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Specification, Estimation and Evaluation of Vector Smooth Transition Autoregressive Models with Applications

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Abstract

We consider a nonlinear vector model called the logistic vector smooth transition autoregressive model. The bivariate single-transition vector smooth transition regression model of Camacho (2004) is generalised to a multivariate and multitransition one. A modelling strategy consisting of specification, including testing linearity, estimation and evaluation of these models is constructed. Nonlinear least squares estimation of the parameters of the model is discussed. Evaluation by misspecification tests is carried out using tests derived in a companion paper. The use of the modelling strategy is illustrated by two applications. In the first one, the dynamic relationship between the US gasoline price and consumption is studied and possible asymmetries in it considered. The second application consists of modelling two well known Icelandic riverflow series, previously considered by many hydrologists and time series analysts.

JEL Classification: C32, C51, C52.

Keywords: Vector STAR model; Modelling nonlinearity; Vector autoregression; Generalized impulse response; Asymmetry; Oil price; River flow.

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1 Introduction

Regime-switching nonlinear models with an observable switch or transition variable have a rather long history. Quandt (1958, 1960) considered a model in which the coefficients of a linear model change at a certain value of an observable stochastic variable. This means that there is at some point an abrupt switch from one regression line to another. Bacon and Watts (1971) generalized this feature such that instead of a switch, the transition from one line to the other is smooth, hence the term "smooth transition". The univariate dynamic counterparts of these models appeared in the time series analysis a few years later. Tong (1978) and Tong and Lim (1980) introduced the threshold autoregressive (TAR) model, whereas Chan and Tong (1986) were the first to consider a smooth transition variant of it. Tsay (1989) constructed a systematic model building strategy for threshold autoregressive models, for an alternative approach, see Strikholm and Teräsvirta (2006) and Teräsvirta, Tjøstheim and Granger (2010, Section 16.4). Teräsvirta (1994) developed a similar strategy for smooth transition autoregressive (STAR) models. The latter work was completed by Eitrheim and Teräsvirta (1996) who derived misspecification tests for STAR models. A coherent modelling strategy for smooth transition regression (STR) models, including misspecification tests, appeared in Teräsvirta (1998). For a recent review, see Teräsvirta, Tjøstheim and Granger (2010, Chapter 3). For a thorough treatment of univariate TAR models, see Tong (1990).

These models are single-equation models. The first nonlinear vector model with an observable switch variable was the vector threshold autoregressive (VTAR) model that Tsay (1998) introduced. The same threshold variable controlled the switch in each equation, and the threshold parameter was also the same. Anderson and Vahid (1998) discussed testing the linear vector autoregressive (VAR) model against a vector smooth transition model. Rothman, van Dijk and Franses (2001) introduced a smooth transition vector error-correction model (STVECM) with a logistic transition function to investigate the Granger-causality hypothesis between money, output, inflation and interest rates. In their model, a single transition function controlled the transition in all equations. Camacho

(2004) considered a bivariate logistic smooth transition model with exogenous regressors, in which each equation could have its own the transition variable. He devised a modelling strategy, consisting of specification, estimation and evaluation stages, for building such models. Recently, Auerbach and Gorodnichenko (2012) estimated a three-dimensional two-regime vector STAR model to investigate the impact of fiscal policy on output. These authors, however, fixed the parameters of the transition function in advance, making their model (the conditional mean) completely linear for the estimation purposes. For a recent survey of vector TAR and STAR models, see Hubrich and Teräsvirta (2013).

In this paper, the previous work is generalized in various ways. Our model can have more than one transition. Exogenous variables are allowed as in Camacho (2004). Seasonality is introduced using seasonal dummies or trigonometric functions. The LVSTAR system can either have the same transition function for all equations or the transition variable may vary from one equation to the next. In the latter case, linear equations are allowed as well.

Following Teräsvirta (1994) and Camacho (2004), a complete modelling strategy is constructed for building LVSTAR models. Linearity and misspecification tests when a single transition variable is controlling the transition in the whole system need special consideration and are discussed in Teräsvirta and Yang (2014). In addition to the misspecification tests for the conditional mean, testing constancy of the error covariance matrix is considered as well. Nonlinear least squares estimation and the problem of finding initial values are discussed in detail. Dynamic properties of estimated models are investigated using generalized impulse response functions, see Koop, Pesaran and Potter (1996). How the modelling strategy works is illustrated by applications to the relationship between gasoline price and consumption in the US and to the daily flow of two Icelandic rivers. Tsay (1998) fitted a bivariate VTAR model with exogenous variables, temperature and precipitation, to the latter pair of time series, and we want to compare our results with his.

The plan of the paper is as follows. The LVSTAR model is introduced in Section 2 and the modelling procedure in Section 3. The specification of the model is the topic of Section 4. Parameter estimation by nonlinear least squares is considered in Section 5 and model evaluation by misspecification tests in Section 6.1. Section 7 contains two empirical applications. Final remarks can be found in Section 8. The relevant proofs are in the Appendices.

2 The statistical framework

The linear vector autoregressive model with k lags (VAR(k)) is defined as follows:

$$\begin{aligned} \mathbf{y}_t &= \mathbf{A}_1'\mathbf{y}_{t-1} + \mathbf{A}_2'\mathbf{y}_{t-2} + ... + \mathbf{A}_k'\mathbf{y}_{t-k} + \mathbf{\Phi}'\mathbf{d}_t + \boldsymbol{\varepsilon}_t \\ &= \mathbf{F}'\mathbf{x}_t + \boldsymbol{\varepsilon}_t, \end{aligned} \tag{2.1}$$

where $\mathbf{F} = (\mathbf{A}_1', ..., \mathbf{A}_k', \mathbf{\Phi}')'$, is a $(kp+q)\times p$ matrix, $\mathbf{x}_t = (\mathbf{y}_{t-1}', ..., \mathbf{y}_{t-k}', \mathbf{d}_t')'$ is a $(kp+q)\times 1$ vector, $\mathbf{y}_t$ is a $p\times 1$ column vector, and each $\mathbf{A}_i$, $i = 1, ..., k$, is a $p\times p$ matrix. Furthermore, $\mathbf{d}_t$ is a $q \times 1$ vector consisting of deterministic components such as intercepts, trends and seasonal dummies, and exogenous variables. Finally, $\mathbf{\Phi}$ is a $q \times p$ matrix containing the coefficients of the elements of $\mathbf{d}_t$. The $p \times 1$ error vector $\boldsymbol{\varepsilon}_t$ is white noise with mean zero and positive definite covariance matrix $\mathbf{\Omega}$.

Throughout the paper, matrices will be denoted by boldface capital letters, and vectors by lowercase boldface ones.

2.1 The vector logistic STAR model

We generalize (2.1) into the logistic vector smooth transition autoregressive (LVSTAR) model. This model has the following representation:

$$\mathbf{y}_t = \mathbf{F}_t'\mathbf{x}_t + \boldsymbol{\varepsilon}_t = \{\sum_{i=1}^{m}(\mathbf{G}_t^{i-1} - \mathbf{G}_t^i)\mathbf{F}_i'\}\mathbf{x}_t + \boldsymbol{\varepsilon}_t, \tag{2.2}$$

where the $p\times 1$ error vector is white noise with mean zero and positive definite covariance matrix $\mathbf{\Omega}$, $\mathbf{F}_i = (\mathbf{A}_{i1}', ..., \mathbf{A}_{ik}', \mathbf{\Phi}_i')'$, $i = 1, ..., m$, is a $(kp + q) \times p$ matrix, and $\mathbf{G}_t^i$ is a diagonal matrix of transition functions:

$$\mathbf{G}_t^i = \text{diag}\left\{\ g(s_{1it}|\gamma_{i1}, c_{i1}),\ ...,\ g(s_{pit}|\gamma_{ip}, c_{ip})\ \right\}, \tag{2.3}$$

for $i = 1, ..., m - 1$, and $\mathbf{G}_t^0 = \mathbf{I}_p$, $\mathbf{G}_t^m = \mathbf{0}$. The diagonal elements of $\mathbf{G}_t^i$ in (2.3) are logistic functions of their transition variables:

$$g(s_{ijt}|\gamma_{ij}, c_{ij}) = (1 + \exp\{-\gamma_{ij}\,(s_{ijt} - c_{ij})\})^{-1},\ \gamma_{ij} > 0, \tag{2.4}$$

for $i = 1, ..., m - 1$ and $j = 1, ..., p$. Some rows of $\mathbf{F}_i'$, $i = 2, ..., m$, may be zero vectors, in which case the corresponding equations are linear. In order to avoid identification problems, the diagonal elements of the corresponding diagonal matrices $\mathbf{G}_t^i$, $i = 2, ..., m$, are assumed to equal one, say. This is also the case for subsets of $\mathbf{F}_i'$, that is, not all equations need to have the same number of transitions.

The function (2.4) is a continuous (for $\gamma_{ij} < \infty$), monotonically increasing sigmoid function of its argument s_{ijt} and bounded between zero and one. We assume that the

transition variable s_{ijt} is a weakly stationary random variable, but it can also be a time trend: $s_{ijt} = t/T$, where T is the number of observations; see for example He et al. (2009). Furthermore, γ_{ij} is the slope or smoothness parameter, determining the shape of the function, or the smoothness of the transition. The parameter c_{ij} is a location parameter determining the midpoint of the transition. When $\gamma_{ij} = 0$, the corresponding equation becomes linear, and when $\gamma_{ij} \to \infty$, the transition in that equation becomes abrupt. In that case, when also $s_{1it} = ... = s_{pit}$, $c_{j1} = ... = c_{jp}$ and $m = 2$, the resulting model is the multivariate TAR model of Tsay (1998).

As an example, consider the case where $m = 2$. Then (2.2) becomes

$$\mathbf{y}_t = \{(\mathbf{I}_p - \mathbf{G}_t^1)\mathbf{F}_1' + \mathbf{G}_t^1\mathbf{F}_2'\}\mathbf{x}_t + \boldsymbol{\varepsilon}_t \tag{2.5}$$

so there is a single transition in each equation of the model. In this case, each location parameter c_{1j}, $j = 1, ..., p$, has a straightforward interpretation. It represents the inflection point in which the transition function has value 1/2, i.e., one is halfway through the transition from $\mathbf{F}_1'$ to $\mathbf{F}_2'$ in the sense that in (2.5) the changing parameter matrix $\mathbf{F}_t' = (1/2)(\mathbf{F}_1' + \mathbf{F}_2')$. When $p = 1$, (2.5) is the univariate logistic STAR model of Teräsvirta (1994).

This type of regime-switching can be convenient for modelling, for example, structural shifts when the transition variable is the normalized time t/T, or business cycle asymmetry where the regimes represent expansions and recessions, see Teräsvirta and Anderson (1992) and Skalin and Teräsvirta (2002) for empirical examples.

The LVSTAR model defined in (2.2) has p different transition functions for each $i = 1, ..., m-1$, and each one can have its own transition variable. If $\gamma_{1i} = \gamma_{2i} = ... = \gamma_{pi} = \gamma_i$, $c_{1i} = c_{2i} = ... = c_{pi} = c_i$, and $s_{1it} = s_{2it} = ... = s_{pit} = s_{it}$, then the i^{th} transition matrix is $\mathbf{G}_t^i = g(s_{it}|\gamma_i, c_i)\mathbf{I}_p$. This may in some applications be a reasonable simplification and reduces the complexity of the model.

Restrictions are required to make the model defined by (2.2), (2.3) and (2.4) identified. In each equation, the likelihood function is invariant to permutations of the order of transitions. This is similar to the so-called label switching problem in mixture models, for discussion see Redner and Walker (1984), Diebolt and Robert (1994) and Richardson and Green (1997). To identify the model, one may assume that the transitions appear in the order their transition variables appear in vector $\mathbf{x}_t$. If two transition functions of the i^{th} equation have the same transition variable, $s_{ijt} = s_{ikt}$, $j \neq k$, identification is achieved by assuming $c_{ij} < c_{ik}$. This generalises to situation in which more than two transition functions have the same transition variable.

The LVSTAR model can be reparameterised as follows:

$$\mathbf{y}_t = (\mathbf{B}_1' + \mathbf{G}_t^1\mathbf{B}_2' + ... + \mathbf{G}_t^{m-1}\mathbf{B}_m')\mathbf{x}_t + \boldsymbol{\varepsilon}_t = \boldsymbol{\Psi}_t'\mathbf{B}'\mathbf{x}_t + \boldsymbol{\varepsilon}_t, \tag{2.6}$$

where $\mathbf{\Psi}_t = \left(\mathbf{I}_p, \mathbf{G}_t^1, ..., \mathbf{G}_t^{m-1}\right)'$ is a $mp \times p$ full rank time-varying matrix. Furthermore, $\mathbf{B} = (\mathbf{B}_1, \mathbf{B}_2, ..., \mathbf{B}_m)$ is a $(kp + q) \times mp$ matrix, where $\mathbf{B}_1 = \mathbf{F}_1$, and $\mathbf{B}_i = \mathbf{F}_i - \mathbf{F}_{i-1}$, $i = 2, ..., m$. The representation (2.2) describes the transition through different extreme regimes $\mathbf{F}_i$, $i = 1, ..., m$, whereas the reparametrised form (2.6) is practical for specification, estimation and evaluation, and hence will be employed hereafter. The special case (2.5) where $m = 2$ has the reparameterised form

$$\mathbf{y}_t = (\mathbf{B}_1' + \mathbf{G}_t^1 \mathbf{B}_2')\mathbf{x}_t + \boldsymbol{\varepsilon}_t.$$

Camacho (2004) considered this representation for modelling bivariate time series.

Moreover, the number of transitions in every equation is not necessarily restricted to be the same. Suppose that equation j has m_j extreme regimes. Let $m = \max(m_1, ..., m_p)$ and formulate the models (2.2) and (2.6). In (2.2), restrict all the j^{th} columns in $\mathbf{F}_i$ to be the same, for all $i \geq m_j$. Equivalently, in (2.6), restrict all the j^{th} columns in $\mathbf{B}_i$ to be zeros, for all $i > m_j$. In the following subsection, we will discuss the details of how to make such restrictions in the following section.

We make the following assumption:

Assumption 2.1. *The sequence of* $\mathbf{y}_t$ *in* (2.2), $t = 1, ..., T$ *is weakly stationary.*

Assumption 2.1 is a high-level assumption. Stationarity of vector nonlinear models is discussed in Saikkonen (2008). Stationarity conditions in his work are valid only for LVSTAR models with a single transition function for all equations. Corresponding results do not seem to exist for the case where each equation has its own transition function.

2.2 Restrictions on linear parameters

In this section we consider possible restrictions on the column space of the linear parameter $\mathbf{B}$, for example, multiple switches between the two extreme regimes, and linearity of certain equations of the system. In some applications it is appropriate to specify the transition function such that the extreme regimes associated with small and large absolute values of $s_t - c$ are identical. This can be achieved within a single-transition model by using the exponential function

$$g(s_{jt}|\gamma_j, c_j) = 1 - \exp\{-\gamma_j \left(s_{jt} - c_j\right)^2\},\ \gamma_j > 0,$$

for applications see Michael et al. (1997), Sarantis (1999), and Peel et al. (2001), or by the second-order logistic function

$$g(s_{jt}|\gamma_j, c_j) = (1 + \exp\{-\gamma_j \left(s_{jt} - c_{j_1}\right)\left(s_{jt} - c_{j_2}\right)\})^{-1},\ \gamma_j > 0,$$

where $c_{j_1} \leq c_{j_2}$ as proposed in Jansen and Teräsvirta (1996). More generally, multiple switches between the two extreme regimes can be described by the general nth-order logistic transition function.

In the framework of the LVSTAR model, assuming multiple switches between the two extreme regimes can also be achieved by imposing restrictions on the parameter matrix $\mathbf{B}$. As an example, consider the 2-transition multivariate model

$$\mathbf{y}_t = (\mathbf{B}_1' + \mathbf{G}_t^1\mathbf{B}_2' + \mathbf{G}_t^2\mathbf{B}_3')\mathbf{x}_t + \boldsymbol{\varepsilon}_t = \boldsymbol{\Psi}_t'\mathbf{B}'\mathbf{x}_t + \boldsymbol{\varepsilon}_t.$$

The assumption of multiple switches between two extreme regimes implies $\mathbf{B}_2 + \mathbf{B}_3 = \mathbf{0}$. This is a more flexible alternative than the previous ones because $\gamma_{i1} \neq \gamma_{i2}$ is allowed.

3 Modelling strategy

Modelling stationary vector time series with LVSTAR models is carried out in a systematic fashion. The modelling strategy may be divided into three stages: specification, estimation, and evaluation. Corresponding procedures for single-equation models exist and have been successfully applied; see for example Box and Jenkins (1970) for ARIMA models, Tsay (1989) and Teräsvirta, Tjøstheim and Granger (2010, Section 16.4) for threshold autoregressive and switching regression models, and Teräsvirta (1998) or Teräsvirta, Tjøstheim and Granger (2010, Section 16.3) for smooth transition regression models. Camacho (2004) designed a modelling strategy for bivariate STAR models, and we generalise it to our family of LVSTAR models. We also consider the special case in which all equations are assumed to have the same transition variable. Tsay (1998) makes this assumption for his two-regime vector threshold autoregressive (VTAR) model. In that model, the whole transition function, i.e., both the threshold variable and thereshold, is the same for all equations of the model.

Estimating a linear stationary VAR model is the first stage in specifying an LVSTAR model. This involves selecting the lag length for the VAR model. Specification consists of testing the linear VAR model against LVSTAR one and, if linearity is rejected, determining the structure of the LVSTAR model. This implies selecting the transition variable(s) and determining the lag structure of the model. The latter means reducing the size of the model by imposing appropriate parameter restrictions. The way linearity is tested depends on the assumptions made about the system. If it is assumed that the LVSTAR model only has a single transition variable, that is, it is the same for all equations, a joint test involving the whole model can be applied. If this assumption is not made, testing and transition variable selection may be carried out equation by equation as in Camacho (2004), see Luukkonen et al. (1988) and Teräsvirta (1994, 1998).

Reducing the size of the model already involves parameter estimation. The parameters of the LVSTAR model are estimated using nonlinear least squares. Numerical aspects of this estimation problem will be considered later. In the linear VAR case, necessary and sufficient conditions for the least squares estimators to be consistent and asymptotically normal exist, but this is not the case for nonlinear LVSTAR model. Stability of the model is a necessary condition, but explicit conditions for stability of the general LVSTAR model do not seem to exist.

Evaluation of the model is done by checking (a necessary condition for) stability numerically and subjecting the estimated model to misspecification tests. These include testing the null hypothesis of no error autocorrelation, the null of no additive nonlinearity, and testing parameter constancy. Constancy of the covariance matrix is tested as well. We now consider these three stages of model building and begin with specification.

4 Specification of the Logistic Vector STAR model

As already mentioned, specification involves testing linearity against the LVSTAR model and selecting the transition functions. Linearity testing requires a test of $m = 1$ against $m = 2$, where m is the number of 'extreme states' and $m - 1$ the number of transitions in the system. Before the estimation, we need to determine m. If $m = 1$ is rejected against $m = 2$, the next step is to test $m = 2$ against $m = 3$. The relevant test is discussed in Teräsvirta and Yang (2014). There is a statistical argument in favour of building the model 'from specific to general': if m is chosen too large, the model will contain unidentified nuisance parameters. This invalidates the asymptotic inference as the parameters of the model cannot be consistently estimated. This is a well known problem, first formulated and discussed by Davies (1977, 1987), see also Watson and Engle (1985), and, later, in the univariate STAR context, by Saikkonen and Luukkonen (1988), Luukkonen et al. (1988) and Teräsvirta (1994). A solution based on constructing the empirical null distribution of the test statistic can be found in Hansen (1996), see also Teräsvirta et al. (2010, Chapter 5) for discussion.

The choice of the transition variables for the LVSTAR model can in some cases be based on economic theory implications. More often, however, economic theory may suggest many potential transition variables. For example, the theory may not be explicit about which lag of a given variable to choose. In the univariate case, a common way is to conduct a linearity test for each potential transition variable and choose the one which produces the strongest rejection measured in the p-value.

In the multivariate case we can conduct linearity tests equation by equation as in Luukkonen et al. (1988), Teräsvirta (1994) and Camacho (2004). For each equation, one

may choose the transition variable that produces the smallest p-value in the test as in the univariate case. If linearity is not rejected for any transition variable in the set, the corresponding equation is assumed to be linear. However, in the multivariate case, if there are several very significant transition variable candidates for some equations, the question is which combination of transition variables to choose for the LVSTAR model. A joint test of linearity against a LVSTAR model in which different equations would have different (predetermined) transition variables would be useful.

To illustrate, consider the p-dimensional single-transition logistic VSTAR model in (2.6):

$$\mathbf{y}_t = \mathbf{B}_1'\mathbf{x}_t + \mathbf{G}_t\mathbf{B}_2'\mathbf{x}_t + \boldsymbol{\varepsilon}_t, \tag{4.1}$$

where the sequence $\{\boldsymbol{\varepsilon}_t\}$ is iid$\mathcal{N}(\mathbf{0}, \boldsymbol{\Omega})$. The null hypothesis of linearity can be written as H_0: $\gamma_j = 0$, $j = 1, ..., p$, and the alternative hypothesis is H_1: at least one $\gamma_j > 0$, $j = 1, ..., p$. The basic idea of the joint test is to replace the transition function by a first-order Taylor expansion. For details, see Teräsvirta and Yang (2014).

If it is assumed *a priori* that potential nonlinearity in the vector system is controlled by a single transition variable, this has to be taken into account in testing linearity. Economic theories or in some cases common sense may suggest this special case. A good example is the investigation of the evolution of different prices of an asset in different markets controlled by the difference between the prices, see Tsay (1998). In that paper, the price difference is the sole transition variable. A joint linearity test against LVSTAR with a single transition variable is therefore advisable, as it makes it possible to control the (asymptotic) size of the test for the whole system. The joint test with a single transition variable is a special case of the joint test with different transition variables, see Teräsvirta and Yang (2014).

Our suggestion for specifying a LVSTAR model with a single transition variable is to conduct the joint linearity tests with each potential transition variable. If none of these tests rejects linearity, one can retain the linear VAR model. If the null hypothesis is rejected for at least one transition variable, we choose the one producing the smallest p-value in the test. Furthermore, we test each equation separately using the selected transition variable in order to find out whether some equations are in fact linear.

5 Estimation of parameters

In this section we consider nonlinear least squares (NLS) estimation of the parameters in the LVSTAR model (2.6). We focus on traditional derivative-based optimization techniques. The model has the parameters $\theta = \{\mathbf{B}, \boldsymbol{\Omega}, \boldsymbol{\Gamma}, \mathbf{C}\}$, where $\mathbf{B} = (\mathbf{B}_1, \mathbf{B}_2, ..., \mathbf{B}_m)$, $\boldsymbol{\Gamma} = \{\gamma_{ij}\}$ and $\mathbf{C} = \{c_{ij}\}$, $i = 1, ..., m-1$, $j = 1, ..., p$. The NLS estimators are obtained

by solving the following minimization problem:

$$\hat{\theta} = \arg\min_{\theta} Q_T(\theta), \tag{5.1}$$

where $Q_T(\theta) = \sum_{t=1}^{T} (\mathbf{y}_t - \mathbf{\Psi}_t'\mathbf{B}'\mathbf{x}_t)' (\mathbf{y}_t - \mathbf{\Psi}_t'\mathbf{B}'\mathbf{x}_t)$.

In practice, the objective function $Q_T(\theta)$ can be rather flat in many directions and meanwhile possess many local optima. Finding a suitable starting-value of θ for the nonlinear optimization algorithm is therefore essential. We employ a grid search algorithm for finding the starting-value. The set of the parameters θ is divided into two subsets: the 'nonlinear parameter set' $\{\mathbf{\Gamma}, \mathbf{C}\}$ and the linear set containing the parameters in $\mathbf{B}$. The basic idea of the grid search is to construct a discrete grid in the parameter space of $\mathbf{\Gamma}$ and $\mathbf{C}$. For each point in the grid, that is, a fixed pair of $\mathbf{\Gamma}$ and $\mathbf{C}$, one estimates the parameters in $\mathbf{B}$. For fixed $\mathbf{\Gamma}$ and $\mathbf{C}$, $\mathbf{B}$ is estimated by linear regression. Such a regression is carried out for all values in the grid. The pair of $\mathbf{\Gamma}$ and $\mathbf{C}$ yielding the smallest sum of squared residuals and the corresponding estimate of $\mathbf{B}$ are selected as the starting-value for the ensuing nonlinear optimization.

Because of the possibility of local optima the grid should be sufficiently dense. It could be necessary to allow it to be more dense in some areas than in others. A dense grid requires a large number of points. But then, the number should not be too large because the computational burden increases with the number of grid points. Finding a proper balance between these conflicting requirements may not always be easy.

Since the error covariance matrix $\mathbf{\Omega}$ does not enter the objective function Q_T, it is convenient to find the initial estimates of the parameters equation by equation. For equation j, let the corresponding parameters be $\mathbf{\Gamma}_j$, $\mathbf{C}_j$ and $\boldsymbol{\beta}_j$, and the corresponding residual sum of squares $Q_{j,T}$. The grid search is carried out as follows:

1. Construct a discrete grid in the parameter space of $\mathbf{\Gamma}_j$ and $\mathbf{C}_j$.

2. For each pair of $\mathbf{\Gamma}_j$ and $\mathbf{C}_j$ in the grid, compute the corresponding $\hat{\boldsymbol{\beta}}_j$ and the residual sum of squares $Q_{j,T}$.

3. Find the smallest $Q_{j,T}$ and choose the corresponding $\hat{\boldsymbol{\beta}}_j$ and the pair of $\mathbf{\Gamma}_j$ and $\mathbf{C}_j$ as suitable starting values.

In constructing the grid, one has to choose combinations of $\mathbf{\Gamma}_j$ and $\mathbf{C}_j$ such that the corresponding transition functions display a sufficient amount of variation in the grid. For example, if the location parameter c in a logistic function is chosen outside the observed support of the transition variable, the corresponding slope parameter γ has to be small enough to compensate for that.

If there is a single transition function for the whole system, the initial estimates are worked out as follows. For fixed $\mathbf{\Gamma}$ and $\mathbf{C}$, the conditional minimizer of Q_T can be obtained by solving the first-order condition equations

$$\sum_{t=1}^{T} \mathbf{x}_t \left(\mathbf{y}_t - \mathbf{\Psi}_t'\mathbf{B}'\mathbf{x}_t\right)' \mathbf{\Psi}_t' = \mathbf{0}$$

that can be rewritten as

$$\sum_{t=1}^{T} \mathbf{x}_t\mathbf{y}_t'\mathbf{\Psi}_t' = \sum_{t=1}^{T} \mathbf{x}_t\mathbf{x}_t'\mathbf{B}\mathbf{\Psi}_t\mathbf{\Psi}_t'. \tag{5.2}$$

The equation (5.2) leads to the following closed form of the NLS estimator of $\mathbf{B}$ conditional on $\mathbf{\Gamma}$ and $\mathbf{C}$:

$$\mathrm{vec}(\hat{\mathbf{B}}) = \left[T^{-1}\sum_{t=1}^{T} \left(\mathbf{\Psi}_t\mathbf{\Psi}_t'\right) \otimes \left(\mathbf{x}_t\mathbf{x}_t'\right)\right]^{-1} \left[T^{-1}\sum_{t=1}^{T} \mathrm{vec}\left(\mathbf{x}_t\mathbf{y}_t'\mathbf{\Psi}_t'\right)\right]. \tag{5.3}$$

Expressed otherwise,

$$\mathrm{vec}(\hat{\mathbf{B}}) = \left(\mathbf{M}'\mathbf{M}\right)^{-1}\mathbf{M}'\mathrm{vec}(\mathbf{Y}') \tag{5.4}$$

where $\mathbf{M} = (\mathbf{\Upsilon}_1, \mathbf{\Upsilon}_2, ..., \mathbf{\Upsilon}_T)'$ is a $Tp \times mp(kp+q)$ matrix and $\mathbf{\Upsilon}_t = \mathbf{\Psi}_t \otimes \mathbf{x}_t$ is an $mp(kp+q) \times p$ matrix. The $T \times p$ residual matrix $\hat{\mathbf{E}} = (\hat{\boldsymbol{\varepsilon}}_1, ..., \hat{\boldsymbol{\varepsilon}}_T)'$ where $\hat{\boldsymbol{\varepsilon}}_t = \mathbf{y}_t - \mathbf{\Psi}_t'\hat{\mathbf{B}}'\mathbf{x}_t$ is a column vector of residuals. The estimated error covariance matrix equals

$$\hat{\mathbf{\Omega}} = T^{-1}\hat{\mathbf{E}}'\hat{\mathbf{E}}.$$

It is seen from (5.4) that $T \geq m(kp+q)$ is a necessary condition for $\mathbf{M}'\mathbf{M}$ to have full rank and, consequently, to ensure that the solution (5.4) is unique.

The grid may be replaced by a heuristic estimation algorithm such as simulated annealing or differential evaluation. Simulations by Schleer (2013) show that these algorithms compare well with grid searches and may in many cases lead directly to the global optimum.

After selecting the starting-values, the parameters can be estimated using NLS. In order to alleviate the computational burden, it is advisable to follow the suggestion of Sollis et al. (1999) made for estimating univariate STAR models. The first iteration consists of re-estimating the parameters in $\mathbf{\Gamma}_j$ and $\mathbf{C}_j$, given the starting values for $\boldsymbol{\beta}_j$. This is done by NLS. Following this, the new value of $\boldsymbol{\beta}_j$, call it $\hat{\boldsymbol{\beta}}_j^{(2)}$, is calculated as in Step 2 of the above algorithm. In the next iteration $\hat{\boldsymbol{\beta}}_j^{(2)}$ is fixed in re-estimating the parameters in $\mathbf{\Gamma}_j$ and $\mathbf{C}_j$. Iteration is continued until convergence. Dividing each iteration into these two steps reduces the dimension of the nonlinear estimation problem and thus saves computation time. If the grid has been dense, the initial step-length of the nonlinear optimization algorithm must be sufficiently short so that optimization with a high probability leads to the local minimum closest to the value found by the grid search. If the

LVSTAR model has same transition function for all equations, the same approach can be used. The difference is that the linear step now consists of estimating a linear VAR model instead of a single equation.

The dimension of the grid can be further reduced for models with more than one transition by keeping the nonlinear parameters from the previously estimated model fixed and restricting the grid search to the new transition. This alleviates the computational burden.

6 Evaluation

6.1 Misspecification tests

Finding out whether the estimated LVSTAR model appears to satisfy the assumptions under which it was estimated is an integral part of model building. We employ the three multivariate misspecification tests constructed in Teräsvirta and Yang (2014). They are the test of no serial correlation test, the test of no additive nonlinearity, and the parameter constancy test. All of them are the extensions of the three univariate misspecification tests developed in Eitrheim and Teräsvirta (1996). The multivariate serial correlation test is derived from the autocorrelation VARMA form, see Godfrey (1988, pp. 117-118). The multivariate additive nonlinearity test is the just the joint linearity test for the case $m > 2$. The multivariate parameter constancy test is a special case of the joint linearity test in which the transition variable is the normalized time $\tau = t/T$.

Lütkepohl (2004) draws attention to the possibility that the error covariance matrix of a vector model may in practice be nonconstant. Testing the constancy of this matrix is therefore an important part of model evaluation. In this work, we use both the test developed in Eklund and Teräsvirta (2007) and the one by Yang (2014). These are tests against multivariate heteroskedasticity but based on different assumptions. The one in Eklund and Teräsvirta (2007) is derived presuming that the correlations of the errors are time-invariant, whereas the test developed by Yang (2014) is based on the assumption that the rotating projection of the covariance matrix is time-invariant. The simulation results in Eklund and Teräsvirta (2007) suggest that the size of the test is only slightly affected by violations of the assumption of constant correlations. The same is true for the assumption of constant rotation projection in Yang (2014).

6.2 Stability of the system

In this paper, the LVSTAR process is assumed weakly stationary. A necessary condition for this is stability. In the linear VAR case, the necessary and sufficient conditions for

exponential stability and weak stationarity coincide. There are no necessary and sufficient conditions for weak stationarity of the vector LSTAR models (such conditions are lacking for the general univariate STAR model as well).

A naive approach to checking weak stationarity consists of checking the roots of the lag polynomials of the extreme regimes and seeing whether or not they lie outside the complex unit disk. However, this would only provide a sufficient condition for weak stationarity. Besides, in practice this is only possible for single-transition models whose equations share the same transition function.

Exponential stability may be studied numerically by generating paths of realisations from the LVSTAR model by switching off the noise, starting from a large number of initial points and seeing whether or not the paths of realisations converge. Convergence to a single stationary point is a necessary condition for exponential stability. Since the method is numerical, the conclusion following from repeated convergence to the same point can only be that the stability assumption is not contradicted by these calculations. If it is, the model may be respecified and re-estimated or abandoned. This diagnostic is employed in the examples of Section 7. The 'histories', i.e., sets of values of the time series in question are natural starting-values for stability calculations.

6.3 Heteroskedasticity-robust tests

Heteroskedasticity is a common feature in both financial and macroeconomic time series of sufficiently high frequency. It has an adverse effect on the empirical size of linearity and misspecification tests. From the results of the tests of the empirical applications in the following section, it can be seen that the tests are much more likely to reject the null of linearity when the covariance matrix is not constant over time. In such circumstances, it is important to have asymptotically valid tests that are reliable in finite samples.

A straightforward way of deriving Lagrange multiplier type tests that are asymptotically heteroskedasticity-robust is to use heteroskedasticity-consistent covariance matrix estimates (HCCME) suggested in White (1980). Nevertheless, the findings in the Monte Carlo experiments in Godfrey and Tremayne (2005) indicated that in small samples, asymptotic critical values are not useful when heteroskedasticity-robust versions of the standard Lagrange multiplier tests are applied. The wild bootstrap turned out to be superior to the tests based on the HCCME in their simulations.

7 Empirical applications

7.1 Gasoline price and consumption

We consider two applications of the LVSTAR model and begin by looking at the interaction between the US real gasoline prices and gasoline consumption. The time series are the monthly real gasoline prices and the monthly real gasoline consumption from February 1973 to December 1998. A rather common, albeit not unanimous, view in the literature is that the effects of energy price shocks on some macroeconomic aggregates are asymmetric. In particular, energy price increases are perceived to have larger effects than energy price decreases. We want to find out whether this is also the case for the relationship between the gasoline price and the gasoline consumption.

This pair of time series has been analysed in Kilian and Vigfusson (2011), KV for short. The original series are transformed into logarithms and differenced, so the two variables to be modelled are approximate monthly growth rates. Figure 1 shows the levels of both series. Let $\mathbf{y}_t = (y_{1,t}, y_{2,t})'$ where $y_{1,t}$ is the real gasoline price change and $y_{2,t}$ the real gasoline consumption change. In order to model the presumed asymmetry, KV censored the price variable in the consumption equation such that it obtained value zero for negative values of the series, that is, $y^+_{1,t-d} = \max(0, y_{1,t-d})$, given a suitable lag d. They estimated the following multivariate VAR(k) model:

$$\begin{aligned} y_{1t} &= a_{10} + \sum_{i=1}^{k} a_{11,i} y_{1,t-i} + \sum_{i=1}^{k} a_{12,i} y_{2,t-i} + \varepsilon_{1t}, \\ y_{2t} &= a_{20} + \sum_{i=0}^{k} a_{21,i} y_{1,t-i} + \sum_{i=1}^{k} a_{22,i} y_{2,t-i} + \sum_{i=0}^{k} f_{21,i} y^+_{1,t-i} + \varepsilon_{2t}, \end{aligned} \tag{7.1}$$

where $\mathsf{E}(\varepsilon_{it}) = 0$, $\mathsf{E}(\varepsilon^2_{it}) = \sigma^2_i$, $i = 1, 2$, and $\mathsf{E}(\varepsilon_{1t}\varepsilon_{2t}) = 0$. The last assumption is needed for identification. The triangular form (7.1) implies weak exogeneity of the price variable, which is not an unreasonable assumption. The focus of KV was on testing the hypothesis $f_{21,i} = 0$, $i = 0, ..., k$, in (7.1). They were interested in possible asymmetry in the effect of a price change on consumption but found no compelling evidence against symmetry.

We begin by fitting a linear VAR model to the series and testing linearity of the two equations against the LVSTAR model. In doing so, we treat lags of both variables as potential transition variables, as we do not know in advance which variable controls potential nonlinearity. It appears that $k = 2$ is a sufficient lag length because the relevant LM test, see Teräsvirta and Yang (2014), does not reject the null hypothesis of no serial correlation. However, the multivariate Lomnicki-Jarque-Bera test in Lütkepohl (2005) strongly rejects the null hypothesis of multivariate normality, and constancy of the error covariance matrix is rejected by the tests of Eklund and Teräsvirta (2007) and Yang

(2014). These rejections may indicate misspecification of the conditional mean, or the presence of outliers in the linear model, or both.

Linearity of the VAR model is tested equation by equation. For both equations, we test using $y_{j,t-d}$, $j = 1, 2$, $d = 1, ..., 6$, as transition variables. Table 1 contains the results. The p-values (multiplied by 100) of Wilks's Λ test, Rao's F test, see Teräsvirta and Yang (2014), and the wild bootstrap LM test are reported. In this application, the first two produce almost identical results, whereas the results of the wild bootstrap tests deviate from them. Linearity is strongly rejected for many transition variables in the first two tests, whereas the same happens in a much smaller number of wild bootstrap tests. The presence of heteroskedasticity is a likely explanation to this difference in performance.

We choose the transition variable according to the p-values of the third-order wild bootstrap tests. This means choosing $y_{1,t-4}$ for the price equation and $y_{1,t-1}$ for the consumption equation. After estimating the two-regime LVSTAR model, the test of no additive nonlinearity indicates that the price equation may have another transition with $y_{1,t-5}$ as the transition variable. But then, one transition seems to be sufficient for the consumption equation. The final LVSTAR model thus consists of a two-transition price equation with transition variables $y_{1,t-4}$ and $y_{1,t-5}$, and a single-transition consumption equation with transition variable $y_{1,t-1}$. Removing the variables corresponding to insignificant parameter estimates and imposing some restrictions based on the discussion in Section 2.2, we end up with the following estimated LVSTAR model:

$$
\begin{aligned}
y_{1,t} &= \underset{0.006}{0.290}\, y_{1,t-1} - \underset{0.006}{0.283}\, y_{1,t-2} + \underset{0.005}{0.139}\, y_{2,t-2} + \underset{0.010}{0.202} \\
&\quad + \left(\underset{0.006}{0.088}\, y_{1,t-1} + \underset{0.003}{0.134}\, y_{2,t-1} + \underset{0.007}{0.283}\, y_{1,t-2} - \underset{0.007}{0.372}\, y_{2,t-2}\right) \\
&\quad \times \left(1 + \exp\left\{-\underset{13.653}{97.24}\left(y_{1,t-4} - \underset{0.026}{0.160}\right)/\sigma_s\right\}\right)^{-1} \\
&\quad + \left(\underset{0.006}{0.463}\, y_{1,t-1} + \underset{0.003}{0.134}\, y_{2,t-1} - \underset{0.007}{0.018}\, y_{1,t-2} + \underset{0.007}{0.105}\, y_{2,t-2} - \underset{0.013}{0.460}\right) \\
&\quad \times \left(1 + \exp\{-\underset{65.695}{94.491}\left(y_{1,t-5} + \underset{0.037}{0.604}\right)/\sigma_s\}\right)^{-1} + \hat{\varepsilon}_{1,t}, \\
y_{2,t} &= -\underset{0.007}{0.214}\, y_{1,t-1} - \underset{0.009}{1.098}\, y_{2,t-1} + \underset{0.006}{0.123}\, y_{1,t-2} - \underset{0.010}{0.249}\, y_{2,t-2} + \underset{0.028}{0.496} \\
&\quad + \left(\underset{0.008}{0.051}\, y_{1,t-1} + \underset{0.010}{0.726}\, y_{2,t-1} - \underset{0.006}{0.123}\, y_{1,t-2} + \underset{0.011}{0.071}\, y_{2,t-2} - \underset{0.029}{0.216}\right) \\
&\quad \times \left(1 + \exp\left\{-\underset{4.086}{21.677}\left(y_{1,t-1} + \underset{0.092}{1.582}\right)/\sigma_s\right\}\right)^{-1} + \hat{\varepsilon}_{2,t}, \qquad (7.2)
\end{aligned}
$$

$$
T = 311,\ \mathrm{tr}\{\hat{\mathbf{\Omega}}_{ML}\} = 6.77,\ \sigma_s = 2.43,
$$

where the figures below the parameter estimates are standard deviation estimates, and σ_s is the sample standard deviation of $y_{1,t}$. The model (7.2) is evaluated by misspecification tests mentioned in Section 6.1, and the results can be seen in Table 2. The model passes almost all the wild bootstrap tests at significance level 0.05. The p-value 0.04 for $y_{1,t-4}$

in the consumption is not small enough to make us reject the null hypothesis of no additive nonlinearity, given the number of observations and the number of tests we conduct. Results of the joint wild bootstrap serial error correlation and parameter constancy tests look fine enough. The results of the tests of constancy of the error covariance matrix (not reported) indicate that there is unmodelled conditional heteroskedasticity in the errors. They explain why the results of the wild bootstrap based tests differ from those tests assuming independent identically distributed errors.

Figure 2 depicts the estimated transition functions for the gasoline price change equation and the consumption change equation. The transition in the consumption equation is smoother than the two transitions in the price equation. As a whole, the observations cover the whole range of values of the transition functions from zero to one. Figure 3 shows the values of the transition functions over time for both equations.

Figure 4 sums up the results of checking the stability of our estimated model, as discussed in Section 6.2. We use all histories in the data set as the initial values, and the paths are shown in Figure 4. The realised price observation sequences converge to -0.302 and the consumption ones to 0.218. No matter which initial values or histories are used, the trajectories converge to the same stationary point.

As is the case in autoregressive models in general, it is not possible to interpret single coefficients of the model. In order to shed light on the question of possible asymmetry of the gasoline prices and consumption to shocks, we compute generalized multivariate impulse functions as suggested in Koop et al. (1996) and represent them using highest density regions (HDR) of Hyndman (1996); see also Teräsvirta et al. (2010, Section 15.3). The HDRs, which in this example are unimodal, are illustrated using boxplots. Responses to positive shocks are shown separately from responses to negative ones.

The lower panel of Figure 5 shows that the response of consumption change to price change shocks is asymmetric. A negative price shock causes a stronger response than a positive one in the sense that the density has greater dispersion in the former case than in the latter. This difference lasts two months before disappearing. Negative price shocks thus cause greater uncertainty in the consumer behaviour than positive ones. The directions are as expected: a negative price shock on the average increases consumption growth, whereas a positive shock decreases it. The response of price to price shocks appears in the upper panel of Figure 5. The two consumption shocks in Figure 6, are symmetric around zero, as the responses to positive and negative shocks are mirror images of each other. Note, however, that given weak exogeneity of the price variable, interpreting these responses is problematic.

7.2 Icelandic river flows

We turn to two daily Icelandic river flow series from the years 1972–1974, in which the flow is measured in cubic metres per second. The series are from the Hydrological Survey of the National Energy Authority of Iceland and were first analyzed using univariate nonlinear models by Tong, Thanoon and Gudmundsson (1985). Tsay (1998) considered the two rivers, Jökulsá eystra and Vatndalsá, jointly and fitted a bivariate threshold autoregressive model with a single threshold to the series.

Tong et al. (1985) describe the rivers and the observation station. Jökulsá is the bigger river of the two, with a large drainage basin that includes a glacier. Vatndalsá has a much smaller drainage area, and some of the flow is due to groundwater. The weather station lies between the two drainage basins at about 650 metres. The temperatures measured there are higher than the ones on the glacier of Jökulsá, which affects the results of modelling. Tong et al. (1985) point out that measuring the rainfall accurately is difficult because of high winds in the area. This may also explain some of the empirical results.

Before modelling, the precipitation series is shifted forward by one day due to the way the rainfall is recorded, see Tong et al. (1985). The flow, precipitation and the temperature are graphed in Figure 7. The flow is strongest in the spring when the snow is melting and slows down in the summer. The spring peak is more pronounced in the Vatndalsá flow than the Jökulsá one, because the drainage area of the latter contains the glacier, which smooths the flow.

We denote the Jökulsá flow by y_{1t}, the Vatndalsá one by y_{2t}, precipitation by x_t and temperature by z_t. We complement our model by a seasonal component that contains an annual half-cycle (a sine function) and an annual cycle. It has the form

$$\delta_1 \sin(n\pi/365) + \delta_2 \sin(2n\pi/365) + \delta_3 \cos(2n\pi/365), \tag{7.3}$$

where n represents the n^{th} day of the year.

We begin by testing linearity. The column 'Lin to R2' in Table 3 reports the results from the wild bootstrap third-order linearity tests. The null hypothesis of linearity is rejected very strongly for both flows and all transition variables; some of the p-values lie below 10^{-16}.* The test results are thus inconclusive, and following Tsay (1998) we select the temperature as the transition variable for both flow equations. Since the flows are heavily autocorrelated, choosing a lag of the flow would have been another possibility. The results of the wild bootstrap test of no additive nonlinearity for the estimated the two-regime (single-transition) LVSTAR model can be found in column 'R2 to R3' of

*The numbers stored in a computer system are not continuous. There exists a positive number ϵ such that $1 + \epsilon \neq 1$, and for any $x < \epsilon$, $1 + x = 1$. This ϵ is the smallest positive floating-point number. In R, $\epsilon = 2.220446e - 16$.

Table 3. They show that the nonlinearity attributed to the temperature has been well described, but there are still many strong rejections of the null hypothesis. The model is thus extended to a three-regime one. Somewhat subjectively, we select the precipitation of lag one x_{t-1} as the new transition variable for both equations. Column 'R3 to R4' in Table 3 contains the results of the tests of no additive nonlinearity. There are still a number of rejections, especially in the Vatndalsá equation. Since we do not want to complicate the model further, however, we terminate the search for transitions.

After removing redundant variables and lags and imposing some other parameter restrictions, the estimated model has the form (7.4)–(7.6).

Before interpreting the results, we evaluate the model. The wild bootstrap tests of no error autocorrelation up to lag 10 can be found in Table 4, and they do not indicate problems. The tests of parameter constancy do not reject the null hypothesis either. These results do not challenge the specified model. Both the results from the error covariance matrix constancy tests in Eklund and Teräsvirta (2007) and Yang (2014) reject strongly, which indicates the presence of heteroskedasticity. This explains the difference between the results of wild bootstrap based tests and the ones assuming i.i.d. errors. A look at Panels (a) and (b) in Figure 7 suggests that there will still be heteroskedasticity in the errors of the estimated LVSTAR model. Indeed, both the test of Eklund and Teräsvirta (2007) and the one by Yang (2014) strongly reject when parameter constancy (homoskedasticity) is tested against heteroskedastic errors.

Next we provide some comments on the estimated model. To begin with, the transition in the Jökulsá equation driven by the temperature, shown in Figure 8, is very smooth. The regime change begins at temperatures around 0°C and is completed when the temperature approaches 10°C. This is due to the glacier. When the temperature increases, so does the flow from the glacier. However, since the glacier is located at $1000 - 1800$ metres, considerably higher than the weather station, the flow increases gradually as a function of the temperature. Vatndalsá without a glacier, has a much more rapid transition driven by the temperature, and the estimate of the location parameter equals 0.3°C, see Figure 9.

Figures 10 and 11 show the transitions driven by the precipitation in the two rivers. Since we are using the same observations for both rivers and the mid-point of transition in Jökulsá is much higher than the one in Vatndalsá it seems that the flow of the small river is affected more rapidly by the rain than that of the big one, which is be reasonable. The transition of the small river driven by the precipitation has a low mid-point at 1.41, suggesting that the response to even small amounts of rain is nonlinear.

$$
\begin{aligned}
y_{1t} \;=\;& \underset{0.002}{0.93} y_{1,t-1} + \underset{0.004}{0.34} y_{1,t-2} - \underset{0.002}{0.21} y_{1,t-3} - \underset{0.028}{0.7}\, y_{1,t-4} + \underset{0.002}{0.21} y_{1,t-5} \\
& + \underset{0.005}{0.43} y_{1,t-6} - \underset{0.002}{0.14} y_{1,t-7} - \underset{0.005}{0.10} y_{1,t-8} - \underset{0.005}{0.08} y_{1,t-10} \\
& + \underset{0.005}{0.08} y_{1,t-12} + \underset{0.005}{0.24} y_{1,t-14} + \underset{0.002}{0.09} y_{1,t-15} \\
& - \underset{0.005}{0.18} y_{2,t-1} - \underset{0.005}{0.05} y_{2,t-5} \\
& + \underset{0.005}{0.26} y_{2,t-7} + \underset{0.002}{0.12} y_{2,t-8} - \underset{0.005}{0.34} y_{2,t-9} - \underset{0.002}{0.07} y_{2,t-10} \\
& + \underset{0.004}{0.07} y_{2,t-13} + \underset{0.003}{0.06} y_{2,t-15} \\
& - \underset{0.002}{0.06} z_{t-1} + \underset{0.068}{4.94} \\
& + \Big(\underset{0.003}{0.06} y_{1,t-1} + \underset{0.019}{3.01} y_{1,t-2} + \underset{0.004}{0.09} y_{1,t-3} - \underset{0.028}{0.7}\, y_{1,t-4} - \underset{0.004}{0.17} y_{1,t-5} \\
& - \underset{0.024}{0.86} y_{1,t-6} - \underset{0.004}{0.07} y_{1,t-7} + \underset{0.023}{1.43} y_{1,t-8} + \underset{0.004}{0.07} y_{1,t-9} - \underset{0.023}{1.1}\, y_{1,t-10} \\
& + \underset{0.025}{0.25} y_{1,t-12} - \underset{0.004}{0.14} y_{1,t-13} - \underset{0.025}{0.05} y_{1,t-14} \\
& - \underset{0.026}{0.96} y_{2,t-1} + \underset{0.027}{0.77} y_{2,t-3} + \underset{0.003}{0.17} y_{2,t-4} + \underset{0.027}{0.83} y_{2,t-5} \\
& + \underset{0.003}{0.07} y_{2,t-6} + \underset{0.026}{0.5}\, y_{2,t-7} - \underset{0.003}{0.39} y_{2,t-8} + \underset{0.003}{0.18} y_{2,t-10} \\
& - \underset{0.022}{0.17} y_{2,t-11} + \underset{0.018}{0.16} y_{2,t-13} + \underset{0.002}{0.09} y_{2,t-14} - \underset{0.011}{1.21} y_{2,t-15} \\
& + \underset{0.003}{0.88} x_{t-1} + \underset{0.005}{0.06} x_{t-2} - \underset{0.003}{0.41} x_{t-3} - \underset{0.009}{0.54} z_t + \underset{0.008}{1.67} z_{t-1} + \underset{0.411}{2.91} \Big) \\
& \times \Big(1 + \exp\Big\{ - \underset{0.030}{0.59} \Big(z_t - \underset{0.191}{4.52} \Big) \Big\} \Big)^{-1} \\
& + \Big(- \underset{0.002}{0.14} y_{1,t-1} - \underset{0.005}{0.46} y_{1,t-2} - \underset{0.003}{0.08} y_{1,t-3} + \underset{0.008}{1.08} y_{1,t-4} + \underset{0.002}{0.07} y_{1,t-5} \\
& - \underset{0.013}{1.1}\, y_{1,t-6} + \underset{0.003}{0.09} y_{1,t-7} + \underset{0.022}{0.72} y_{1,t-8} - \underset{0.005}{0.25} y_{1,t-9} + \underset{0.022}{0.78} y_{1,t-10} \\
& + \underset{0.005}{0.22} y_{1,t-11} + \underset{0.017}{0.19} y_{1,t-12} - \underset{0.013}{0.06} y_{1,t-14} + \underset{0.004}{0.11} y_{1,t-15} \\
& - \underset{0.012}{0.72} y_{2,t-1} - \underset{0.004}{0.26} y_{2,t-2} + \underset{0.013}{0.5}\, y_{2,t-3} + \underset{0.004}{0.14} y_{2,t-4} \\
& - \underset{0.005}{0.07} y_{2,t-6} - \underset{0.023}{0.7}\, y_{2,t-7} + \underset{0.005}{0.07} y_{2,t-8} + \underset{0.02}{0.95} y_{2,t-9} + \underset{0.005}{0.14} y_{2,t-10} \\
& - \underset{0.014}{0.98} y_{2,t-11} - \underset{0.006}{0.24} y_{2,t-12} + \underset{0.018}{0.24} y_{2,t-13} + \underset{0.013}{0.52} y_{2,t-15} \\
& + \underset{0.002}{0.35} x_{t-1} - \underset{0.005}{0.17} z_t + \underset{0.179}{10.45} \Big) \\
& \times \Big(1 + \exp\Big\{ - \underset{1.564}{12.77} \Big(x_{t-1} - \underset{0.017}{5.17} \Big) \Big\} \Big)^{-1} \\
& - \underset{0.201}{4.01} \sin\left(\frac{n\pi}{365} \right) + \underset{0.027}{0.53} \sin\left(\frac{2n\pi}{365} \right) - \underset{0.110}{2.16} \cos\left(\frac{2n\pi}{365} \right) + \hat{\varepsilon}_{1,t}, \qquad (7.4)
\end{aligned}
$$

$$
\begin{aligned}
y_{2t} \;=\;& \underset{0.001}{-0.12} y_{1,t-1} + \underset{0.002}{0.71} y_{1,t-2} \\
& + \underset{0.003}{0.23} y_{1,t-6} - \underset{0.002}{0.06} y_{1,t-8} - \underset{0.002}{0.14} y_{1,t-10} \\
& + \underset{0.003}{0.1}\, y_{1,t-12} - \underset{0.002}{0.15} y_{1,t-14} \\
& + \underset{0.002}{0.05} y_{2,t-1} \\
& - \underset{0.001}{0.02} y_{2,t-6} + \underset{0.002}{0.08} y_{2,t-9} \\
& - \underset{0.002}{0.09} y_{2,t-11} + \underset{0.001}{0.06} y_{2,t-15} \\
& + \underset{0.005}{0.31} x_{t-2} + \underset{0.036}{3.09} \\
& + \Big(\underset{0.001}{0.1}\, y_{1,t-1} + \underset{0.002}{0.62} y_{1,t-2} - \underset{0.003}{0.64} y_{1,t-4} \\
& + \underset{0.003}{0.12} y_{1,t-6} + \underset{0.003}{0.12} y_{1,t-8} + \underset{0.003}{0.04} y_{1,t-10} \\
& - \underset{0.001}{0.05} y_{1,t-11} - \underset{0.003}{0.12} y_{1,t-12} + \underset{0.003}{0.19} y_{1,t-14} \\
& - \underset{0.003}{0.19} y_{2,t-1} + \underset{0.003}{0.2}\, y_{2,t-5} \\
& - \underset{0.003}{0.27} y_{2,t-7} - \underset{0.003}{0.09} y_{2,t-9} \\
& + \underset{0.003}{0.35} y_{2,t-11} - \underset{0.003}{0.23} y_{2,t-13} \\
& + \underset{0.001}{0.05} x_{t-1} - \underset{0.001}{0.04} x_{t-2} + \underset{0.001}{0.06} x_{t-3} - \underset{0.051}{2.49} \Big) \\
& \times \left(1 + \exp\left\{ \underset{0.411}{-2.63} \left(z_t - \underset{0.066}{0.31} \right) \right\} \right)^{-1} \\
& + \Big(\underset{0.001}{0.05} y_{1,t-1} + \underset{0.002}{0.09} y_{1,t-2} - \underset{0.001}{0.07} y_{1,t-3} - \underset{0.003}{0.1}\, y_{1,t-4} + \underset{0.001}{0.02} y_{1,t-5} \\
& - \underset{0.003}{0.1}\, y_{1,t-6} + \underset{0.003}{0.36} y_{1,t-10} \\
& - \underset{0.003}{0.34} y_{1,t-12} + \underset{0.003}{0.67} y_{1,t-14} \\
& - \underset{0.003}{0.74} y_{2,t-1} + \underset{0.003}{0.63} y_{2,t-3} - \underset{0.003}{0.83} y_{2,t-5} \\
& + \underset{0.004}{0.68} y_{2,t-7} - \underset{0.004}{0.28} y_{2,t-9} \\
& - \underset{0.003}{0.14} y_{2,t-11} + \underset{0.003}{0.38} y_{2,t-13} - \underset{0.002}{0.18} y_{2,t-15} \\
& + \underset{0.001}{0.05} x_{t-1} - \underset{0.005}{0.33} x_{t-2} - \underset{0.04}{1.82} \Big) \\
& \times \left(1 + \exp\left\{ \underset{4.122}{-9.37} \left(x_{t-1} - \underset{0.081}{1.41} \right) \right\} \right)^{-1} \\
& - \underset{0.110}{0.72} \sin\left(\frac{n\pi}{365} \right) + \underset{0.020}{0.45} \sin\left(\frac{2n\pi}{365} \right) - \underset{0.067}{0.28} \cos\left(\frac{2n\pi}{365} \right) + \hat{\varepsilon}_{2,t}, \qquad (7.5)
\end{aligned}
$$

$$
\hat{\boldsymbol{\Omega}} = \begin{bmatrix} 15.18 & 0.44 \\ 0.44 & 1.95 \end{bmatrix}, \qquad (7.6)
$$

and $\mathrm{tr}\{\hat{\boldsymbol{\Omega}}\} = 17.13$.

It is seen from the estimates of the coefficients of the lagged flow that the flow does increase with the temperature as is expected. A seemingly mysterious detail is that there are cross-effects, although there is no linkage between the rivers. Tsay (1998) also found these cross-effects in his Vector TAR model. We shall return to this point in the discussion of generalised impulse responses.

The temperature itself does enter the Jökulsá flow equation, but the coefficients are hard to interpret. This may be due to the fact that there is only one weather station, and its readings may therefore not be useful other than in the transition function. The same is true for Vatndalsá.

As in the preceding example, in order to illustrate the dynamic behaviour of the estimated model we estimate generalised impulse response functions. For the transition variable of the temperature, we define two separate sets of histories: one when the temperature $z_t \geq 0.3$ at the moment of shock, and the other when $z_t < 0.3$. This value equals the estimate of the location parameter in the transition function of the Vatndalsá equation. It also represents the point in which melting of snow has begun (the flow has begun to increase) in parts of the Jökulsá drainage basin. The shocks are divided to positive and negative ones. This gives eight different types of shocks according to the flow shocked, the temperature, and the sign of the shock. As in the previous application, the impulse response functions are described using HDRs and boxplots. The results are in Figures 12 - 15.

The effects of shocks generally last longer in the 'summer', $z_t \geq 0.3$, than in the 'winter', $z_t < 0.3$. Although the mode of the HDR converges to zero quite quickly in the summer, the densities shrink towards a point much more slowly than in the winter. Most of the responses are close to being symmetric: the most pronounced asymmetry can be found in the flow of Vatndalsá in the summer, see Figure 15. A negative shock to that flow causes a much stronger response than a positive one. The most puzzling finding is the strong response of the Jökulsá flow to shocks to the Vatndalsá equation in the summer in Figure 14. It is also asymmetric and even stronger than the response of the flow of this river to own shocks. The HDRs of the latter can be found in Figure 12. But then, as Figure 13 shows, the flow of Vatndalsá is not much affected by shocks to the flow of Jökulsá.

Since the rivers are not connected, the effects of shocks to the Vatndalsá flow to the flow of Jökulsá have no physical explanation. It seems plausible to think, however, that change in the flow of the former is a leading indicator that precedes a corresponding change in the flow of the latter. Since Vatndalsá has a smaller drainage area that is located at a lower altitude than that of Jökulsá and has no glacier, changes in precipitation and how snow is melting there signal corresponding changes in the flow of the larger river. In the

summer when the snow has melted in the Vatndalsá drainage base and the flow slowed down, one would also expect a reduced flow in this river. Interestingly, the response begins with a one-day lag, which may be due to the inertia caused by the glacier. That the opposite cross-effect is very small may be explained by the size difference of these rivers. Shocking a big flow (Jökulsá when the snow is melting, say) does not have a similar dynamic indicator effect on the flow of the smaller river. Finally, these responses are almost negligible when the temperatures are below freezing.

For the transition variable of the precipitation, we also define two separate sets of histories: one when the precipitation $x_{t-1} > 5$ at the moment of shock, and the other when $x_{t-1} < 1.4$. The remaining histories are discarded. For space reasons, some of the HDRs are not shown, but they are available at
`http://creates.au.dk/research/research-papers/supplementary-downloads/`.

8 Concluding remarks

We generalise previous versions of vector LSTAR or LSTR models to the case in which the model can have a different transition variable or variables for each equation and the model can at the same time contain more than one transition. This is the case in our first application. In the second one, the transition variables are the same, but the transition functions are not restricted to be identical. In this application, relaxing the restriction of identical transitions turns out to be important. We devise a modelling strategy for this class of nonlinear models, consisting of specification, including testing linearity, estimation and evaluation stages. Parameter estimation is carried out by nonlinear least squares, and a major tool at the evaluation stage is a set of misspecification tests as in Teräsvirta (1996), Teräsvirta (1998) and Camacho (2004). The dynamic behaviour of the model is characterised by generalised impulse response functions. The two applications show how the strategy works in practice and how the estimated model can be interpreted using impulse responses and highest density regions.

In this work the LVSTAR process is assumed stationary, unless the transition variable is time, but generalizing the approach to nonstationary linearly cointegrated series, as in Rothman et al. (2001), appears straightforward. This is true as long as the short-run dynamic behaviour of the model, including the drift towards the equilibrium, is characterized using nonlinearity of STAR type. Specifying and accommodating a nonlinear equilibrium correction relation is from a statistical point of view a much more complicated problem. Some discussion can be found in Ripatti and Saikkonen (2001). We leave these extensions to further work.

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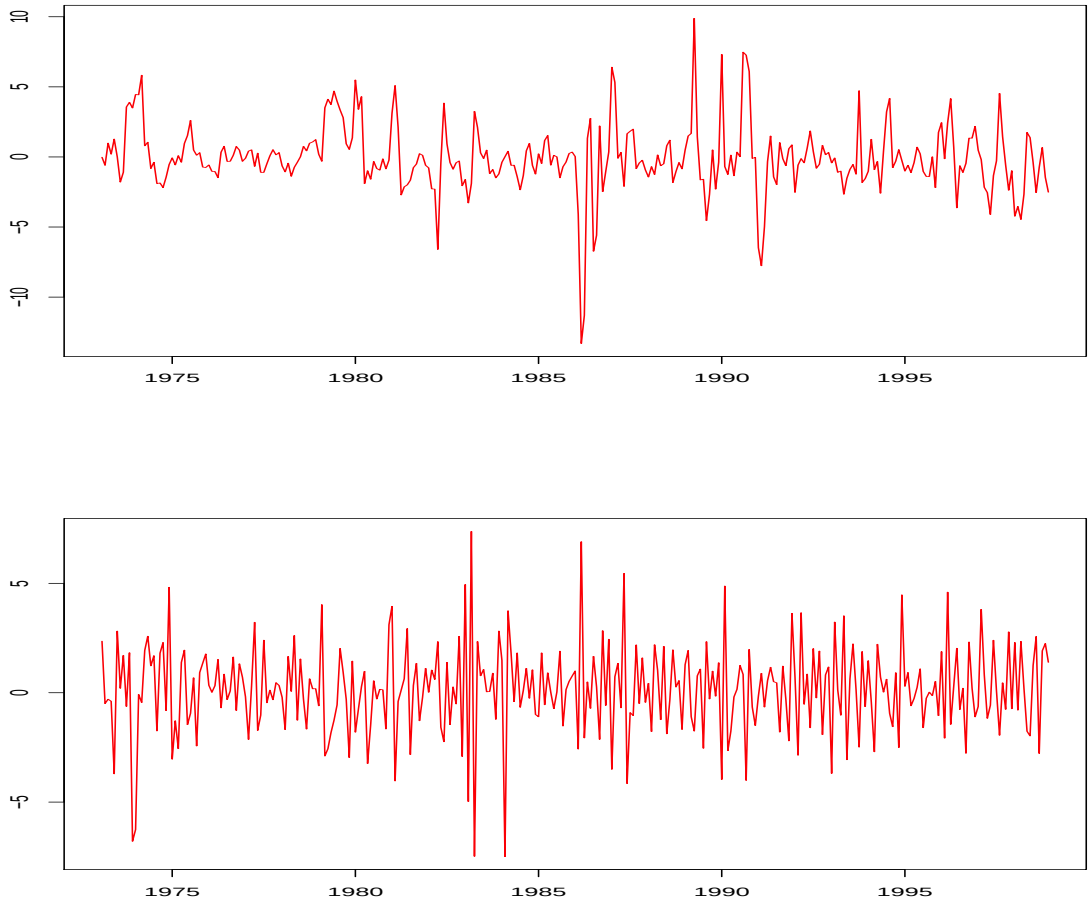


Figure 1: Top panel: Monthly growth rate of gasoline price (log difference); Bottom panel: Monthly growth rate of gasoline consumption (log difference), both from February 1973 to December 1998.

Table 1: p-values of linearity tests for the VAR(2) model against the LVSTAR model in Section 7.1. The upper part is the linearity tests in the gasoline price change equation, and the lower part is the linearity tests in the gasoline consumption change equation. 3^e is the third-order parsimonious test. *trans. variable* stands for transition variables, WLK for Wilks's statistic, RAO for Rao's statistic and WB for wild bootstrap. All p-values are multiplied by 100.

trans. variable		$y_{1,t-1}$	$y_{1,t-2}$	$y_{1,t-3}$	$y_{1,t-4}$	$y_{1,t-5}$	$y_{1,t-6}$	$y_{2,t-1}$	$y_{2,t-2}$	$y_{2,t-3}$	$y_{2,t-4}$	$y_{2,t-5}$	$y_{2,t-6}$	t/T
Order	Test	Gasoline price change equation												
1	WLK	0.10	0.14	0.25	0.00	0.01	31.82	0.86	9.41	6.65	0.00	54.91	17.45	7.11
	RAO	0.10	0.14	0.25	0.00	0.01	31.83	0.86	9.42	6.66	0.00	54.92	17.46	7.12
	WB	2.68	2.52	18.12	4.66	6.42	11.26	23.92	24.40	69.00	19.24	50.22	30.16	1.40
3	WLK	0.00	0.69	0.06	0.05	0.02	27.26	22.50	7.30	30.66	0.17	39.20	54.55	14.98
	RAO	0.00	0.71	0.06	0.05	0.02	27.53	22.75	7.43	30.95	0.17	39.50	54.83	15.19
	WB	5.38	26.44	2.28	0.62	2.30	56.80	54.64	11.34	88.98	3.46	88.08	49.72	4.32
3^e	WLK	0.24	0.02	0.22	0.00	0.01	42.27	1.64	16.67	9.05	0.01	58.84	23.23	9.03
	RAO	0.25	0.02	0.22	0.00	0.01	42.29	1.64	16.69	9.06	0.01	58.86	23.25	9.04
	WB	5.38	2.24	7.52	1.72	5.92	34.50	36.06	12.62	78.42	12.62	55.68	35.72	3.86
Order	Test	Gasoline consumption change equation												
1	WLK	3.24	0.22	0.09	21.41	21.42	23.23	0.00	19.77	20.23	44.91	44.71	93.13	0.07
	RAO	3.25	0.22	0.09	21.42	21.43	23.24	0.00	19.79	20.24	44.93	44.73	93.13	0.07
	WB	1.06	0.24	0.10	32.74	82.66	15.64	4.84	36.34	50.28	70.72	28.70	91.46	2.52
3	WLK	0.00	0.43	0.28	19.86	6.46	56.26	0.00	17.24	10.58	73.68	81.04	17.04	0.00
	RAO	0.00	0.44	0.29	20.10	6.58	56.54	0.01	17.47	10.75	73.89	81.21	17.26	0.00
	WB	0.04	0.56	0.12	3.36	14.34	28.22	4.30	47.88	27.14	78.80	44.62	53.12	0.82
3^e	WLK	4.30	0.27	0.11	32.94	19.12	16.65	0.01	27.48	31.90	60.63	55.47	58.99	0.08
	RAO	4.31	0.27	0.11	32.96	19.14	16.67	0.01	27.50	31.92	60.65	55.49	59.01	0.08
	WB	2.52	0.26	0.04	42.00	49.48	14.28	13.78	43.94	70.38	39.30	34.86	97.14	5.28

Table 2: p-values of misspecification tests of the estimated model (7.2). GP stands for gasoline price change equation, and GC stands for gasoline consumption change equation. The tests of no additive nonlinearity are based on the third-order Taylor expansion. WLK stands for Wilks's statistic, RAO for Rao's statistic and WB for wild bootstrap. All p-values are multiplied by 100.

Joint error serial correlation											Joint parameter constancy			
lags(J)	1	2	3	4	5	6	7	8	9	10	eq.	GP	GC	P&C
WLK	70.93	9.17	19.99	35.31	25.43	36.51	18.43	25.74	5.05	16.05	WLK	52.74	10.53	27.37
RAO	70.93	9.17	20.00	35.32	25.45	36.54	18.47	25.80	5.09	16.14	RAO	54.34	11.45	29.48
WB	90.00	11.00	32.50	50.30	35.90	72.30	60.10	60.10	24.50	57.20	WB	28.80	35.60	46.60

No additive nonlinearity														
eq.	test	$y_{1,t-1}$	$y_{1,t-2}$	$y_{1,t-3}$	$y_{1,t-4}$	$y_{1,t-5}$	$y_{1,t-6}$	$y_{2,t-1}$	$y_{2,t-2}$	$y_{2,t-3}$	$y_{2,t-4}$	$y_{2,t-5}$	$y_{2,t-6}$	t/T
GP	WLK	3.46	17.85	0.41	1.75	1.59	20.85	8.75	78.71	66.47	0.45	84.65	7.41	86.43
	RAO	3.47	17.88	0.42	1.76	1.60	20.88	8.77	78.73	66.50	0.45	84.66	7.43	86.45
	WB	45.10	73.40	13.10	25.20	24.10	67.00	12.40	70.00	89.30	4.60	76.90	67.60	40.90
GC	WLK	5.81	64.24	37.79	4.63	11.24	8.10	93.22	31.05	34.43	74.13	26.96	60.22	6.29
	RAO	5.83	64.27	37.83	4.65	11.26	8.12	93.23	31.09	34.47	74.15	26.99	60.25	6.31
	WB	63.00	27.00	50.00	4.00	6.30	9.50	67.00	31.20	11.20	59.40	13.20	88.40	37.90

Table 3: p-values of the wild bootstrap version of the third-order LM-type misspecification tests for the estimated model (7.4) and (7.5) against additive nonlinearity. *Jokul* stands for the Jökulsá flow equation, and *Vatn* for the Vatnsdalsá flow equation. Only the wild bootstrap results from the third-order linearity tests are reported. All p-values are multiplied by 100.

	Lin to R2		R2 to R3		R3 to R4	
Trans. variable	Jokul	Vatn	Jokul	Vatn	Jokul	Vatn
$y_{1,t-1}$	0.00	3.06	0.10	0.00	0.00	0.30
$y_{1,t-2}$	0.00	4.38	0.10	0.00	0.40	0.00
$y_{1,t-3}$	0.00	1.70	0.80	0.00	7.40	0.00
$y_{1,t-4}$	0.00	15.82	3.70	0.10	10.80	0.90
$y_{1,t-5}$	0.00	59.20	0.60	0.10	16.80	1.20
$y_{1,t-6}$	0.00	7.58	5.70	0.00	42.50	0.80
$y_{1,t-7}$	0.00	10.60	0.10	0.00	6.70	0.10
$y_{2,t-1}$	1.46	0.00	0.40	0.00	0.00	0.00
$y_{2,t-2}$	2.18	0.16	0.30	0.00	7.90	0.00
$y_{2,t-3}$	1.54	2.14	0.70	0.00	3.90	0.00
$y_{2,t-4}$	5.36	13.24	3.60	0.00	12.20	0.00
$y_{2,t-5}$	6.40	33.44	5.50	0.00	1.90	0.20
$y_{2,t-6}$	4.16	12.80	3.30	0.00	4.60	0.10
$y_{2,t-7}$	9.28	7.76	9.30	0.00	26.00	0.00
x_{t-1}	0.00	0.02	0.10	0.10	0.20	1.40
x_{t-2}	0.18	0.00	11.90	0.00	0.10	1.60
x_{t-3}	8.68	1.04	0.30	0.00	17.40	65.00
z_t	0.00	0.00	82.60	10.60	53.70	80.50

Table 4: p-values of the wild bootstrap version of the LM-type joint serial correlation tests of the estimated model (7.4) and (7.5). The sample size of the wild bootstrapping is 1000. The p-values are all multiplied by 100.

Joint error serial correlation										
lag(J)	1	2	3	4	5	6	7	8	9	10
$100\times p$-value	36.60	1.20	13.50	14.00	27.60	3.40	12.40	12.10	24.10	10.70

Table 5: p-values of the wild bootstrap version of the partial and joint parameter constancy tests of the estimated model (7.4) and (7.5). All p-values are multiplied by 100.

Partial and joint parameter constancy			
test	Jokul	Vatns	Joint
$100\times p$-value	5.30	45.4	9.70

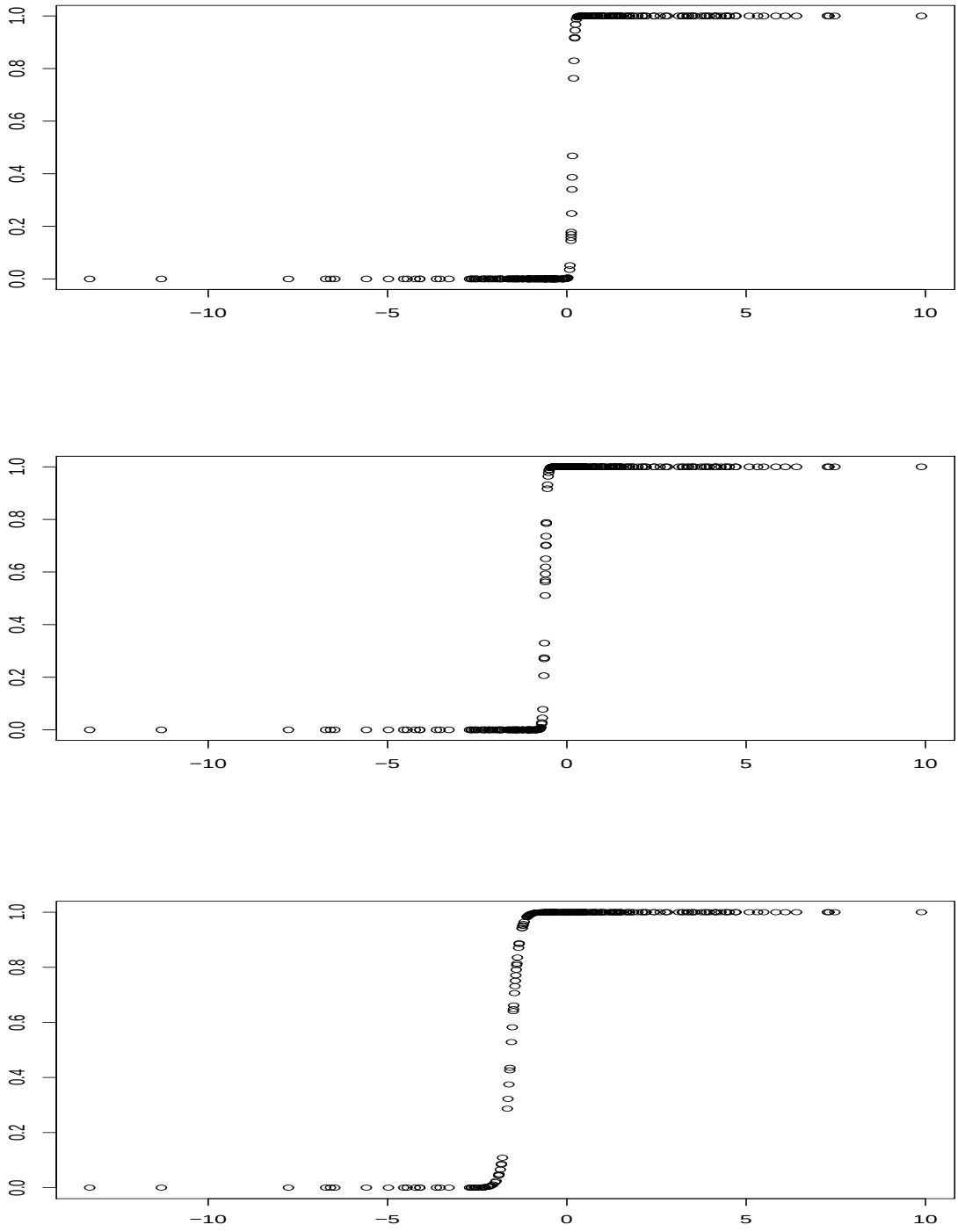


Figure 2: Transition functions for the gasoline price change equation and the consumption change equation. Top panel: the first transition driven by $y_{1,t-4}$ in the price equation; Mid-panel: the second transition driven by $y_{1,t-5}$ in the price equation; Bottom panel: the transition driven by $y_{1,t-1}$ in the consumption equation. Each circle represents an observation.

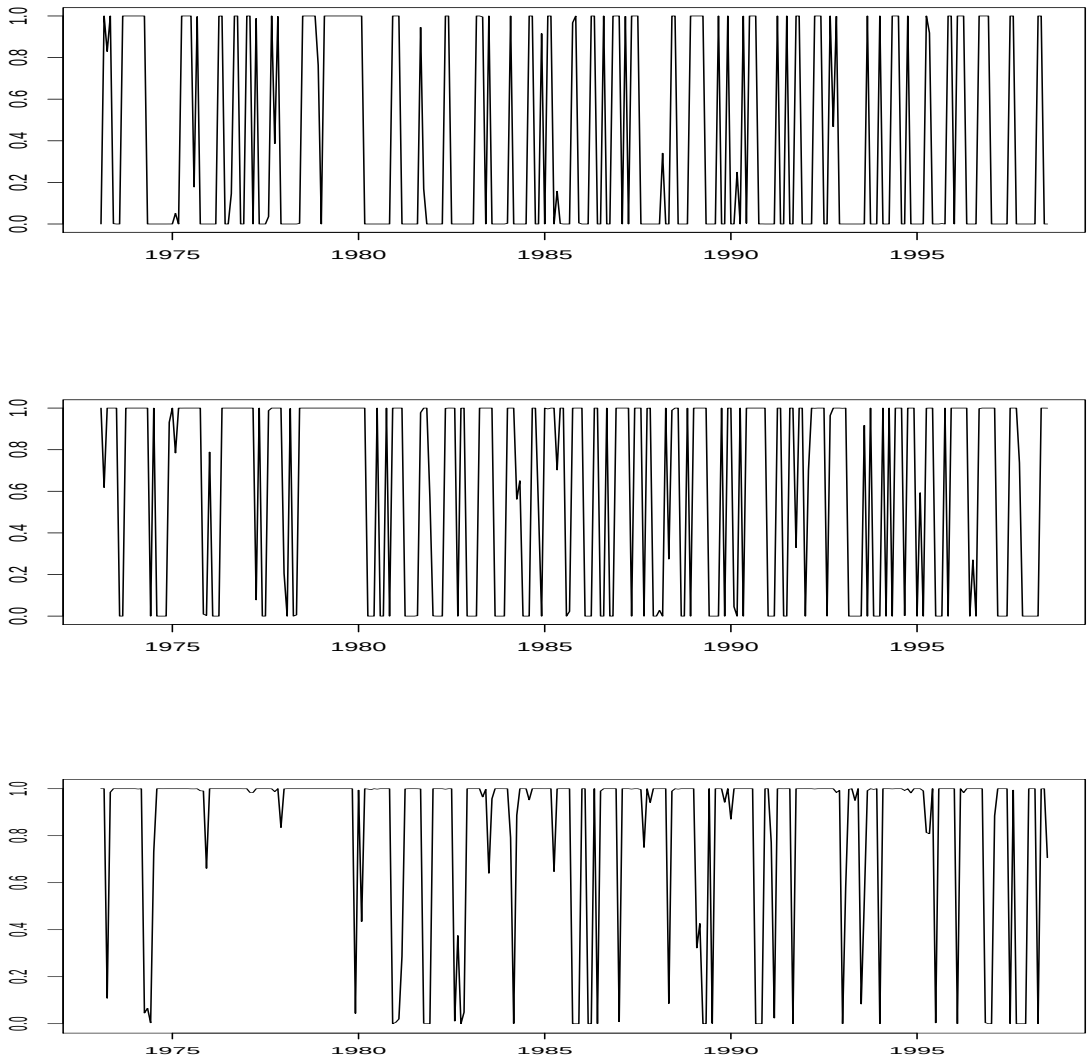


Figure 3: Values of the transition functions over time. Top panel: the transition driven by $y_{1,t-4}$ in the price equation; Mid-panel: the transition driven by $y_{1,t-5}$ in the price equation; Bottom panel: the transition driven by $y_{1,t-1}$ in the consumption equation.

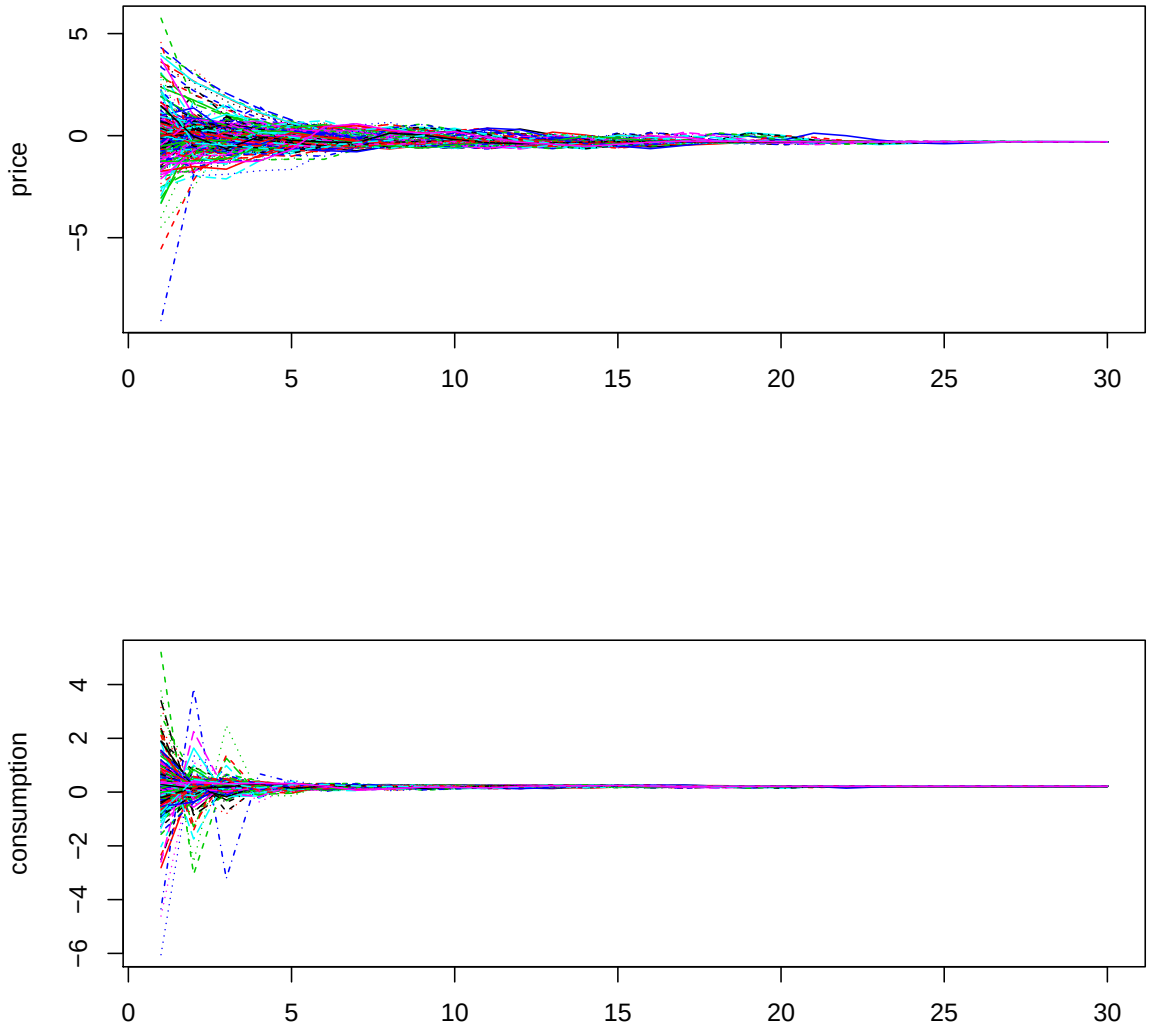


Figure 4: Stability of the two processes. Top panel: the price change equation, stable point= -0.302. Bottom panel: the consumption change equation, stable point= 0.218.

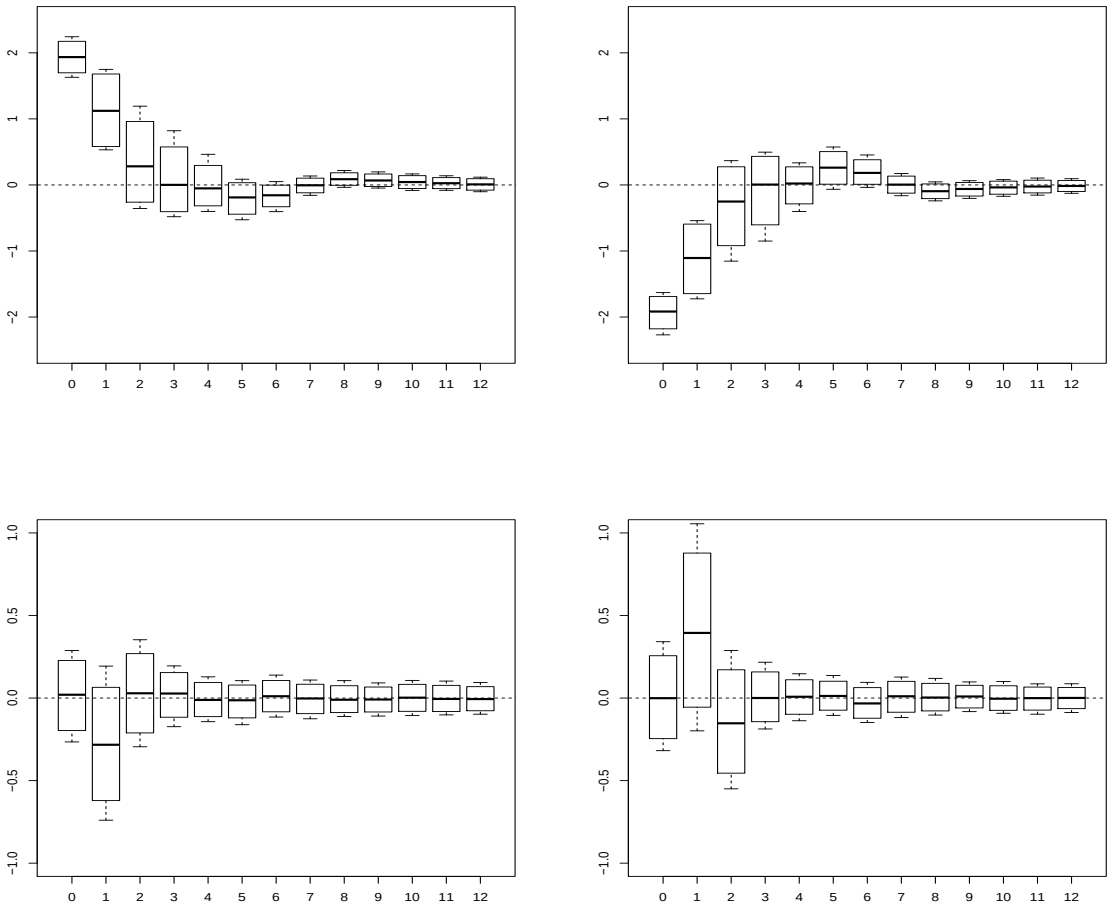


Figure 5: Boxplots of generalized impulse response functions of the LVSTAR model (7.2) represented with 50 percent (box) and 80 percent (whisker) highest density regions. Top left panel: gasoline price percent change to **positive** price shocks; Top right panel: gasoline price percent change to **negative** price shocks. Bottom left panel: gasoline consumption percent change to **positive** price shocks; Bottom right panel: gasoline consumption percent change to **negative** price shocks.

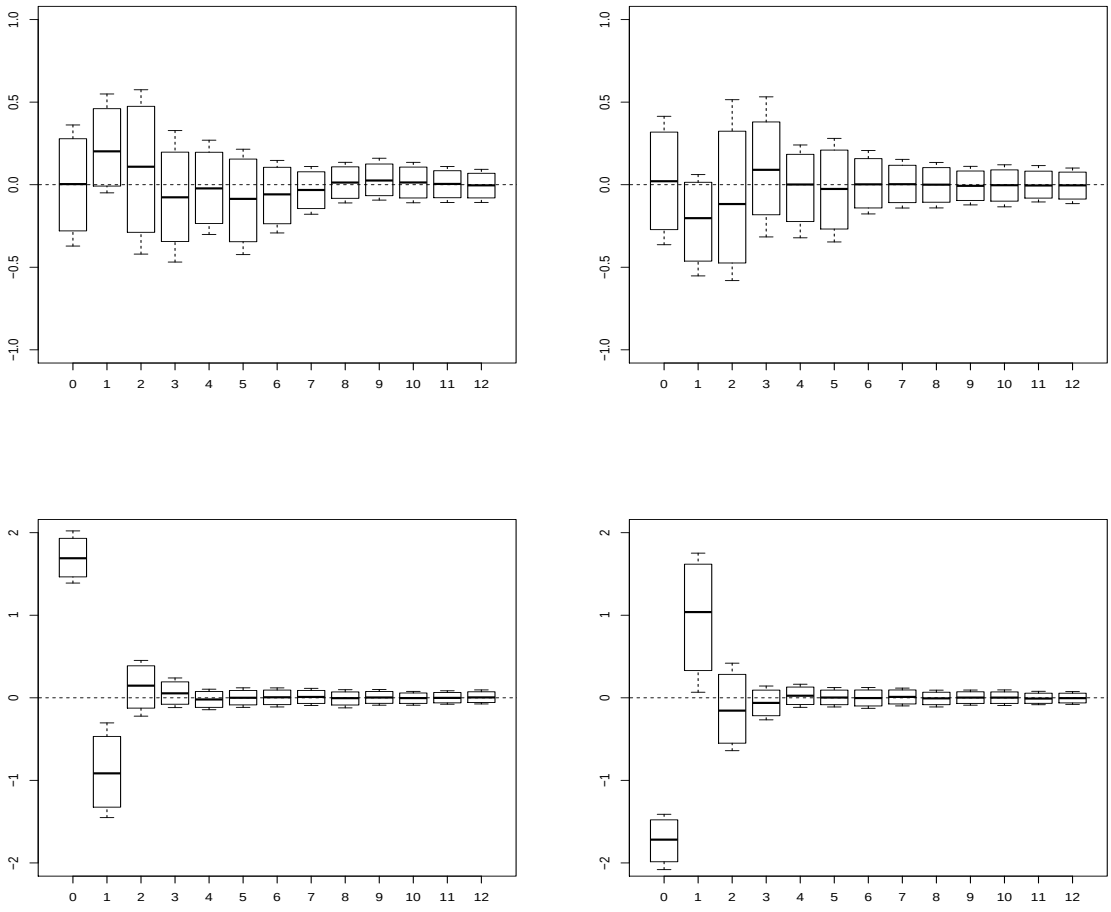
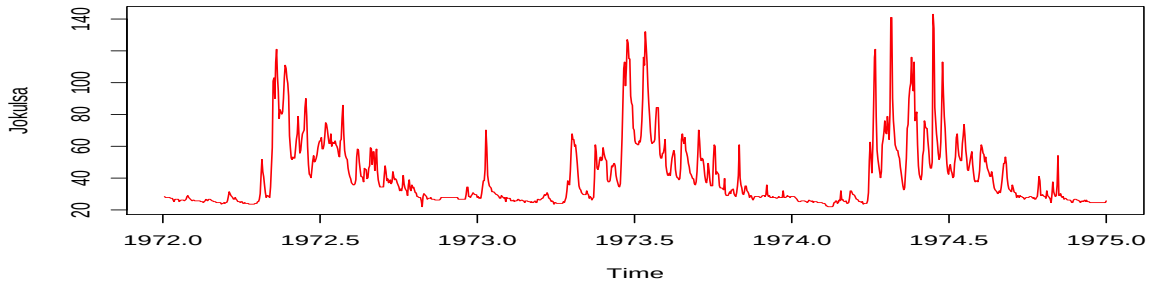
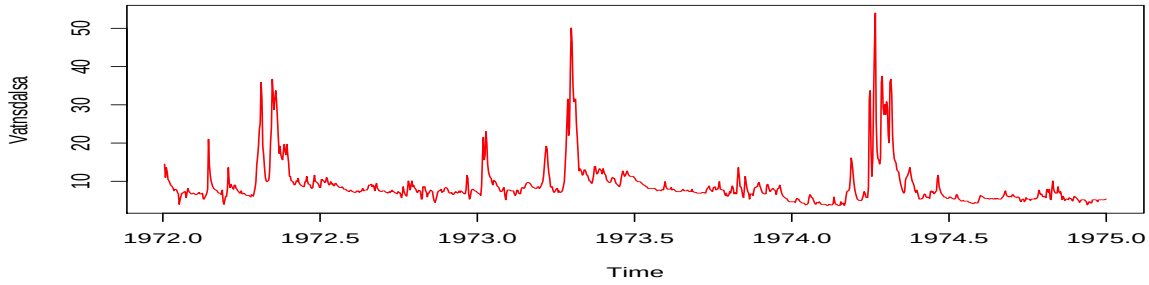


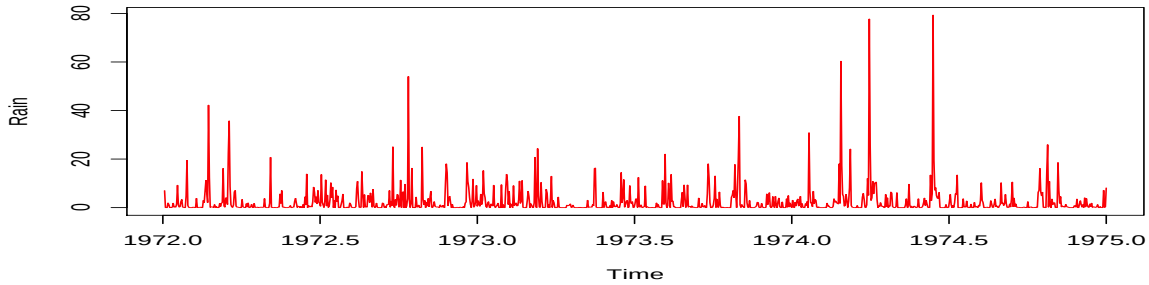
Figure 6: Boxplots of generalized impulse response functions of the LVSTAR model (7.2) represented with 50 percent (box) and 80 percent (whisker) highest density regions. Top left panel: gasoline price percent change to **positive** consumption shocks; Top right panel: gasoline price percent change to **negative** consumption shocks. Bottom left panel: gasoline consumption percent change to **positive** consumption shocks; Bottom right panel: gasoline consumption percent change to **negative** consumption shocks.



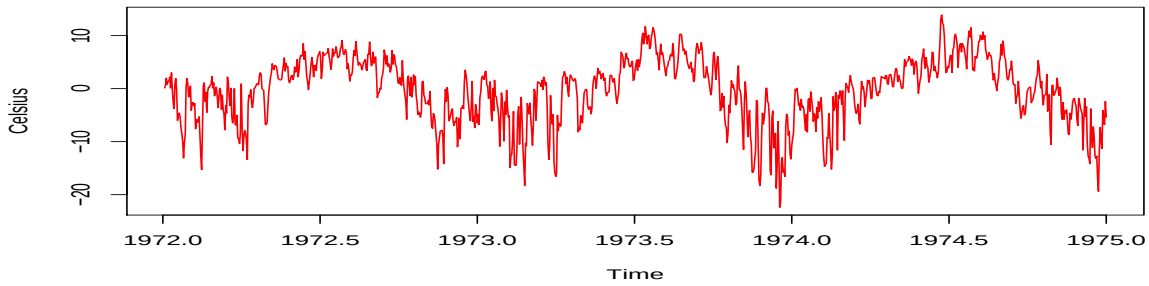
(a) The flow of Jökulsá Eystri, $m^3 s^{-1}$



(b) The flow of Vatnsdalsá, $m^3 s^{-1}$

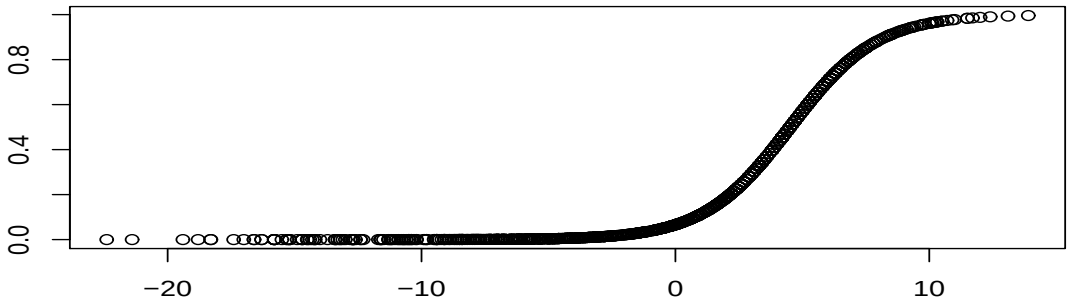


(c) Precipitation in millimetres

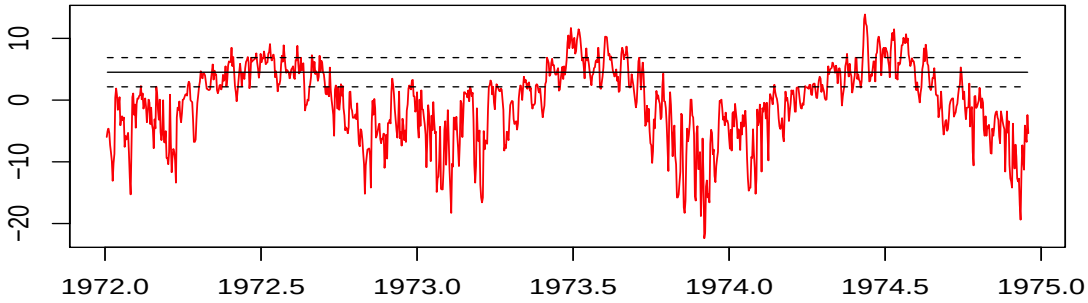


(d) Temperature in centigrades °C

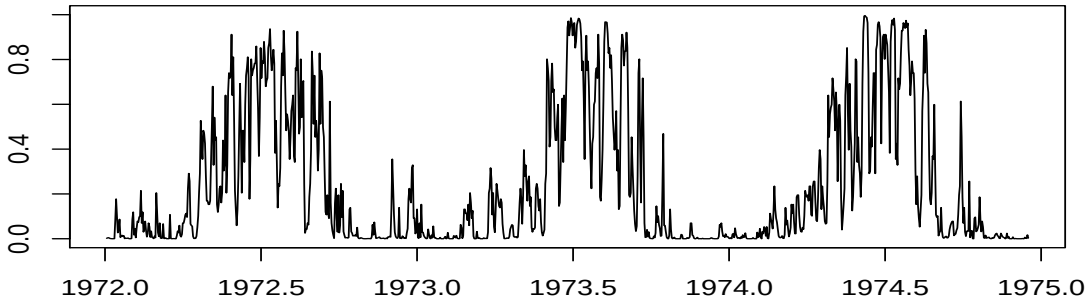
Figure 7: Daily river flow, precipitation and temperature series, 1972-1974.



(a) The estimated transition function. Each circle represents an observation.

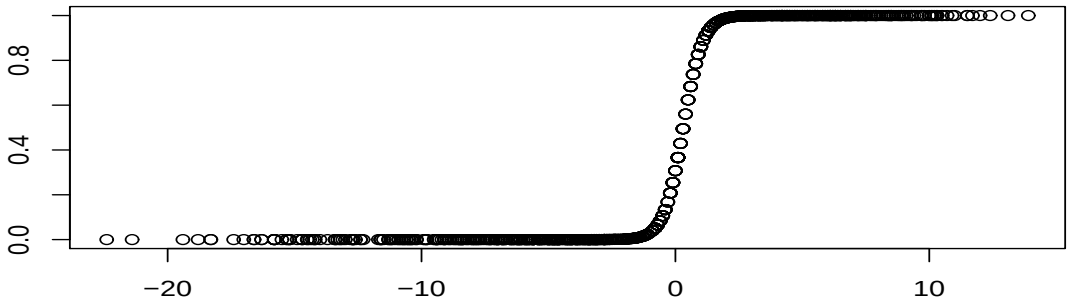


(b) Transition variable; the inflection point (solid line) and the interval for the smooth transition (dashed lines).

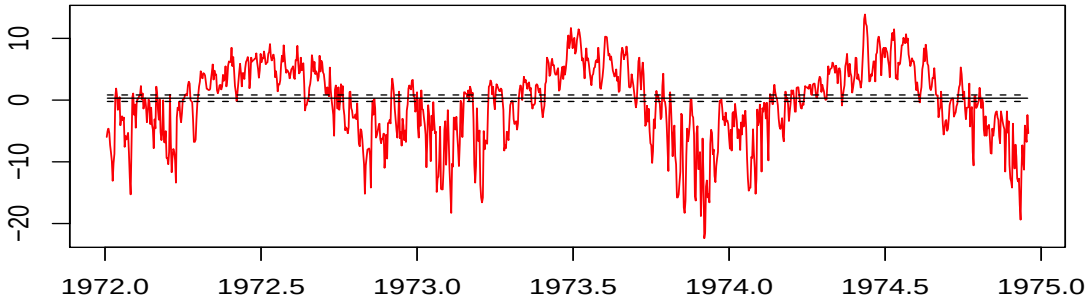


(c) Values of the transition function over time.

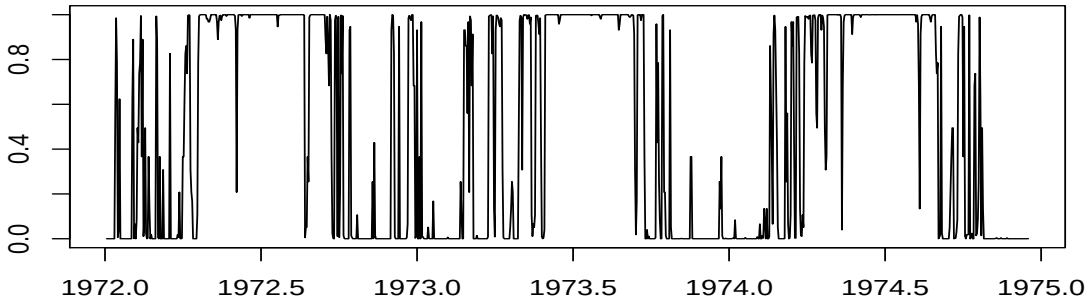
Figure 8: Transitions of the Jökulsá flow driven by the temperature.



(a) Estimated transition function. Each circle represents an observation.

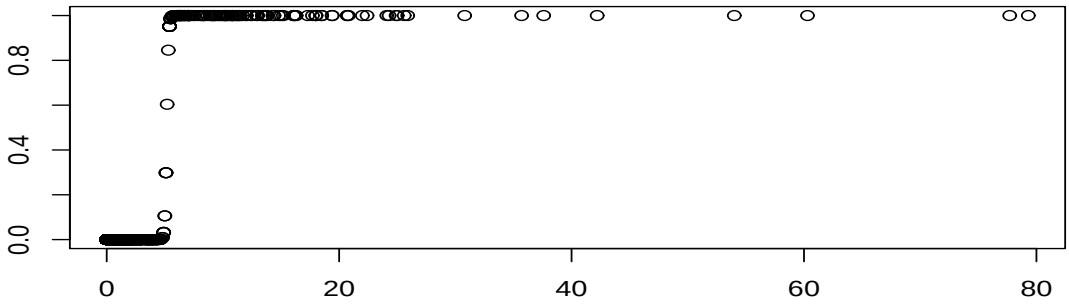


(b) Transition variable; the inflection point (solid line) and the interval for the smooth transition (dashed lines).

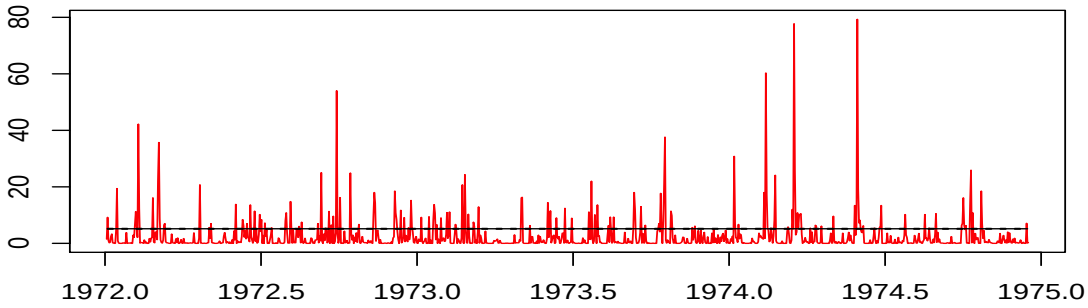


(c) Values of the transition function over time.

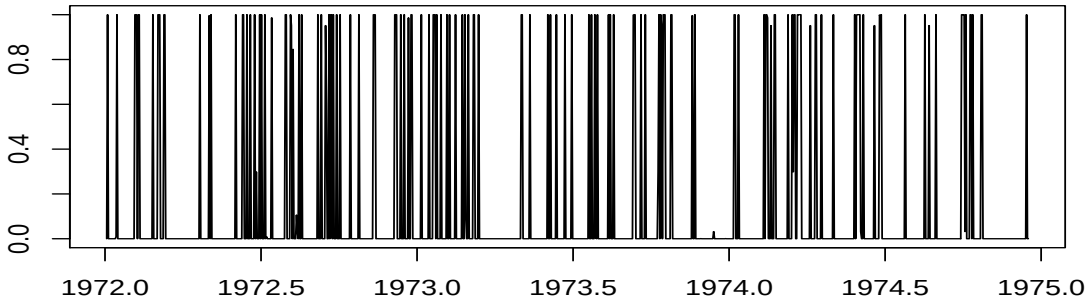
Figure 9: Transitions of the Vatnsdalsá flow driven by the temperature.



(a) Estimated transition function. Each circle represents an observation.

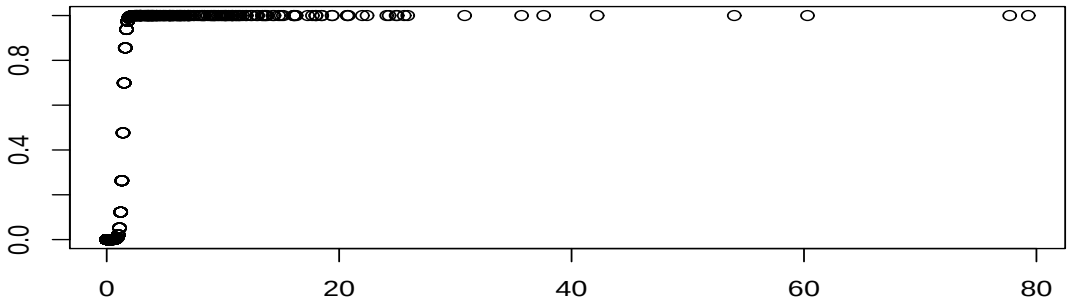


(b) Transition variable; the inflection point (solid line) and the interval for the smooth transition (dashed lines).

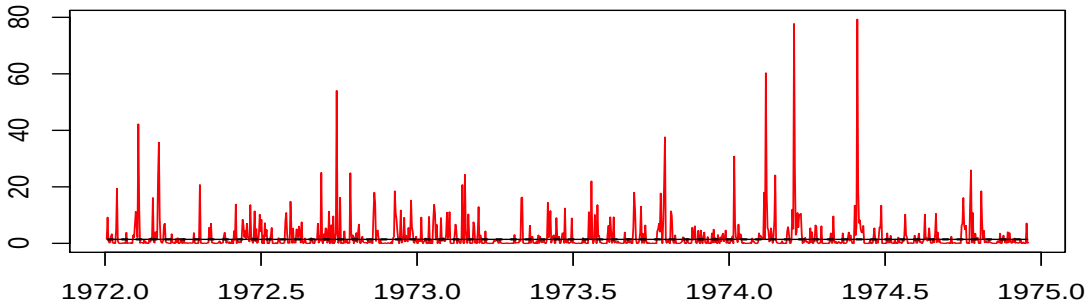


(c) Values of the transition function over time.

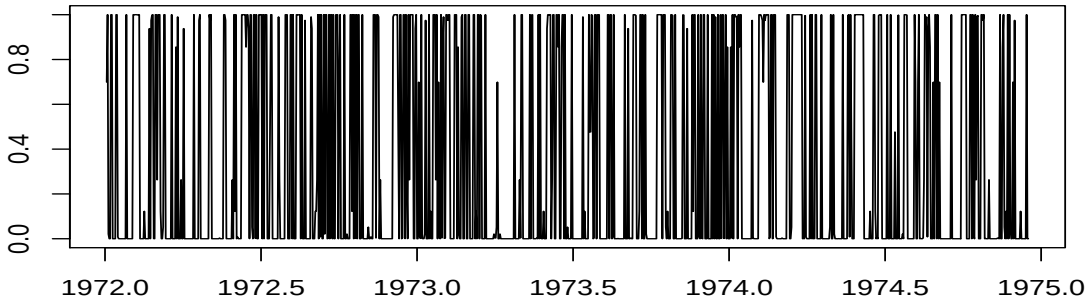
Figure 10: Transitions of the Jökulsá flow driven by the precipitation.



(a) Estimated transition function. Each circle represents an observation.

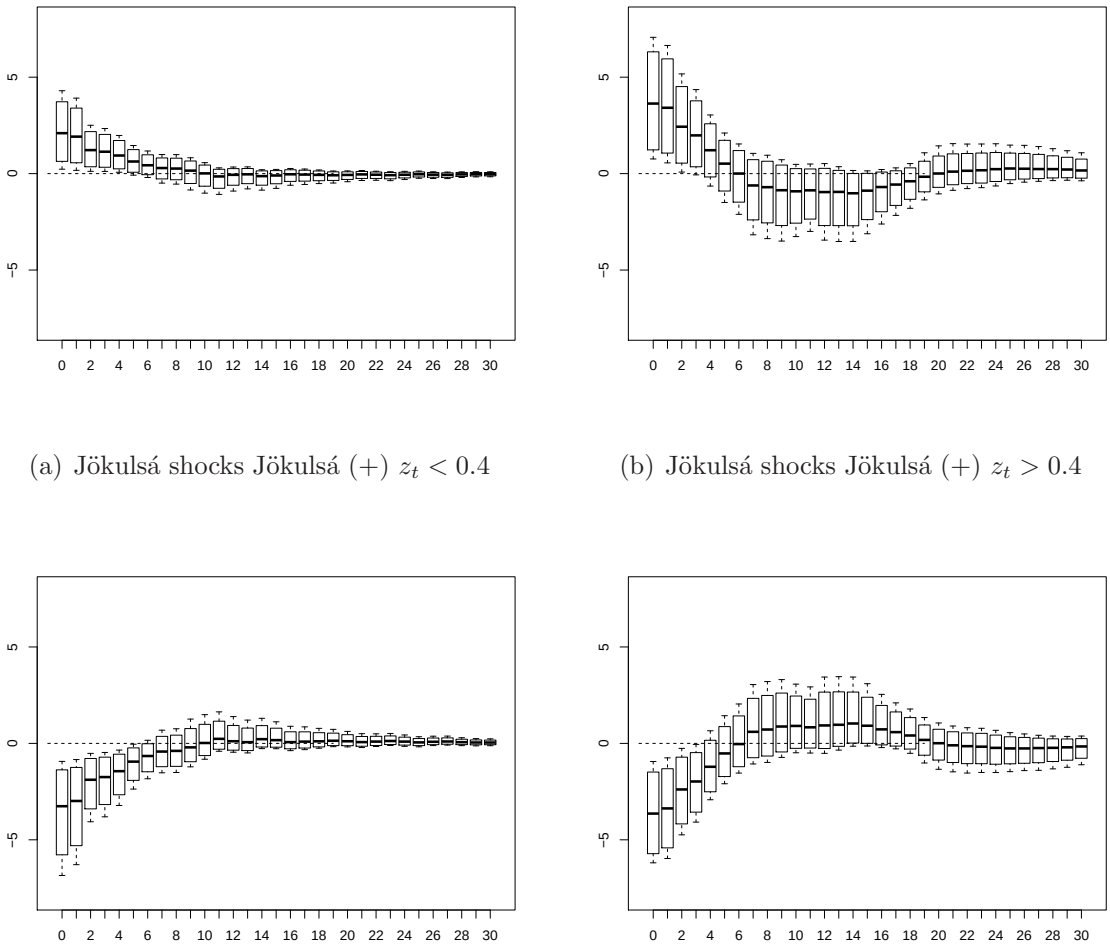


(b) Transition variable; the inflection point (solid line) and the interval for the smooth transition (dashed lines).



(c) Values of the transition function over time.

Figure 11: Transitions of the Vatnsdalsá flow driven by the precipitation.



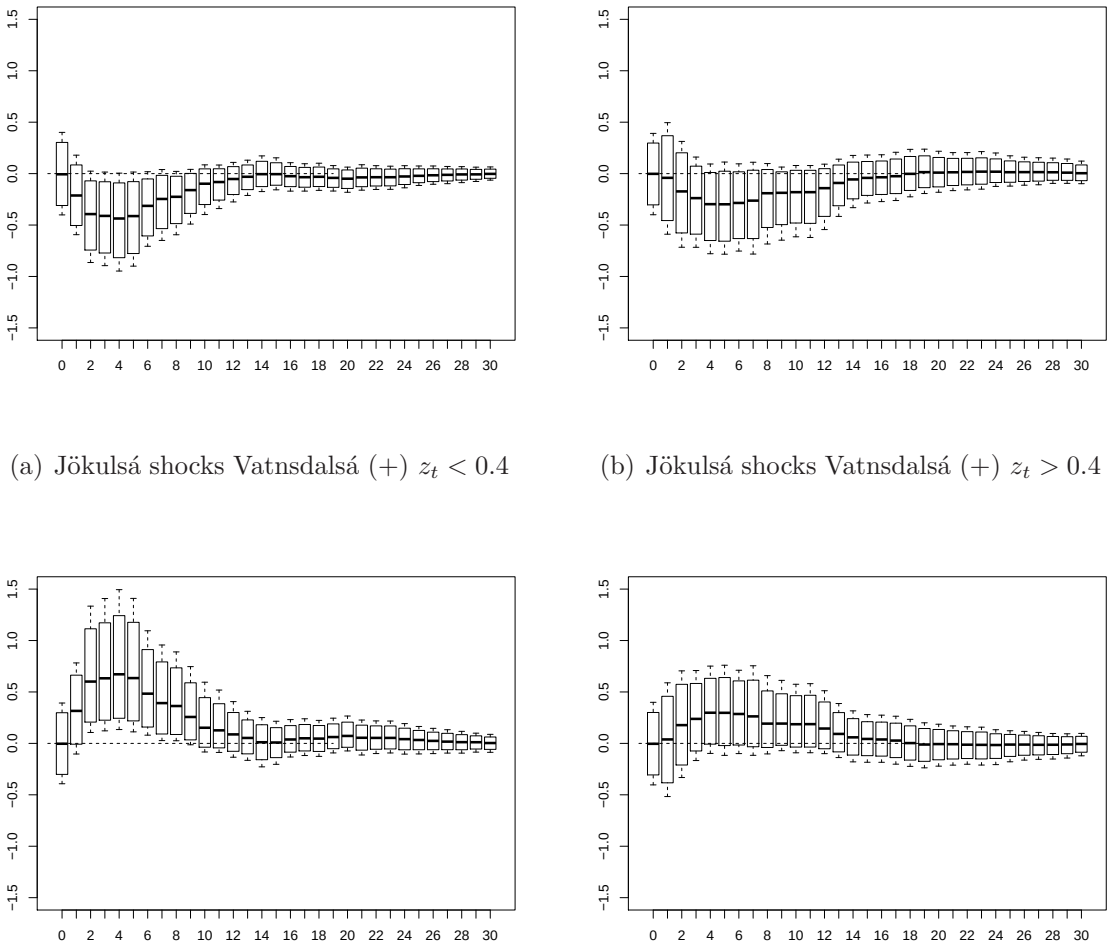
(a) Jökulsá shocks Jökulsá (+) $z_t < 0.4$

(b) Jökulsá shocks Jökulsá (+) $z_t > 0.4$

(c) Jökulsá shocks Jökulsá (-) $z_t < 0.4$

(d) Jökulsá shocks Jökulsá (-) $z_t > 0.4$

Figure 12: Boxplots of generalized impulse response functions of the LVSTAR model (7.4) and (7.5) represented with 50 percent (box) and 80 percent (whisker) highest density regions. Top left panel: Jökulsá to **positive** Jökulsá shocks below freezing; Top right panel: The same above freezing. Bottom left panel: Jökulsá to **negative** Jökulsá shocks below freezing; Bottom right panel: The same above freezing.



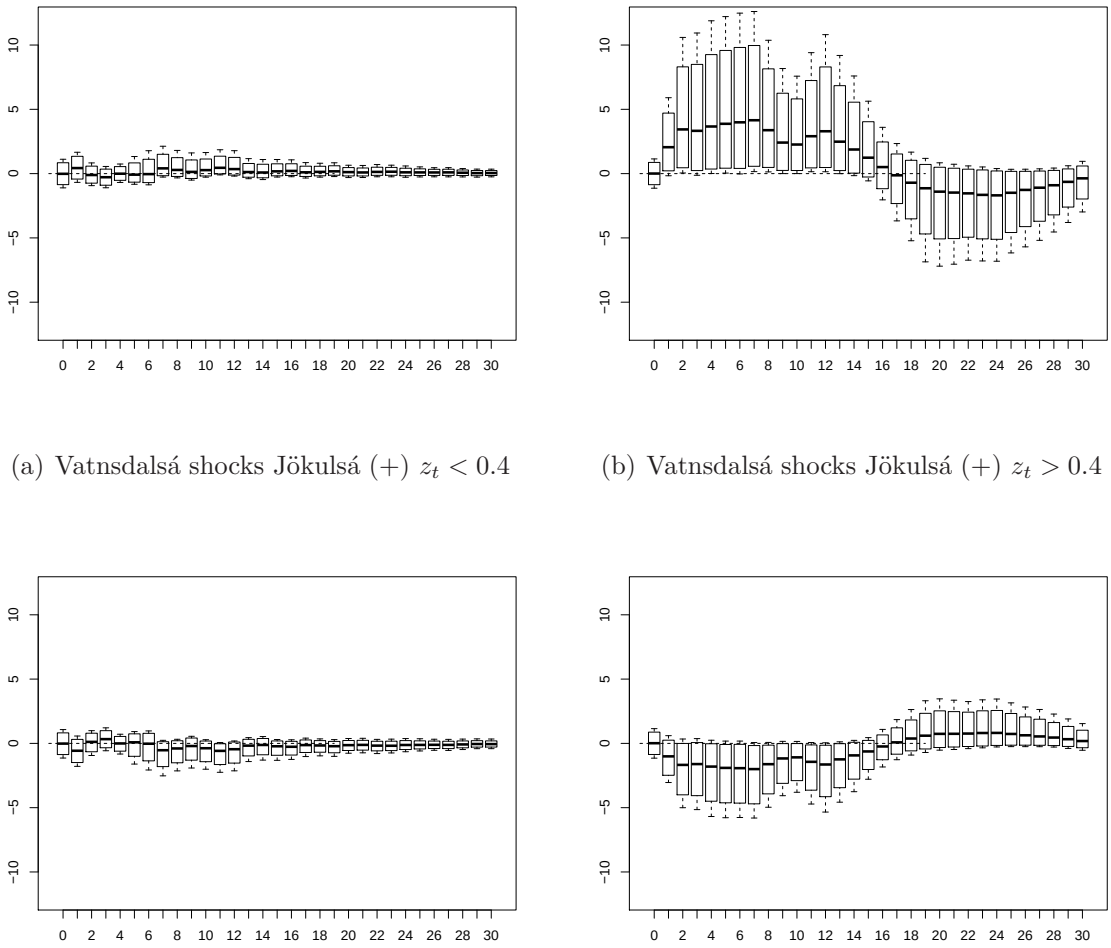
(a) Jökulsá shocks Vatnsdalsá (+) $z_t < 0.4$

(b) Jökulsá shocks Vatnsdalsá (+) $z_t > 0.4$

(c) Jökulsá shocks Vatnsdalsá (-) $z_t < 0.4$

(d) Jökulsá shocks Vatnsdalsá (-) $z_t > 0.4$

Figure 13: Boxplots of generalized impulse response functions of the LVSTAR model (7.4) and (7.5) represented with 50 percent (box) and 80 percent (whisker) highest density regions. Top left panel: Vatnsdalsá to **positive** Jökulsá shocks below freezing; Top right panel: The same above freezing. Bottom left panel: Vatnsdalsá to **negative** Jökulsá shocks below freezing; Bottom right panel: The same above freezing.



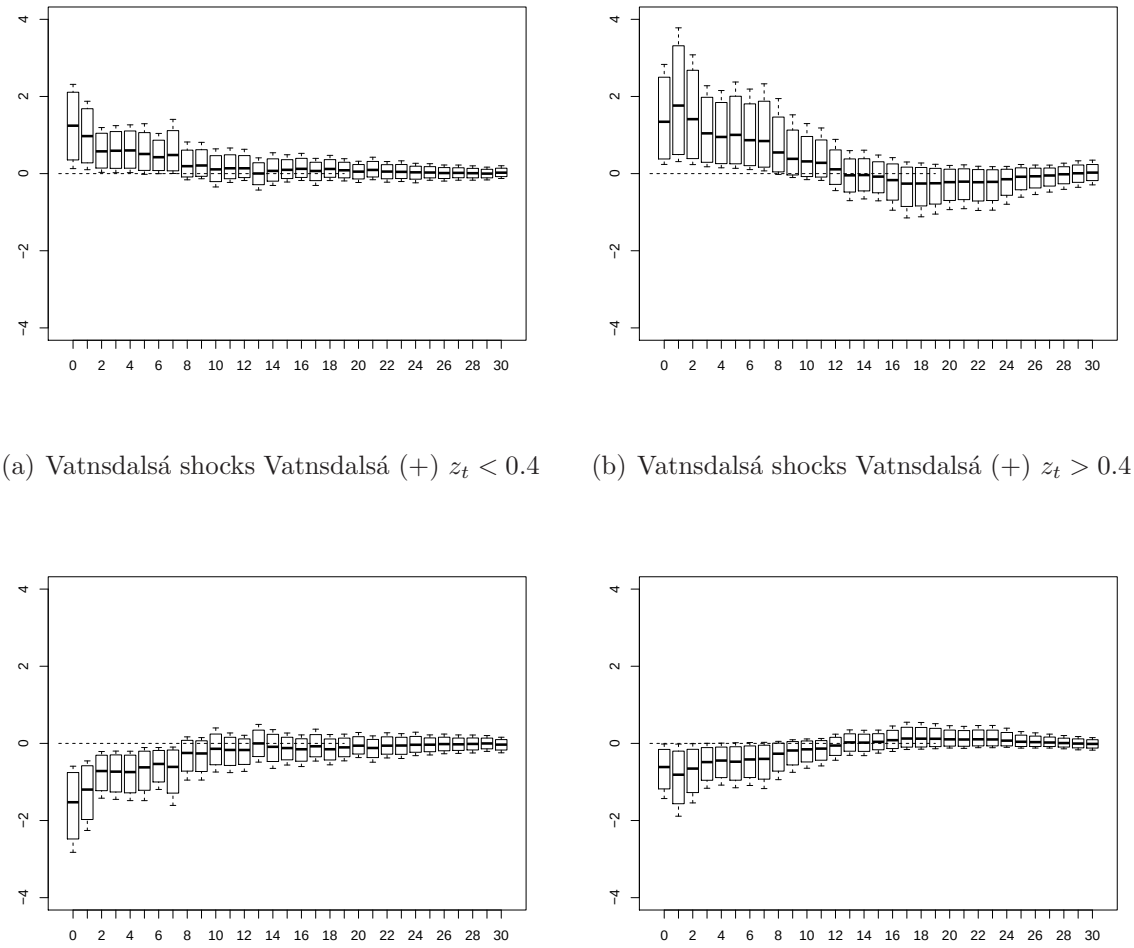
(a) Vatnsdalsá shocks Jökulsá (+) $z_t < 0.4$

(b) Vatnsdalsá shocks Jökulsá (+) $z_t > 0.4$

(c) Vatnsdalsá shocks Jökulsá (-) $z_t < 0.4$

(d) Vatnsdalsá shocks Jökulsá (-) $z_t > 0.4$

Figure 14: Boxplots of generalized impulse response functions of the LVSTAR model (7.4) and (7.5) represented with 50 percent (box) and 80 percent (whisker) highest density regions. Top left panel: Jökulsá to **positive** Vatnsdalsá shocks below freezing; Top right panel: The same above freezing. Bottom left panel: Jökulsá to **negative** Vatnsdalsá shocks below freezing; Bottom right panel: The same above freezing.



(a) Vatnsdalsá shocks Vatnsdalsá (+) $z_t < 0.4$

(b) Vatnsdalsá shocks Vatnsdalsá (+) $z_t > 0.4$

(c) Vatnsdalsá shocks Vatnsdalsá (-) $z_t < 0.4$

(d) Vatnsdalsá shocks Vatnsdalsá (-) $z_t > 0.4$

Figure 15: Boxplots of generalized impulse response functions of the LVSTAR model (7.4) and (7.5) represented with 50 percent (box) and 80 percent (whisker) highest density regions. Top left panel: Vatnsdalsá to **positive** Vatnsdalsá shocks below freezing; Top right panel: The same above freezing. Bottom left panel: Vatnsdalsá to **negative** Vatnsdalsá shocks below freezing; Bottom right panel: The same above freezing.

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