probability. It follows that

$$S_h^{n*}\to^{st}N\left(0,\nabla' h\left(vec\left(\int_0^1\Sigma_s ds\right)\right)\tilde{V}\nabla h\left(vec\left(\int_0^1\Sigma_s ds\right)\right)\right)$$

in probability, given Theorem **3.1.** The result follows from Polya's theorem (see, e.g., Serfling (1980)), given that the normal distribution is continuous.

Auxilliary Lemmas

As in Jacod et al. (2009), we assume in the following that the processes a, σ and X are bounded processes satisfying (1) with a and σ adapted càdlàg processes. As Jacod et al. (2009) explain, this assumption simplifies the mathematical derivations without loss of generality (by a standard localization procedure detailed in Jacod (2008)). Formally, we derive our results under the following assumption.

Assumption 4. X satisfies equation (2) with a and σ adapted càdlàg processes such that a, σ, and X are bounded processes (implying that α is also bounded).

Notation

We introduce the following additional notation associated with the pre-averaged weighting function g. Let

$$\begin{aligned}\phi_1(s) &= \int_s^1 g'(u)\, g'(u-s)\, du,\ \phi_2(s) = \int_s^1 g(u)\, g(u-s)\, du,\ \Phi_{ij} = \int_0^1 \phi_i(s)\, \phi_j(s)\, ds, \\ \text{and for } i &= 1,2,\ \psi_i = \phi_i(0).\end{aligned}$$

We also let

$$\begin{aligned}\Lambda_{kl,k'l'}(s) &= \Sigma_{kk'}(s)\, \Sigma_{ll'}(s) + \Sigma_{kl'}(s)\, \Sigma_{lk'}(s) \\ \Theta_{kl,k'l'}(s) &= \Sigma_{kk'}(s)\, \Psi_{ll'}(s) + \Sigma_{kl'}(s)\, \Psi_{k'l}(s) + \Sigma_{kl'}(s)\, \Psi_{kl'}(s) + \Sigma_{ll'}(s)\, \Psi_{kk'}(s) \\ \Upsilon_{kl,k'l'} &= \Psi_{kk'}(s)\, \Psi_{ll'}(s) + \Psi_{kl'}(s)\, \Psi_{lk'}(s).\end{aligned}$$

Lemma 7.1. *Suppose (2) and Assumptions 1-4 hold. Furthermore suppose that $\widehat{\Gamma}^n$ is given by (26) as well and let $1/2 < \delta_2 < 2/3$. Then we have*

$$V^n_{kl,k'l'} = \frac{\tau_n^2}{2} \sum_{\alpha=1}^{J_n-1} \left(\mathcal{Z}^n_{kl}(\alpha) - \mathcal{Z}^n_{kl}(\alpha+1)\right)\left(\mathcal{Z}^n_{k'l'}(\alpha) - \mathcal{Z}^n_{k'l'}(\alpha+1)\right) \to^P V_{kl,k'l'}.$$

where $\tau_n = n^{1/4}$

$$\mathcal{Z}^n_{kl}(\alpha) = \frac{1}{\psi_2 k_n} \sum_{i=1}^{b_n} \bar{Y}^k_{\frac{i-1+(\alpha-1)b_n}{n}}\, \bar{Y}^l_{\frac{i-1+(\alpha-1)b_n}{n}} \text{ and}$$

$$V_{kl,k'l'} = \frac{2}{\psi_2^2} \Phi_{22}\theta \int_0^1 \left(\Lambda_{kl,k'l'}(s)\, ds + \frac{2\Phi_{12}}{\theta} \int_0^1 \Theta_{kl,k'l'}(s)\, ds + \frac{\Phi_{11}}{\theta^3} \Upsilon_{kl,k'l'}\right) \equiv \int_0^1 \varsigma(s)\, ds. \tag{39}$$

Lemma 7.2. *Suppose (2) and Assumptions 1-4 hold. Furthermore suppose that $\widehat{\Gamma}^n = \left(\widehat{\Gamma}^n_{kl}\right)_{1\le k,l\le d}$, where $\widehat{\Gamma}^n_{kl}$ is given by (28) as well and let $1/2 < \delta_2 < 2/3$. Then we have*

$$V^n_{kl,k'l'} = \frac{\tau_n^2}{2} \sum_{\alpha=1}^{J_n-1} \left(\mathcal{Z}^n_{kl}(\alpha) - \mathcal{Z}^n_{kl}(\alpha+1)\right)\left(\mathcal{Z}^n_{k'l'}(\alpha) - \mathcal{Z}^n_{k'l'}(\alpha+1)\right) \to^P V_{kl,k'l'}.$$

where $\tau_n = n^{1/4}$

$$\mathcal{Z}^n_{kl}(\alpha) = \sum_{t^k_i \in B_n(\alpha)} \sum_{j=0}^{n_l - k_n + 1} \bar{Y}^k_{t^k_i} \bar{Y}^l_{t^l_j} 1_{A^{kl}_{ij}}, \tag{40}$$

and $V_{kl,k'l'}$ *is given in Theorem 3.4 of Christensen et al. (2013).*

Proof of Lemma 7.1. The proof follows closely that for Theorem 4.1 of Christensen et al. (2013), however for completeness, we present the relevant details. Given the definition of $V^n_{kl,k'l'}$ and $\mathcal{Z}^n_{kl}(\alpha)$, after adding and substracting appropriately, we can write

$$\begin{aligned} V^n_{kl,k'l'} &= \frac{\sqrt{n}}{2}\left(\sum_{\alpha=1}^{J_n-1} 2\mathcal{Z}^n_{kl}(\alpha)\mathcal{Z}^n_{k'l'}(\alpha) - \left(\sum_{\alpha=1}^{J_n-1} \mathcal{Z}^n_{kl}(\alpha)\mathcal{Z}^n_{k'l'}(\alpha+1) + \sum_{\alpha=1}^{J_n-1}\mathcal{Z}^n_{kl}(\alpha+1)\mathcal{Z}^n_{k'l'}(\alpha)\right)\right) \\ &\quad + \frac{\sqrt{n}}{2}\left(\mathcal{Z}^n_{kl}(J_n)\mathcal{Z}^n_{k'l'}(J_n) - \mathcal{Z}^n_{kl}(1)\mathcal{Z}^n_{k'l'}(1)\right) \\ &= L^n_{kl,k'l'} + R^n_{kl,k'l'}. \end{aligned}$$

where the remainder term is

$$\begin{aligned} R^n_{kl,k'l'} &= \frac{\sqrt{n}}{2}\left(\mathcal{Z}^n_{kl}(J_n)\mathcal{Z}^n_{k'l'}(J_n) - \mathcal{Z}^n_{kl}(1)\mathcal{Z}^n_{k'l'}(1)\right) \\ &= O_P\left(n^{-\frac{3}{2}} b_n^2\right) = O_P\left(\left(\frac{b_n}{n^{3/4}}\right)^2\right) \\ &= o_P(1), \end{aligned}$$

so long as $\delta_2 < 3/4$, where we used the definitions of $\mathcal{Z}^n_{kl}(\alpha) = \frac{1}{\psi_2 k_n}\sum_{i=1}^{b_n} \bar{Y}^k_{\frac{i-1+(\alpha-1)b_n}{n}} \bar{Y}^l_{\frac{i-1+(\alpha-1)b_n}{n}}$, the Cauchy-Schwartz inequality, the fact that under Assumption 4 for some $q > 0$, $E\left(\left|\bar{Y}^k_{\frac{i}{n}}\right|^q\right) \leq Kn^{-q/4}$ uniformly in i (cf. Lemma 6.2 of Christensen et al. (2013)). Next we show that the leading term is such that

$$p\lim_{n\to\infty} L^n_{kl,k'l'} = V_{kl,k'l'}, \text{ for } 1 \leq k, k'l,' l' \leq d. \tag{41}$$

It is obviously enough to prove the result for the unsymmetrized estimator

$$\widetilde{L}^n_{kl,k'l'} = \sqrt{n}\sum_{\alpha=1}^{J_n-1}\left(\mathcal{Z}^n_{kl}(\alpha)\mathcal{Z}^n_{k'l'}(\alpha) - \mathcal{Z}^n_{kl}(\alpha)\mathcal{Z}^n_{k'l'}(\alpha+1)\right).$$

Next, we introduce two approximating version of $\bar{B}(l,r)_j$ first, namely

$$\widetilde{\mathcal{Z}}^n_{kl}(\alpha) = \frac{1}{\psi_2 k_n}\sum_{i=1}^{b_n} \widetilde{Y}^k_{\frac{i-1+(\alpha-1)b_n}{n}} \widetilde{Y}^l_{\frac{i-1+(\alpha-1)b_n}{n}},$$

$$\widehat{\mathcal{Z}}^n_{kl}(\alpha) = \frac{1}{\psi_2 k_n}\sum_{i=1}^{b_n} \widetilde{Y}^k_{\frac{i-1+\alpha b_n}{n}} \widetilde{Y}^l_{\frac{i-1+\alpha b_n}{n}},$$

where we have set $\widetilde{Y}^k{}_{\frac{i}{n}} = \bar{\epsilon}_{\frac{i}{n}} + \sum_{\nu=1}^{d} \sigma^{k\nu}_{\frac{(\alpha-1)b_n}{n}} \overline{W^\nu_{\frac{i}{n}}}^k$, for $\frac{(\alpha-1)b_n}{n} \leq \frac{i}{n} < \frac{\alpha b_n}{n}$. Indeed we will show that the error due to replacing $\bar{Y}^k_{\frac{i}{n}}$ by $\widetilde{Y}^k{}_{\frac{i}{n}}$ is small and will not affect our theoretical results, since σ is assumed to be an Ito semimartingale itself. We have that, for $\frac{(\alpha-1)b_n}{n} \leq \frac{i}{n} < \frac{\alpha b_n}{n}$

$$\begin{aligned} E\left(\left|\bar{Y}^k_{\frac{i}{n}} - \widetilde{Y}^k_{\frac{i}{n}}\right|\right) &= E\left|\sum_{j=1}^{k_n} g\left(\frac{j}{k_n}\right)\int_{\frac{i+j-1}{n}}^{\frac{i+j}{n}} a^k_s ds + \sum_{j=1}^{k_n} g\left(\frac{j}{k_n}\right)\int_{\frac{i+j-1}{n}}^{\frac{i+j}{n}} \left(\sigma^{k\nu}_s - \sigma^{k\nu}_{\frac{(\alpha-1)b_n}{n}}\right) dW^\nu_s\right| \\ &\leq K\left(\frac{k_n}{n} + \left(\sum_{j=1}^{k_n} g^2\left(\frac{j}{k_n}\right)\sum_{\nu=1}^{d} E\left|\int_{\frac{i+j-1}{n}}^{\frac{i+j}{n}} \left(\sigma^{k\nu}_s - \sigma^{k\nu}_{\frac{(\alpha-1)b_n}{n}}\right) dW^\nu_s\right|^2\right)^{1/2}\right) \\ &\leq K\left(\frac{k_n}{n} + \left(\frac{k_n}{n}\frac{b_n}{n}\right)^{1/2}\right) \leq K\frac{(k_n b_n)^{1/2}}{n}. \end{aligned}$$

Note also that $E\left(\left|\mathcal{Z}^n_{kl}(\alpha)\right|\right) \leq K\frac{b_n}{n}$, thus it follows that

$$\begin{aligned} E\left(\left|\mathcal{Z}^n_{kl}(\alpha) - \widetilde{\mathcal{Z}}^n_{kl}(\alpha)\right|\right) &\leq K b_n\left(\frac{(k_n b_n)^{1/2}}{n}\left(\frac{1}{\sqrt{k_n}}\right)^{\frac{(l+r)}{4}-1}\right) \\ &\leq K\left(\frac{b_n}{n}\right)^{3/2}, \end{aligned}$$

similarly for $\widehat{\mathcal{Z}}^n_{kl}(\alpha)$, we have $E\left(\left|\mathcal{Z}^n_{kl}(\alpha) - \widehat{\mathcal{Z}}^n_{kl}(\alpha)\right|\right) \leq K\left(\frac{b_n}{n}\right)^{3/2}$. So by using the fact that $\delta < 2/3$ we obtain $\widetilde{L}^n_{kl,k'l'} - \widehat{L}^n_{kl,k'l'} = o_P(1)$, where

$$\widehat{L}^n_{kl,k'l'} = \sqrt{n}\sum_{j=1}^{J_n-1}\left(\widehat{\mathcal{Z}}^n_{kl}(\alpha)\widehat{\mathcal{Z}}^n_{k'l'}(\alpha) - \widehat{\mathcal{Z}}^n_{kl}(\alpha)\widetilde{\mathcal{Z}}^n_{k'l'}(\alpha+1)\right).$$

Then it is simple to deduce that

$$\begin{aligned} \sqrt{n}\left|\sum_{\alpha=1}^{J_n-1} E\left(\widehat{\mathcal{Z}}^n_{kl}(\alpha)\widehat{\mathcal{Z}}^n_{k'l'}(\alpha) - E\left(\widehat{\mathcal{Z}}^n_{kl}(\alpha)\widehat{\mathcal{Z}}^n_{k'l'}(\alpha)\,|\mathcal{F}^n_{\frac{(\alpha-1)b_n}{n}}\right)\right)\right| &\leq K\frac{b_n^{3/2}}{n}, \\ \sqrt{n}\left|\sum_{\alpha=1}^{J_n-1}\left(\widehat{\mathcal{Z}}^n_{kl}(\alpha)\widetilde{\mathcal{Z}}^n_{k'l'}(\alpha+1) - E\left(\widehat{\mathcal{Z}}^n_{kl}(\alpha)\widetilde{\mathcal{Z}}^n_{k'l'}(\alpha+1)\,|\mathcal{F}^n_{\frac{(\alpha-1)b_n}{n}}\right)\right)\right| &\leq K\frac{b_n^{3/2}}{n}, \end{aligned}$$

by conditional independence, and now we are left with

$$\widehat{L}^n_{kl,k'l'} = \sqrt{n}\sum_{\alpha=1}^{J_n-1}\left(\widehat{\mathcal{Z}}^n_{kl}(\alpha)\widetilde{\mathcal{Z}}^n_{k'l'}(\alpha) - \widehat{\mathcal{Z}}^n_{kl}(\alpha)\widetilde{\mathcal{Z}}^n_{k'l'}(\alpha+1)\,|\mathcal{F}^n_{\frac{(\alpha-1)b_n}{n}}\right) + o_P(1).$$

From the same arguments as in Podolskij and Vetter (2010) and using $\delta_2 > 1/2$, we obtain

$$\begin{aligned} &\sqrt{n}\left(\widehat{\mathcal{Z}}^n_{kl}(\alpha)\widetilde{\mathcal{Z}}^n_{k'l'}(\alpha) - \widehat{\mathcal{Z}}^n_{kl}(\alpha)\widetilde{\mathcal{Z}}^n_{k'l'}(\alpha+1)\,|\mathcal{F}^n_{\frac{(\alpha-1)b_n}{n}}\right) \\ &= \int_{\frac{(\alpha-1)b_n}{n}}^{\frac{\alpha b_n}{n}} \varsigma(s)\,ds + o\left(\frac{b_n}{n}\right), \end{aligned}$$

uniformly in α, where we use $V^n_{kl,k'l'} = \int_0^1 \varsigma(s)\,ds$ with the process ς given by the right hand side of

(39) thus we have

$$\widehat{L}^n_{kl,k'l'} = \int_0^1 \varsigma(s)\,ds + o_P(1)$$

and the proof is complete.

Proof of Lemma 7.2. Given the definitions of $V^n_{kl,k'l'}$ and $\mathcal{Z}^n_{kl}(\alpha)$, after adding and substracting appropriately, we get that

$$\begin{aligned} V^n_{kl,k'l'} &= \frac{\sqrt{n}}{2}\left(\sum_{\alpha=1}^{J_n-1}\left(2\mathcal{Z}^n_{kl}(\alpha)\,\mathcal{Z}^n_{k'l'}(\alpha) - \mathcal{Z}^n_{kl}(\alpha)\,\mathcal{Z}^n_{k'l'}(\alpha+1) - \mathcal{Z}^n_{kl}(\alpha+1)\,\mathcal{Z}^n_{k'l'}(\alpha)\right)\right) \\ &\quad + \frac{\sqrt{n}}{2}\left(\mathcal{Z}^n_{kl}(J_n)\,\mathcal{Z}^n_{k'l'}(J_n) - \mathcal{Z}^n_{kl}(1)\,\mathcal{Z}^n_{k'l'}(1)\right) \\ &\equiv L^n_{kl,k'l'} + R^n_{kl,k'l'}. \end{aligned}$$

where the remainder term is

$$\begin{aligned} R^n_{kl,k'l'} &= \frac{\sqrt{n}}{2}\left(\mathcal{Z}^n_{kl}(J_n)\,\mathcal{Z}^n_{k'l'}(J_n) - \mathcal{Z}^n_{kl}(1)\,\mathcal{Z}^n_{k'l'}(1)\right) \\ &= O_P\left(n^{-\frac{3}{2}}b_n^2\right) = O_P\left(\left(\frac{b_n}{n^{3/4}}\right)^2\right) = o_P(1), \end{aligned}$$

so long as $\delta_2 < 3/4$, where we used the definitions of $\mathcal{Z}^n_{kl}(\alpha) = \sum_{t_i^k \in B_n(\alpha)} \sum_{j=0}^{n_l-k_n+1} \bar{Y}^k_{t_i^k}\bar{Y}^l_{t_j^l} 1_{A^{kl}_{ij}}$, the Cauchy-Schwartz inequality, the fact that for some $q > 0$, $E\left(\left|\bar{Y}^k_{\frac{i}{n}}\right|^q\right) \leq K n^{-q/4}$ uniformly in i (cf. Lemma 6.2 of Christensen et al. (2013)). Thus result follows since $L^n_{kl,k'l'}$ is exactly the consistent estimator of $V_{kl,k'l'}$ proposed by Christensen et al. (2013) (cf. Theorem 4.1).

References

[1] Andersen, T.G., T. Bollerslev, F.X. Diebold and P. Labys (2003). "Modeling and forecasting realized volatility," *Econometrica*, 71, 529-626.

[2] Aït-Sahalia, Y., J. Fan, and D. Xiu (2010). "High frequency covariance estimates with noisy and asynchronous financial data," *Journal of the American Statistical Association*, 105, 1504-1517.

[3] Bandi, F., and J. Russell (2006). "Separating microstructure noise from volatility," *Journal of Financial Economics*, 79(3), 655-692.

[4] Bandi, F., and J. Russell (2011). "Market microstructure noise, integrated variance estimators, and the accuracy of asymptotic approximations," *Journal of Econometrics*, 160, 145-159.

[5] Barndorff-Nielsen, O., S. E. Graversen, J. Jacod, M. Podolskij, and N. Shephard, (2006). "A central limit theorem for realised power and bipower variations of continuous semimartingales." In Y. Kabanov, R. Lipster, and J. Stoyanov (Eds.), From Stochastic Analysis to Mathematical Finance, Festschrift for Albert Shiryaev, 33 - 68. Springer.

[6] Barndorff-Nielsen, O., P. Hansen, A. Lunde, and N. Shephard (2008). "Designing realised kernels to measure the ex-post variation of equity prices in the presence of noise," *Econometrica*, 76, 1481-1536.

[7] Barndorff-Nielsen, O., P. Hansen, A. Lunde, and N. Shephard (2011). "Multivariate realised kernels: consistent positive semi-definite estimators of the covariation of equity prices with noise and non-synchronous trading," *Journal of Econometrics*, 162, 149-169.

[8] Barndorff-Nielsen, O. and N. Shephard, (2004a). "Econometric analysis of realised covariation: high frequency based covariance, regression and correlation in financial economics," *Econometrica*, 72, 885-925.

[9] Barndorff-Nielsen, O.E. and N. Shephard (2004b). "Measuring the Impact of Jumps in Multivariate Price Processes Using Bipower Covariation," *Working paper, Oxford University.*

[10] Barndorff-Nielsen, O. and N. Shephard, (2006). "Econometrics of testing for jumps in financial economics using bipower variation," *Journal of Financial Econometrics*, 4, 1-30.

[11] Bibinger, M. (2011). "Efficient covariance estimation for asynchronous noisy high-frequency data," *Scandinavian Journal of Statistics*, 38, 23-45.

[12] Bibinger, M., N. Hautsch, P. Malec, and M. Reis (2014). "Estimating the quadratic covariation matrix from noisy observations: local method of moments and efficiency," *Annals of Statistics*, Forthcoming.

[13] Bollerslev, T. and V. Todorov, (2010). "Jumps and betas: a new theoretical framework for disentangling and estimating systematic risks," *Journal of Econometrics*, 157, 220-235.

[14] Boudt, K., C. Croux, and S. Laurent (2011)."Outlyingness weighted quadratic covariation," *Journal of Financial Econometrics*, 9(4), 657-684.

[15] Bühlmann, P. and H. R. Künsch (1995): "The blockwise bootstrap for general parameters of a stationary time series," *Scandinavian Journal of Statistics*, 22(1), 35-54.

[16] Christensen, K., S. Kinnebrock, and M. Podolskij (2010). "Pre-averaging estimators of the ex-post covariance matrix in noisy diffusion models with non-synchronous data," *Journal of Econometrics*, 159, 116-133.

[17] Christensen, K., M. Podolskij, and M. Vetter (2013). "On covariation estimation for multivariate continuous Ito semimartingales with noise in non-synchronous observation schemes," *Journal of Multivariate Analysis*, 120, 59-84.

[18] Corsi, F., S. Peluso, and F. Audrino (2014). "Missing asynchronicity: a Kalman-EM approach to multivariate realized covariance estimation,"*Journal of Applied Econometrics*, Forthcoming.

[19] Delbaen, F., and W. Schachermayer, (1994). "A general version of the fundamental theorem of asset pricing," *Mathematische Annalen*, 300, 463-520.

[20] Diebold, F.X. and Strasser, G.H. (2013). "On the correlation structure of microstructure noise: a financial economic approach," *Review of Economics Studies*, 80, 1304-1337.

[21] Dovonon, P., Gonçalves, S., Hounyo, U. and N. Meddahi (2014). "Bootstrapping high-frequency jump tests," manuscript.

[22] Dovonon, P., Gonçalves, S. and N. Meddahi, (2013). "Bootstrapping realized multivariate volatility measures," *Journal of Econometrics*, 172, 49-65.

[23] Epps, T. W., (1979). "Comovements in stock prices in the very short run," *Journal of the American Statistical Association*, 74(366), 291-298.

[24] Gonçalves, S. and N. Meddahi (2009). "Bootstrapping realized volatility," *Econometrica*, 77(1), 283-306.

[25] Gonçalves, S., Hounyo, U. and N. Meddahi (2014). "Bootstrap inference for pre-averaged realized volatility based on non-overlapping returns," to appear in Journal of Financial Econometrics.

[26] Gonçalves, S., and H. White (2002). "The bootstrap of the mean for dependent heterogeneous arrays," *Econometric Theory*, 18, 1367-1384.

[27] Hansen, P.R. and A. Lunde, (2006). "Realized variance and market microstructure noise," *Journal of Business and Economics Statistics*, 24, 127-161.

[28] Hasbrouck, J. (2007). "Empirical Market Microstructure," *Oxford University Press.*

[29] Hautsch N., and Podolskij, M., (2013). "Pre-averaging based estimation of quadratic variation in the presence of noise and jumps: Theory, Implementation, and Empirical Evidence," *Journal of Business and Economic Statistics*, 31(2), 165-183.

[30] Hayashi, T., and N. Yoshida, (2005). "On covariance estimation of non-synchronously observed diffusion processes," *Bernoulli*, 11, 359-379.

[31] Hounyo, U. (2013). "Bootstrapping realized volatility and realized beta under a local Gaussianity assumption," Research paper 2013-30, CREATES, Aarhus University.

[32] Hounyo, U. , Gonçalves, S., and N. Meddahi (2013). "Bootstrapping pre-averaged realized volatility under market microstructure noise," Research paper 2013-28, CREATES, Aarhus University.

[33] Huang, X. and G. Tauchen (2005). "The relative contribution of jumps to total price variance," *Journal of Financial Econometrics*, 3(4), 456-499.

[34] Jacod, J. (2008). "Asymptotic properties of realized power variations and related functionals of semimartingales," *Stochastic Processes and Their Applications*, 118, 517-559.

[35] Jacod, J., Y. Li, P. Mykland, M. Podolskij, and M. Vetter (2009). "Microstructure noise in the continuous case: the pre-averaging approach," *Stochastic Processes and Their Applications*, 119, 2249-2276.

[36] Jacod, J. and A.N. Shiryaev (2003). "Limit Theorems for Stochastic Processes,"*2nd ed. Springer-Verlag, Berlin.*

[37] Jacod, J. and V. Todorov, (2009). "Testing for Common Arrivals of Jumps for Discretely Observed Multi-dimensional Processes," *Annals of Statistics*, 37, 1792-1838.

[38] Kalnina, I. (2011). "Subsampling high frequency data," *Journal of Econometrics*, 161(2), 262-283.

[39] Katz, M.L., (1963). "Note on the Berry-Esseen theorem," *Annals of Mathematical Statistics*, 34, 1107-1108.

[40] Künsch, H.R. (1989). "The jackknife and the bootstrap for general stationary observations," *Annals of Statistics* 17, 1217-1241.

[41] Liu, R.Y. (1988). "Bootstrap procedure under some non-i.i.d. models," *Annals of Statistics* 16, 1696-1708.

[42] Liu, C. and C. Y. Tang (2014). "A quasi-maximum likelihood approach to covariance matrix with high frequency data," *Journal of Econometrics. Forthcoming.*

[43] Mancini, C. and F. Gobbi (2012). "Identifying the Brownian covariation from the co-jumps given discrete observations," *Econometric Theory*, 28, 249–273.

[44] Mykland, P. (2012). "A Gaussian calculus for inference from high frequency data," *Annals of Finance*, 8 235-258.

[45] Mykland, P.A. and L. Zhang (2009). "Inference for continous semimartingales observed at high frequency," *Econometrica*, 77, 1403-1455.

[46] Park, S., and O. Linton (2012). "Estimating the quadratic covariation matrix for an asynchronously observed continuous time signal masked by additive noise," *FMG Discussion Papers* 703.

[47] Podolskij, M., and M. Vetter (2009). "Estimation of volatility functionals in the simultaneous presence of microstructure noise and jumps," *Bernoulli*, 15(3), 634-658.

[48] Podolskij, M., and M. Vetter (2010). "Understanding limit theorems for semimartingales: a short survey," *Statistica Neerlandica*, 64, 329-351.

[49] Politis, D. N. and Romano, J. P. (1992). "A general resampling scheme for triangular arrays of α-mixing random variables," *Annals of Statistics*, 20, 1985-2007.

[50] Politis, D.N., Romano, J.P., Wolf, M. (1999). "Subsampling," Springer-Verlag, New York.

[51] Serfling, R.J., (1980). "Approximation theorems of mathematical statistics," Wiley, New York.

[52] Shephard, N. Xiu, D. (2014). “Econometric analysis of multivariate realised QML: estimation of the covariation of equity prices under asynchronous trading,” *Working paper No. 12-14, The University of Chicago Booth School of Business.*

[53] Varneskov, R. T. (2014). “Flat-top realized kernel estimation of quadratic covariation with non-synchronous and noisy asset prices,” *Unpublished Manuscript, Aarhus University.*

[54] Voev, V. and A. Lunde, (2007). “Integrated covariance estimation using high-frequency data in the presence of noise,” *Journal of Financial Econometrics*, 5, 68-104.

[55] Wu, C.F.J., (1986). “Jackknife, bootstrap and other resampling methods in regression analysis,” *Annals of Statistics 14*, 1261-1295.

[56] Xiu, D. (2010). “Quasi-maximum likelihood estimation of volatility with high frequency data,” *Journal of Econometrics*, 159, 235-250.

[57] Zhang, L. (2006). “Efficient estimation of stochastic volatility using noisy observations: a multi-scale approach,” *Bernoulli*, 12, 1019-1043.

[58] Zhang, L. (2011). “Estimating covariation: Epps effect, microstructure noise,” *Journal of Econometrics*, 160, 33-47.

[59] Zhang, L, P.A. Mykland, and Y. Aït-Sahalia (2005). “A tale of two time-scales: determining integrated volatility with noisy high frequency data,” *Journal of the American Statistical Association*, 100, 1394-1411.

[60] Zhang, L., Mykland, P. and Y. Aït-Sahalia (2011). “Edgeworth expansions for realized volatility and related estimators,” *Journal of Econometrics*, 160, 190-203.

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