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CREATES Research Paper 2013-40

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Polynomial Regressions and Nonsense Inference

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November, 2013

Abstract

Polynomial specifications are widely used, not only in applied economics, but also in epidemiology, physics, political analysis, and psychology, just to mention a few examples. In many cases, the data employed to estimate such specifications are time series that may exhibit stochastic nonstationary behavior. We extend Phillips' (1986) results by proving an inference drawn from polynomial specifications, under stochastic nonstationarity, is misleading unless the variables cointegrate. We use a generalized polynomial specification as a vehicle to study its asymptotic and finite-sample properties. Our results, therefore, lead to a call to be cautious whenever practitioners estimate polynomial regressions.

Keywords: Polynomial Regression; misleading Inference; Integrated Processes.

JEL Classification: C12, C15, C22

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[‡] The first draft of the article was written while the author was visiting the Center for Research and Teaching in Economics (CIDE). He gratefully acknowledges Alejandro López-Feldman for his support. The author acknowledges support from CREATES - Center for Research in Econometric Analysis of Time Series (DNRF78), funded by the Danish National Research Foundation.

1 Introduction

There is some research on the effects of nonstationarity of the variables on nonlinear relationships (spurious inference on linear specifications was uncovered by Granger and Newbold (1974), and later explained by Phillips (1986)). Lee, Kim, and Newbold (2005) show (both, in finite samples and asymptotically) that six nonlinear tests,[1] when applied to independent random walks, tend to identify spurious (non existing) nonlinear relationships.[2] O'Brien (2008) extends these results by studying the behavior of two additional tests; the BDS test and another one proposed by Peña and Rodriguez (2005); he finds that the former also yields results that don't make sense, whilst the latter proves to have good power properties even in small samples. Wagner (2012) studies the properties of the non-parametric Phillip's unit root test applied to polynomials of integrated processes, and concludes, broadly speaking, that the tests does not possess an asymptotic nuisance-parameter-free distribution except under very specific conditions.

To the best of our knowledge, the "nonlinear relationship-spurious inference" literature (briefly sketched earlier) focuses on statistical tests rather than polynomial regressions. The latter are used to linearly relate the dependent variable to a k^{th} order polynomial on an independent variable x. Such regressions therefore fit (through ordinary least squares, ols) a nonlinear relationship between a polynomial on the independent variable and the conditional mean of y.

These specifications can be traced back to the nineteenth century, to impute series (see Gergonne (1815)). Despite its old age, polynomial regressions remain widely used in a large number of scientific fields which include, epidemiology/disease progression (Chatterjee and Sarkar (2009)), geophysics (Verma (2009)), physics (Barker, Street-Perrott, Leng, Greenwood, Swain, Perrott, Telford, and Ficken, 2001, p. 2310), political analysis (Green, Leong, Kern, Gerber, and Larimer (2009)), psychology Shanock, Baran, Gentry, Pattison, and Heggestad (2010), and of course, statistics. Splines regression models (cubic splines, for example) can be used to smooth / impute series.

In empirical economics, polynomial specifications can be found in many subfields, such as, financial economics (Ioannidis, Peel, and Peel (2003); Ferrer, González, and Soto (2010)), labour economics (Leonardi and Pica (2013); Straka

[1]Five of these tests are well known: RESET test, McLeod and Li test, Keenan test, Neural Network test, and White's information matrix. The last one was proposed by Hamilton (2001).

[2]It is noteworthy that de Jong (2003) studied the spurious regression phenomenon under stochastic nonstationarity when the logarithms of independent I(1) variables are used. Logarithmic transformations are commonly used in applied studies to deal with nonlinearity.

(1993)), agricultural economics (Ackello-Ogutu, Paris, and Williams (1985)), macroeconomics (exchange rates, Darvas (2008)) and environmental economics (Auffhammer and Kellogg (2011); Kellenberg (2012)). An evocative example can be found in the empirical research dealing with the Kuznets curve, and the environmental Kuznets curve; the inverse U-shaped relationship between the variables is typically specified as the dependent variable regressed on the independent and its square (see Grossman and Krueger (1993); Labson and Crompton (1993); it is noteworthy that Kuznetz specifications usually employ even-order polynomials).

Even though polynomial regressions remain an important empirical tool, we could not find in the literature any attempt to study their properties when the variables behave as independent nonstationary processes. This might be so because the effect of nonstationarity is rather intuitive and econometricians, at least those familiar with the spurious regression, could speculate that t-ratios diverge and the R^2 does not collapse. However, many researchers in diverse fields seem to be unaware of this possibility.

In this paper, we confirm that an inference drawn from a polynomial regression, when the variables are generated as independent integrated processes, is misleading.[3] We provide evidence that generalizes Phillip's results in two new directions: (i) We allow for the exponent of the variables, both explanatory and dependent in a bivariate regression, take any natural number; (ii) we allow for an arbitrary (natural number) order for the polynomial in x in a k-variate regression. The main objective of this work is to warn practitioners about the considerable risks of spurious inference when powers of a nonstationary variable are used as regressors.

This paper is organized in a very simple manner. Next section presents the data-generating processes (DGPs) and the main results, divided in two theorems. A small Monte Carlo shows that the asymptotics are a sufficiently accurate representation of finite sample behavior of the regressions.

2 Asymptotics of polynomial regressions

The variables, both dependent and independent, are generated as independent driftless unit roots,

$$z_t = z_{t-1} + u_{z,t}, \tag{1}$$

[3]When the variables cointegrate, inference drawn from such a specification is no longer misleading.

for $z = x, y$. The innovations, $u_{y,t}$ and $u_{x,t}$ are independent of each other and obey the conditions stated in Phillips's (1986, p. 313) Assumption 1. We use these variables to estimate the following specification:

$$y_t^m = \alpha + \beta x_t^k + u_t, \tag{2}$$

where $m, k \in \mathbb{N}$. A word on notation; the symbol $\xrightarrow{D}$ denotes weak convergence and, for simplicity, $W_z \equiv W_z(r)$, for $z = x, y$, denotes a Wiener standard process. The stochastic integral $\int_0^1$ is written as $\int$.

Theorem 1. Let $\{y_t\}_{t=1}^{\infty}$, and $\{x_t\}_{t=1}^{\infty}$ be independently generated by eq. (1). Estimate by ols specification (2). Then, as $T \to \infty$:

1. $T^{-\frac{m}{2}}\hat{\alpha} \xrightarrow{D} \sigma_y^m \left[\frac{\int w_y^m \int w_x^{2k} - \int w_x^k w_y^m \int w_x^k}{\int w_x^{2k} - \left(\int w_x^k\right)^2}\right]$,

2. $T^{-\frac{1}{2}(m-k)}\hat{\beta} \xrightarrow{D} \frac{\sigma_y^m}{\sigma_x^k}\left[\frac{\int w_x^k w_y^m - \int w_x^k \int w_y^m}{\int w_x^{2k} - \left(\int w_x^k\right)^2}\right] \equiv \frac{\sigma_y^m}{\sigma_x^k}\widetilde{\beta}$,

3. $T^{-\frac{1}{2}}t_{\hat{\beta}} \xrightarrow{D} \frac{\int w_x^k w_y^m - \int w_x^k \int w_y^m}{\left[\left(\int w_x^{2k} - \left(\int w_x^k\right)^2\right)\left(\int w_y^{2m} - \left(\int w_y^m\right)^2 - \left(\int w_x^k w_y^m - \int w_x^k \int w_y^m\right)^2\right)\right]^{\frac{1}{2}}}$,

4. $R^2 \xrightarrow{D} \widetilde{\beta}^2 \left[\frac{\int w_x^{2k} - \left(\int w_x^k\right)^2}{\int w_y^{2m} - \left(\int w_y^m\right)^2}\right]$.

Proof: see appendix A.

Note that all these results are an extension of Phillips (1986),[4] which implies that, no matter what power does the practitioner applies to the variables, the spurious regression phenomenon remains identical. That said, a more interesting specification should allow for a more complete polynomial of the independent variable, as in

$$y_t = \beta_0 + \beta_1 x_t + \beta_2 x_t^2 + \cdots + \beta_k x_t^k + u_t, \tag{3}$$

where $k \in \mathbb{N}$. In this case, ols estimates still generate a spurious regression:

Theorem 2. Let $\{y_t\}_{t=1}^{\infty}$ and $\{x_t\}_{t=1}^{\infty}$, be independently generated by eq. (1). Estimate by ols specification (3). Then, as $T \to \infty$:

[4] It is noteworthy to mention that, for $k = m = 1$, our results are exactly those of Phillips (1986).

1.
$$\begin{bmatrix} T^{-\frac{1}{2}}\hat{\beta}_0 \\ \hat{\beta}_1 \\ T^{\frac{1}{2}}\hat{\beta}_2 \\ \vdots \\ T^{\frac{1}{2}(k-1)}\hat{\beta}_k \end{bmatrix} \overset{D}{\to} \sigma_y \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_x & 0 & \cdots & 0 \\ 0 & 0 & \sigma_x^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_x^k \end{pmatrix}^{-1} \times$$
$$\begin{pmatrix} 1 & \int w_x & \int \omega_x^2 & \cdots & \int w_x^k \\ \int w_x & \int \omega_x^2 & \int \omega_x^3 & \cdots & \int \omega_x^{k+1} \\ \int \omega_x^2 & \int \omega_x^3 & \cdots & \cdots & \int \omega_x^{k+2} \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ \int w_x^k & \cdots & \cdots & \cdots & \int \omega_x^{2k} \end{pmatrix}^{-1} \begin{pmatrix} \int w_y \\ \int \omega_x \omega_y \\ \int \omega_x^2 \omega_y \\ \vdots \\ \int \omega_x^k \omega_y \end{pmatrix}.$$

2. $S^2 = Op(T)$ where $S^2 = T^{-1}\sum_{t=1}^{T}\left(y_t - \sum_{i=0}^{k}\hat{\beta}_i x_t^i\right)^2$.

3. $t_{\hat{\beta}_i} = Op\left(T^{\frac{1}{2}}\right)$ for $i = 0, 1, 2, \ldots, k$.

Proof: see appendix B.

Note the linear pattern in the order of convergence of the parameters; whilst the constant term, $\hat{\beta}_0$ diverges at rate $T^{\frac{1}{2}}$, $\hat{\beta}_1$ neither diverges, nor collapses, $\hat{\beta}_2$ collapses at rate $T^{-\frac{1}{2}}$, and so on. Even though, all the t-ratios of the estimated parameters diverge at the usual rate $T^{\frac{1}{2}}$. In both theorems, the convergence rate of the t-ratios associated with the estimates diverge. This implies that, for a sufficiently large sample, the null hypothesis that the parameters are equal to zero will eventually be rejected. Finite sample evidence suggests that this actually occurs in even rather small samples of $100 - 500$ observations (Table 1).

3 Concluding remarks

In this paper we extended the results of what is known as spurious inference by studying the asymptotic and finite-sample behavior of the t-ratios in an OLS-estimated regression where the dependent variable and/or the explanatory variable are nonlinearly tranformed by means of a polynomial. When the variables are independent and stochastically nonstationary, the inference based on OLS estimates

Table 1: **Rejection rates of t-ratios**

T	k	Specification (2)			Specification (3)			
		m			with k=4			
		1	2	3	β_1	β_2	β_3	β_4
	1	0.77	0.71	0.71				
100	2	0.71	0.66	0.65	0.46	0.35	0.33	0.31
	3	0.72	0.66	0.66				
	1	0.85	0.82	0.82				
250	2	0.81	0.78	0.78	0.64	0.56	0.52	0.50
	3	0.82	0.78	0.78				
	1	0.89	0.87	0.87				
500	2	0.86	0.84	0.84	0.73	0.67	0.64	0.63
	3	0.88	0.84	0.84				

Rejection rates of t-ratio associated to: (i) for specification (2), $\hat{\beta}$; (ii) for specification (3) all β's. DGP parameters: $u_{z,t} \sim iid\mathcal{N}(0,1)$, for $z = x, y$. The code of this Monte Carlo experiment is available as supplementary material.

is misleading (our results concern pure I(1) processes, but provide a natural guide to future research; near-integration, I(2) and broken linear trend processes should be further studied). This result should be understood as a call to be cautious whenever practitioners estimate polynomial regressions.

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A Proof of Theorem 1

Proof. In order to get all of the results we use the asymptotic results provided in Giles (2007):

1. $T^{-\frac{1}{2}(k+2)} \sum \xi_{z,t-1}^{k} \xrightarrow{D} \sigma_z^k \int_0^1 \left[\omega_z(r)\right]^k dr$,
2. $T^{-\frac{1}{2}(k+m+2)} \sum \xi_{x,t-1}^{k} \xi_{y,t-1}^{m} \xrightarrow{D} \sigma_x^k \sigma_y^m \int w_x^k w_y^m$.

We now define the rates of convergences of ols estimates ($\sum$ is short for $\sum_{t=1}^{T}$).

$$
\begin{aligned}
\begin{bmatrix} \hat{\alpha} \\ \\ \hat{\beta} \end{bmatrix} &= \begin{bmatrix} T & \sum x_t^k \\ \\ \sum x_t^k & \sum x_t^{2k} \end{bmatrix}^{-1} \begin{bmatrix} \sum y_t^m \\ \\ \sum x_t^k y_t^m \end{bmatrix}, \\
&= \begin{bmatrix} O(T) & Op\left(T^{\frac{1}{2}(k+2)}\right) \\ \\ Op\left(T^{\frac{1}{2}(k+2)}\right) & Op\left(T^{k+1}\right) \end{bmatrix}^{-1} \begin{bmatrix} Op\left(T^{\frac{1}{2}(m+2)}\right) \\ \\ Op\left(T^{\frac{1}{2}(m+k+2)}\right) \end{bmatrix}.
\end{aligned}
$$

By simple algebra, we get:

$$
\begin{bmatrix} \hat{\alpha} \\ \\ \hat{\beta} \end{bmatrix} = \begin{bmatrix} Op\left(T^{\frac{m}{2}}\right) \\ \\ Op\left(T^{\frac{1}{2}(m-k)}\right) \end{bmatrix}.
$$

Therefore:

$$\begin{bmatrix} T^{-\frac{m}{2}}\ \hat{\alpha} \\ \\ T^{-\frac{1}{2}(m-k)}\ \hat{\beta} \end{bmatrix} \overset{D}{\to} \begin{bmatrix} 1 & \sigma_x^k \int w_x^k \\ \\ \sigma_x^k \int w_x^k & \sigma_x^{2k} \int w_x^{2k} \end{bmatrix}^{-1} \begin{bmatrix} \sigma_y^m \int w_y^m \\ \\ \sigma_y^m \sigma_x^k \int w_x^k w_y^m \end{bmatrix}$$

$$= \frac{1}{\sigma_x^{2k}\left(\int w_x^{2k} - \left(\int w_x^k\right)^2\right)} \begin{bmatrix} \sigma_x^{2k} \int w_x^{2k} & -\sigma_x^k \int w_x^k \\ \\ -\sigma_x^k \int w_x^k & 1 \end{bmatrix} \times \begin{bmatrix} \sigma_y^m \int w_y^m \\ \\ \sigma_y^m \sigma_x^k \int w_x^k w_y^m \end{bmatrix},$$

$$= \frac{1}{\sigma_x^{2k}(\int w_x^{2k} - \left(\int w_x^k\right)^2)} \begin{bmatrix} \sigma_y^m \sigma_x^{2k} \left(\int w_y^m \int w_x^{2k} - \int w_x^k w_y^m \int w_x^k\right) \\ \\ \sigma_y^m \sigma_x^k (\int w_x^k w_y^m - \int w_x^k \int w_y^m) \end{bmatrix}.$$

Finally,

$$\begin{bmatrix} T^{-\frac{m}{2}}\ \hat{\alpha} \\ \\ T^{-\frac{1}{2}(m-k)}\ \hat{\beta} \end{bmatrix} \overset{D}{\to} \begin{bmatrix} \sigma_y^m \left[\frac{\int w_y^m \int w_x^{2k} - \int w_x^k w_y^m \int w_x^k}{\int w_x^{2k} - \left(\int w_x^k\right)^2}\right] \\ \\ \frac{\sigma_y^m}{\sigma_x^k} \left[\frac{\int w_x^k w_y^m - \int w_x^k \int w_y^m}{\int w_x^{2k} - \left(\int w_x^k\right)^2}\right] \end{bmatrix} \text{as } T \to \infty, \quad (4)$$

which proves results 1 and 2 in Theorem 1.

Let $\tilde{\beta} \equiv \frac{\int w_x^k w_y^m - \int w_x^k \int w_y^m}{\int w_x^{2k} - \left(\int w_x^k\right)^2}$; following Phillips (1986), to get $t_{\hat{\beta}}$, we define $S^2 = T^{-1} \sum \left(y_t^m - \hat{\alpha} - \hat{\beta} x_t^k\right)^2$. Then,

$$T^{-m} S^2 = T^{-(m+1)} \sum \left[(y_t^m - \bar{y}) - \hat{\beta}\left(x_t^k - \bar{x}\right)\right]^2,$$

$$= T^{-(m+1)} \sum (y_t^m - \bar{y})^2 - \hat{\beta}^2\, T^{-(m+1)} \sum \left(x_t^k - \bar{x}\right)^2,$$

$$T^{-m} S^2 \overset{D}{\to} \sigma_y^{2m} \left[\int w_y^{2m} - \left(\int w_y^m\right)^2 - \tilde{\beta}^2 \left(\int w_x^{2k} - \left(\int w_x^k\right)^2\right)\right] \quad (5)$$

as $T \to \infty$.

Then, we use equations (4) and (5) to get,

$$\begin{aligned}
T^{-\frac{1}{2}} t_{\hat{\beta}} &= \frac{\hat{\beta}}{T^{\frac{1}{2}} S_{\hat{\beta}}}, \\
&= \frac{\hat{\beta}}{T^{\frac{1}{2}} S \left[\sum \left(x_t^k - \bar{x} \right)^2 \right]^{-\frac{1}{2}}}, \\
&= \frac{\left(T^{-\frac{1}{2}(m-k)} \right) \hat{\beta} T \left(T^{-(k+1)} \sum \left(x_t^k - \bar{x} \right)^2 \right)^{\frac{1}{2}}}{T \left(T^{-\frac{m}{2}} S \right)}, \\
T^{-\frac{1}{2}} t_{\hat{\beta}} &\xrightarrow{D} \frac{\frac{\sigma_y^m}{\sigma_x^k} \tilde{\beta} \sigma_x^k \left[\int w_x^{2k} - \left(\int w_x^k \right)^2 \right]^{\frac{1}{2}}}{\sigma_y^m \left[\int w_y^{2m} - \left(\int w_y^m \right)^2 - \tilde{\beta}^2 \left(\int w_x^{2k} - \left(\int w_x^k \right)^2 \right) \right]^{\frac{1}{2}}} \quad \text{as } T \to \infty.
\end{aligned}$$

Then, after simple algebra we get,

$$T^{-\frac{1}{2}} t_{\hat{\beta}} \xrightarrow{D} \frac{\int w_x^k w_y^m - \int w_y^m \int w_x^k}{\left[\left(\int w_x^{2k} - \left(\int w_x^k \right)^2 \right) \left(\int w_y^{2m} - \left(\int w_y^m \right)^2 - \left(\int w_x^k w_y^m - \int w_x^k \int w_y^m \right)^2 \right) \right]^{\frac{1}{2}}}, \quad \text{as } T \to \infty,$$

proving result 3 of Theorem 1.

Finally, the asymptotic nonstandard distribution of R^2 is given by:

$$\begin{aligned}
R^2 &= \frac{\sum (\hat{y}_t^m - \bar{y})^2}{\sum (y_t^m - \bar{y})^2}, \\
&= \frac{\hat{\beta}^2 \, T^{-(m-k)} T^{-(k+1)} \sum \left(x_t^k - \bar{x} \right)^2}{T^{-(m+1)} \sum (y_t^m - \bar{y})^2}, \\
R^2 &\xrightarrow{D} \frac{\tilde{\beta}^2 \left[\int w_x^{2k} - \left(\int w_x^k \right)^2 \right]}{\int w_y^{2m} - \left(\int w_y^m \right)^2}, \quad \text{as } T \to \infty.
\end{aligned}$$

This proves the last result of theorem 1. □

B Proof of Theorem 2.

Proof. The polynomial specification (3) has the following ols estimators:

$$\begin{bmatrix} \hat\beta_0 \\ \hat\beta_1 \\ \hat\beta_2 \\ \vdots \\ \hat\beta_k \end{bmatrix} = \begin{pmatrix} T & \sum x_t & \sum x_t^2 & \cdots & \sum x_t^k \\ \sum x_t & \sum x_t^2 & \sum x_t^3 & \cdots & \sum x_t^{k+1} \\ \sum x_t^2 & \sum x_t^3 & \cdots & \cdots & \sum x_t^{k+2} \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ \sum x_t^k & \cdots & \cdots & \cdots & \sum x_t^{2k} \end{pmatrix}^{-1} \begin{pmatrix} \sum y_t \\ \sum x_t y_t \\ \sum x_t^2 y_t \\ \vdots \\ \sum x_t^k y_t \end{pmatrix},$$

or $\hat{B} = \Sigma_{xx}^{-1}\Sigma_{xy}$, for short. To obtain the rates of convergences of the ols estimates, note that Σ_{xx}^{-1} is a Hankel matrix. The orders of convergence of each element in such a matrix are given by:

$$\begin{pmatrix} O(T) & Op\left(T^{\frac{3}{2}}\right) & Op\left(T^2\right) & \cdots & Op\left(T^{\frac{1}{2}(k+2)}\right) \\ Op\left(T^{\frac{3}{2}}\right) & Op\left(T^2\right) & Op\left(T^{\frac{5}{2}}\right) & \cdots & Op\left(T^{\frac{1}{2}(k+3)}\right) \\ Op\left(T^2\right) & Op\left(T^{\frac{5}{2}}\right) & \cdots & \cdots & Op\left(T^{\frac{1}{2}(k+4)}\right) \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ Op\left(T^{\frac{1}{2}(k+2)}\right) & \cdots & \cdots & \cdots & Op\left(T^{k+1}\right) \end{pmatrix}.$$

The Hankel matrix can be inverted using some results from linear algebra theory (spectral decomposition).[5] That said, we are not interested on computing the exact inverse, but rather to use cases with $k = 1, 2, 3, \ldots$. For specification

[5]Also, while it would be possible to analyze some interesting properties of Hankel matrices given by Hannan and Deistler (2012) or Golub and Van Loan (2012) *inter alia*, there are some numerical algorithms like Trench (1965), or Kumar and Alsaleh (1996) for polynomial regressions, that work with a Hankel matrix.

(3), it is straightforward to see that:

$$\Sigma_{xx}^{-1}=\begin{pmatrix} O(T^{-1}) & Op\left(T^{-\frac{3}{2}}\right) & Op\left(T^{-2}\right) & \cdots & Op\left(T^{-\frac{1}{2}(k+2)}\right) \\ Op\left(T^{-\frac{3}{2}}\right) & Op\left(T^{-2}\right) & Op\left(T^{-\frac{5}{2}}\right) & \cdots & Op\left(T^{-\frac{1}{2}(k+3)}\right) \\ Op\left(T^{-2}\right) & Op\left(T^{-\frac{5}{2}}\right) & \cdots & \cdots & Op\left(T^{-\frac{1}{2}(k+4)}\right) \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ Op\left(T^{-\frac{1}{2}(k+2)}\right) & \cdots & \cdots & \cdots & Op\left(T^{-(k+1)}\right) \end{pmatrix},$$

which is again a Hankel matrix. We follow Hamilton (1994) to obtain the orders of convergence and the asymptotic distributions of ols estimates. We first define the following matrices:

$$\gamma_1=\begin{pmatrix} T^{-\frac{1}{2}} & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & T^{\frac{1}{2}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & T^{\frac{1}{2}(k-1)} \end{pmatrix}, \tag{6}$$

and

$$\gamma_2=\begin{pmatrix} T^{\frac{3}{2}} & 0 & 0 & \cdots & 0 \\ 0 & T^{2} & 0 & \cdots & 0 \\ 0 & 0 & T^{\frac{5}{2}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & T^{\frac{1}{2}(k+3)} \end{pmatrix}. \tag{7}$$

Then, using matrices (6) and (7) we have $\gamma_1 \hat{B} = \gamma_1 \left(\sum_{xx}^{-1}\right) \gamma_2 \gamma_2^{-1} \left(\sum_{xy}\right)$. Finally, we get: $\gamma_1 \hat{B} = \left\{\gamma_1^{-1} \sum_{xx}^{-1} \gamma_2^{-1}\right\}^{-1} \left\{\gamma_2^{-1} \left(\sum_{xy}\right)\right\}$.

We have:

$$\begin{bmatrix} T^{-\frac{1}{2}} & & & & \\ & 1 & & & \\ & & T^{\frac{1}{2}} & & \\ & & & \ddots & \\ & & & & T^{\frac{1}{2}(k-1)} \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_k \end{bmatrix} =$$

$$\left\{ \begin{bmatrix} T^{\frac{1}{2}} & & & & \\ & 1 & & & \\ & & T^{-\frac{1}{2}} & & \\ & & & \ddots & \\ & & & & T^{-\frac{1}{2}(k-1)} \end{bmatrix} \begin{bmatrix} T & \sum x_t & \sum x_t^2 & \cdots & \sum x_t^k \\ \sum x_t & \sum x_t^2 & \sum x_t^3 & \cdots & \sum x_t^{k+1} \\ \sum x_t^2 & \sum x_t^3 & \cdots & \cdots & \sum x_t^{k+2} \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ \sum x_t^k & \cdots & \cdots & \cdots & \sum x_t^{2k} \end{bmatrix} \right.$$

$$\left. \begin{bmatrix} T^{-\frac{3}{2}} & & & & \\ & T^{-2} & & & \\ & & T^{-\frac{5}{2}} & & \\ & & & \ddots & \\ & & & & T^{-\frac{1}{2}(k+3)} \end{bmatrix} \right\}^{-1} \times \begin{bmatrix} T^{-\frac{3}{2}} & & & & \\ & T^{-2} & & & \\ & & T^{-\frac{5}{2}} & & \\ & & & \ddots & \\ & & & & T^{-\frac{1}{2}(k+3)} \end{bmatrix} \tag{8}$$

$$\begin{pmatrix} \sum y_t \\ \sum x_t \sum y_t \\ \sum x_t^2 \sum y_t \\ \vdots \\ \sum x_t^k \sum y_t \end{pmatrix}$$

We then multiply the matrices of equation (8):

$$\begin{bmatrix} T^{-\frac{1}{2}}\hat{\beta}_0 \\ \hat{\beta}_1 \\ T^{\frac{1}{2}}\hat{\beta}_2 \\ \vdots \\ T^{\frac{1}{2}(k-1)}\hat{\beta}_k \end{bmatrix} \xrightarrow{D} \begin{pmatrix} 1 & \sigma_x \int w_x & \sigma_x^2 \int \omega_x^2 & \cdots & \sigma_x^k \int w_x^k \\ \sigma_x \int w_x & \sigma_x^2 \int \omega_x^2 & \sigma_x^3 \int \omega_x^3 & \cdots & \sigma_x^{k+1} \int \omega_x^{k+1} \\ \sigma_x^2 \int \omega_x^2 & \sigma_x^3 \int \omega_x^3 & \cdots & \cdots & \sigma_x^{k+2} \int \omega_x^{k+2} \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ \sigma_x^k \int w_x^k & \cdots & \cdots & \cdots & \sigma_x^{2k} \int \omega_x^{2k} \end{pmatrix}^{-1} \times \begin{pmatrix} \sigma_y \int w_y \\ \sigma_x\sigma_y \int \omega_x\omega_y \\ \sigma_x^2\sigma_y \int \omega_x^2\omega_y \\ \vdots \\ \sigma_x^k\sigma_y \int \omega_x^k\omega_y \end{pmatrix}. \quad (9)$$

Finally, we factor the variances (σ_x, σ_y) from equation (9):

$$\begin{bmatrix} T^{-\frac{1}{2}}\hat{\beta}_0 \\ \hat{\beta}_1 \\ T^{\frac{1}{2}}\hat{\beta}_2 \\ \vdots \\ T^{\frac{1}{2}(k-1)}\hat{\beta}_k \end{bmatrix} \xrightarrow{D} \sigma_y \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_x & 0 & \cdots & 0 \\ 0 & 0 & \sigma_x^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_x^k \end{pmatrix}^{-1} \times$$

$$\begin{pmatrix} 1 & \int w_x & \int \omega_x^2 & \cdots & \int w_x^k \\ \int w_x & \int \omega_x^2 & \int \omega_x^3 & \cdots & \int \omega_x^{k+1} \\ \int \omega_x^2 & \int \omega_x^3 & \cdots & \cdots & \int \omega_x^{k+2} \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ \int w_x^k & \cdots & \cdots & \cdots & \int \omega_x^{2k} \end{pmatrix}^{-1} \begin{pmatrix} \int w_y \\ \int \omega_x\omega_y \\ \int \omega_x^2\omega_y \\ \vdots \\ \int \omega_x^k\omega_y \end{pmatrix},$$

which proves eq. (1) in Theorem 2.

To obtain the asymptotics of the k t-ratios, note that the order of convergence of the estimated variance, S^2,

$$S^2 = T^{-1} \sum_{t=1}^{T} \left(y_t - \hat{\beta}_0 - \hat{\beta}_1 x_t - \hat{\beta}_2 x_t^2 - \cdots - \hat{\beta}_k x_t^k \right)^2, \quad (10)$$

is always equal to T^2. To see this, we expand expression (10) and analyze the convergence order of each element given by the previous result:

$$
\begin{aligned}
S^2 &= T^{-1}\sum_{t=1}^{T}\left(y_t-\hat{\beta}_0-\hat{\beta}_1x_t-\hat{\beta}_2x_t^2-\cdots-\hat{\beta}_kx_t^k\right)^2, \\
&= T^{-1}\Bigg\{\underbrace{\sum y_t^2}_{Op(T^2)}+\underbrace{\sum\hat{\beta}_0^2}_{Op(T^2)}+\underbrace{\hat{\beta}_1^2\sum x_t^2}_{Op(T^2)}+\underbrace{\hat{\beta}_2^2\sum x_t^4}_{Op(T^2)}+\cdots+\underbrace{\hat{\beta}_k^2\sum x_t^{2k}}_{Op(T^2)} \\
&\quad -2\left[\underbrace{\hat{\beta}_0\sum y_t}_{Op(T^2)}-\underbrace{\hat{\beta}_1\sum y_tx_t}_{Op(T^2)}+\underbrace{\hat{\beta}_2\sum y_tx_t^2}_{Op(T^2)}+\underbrace{\hat{\beta}_3\sum y_tx_t^3}_{Op(T^2)}-\cdots-\underbrace{\hat{\beta}_k\sum y_tx_t^k}_{Op(T^2)}\right] \\
&\quad +2\underbrace{\hat{\beta}_0}_{Op(T^{\frac{1}{2}})}\left[\underbrace{\hat{\beta}_1\sum x_t}_{Op(T^{\frac{3}{2}})}+\underbrace{\hat{\beta}_2\sum x_t^2}_{Op(T^{\frac{3}{2}})}+\underbrace{\hat{\beta}_3\sum x_t^3}_{Op(T^{\frac{3}{2}})}+\underbrace{\hat{\beta}_4\sum x_t^4}_{Op(T^{\frac{3}{2}})}+\cdots+\underbrace{\hat{\beta}_k\sum x_t^k}_{Op(T^{\frac{3}{2}})}\right] \\
&\quad +2\underbrace{\hat{\beta}_1}_{Op(1)}\left[\underbrace{\hat{\beta}_2\sum x_t^3}_{Op(T^2)}+\underbrace{\hat{\beta}_3\sum x_t^4}_{Op(T^2)}+\underbrace{\hat{\beta}_4\sum x_t^5}_{Op(T^2)}+\underbrace{\hat{\beta}_5\sum x_t^5}_{Op(T^2)}+\cdots+\underbrace{\hat{\beta}_k\sum x_t^{k+1}}_{Op(T^2)}\right] \\
&\quad \vdots \\
&\quad +2\underbrace{\hat{\beta}_{k-1}\hat{\beta}_k\sum x_t^{2k-1}}_{Op(T^2)}\Bigg\}.
\end{aligned}
$$

Therefore, $S^2 = Op(T)$ which proves result 2 of Theorem 2.

Finally, $t_{\hat{\beta}_k} = \frac{\hat{\beta}_k}{\left[S^2\sum_{kk}^{-1}(k,k)\right]^{\frac{1}{2}}} = \frac{Op\left(T^{\frac{1}{2}(k-1)}\right)}{\left[Op(T)Op\left(T^{-(k+1)}\right)\right]^{\frac{1}{2}}} = \frac{Op\left(T^{\frac{1}{2}(k-1)}\right)}{\left[Op\left(T^{-\frac{k}{2}}\right)\right]} = Op\left(T^{\frac{1}{2}}\right)$.

This proves Theorem 2. □

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