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Edgeworth expansion for functionals of continuous diffusion processes *

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Abstract

This paper presents new results on the Edgeworth expansion for high frequency functionals of continuous diffusion processes. We derive asymptotic expansions for weighted functionals of the Brownian motion and apply them to provide the Edgeworth expansion for power variation of diffusion processes. Our methodology relies on martingale embedding, Malliavin calculus and stable central limit theorems for semimartingales. Finally, we demonstrate the density expansion for studentized statistics of power variations.

Keywords: diffusion processes, Edgeworth expansion, high frequency observations, power variation.

JEL Classification: C10, C13, C14.

1 Introduction

Edgeworth expansions have been widely investigated by probabilists and statisticians in various settings. Nowadays, there exists a vast amount of literature on Edgeworth expansions in the case of independent random variables (cf. [6]), weakly dependent variables (cf. [9]) or in the framework of martingales ([20], [24]). We refer to classical books [6], [10] and [18] for a comprehensive theory of asymptotic expansions and their applications. We remark that those authors mainly deal with Edgeworth expansions associated with a normal limit.

In the framework of high frequency data (or infill asymptotics), which refers to the sampling scheme in which the time step between two consecutive observations converges

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to zero while the time span remains fixed, a mixed normal limit appears as a typical asymptotic distribution. In the last years a lot of research has been devoted to limit theorems for high frequency observations of diffusion processes or Itô semimartingales, see e.g. [4, 11, 12, 15] among many others. Such limit theorems find manifold applications in parametric and semiparametric inference for diffusion models, estimation of quadratic variation and related objects (see e.g. [5, 19]), testing approaches for semimartingales (see e.g. [1, 8]) or numerical analysis (see e.g. [13]). While asymptotic mixed normality of high frequency functionals has been proved in various settings, the Edgeworth expansions associated with mixed normal limits have not been considered.

In this paper we present the asymptotic expansion for high frequency statistics of continuous diffusion processes. More precisely, we study the Edgeworth expansion of weighted functionals of Brownian motion, where the weight arises from a continuous SDE, and apply the asymptotic results to power variations of continuous SDE's. Finally, we will obtain the density expansion for a studentized version of the power variation.

Our approach is based on the recent work of Yoshida [26], who uses a martingale embedding method to obtain the asymptotic expansion of the characteristic function associated with a mixed normal limit. In a second step the asymptotic density expansion is achieved via the Fourier inversion. Let us briefly sketch the main concepts of [26]. We are given a functional Z_n, which admits the decomposition

$$Z_n = M_n + r_n N_n,$$

where M_n is a leading term, r_n is a deterministic sequence with $r_n \to 0$ and N_n is some tight sequence of random variables. Here M_n is a terminal value of a *continuous* martingale $(M_t^n)_{t\in[0,1]}$, which converges to a mixed normal limit in the functional sense. Under various technical conditions, including Malliavin differentiability of the involved objects, joint stable convergence of (M_n, N_n) and estimates of the tail behaviour of the characteristic function, the paper [26] demonstrated the Edgeworth expansion for the density of Z_n (and, more generally, for the density of the pair (Z_n, F_n), where F_n is another functional usually used for studentization). The asymptotic theory has been applied to quadratic functionals M_n in [26, 27]. We would also like to refer to a related work of [24], where a martingale expansion in the case of normal limits has been presented. It was applied to the Edgeworth expansion for an ergodic diffusion process and an estimator of the volatility parameter (cf. [7]).

Although the paper [26] presents a general theory, its particular application to typical functionals of continuous diffusion processes is by far not straightforward. When dealing with commonly used high frequency statistics, such as e.g. power variations, we are confronted with several levels of complications, which we list below:

(i) The computation of the second order term N_n in the decomposition of Z_n appears to be rather involved (cf. Theorem 4.3). This stochastic second order expansion requires a very precise treatment of the functional Z_n.

(ii) The joint asymptotic mixed normality of the vector (M^n, N_n, F_n, C^n), where C^n is the quadratic variation process associated with the martingale M^n and F_n is an external functional mentioned above, is required for the Edgeworth expansion (cf. Theorem 5.1).

The proof of such results relies on stable limit theorems for semimartingales (cf. Theorems 8.1, 8.2 and 4.4).

(iii) Another ingredients of Edgeworth expansion are the adaptive random symbol $\underline{\sigma}$ and the anticipative random symbol $\overline{\sigma}$ (see [26] or Section 2 for the definition of random symbols). While the adaptive random symbol $\underline{\sigma}$ is given explicitly using the results of (ii), the anticipative random symbol $\overline{\sigma}$ is defined in an implicit way. We will show how this symbol can be determined in Sections 3.3 and 3.4. For this purpose we will apply the Wiener chaos expansion and the duality between the kth Malliavin derivative D^k and its adjoint δ^k.

(iv) Checking the technical conditions presented in Section 2.3 is another difficult task. In particular, we need to show the existence of densities and to analyze the tail behaviour of the characteristic function. This part involves many elements of Malliavin calculus (cf. Section 3.5 and 3.6).

We see that the derivation of the Edgeworth expansion relies on a combination of various fields of stochastic calculus, such as limit theorems for semimartingales, Malliavin calculus and martingale methods. These steps require a completely new treatment in the power variation case, compared with those in simple quadratic functionals.

The paper is organised as follows. In Section 2 we recall some results of [26] and demonstrate an application to simple quadratic functionals. Section 3 is devoted to functionals of Brownian motion with random weights. We will deal with the treatment of the steps (i)-(iv), although the second order term N_n remains absent. In section 4 we show the asymptotic theory for the class of generalized power variations of continuous SDE's. In particular, we will determine the asymptotic behaviour of the second order term N_n. Section 5 combines the results of Sections 3 and 4, and we obtain an Edgeworth expansion for the power variation case. In Section 6 we deduce the formula for the asymptotic density associated with a studentized version of power variation, which is probably most useful for applications. Section 7 is devoted to the derivation of the second order term N_n. Finally, Appendix collects the proofs of limit theorems for semimartingales, which are suitable for functionals considered in this paper.

2 Asymptotic expansion associated with mixed normal limit

As we are applying various techniques from Malliavin calculus and stable central limit theorems for semimartingales, we start by introducing some notation.

(a) Throughout the paper Δ_n denotes a sequence of positive real numbers with $\Delta_n \to 0$ and such that $1/\Delta_n$ is an integer. For the observation times $i\Delta_n$, $i \in \mathbb{N}$, we use a shorthand notation $t_i := i\Delta_n$. For any function $f : \mathbb{R} \to \mathbb{R}$ we denote by $f^{(k)}$ its kth derivative; for a function $f : \mathbb{R}^2 \to \mathbb{R}$ and $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ the operator d^α is defined via $d^\alpha = d_{x_1}^{\alpha_1} d_{x_2}^{\alpha_2}$, where $d_{x_i}^k f$, $i = 1, 2$, denotes the kth partial derivative of f. The set $C_p^k(\mathbb{R})$ (resp. $C_b^k(\mathbb{R})$) denotes the space of k times differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that all derivatives up to order k have polynomial growth (resp. are bounded). Finally, $\mathtt{i} := \sqrt{-1}$.

(b) The set $\mathbb{L}^q$ denotes the space of random variables with finite qth moment; the corresponding $\mathbb{L}^q$-norms are denoted by $\|\cdot\|_{\mathbb{L}^q}$. The notation $Y_n \xrightarrow{d_{st}} Y$ (resp. $Y_n \xrightarrow{\mathbb{P}} Y$, $Y_n \xrightarrow{d} Y$) stands for stable convergence (resp. convergence in probability, convergence in law).

(c) We now introduce some notions of Malliavin calculus (we refer to the books of Ikeda and Watanabe [23] and Nualart [21] for a detailed exposition of Malliavin calculus). Set $\mathbb{H} = \mathbb{L}^2([0,1], dx)$ and let $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ denote the usual scalar product on $\mathbb{H}$. We denote by D^k the kth Malliavin derivative operator and by δ^k its unbounded adjoint (also called Skrokhod integral of order k). The space $\mathbb{D}_{k,q}$ is the completion of the set of smooth random variables with respect to the norm

$$\|Y\|_{k,q} := \left(\mathbb{E}[|Y|^q] + \sum_{m=1}^{k} \mathbb{E}[\|D^m Y\|^q_{\mathbb{H}^{\otimes m}}] \right)^{1/q}.$$

For any smooth d-dimensional random variable Y the Malliavin matrix is defined via $\sigma_Y := (\langle DY_i, DY_j \rangle_{\mathbb{H}})_{1 \le i,j \le d}$. We sometimes write $\Delta_Y := \det \sigma_Y$ for the determinant of the Malliavin matrix. Finally, we set $\mathbb{D}_{k,\infty} = \cap_{q \ge 2} \mathbb{D}_{k,q}$.

We start this section by reviewing the theoretical results from [26], which concern the Edgeworth expansion associated with a mixed normal limit. On a filtered Wiener space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, \mathbb{P})$ we consider a one-dimensional functional Z_n, which admits the decomposition

$$Z_n = M_n + r_n N_n, \tag{2.1}$$

where r_n is a deterministic sequence with $r_n \to 0$ and N_n is some tight sequence of random variables (in this paper we will have $r_n = \Delta_n^{1/2}$). We assume that the leading term M_n is a terminal value of some continuous $(\mathcal{F}_t)$-martingale $(M_t^n)_{t\in[0,1]}$, i.e. $M_n = M_1^n$. In this paper we are interested in cases where M_n (and so Z_n) converges stably in law to a mixed normal variable M (stable convergence has been originally introduced in [22]). This means:

$$M_n \xrightarrow{d_{st}} M, \tag{2.2}$$

where the random variable M is defined on an extension $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ of the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and, conditionally on $\mathcal{F}$, M has a normal law with mean 0 and conditional variance C. In this case we use the notation

$$M \sim MN(0, C).$$

We recall that a sequence of random variables $(Y_n)_{n\in\mathbb{N}}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in a metric space E is said to converge stably with limit Y, written $Y_n \xrightarrow{d_{st}} Y$, where Y is defined on an extension $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ of the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$, iff for any bounded, continuous function g and any bounded $\mathcal{F}$-measurable random variable X it holds that

$$\mathbb{E}[g(Y_n)X] \to \overline{\mathbb{E}}[g(Y)X], \quad n \to \infty. \tag{2.3}$$

For statistical applications it is not sufficient to consider the Edgeworth expansion of the law of Z_n. It is much more adequate to study the asymptotic expansion for the pair

(Z_n, F_n), where F_n is another functional which converges in probability:

$$F_n \xrightarrow{\mathbb{P}} F.$$

When F_n is a consistent estimator of the conditional variance C (i.e. $F = C$), which is the most important application, we would obtain by the properties of stable convergence:

$$\frac{Z_n}{\sqrt{F_n}} \xrightarrow{d} \mathcal{N}(0,1).$$

In this case the asymptotic expansion of the law of (Z_n, F_n) would imply the Edgeworth expansion for the studentized statistic $Z_n/\sqrt{F_n}$.

We consider the stochastic processes $(M_t)_{t\in[0,1]}$ and $(C_t^n)_{t\in[0,1]}$ with

$$M = M_1, \qquad C_t = \langle M \rangle_t, \qquad C_t^n = \langle M^n \rangle_t, \qquad C_n = \langle M^n \rangle_1. \tag{2.4}$$

Here the process $(M_t)_{t\in[0,1]}$, defined on $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$, represents the stable limit of the continuous $(\mathcal{F}_t)$-martingale $(M_t^n)_{t\in[0,1]}$, while C^n denotes the quadratic variation process associated with M^n. Now, let us set

$$\widehat{C}_n = r_n^{-1}(C_n - C), \tag{2.5}$$

$$\widehat{F}_n = r_n^{-1}(F_n - F). \tag{2.6}$$

Apart from various technical conditions, presented in the Section 2.3, our main assumption will be the following:

(A1) **(i)** $(M_\cdot^n, N_n, \widehat{C}_n, \widehat{F}_n) \xrightarrow{d_{st}} (M_\cdot, N, \widehat{C}, \widehat{F})$.

(ii) $M_t \sim MN(0, C_t)$.

In order to present an Edgeworth expansion for the pair (Z_n, F_n) we need to define two *random symbols* $\underline{\sigma}$ and $\overline{\sigma}$, which play a crucial role in what follows. We call $\underline{\sigma}$ the adaptive (or classical) random symbol and $\overline{\sigma}$ the anticipative random symbol.

2.1 The classical random symbol $\underline{\sigma}$

Let $\widetilde{\mathcal{F}} = \mathcal{F} \vee \sigma(M)$. We take a random function $\widetilde{C}(z)$ such that

$$\widetilde{C}(M) = \mathbb{E}[\widehat{C}|\widetilde{\mathcal{F}}]. \tag{2.7}$$

In the same way we define the variables $\widetilde{F}(z)$ and $\widetilde{N}(z)$ such that

$$\widetilde{F}(M) = \mathbb{E}[\widehat{F}|\widetilde{\mathcal{F}}], \qquad \widetilde{N}(M) = \mathbb{E}[N|\widetilde{\mathcal{F}}].$$

Remark 2.1. Due to Assumption (A1) (i) we have the pointwise stable convergence $(M_n, N_n, \widehat{C}_n, \widehat{F}_n) \xrightarrow{d_{st}} (M, N, \widehat{C}, \widehat{F})$. Usually, the limit $(M, N, \widehat{C}, \widehat{F})$ is jointly mixed normal with expectation $\mu \in \mathbb{R}^4$ (and $\mu_1 = 0$) and conditional covariance matrix $\Sigma \in \mathbb{R}^{4\times 4}$. We deduce, for instance, that

$$\widetilde{N}(M) = \mu_2 + \frac{\Sigma_{12}}{\Sigma_{11}} M.$$

Consequently, we have $\widetilde{N}(z) = \mu_2 + \frac{\Sigma_{12}}{\Sigma_{11}} z$. The quantities $\widetilde{C}(z)$ and $\widetilde{F}(z)$ are computed similarly. □

Now, the adaptive random symbol $\underline{\sigma}$ is defined by

$$\underline{\sigma}(z, \mathtt{i}u, \mathtt{i}v) = \frac{(\mathtt{i}u)^2}{2}\widetilde{C}(z) + \mathtt{i}u\widetilde{N}(z) + \mathtt{i}v\widetilde{F}(z). \tag{2.8}$$

Notice that $\underline{\sigma}$ is a second order polynomial in $(\mathtt{i}u, \mathtt{i}v)$. The random symbol $\underline{\sigma}(z, \mathtt{i}u, \mathtt{i}v)$ is called classical, because it appears already in the martingale expansion in the central limit theorem ([24, 25]), i.e. in the case where C is a deterministic constant. In contrast, the anticipative random symbol $\overline{\sigma}$, which will be defined in the next subsection, is due to the mixed normality of the limit. In fact, it disappears if C is non-random.

2.2 The anticipative random symbol $\overline{\sigma}$

The second random symbol $\overline{\sigma}$ is given in an implicit way. Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ with $|\alpha| = \alpha_1 + \alpha_2$. Set

$$\partial^\alpha = \mathtt{i}^{-|\alpha|} d^\alpha.$$

We define the quantity Φ_n by

$$\Phi_n(u, v) = \mathbb{E}\left[\exp\Big(-\frac{u^2}{2}C + \mathtt{i}vF\Big)\Big(\mathcal{E}(\mathtt{i}uM^n)_1 - 1\Big)\psi_n\right],$$

where $\mathcal{E}(H)_t$ denotes the exponential martingale associated with a continuous martingale H, i.e.

$$\mathcal{E}(H)_t = \exp\Big(H_t - \frac{1}{2}\langle H\rangle_t\Big) = 1 + \int_0^t \mathcal{E}(H)_s dH_s,$$

and the random variable ψ_n plays a role of a threshold that ensures the integrability of the above expression, whose precise definition is given in Section 2.3 below. In particular, ψ_n converges to 1 in probability.

Remark 2.2. Recalling the definition of the exponential martingale $\mathcal{E}(\mathtt{i}uM^n)$, we observe that $\Phi_n(u, v)$ is closely related to the joint characteristic function of (M_n, F). Condition (A5) of Section 2.3 specifies the tail behaviour of $\Phi_n(u, v)$. When $C = F$ is deterministic, i.e. we are in the framework of a standard central limit theorem, the truncation ψ_n can be dropped and we obtain that $\Phi_n(u, v) = 0$, since $(\mathcal{E}(\mathtt{i}uM^n)_t - 1)_{t\in[0,1]}$ is a martingale with mean 0. □

Now, we assume that the limit $\Phi^\alpha(u, v) := \lim_{n\to\infty} r_n^{-1}\partial^\alpha\Phi_n(u, v)$ (if it exists) admits the representation

$$\Phi^\alpha(u, v) = \partial^\alpha \mathbb{E}\left[\exp\Big(-\frac{u^2}{2}C + \mathtt{i}vF\Big)\overline{\sigma}(\mathtt{i}u, \mathtt{i}v)\right], \qquad (u, v) \in \mathbb{R}^2, \tag{2.9}$$

where the random symbol $\overline{\sigma}(\mathtt{i}u,\mathtt{i}v)$ has the form

$$\overline{\sigma}(\mathtt{i}u,\mathtt{i}v) = \sum_j \bar{c}_j(\mathtt{i}u)^{m_j}(\mathtt{i}v)^{n_j} \qquad \text{(finite sum)} \tag{2.10}$$

with $\bar{c}_j \in \mathbb{D}_{l,\infty}$ for a certain $l \in \mathbb{N}$ (cf. assumption (A4) in Section 2.3). We remark that $\overline{\sigma}(\mathtt{i}u,\mathtt{i}v)$ is a polynomial with random coefficients.

2.3 Assumptions and truncation functionals

In this subsection we state the conditions $(\mathrm{A2})_\ell$, (A3), $(\mathrm{A4})_{\ell,\mathfrak{n}}$, (A5) required in Theorem 2.3 below. Localization techniques will be essential to carry out the computations rigorously. We introduce a functional s_n for this purpose.

(A2)$_\ell$ (i) $F \in \mathbb{D}_{\ell+1,\infty}$ and $C \in \mathbb{D}_{\ell,\infty}$.

(ii) $M_n \in \mathbb{D}_{\ell+1,\infty}$, $F_n \in \mathbb{D}_{\ell+1,\infty}$, $C_n \in \mathbb{D}_{\ell,\infty}$, $N_n \in \mathbb{D}_{\ell+1,\infty}$ and $s_n \in \mathbb{D}_{\ell,\infty}$. Moreover,

$$\sup\Big\{\|M_n\|_{\ell+1,p} + \|\widehat{C}_n\|_{\ell,p} + \|\widehat{F}_n\|_{\ell+1,p} + \|N_n\|_{\ell+1,p} + \|s_n\|_{\ell,p}\Big\} < \infty.$$

for every $p \geq 2$.

(A3) (i) $\mathbb{P}\left[\Delta_{(M_n,F)} < s_n\right] = O(r_n^{1+\kappa})$ for some positive constant κ. Recall that $\Delta_{(M_n,F)}$ denotes the determinant of the Malliavin matrix of (M_n, F).

(ii) For every $p \geq 2$,

$$\limsup_{n\to\infty} \mathbb{E}\big[s_n^{-p}\big] < \infty,$$

and moreover $C^{-1} \in \mathbb{L}^\infty$.

(A4)$_{\ell,\mathfrak{n}}$ (i) $\widetilde{C}(z)$, $\widetilde{N}(z)$ and $\widetilde{F}(z)$ are random polynomials with coefficients in $\mathbb{D}_{4,\infty}$.

(ii) The random symbol $\overline{\sigma}$ admits a representation

$$\overline{\sigma}(\mathtt{i}u,\mathtt{i}v) = \sum_j \bar{c}_j(\mathtt{i}u)^{m_j}(\mathtt{i}v)^{n_j} \qquad \text{(finite sum)},$$

where the numbers $n_j \in \mathbb{N}$ satisfy $n_j \leq \mathfrak{n}$ and $\bar{c}_j \in \mathbb{D}_{\ell,\infty}$.

(A5) For some $q \in (1/3, 1/2)$,

$$\sup_n \sup_{(u,v)\in\Lambda_n^0(2,q)} |(u,v)|^3 r_n^{-1/2} |\Phi_n^\alpha(u,v)| \;<\; \infty$$

for every $\alpha \in \mathbb{Z}_+^2$, where $\Lambda_n^0(2,q) = \{(u,v) \in \mathbb{R}^2; |(u,v)| \leq r_n^{-q/2}\}$.

Truncation techniques will play an essential role in derivation of the asymptotic expansion. We shall construct a truncation functional ψ_n, which has been introduced in the definition of $\Phi_n(u,v)$, below so that it gives uniform convergence of C_t^n and the non-degeneracy of (Z_n, F_n). Let $\psi \in C^\infty([0,1])$ be a real-valued function with $\psi(x) = 1$ for $|x| \leq 1/2$ and $\psi(x) = 0$ for $|x| \geq 1$. Recalling that $C_1 = C$, we define a random variable ξ_n by

$$\xi_n = 10^{-1}\Delta_n^{-c}(C_1^n - C)^2 + 2\big[1 + 4\Delta_{(M_1^n, C)} s_n^{-1}\big]^{-1} + \Delta_n^{c_1} C^2 \qquad (2.11)$$
$$+ L^* \int_{[0,1]^2} \left(\frac{\Delta_n^{-q}|C_t^n - C_t - C_s^n + C_s|}{|t-s|^{3/8}} \right)^8 dtds,$$

where L^* is a sufficiently large constant, $c_1 > 0$, c satisfies $2q < c < 1$ and the constant q is defined in (A5). Define the 2×2 random matrix R_n' by

$$R_n' = \sigma_{Q_n}^{-1}\big(r_n \langle DQ_n, DR_n\rangle_{\mathbb{H}} + r_n \langle DR_n, DQ_n\rangle_{\mathbb{H}} + r_n^2 \langle DR_n, DR_n\rangle_{\mathbb{H}}\big),$$

where $Q_n = (M_n, F)$ and $R_n = (N_n, \widehat{F}_n)$. Obviously

$$\sigma_{(Z_n, F_n)} = \sigma_{Q_n}(I_2 + R_n'), \qquad (2.12)$$

where I_2 is the 2×2 identity matrix. Let $\xi_n' = r_n^{-1}|R_n'|^2$. We define ψ_n by

$$\psi_n = \psi(\xi_n)\psi(\xi_n'). \qquad (2.13)$$

2.4 The asymptotic expansion of the density of (Z_n, F_n)

We set

$$\sigma = \underline{\sigma} + \overline{\sigma}. \qquad (2.14)$$

We remark that due to the definition of $\underline{\sigma}$ and $\overline{\sigma}$ the random symbol σ admits the representation

$$\sigma(z, \mathtt{i}u, \mathtt{i}v) = \sum_j c_j(z)(\mathtt{i}u)^{m_j}(\mathtt{i}v)^{n_j} \qquad \text{(finite sum)} \qquad (2.15)$$

for some $c_j(z) \in \cap_{p>1}\mathbb{L}^p$. The approximative density of (Z_n, F_n) is defined as

$$p_n(z,x) = \mathbb{E}[\phi(z;0,C)|F = x]p^F(x) \qquad (2.16)$$
$$+ \; r_n \sum_j (-d_z)^{m_j}(-d_x)^{n_j}\Big(\mathbb{E}\left[c_j(z)\phi(z;0,C)|F = x\right] p^F(x)\Big),$$

where p^F denotes the density of F and $\phi(\cdot; a, b^2)$ is the density of $\mathcal{N}(a, b^2)$-distribution. Obviously, we will require certain regularity conditions in terms of Malliavin calculus in order to validate the existence of the density p^F and the derivatives in (2.16) as well as to validate the estimate of the approximation error.

For any integrable function $h:\mathbb{R}^2\to\mathbb{R}$ we set

$$\Delta_n(h)=\left|\mathbb{E}[h(Z_n,F_n)]-\int h(z,x)p_n(z,x)dzdx\right|. \tag{2.17}$$

The next theorem has been proven in [26].

Theorem 2.3. *Let $\ell=5\vee 2[(\mathfrak{n}+3)/2]$ with $\mathfrak{n}=\max_j n_j$, where the integers n_j are defined at (2.15). Define the set $\mathcal{E}(K,\gamma)=\{h:\mathbb{R}^2\to\mathbb{R}|\ h$ is measurable and $|h(z,x)|\le K(|z|+|x|)^\gamma\}$ for some $K,\gamma>0$. Under the assumptions (A1), (A2)$_\ell$, (A3), (A4)$_{\ell,\mathfrak{n}}$ and (A5), we have that*

$$\sup_{h\in\mathcal{E}(K,\gamma)}\Delta_n(h)=o(r_n). \tag{2.18}$$

In the following subsection we will explain how this result applies to weighted quadratic functionals of a Brownian motion.

2.5 A useful example

We start by applying the result of Theorem 2.3 to a simple example, which however gives a first intuition how the main quantities are computed. Let $(W_t)_{t\in[0,1]}$ be a standard one-dimensional Brownian motion and consider the weighted functional

$$M_n=\Delta_n^{-1/2}\sum_{i=1}^{1/\Delta_n}a(W_{t_{i-1}})\Big(|\Delta_i^nW|^2-\Delta_n\Big),\qquad \Delta_i^nW=W_{t_i}-W_{t_{i-1}}, \tag{2.19}$$

where $a\in C_p^\infty(\mathbb{R})$ and $\Delta_n\to 0$ (recall that $1/\Delta_n$ is an integer). In this section we demonstrate how the Edgeworth expansion is computed for M_n. Such quadratic functionals have been already discussed in details in [26, 27]. For this reason we will dispense with the exact derivation at certain steps of the proof; in particular, we will not show conditions (A2)$_\ell$-(A5) at this stage. We start with the asymptotic properties of the quadratic variation process C^n.

2.5.1 Asymptotic properties of C^n

The Itô formula implies the identity

$$|\Delta_i^nW|^2-\Delta_n=2\int_{t_{i-1}}^{t_i}(W_s-W_{t_{i-1}})dW_s,$$

and we conclude that M_n is a terminal value of the continuous $(\mathcal{F}_t)$-martingale

$$M_t^n=2\Delta_n^{-1/2}\sum_{i\ge1}a(W_{t_{i-1}})\int_{t_{i-1}\wedge t}^{t_i\wedge t}(W_s-W_{t_{i-1}})dW_s, \tag{2.20}$$

i.e. $M_n=M_1^n$. We also remark that

$$M_t^n=\int_0^t b_s^ndW_s,\qquad b_s^n=2\Delta_n^{-1/2}a(W_{\Delta_n[s/\Delta_n]})(W_s-W_{\Delta_n[s/\Delta_n]}), \tag{2.21}$$

and thus

$$C_t^n = \langle M^n \rangle_t = 4\Delta_n^{-1} \int_0^t a^2(W_{\Delta_n[s/\Delta_n]})(W_s - W_{\Delta_n[s/\Delta_n]})^2 ds. \qquad (2.22)$$

Theorem 8.1 of Appendix implies that

$$C_t^n \xrightarrow{\mathbb{P}} C_t = 2\int_0^t a^2(W_s)ds.$$

Remark 2.4. Notice that Theorems 8.1 and 8.2 of Appendix are formulated for bounded weight functions a. However, as demonstrated in Section 3 of [4], all ingredients involved in our analysis (in particular, the function a in this case) can be assumed to be bounded w.l.o.g by a localization technique when proving Theorems 8.1 and 8.2. □

In this example we will consider

$$F_n = 2\Delta_n \sum_{i=1}^{1/\Delta_n} a^2(W_{t_{i-1}}), \qquad (2.23)$$

which is a Riemann sum approximation of $C = C_1$. We clearly have the convergence in probability

$$F_n \xrightarrow{\mathbb{P}} C = 2\int_0^1 a^2(W_s)ds.$$

Here and throughout the paper the functional stable convergence $(M_\cdot^n, N_n, \widehat{C}_n, \widehat{F}_n) \xrightarrow{d_{st}} (M_\cdot, N, \widehat{C}, \widehat{F})$ follows directly from the general result of Theorem 8.2 in Appendix, so the assumption (A1) will be always satisfied. However, for the computation of the random symbol $\underline{\sigma}$ we only require the pointwise stable convergence $(M_n, N_n, \widehat{C}_n, \widehat{F}_n) \xrightarrow{d_{st}} (M, N, \widehat{C}, \widehat{F})$, which we will present from now on. The first convergence of the following proposition is a straightforward consequence of [4, Section 8].

Proposition 2.5. *It holds that*

$$\Delta_n^{-1/2}(F_n - C) \xrightarrow{\mathbb{P}} 0.$$

Furthermore, we obtain the stable convergence

$$(M_n, \widehat{C}_n) \xrightarrow{d_{st}} (M, \widehat{C}) \sim MN(0, \Sigma) \qquad \textit{with} \qquad \Sigma = \int_0^1 \Sigma_s ds,$$

where the matrix Σ_s is defined by

$$\Sigma_s^{11} = 2a^2(W_s), \qquad \Sigma_s^{22} = \frac{16}{3}a^4(W_s), \qquad \Sigma_s^{12} = \Sigma_s^{21} = \frac{8}{3}a^3(W_s).$$

2.5.2 Computation of $\underline{\sigma}$ and $\overline{\sigma}$

Now, we start with the computation of the random symbol σ. We remark that the following computations are rather typical. The adaptive random symbol $\underline{\sigma}$ is given by

$$\underline{\sigma}(z, \mathtt{i}u, \mathtt{i}v) = \frac{2z(\mathtt{i}u)^2 \int_0^1 a^3(W_s)ds}{3\int_0^1 a^2(W_s)ds} =: z(\mathtt{i}u)^2 \mathcal{C}_1 \tag{2.24}$$

due to Remark 2.1 and $\widetilde{N}(z) = \widetilde{F}(z) = 0$.

We turn our attention to $\overline{\sigma}$. We will not give a rigorous proof, as the details can be found in [26, 27]. Instead we are aiming to present the most important steps of the derivation (in Section 3 we will treat a more general type of functionals in a detailed manner). Recall that in our case it holds that $F = C$. We set

$$e_t^n(u) = \mathcal{E}(\mathtt{i}uM^n)_t, \qquad \Psi(u,v) = \exp\left((-\frac{u^2}{2} + \mathtt{i}v)C\right) \tag{2.25}$$

As $e_t^n(u)$ is a continuous exponential martingale we have that

$$\Phi_n(u,v) = \mathbb{E}\left[\Psi(u,v)\int_0^1 e_t^n(u)d(\mathtt{i}uM_t^n)\psi_n\right],$$

for the truncation functional ξ_n defined in Section 2.3. The variable $\Delta_n^{-1/2}\Phi_n(u,v)$ has the decomposition:

$$\Delta_n^{-1/2}\Phi_n(u,v) \quad = \quad \check{\mathfrak{A}}_n(u,v) + \hat{\mathfrak{A}}_n(u,v)$$

with

$$\check{\mathfrak{A}}_n(u,v) = \Delta_n^{-1/2}\sum_{i=1}^{1/\Delta_n}\mathbb{E}\left[\Psi(u,v)\int_{t_{i-1}}^{t_i} e_{t_{i-1}}^n(u)d(\mathtt{i}uM_t^n)\psi_n\right],$$

$$\hat{\mathfrak{A}}_n(u,v) = \Delta_n^{-1/2}\sum_{i=1}^{1/\Delta_n}\mathbb{E}\left[\Psi(u,v)\int_{t_{i-1}}^{t_i} (e_t^n(u) - e_{t_{i-1}}^n(u))d(\mathtt{i}uM_t^n)\psi_n\right].$$

We will see that $\check{\mathfrak{A}}_n(u,v)$ is the dominating term, while $\hat{\mathfrak{A}}_n(u,v)$ turns out to be negligible. Setting for simplicity $a_t := a(W_t)$, we obtain

$$\check{\mathfrak{A}}_n(u,v) \quad = \quad \mathtt{i}u\Delta_n^{-1}\sum_{i=1}^{1/\Delta_n}\mathbb{E}\left[\delta^2(1_{I_i^n}^{\otimes 2}) \times \Psi(u,v)a_{t_{i-1}}\psi_n e_{t_{i-1}}^n(u)\right],$$

where $I_i^n = 1_{(t_{i-1},t_i]}$ and δ^2 denotes the Skorokhod integral of order two. Now, we recall the integration by parts (or duality) formula (see e.g. [21]): For any $k \in \mathbb{N}$ and $w \in \mathrm{Dom}\ \delta^k$ and any smooth random variable $Y \in \mathbb{D}_{k,2}$, it holds that

$$\mathbb{E}[\delta^k(w)Y] = \mathbb{E}[\langle w, D^kY\rangle_{\mathbb{H}^{\otimes k}}]. \tag{2.26}$$

Applying the duality formula (2.26) we conclude that

$$\begin{aligned}
&\check{\mathfrak{A}}_n(u,v) \\
&= \mathtt{i}u\Delta_n^{-1}\sum_{i=1}^{1/\Delta_n}\mathbb{E}\left[\int_0^1\int_0^1 1_{I_i^n}^{\otimes 2}(s_1,s_2)D_{s_1,s_2}\big(\Psi(u,v)a_{t_{i-1}}\psi_n e_{t_{i-1}}^n(u)\big)ds_1ds_2\right] \\
&= \mathtt{i}u\Delta_n^{-1}\sum_{i=1}^{1/\Delta_n}\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{t_i} ds_1ds_2\,\mathbb{E}\left[a_{t_{i-1}}e_{t_{i-1}}^n(u)D_{s_1,s_2}\big(\Psi(u,v)\psi_n\big)\right] \\
&= \mathtt{i}u\Delta_n^{-1}\sum_{i=1}^{1/\Delta_n}\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{t_i} ds_1ds_2\,\mathbb{E}\left[a_{t_{i-1}}e_{t_{i-1}}^n(u)\psi_n D_{s_1,s_2}\big(\Psi(u,v)\big)\right]+o(1),
\end{aligned}$$

where the term $o(1)$ is explained by the fact that $D\psi_n\to 0$ since $\limsup_{n\to\infty}|\xi_n|<1/2$. Recalling again that $\psi_n \stackrel{\mathbb{P}}{\longrightarrow} 1$, we obtain

$$\lim_{n\to\infty}\check{\mathfrak{A}}_n(u,v) = \int_0^1 \mathbb{E}\left[\mathtt{i}ua_te_t(u)D_{t,t}\Psi(u,v)\right]dt$$

by the functional stable convergence $M^n \stackrel{d_{st}}{\longrightarrow} M$ entailing

$$e_t^n(u) \stackrel{d_{st}}{\longrightarrow} e_t(u)=\exp\Big(\mathtt{i}uM_t+\frac{u^2}{2}C_t\Big).$$

The process $e_t(u)$ is again an exponential martingale and we have

$$\overline{\mathbb{E}}[e_t(u)|\mathcal{F}]=1$$

for all $t\geq 0$, $u\in\mathbb{R}$. Hence,

$$\begin{aligned}
\lim_{n\to\infty}\check{\mathfrak{A}}_n(u,v) &= \int_0^1 \mathbb{E}\left[\mathtt{i}ua_se_s(u)D_{s,s}\Psi(u,v)\right]ds \\
&= \mathbb{E}\big[\Psi(u,v)\,\mathtt{i}u(4l^2\mathcal{C}_2+2l\mathcal{C}_3)\big]
\end{aligned}$$

with $l=-\frac{u^2}{2}+\mathtt{i}v$ and

$$\begin{aligned}
\mathcal{C}_2 &= \int_0^1 a(W_s)\Big(\int_s^1 (a^2)'(W_x)dx\Big)^2 ds \\
\mathcal{C}_3 &= \int_0^1 a(W_s)\Big(\int_s^1 (a^2)''(W_x)dx\Big) ds
\end{aligned}$$

In a similar way, we obtain the representation

$$\begin{aligned}
\hat{\mathfrak{A}}_n(u,v) &= 2\mathtt{i}u\Delta_n^{-1}\sum_{i=1}^{1/\Delta_n}\int_{t_{i-1}}^{t_i}ds_1\int_{t_{i-1}}^{s_1}ds_2 \\
&\quad\times\mathbb{E}\left[D_{s_2}\bigg(D_{s_1}\Big\{\Psi(u,v)\psi_n a_{t_{i-1}}\times(e_{s_1}^n(u)-e_{t_{i-1}}^n(u))\Big\}\bigg)\right] \\
&\to 0
\end{aligned}$$

as $n \to \infty$ for every (u, v) by using $\mathbb{L}^p$-continuity of the densities of the derivatives of $e^n_{s_1}(u) - e^n_{t_{i-1}}(u)$. Putting things together we found that

$$\lim_{n\to\infty} \Delta_n^{-1/2}\Phi_n(u,v) = \int_0^1 \mathbb{E}\left[\mathtt{i}u a_s e_s(u) D_{s,s}\Psi(u,v)\right] ds = \mathbb{E}\big[\Psi(u,v)\, \mathtt{i}u(4l^2\mathcal{C}_2 + 2l\mathcal{C}_3)\big].$$

The corresponding identity for higher order derivatives of $\Phi_n(u, v)$ is proved similarly. Thus, the anticipative random symbol is given by

$$\overline{\sigma}(\mathtt{i}u, \mathtt{i}v) = \mathtt{i}u(-u^2 + 2\mathtt{i}v)^2\mathcal{C}_2 + \mathtt{i}u(-u^2 + 2\mathtt{i}v)\mathcal{C}_3.$$

Remark 2.6. Recall that the statistic M_n depends on $H_2(\Delta_i^n W/\sqrt{\Delta_n})$, where H_2 is the second Hermite polynomial. In this case the anticipative random symbol $\overline{\sigma}$ is non-degenerate as we have just proved. In Section 3 we will show the following fact: Elements of higher order chaos, i.e. $H_m(\Delta_i^n W/\sqrt{\Delta_n})$ with $m \geq 3$, lead to $\overline{\sigma} = 0$. In other words, for the computation of the anticipative random symbol $\overline{\sigma}$ of a weighted functional M_n based on $f(\Delta_i^n W/\sqrt{\Delta_n})$, where f is a measurable function with Hermite rank at least 2, only the projection of $f(\Delta_i^n W/\sqrt{\Delta_n})$ onto the second order Wiener chaos matters. □

Now, the full random symbol is

$$\sigma(z, \mathtt{i}u, \mathtt{i}v) = z(\mathtt{i}u)^2\mathcal{C}_1 + \mathtt{i}u(-u^2 + 2\mathtt{i}v)^2\mathcal{C}_2 + \mathtt{i}u(-u^2 + 2\mathtt{i}v)\mathcal{C}_3.$$

Therefore the approximative density $p_n(z, x)$ of (M_n, F_n) is given as (recall that $F = C$)

$$\begin{aligned} p_n(z,x) = {} & \phi(z;0,x)p^C(x) + \Delta_n^{1/2}\Big(d_z^2\{z\phi(z;0,x)\}p^C(x)\mathbb{E}[\mathcal{C}_1|C=x] \\ & -d_z(d_z^2 - 2d_x)^2\{\mathbb{E}[\mathcal{C}_2\phi(z;0,x)|C=x]p^C(x)\} \\ & -d_z(d_z^2 - 2d_x)\{\mathbb{E}[\mathcal{C}_3\phi(z;0,x)|C=x]p^C(x)\}\Big). \end{aligned}$$

3 Functionals of Brownian motion with random weights

In this section we go one step further by considering general weighted functionals of a Brownian motion with weights depending on a given stochastic differential equation, and we shall derive an expansion formula. Here the stochastic second order term N_n is still absent. In later sections, we will meet an expansion with non-vanishing N_n when considering the power variations of diffusion processes. However, we will solve two essential problems in this general but concrete situation, that is, identification of the anticipative random symbol in this model, and proof of the nondegeneracy of the functionals.

On a given Wiener space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, \mathbb{P})$ we consider a 1-dimensional stochastic differential equation of the form

$$dX_t = b^{[1]}(X_t)dW_t + b^{[2]}(X_t)dt, \tag{3.1}$$

where X_0 is a bounded random variable, $b^{[1]}, b^{[2]} : \mathbb{R} \to \mathbb{R}$ are two deterministic functions and W is a standard Brownian motion. Sometimes we will use the notation

$$b_t^{[1]} = b^{[1]}(X_t), \qquad b_t^{[2]} = b^{[2]}(X_t).$$

The somewhat unusual notation $b^{[1]}$, $b^{[2]}$ refers to the fact that the diffusion term $b^{[1]}$ dominates the drift term $b^{[2]}$ in all asymptotic expansions (so $b^{[1]}$ is the first order term and $b^{[2]}$ is the second order term). Under standard smoothness conditions the processes $b_t^{[k]}$, $k = 1, 2$, also satisfy a SDE of the type (3.1) by Itô formula; in this case we denote by $b_t^{[k.1]}$ (resp. $b_t^{[k.2]}$) the diffusion term (resp. the drift term) of $b_t^{[k]}$. In the same manner we introduce the processes $b_t^{[k_1 \dots k_d]}$, $k_1, \dots, k_d = 1, 2$, recursively. We will assume that $b^{[1]}$ and $b^{[2]}$ are in $C_{b,1}^{\infty}(\mathbb{R})$. [1]

In this section we consider weighted functionals of the Brownian motion of the type

$$M_n = \Delta_n^{1/2} \sum_{i=1}^{1/\Delta_n} a(X_{t_{i-1}}) f\Big(\frac{\Delta_i^n W}{\sqrt{\Delta_n}}\Big), \tag{3.2}$$

where $a \in C_p^{\infty}(\mathbb{R})$ and $f \in C_p^{11}(\mathbb{R})$. Since f has polynomial growth, it holds that $\mathbb{E}[f^2(Z)] < \infty$ with $Z \sim \mathcal{N}(0,1)$. Consequently, the function f exhibits a Hermite expansion. We assume that the function f has the form

$$f(x) = \sum_{k=2}^{\infty} \lambda_k H_k(x) \qquad \text{in } \mathbb{L}^2(\mathbb{R}; \phi(x;0,1)dx) \tag{3.3}$$

with

$$\lambda_k = \frac{\mathbb{E}[f(Z)H_k(Z)]}{k!}, \quad Z \sim \mathcal{N}(0,1),$$

where H_k is the kth Hermite polynomial, i.e. $H_0(x) = 1$ and

$$H_k(x) = (-1)^k e^{\frac{x^2}{2}} \frac{d^k}{dx^k}(e^{-\frac{x^2}{2}}), \qquad k \geq 1.$$

In particular, the Hermite rank of the function f is at least 2 and $\mathbb{E}[f(Z)] = 0$ for $Z \sim \mathcal{N}(0,1)$. We will see later that the Hermite rank 1 would not lead to the asymptotic mixed normal distribution with conditional mean 0. In this section, we will consider

$$F_n = \Delta_n \mathrm{Var}[f(Z)] \sum_{i=1}^{1/\Delta_n} a^2(X_{t_{i-1}}), \tag{3.4}$$

which is a Riemann sum approximation of $C = \langle M \rangle_1$, as the reference variable. It is easy to see the following properties (cf. Proposition 2.5).

[1] The set of smooth functions such that each derivative of positive order is bounded.

Proposition 3.1. *It holds that*

$$F_n \xrightarrow{\mathbb{P}} C = Var[f(Z)] \int_0^1 a^2(X_s)ds$$

and

$$\Delta_n^{-1/2}(F_n - C) \xrightarrow{\mathbb{P}} 0$$

as $n \to \infty$.

3.1 A limit theorem for $(M_n, \widehat{C}_n)$ and the adaptive random symbol

First, we note that for $H = f\Big(\frac{\Delta_i^n W}{\sqrt{\Delta_n}}\Big)$ it holds

$$H = \int_0^1 \mathbb{E}[D_s H|\mathcal{F}_s]dW_s,$$

which is the Clark-Ocone formula. Consequently, we deduce the identity

$$f\Big(\frac{\Delta_i^n W}{\sqrt{\Delta_n}}\Big) = \Delta_n^{-1/2} \int_{t_{i-1}}^{t_i} \mathbb{E}\Big[f'\Big(\frac{\Delta_i^n W}{\sqrt{\Delta_n}}\Big)|\mathcal{F}_s\Big]dW_s.$$

Thus, we naturally have a continuous square-integrable $(\mathcal{F}_t)$-martingale $M^n = (M_t^n)_{t\in[0,1]}$ given by

$$M_t^n = \int_0^t b_s^n dW_s, \qquad b_s^n = a(X_{\Delta_n[s/\Delta_n]})\mathbb{E}\Big[f'\Big(\frac{W_{\Delta_n[s/\Delta_n]+\Delta_n} - W_{\Delta_n[s/\Delta_n]}}{\sqrt{\Delta_n}}\Big)|\mathcal{F}_s\Big] \tag{3.5}$$

and we deduce that

$$C_t^n = \langle M^n\rangle_t = \int_0^t a^2(X_{\Delta_n[s/\Delta_n]})\mathbb{E}^2\Big[f'\Big(\frac{W_{\Delta_n[s/\Delta_n]+\Delta_n} - W_{\Delta_n[s/\Delta_n]}}{\sqrt{\Delta_n}}\Big)|\mathcal{F}_s\Big]ds. \tag{3.6}$$

From this identity we obtain the convergence (see Theorem 8.1 in Appendix)

$$C_t^n \xrightarrow{\mathbb{P}} C_t = \mathrm{Var}[f(Z)] \int_0^t a^2(X_s)ds.$$

By Theorem 8.2 of Appendix we deduce the following result.

Proposition 3.2. *It holds that*

$$(M_n, \widehat{C}_n) \xrightarrow{d_{st}} (M, \widehat{C}) \sim MN(0, \Sigma) \qquad \textit{with} \qquad \Sigma = \int_0^1 \Sigma_s ds,$$

where the matrix Σ_s *is defined by*

$$\Sigma_s^{11} = Var[f(Z)]\ a^2(X_s), \qquad \Sigma_s^{22} = \Gamma_1 a^4(X_s), \qquad \Sigma_s^{12} = \Sigma_s^{21} = \Gamma_2 a^3(X_s),$$

with

$$\Gamma_1 = Var\left[\int_0^1 \mathbb{E}^2[f'(W_1)|\mathcal{F}_s]ds\right],$$

$$\Gamma_2 = Cov\left[f(W_1), \int_0^1 \mathbb{E}^2[f'(W_1)|\mathcal{F}_s]ds\right].$$

Notice that the stable convergence in the above proposition does not hold if f has Hermite rank 1, since in this case the process $(v_s)_{s\geq 0}$ defined in Theorem 8.2 is not identically 0. As in the previous section we immediately obtain the adaptive random symbol

$$\underline{\sigma}(z,\mathtt{i}u,\mathtt{i}v) = \frac{4z(\mathtt{i}u)^2\int_0^1 a^3(X_s)ds}{3\mathrm{Var}[f(Z)]\int_0^1 a^2(X_s)ds} =: z(\mathtt{i}u)^2\mathcal{C}_1. \tag{3.7}$$

3.2 Setting s_n

We need to define the functionals s_n (and consequently ξ_n) to go further. We set $\beta(x) := Var[(f(Z))]a(x)^2$ with $Z\sim\mathcal{N}(0,1)$ and $a_t := a(X_t)$. Let

$$\sigma_{22}(t) = \int_0^t \left[\int_r^1 \beta'_s D_r X_s ds\right]^2 dr.$$

Define a matrix $\tilde{\sigma}(n,t)$ by

$$\tilde{\sigma}(n,t) = \begin{bmatrix} \tilde{\sigma}_{11}(n,t) & \tilde{\sigma}_{12}(n,t) \\ \tilde{\sigma}_{12}(n,t) & \sigma_{22}(t) \end{bmatrix}$$

with

$$\begin{aligned}\tilde{\sigma}_{11}(n,t) &= \Delta_n \sum_{i:t_i\leq t}\left[a_{t_{i-1}} f'(\Delta_n^{-1/2}\Delta_i^n W)\right]^2 \\ &\quad + \sum_{i:t_i\leq t}\int_{t_{i-1}}^{t_i}\left[\Delta_n^{1/2}\sum_{k=i+1}^n a'_{t_{k-1}} f(\Delta_n^{-1/2}\Delta_k^n W)1_{\{t_k\leq t\}} D_r X_{t_{k-1}}\right]^2 dr\end{aligned}$$

and

$$\tilde{\sigma}_{12}(n,t) = \sum_{i:t_i\leq t}\int_{t_{i-1}}^{t_i}\left(\left[\Delta_n^{1/2}\sum_{k=i+1}^n a'_{t_{k-1}} f(\Delta_n^{-1/2}\Delta_k^n W)1_{\{t_k\leq t\}} D_r X_{t_{k-1}}\right]\int_r^1 \beta'_s D_r X_s ds\right) dr$$

for $t\in\Pi^n$. Define s_n by

$$s_n = \frac{1}{2}\det\left[\tilde{\sigma}\left(n,\frac{1}{2}\right)+\psi\left(\frac{m_n}{2\mathsf{c}_1}\right)I_2\right],$$

where I_2 is the 2×2 unit matrix, $\psi:\mathbb{R}\to[0,1]$ is a smooth function such that $\psi(x)=1$ if $|x|\leq 1/2$ and $\psi(x)=0$ if $|x|\geq 1$, c_1 is a positive number, and

$$m_n = \Delta_n\sum_{i=1}^{[1/2\Delta_n]}\left[f'(\Delta_n^{-1/2}\Delta_i^n W)\right]^2.$$

We will later show that the random variable s_n satisfies assumption (A3). We define ξ_n using s_n as in Section 2.3.

3.3 Decompositions of the torsion

In this subsection we present some preparatory decompositions for the computation of $\overline{\sigma}$. Recall that $f \in C_p^{11}(\mathbb{R})$ and it admits the Hermite expansion $f(x) = \sum_{k=2}^{\infty} \lambda_k H_k(x)$. Consequently, it holds that $\sum_{k=2}^{\infty} k! k^{11} \lambda_k^2 < \infty$.

The martingale $M^n = (M_t)_{t\in[0,1]}$ admits the local chaos expansion

$$M_t^n = \Delta_n^{1/2} \sum_{i\geq 1} a_{t_{i-1}} \sum_{k=2}^{\infty} k! \lambda_k \Delta_n^{-k/2} \int_{t_{i-1}\wedge t}^{t_i \wedge t} \int_{t_{i-1}}^{s_1} \cdots \int_{t_{i-1}}^{s_{k-1}} dW_{s_k} \cdots dW_{s_2} dW_{s_1} \quad (3.8)$$

Obviously, each infinite sum in (3.8) is well defined as an $\mathbb{L}^2$-limit when $k \to \infty$. Since

$$H_k(\Delta_n^{-1/2} \Delta_i^n W) = k! \Delta_n^{-k/2} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{s_1} \cdots \int_{t_{i-1}}^{s_{k-1}} dW_{s_k} \cdots dW_{s_2} dW_{s_1},$$

we find (3.2) again. We recall that for each (n, u), the random variable $\sup_{t\in[0,1]} |e^n(u)|$ is bounded uniformly in ω under the truncation by ψ_n. Thus, one can identify $e_t^n(u)$ with a stopped $e_{\tau_n \wedge t}^n(u)$ by some stopping time $\tau_n = \tau_n(u)$ that makes the stopped process bounded uniformly in ω for every n (but not uniformly in n). This remark ensures that variables are in the domain of the Skorokhod integral. [2]

Since the infinite sums in k of (3.8) are also limits of $\mathbb{L}^2$-martingales, we can validate the exchange of the limit and the sum, and then use the duality between the Skorokhod

[2] A smooth truncation is possible to construct so as to make irregularity of the stopping time disappear completely on the remaining event. On the other hand, it is also true that one can go without introducing such τ_n explicitly thanks to ψ_n if the functional $D_{s_2}(e_{s_2}^n D_{s_1}(\Psi(u, v)\psi_n \cdots))$ in the following expressions is expanded and interpreted naturally as $e_{s_2}^n \Psi(u, v) \times \cdots$. This is always possible because, for every n, by some smooth truncation that causes $C_1^n \leq A$ locally, the duality operation becomes valid and then the limit $A \to \infty$ gives the formula in expanded form.

integral and the derivative operator D (cf. (2.26)) to carry out

$$
\begin{aligned}
&\sum_{i=1}^{1/\Delta_n} \mathbb{E}\Big[\int_{t_{i-1}}^{t_i} e_t^n(u) dM_t^n\, \Psi(u,v)\psi_n\Big] \\
= \ & \Delta_n^{1/2} \sum_{i=1}^{1/\Delta_n} \sum_{k=2}^{\infty} k!\lambda_k \Delta_n^{-k/2} \mathbb{E}\Big[\int_{t_{i-1}}^{t_i} e_{s_1}^n(u) \Big(\int_{t_{i-1}}^{s_1} \cdots \int_{t_{i-1}}^{s_{k-1}} dW_{s_k} \cdots dW_{s_2}\Big) dW_{s_1}\, \Psi(u,v)\psi_n a_{t_{i-1}}\Big] \\
= \ & \Delta_n^{1/2} \sum_{i=1}^{1/\Delta_n} \sum_{k=2}^{\infty} k!\lambda_k \Delta_n^{-k/2} \\
& \times \mathbb{E}\Big[\int_0^1 e_{s_1}^n(u) \Big(\int_{t_{i-1}}^{s_1} \cdots \int_{t_{i-1}}^{s_{k-1}} dW_{s_k} \cdots dW_{s_2}\Big) 1_{I_i^n}(s_1) D_{s_1}(\Psi(u,v)\psi_n a_{t_{i-1}}) ds_1\Big] \\
= \ & \Delta_n^{1/2} \sum_{i=1}^{1/\Delta_n} \sum_{k=2}^{\infty} k!\lambda_k \Delta_n^{-k/2} \int_{t_{i-1}}^{t_i} ds_1 \\
& \times \mathbb{E}\Big[\int_{t_{i-1}}^{s_1} \Big(\int_{t_{i-1}}^{s_2} \cdots \int_{t_{i-1}}^{s_{k-1}} dW_{s_k} \cdots dW_{s_3}\Big) dW_{s_2} e_{s_1}^n(u) D_{s_1}(\Psi(u,v)\psi_n a_{t_{i-1}})\Big] \\
= \ & \Delta_n^{1/2} \sum_{i=1}^{1/\Delta_n} \sum_{k=2}^{\infty} k!\lambda_k \Delta_n^{-k/2} \int_{t_{i-1}}^{t_i} ds_1 \int_{t_{i-1}}^{s_1} ds_2 \\
& \times \mathbb{E}\Big[\int_{t_{i-1}}^{s_2} \Big(\int_{t_{i-1}}^{s_3} \cdots \int_{t_{i-1}}^{s_{k-1}} dW_{s_k} \cdots dW_{s_4}\Big) dW_{s_3} D_{s_2}\Big(e_{s_1}^n(u) D_{s_1}(\Psi(u,v)\psi_n a_{t_{i-1}})\Big)\Big].
\end{aligned} \tag{3.9}
$$

Applying the duality once again, we obtain the decomposition

$$
\begin{aligned}
\mathfrak{A}_n(u,v) \ := \ & \Delta_n^{-1/2} \sum_{i=1}^{1/\Delta_n} \mathbb{E}\Big[\int_{t_{i-1}}^{t_i} e_t^n(u) dM_t^n\, \Psi(u,v)\psi_n\Big] \\
= \ & 2 \sum_{i=1}^{1/\Delta_n} \lambda_2 \Delta_n^{-1} \int_{t_{i-1}}^{t_i} ds_1 \int_{t_{i-1}}^{s_1} ds_2 \mathbb{E}\Big[D_{s_2}\Big(e_{s_1}^n(u) D_{s_1}(\Psi(u,v)\psi_n a_{t_{i-1}})\Big)\Big] \\
& + \sum_{i=1}^{1/\Delta_n} \sum_{k=3}^{\infty} k!\lambda_k \Delta_n^{-k/2} \int_{t_{i-1}}^{t_i} ds_1 \int_{t_{i-1}}^{s_1} ds_2 \int_{t_{i-1}}^{s_2} ds_3 \\
& \times \mathbb{E}\Big[\int_{t_{i-1}}^{s_3} \cdots \int_{t_{i-1}}^{s_{k-1}} dW_{s_k} \cdots dW_{s_4} \times D_{s_3}\Big\{D_{s_2}\Big(e_{s_1}^n(u) D_{s_1}(\Psi(u,v)\psi_n a_{t_{i-1}})\Big)\Big\}\Big] \\
= \ & \ddot{\mathfrak{A}}_n(u,v) + \dddot{\mathfrak{A}}_n(u,v),
\end{aligned}
$$

where

$$\ddot{\mathfrak{A}}_n(u,v) = 2\sum_{i=1}^{1/\Delta_n} \lambda_2 \Delta_n^{-1} \int_{t_{i-1}}^{t_i} ds_1 \int_{t_{i-1}}^{s_1} ds_2 \mathbb{E}\Big[D_{s_2}\Big(e^n_{s_1}(u) D_{s_1}(\Psi(u,v)\psi_n a_{t_{i-1}})\Big)\Big],$$

$$\begin{aligned}\dddot{\mathfrak{A}}_n(u,v) = &\sum_{i=1}^{1/\Delta_n} \Delta_n^{-3/2} \int_{t_{i-1}}^{t_i} ds_1 \int_{t_{i-1}}^{s_1} ds_2 \int_{t_{i-1}}^{s_2} ds_3 \\ &\times \mathbb{E}\Big[\Big(\sum_{k=3}^{\infty} k!\lambda_k \Delta_n^{-(k-3)/2} \int_{t_{i-1}}^{s_3} \cdots \int_{t_{i-1}}^{s_{k-1}} dW_{s_k} \cdots dW_{s_4}\Big) \\ &\times D_{s_3}\Big\{D_{s_2}\Big(e^n_{s_1}(u) D_{s_1}(\Psi(u,v)\psi_n a_{t_{i-1}})\Big)\Big\}\Big].\end{aligned}$$

Here we used three times Malliavin differentiability of the objects. We remark that the first term $\ddot{\mathfrak{A}}_n(u,v)$, which is associated with the second order Wiener chaos, is a dominating quantity, while $\dddot{\mathfrak{A}}_n(u,v)$ will turn out to be negligible (cf. Remark 2.6).

3.4 Identification of the anticipative random symbol

We shall specify the limit of $\mathfrak{A}_n(u,v)$. First,

$$\begin{aligned}|\dddot{\mathfrak{A}}_n(u,v)| \le\ & \sum_{i=1}^{1/\Delta_n} \Delta_n^{-3/2} \int_{t_{i-1}}^{t_i} ds_1 \int_{t_{i-1}}^{s_1} ds_2 \int_{t_{i-1}}^{s_2} ds_3 \\ &\times \Big\| \sum_{k=3}^{\infty} k!\lambda_k \Delta_n^{-(k-3)/2} \int_{t_{i-1}}^{s_3} \cdots \int_{t_{i-1}}^{s_{k-1}} dW_{s_k} \cdots dW_{s_4} \Big\|_{\mathbb{L}^2} \\ &\times \Big\| D_{s_3}\Big\{D_{s_2}\Big(e^n_{s_1}(u) D_{s_1}(\Psi(u,v)\psi_n a_{t_{i-1}})\Big)\Big\}\Big\|_{\mathbb{L}^2} \\ \le\ & \Delta_n^{1/2} \sqrt{\sum_{k=3}^{\infty} k! k^3 \lambda_k^2} \times \sup_{\substack{n\in\mathbb{N},\ s_1,s_2,s_3\in[0,1] \\ t_{i-1}<s_3<s_2<s_1\le t_i}} \Big\| D_{s_3}\Big\{D_{s_2}\Big(e^n_{s_1}(u) D_{s_1}(\Psi(u,v)\psi_n a_{t_{i-1}})\Big)\Big\}\Big\|_{\mathbb{L}^2} \\ \to\ & 0\end{aligned}$$

as $n\to\infty$ for every (u,v), since the above supremum is bounded due to assumption $(A2)_8$, and product and chain rule for the Malliavin derivative.

Next, we will treat $\ddot{\mathfrak{A}}_n(u,v)$. We deform it as $\ddot{\mathfrak{A}}_n(u,v) = \check{\mathfrak{A}}_n(u,v) + \hat{\mathfrak{A}}_n(u,v)$ with

$$\check{\mathfrak{A}}_n(u,v) = 2\sum_{i=1}^{1/\Delta_n} \lambda_2 \Delta_n^{-1} \int_{t_{i-1}}^{t_i} ds_1 \int_{t_{i-1}}^{s_1} ds_2 \mathbb{E}\Big[a_{t_{i-1}} e^n_{t_{i-1}}(u) D_{s_2}\big(D_{s_1}(\Psi(u,v)\psi_n)\big)\Big],$$

thanks to $D_s a_{t_{i-1}} = 0$ and $D_s e^n_{t_{i-1}}(u) = 0$ for $s > t_{i-1}$, and

$$\hat{\mathfrak{A}}_n(u,v) = 2\sum_{i=1}^{1/\Delta_n} \lambda_2 \Delta_n^{-1} \int_{t_{i-1}}^{t_i} ds_1 \int_{t_{i-1}}^{s_1} ds_2 \mathbb{E}\Big[D_{s_2}\big(e^n_{s_1}(u) - e^n_{t_{i-1}}(u)\big) \times D_{s_1}(\Psi(u,v)\psi_n a_{t_{i-1}})\Big].$$

Then by continuity of $e^n_\cdot(u)$ in $\mathbb{D}_{1,p}$ (see again (A2)), we conclude $\hat{\mathfrak{A}}_n(u,v) \to 0$ as $n \to \infty$ for every (u,v). Since $e^n_{s_1}\Psi(u,v)$ is bounded under truncation by ψ_n or even by its derivative, the $\mathbb{L}^p$-continuity of the objects yields

$$\check{\mathfrak{A}}_n(u,v) \quad \to \quad \lambda_2 \mathbb{E}\Big[\int_0^1 a_t \exp\big(\mathtt{i}uM_t + \frac{1}{2}u^2 C_t\big) D_t D_t \Psi(u,v) dt\Big], \tag{3.10}$$

where $D_t D_t \Psi(u,v) = \lim_{s\uparrow t} D_s D_t \Psi(u,v)$. It should be noted that the integrability and this limiting procedure are valid because $D_t D_t \Psi(u,v) = \Psi(u,v) A_t$ with a sum A_t of regular variables, and

$$\text{ess sup}_\omega \sup_{t\in[0,1]} (C^n_t - C_1) 1_{\{|\xi_n|<1\}} \le \Delta_n^{c/2} \le 1 < \infty$$

for all n, due to $C^n_t \le C^n_1$ and the construction of the quantity ξ_n in Section 2.3. Furthermore,

$$\begin{aligned} & \lambda_2 \mathbb{E}\Big[\int_0^1 a_t \exp\big(\mathtt{i}uM_t + \frac{1}{2}u^2 C_t\big) D_t D_t \Psi(u,v) dt\Big] \\ = \ & \lambda_2 \mathbb{E}\Big[\int_0^1 a_t \mathbb{E}\big[\exp\big(\mathtt{i}uM_t\big)|\mathcal{F}\big]\big\{\exp\big(\frac{1}{2}u^2 C_t\big) \times \Psi(u,v)\big\} A_t dt\Big] \\ = \ & \lambda_2 \mathbb{E}\Big[\int_0^1 a_t D_t D_t \Psi(u,v) dt\Big]. \end{aligned}$$

Consequently, for

$$\Phi^\alpha_n(u,v) \quad = \quad \mathtt{i}^{-|\alpha|} d^\alpha_{(u,v)} \mathbb{E}\big[L^n_1(u)\Psi(u,v)\psi_n\big] = \mathbb{E}\Big[\mathtt{i}u \int_0^1 e^n_t(u) dM^n_t \, \Psi(u,v)\psi_n\Big],$$

where $L^n_t(u) = e^n_t(u) - 1$, we obtain

$$\begin{aligned} \tilde{\Phi}^\alpha(u,v) &= \lim_{n\to\infty} \Delta_n^{-1/2} \Phi^\alpha_n(u,v) \\ &= \lim_{n\to\infty} \mathtt{i}^{-|\alpha|} d^\alpha_{(u,v)} \big(\mathtt{i}u \mathfrak{A}_n(u,v)\big) \\ &= \lambda_2 \mathtt{i}^{-|\alpha|} d^\alpha_{(u,v)} \mathbb{E}\Big[\int_0^1 \mathtt{i}u a_t D_t D_t \Psi(u,v) dt\Big] \\ &= \lambda_2 \mathtt{i}^{-|\alpha|} d^\alpha_{(u,v)} \mathbb{E}\Big[\Psi(u,v) \cdot \int_0^1 \mathtt{i}u a_t \Big((-\frac{u^2}{2} + \mathtt{i}u)^2 (D_t C)^2 + (-\frac{u^2}{2} + \mathtt{i}u) D_t D_t C\Big) dt\Big] \end{aligned}$$

Therefore,

$$\overline{\sigma}(\mathtt{i}u, \mathtt{i}v) \quad = \quad \lambda_2 \int_0^1 \mathtt{i}u a_t \Big((-\frac{u^2}{2} + \mathtt{i}u)^2 (D_t C)^2 + (-\frac{u^2}{2} + \mathtt{i}u) D_t D_t C\Big) dt. \tag{3.11}$$

We recall that the process DX_t is given as the solution of the SDE

$$D_s X_t \quad = \quad b^{[1]}(X_s) + \int_s^t (b^{[2]})'(X_u) D_s X_u du + \int_s^t (b^{[1]})'(X_u) D_s X_u dW_u$$

for $s \le t$ (and 0 when $s > t$), and

$$\begin{aligned} D_r D_s X_t &= (b^{[1]})'(X_s) D_r X_s + \int_x^t (b^{[2]})''(X_u) D_r X_u D_s X_u du + \int_s^t (b^{[2]})'(X_u) D_r D_s X_u du \\ &\quad + \int_s^t (b^{[1]})''(X_u) D_r X_u D_s X_u dW_u + \int_s^t (b^{[1]})'(X_u) D_r D_s X_u dW_u \end{aligned}$$

for $r < s \le t$. Then (3.11) implies the identity.

$$\overline{\sigma}(\mathtt{i}u, \mathtt{i}v) = \mathtt{i}u\lambda_2\Big((-\frac{u^2}{2} + \mathtt{i}v)^2 \mathrm{Var}^2[f(Z)]\mathcal{C}_2 + (-\frac{u^2}{2} + \mathtt{i}v)\mathrm{Var}[f(Z)](\mathcal{C}_3 + \mathcal{C}_4)\Big) \tag{3.12}$$

with

$$\begin{aligned} \mathcal{C}_2 &= \int_0^1 a(X_s)\Big(\int_s^1 (a^2)'(X_u) D_s X_u du\Big)^2 ds \\ \mathcal{C}_3 &= \int_0^1 a(X_s)\Big(\int_s^1 (a^2)''(X_u)(D_s X_u)^2 du\Big) ds \\ \mathcal{C}_4 &= \int_0^1 a(X_s)\Big(\int_s^1 (a^2)'(X_u) D_s D_s X_u du\Big) ds. \end{aligned}$$

Now, having obtained the full random symbol $\sigma = \underline{\sigma} + \overline{\sigma}$ and hence the density $p_n(z, x)$ for σ, we can formulate the following statement, which generalizes the results of [26, Theorem 6] and [27, Theorem 1] on the quadratic form to the weighted power variation of Brownian motion.

Theorem 3.3. *Let $b^{[1]}, b^{[2]} \in C^\infty_{b,1}(\mathbb{R})$, $a \in C^\infty_p(\mathbb{R})$ and $f \in C^{11}_p(\mathbb{R})$. Let the functional F_n be given by (3.4). Define $\beta(x) = Var[f(Z)]\, a(x)^2$ for a standard normal random variable Z. Assume that the following conditions are satisfied:*

(C1) $\inf_x |b^{[1]}(x)| > 0$ *and* $\inf_x |a(x)| > 0$.

(C2) $\sum_{k=1}^\infty |\beta^{(k)}(X_0)| > 0$.

Then for any positive numbers K and γ, it holds that

$$\sup_{h \in \mathcal{E}(K,\gamma)} \Big|\mathbb{E}[h(M_n, F_n)] - \int h(z, x) p_n(z, x) dz dx\Big| = o(\sqrt{\Delta_n})$$

as $n \to \infty$, where the set $\mathcal{E}(K, \gamma)$ was defined in Theorem 2.3.

In the rest of this section, we will prove Theorem 3.3. We will verify conditions (A1), $(A2)_\ell$, (A3), $(A4)_{\ell,\mathfrak{n}}$ and (A5) of Theorem 2.3 for $\ell = 10$. The conditions of Theorem 3.3 trivially imply (A1) and $(A2)_\ell$. We already have $(A4)_{\ell,\mathfrak{n}}$. In the following subsections we concentrate on proving (A3) and (A5).

3.5 Estimate of the characteristic functions

We shall now show condition (A5) of Section 2.3 under the assumptions of Theorem 3.3, namely

$$\sup_n \sup_{(u,v)\in\Lambda_n^0(2,q)} |(u,v)|^3 \Delta_n^{-1/2} |\Phi_n^\alpha(u,v)| \quad < \quad \infty \tag{3.13}$$

for

$$\Phi_n^\alpha(u,v) \quad = \quad \mathtt{i}^{-|\alpha|} d^\alpha_{(u,v)} \mathbb{E}[L_1^n(u)\Psi(u,v)\psi_n], \qquad L_t^n(u) = e_t^n(u) - 1.$$

We apply the duality formula twice and use nondegeneracy of the Malliavin matrix of (M_t^n, F) together with that of $C - C_t$, in the expression

$$\Phi_n^\alpha(u,v) \quad = \quad \mathtt{i}^{-|\alpha|} d^\alpha_{(u,v)} \mathbb{E}\Big[\int_0^1 e_t^n(u) d(\mathtt{i}uM_t^n)\, \Psi(u,v)\psi_n\Big].$$

For this purpose, the representation (3.9) is useful. By the $\mathbb{L}^2$-convergence, we see that

$$\begin{aligned}
&\sum_{i=1}^{1/\Delta_n} \mathbb{E}\Big[\int_{t_{i-1}}^{t_i} e_t^n(u) dM_t^n\, \Psi(u,v)\psi_n\Big] \\
&= \Delta_n^{1/2} \sum_{i=1}^{1/\Delta_n} \sum_{k=2}^{\infty} k!\lambda_k \Delta_n^{-k/2} \int_{t_{i-1}}^{t_i} ds_1 \int_{t_{i-1}}^{s_1} ds_2 \\
&\quad \times \mathbb{E}\Big[\int_{t_{i-1}}^{s_2}\int_{t_{i-1}}^{s_3}\cdots\int_{t_{i-1}}^{s_{k-1}} dW_{s_k}\cdots dW_{s_4}dW_{s_3}\, a_{t_{i-1}} D_{s_2}\Big(e_{s_1}^n(u) D_{s_1}(\Psi(u,v)\psi_n)\Big)\Big] \\
&= \Delta_n^{-1/2} \sum_{i=1}^{1/\Delta_n} \int_{t_{i-1}}^{t_i} ds_1 \int_{t_{i-1}}^{s_1} ds_2 \\
&\quad \times \mathbb{E}\Big[\sum_{k=2}^{\infty} k!\lambda_k \Delta_n^{-(k-2)/2} \int_{t_{i-1}}^{s_2}\int_{t_{i-1}}^{s_3}\cdots\int_{t_{i-1}}^{s_{k-1}} dW_{s_k}\cdots dW_{s_4}dW_{s_3}\, a_{t_{i-1}} D_{s_2}\Big(e_{s_1}^n(u) D_{s_1}(\Psi(u,v)\psi_n)\Big)\Big] \\
&= \Delta_n^{-1/2} \sum_{i=1}^{1/\Delta_n} \int_{t_{i-1}}^{t_i} ds_1 \int_{t_{i-1}}^{s_1} ds_2\, \mathbb{E}\Big[f^\dagger_{n,i,s_2}\, a_{t_{i-1}} D_{s_2}\Big(e_{s_1}^n(u) D_{s_1}(\Psi(u,v)\psi_n)\Big)\Big],
\end{aligned}$$

where

$$f^\dagger_{n,i,s_2} \quad = \quad \sum_{k=2}^{\infty} k!\lambda_k \Delta_n^{-(k-2)/2} \int_{t_{i-1}}^{s_2}\int_{t_{i-1}}^{s_3}\cdots\int_{t_{i-1}}^{s_{k-1}} dW_{s_k}\cdots dW_{s_4}dW_{s_3},$$

and consequently reach the representation

$$\mathtt{i}u \sum_{i=1}^{1/\Delta_n} \mathbb{E}\Big[\int_{t_{i-1}}^{t_i} e_t^n(u) dM_t^n\, \Psi(u,v)\psi_n\Big] \quad = \quad \Delta_n^{-1/2} \sum_{i=1}^{1/\Delta_n} \int_{t_{i-1}}^{t_i} ds_1 \int_{t_{i-1}}^{s_1} ds_2\, E_i^n(u,v)_{s_1,s_2} \tag{3.14}$$

where

$$E_i^n(u,v)_{s_1,s_2} \quad = \quad iu\,\mathbb{E}\Big[f^{\dagger}_{n,i,s_2}\,a_{t_{i-1}}D_{s_2}\Big(e^n_{s_1}(u)D_{s_1}(\Psi(u,v)\psi_n)\Big)\Big]. \tag{3.15}$$

Let

$$\mathbb{E}^n_s(u,v) \quad = \quad e^n_s(u)\Psi(u,v).$$

Then $\mathbb{E}_s(u,v)$ has the FGH-decomposition (cf. [26, page 22]):

$$\mathbb{E}^n_s(u,v) \quad = \quad \mathbb{F}^n_s(u,v)\mathbb{G}_s(u)\mathbb{H}^n_s(u)$$

with

$$\begin{aligned}
\mathbb{F}^n_s(u,v) &= \exp\big(\mathtt{i}uM^n_s+\mathtt{i}vC_1\big), \qquad C_1=C,\\
\mathbb{G}_s(u) &= \exp\Big(-\frac{1}{2}u^2(C_1-C_s)\Big),\\
\mathbb{H}^n_s(u) &= \exp\Big(\frac{1}{2}u^2(C^n_s-C_s)\Big).
\end{aligned}$$

From (3.15) and the FGH-decomposition,

$$E_i^n(u,v)_{s_1,s_2} \quad = \quad \mathbb{E}\Big[\mathbb{F}^n_{s_1}(u,v)\mathbb{G}_{s_1}(u)\mathbb{H}^n_{s_1}(u)\psi^n_{s_1,s_2}(u,v)f^{\dagger}_{n,i}\,a_{t_{i-1}}\Big], \tag{3.16}$$

where

$$\begin{aligned}
\psi^n_{s_1,s_2}(u,v) &= \mathtt{i}u\,\big(e^n_{s_1}(u)\Psi(u,v)\big)^{-1}D_{s_2}\big(e^n_{s_1}(u)D_{s_1}(\Psi(u,v)\psi_n)\big)\\
&= \Big\{\psi_n\big(-\frac{u^2}{2}+\mathtt{i}v\big)D_{s_1}C_1+D_{s_1}\psi_n\Big\}\mathtt{i}u\Big(\mathtt{i}uD_{s_2}M^n_{s_1}+\frac{u^2}{2}D_{s_2}C^n_{s_1}\Big)\\
&\quad+\psi_n\mathtt{i}u\big(-\frac{u^2}{2}+\mathtt{i}v\big)^2(D_{s_2}C_1)(D_{s_1}C_1)\\
&\quad+2(D_{s_2}\psi_n)\mathtt{i}u\big(-\frac{u^2}{2}+\mathtt{i}v\big)D_{s_1}C_1+D_{s_2}D_{s_1}\psi_n\,\mathtt{i}u.
\end{aligned}$$

Suppose that the following condition, which we will prove in the next subsection, is satisfied for $\ell=10$:

(C2$^\flat$) The variables s_n ($n\in\mathbb{N}$) satisfy the following conditions.

- **(i)** $\sup_{t\geq\frac{1}{2}}\mathbb{P}\big[\det\sigma_{(M^n_t,C_1)}<s_n\big]=O(\Delta_n^{4/3+\varepsilon})$ as $n\to\infty$ for some $\varepsilon>0$.
- **(ii)** $\limsup_{n\to\infty}\mathbb{E}[s_n^{-p}]<\infty$ for every $p>1$.
- **(iii)** $\limsup_{n\to\infty}\|s_n\|_{\ell,p}<\infty$ for every $p\geq 2$.

Note that condition (C2$^\flat$) immediately implies (A3). Now following the (a)-(h) procedure of [26, page 22] and the argument of the proof of Theorem 4 therein, we can obtain

$$\sup_n \sup_{i=1,\dots,n} \sup_{s_1,s_2:t_{i-1}<s_1<s_2\le t_i} \sup_{(u,v)\in\Lambda_n^0(2,q)} |(u,v)|^3 \big|E_i^n(u,v)_{s_1,s_2}\big| \ < \ \infty \tag{3.17}$$

by applying the integration-by-parts formula at most 8 times. More precisely, we introduce a new truncation

$$\psi_{n,s_1} \ = \ \psi\Bigg(2\big[1+4\Delta_{(M^n_{s_1},C)}s_n^{-1}\big]^{-1}\Bigg),$$

which will be used when the integration-by-parts formula for $(M^n_{s_1},C)$ is applied for $s_1 \ge 1/2$. We have the decomposition of $E_i^n(u,v)_{s_1,s_2}$ expressed by (3.16):

$$\begin{aligned} & E_i^n(u,v)_{s_1,s_2} \\ = \ & \mathbb{E}\Bigg[\mathbb{F}^n_{s_1}(u,v)\mathbb{G}_{s_1}(u)\mathbb{H}^n_{s_1}(u)\psi^n_{s_1,s_2}(u,v)\psi^n_{s_1}f^\dagger_{n,i}\,a_{t_{i-1}}\Bigg] + R_{n,s_1,s_2}(u,v) \end{aligned}$$

with

$$|R_{n,s_1,s_2}(u,v)| \ \le \ K\Delta_n^{-5q/2}\sup_{s'}\|1-\psi^n_{s'}\|_{\mathbb{L}^p}$$

for all n, s_1 and restricted (u,v), where K a constant. The right-hand side can be shown to be of order $o(\Delta_n^{3q/2})$ for sufficiently small numbers $q > 1/3$ (cf. assumption (A5)) and $p > 1$. Then, as already noticed, we can follow the (a)-(h) procedure of [26], by using the FGH-decomposition, but with $\psi(\xi_n)\psi_{n,s_1}$ for truncation, to obtain (3.17).

Finally, we obtain (3.13) for $\alpha = 0$ from (3.17). When $\alpha \neq 0$, the argument of the proof is essentially the same as above. As a conclusion, (3.13) (and consequently (A5)) holds for every α under the assumptions $(C1)$ and $(C2^\flat)$.

Obviously, condition (A3) is valid under $(C1)$ and $(C2^\flat)$. In particular, the non-degeneracy of C simply follows from $\inf_x |a(x)| > 0$. Thus, we are left to proving condition $(C2^\flat)$.

3.6 Proof of (C2$^\flat$)

We shall now prove that condition (C2$^\flat$) holds under the assumptions of Theorem 3.3. Recall that

$$M_t^n \ = \ \Delta_n^{1/2}\sum_{i:t_i\le t} a(X_{t_{i-1}})f(\Delta_n^{-1/2}\Delta_i^n W)$$

for $t \in \Pi^n = \{t_i\}$. We deduce that

$$\begin{aligned} D_r M_t^n &= \sum_{i:t_i \le t} a_{t_{i-1}} f'(\Delta_n^{-1/2}\Delta_i^n W) 1_{(t_{i-1},t_i]}(r) \\ &\quad + \Delta_n^{1/2} \sum_{i:t_i \le t} a'_{t_{i-1}} D_r X_{t_{i-1}} f(\Delta_n^{-1/2}\Delta_i^n W) 1_{\{r \le t_{i-1}\}} \\ &= \sum_{i:t_i \le t} \Big[a_{t_{i-1}} f'(\Delta_n^{-1/2}\Delta_i^n W) \\ &\quad + \Delta_n^{1/2} \sum_{k=i+1}^{n} a'_{t_{k-1}} f(\Delta_n^{-1/2}\Delta_k^n W) 1_{\{t_k \le t\}} D_r X_{t_{k-1}} \Big] 1_{(t_{i-1},t_i]}(r) \end{aligned}$$

for $t \in \Pi^n$, where $\sum_{k=n+1}^{n} \cdots = 0$. Hence

$$\begin{aligned} \sigma_{11}(n,t) &:= \sigma_{M_t^n} \\ &= \sum_{i:t_i \le t} \int_{t_{i-1}}^{t_i} \Big[a_{t_{i-1}} f'(\Delta_n^{-1/2}\Delta_i^n W) \\ &\quad + \Delta_n^{1/2} \sum_{k=i+1}^{n} a'_{t_{k-1}} f(\Delta_n^{-1/2}\Delta_k^n W) 1_{\{t_k \le t\}} D_r X_{t_{k-1}} \Big]^2 dr \end{aligned}$$

for $t \in \Pi^n$. We have $C_t = \int_0^t \beta(X_s)ds$. Since

$$D_r C_t = \int_r^t \beta'_s D_r X_s ds, \qquad t \in [0,1],$$

we obtain

$$\begin{aligned} \sigma_{12}(n,t) &:= \langle DM^n, DC \rangle_{\mathbb{H}} \\ &= \sum_{i:t_i \le t} \int_{t_{i-1}}^{t_i} \Big(\Big[a_{t_{i-1}} f'(\Delta_n^{-1/2}\Delta_i^n W) \\ &\quad + \Delta_n^{1/2} \sum_{k=i+1}^{n} a'_{t_{k-1}} f(\Delta_n^{-1/2}\Delta_k^n W) 1_{\{t_k \le t\}} D_r X_{t_{k-1}} \Big] \int_r^1 \beta'_s D_r X_s ds \Big) dr \end{aligned}$$

for $t \in \Pi^n$. The Malliavin matrix of (M_t^n, C) is

$$\sigma_{(M_t^n, C)} = \begin{bmatrix} \sigma_{11}(n,t) & \sigma_{12}(n,t) \\ \sigma_{12}(n,t) & \sigma_{22}(1) \end{bmatrix}$$

for $t \in \Pi^n$. Let

$$\sigma(n,t) = \begin{bmatrix} \sigma_{11}(n,t) & \sigma_{12}(n,t) \\ \sigma_{12}(n,t) & \sigma_{22}(t) \end{bmatrix}.$$

By the Clark-Ocone representation formula, we have

$$f'(\Delta_n^{-1/2}\Delta_i^n W) = \Delta_n^{-1/2} \int_{t_{i-1}}^{t_i} a_{n,i}(s) dW_s$$

with

$$a_{n,i}(s) \quad = \quad \Delta_n^{1/2}\mathbb{E}\Big[D_s\Big(f'(\Delta_n^{-1/2}\Delta_i^n W)\Big) \mid \mathcal{F}_s\Big],$$

and moreover

$$\begin{aligned} a_{n,i}(s) &= \mathbb{E}\big[f''(\Delta_n^{-1/2}\Delta_i^n W) \mid \mathcal{F}_s\big] 1_{(t_{i-1},t_i]}(s) \\ &= g_s\big(\Delta_n^{-1/2}(W_s - W_{t_{i-1}})\big)\, 1_{(t_{i-1},t_i]}(s) \end{aligned}$$

with

$$g_r(z) \quad = \quad \int f''\Big(z + \sqrt{\frac{t_i - r}{\Delta_n}}\, x\Big)\phi(x;0,1)dx$$

for $r \in (t_{i-1}, t_i]$. Then obviously,

$$\sup_{\substack{s\in(t_{i-1},t_i]\\ i=1,\dots,n\\ n\in\mathbb{N}}} \|a_{n,i}(s)\|_{9,p} < \infty$$

for every $p > 1$. In the same way, we see that

$$f(\Delta_n^{-1/2}\Delta_i^n W) \quad = \quad \Delta_n^{-1/2}\int_{t_{i-1}}^{t_i} \alpha_{n,i}(s)dW_s$$

with some predictable processes $\alpha_{n,i}(s)$ satisfying

$$\sup_{\substack{s\in(t_{i-1},t_i]\\ i=1,\dots,n\\ n\in\mathbb{N}}} \|\alpha_{n,i}(s)\|_{10,p} < \infty$$

for every $p > 1$. By Lemma 5 of [26],

$$\begin{aligned} &\Big\|\sum_{i:t_i\le t}\int_{t_{i-1}}^{t_i} a_{t_{i-1}} f'(\Delta_n^{-1/2}\Delta_i^n W)\Delta_n^{1/2}\sum_{k=i+1}^{n} a'_{t_{k-1}} f(\Delta_n^{-1/2}\Delta_k^n W)1_{\{t_k\le t\}}D_r X_{t_{k-1}}dr\Big\|_{\mathbb{L}^9} \\ =\ & \Big\|\Delta_n\sum_{i:t_i\le t}\Big[a_{t_{i-1}}\Big(\Delta_n^{-1/2}\int_{t_{i-1}}^{t_i} a_{n,i}(s_1)dW_{s_1}\Big) \\ &\times\Big(\Delta_n^{1/2}\sum_{k=i+1}^{n}\Big\{\int_{t_{i-1}}^{t_i} a'_{t_{k-1}}1_{\{t_k\le t\}}\Delta_n^{-1}D_r X_{t_{k-1}}dr\Big\}\Delta_n^{-1/2}\int_{t_{k-1}}^{t_k}\alpha_{n,i}(s)dW_s\Big)\Big]\Big\|_{\mathbb{L}^9} \\ =\ & O(\Delta_n^{1/2}) \end{aligned}$$

for $t \in \Pi^n$. Hence

$$\sup_{n\in\mathbb{N}}\sup_{t\in\Pi^n}\|\sigma_{11}(n,t) - \tilde\sigma_{11}(n,t)\|_{\mathbb{L}^9} \quad = \quad O(\Delta_n^{1/2})$$

as $n\to\infty$, where the term $\tilde{\sigma}_{11}(n,t)$ is defined in Section 3.2. Furthermore, by the same lemma, we have

$$
\begin{aligned}
&\sup_{t\in\Pi^n}\left\|\sum_{i:t_i\le t}\int_{t_{i-1}}^{t_i}\left(a_{t_{i-1}}f'(\Delta_n^{-1/2}\Delta_i^nW)\int_r^1\beta_s'D_rX_sds\right)dr\right\|_{\mathbb{L}^{10}}\\
=&\sup_{t\in\Pi^n}\left\|\Delta_n\sum_{i:t_i\le t}a_{t_{i-1}}\left(\Delta_n^{-1/2}\int_{t_{i-1}}^{t_i}a_{n,i}(s_1)dW_{s_1}\right)\left(\Delta_n^{-1}\int_{t_{i-1}}^{t_i}\left(\int_r^1\beta_s'D_rX_sds\right)dr\right)\right\|_{\mathbb{L}^9}\\
=&\ O(\Delta_n^{1/2}).
\end{aligned}
$$

Therefore

$$
\sup_{t\in\Pi^n}\|\sigma_{12}(n,t)-\tilde{\sigma}_{12}(n,t)\|_{\mathbb{L}^9} \quad=\quad O(\Delta_n^{1/2}).
$$

From these estimates,

$$
\sup_{t\in\Pi^n}\|\sigma(n,t)-\tilde{\sigma}(n,t)\|_{\mathbb{L}^9} \quad=\quad O(\Delta_n^{1/2}).
$$

One has

$$
\begin{aligned}
\det\tilde{\sigma}(n,t) &= \tilde{\sigma}_{11}(n,t)\sigma_{22}(t)-\tilde{\sigma}_{12}(n,t)^2\\
&\ge \Delta_n\sum_{i:t_i\le t}\left[a_{t_{i-1}}f'(\Delta_n^{-1/2}\Delta_i^nW)\right]^2\times\sigma_{22}(t)\\
&\ge \inf_x|a(x)|^2\,m_n\,\sigma_{22}(t)
\end{aligned}
\tag{3.18}
$$

for $t\in\Pi^n$, where the random variable m_n is defined in Section 3.2.

Now, we shall verify $(C2^\flat)$. Checking $(C2^\flat)$ (iii) is not difficult if one estimates the $\mathbb{H}^{\otimes m}$-norms of $D_{r_1,\dots,r_m}$-derivative of the objects, in part with the aid of the Burkholder inequality.

For $(C2^\flat)$ (ii), it suffices to show

$$
\limsup_{n\to\infty}\mathbb{E}[1_{\{m_n\ge \mathsf{c}_1\}}(\det\tilde{\sigma}(n,1/2))^{-p}] \quad<\quad\infty \tag{3.19}
$$

for every $p>1$ since $s_n\ge 1/2$ when $m_n<\mathsf{c}_1$. Consider the two-dimensional stochastic process $\bar{X}_t=(X_t^{(1)},X_t^{(2)})$ defined by the stochastic integral equations with smooth coefficients

$$
\bar{X}_t \quad=\quad \bar{X}_0+\int_0^tV_1(\bar{X}_s)\circ dW_s+\int_0^tV_0(\bar{X}_s)ds, \tag{3.20}
$$

for $t\in[0,1]$, where the first integral is given in the Stratonovich sense and

$$
V_1(x)=\begin{bmatrix} b^{[1]}(x^1)\\ 0\end{bmatrix},\qquad V_0(x)=\begin{bmatrix}\tilde{b}^{[2]}(x^1)\\ \beta(x^1)\end{bmatrix}
$$

for $x = (x^1, x^2)$, $\tilde{b}^{[2]} = b^{[2]} - 2^{-1}b^{[1]}(b^{[1]})'$. Under $(C2)$, the system (3.20) satisfies the Hörmander condition

$$\text{Lie}[V_0; V_1](x^1, 0) = \mathbb{R}^2 \quad (\forall x^1 \in \text{supp}\mathcal{L}\{X_0\}),$$

the Lie algebra generated by V_1 and V_0, and as a result, for any $t \in (0, 1]$ and $p > 1$, there exists a constant K_p such that

$$\sup_{\mathbf{v}\in\mathbb{R}^2:|\mathbf{v}|=1} \mathbb{P}\left[\mathbf{v}^\star \int_0^t \bar{Y}_s^{-1} V(\bar{X}_s) V(\bar{X}_s)^\star (\bar{Y}_s^{-1})^\star ds\, \mathbf{v} \le \varepsilon\right] \le K_p \varepsilon^p$$

for all $\varepsilon \in (0, 1)$. Here $\bar{Y}_t$ denotes a unique solution of the variational equation corresponding to (3.20). See Kusuoka and Stroock [16, 17], Ikeda and Watanabe [23], Nualart [21] for the nondegeneracy argument. Since both $\bar{Y}_1$ and $\bar{Y}_1^{-1}$ are bounded in $\cap_{p>1}\mathbb{L}^p$, we have

$$\sup_{\mathbf{v}\in\mathbb{R}^2:|\mathbf{v}|=1} \mathbb{P}\left[\mathbf{v}^\star \int_0^t \bar{Y}_1\bar{Y}_s^{-1} V(\bar{X}_s) V(\bar{X}_s)^\star (\bar{Y}_s^{-1})^\star \bar{Y}_1^\star ds\, \mathbf{v} \le \varepsilon\right] \le K_p' \varepsilon^p$$

form some constant $K_p' > 0$, and in particular this implies

$$\mathbb{P}\big[\sigma_{22}(t) \le \varepsilon\big] \le K_p' \varepsilon^p$$

for all $\varepsilon \in (0, 1]$. This inequality gives

$$\sigma_{22}(t)^{-1} \in \bigcap_{p>1} \mathbb{L}^p$$

for every $t \in (0, 1]$, and consequently, in view of (3.18), we obtained (3.19) and hence $(C2^\flat)$ (ii) for arbitrary $\mathsf{c}_1 > 0$.

Finally,

$$\begin{aligned}
& \sup_{t\ge\frac{1}{2}} \mathbb{P}[\det \sigma_{(M_t^n, C_1)} < s_n] \\
\le\ & \sup_{t\ge\frac{1}{2}} \mathbb{P}[\det \sigma(n,t) < s_n] \\
\le\ & \sup_{t\in\Pi^n:t\ge\frac{1}{2}} \mathbb{P}[\det \sigma(n,t) < 1.5 s_n] + \sup_{s,t:|t-s|\le\Delta_n} \mathbb{P}\Big[|\det \sigma(n,t) - \det \sigma(n,s)| > 0.5 s_n\Big] \\
\le\ & \mathbb{P}[\det \sigma(n, 1/2) < 1.5 s_n] + O(\Delta_n^{1.35}) \\
\le\ & \mathbb{P}[\det \tilde{\sigma}(n, 1/2) < 2 s_n] + \mathbb{P}[|\det \sigma(n,1/2) - \det \tilde{\sigma}(n,1/2)| > 0.5 s_n] + O(\Delta_n^{1.35}) \\
\le\ & \mathbb{P}[m_n > 2\mathsf{c}_1,\ \det \tilde{\sigma}(n,1/2) < 2 s_n] + \mathbb{P}[m_n \le 2\mathsf{c}_1] \\
& + \Delta_n^{-3/19}\mathbb{E}[|\det \sigma(n,1/2) - \det \tilde{\sigma}(n,1/2)|^3] + 2^{5\times 19/3}\Delta_n^{5/3}\mathbb{E}[s_n^{-5\times 19/3}] + O(\Delta_n^{1.35}) \\
=\ & O(\Delta_n^{51/38})
\end{aligned}$$

as $n \to \infty$ if we take $\mathsf{c}_1 < \mathbb{E}[f'(Z)^2]/2$. Thus we have verified $(C2^\flat)$ (i), which completes the proof. □

4 Stochastic expansion of generalized power variation of diffusions

Hereafter we will concentrate on the stochastic expansion of the type (2.1) for the class of generalized power variation. The results of this section are necessary for the derivation of the Edgeworth expansion for power variation, which is presented in Section 5, but they might be also useful for other expansion problems in high frequency framework. We again consider a one-dimensional diffusion process $X = (X_t)_{t\in[0,1]}$ satisfying the stochastic differential equation

$$dX_t = b^{[1]}(X_t)dW_t + b^{[2]}(X_t)dt.$$

Our aim is to study the stochastic expansion of generalized power variations of the form

$$V_n(f) = \Delta_n \sum_{i=1}^{1/\Delta_n} f\Big(\frac{\Delta_i^n X}{\sqrt{\Delta_n}}\Big), \qquad \Delta_i^n X = X_{t_i} - X_{t_{i-1}}, \tag{4.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given *even* function, i.e. $f(x) = f(-x)$ for all $x \in \mathbb{R}$. This type of functionals play a very important role in mathematical finance, where they are used for various estimation and testing procedures; see e.g. [4], [5], [8] and [12] among many others. The most classical subclass of statistics (4.1) are power variations, which correspond to functions of the form $f(x) = |x|^p$; we will concentrate on Edgeworth expansion of power variations in the next section. We introduce the notation

$$\rho_x(f) = \mathbb{E}[f(xZ)], \qquad x \in \mathbb{R},\ Z \sim \mathcal{N}(0,1) \tag{4.2}$$

whenever the latter is finite. Now, let us recall the law of large numbers and the central limit theorem for the functional $V_n(f)$ derived in [4].

Theorem 4.1. *(i) Assume that $b^{[1]}, b^{[2]} \in C(\mathbb{R})$ and $f \in C_p(\mathbb{R})$. Then it holds that*

$$V_n(f) \xrightarrow{\mathbb{P}} V(f) = \int_0^1 \rho_{b_s^{[1]}}(f)ds. \tag{4.3}$$

(ii) If moreover $b^{[1]} \in C^2(\mathbb{R})$ and $f \in C_p^1(\mathbb{R})$ we obtain the stable convergence

$$\Delta_n^{-1/2}\Big(V_n(f) - V(f)\Big) \xrightarrow{d_{st}} M \sim MN\left(0, \int_0^1 \rho_{b_s^{[1]}}(f^2) - \rho^2_{b_s^{[1]}}(f)ds\right). \tag{4.4}$$

Remark 4.2. Recall that due to the Itô formula the assumption $b^{[1]} \in C^2(\mathbb{R})$ implies that the process $b_t^{[1]}$ satisfies a SDE of the form (3.1). Thus, $b_t^{[1]}$ is an Itô semimartingale, which is usually required for proving (4.4) (see e.g. [4]). □

Now, we derive the second order stochastic expansion associated with the central limit theorem (4.4). Let us introduce the notation

$$\alpha_i^n = \Delta_n^{-1/2} b_{t_{i-1}}^{[1]} \Delta_i^n W, \tag{4.5}$$

which serves as an approximation of the increment $\Delta_i^n X/\sqrt{\Delta_n}$. One of the main results of this section is the following theorem. We remark that this result might be of independent interest for other expansion problems in probability and statistics.

Theorem 4.3. *Assume that* $b^{[2]} \in C^2(\mathbb{R})$, $b^{[1]} \in C^4(\mathbb{R})$ *and* $f \in C_p^2(\mathbb{R})$. *Then we obtain the stochastic expansion*

$$\widetilde{V}_n(f) := \Delta_n^{-1/2}\Big(V_n(f) - V(f)\Big) = M_n + \Delta_n^{1/2} N_n + o_{\mathbb{P}}(\Delta_n^{1/2}) \tag{4.6}$$

with

$$M_n = \Delta_n^{1/2} \sum_{i=1}^{1/\Delta_n} \Big(f(\alpha_i^n) - \rho_{b^{[1]}_{t_{i-1}}}\Big), \tag{4.7}$$

and $N_n = \sum_{k=1}^5 N_{n,k}$

$$\begin{aligned}
N_{n,1} &= \Delta_n^{1/2} \sum_{i=1}^{1/\Delta_n} f'(\alpha_i^n)\Big(b^{[2]}_{t_{i-1}} + \frac{1}{2} b^{[1.1]}_{t_{i-1}} H_2(\Delta_i^n W/\sqrt{\Delta_n})\Big), \\
N_{n,2} &= \Delta_n^{-1/2} \sum_{i=1}^{1/\Delta_n} f'(\alpha_i^n)\Big(b^{[2.1]}_{t_{i-1}} \int_{t_{i-1}}^{t_i} (W_s - W_{t_{i-1}}) ds \\
&+ b^{[1.2]}_{t_{i-1}} \int_{t_{i-1}}^{t_i} \{s - t_{i-1}\} dW_s + \frac{\Delta_n^{3/2} b^{[1.1.1]}_{t_{i-1}}}{6} H_3(\Delta_i^n W/\sqrt{\Delta_n})\Big), \\
N_{n,3} &= \frac{\Delta_n}{2} \sum_{i=1}^{1/\Delta_n} f''(\alpha_i^n)\Big(b^{[2]}_{t_{i-1}} + \frac{1}{2} b^{[1.1]}_{t_{i-1}} H_2(\Delta_i^n W/\sqrt{\Delta_n})\Big)^2, \\
N_{n,4} &= \frac{1}{2\Delta_n} \sum_{i=1}^{1/\Delta_n} \Big(- \rho''_{b^{[1]}_{t_{i-1}}}(f) |b^{[1.1]}_{t_{i-1}}|^2 \int_{t_{i-1}}^{t_i} (W_s - W_{t_{i-1}})^2 ds \\
&- \Delta_n^2 \rho'_{b^{[1]}_{t_{i-1}}}(f) b^{[1.2]}_{t_{i-1}}\Big), \\
N_{n,5} &= -\Delta_n^{-1} \sum_{i=1}^{1/\Delta_n} \rho'_{b^{[1]}_{t_{i-1}}}(f) b^{[1.1]}_{t_{i-1}} \int_{t_{i-1}}^{t_i} (W_s - W_{t_{i-1}}) ds,
\end{aligned} \tag{4.8}$$

where $(H_k)_{k \geq 0}$ *denote the Hermite polynomials and the processes* $b_t^{[k_1 \ldots k_d]}$ *were defined in Section 3.*

Proof. See Section 7. □

To describe the limits of the quantities $N_{n,k}$, $1 \leq k \leq 5$, we need to introduce some further notation.

Notation. We introduce the functions $g_k : \mathbb{R}^6 \to \mathbb{R}$, $1 \le k \le 5$, as follows:

$$\begin{aligned}
g_1(x_1,\ldots,x_6) &= \mathbb{E}\Big[Uf'(x_2U)\Big(x_1+\frac{1}{2}x_5H_2(U)\Big)-\rho'_{x_2}(f)x_5UV\Big] \\
g_2(x_1,\ldots,x_6) &= \mathbb{E}\Big[f'(x_2U)\Big((x_3+x_4)V+\frac{1}{6}x_6H_3(U)\Big)\Big] \\
g_3(x_1,\ldots,x_6) &= \frac{1}{2}\mathbb{E}\Big[f''(x_2U)\Big(x_1+\frac{1}{2}x_5H_2(U)\Big)^2\Big] \\
g_4(x_1,\ldots,x_6) &= -\frac{1}{4}\rho''_{x_2}(f)x_5^2-\frac{1}{2}\rho'_{x_2}(f)x_4 \\
g_5(x_1,\ldots,x_6) &= \mathbb{E}\Big[\Big\{f'(x_2U)\Big(x_1+\frac{1}{2}x_5H_2(U)\Big)-\rho'_{x_2}(f)x_5V\Big\}^2\Big]
\end{aligned}$$

with

$$(U,V)\sim\mathcal{N}_2\left(0,\begin{pmatrix}1 & 1/2\\ 1/2 & 1/3\end{pmatrix}\right).$$

Remark 4.4. Theorem 8.1 implies the convergence in probability

$$N_{n,k}\xrightarrow{\mathbb{P}}N_k=\int_0^1 g_k(b_s^{[2]},b_s^{[1]},b_s^{[2.1]},b_s^{[1.2]},b_s^{[1.1]},b_s^{[1.1.1]})ds, \qquad k=2,3,4 \tag{4.9}$$

under the assumptions of Theorem 4.3. The terms $N_{n,1}$ and $N_{n,5}$ converge stably in law due to Theorem 8.2; their asymptotic distributions will be specified later. □

Remark 4.5. The fact that we consider the drift and volatility processes of the type $b_s^{[k]} = b^{[k]}(X_s)$ is not essential for developing the stochastic expansion of Theorem 4.3. In general the processes $b_s^{[k_1\ldots k_l]}$ that appear in Theorem 4.3 may depend on different Brownian motions, which are not perfectly correlated with W that drives the process X. In this case a similar stochastic expansion can be deduced; however, it will contain additional terms, which are due to new Brownian motions. □

In the next section we will require a consistent estimator of the asymptotic variance of M_n, i.e.

$$C=\int_0^1 \rho_{b_s^{[1]}}(f^2)-\rho^2_{b_s^{[1]}}(f)ds.$$

A rather natural one is given by

$$F_n=\Delta_n\sum_{i=1}^{1/\Delta_n} f^2\Big(\frac{\Delta_i^nX}{\sqrt{\Delta_n}}\Big)-f\Big(\frac{\Delta_i^nX}{\sqrt{\Delta_n}}\Big)f\Big(\frac{\Delta_{i+1}^nX}{\sqrt{\Delta_n}}\Big) \tag{4.10}$$

We remark that F_n is a feasible statistic in contrast to the Riemann sum approximation defined at (2.23). The next theorem, which follows from the combination of central limit theorems presented in [4] and Theorem 8.2, describes the joint asymptotic distribution of (M_n,F_n,N_n). This result is crucial for the derivation of the Edgeworth expansion.

Theorem 4.6. *Assume that conditions of Theorem 4.3 are satisfied. Then we obtain the stable convergence*

$$\Big(M_n, \Delta_n^{-1/2}(F_n - C), N_n\Big) \xrightarrow{d_{st}} (M, \widehat{F}, N) \sim MN\Big(\mu, \int_0^1 \Xi_s ds\Big),$$

where the matrix Ξ_s is given as

$$\begin{aligned}
\Xi_s^{11} &= \rho_{b_s^{[1]}}(f^2) - \rho^2_{b_s^{[1]}}(f) \\
\Xi_s^{12} &= \Xi_s^{21} = \rho_{b_s^{[1]}}(f^3) - 3\rho_{b_s^{[1]}}(f^2)\rho_{b_s^{[1]}}(f) + 2\rho^3_{b_s^{[1]}}(f) \\
\Xi_s^{22} &= \rho_{b_s^{[1]}}(f^4) - 4\rho_{b_s^{[1]}}(f^3)\rho_{b_s^{[1]}}(f) + 6\rho_{b_s^{[1]}}(f^2)\rho^2_{b_s^{[1]}}(f) - 3\rho^4_{b_s^{[1]}}(f), \\
\Xi_s^{33} &= (g_5 - g_1^2)(b_s^{[2]}, b_s^{[1]}, b_s^{[2.1]}, b_s^{[1.2]}, b_s^{[1.1]}, b_s^{[1.1.1]}),
\end{aligned}$$

and $\Xi_s^{13} = \Xi_s^{23} = 0$, and $\mu_1 = \mu_2 = 0$,

$$\mu_3 = \int_0^1 g_1(b_s^{[2]}, b_s^{[1]}, b_s^{[2.1]}, b_s^{[1.2]}, b_s^{[1.1]}, b_s^{[1.1.1]})dW_s + \sum_{k=2}^4 N_k.$$

5 Asymptotic expansion for the power variation

Now we have all instruments at hand to obtain the Edgeworth expansion for the case of power variation $V_n(f_p)$ with

$$f_p(x) = |x|^p,$$

which is our leading example. As we mentioned in Section 4, this would be the most important class of functionals in mathematical finance. In order to obtain the Edgeworth expansion for power variation, we will combine the results of Sections 3 and 4. Applying Theorem 4.3 to the function f_p we see that the martingale part M_n is given as

$$M_n = \Delta_n^{1/2} \sum_{i=1}^{1/\Delta_n} |b^{[1]}(X_{t_{i-1}})|^p \Big(\Big|\frac{\Delta_i^n W}{\sqrt{\Delta_n}}\Big|^p - m_p\Big)$$

with $m_p = \mathbb{E}[|\mathcal{N}(0,1)|^p]$. In particular, M_n is a weighted power variation studied in Section 3. Consequently, we can apply the results of Section 3 with

$$a(x) = |b^{[1]}(x)|^p, \qquad f(x) = f_p(x) - m_p \qquad \text{and} \quad p \in 2\mathbb{N} \cup (11, \infty).$$

Now, we will compute all quantities from previous sections required for the Edgeworth expansion. First, we obtain the Hermite expansion

$$f(x) = \sum_{k=2}^{\infty} \lambda_k H_k(x)$$

with $\lambda_k = 0$ if k is odd (because f is an even function), and

$$\lambda_2 = \frac{m_{p+2} - m_p}{2}.$$

We start with the computation of the random symbol $\underline{\sigma}$. Here we mainly need to determine the functions $g_1, \ldots, g_5$ defined in Section 4. We observe that, for any $k \geq 0$ with $k < p$,

$$f_p^{(k)}(x) = \operatorname{sgn}(x)^k p(p-1)\cdots(p-k+1)|x|^{p-k}, \qquad \rho_x(f_p) = m_p|x|^p.$$

Now, a straightforward calculation gives the identities

$$\begin{aligned}
g_1(x_1,\ldots,x_6) &= p \operatorname{sgn}(x_2)|x_2|^{p-1}\Big(x_1 m_p + \frac{1}{2}x_5(m_{p+2} - 2m_p)\Big)\\
g_2(x_1,\ldots,x_6) &= p \operatorname{sgn}(x_2)|x_2|^{p-1}\Big(\frac{1}{2}(x_3+x_4)m_p + \frac{1}{6}x_6(m_{p+2} - m_p)\Big)\\
g_3(x_1,\ldots,x_6) &= \frac{p(p-1)}{2}|x_2|^{p-2}\Big(x_1^2 m_{p-2} + x_1x_5(m_p - m_{p-2}) + \frac{x_5^2}{4}(m_{p+2} - 2m_p + m_{p-2})\Big)\\
g_4(x_1,\ldots,x_6) &= \frac{p}{4}m_p\Big(-(p-1)|x_2|^{p-2}x_5^2 - 2x_4 \operatorname{sgn}(x_2)|x_2|^{p-1}\Big)\\
g_5(x_1,\ldots,x_6) &= p^2|x_2|^{2p-2}\Big(x_1^2 m_{2p-2} + x_1x_5(m_{2p} - m_{2p-2}) + \frac{x_5^2}{4}(m_{2p+2} - 2m_{2p} + m_{2p-2})\\
&+ \frac{x_5^2}{3}m_p^2 - x_5 m_p(x_1 m_p + \frac{x_5}{2}[m_{p+2} - m_p])\Big)
\end{aligned}$$

As in the previous section we consider the quantity

$$F_n = \Delta_n \sum_{i=1}^{1/\Delta_n} f_{2p}\Big(\frac{\Delta_i^n X}{\sqrt{\Delta_n}}\Big) - f_p\Big(\frac{\Delta_i^n X}{\sqrt{\Delta_n}}\Big) f_p\Big(\frac{\Delta_{i+1}^n X}{\sqrt{\Delta_n}}\Big)$$

as a consistent estimator of C. We obtain the following result, which again follows from Theorem 8.2.

Theorem 5.1. *Assume that conditions of Theorem 4.3 are satisfied. Then we obtain the stable convergence*

$$\Big(M_n, \Delta_n^{-1/2}(F_n - C), N_n, \Delta_n^{-1/2}(C_n - C)\Big) \xrightarrow{d_{st}} (M, \widehat{F}, N, \widehat{C}) \sim MN\Big(\mu, \int_0^1 \Xi_s ds\Big),$$

where the entries Ξ_s^{ij}, $1 \leq i,j \leq 3$, of the matrix $\Xi_s \in \mathbb{R}^{4\times 4}$ and μ_j, $1 \leq j \leq 3$ of the vector $\mu \in \mathbb{R}^4$ are given in Theorem 4.6, and $\mu_4 = \Xi_s^{34} = 0$,

$$\begin{aligned}
\Xi_s^{14} &= \Xi_s^{41} = \Gamma_2\ |b^{[1]}(X_s)|^{3p},\\
\Xi_s^{24} &= \Xi_s^{42} = \overline{\Gamma}\ |b^{[1]}(X_s)|^{4p},\\
\Xi_s^{44} &= \Gamma_1\ |b^{[1]}(X_s)|^{4p},
\end{aligned}$$

where the constants Γ_1, Γ_2 *are given in Proposition 3.2 and* $\overline{\Gamma}$ *is defined as*

$$\overline{\Gamma} = Cov\left[f_{2p}(W_1), \int_0^1 \mathbb{E}^2[f_p'(W_1)|\mathcal{F}_s]ds\right] - 2Cov\left[f_p(W_1)f_p(W_2-W_1), \int_0^1 \mathbb{E}^2[f_p'(W_1)|\mathcal{F}_s]ds\right].$$

As a consequence of Theorem 5.1 and Remark 2.1 we conclude that

$$\underline{\sigma}(z, \mathtt{i}u, \mathtt{i}v) = (\mathtt{i}u)^2\mathcal{H}_1(z) + \mathtt{i}u\mathcal{H}_2 + \mathtt{i}v\mathcal{H}_3(z) \tag{5.1}$$

with

$$\mathcal{H}_1(z) = z\frac{\int_0^1 \Xi_s^{14}ds}{2\int_0^1 \Xi_s^{11}ds}, \qquad \mathcal{H}_2 = \mu_3, \qquad \mathcal{H}_3(z) = z\frac{\int_0^1 \Xi_s^{12}ds}{\int_0^1 \Xi_s^{11}ds}.$$

It should be noted that $\underline{\sigma}$ of (5.1) is essentially the same but different from $\underline{\sigma}$ of (3.7) since the reference functional F_n is now defined by (4.10) not by (3.4) while the limits of both coincide with each other and the ways of derivation of two adaptive random symbols are the same except for $\widehat{F}$. Using the results of Section 3 we immediately obtain the anticipative random symbol

$$\overline{\sigma}(\mathtt{i}u, \mathtt{i}v) = \mathtt{i}u\Big(\mathtt{i}v - \frac{u^2}{2}\Big)^2\mathcal{H}_4 + \mathtt{i}u\Big(\mathtt{i}v - \frac{u^2}{2}\Big)\mathcal{H}_5 \tag{5.2}$$

with

$$\mathcal{H}_4 = \lambda_2(m_{2p} - m_p^2)^2\mathcal{C}_2, \qquad \mathcal{H}_5 = \lambda_2(m_{2p} - m_p^2)(\mathcal{C}_3 + \mathcal{C}_4),$$

where

$$\begin{aligned}
\mathcal{C}_2 &= \int_0^1 |b^{[1]}(X_s)|^p\Big(\int_s^1 \Big(|b^{[1]}|^{2p}\Big)'(X_u)D_sX_udu\Big)^2ds,\\
\mathcal{C}_3 &= \int_0^1 |b^{[1]}(X_s)|^p\Big(\int_s^1 \Big(|b^{[1]}|^{2p}\Big)''(X_u)(D_sX_u)^2du\Big)ds,\\
\mathcal{C}_4 &= \int_0^1 |b^{[1]}(X_s)|^p\Big(\int_s^1 \Big(|b^{[1]}|^{2p}\Big)'(X_u)D_sD_sX_udu\Big)ds.
\end{aligned}$$

In the power variation case, $a(x) = |b^{[1]}(x)|^p$ and we assumed in $(C1)$ that $a(x)$ is bounded away from zero. So, in our situation, $a(x)$ is smooth in a neighborhood of X_0. By a certain large deviation argument, we may assume that $a(x)$ is smooth and even having bounded derivatives, from the beginning, at least in the proof of asymptotic nondegeneracy.

From the above argument, we obtain an asymptotic expansion for the power variation. Recall $\widetilde{V}_n(f) = \Delta_n^{-1/2}\big(V_n(f) - V(f)\big)$.

Theorem 5.2. *Let* $b^{[1]}, b^{[2]} \in C_{b,1}^\infty(\mathbb{R})$ *and* $f_p(x) = |x|^p$ *with* $p \in 2\mathbb{N} \cup (13, \infty)$*. Assume that* $\inf_x |b^{[1]}(x)| > 0$*,* $\sum_{k=1}^\infty |(b^{[1]})^{(k)}(X_0)| > 0$ *and let the functional* F_n *be given by (3.4). Then for the density* $p_n(z,x)$ *corresponding to the random symbol* σ *determined by (5.1) and (5.2), it holds that*

$$\sup_{h\in\mathcal{E}(K,\gamma)}\left|\mathbb{E}[h(\widetilde{V}_n(f_p), F_n)] - \int h(z,x)p_n(z,x)dzdx\right| = o\big(\sqrt{\Delta_n}\,\big)$$

as $n \to \infty$*, for any positive numbers* K *and* γ*.*

Theorem 5.2 is proved by applying Theorems 3.3 and 5.1. In the present situation, N_n involves f'' and that is the reason why the number 13 appears. However, it would be possible to reduce it to 11 if the estimations related with N_n-part is refined, though we do not pursue this point in this article.

Theorem 5.2 and the corresponding Edgeworth expansion for the studentized statistics at (6.1) are the main results of this paper. In particular, these asymptotic expansions can be applied to distribution analysis of various statistics in financial mathematics as power variation type estimators are frequently used in this field. Another potential area of application is Euler approximation of continuous SDE's of the form (3.1). As is well-known from [13], the Euler approximation scheme is asymptotically mixed normal and its limit depends on the asymptotic theory for quadratic variation. Thus, our Edgeworth expansion results can be potentially applied to numerical analysis of SDE's to obtain a more precise formula for the error distribution.

Remark 5.3. As we mentioned above, we can combine the results of Sections 3 and 4, because we consider the power function $f_p(x) = |x|^p$. In this case the dominating part M_n is a weighted power variation in the sense of Section 3. The case of a general *even* function f is more complicated. The results of 4 still apply, but the computations of the random symbol $\overline{\sigma}$ is more involved. Let us shortly sketch the idea how $\overline{\sigma}$ can be obtained. Recall that in the general case the term M_n is given as

$$M_n = \Delta_n^{1/2} \sum_{i=1}^{1/\Delta_n} \Big(f(\alpha_i^n) - \rho_{b^{[1]}_{t_{i-1}}} \Big)$$

(see Theorem 4.3). As in Section 3 we therefore need to compute the projection onto the second order Wiener chaos of the quantity

$$f\Big(\Delta_n^{-1/2} b^{[1]}_{t_{i-1}} \Delta_i^n W\Big) - \rho_{b^{[1]}_{t_{i-1}}}.$$

For this purpose we use the following multiplication formula (see [3])

$$H_k(\gamma x) = \sum_{i=0}^{[k/2]} 2^{-i} \gamma^{k-2i} (\gamma^2 - 1)^i \binom{k}{2i} \frac{(2i)!}{i!} H_{k-2i}(x), \qquad \gamma \in \mathbb{R}.$$

Under the assumptions of Section 4, the function f admits the Hermite expansion $f(x) = \sum_{k=0}^{\infty} \lambda_{2k} H_{2k}(x)$ (since f is even). Hence, we deduce that

$$f(\gamma x) = \sum_{k=0}^{\infty} \lambda_{2k} \left(\sum_{i=0}^{k} 2^{-i} \gamma^{2k-2i} (\gamma^2 - 1)^i \binom{2k}{2i} \frac{(2i)!}{i!} H_{2k-2i}(x) \right).$$

We conclude that the projection of $f(\alpha_i^n) - \rho_{b^{[1]}_{t_{i-1}}}$ onto the second order Wiener chaos is given by

$$\sum_{k=1}^{\infty} 2^{-k+1} \lambda_{2k} |b^{[1]}_{t_{i-1}}|^2 (|b^{[1]}_{t_{i-1}}|^2 - 1)^{k-1} \binom{2k}{2} \frac{(2(k-1))!}{(k-1)!} \; H_2(\Delta_n^{-1/2} \Delta_i^n W).$$

Using this identity one can compute $\overline{\sigma}$ as in Section 3. However, we dispense with the exact exposition. □

6 Studentization

As we mentioned in the beginning, we are mainly interested in the Edgeworth expansion connected with standard central limit theorem

$$\frac{Z_n}{\sqrt{F_n}} \xrightarrow{d} \mathcal{N}(0,1).$$

where F_n is a consistent estimator of C defined in (4.10). In the following we present such an Edgeworth expansion for the case of power variation discussed in the previous section. First of all, we remark that the random symbol $\sigma(z, \mathtt{i}u, \mathtt{i}v)$ is given as

$$\sigma(z, \mathtt{i}u, \mathtt{i}v) = \sum_{j=1}^{8} c_j(z)(\mathtt{i}u)^{m_j}(\mathtt{i}v)^{n_j},$$

where

$$\begin{array}{llllll} m_1 = 1, & n_1 = 0, & c_1(z) = \mathcal{H}_2, & m_2 = 0, & n_2 = 1, & c_2(z) = \mathcal{H}_3(z) \\ m_3 = 2, & n_3 = 0, & c_3(z) = \mathcal{H}_1(z), & m_4 = 1, & n_4 = 1, & c_4(z) = \mathcal{H}_5 \\ m_5 = 3, & n_5 = 0, & c_5(z) = \frac{1}{2}\mathcal{H}_5, & m_6 = 1, & n_6 = 2, & c_6(z) = \mathcal{H}_4 \\ m_7 = 3, & n_7 = 1, & c_7(z) = \mathcal{H}_4, & m_8 = 5, & n_8 = 0, & c_8(z) = \frac{1}{4}\mathcal{H}_4. \end{array}$$

As a consequence, we obtain the following decomposition for the density $p_n(z,x)$ of (Z_n, F_n):

$$p_n(z,x) = \phi(z;0,x)p^C(x) + \Delta_n^{1/2} \sum_{j=1}^{8} p_j(z,x)$$

with

$$p_j(z,x) = (-d_z)^{m_j}(-d_x)^{n_j}\Big(\phi(z;0,x)p^C(x)\mathbb{E}\left[c_j(z)|C=x\right]\Big), \qquad 1 \le j \le 8.$$

We start with the following observation. Let Π be a finite measure on $\mathbb{R}$ with density π, such that all moments of Π are finite. Then it trivially holds that

$$\lim_{x\to\infty} |x|^k \pi(x) = 0, \qquad \lim_{x\to-\infty} |x|^k \pi(x) = 0 \qquad k \ge 0.$$

Given that the density π is a C^k function and g is a polynomial, we also have

$$\int_{\mathbb{R}} g^{(k)}(x)\pi(x)dx = (-1)^k \int_{\mathbb{R}} g(x)\pi^{(k)}(x)dx$$

by induction. Let g be an arbitrary polynomial and $\kappa(x) = \mathbb{E}[H|C = x]p^C(x)$ for an integrable random variable H, and note that

$$\int_{\mathbb{R}} m(x)\kappa(x)dx = \mathbb{E}[m(C)H],$$

whenever the integral makes sense. We define the polynomials $q_{\beta,v}(z,x)$ via

$$d_x^\beta g(z/\sqrt{x}) = \sum_{v\le\beta} q_{\beta,v}(z/\sqrt{x},1/\sqrt{x})g^{(v)}(z/\sqrt{x}),$$

where $g^{(v)}$ denotes the vth derivative of g. Let $(\alpha,\beta)\in\mathbb{N}_0^2$. Then it holds that

$$\begin{aligned}
&\int_{\mathbb{R}^2} g\Big(\frac{z}{\sqrt{x}}\Big) d_z^\alpha d_x^\beta\Big[\phi(z;0,x)\kappa(x)\Big]dzdx = (-1)^\beta\int_{\mathbb{R}^2} d_x^\beta g\Big(\frac{z}{\sqrt{x}}\Big)d_z^\alpha\phi(z;0,x)\kappa(x)dzdx\\
&= (-1)^\beta\int_{\mathbb{R}^2}\sum_{v\le\beta} q_{\beta,v}\Big(\frac{z}{\sqrt{x}},\frac{1}{\sqrt{x}}\Big)g^{(v)}\Big(\frac{z}{\sqrt{x}}\Big)d_z^\alpha\phi(z;0,x)\kappa(x)dzdx\\
&= (-1)^\beta\int_{\mathbb{R}^2}\sum_{v\le\beta} q_{\beta,v}\Big(y,\frac{1}{\sqrt{x}}\Big)g^{(v)}(y)x^{-\alpha/2}d_y^\alpha\phi(y;0,1)\kappa(x)dydx\\
&= (-1)^\beta\int_{\mathbb{R}} g(y)\sum_{v\le\beta}(-1)^v d_y^v\left\{d_y^\alpha\phi(y;0,1)\int_{\mathbb{R}} q_{\beta,v}\Big(y,\frac{1}{\sqrt{x}}\Big)x^{-\alpha/2}\kappa(x)dx\right\}dy\\
&= \int_{\mathbb{R}} g(y)\sum_{v\le\beta}(-1)^{\beta+v}d_y^v\left\{d_y^\alpha\phi(y;0,1)\mathbb{E}\Big[HC^{-\alpha/2}q_{\beta,v}(y,C^{-1/2})\Big]\right\}dy.
\end{aligned}$$

Clearly, the above identity will enable us to compute the Edgeworth expansion for the studentized statistic $Z_n/\sqrt{F_n}$. We need to determine the polynomials $q_{\beta,v}$ for $\beta=0,1,2$:

$$\begin{aligned}
&q_{0,0}(a,b)=1,\\
&q_{1,0}(a,b)=0, \qquad q_{1,1}(a,b)=-\frac{1}{2}ab^2,\\
&q_{2,0}(a,b)=0, \qquad q_{2,1}(a,b)=\frac{3}{4}ab^4, \qquad q_{2,2}(a,b)=\frac{1}{4}a^2b^4.
\end{aligned}$$

Recall the identity $d_y^\alpha\phi(y;0,1)=(-1)^\alpha H_\alpha(y)\phi(y;0,1)$ and

$$H_1(x)=x, \qquad H_3(x)=x^3-3x, \qquad H_5(x)=x^5-10x^3+15x.$$

A straightforward computation shows that

$$\begin{aligned}
\int_{\mathbb{R}^2} g\Big(\frac{z}{\sqrt{x}}\Big)p_1(z,x)dzdx &= \mathbb{E}[\mathcal{H}_2C^{-1/2}]\int_{\mathbb{R}} g(y)y\phi(y;0,1)dy,\\
\int_{\mathbb{R}^2} g\Big(\frac{z}{\sqrt{x}}\Big)\sum_{j=4}^5 p_j(z,x)dzdx &= -\frac{1}{2}\mathbb{E}[\mathcal{H}_5C^{-3/2}]\int_{\mathbb{R}} g(y)y\phi(y;0,1)dy,\\
\int_{\mathbb{R}^2} g\Big(\frac{z}{\sqrt{x}}\Big)\sum_{j=6}^8 p_j(z,x)dzdx &= \frac{3}{4}\mathbb{E}[\mathcal{H}_4C^{-5/2}]\int_{\mathbb{R}} g(y)y\phi(y;0,1)dy.
\end{aligned}$$

The corresponding computation for the terms $p_2(z,x)$ and $p_3(z,x)$ has to be performed separately, since the random variables c_2 and c_3 depend on z. Recall that the quantities $\mathcal{H}_1(z)$ and $\mathcal{H}_3(z)$ are linear in z, i.e. $\mathcal{H}_1(z) = z\widetilde{\mathcal{H}}_1$, $\mathcal{H}_3(z) = z\widetilde{\mathcal{H}}_3$. We deduce as above (here $\kappa(x) = \mathbb{E}[\widetilde{\mathcal{H}}_3|C = x]p^C(x)$)

$$\int_{\mathbb{R}^2} g\Big(\frac{z}{\sqrt{x}}\Big)p_2(z,x)dzdx = -\int_{\mathbb{R}^2} zg\Big(\frac{z}{\sqrt{x}}\Big)d_x\Big[\phi(z;0,x)\kappa(x)\Big]dzdx$$

$$= \int_{\mathbb{R}} g(y)d_y\Big\{y\phi(y;0,1)\mathbb{E}\Big[\widetilde{\mathcal{H}}_3 q_{1,1}(y,C^{-1/2})C^{1/2}\Big]\Big\}\,dy$$

$$= \frac{1}{2}\mathbb{E}[\widetilde{\mathcal{H}}_3 C^{-1/2}]\int_{\mathbb{R}} g(y)\phi(y;0,1)(2y-y^3)dy.$$

Finally, we obtain that (here $\kappa(x) = \mathbb{E}[\widetilde{\mathcal{H}}_1|C = x]p^C(x)$)

$$\int_{\mathbb{R}^2} g\Big(\frac{z}{\sqrt{x}}\Big)p_3(z,x)dzdx = \int_{\mathbb{R}^2} g\Big(\frac{z}{\sqrt{x}}\Big)d_z^2\Big[z\phi(z;0,x)\kappa(x)\Big]dzdx$$

$$= \int_{\mathbb{R}^2} x^{-1}g''(y)y\phi(y;0,1)\kappa(x)dydx$$

$$= \mathbb{E}[\widetilde{\mathcal{H}}_1 C^{-1/2}]\int_{\mathbb{R}} g(y)d_y^2[y\phi(y;0,1)]dy = \mathbb{E}[\widetilde{\mathcal{H}}_1 C^{-1/2}]\int_{\mathbb{R}} g(y)H_3(y)\phi(y;0,1)dy.$$

Combining the above results, we deduce the Edgeworth expansion for the density of $Z_n/\sqrt{F_n}$

$$p^{Z_n/\sqrt{F_n}}(y) = \phi(y;0,1) + \Delta_n^{1/2}\phi(y;0,1)\Big(y\Big\{\mathbb{E}[\mathcal{H}_2 C^{-1/2}] - \frac{1}{2}\mathbb{E}[\mathcal{H}_5 C^{-3/2}] \tag{6.1}$$

$$+\frac{3}{4}\mathbb{E}[\mathcal{H}_4 C^{-5/2}] + \mathbb{E}[\widetilde{\mathcal{H}}_3 C^{-1/2}] - 3\mathbb{E}[\widetilde{\mathcal{H}}_1 C^{-1/2}]\Big\} + y^3\Big\{\mathbb{E}[\widetilde{\mathcal{H}}_1 C^{-1/2}] - \frac{1}{2}\mathbb{E}[\widetilde{\mathcal{H}}_3 C^{-1/2}]\Big\}\Big),$$

which is one of the main statements of the paper.

Remark 6.1. In practice the application of the asymptotic expansion at (6.1) requires the knowledge of the coefficients of the type $b^{[k_1\ldots k_d]}$ (cf. (4.9)). While the volatility related processes $b^{[1]}$, $b^{[1.1]}$, $b^{[1.1.1]}$ can be estimated from high frequency data X_{t_i}, the drift related processes $b^{[2]}$, $b^{[2.1]}$, $b^{[1.2]}$ can't be consistently estimated on a fixed time span. Thus, the applicability of the Edgeworth expansion at (6.1) relies on the knowledge of the drift related coefficients or their estimation on an infinite time span. □

Example 6.2. (*Classical Edgeworth expansion*) In this example we compare the classical Edgeworth expansion with the result derived in (6.1). Let $(Y_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with mean μ and variance σ^2. Define $S_n = n^{-1/2}\sum_{i=1}^n \sigma^{-1}(Y_i - \mu)$. Then the Edgeworth expansion of the density of S_n is given as

$$\phi(y;0,1) + \frac{\kappa_3}{6\sigma^3\sqrt{n}}\phi(y;0,1)H_3(y),$$

where κ_3 denotes the third cumulant of the law of Y_1. Let us now consider the quantity M_n from (3.2) with $a \equiv 1$ and $\Delta_n = n^{-1}$, i.e.

$$M_n = n^{-1/2} \sum_{i=1}^{n} f(\sqrt{n}\Delta_i^n W) \qquad \text{with} \qquad \mathbb{E}[f(W_1)] = 0.$$

Due to self-similarity of the Brownian motion, we are in the classical setting of i.i.d. observations. In this case $C = \sigma^2 = \mathbb{E}[f^2(W_1)]$, and if we set $F_n \equiv C$, we obtain from (6.1):

$$\phi(y; 0, 1) + \frac{1}{\sqrt{n}} \phi(y; 0, 1) H_3(y) \widetilde{\mathcal{H}}_1 C^{-1/2}$$

as the approximative density, since all quantities in (6.1) are 0 except C and $\widetilde{\mathcal{H}}_1$. We now show that the quantities $\widetilde{\mathcal{H}}_1 C^{-1/2}$ and $\frac{\kappa_3}{6\sigma^3}$ are indeed equal. Recall from the previous section that

$$\widetilde{\mathcal{H}}_1 = \frac{\int_0^1 \Xi_s^{14} ds}{2 \int_0^1 \Xi_s^{11} ds}, \qquad \int_0^1 \Xi_s^{11} ds = C, \qquad \int_0^1 \Xi_s^{14} ds = \mathbb{E}\left[f(W_1) \int_0^1 \mathbb{E}^2[f'(W_1)|\mathcal{F}_s] ds \right].$$

Hence, we just need to prove the identity

$$\kappa_3 = 3\mathbb{E}\left[f(W_1) \int_0^1 \mathbb{E}^2[f'(W_1)|\mathcal{F}_s] ds \right].$$

But $\kappa_3 = \mathbb{E}[f^3(W_1)]$ and Itô formula implies that

$$\begin{aligned} 3\mathbb{E}\left[f(W_1) \int_0^1 \mathbb{E}^2[f'(W_1)|\mathcal{F}_s] ds \right] &= 3\mathbb{E}\left[\int_0^1 \mathbb{E}[f(W_1)|\mathcal{F}_s] \mathbb{E}^2[f'(W_1)|\mathcal{F}_s] ds \right] \\ &= \mathbb{E}[f^3(W_1)] \end{aligned}$$

due to the identity $f(W_1) = \int_0^1 \mathbb{E}[f'(W_1)|\mathcal{F}_s] dW_s$. □

7 Proofs

7.1 A stochastic expansion

Below, we denote by K a generic positive constant, which may change from line to line. We also write K_p if the constant depends on an external parameter p.

Proof of Theorem 4.3: First, we remark that all processes of the type $(b_s^{[k_1 \dots k_m]})_{s \geq 0}$ ($k_j \in \{1, 2\}$), which we consider below, are continuous and so locally bounded. Applying the localization technique described in Section 3 of [4] we can assume w.l.o.g. that these processes are bounded in (ω, s), which we do from now on. We decompose

$$\Delta_n^{-1/2} \Big(V_n(f) - V(f) \Big) = M_n + R_n^{(1)} + R_n^{(2)}$$

with

$$R_n^{(1)} = \Delta_n^{-1/2}\Big(V_n(f) - \Delta_n \sum_{i=1}^{1/\Delta_n} f(\alpha_i^n)\Big),$$

$$R_n^{(2)} = \Delta_n^{-1/2}\Big(\Delta_n \sum_{i=1}^{1/\Delta_n} \rho_{b^{[1]}_{t_{i-1}}} - V(f)\Big).$$

We start with the asymptotic expansion of the quantity $R_n^{(2)}$. Due to Burkholder inequality any process Y of the form (3.1) with bounded coefficients $b^{[1]}$, $b^{[2]}$ satisfies the inequality

$$\mathbb{E}[|Y_t - Y_s|^p] \le C_p |t-s|^{p/2} \tag{7.1}$$

for any $p \ge 0$. In particular, this inequality holds for the processes $b^{[1]}$, $b^{[2]}$, $b^{[2.2]}$, $b^{[2.1]}$, $b^{[1.2]}$, $b^{[1.1]}$ as they are diffusion processes (due to Itô formula). Applying (7.1) and the Taylor expansion we deduce that

$$\begin{aligned} R_n^{(2)} &= \Delta_n^{-1/2} \sum_{i=1}^{1/\Delta_n} \int_{t_{i-1}}^{t_i} \{\rho_{b^{[1]}_{t_{i-1}}} - \rho_{b^{[1]}_s}\} ds \\ &= \Delta_n^{-1/2} \sum_{i=1}^{1/\Delta_n} \int_{t_{i-1}}^{t_i} \{\rho'_{b^{[1]}_{t_{i-1}}}(b^{[1]}_{t_{i-1}} - b^{[1]}_s) - \frac{1}{2}\rho''_{b^{[1]}_{t_{i-1}}}(b^{[1]}_{t_{i-1}} - b^{[1]}_s)^2\} ds \\ &+ o_{\mathbb{P}}(\Delta_n^{1/2}) \\ &=: R_n^{(2.1)} + R_n^{(2.2)} + o_{\mathbb{P}}(\Delta_n^{1/2}). \end{aligned}$$

Recall that

$$b^{[1]}_t = b^{[1]}_0 + \int_0^t b^{[1.2]}_s ds + \int_0^t b^{[1.1]}_s dW_s.$$

We conclude the identity

$$\begin{aligned} R_n^{(2.2)} &= -\frac{\Delta_n^{-1/2}}{2} \sum_{i=1}^{1/\Delta_n} \rho''_{b^{[1]}_{t_{i-1}}} \int_{t_{i-1}}^{t_i} (b^{[1]}_{t_{i-1}} - b^{[1]}_s)^2 ds \\ &= -\frac{\Delta_n^{-1/2}}{2} \sum_{i=1}^{1/\Delta_n} \rho''_{b^{[1]}_{t_{i-1}}} |b^{[1.1]}_{t_{i-1}}|^2 \int_{t_{i-1}}^{t_i} (W_{t_{i-1}} - W_s)^2 ds + o_{\mathbb{P}}(\Delta_n^{1/2}) \\ &=: \Delta_n^{1/2}(N^{(1)}_{n,4} + o_{\mathbb{P}}(1)). \end{aligned}$$

For the term $R_n^{(2.1)}$ we obtain the decomposition

$$\begin{aligned}
R_n^{(2.1)} &= -\Delta_n^{-1/2} \sum_{i=1}^{1/\Delta_n} \rho'_{b^{[1]}_{t_{i-1}}} \int_{t_{i-1}}^{t_i} \Big(\int_{t_{i-1}}^{s} b_u^{[1.2]} du + \int_{t_{i-1}}^{s} b_u^{[1.1]} dW_u \Big) ds \\
&= -\Delta_n^{-1/2} \sum_{i=1}^{1/\Delta_n} \rho'_{b^{[1]}_{t_{i-1}}} \Big(\frac{\Delta_n^2}{2} b_{t_{i-1}}^{[1.2]} + b_{t_{i-1}}^{[1.1]} \int_{t_{i-1}}^{t_i} (W_s - W_{t_{i-1}}) ds \Big) \\
&+ o_{\mathbb{P}}(\Delta_n^{1/2}) \\
&=: \Delta_n^{1/2} (N_{n,4}^{(2)} + N_{n,5} + o_{\mathbb{P}}(1)).
\end{aligned}$$

We remark that

$$N_{n,4} = N_{n,4}^{(1)} + N_{n,4}^{(2)}.$$

The treatment of the quantity $R_n^{(1)}$ is a bit more involved. We apply again (7.1) and Taylor expansion:

$$\begin{aligned}
R_n^{(1)} &= \Delta_n^{1/2} \sum_{i=1}^{1/\Delta_n} \Big(f'(\alpha_i^n) \{ \frac{\Delta_i^n X}{\sqrt{\Delta_n}} - \alpha_i^n \} + \frac{1}{2} f''(\alpha_i^n) \{ \frac{\Delta_i^n X}{\sqrt{\Delta_n}} - \alpha_i^n \}^2 \Big) + o_{\mathbb{P}}(\Delta_n^{1/2}) \\
&=: R_n^{(1.1)} + R_n^{(1.2)} + o_{\mathbb{P}}(\Delta_n^{1/2}).
\end{aligned}$$

For the term $R_n^{(1.2)}$ we obtain the decomposition

$$\begin{aligned}
R_n^{(1.2)} &= \frac{\Delta_n^{-1/2}}{2} \sum_{i=1}^{1/\Delta_n} f''(\alpha_i^n) \Big(\int_{t_{i-1}}^{t_i} b_s^{[2]} ds + \int_{t_{i-1}}^{t_i} b_s^{[1]} - b_{t_{i-1}}^{[1]} dW_s \Big)^2 \\
&= \frac{\Delta_n^{-1/2}}{2} \sum_{i=1}^{1/\Delta_n} f''(\alpha_i^n) \Big(\Delta_n b_{t_{i-1}}^{[2]} + b_{t_{i-1}}^{[1.1]} \int_{t_{i-1}}^{t_i} (W_s - W_{t_{i-1}}) dW_s \Big)^2 + o_{\mathbb{P}}(\Delta_n^{1/2}) \\
&= \frac{\Delta_n^{3/2}}{2} \sum_{i=1}^{1/\Delta_n} f''(\alpha_i^n) \Big(b_{t_{i-1}}^{[2]} + \frac{1}{2} b_{t_{i-1}}^{[1.1]} H_2 \Big(\frac{\Delta_i^n W}{\sqrt{\Delta_n}} \Big) \Big)^2 + o_{\mathbb{P}}(\Delta_n^{1/2}) \\
&= \Delta_n^{1/2} (N_{n,3} + o_{\mathbb{P}}(1)).
\end{aligned}$$

The quantity $R_n^{(1.1)}$ is decomposed as

$$\begin{aligned}
R_n^{(1.1)} &= \sum_{i=1}^{1/\Delta_n} f'(\alpha_i^n) \Big(\int_{t_{i-1}}^{t_i} b_s^{[2]} ds + \int_{t_{i-1}}^{t_i} \{ b_s^{[1]} - b_{t_{i-1}}^{[1]} \} dW_s \Big) \\
&= R_n^{(1.1.1)} + R_n^{(1.1.2)}
\end{aligned}$$

with

$$R_n^{(1.1.1)} = \Delta_n \sum_{i=1}^{1/\Delta_n} f'(\alpha_i^n)\Big(b_{t_{i-1}}^{[2]} ds + \frac{1}{2} b_{t_{i-1}}^{[1.1]} H_2\Big(\frac{\Delta_i^n W}{\sqrt{\Delta_n}}\Big)\Big),$$

$$R_n^{(1.1.2)} = \sum_{i=1}^{1/\Delta_n} f'(\alpha_i^n)\Big(\int_{t_{i-1}}^{t_i} \{b_s^{[2]} - b_{t_{i-1}}^{[2]}\} ds$$

$$+ \int_{t_{i-1}}^{t_i} \Big(\int_{t_{i-1}}^{s} b_u^{[1.2]} du + \int_{t_{i-1}}^{s} \{b_u^{[1.1]} - b_{t_{i-1}}^{[1.1]}\} dW_u\Big) dW_s\Big).$$

We remark that

$$R_n^{(1.1.1)} = \Delta_n^{1/2} N_{n,1}.$$

Since f' is an odd function (because f is even) we deduce that

$$R_n^{(1.1.2)} = \sum_{i=1}^{1/\Delta_n} f'(\alpha_i^n)\Big(b_{t_{i-1}}^{[2.1]} \int_{t_{i-1}}^{t_i} (W_s - W_{t_{i-1}}) ds + \frac{\Delta_n^{3/2} b_{t_{i-1}}^{[1.1.1]}}{6} H_3\Big(\frac{\Delta_i^n W}{\sqrt{\Delta_n}}\Big)$$

$$+ b_{t_{i-1}}^{[1.2]} \int_{t_{i-1}}^{t_i} (s - t_{i-1}) dW_s\Big) + o_{\mathbb{P}}(\Delta_n^{1/2}).$$

As

$$R_n^{(1.1.2)} = \Delta_n^{1/2}(N_{n,2} + o_{\mathbb{P}}(1)),$$

we are done. □

8 Appendix

In this subsection we present a law of large numbers and a multivariate functional stable convergence theorem, which is frequently used in this paper. For any $k = 1, \ldots, d$, let $g_k : C([0,1]) \to \mathbb{R}$ be a measurable function with polynomial growth, i.e.

$$|g_k(x)| \le K(1 + \|x\|_\infty^p),$$

for some $K > 0$, $p > 0$ and $\|x\|_\infty = \sup_{z\in[0,1]} |x(z)|$. In most cases g_k will be a function of $x(1)$; the path-dependent version is only required to account for the asymptotic behaviour of the functional C_n. Let $(a_s)_{s\ge 0}$ be an $\mathbb{R}^d$-valued, $(\mathcal{F}_s)$-adapted, continuous and bounded stochastic process. Our first result is the following theorem.

Theorem 8.1. *Let $g : \mathbb{R}^d \times C([0,1]) \to \mathbb{R}$ be a measurable function with polynomial growth in the last variable and $a = (a_1, \ldots, a_d)$. Then it holds that*

$$\Delta_n \sum_{i=1}^{1/\Delta_n} g\Big(a_{t_{i-1}}, \Delta_n^{-1/2}\{W_{t_{i-1}+s\Delta_n} - W_{t_{i-1}}\}_{0\le s\le 1}\Big) \xrightarrow{\mathbb{P}} \int_0^1 \rho(a_s, g) ds$$

with

$$\rho(z, g) := \mathbb{E}[g(z, \{W_s\}_{0\le s\le 1})], \qquad z \in \mathbb{R}^d.$$

Proof of Theorem 8.1: Since $\Delta_n^{-1/2}\{W_{t_{i-1}+s\Delta_n} - W_{t_{i-1}}\}_{0\le s\le 1} \overset{d}{=} \{W_s\}_{0\le s\le 1}$. we obtain that

$$\Delta_n \sum_{i=1}^{1/\Delta_n} \mathbb{E}\Big[g\Big(a_{t_{i-1}}, \Delta_n^{-1/2}\{W_{t_{i-1}+s\Delta_n} - W_{t_{i-1}}\}_{0\le s\le 1}\Big)|\mathcal{F}_{t_{i-1}}\Big]$$

$$= \Delta_n \sum_{i=1}^{1/\Delta_n} \rho(a_{t_{i-1}}, g) \xrightarrow{\mathbb{P}} \int_0^1 \rho(a_s, g)ds.$$

On the other hand, we deduce that

$$\Delta_n \sum_{i=1}^{1/\Delta_n} g\Big(a_{t_{i-1}}, \Delta_n^{-1/2}\{W_{t_{i-1}+s\Delta_n} - W_{t_{i-1}}\}_{0\le s\le 1}\Big)$$

$$-\Delta_n \sum_{i=1}^{1/\Delta_n} \mathbb{E}\Big[g\Big(a_{t_{i-1}}, \Delta_n^{-1/2}\{W_{t_{i-1}+s\Delta_n} - W_{t_{i-1}}\}_{0\le s\le 1}\Big)|\mathcal{F}_{t_{i-1}}\Big] \xrightarrow{\mathbb{P}} 0,$$

because

$$\Delta_n^2 \sum_{i=1}^{1/\Delta_n} \mathbb{E}\Big[g^2\Big(a_{t_{i-1}}, \Delta_n^{-1/2}\{W_{t_{i-1}+s\Delta_n} - W_{t_{i-1}}\}_{0\le s\le 1}\Big)|\mathcal{F}_{t_{i-1}}\Big] \xrightarrow{\mathbb{P}} 0.$$

This completes the proof. □

Next, we consider a sequence of d-dimensional processes $Y_t^n = (Y_{1,t}^n, \dots, Y_{d,t}^n)$ defined via

$$\begin{aligned} Y_{k,t}^n &= \Delta_n^{1/2} \sum_{i=1}^{[t/\Delta_n]} a_{t_{i-1}}^k \Big[g_k\Big(\Delta_n^{-1/2}\{W_{t_{i-1}+s\Delta_n} - W_{t_{i-1}}\}_{0\le s\le 1}\Big) \\ &- \mathbb{E}g_k\Big(\Delta_n^{-1/2}\{W_{t_{i-1}+s\Delta_n} - W_{t_{i-1}}\}_{0\le s\le 1}\Big)\Big], \qquad k = 1, \dots, d. \end{aligned}$$

The stable convergence of Y^n is as follows.

Theorem 8.2. *It holds that*

$$Y_t^n \xrightarrow{d_{st}} Y_t = \int_0^t v_s dW_s + \int_0^t (w_s - v_s v_s^\star)^{1/2} dW_s',$$

where the functional convergence is stable in law, W' is a d-dimensional Brownian motion independent of $\mathcal{F}$, and the processes $(v_s)_{s\ge 0}$ in $\mathbb{R}^d$ and $(w_s)_{s\ge 0}$ in $\mathbb{R}^{d\times d}$ are defined as

$$\begin{aligned} v_s^k &= a_s^k \mathbb{E}\Big[g_k(\{W_s\}_{0\le s\le 1})W_1\Big], \\ w_s^{kl} &= a_s^k a_s^l cov\Big[g_k(\{W_s\}_{0\le s\le 1}), g_l(\{W_s\}_{0\le s\le 1})\Big], \end{aligned}$$

with $1 \le k, l \le d$. In particular, it holds that $\int_0^t w_s^{1/2} dW_s' \sim MN\left(0, \int_0^t w_s ds\right)$.

Proof of Theorem 8.2: We write $Y_t^n = \sum_{i=1}^{[t/\Delta_n]} \chi_i^n$ with

$$\begin{aligned}\chi_{i,k}^n &= \Delta_n^{1/2} a_{t_{i-1}}^k \Big[g_k\Big(\Delta_n^{-1/2}\{W_{t_{i-1}+s\Delta_n} - W_{t_{i-1}}\}_{0\le s\le 1}\Big) \\ &- \mathbb{E} g_k\Big(\Delta_n^{-1/2}\{W_{t_{i-1}+s\Delta_n} - W_{t_{i-1}}\}_{0\le s\le 1}\Big)\Big], \qquad k = 1,\dots,d.\end{aligned}$$

According to Theorem IX.7.28 of [14] we need to show that

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}[\chi_{i,k}^n \chi_{i,l}^n | \mathcal{F}_{t_{i-1}}] \stackrel{\mathbb{P}}{\longrightarrow} \int_0^t w_s^{kl} ds, \tag{8.1}$$

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}[\chi_{i,k}^n \Delta_i^n W | \mathcal{F}_{t_{i-1}}] \stackrel{\mathbb{P}}{\longrightarrow} \int_0^t v_s^k ds, \tag{8.2}$$

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}[|\chi_{i,k}^n|^2 1_{\{|\chi_{i,k}^n|>\epsilon\}} | \mathcal{F}_{t_{i-1}}] \stackrel{\mathbb{P}}{\longrightarrow} 0 \qquad \forall \epsilon > 0, \tag{8.3}$$

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}[\chi_{i,k}^n \Delta_i^n Q | \mathcal{F}_{t_{i-1}}] \stackrel{\mathbb{P}}{\longrightarrow} 0, \tag{8.4}$$

where $1 \le k, l \le d$ and the last condition must hold for all bounded continuous martingales Q with $[W, Q] = 0$. Conditions (8.1) and (8.2) are obvious since

$$\Delta_n^{-1/2}\{W_{t_{i-1}+s\Delta_n} - W_{t_{i-1}}\}_{0\le s\le 1} \stackrel{d}{=} \{W_s\}_{0\le s\le 1}.$$

Condition (8.3) follows from

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}[|\chi_{i,k}^n|^2 1_{\{|\chi_{i,k}^n|>\epsilon\}} | \mathcal{F}_{t_{i-1}}] \le \epsilon^{-2} \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}[|\chi_{i,k}^n|^4 | \mathcal{F}_{t_{i-1}}] \le K\Delta_n \to 0,$$

which holds since the process a is bounded and g_k is of polynomial growth. In order to prove the last condition, we use the Itô-Clark representation theorem

$$\begin{aligned}& g_k\Big(\Delta_n^{-1/2}\{W_{t_{i-1}+s\Delta_n} - W_{t_{i-1}}\}_{0\le s\le 1}\Big) \\ & - \mathbb{E} g_k\Big(\Delta_n^{-1/2}\{W_{t_{i-1}+s\Delta_n} - W_{t_{i-1}}\}_{0\le s\le 1}\Big) = \int_{t_{i-1}}^{t_i} \eta_{k,s}^n dW_s\end{aligned}$$

for some predictable process η_k^n. Itô isometry implies the identity

$$\mathbb{E}[\chi_{i,k}^n \Delta_i^n Q | \mathcal{F}_{t_{i-1}}] = \Delta_n^{1/2} a_{t_{i-1}}^k \mathbb{E}\left[\int_{t_{i-1}}^{t_i} \eta_{k,s}^n d[W,Q]_s | \mathcal{F}_{t_{i-1}}\right] = 0.$$

This completes the proof of Theorem 8.2. □

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