

Assessing Relative Volatility/Intermittency/Energy Dissipation

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Abstract

We introduce the notion of relative volatility/intermittency and demonstrate how relative volatility statistics can be used to estimate consistently the temporal variation of volatility/intermittency even when the data of interest are generated by a non-semimartingale, or a Brownian semistationary process in particular. While this estimation method is motivated by the assessment of relative energy dissipation in empirical data of turbulence, we apply it also to energy price data. Moreover, we develop a probabilistic asymptotic theory for relative power variations of Brownian semistationary processes and Itô semimartingales and discuss how it can be used for inference on relative volatility/intermittency.

Keywords: Brownian semistationary process, energy dissipation, intermittency, power variation, turbulence, volatility.

JEL Classification: C10, C14

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1 Introduction

The concept of (stochastic) *Volatility* is of central importance in many fields of science. In some of these the term *Intermittency* is used instead of volatility. Thus volatility/intermittency has a central role in mathematical finance and financial econometrics [BNS10], in turbulence, rain and cloud studies [LS06, Way06] and other aspects of environmental science [PA12], in relation to nanoscale emitters [FVKJ13], magnetohydrodynamics [MP09], and to liquid mixtures of chemicals [Sre04], and last but not least in the physics of fusion plasmas [CSVMAV00]. In turbulence the key concept of *Energy Dissipation* is subsumed under that of intermittency¹.

Speaking generally, volatility/intermittency is taken to mean that the phenomenon under study exhibits more variation than expected; that is, more than the most basic type of random influence² envisaged.

Hence volatility/intermittency is a *relative* concept, and its meaning depends on the particular setting under investigation. Once that meaning is clarified the question is how to assess the volatility/intermittency empirically and then to describe it in stochastic terms and incorporate it in a suitable probabilistic model.

Such ‘additional’ random fluctuations generally vary, in time and/or in space, in regard to *Intensity* (*activity rate* and *duration*) and *Amplitude*. Typically the volatility/intermittency may be further classified into continuous and discrete (i.e. jumps) elements, and long and short term effects.

In finance the investigation of volatility is well developed and many of the procedures of probabilistic and statistical analysis applied are similar to those of relevance in turbulence, for instance in regard to multipower variations, particularly quadratic and bipower variations and variation ratios.

Other important issues concern the modelling of propagating stochastic volatility/intermittency fields and the question of predictability of volatility/intermittency.

This paper introduces a concept of *realised relative volatility/intermittency* and hence of *realised relative energy dissipation*, the ultimate purpose of which is to assess the relative volatility/intermittency or energy dissipation in arbitrary subregions of a region C of space-time relative to the total volatility/intermittency/energy in C .

We start the further discussion by describing, in Section 2, how energy dissipation in turbulence is defined and traditionally assessed. This is followed by a brief outline of some results from the theory of *Brownian semistationary* (\mathcal{BSS}) processes that are pertinent for the main topic of the present paper. The definition of realised relative volatility/intermittency/energy is given in Section 3. For concreteness and because of its particular importance we focus on realised relative energy dissipation. Asymptotic probabilistic properties — consistency and a functional central limit theorem — for realised

¹For discussions of intermittency and energy dissipation in turbulence see [Fri95], Chapter 6 and [Tsi09], Chapter 7, cf. also the illustration on p. 20 of the latter reference.

²Often thought of as Gaussian.

relative power variations are derived in Section 4. Applications to data on turbulence and energy prices are presented in Section 5. Section 6 concludes.

2 Energy dissipation in turbulence

In a purely spatial setting the *energy dissipation* of a homogenous and isotropic turbulent field is (up to an ignorable constant involving viscosity)

$$\varepsilon(x) = \sum_{i,j=1}^3 \left(\frac{\partial y_i}{\partial x_j} + \frac{\partial y_j}{\partial x_i} \right)^2, \quad (1)$$

where y_i denotes the velocity at the spatial position $x \in \mathbb{R}^3$. The *coarse grained energy dissipation* over a region C in \mathbb{R}^3 is then given by

$$\bar{\varepsilon}(C) = \frac{1}{\mu(C)} \int_C \varepsilon(x) dx.$$

Furthermore, if only measurements of the velocity component in the main direction x_1 of the flow are considered one defines the *surrogate energy dissipation* as

$$\varepsilon(x) = \left(\frac{\partial y_1}{\partial x_1} \right)^2.$$

By Taylor's frozen field hypothesis [Tay38], this may then be reinterpreted as the *timewise surrogate energy dissipation*

$$\varepsilon(t) = \left(\frac{\partial y_t}{\partial t} \right)^2,$$

which would be the relevant quantity in case the measurements were of the same, main, component of the velocity but now as a function of time rather than of position.

Associated to this is the coarse grained energy dissipation corresponding to the interval $[t, t + u]$ and given by

$$\bar{\varepsilon}(u) = \frac{\varepsilon^+(t + u) - \varepsilon^+(t)}{u},$$

where

$$\varepsilon^+(t) = \int_0^t \left(\frac{dy_s}{dt} \right)^2 ds. \quad (2)$$

Supposing that the velocity y_t has been observed over the interval $[0, T]$ at times $0, \delta, \dots, [T/\delta]\delta$, when it comes to estimating $\varepsilon^+(t)$, as given by (2), this is traditionally done by taking the normalised realised quadratic variation $[y_\delta]_t/\delta$, where

$$[y_\delta]_t = \sum_{j=1}^{[t/\delta]} (y_{j\delta} - y_{(j-1)\delta})^2.$$

Correspondingly, the coarse grained energy dissipation over $[0, T]$ is estimated by

$$\hat{\varepsilon}(T) = \frac{[y_\delta]_T}{T}.$$

The definitions (1) and (2) of course assume that the sample path y is differentiable. On the other hand, going back to Kolmogorov, it is broadly recognised that turbulence can

only be comprehensibly understood by viewing it as a random phenomenon.³ Accordingly, y should be viewed as a stochastic process, henceforth denoted Y , and it is not realistic to assume that its sample paths are differentiable. Thus a broader setting for the analysis of the energy dissipation in Y is called for, and in the following we propose and discuss such a setting.

A Brownian semistationary (\mathcal{BSS}) process, as introduced in [BNSch09], may be used as a model for the timewise development of the velocity at a fixed point in space and in the main direction of the flow in a homogeneous and isotropic turbulent field. For focus and illustration we shall consider cases where Y is a stationary \mathcal{BSS} process,

$$Y_t = \mu + \int_{-\infty}^t g(t-s)\sigma_s dB_s + \int_{-\infty}^t q(t-s)\sigma_s^2 ds, \quad (3)$$

where g and q are deterministic kernel functions, B is Brownian motion and σ is a stationary process expressing the volatility/intermittency of the process. In that context the gamma form

$$g(t) = ct^{\nu-1}e^{-\lambda t} \quad (4)$$

of the kernel g has a special role. In particular, if $\nu = \frac{5}{6}$ and σ is square integrable, then the autocorrelation function of Y is identical to von Kármán's autocorrelation function [vKar48] for ideal turbulence.

In relation to the \mathcal{BSS} process (3) with gamma kernel (4) a central question is that of determining σ^2 from Y . In case the process is a semimartingale the answer is given by the quadratic variation of Y ; in fact, then $[Y]_t = \sigma_t^{2+}$, where

$$\sigma_t^{2+} = \int_0^t \sigma_s^2 ds \quad (5)$$

is the accumulated quadratic volatility over the interval $[0, t]$. However, in the cases of most interest for turbulence, that is $\nu \in (\frac{1}{2}, 1) \cup (1, \frac{3}{2})$ the process Y is not a semimartingale and in order to determine σ^{2+} by a limiting procedure from the realised quadratic variation

$$[Y_\delta]_t = \sum_{j=1}^{\lfloor t/\delta \rfloor} (Y_{j\delta} - Y_{(j-1)\delta})^2, \quad t \geq 0,$$

the latter has to be normalised by a factor depending on δ and ν . Specifically, as shown in [BNSch09], this factor is $\delta c(\delta)^{-2}$ where

$$c(\delta) = \mathbf{E}\{(G_\delta - G_0)^2\}^{1/2} \quad (6)$$

is defined using the Gaussian core

$$G_t = \int_{-\infty}^t g(t-s)dB_s$$

of the process Y . Then have that, as $\delta \rightarrow 0$,

$$\frac{\delta}{c(\delta)^2} [Y_\delta]_t \xrightarrow{p} \sigma_t^{2+}. \quad (7)$$

Using this result for estimation of σ_t^{2+} requires either that ν is known or that a sufficiently accurate estimate of ν can be found. The latter question has led to detailed studies of the application of power and multipower variations to estimation of ν , see [BNCP12, BNCP13, CHPP13].

³See [Kol41a, Kol41b, Kol41c, Kol62] and, for a recent overview, [Tsi09].

3 Realised relative V/I/E

Supposing again that the velocity Y_t has been observed at times $0, \delta, \dots, \lfloor T/\delta \rfloor \delta$, we are interested in the *relative energy dissipation* of Y over any subinterval $[t, t+u]$ of $[0, T]$, i.e.,

$$\frac{\varepsilon^+(t+u) - \varepsilon^+(t)}{\varepsilon^+(T)}, \quad (8)$$

where $\varepsilon^+(T)$ is the energy dissipation in $[0, T]$. Within the turbulence literature, this definition of the relative energy dissipation is strongly related to the definition of a *multiplier* in the cascade picture of the transport of energy from large to small scales (see [CSG08] and references therein).

We now introduce the concept of realised relative energy dissipation. Specifically, whether Y_t is deterministic and differentiable or an arbitrary stochastic process we define the *realised relative energy dissipation* over the subinterval $[t, t+u]$ as

$$R_\delta^+(t, t+u) = \widetilde{[Y_\delta]_{t+u, T}} - \widetilde{[Y_\delta]_{t, T}},$$

where

$$\widetilde{[Y_\delta]_{t, T}} = \frac{[Y_\delta]_t}{[Y_\delta]_T}$$

is the *realised relative quadratic variation* of Y . We note that the quantity $R_\delta^+(t, t+u)$ is entirely empirically based.

In the “classical” case of turbulence, where Y_t is differentiable, as $\delta \rightarrow 0$ we have

$$[Y_\delta]_{t+u} - [Y_\delta]_t \sim \delta(\varepsilon^+(t+u) - \varepsilon^+(t))$$

and hence, as $\delta \rightarrow 0$,

$$R_\delta^+(t, t+u) \rightarrow \frac{\varepsilon^+(t+u) - \varepsilon^+(t)}{\varepsilon^+(T)},$$

i.e., the limit equals the relative energy dissipation (8).

Now suppose that Y is a stationary \mathcal{BSS} process (3) with gamma kernel (4) and $\nu > \frac{1}{2}$, as it needed for the stochastic integral to exist. Then, if $\nu > \frac{3}{2}$, Y has continuous differentiable sample paths, i.e. we are essentially in the “classical” situation. If $\nu = 1$ the process Y is a semimartingale and the realised quadratic variation $[Y_\delta]$ converges to the quadratic variation $[Y]$, that is

$$[Y_\delta]_t \xrightarrow{p} [Y]_t = \sigma_t^{2+},$$

where σ_t^{2+} is the accumulated quadratic volatility/intermittency (5). Consequently, for the realised relative energy dissipation we have

$$R_\delta^+(t, t+u) \xrightarrow{p} \frac{[Y]_{t+u} - [Y]_t}{[Y]_T} = \frac{\sigma_{t+u}^{2+} - \sigma_t^{2+}}{\sigma_T^{2+}},$$

i.e., the limit is the relative accumulated squared volatility/intermittency.

Finally, suppose that $\nu \in (\frac{1}{2}, 1) \cup (1, \frac{3}{2})$, i.e., we are in the non-semimartingale case and the sample paths are Hölder continuous of order $\nu - 1/2$. Then, subject to a mild condition on q (see Appendix C for a result covering the case where q is of the gamma form), we have again, as $\delta \rightarrow 0$, that

$$R_\delta^+(t, t+u) \xrightarrow{p} \frac{\sigma_{t+u}^{2+} - \sigma_t^{2+}}{\sigma_T^{2+}}$$

although

$$[Y_\delta]_t \propto \begin{cases} \delta^{-2(1-\nu)} \rightarrow \infty & \text{if } \nu \in (\frac{1}{2}, 1) \\ \delta^{2(\nu-1)} \rightarrow 0 & \text{if } \nu \in (1, \frac{3}{2}) \end{cases}.$$

This follows directly from limiting results in [BNSch09] and [BNCP12]. In view of these results, in the turbulence context we view the limit of R_δ^+ as the relative energy dissipation.

Remark 1. As mentioned in Section 2, use of the original assessment procedure (7) requires determination of the degree of freedom/smoothness parameter ν . The realised relative quadratic variation $\widehat{[Y_\delta]_{t,T}}$ is entirely empirically determined and its consistency does not rely on inference on ν .

4 Probabilistic asymptotic theory of realised relative power variations

We develop now a probabilistic asymptotic theory for *realised relative power variations*, going slightly beyond the earlier discussion of quadratic variations and energy dissipation. To highlight the robustness of realised relative power variations to model misspecification, we consider both a \mathcal{BSS} process and an Itô semimartingale as the underlying process. While we limit the discussion to *power* variations for the sake of simpler exposition, our results can be easily extended to *multipower* variations.

4.1 Probabilistic setup and consistency

Let us consider a stochastic process $Y = \{Y_t\}_{t \geq 0}$, defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, P)$ via the decomposition

$$Y_t = A_t + X_t,$$

where $A = \{A_t\}_{t \geq 0}$ is a process that allows for skewness in the distribution of Y_t . The process A is assumed to fulfill one of two negligibility conditions, viz. (10) and (13) given below (Appendix C presents more concrete criteria that can be used to check these conditions). Given a standard Brownian motion $B = \{B_t\}_{t \in \mathbb{R}}$ and a càglàd process $\sigma = \{\sigma_t\}_{t \in \mathbb{R}}$, adapted to the natural filtration of B , we allow for the following two specifications of the process $X = \{X_t\}_{t \geq 0}$.

(I) X is a local Brownian martingale given by

$$X_t = \int_0^t \sigma_s dB_s.$$

(II) X is a \mathcal{BSS} process given by

$$X_t = \int_{-\infty}^t g(t-s) \sigma_s dB_s,$$

where $g : (0, \infty) \rightarrow \mathbb{R}$ is a square integrable weight function such that

$$\int_{-\infty}^t g(t-s)^2 \sigma_s^2 ds < \infty \quad \text{a.s.}$$

for all $t \geq 0$.

On the one hand, by choosing A to be absolutely continuous in the case (I), we see that this framework includes rather general Brownian semimartingales. On the other hand, in the case (II) the process Y is typically a non-semimartingale, as discussed above.

Recall that for any $p > 0$, the p -th order realised power variation of the process Y with lag $\delta > 0$ is given by

$$[Y_\delta]_t^{(p)} = \sum_{j=1}^{\lfloor t/\delta \rfloor} |Y_{j\delta} - Y_{(j-1)\delta}|^p, \quad t \geq 0.$$

The power variations $[A_\delta]^{(p)}$ and $[X_\delta]^{(p)}$ are, of course, defined analogously. Similarly to earlier literature on power and multipower variations of \mathcal{BSS} processes, [BNCP12, BNCP13, CHPP13], we assume that the kernel function g behaves like $t^{\nu-1}$ near zero for some $\nu \in (\frac{1}{2}, 1) \cup (1, \frac{3}{2})$, or more precisely that

$$g(t) = t^{\nu-1} L_g(t),$$

where L_g is slowly varying at zero, which implies that X is not a semimartingale in the case (II). Then, under some further regularity conditions⁴ on g , which are satisfied when g is the gamma kernel (4), and defining the normalisation factor $c(\delta)$ by (6), we have

$$\frac{\delta}{c(\delta)^p} [X_\delta]_t^{(p)} \xrightarrow{\delta \rightarrow 0} m_p \sigma_t^{p+}, \quad (9)$$

where $\sigma_t^{p+} = \int_0^t |\sigma_s|^p ds$ and $m_p = \mathbb{E}\{|\xi|^p\}$ for $\xi \sim N(0, 1)$, by Theorem 3.1 of [CHPP13]. In the case (I), setting $c(\delta) = \sqrt{\delta}$, the convergence (9) holds without any additional assumptions, e.g., by Theorem 2.2 of [BGJPS06]. Additionally, note that the convergence (9) holds also when X is replaced with Y , provided that

$$\frac{\delta}{c(\delta)^p} [A_\delta]_t^{(p)} \xrightarrow{\delta \rightarrow 0} 0. \quad (10)$$

For fixed time horizon $T > 0$, we introduce the p -th order *realised relative power variation* process over $[0, T]$ by

$$\widetilde{[Y_\delta]_{t,T}}^{(p)} = \frac{[Y_\delta]_t^{(p)}}{[Y_\delta]_T^{(p)}}, \quad 0 \leq t \leq T.$$

The relative power variation has the following evident consistency property.

Theorem 2. *If (9) and (10) hold, then*

$$\widetilde{[Y_\delta]_{t,T}}^{(p)} \xrightarrow{\delta \rightarrow 0} \widetilde{\sigma}_{t,T}^{p+} = \frac{\sigma_t^{p+}}{\sigma_T^{p+}}$$

uniformly in $t \in [0, T]$.

Remark 3. The relative integrated volatility/intermittency $\widetilde{\sigma}_{t,T}^{p+}$ can be seen as a “cumulative distribution function” of volatility/intermittency on $[0, T]$; we have $\widetilde{\sigma}_{0,T}^{p+} = 0$, $\widetilde{\sigma}_{T,T}^{p+} = 1$, and $t \mapsto \widetilde{\sigma}_{t,T}^{p+}$ is non-decreasing.

Remark 4. Theorem 2 can be generalized to a spatial setting, applying specifically to ambit fields driven by white noise, using the asymptotic results in [Pak13].

⁴See [CHPP13]. Most importantly, they include the assumption that $c(\delta) = \delta^{\nu-1/2} L_c(\delta)$, where L_c is slowly varying at zero.

4.2 Central limit theorem

Recall first that random elements U_1, U_2, \dots in some metric space \mathbb{U} *converge stably* (in law) to a random element U in \mathbb{U} , defined on an extension $(\Omega', \mathcal{F}', P')$ of the underlying probability space (Ω, \mathcal{F}, P) , if

$$\mathbb{E}\{f(U_n)V\} \xrightarrow[n \rightarrow \infty]{} \mathbb{E}'\{f(U)V\}$$

for any bounded, continuous function $f : \mathbb{U} \rightarrow \mathbb{R}$ and bounded random variable V on (Ω, \mathcal{F}, P) . We denote stable convergence by $\xrightarrow{\text{st}}$. Clearly, stable convergence implies ordinary convergence in law, while the converse is not true. This mode of convergence was introduced by Rényi [Ren63].

Remark 5. The usefulness of stable convergence can be illustrated by the following example that is pertinent to the asymptotic results below. Suppose that $U_n \xrightarrow{\text{st}} \theta\xi$ in \mathbb{R} as $n \rightarrow \infty$, where $\xi \sim N(0, 1)$ and θ is a positive random variable independent of ξ . In other words, U_n follows asymptotically a mixed Gaussian law with mean zero and conditional variance θ^2 . If $\hat{\theta}_n$ is a positive, consistent estimator of θ , i.e., $\hat{\theta}_n \xrightarrow{P} \theta$ as $n \rightarrow \infty$, then the stable convergence of U_n allows us to deduce that $U_n/\hat{\theta}_n \xrightarrow{d} N(0, 1)$. We refer to [Ren63, AE78] and [JS03, pp. 512–518] for more information on stable convergence.

Let us write $D([0, T])$ for the space of càdlàg functions from $[0, T]$ to \mathbb{R} , endowed with the usual Skorohod metric [JS03, Chapter V]. (Recall, however, that convergence to a continuous function in this metric is equivalent to uniform convergence.) Under slightly strengthened assumptions, the realised power variation of X satisfies a stable central limit theorem of the form

$$\delta^{-1/2} \left(\frac{\delta}{c(\delta)^p} [X_\delta]_t^{(p)} - m_p \sigma_t^{p+} \right) \xrightarrow[\delta \rightarrow 0]{\text{st}} \sqrt{\lambda_{X,p}} \int_0^t |\sigma_s|^p dW_s \quad \text{in } D([0, T]), \quad (11)$$

where $\lambda_{X,p} > 0$ is a deterministic constant and $\{W_t\}_{t \in [0, T]}$ a standard Brownian motion, independent of \mathcal{F} , defined on an extension of (Ω, \mathcal{F}, P) . Indeed, in the case (I) the convergence (11) holds with $\lambda_{X,p} = m_{2p} - m_p^2$, provided that σ is an Itô semimartingale (see Theorem 2.4 of [BGJPS06]). Moreover, we have (11) also in the case (II) if we make the restriction $\nu \in (\frac{1}{2}, 1)$ — the situation of most interest concerning turbulence — and assume that σ satisfies a Hölder condition in expectation (see Theorem 3.2 of [CHPP13]). Then, in contrast to the semimartingale case,

$$\lambda_{X,p} = \lambda_p(\nu) > m_{2p} - m_p^2, \quad (12)$$

where $\lambda_p : (\frac{1}{2}, 1) \rightarrow (0, \infty)$ is a continuous function defined using the correlation structure of *fractional Brownian noise* (see Appendix B for the definition and proof of continuity). Analogously to (10), the convergence (11) extends to the power variation of Y if

$$\frac{\sqrt{\delta}}{c(\delta)^p} [A_\delta]_t^{(p)} \xrightarrow[\delta \rightarrow 0]{p} 0. \quad (13)$$

The realised relative power variation of Y satisfies the following central limit theorem, which is an immediate consequence of Lemma 10 in Appendix A.

Theorem 6. *If (11) and (13) hold, then*

$$\delta^{-1/2} \left(\widetilde{[Y_\delta]_{t,T}^{(p)}} - \widetilde{\sigma}_{t,T}^{p+} \right) \xrightarrow[\delta \rightarrow 0]{\text{st}} \frac{\sqrt{\lambda_{X,p}}}{m_p \int_0^T |\sigma_s|^p ds} \left(\int_0^t |\sigma_s|^p dW_s - \widetilde{\sigma}_{t,T}^{p+} \int_0^T |\sigma_s|^p dW_s \right) \quad (14)$$

in $D([0, T])$, where $\lambda_{X,p}$ and W are as in (11).

Remark 7. In the case (II) the restriction $\nu \in (\frac{1}{2}, 1)$ can be relaxed when one considers power variations defined using *second* or *higher order differences* of Y (cf. [BNCP13, CHPP13]). Then, (11) holds for all $\nu \in (\frac{1}{2}, 1) \cup (1, \frac{3}{2})$. As Theorems 2 and 6 do not depend on the type of differences used in the power variation, they obviously apply also in this case.

Conditional on \mathcal{F} , the limiting process on the right-hand side of (14) is a Gaussian bridge. In particular, its (unconditional) marginal law at time $t \in [0, T]$ is mixed Gaussian with mean zero and conditional variance

$$\frac{\lambda_{X,p}}{(m_p \sigma_T^{p+})^2} \left((1 - \tilde{\sigma}_{t,T}^{p+})^2 \sigma_t^{2p+} + (\tilde{\sigma}_{t,T}^{p+})^2 (\sigma_T^{2p+} - \sigma_t^{2p+}) \right). \quad (15)$$

Note also that when σ is constant, the limiting process reduces to a Brownian bridge. In effect, the result is analogous to Donsker's theorem for empirical cumulative distribution functions (see, e.g., [Kos08] for an overview of such results).

Clearly, we may estimate the asymptotic variance (15) consistently using

$$V_t(\delta) = \frac{\lambda_{X,p}}{\delta \cdot m_{2p} \cdot ([Y_\delta]_T^{(p)})^2} \left(\left(1 - \widetilde{[Y_\delta]_{t,T}}^{(p)}\right)^2 [Y_\delta]_t^{(2p)} + \left(\widetilde{[Y_\delta]_{t,T}}^{(p)}\right)^2 ([Y_\delta]_T^{(2p)} - [Y_\delta]_t^{(2p)}) \right).$$

In the case (II) the estimator $V_t(\delta)$ is not feasible as such since ν appears as a nuisance parameter in $\lambda_{X,p} = \lambda_p(\nu)$. However, we may replace $\lambda_{X,p}$ with $\lambda_p(\hat{\nu}_\delta)$, where $\hat{\nu}_\delta$ is any consistent estimator of ν based on the observations $Y_0, Y_\delta, \dots, Y_{\lfloor T/\delta \rfloor \delta}$ (they have been developed in [BNCP12, BNCP13, CHPP13]). Using the properties of stable convergence, we obtain the following feasible central limit theorem.

Proposition 8. *If (11) and (13) hold, then for any $t \in (0, T)$,*

$$\frac{\delta^{-1/2} \left(\widetilde{[Y_\delta]_{t,T}}^{(p)} - \tilde{\sigma}_{t,T}^{p+} \right)}{\sqrt{V_t(\delta)}} \xrightarrow[\delta \rightarrow 0]{d} N(0, 1).$$

4.3 Inference on realised relative V/I/E

Proposition 8 can be used to construct approximative, pointwise confidence intervals for the relative volatility/intermittency $\tilde{\sigma}_{t,T}^{p+}$. Since, by construction, $\tilde{\sigma}_{t,T}^{p+}$ assumes values in $[0, 1]$, it is reasonable to constrain the confidence interval to be a subset of $[0, 1]$. Thus, we define for any $a \in (0, 1)$ the corresponding $(1 - a) \cdot 100\%$ confidence interval as

$$\left[\max \left\{ \widetilde{[Y_\delta]_{t,T}}^{(p)} - z_{1-a/2} \cdot \sqrt{\delta V_t(\delta)}, 0 \right\}, \min \left\{ \widetilde{[Y_\delta]_{t,T}}^{(p)} + z_{1-a/2} \cdot \sqrt{\delta V_t(\delta)}, 1 \right\} \right],$$

where $z_{1-a/2} > 0$ is the $1 - \frac{a}{2}$ -quantile of the standard Gaussian distribution.

Another application of the central limit theory is a non-parametric *homoskedasticity test* that is similar in nature to Kolmogorov–Smirnov and Cramér–von Mises goodness-of-fit tests for empirical distribution functions. This extends the homoskedasticity tests proposed by Dette, Podolskij and Vetter [DPV06] and Dette and Podolskij [DP08] to a non-semimartingale setting. To formulate our test, we introduce the hypotheses

$$\begin{cases} H_0 : \sigma_t = \sigma_0 \text{ for all } t \in [0, T], \\ H_1 : \sigma_t \neq \sigma_0 \text{ for some } t \in [0, T]. \end{cases}$$

Theorem 6 implies that under H_0 ,

$$\delta^{-1/2} \left(\widetilde{[Y_\delta]_{t,T}^{(p)}} - \frac{t}{T} \right) \xrightarrow[\delta \rightarrow 0]{st} \frac{\sqrt{\lambda_{X,p}}}{m_p \cdot T} \left(W_t - \frac{t}{T} W_T \right). \quad (16)$$

It is intuitively appealing to use the L^1 -norm to assess the distance of realised relative power variations — although other metrics or norms could also be used. Thus, we define the test statistic as

$$S_\delta = \frac{m_p}{\sqrt{\delta \cdot T \cdot \lambda_{X,p}}} \int_0^T \left| \widetilde{[Y_\delta]_{t,T}^{(p)}} - \frac{t}{T} \right| dt. \quad (17)$$

Also in (17), we may use $\lambda_p(\hat{\nu}_\delta)$ instead of $\lambda_{X,p}$ in the case (II). By (16) and the scaling properties of Brownian motion, we have under H_0 ,

$$S_\delta \xrightarrow[\delta \rightarrow 0]{st} \int_0^1 |W_s^*| ds, \quad (18)$$

where $\{W_t^*\}_{t \in [0,1]}$ is a standard Brownian bridge independent of \mathcal{F} , while under H_1 the statistic S_δ diverges to infinity. The cumulative distribution function of the functional on the right-hand side of (18) has been derived by Johnson and Killeen [JK83] and it can be stated using the Airy function, which allows for a straightforward numerical implementation.

5 Applications to empirical data

5.1 Brookhaven turbulence data

We apply the methodology developed above first to data of turbulence. The data consist of a time series of the main component of a turbulent velocity vector, measured at a fixed position in the atmospheric boundary layer using a hotwire anemometer, during an approximately 66 minutes long observation period at sampling frequency of 5 kHz (i.e. 5000 observations per second). The measurements were made at *Brookhaven National Laboratory* (Long Island, NY), and a comprehensive account of the data can be found in [Drh00].

As a first illustration, we study the observations up to time horizon $T = 800$ milliseconds. Using the smallest possible lag, $\delta = 0.2$ ms, this amounts to 4000 observations. Figure 1(a) displays the squared increments corresponding to these observations. As a comparison, the same time horizon is captured in Figure 1(b) but with lag $\delta = 0.8$ ms. Figure 1(c) compares the associated accumulated realised relative energy dissipations/quadratic variations. The graphs for these two lags show very similar behaviour, exhibiting how the total time interval is divided into a sequence of intervals over which the slope of the energy dissipation is roughly constant. On the other hand, the amplitudes of the volatility/intermittency are of the same order in the whole observation interval.

To be able to draw inference on relative volatility/intermittency using the data, we need to address two issues. Firstly, for this time series, the lags $\delta = 0.2$ ms and $\delta = 0.8$ ms are *below* the so-called *inertial range* of turbulence, where a *BSS* process with a gamma kernel — a model of ideal turbulence — provides an accurate description of the data (see [CHPP13], where the same data are analysed). Secondly, the data were digitised using a 12-bit analog-to-digital converter. Thus, the measurements can assume at most $2^{12} = 4096$ different values, and due to the resulting discretisation error, a non-negligible amount of the increments are in fact equal to zero (roughly 20 % of all increments). These

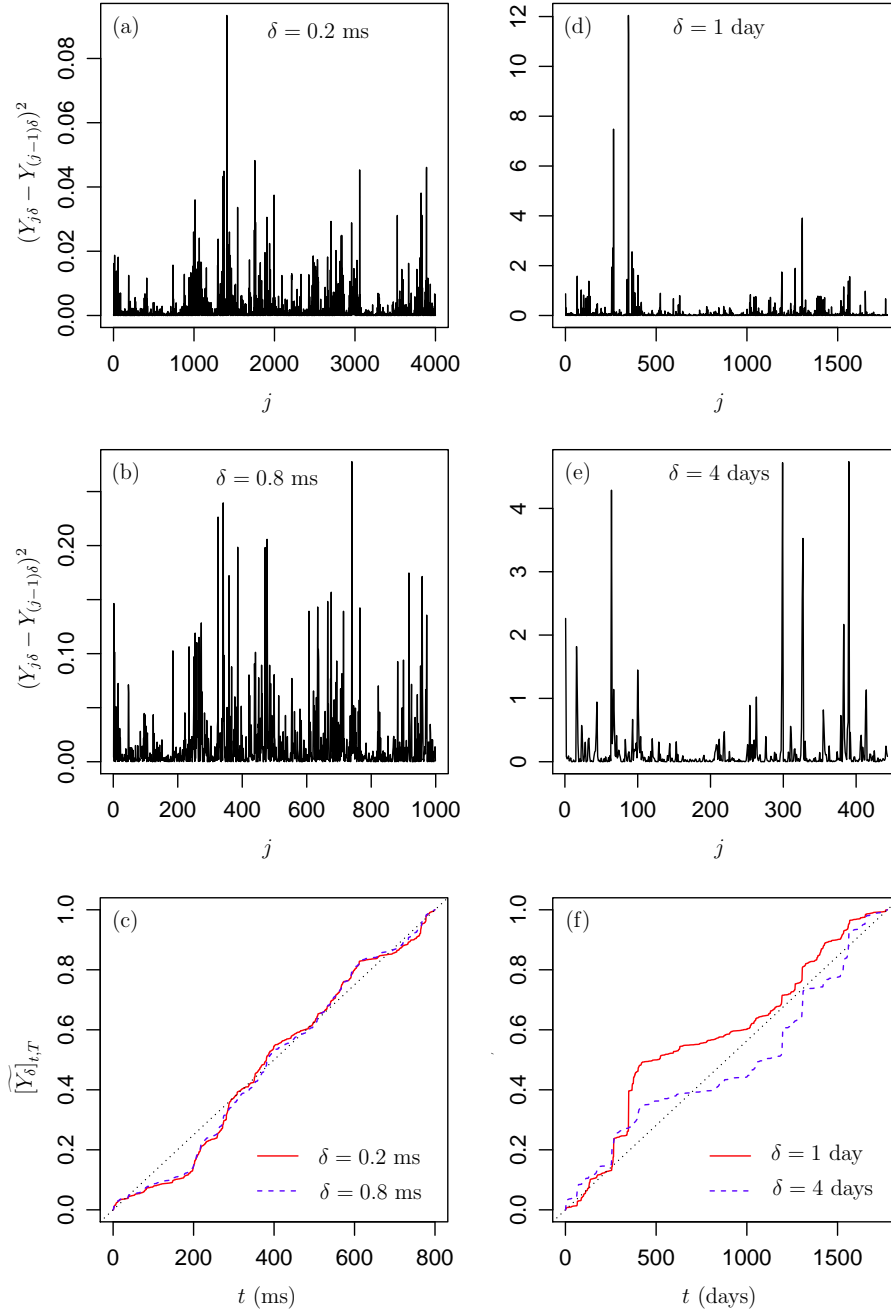


Figure 1: Left panel, Brookhaven turbulence data: (a) The squared increment process with lag $\delta = 0.2$ ms over the time horizon $T = 800$ ms. (b) The squared increment process with lag $\delta = 0.8$ ms over the same time horizon $T = 800$ ms. (c) The realised relative energy dissipation/quadratic variations corresponding to $\delta = 0.2$ ms and $\delta = 0.8$ ms, and the same time horizon, $T = 800$ ms, as in plots (a) and (b). Right panel, logarithmic EEX electricity spot prices: (d) The squared increment process with lag $\delta = 1$ day over the time horizon $T = 1775$ days. (e) The squared increment process with lag $\delta = 4$ days over the same time horizon $T = 1775$ days. (f) The realised relative quadratic variation corresponding to $\delta = 1$ day and $\delta = 4$ days and the same time horizon, $T = 1775$ days, as in plots (d) and (e).

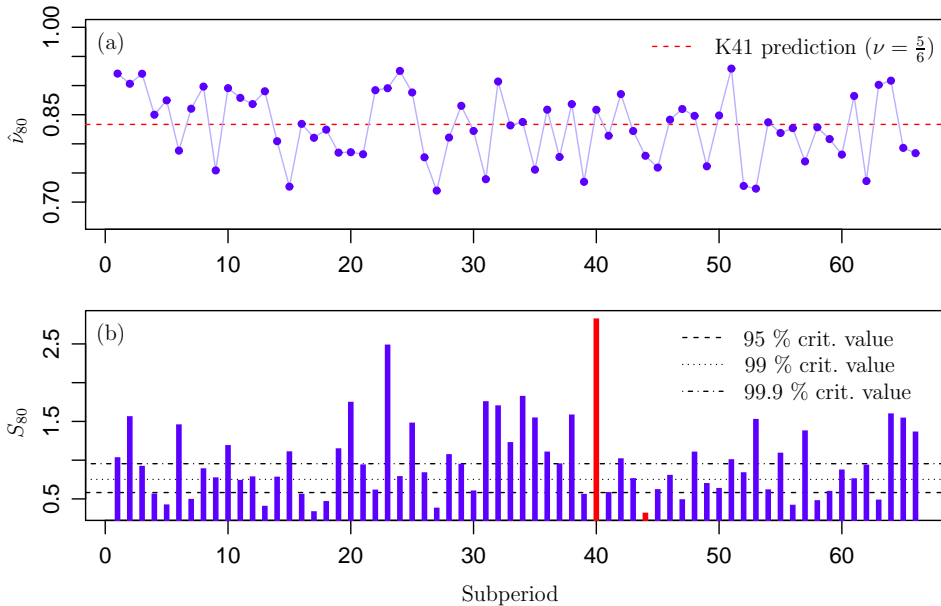


Figure 2: Brookhaven turbulence data: (a) Estimates of ν , using the change-of-frequency method and lag $\delta = 80$ ms, for each one-minute subperiod and the value predicted by Kolmogorov’s (K41) scaling law. (b) Test statistics and critical values for the constancy of σ for each subperiod. The red bars indicate the 40th and 44th subperiods, which are analysed in more detail in Figure 3.

discretisation errors are bound to bias the estimation of the parameter ν , which is needed for the inference methods. We mitigate these issues by *subsampling*, namely, we apply the inference methods using a considerably longer lag, $\delta = 80$ ms, which is near the lower bound of the inertial range for this time series [CHPP13, Figure 1].

We divide the time series into 66 non-overlapping one-minute-long subperiods, testing the constancy of σ , i.e., the null hypothesis H_0 , within each subperiod. Figure 2(a) displays the estimates of ν for each subperiod using the *change-of-frequency* method [BNCP13, CHPP13]. All of the estimates belong to the interval $(\frac{1}{2}, 1)$ and they are scattered around the value $\nu = \frac{5}{6}$ predicted by Kolmogorov’s (K41) scaling law of turbulence [Kol41a, Kol41c]. The homoskedasticity test statistics and critical values, derived using (18), in Figure 2(b) indicate that the null hypothesis of constancy of σ is typically rejected. To understand what kind of intermittency the test is detecting, we look into the extremal cases, the 40th and 44th subperiods (red bars). To this end, we plot the realised relative energy dissipations, with $\delta = 80$ ms, during the 40th and 44th subperiods in Figure 3(a) and (b), respectively. We also include the pointwise confidence intervals, the p-values of the homoskedasticity test, and as a reference, the realised relative energy dissipations using the smallest possible lag $\delta = 0.2$ ms. While the realised relative energy dissipations exhibit a slight discrepancy between the lags $\delta = 80$ ms and $\delta = 0.2$ ms, it is clear that 40th subperiod contains significant intermittency, whereas during the 44th subperiod, the (accumulated) realised relative energy dissipation grows nearly linearly, apart from the final 10 seconds.

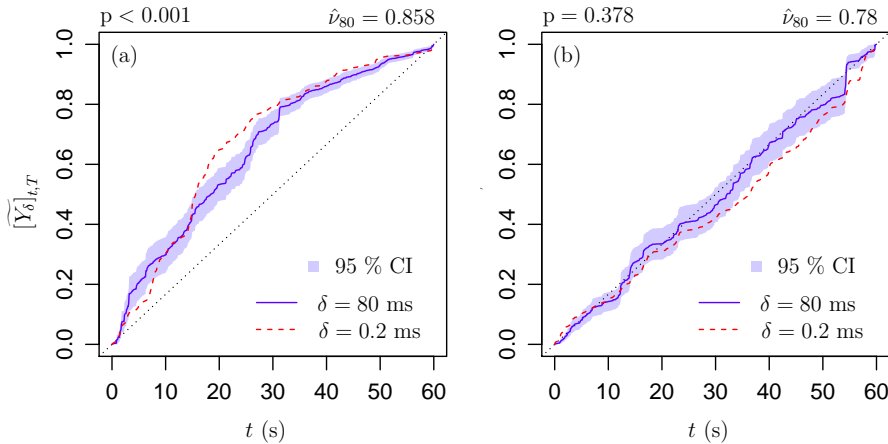


Figure 3: Brookhaven turbulence data: Realised relative energy dissipation during the 40th (a) and 44th (b) subperiods with $\delta = 80$ ms and $\delta = 0.2$ ms. Additionally, p-values for the hypothesis H_0 , estimates of ν using the change-of-frequency method, and 95% pointwise confidence intervals, all using the lag $\delta = 80$ ms.

5.2 EEX electricity spot prices

We also briefly exemplify the concept of relative volatility using electricity spot price data from the *European Energy Exchange* (EEX). Specifically, we consider deseasonalised daily *Phelix* peak load data (that is, the daily averages of the hourly spot prices of electricity delivered between 8 AM and 8 PM) with delivery days ranging from January 1, 2002 to October 21, 2008. Weekends are not included in the peak load data, and in total we have 1775 observations. This time series was studied in [BNBV12] and the deseasonalisation method is explained therein. As usual, we consider here *logarithmic* prices.

Figure 1(d) shows the squared increments up to the total time horizon $T = 1775$ days with lag $\delta = 1$ day. The same time horizon is captured in Figure 1(e) but with a resolution $\delta = 4$ days. Figure 1(f) compares the corresponding accumulated realised relative quadratic variations. The results for these two lags do not show the same similarity as with the turbulence data (Figure 1(a–b)). Judging by eye, we observe that the intensity of the volatility is changing with lag δ . This lag dependence is also observed in the amplitudes, again in contrast to the figures on the left hand side. (However, more quantitative investigation of such amplitude/density arguments is outside the scope of the present paper.) The dependence of the estimation results on the lag δ is, at least partly, explained by the relatively low sampling frequency of the data. With $\delta = 1$ day, the increments are dominated by a few exceptional observations (which may correspond to jumps or intraday volatility bursts). Choosing $\delta = 4$ days reduces the contribution of these observations since the time series exhibits significant first-order autocorrelation [BNBV12, Figure 1].

Remark 9. It was shown in [BNBV12] that by suitably choosing both g and q to be of gamma type it is possible to construct a \mathcal{BSS} process with *normal inverse Gaussian* one-dimensional marginal law, which corresponds closely to the empirics for the time series of log spot prices considered. Moreover, the estimated value of the smoothness parameter ν for this time series falls in the interval $(\frac{1}{2}, 1)$.

6 Conclusion

The definition of realised relative energy dissipation introduced in this paper applies to any continuous time, real valued process Y . An extension to vector valued processes is an issue of interest, in particular in relation to the definition (1) of the energy dissipation in three-dimensional turbulent fields.

The extent to which the realised volatility/intermittency/energy is an empirical counterpart of what can be conceived theoretically as relative volatility/intermittency/energy depends on the model under consideration. As discussed above this is the case, in particular, both under Brownian semimartingales, as these occur widely in mathematical finance and financial econometrics, and under stationary \mathcal{BSS} processes.

In the timewise stationary setting, the realised relative energy dissipation is a parameter free statistic which provides estimates of the relative energy in subintervals of the full observation range, by relating the quadratic variation over each subinterval to the total realised energy for the entire range. It provides robust estimates of the relative accumulated energy as this develops over time and is intimately connected to the concepts of volatility/intermittency and energy dissipation as these occur in statistical turbulence and in finance. This was illustrated in connection to the class of \mathcal{BSS} processes with g of the gamma form.

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A Stable convergence of normalised processes

Theorem 6 is a corollary of the following simple result concerning the stable convergence of a process that has been normalised by its terminal value.

Lemma 10. *Let X, X^1, X^2, \dots be random elements in $D([0, T])$, with non-decreasing sample paths, defined on (Ω, \mathcal{F}, P) and let ξ be a random element in $D([0, T])$ defined on an extension $(\Omega', \mathcal{F}', P')$ of (Ω, \mathcal{F}, P) . Suppose, moreover, that almost surely $\xi \in C([0, T])$, $X \in C([0, T])$, $X_T^n \neq 0$ for any $n \in \mathbb{N}$, and $X_T \neq 0$. If*

$$\sqrt{n}(X_t^n - X_t) \xrightarrow[n \rightarrow \infty]{\text{st}} \xi_t \quad \text{in } D([0, T]), \quad (19)$$

then

$$\sqrt{n} \left(\frac{X_t^n}{X_T^n} - \frac{X_t}{X_T} \right) \xrightarrow[n \rightarrow \infty]{\text{st}} \frac{1}{X_T} \left(\xi_t - \frac{X_t}{X_T} \xi_T \right) \quad \text{in } D([0, T]).$$

Proof. Since X^n and X have non-decreasing paths and $X \in C([0, T])$, we have

$$\sup_{0 \leq t \leq T} \left| \frac{X_t^n}{X_T^n} - \frac{X_t}{X_T} \right| \leq \frac{2}{|X_T|} \sup_{0 \leq t \leq T} |X_t^n - X_t| \xrightarrow[n \rightarrow \infty]{p} 0$$

by (19). Due to the properties of stable convergence, we obtain

$$\left(\sqrt{n}(X_t^n - X_t), \frac{X_t^n}{X_T^n} \right) \xrightarrow[n \rightarrow \infty]{\text{st}} \left(\xi_t, \frac{X_t}{X_T} \right) \quad \text{in } D([0, T])^2. \quad (20)$$

Let us now consider the decomposition

$$\sqrt{n} \left(\frac{X_t^n}{X_T^n} - \frac{X_t}{X_T} \right) = \frac{1}{X_T} \left(\sqrt{n}(X_t^n - X_t) - \sqrt{n}(X_T^n - X_T) \frac{X_t^n}{X_T^n} \right).$$

Using again the fact that convergence to a continuous function in $D([0, T])$ is equivalent to uniform convergence, it follows that the map $(x, y) \mapsto x - x(T)y$ from $D([0, T])^2$ to $D([0, T])$ is continuous on $C([0, T])^2$. Since $\xi \in C([0, T])$ and $X \in C([0, T])$ a.s., the assertion follows from (20) and the properties of stable convergence. \square

B Continuity of the correlation structure of fractional Brownian noise

The function $\lambda_p : (\frac{1}{2}, 1) \rightarrow (0, \infty)$ that appears in (12) is defined by

$$\lambda_p(\nu) = \sum_{l=2}^{\infty} l! a_l^2 \left(1 + 2 \sum_{j=1}^{\infty} \rho_\nu(j)^l \right),$$

where a_2, a_3, \dots are the coefficients in the expansion of the function $u_p(x) = |x|^p - m_p$, $x \in \mathbb{R}$, in second and higher-order Hermite polynomials $x^2 - 1, x^3 - 3x, \dots$, satisfying $\sum_{l=2}^{\infty} l! a_l^2 < \infty$. In the most important case $p = 2$ we have, clearly, $a_2 = 1$ and $a_l = 0$ for all $l > 2$. The sequence $(\rho_\nu(j))_{j=1}^{\infty}$ coincides with the correlation function of *fractional Brownian noise* with Hurst parameter $\nu - 1/2$, namely

$$\rho_\nu(j) = \frac{1}{2} ((j+1)^{2\nu-1} - 2j^{2\nu-1} + (j-1)^{2\nu-1}), \quad j \geq 1.$$

The following lemma justifies the use of $\lambda_p(\hat{\nu}_\delta)$ as a consistent estimator of $\lambda_{X,p}$ in the case (II).

Lemma 11. *The function λ_p is continuous.*

Proof. Let us first show that for any $l \geq 2$, the map

$$\nu \mapsto \sum_{j=1}^{\infty} \rho_\nu(j)^l \tag{21}$$

is continuous on $(\frac{1}{2}, 1)$. Note that $\nu \mapsto \rho_\nu(j)$ is clearly continuous for any $j \geq 1$ and, moreover, that $|\rho_\nu(j)|^l \leq \rho_\nu(j)^2$ since $|\rho_\nu(j)| \leq 1$. Following [DM03, p. 419], write

$$\rho_\nu(j) = \frac{1}{2} j^{2\nu-1} g_\nu(1/j), \tag{22}$$

where $g_\nu(x) = (1+x)^{2\nu-1} - 2 + (1-x)^{2\nu-1}$, $x \in \mathbb{R}$. A straightforward computation shows that $g_\nu(0) = 0$, $g'_\nu(0) = 0$, and

$$g''_\nu(x) = (2\nu - 2)(2\nu - 1)((1+x)^{2\nu-3} + (1-x)^{2\nu-3}).$$

In particular, we may define $C = \sup_{\nu \in (1/2, 1)} \sup_{x \in [0, 1/2]} |g''_\nu(x)| < \infty$, whence $|g_\nu(x)| \leq Cx^2$ for all $\nu \in (\frac{1}{2}, 1)$ and $x \in [0, \frac{1}{2}]$. Applying this bound to (22) implies that

$$|\rho_\nu(j)| \leq Cj^{2\nu-3} \leq \frac{C}{j}, \quad j \geq 2,$$

for any $\nu \in (\frac{1}{2}, 1)$. The continuity of the function in (21) follows now from Lebesgue's dominated convergence theorem. Finally, the actual assertion follows by invoking the bounds

$$\left| \sum_{j=1}^{\infty} \rho_{\nu}(j)^l \right| \leq \sum_{j=1}^{\infty} |\rho_{\nu}(j)|^2 \leq C \left(1 + \sum_{j=2}^{\infty} \frac{1}{j^2} \right) < \infty,$$

which hold for all $l \geq 2$, and Lebesgue's dominated convergence theorem again. \square

C Sufficient conditions for the negligibility of the skewness term

Suppose first that the process $A = \{A_t\}_{t \geq 0}$ is given by

$$A_t = \mu + \int_0^t a_s ds,$$

where $\mu \in \mathbb{R}$ is a constant and the process $\{a_t\}_{t \geq 0}$ is measurable and locally bounded. Then we can establish rather simple conditions for its negligibility in the asymptotic results for power variations. By Jensen's inequality, we have for any $p \geq 1$, $s \geq 0$, and $t \geq 0$,

$$|A_s - A_t|^p \leq C_a \cdot |s - t|^p,$$

where $C_a > 0$ is a random variable that depends locally on the path of a . Thus, the condition (10) holds whenever

$$\frac{\delta}{c(\delta)} \rightarrow 0, \tag{23}$$

and (13) holds if

$$\frac{\delta^{p-1/2}}{c(\delta)^p} \rightarrow 0. \tag{24}$$

Notably, in the semimartingale case (I), i.e. $c(\delta) = \sqrt{\delta}$, (23) is always true and (24) holds if $p > 1$.

Suppose now, instead, that A follows

$$A_t = \mu + \int_{-\infty}^t q(t-s)a_s ds, \tag{25}$$

where q is the gamma kernel

$$q(t) = c't^{\eta-1}e^{-\rho t}$$

for some $c' > 0$, $\eta > 0$, and $\rho > 0$. We assume that the process $\{a_t\}_{t \in \mathbb{R}}$ is measurable, locally bounded, and satisfies

$$A_t^* = \sup_{0 \leq u \leq t} \int_{-\infty}^u q(u-s)|a_s| ds < \infty \quad \text{a.s.} \tag{26}$$

for any $t \geq 0$, which is true, e.g., when the auxiliary process $\int_{-\infty}^u q(u-s)|a_s| ds$, $u \geq 0$, has a càdlàg or continuous modification.

Lemma 12. *Suppose that A is given by (25) and that (26) holds. Then in the case (II) the condition (10) holds if $\min\{\eta, 1\} > \nu - \frac{1}{2}$ and (13) holds if $\min\{\eta, 1\} > \nu - \frac{p-1}{2p}$. The corresponding restrictions in the case (I) can be obtained by setting $\nu = 1$ above.*

Proof. Let us first look into the properties of q . For the sake of simpler notation, we make the innocuous assumption that $c' = 1$. Since

$$q'(t) = \left(\frac{\eta - 1}{t} - \rho \right) q(t), \quad (27)$$

we find that q is decreasing when $\eta \leq 1$. When $\eta > 1$, q is increasing on $(0, \frac{\eta-1}{\rho})$ and decreasing on $(\frac{\eta-1}{\rho}, \infty)$.

Let $t \geq 0$ be fixed, $\delta \in (0, 1)$, and let $j \geq 1$ be such that $j\delta \leq t$. Below, all big O estimates hold uniformly in such j . We consider the decomposition

$$\begin{aligned} A_{j\delta} - A_{(j-1)\delta} &= \int_{(j-1)\delta}^{j\delta} q(j\delta - s) a_s ds + \int_{(j-2)\delta}^{(j-1)\delta} (q(j\delta - s) - q((j-1)\delta - s)) a_s ds \\ &\quad + \int_{s^*}^{(j-2)\delta} (q(j\delta - s) - q((j-1)\delta - s)) a_s ds \\ &\quad + \int_{-\infty}^{s^*} (q(j\delta - s) - q((j-1)\delta - s)) a_s ds \\ &= I_\delta^1 + I_\delta^2 + I_\delta^3 + I_\delta^4, \end{aligned}$$

where

$$s^* = -\max \left\{ \frac{\eta - 1}{\rho}, 1 \right\}.$$

When $\eta \geq 1$, q is bounded and we have $|I_\delta^1 + I_\delta^2| = a_t^* O(\delta)$, where

$$a_t^* = \sup_{s^* \leq s \leq t} |a_s| < \infty \quad \text{a.s.},$$

and when $\eta < 1$, we find that

$$|I_\delta^1 + I_\delta^2| \leq 2a_t^* \int_0^\delta q(s) ds = a_t^* O(\delta^\eta).$$

Next, we want to show that

$$|I_\delta^3| = a_t^* O(\delta^{\min\{\eta, 1\}}). \quad (28)$$

In the case $\eta \geq 2$ the derivative q' is bounded and (28) is immediate. Suppose that $\eta < 2$. Then, $|q'(t)| \leq Ct^{\eta-2}$ on any finite interval, where $C > 0$ depends on the interval. Using the mean value theorem, we obtain

$$|I_\delta^3| \leq Ca_t^* \delta \int_{s^*}^{(j-2)\delta} ((j-1)\delta - s)^{\eta-2} ds,$$

which implies (28). To bound $|I_\delta^4|$, note that, by (27), $|q'(t)| \leq C'q(t)$ for all $t \geq -s^*$, where $C' > 0$ is a constant. For any $s < s^*$, we have $(j-1)\delta - s > \frac{\eta-1}{\rho}$. Thus, by the mean value theorem,

$$|(q(j\delta - s) - q((j-1)\delta - s))| \leq C'q((j-1)\delta - s)\delta$$

and, consequently,

$$|I_\delta^4| \leq C'\delta \int_{-\infty}^{(j-1)\delta} q((j-1)\delta - s) |a_s| ds = A_t^* O(\delta).$$

Collecting the estimates, we have

$$|A_{j\delta} - A_{(j-1)\delta}| = \max\{a_t^*, A_t^*\}O(\delta^{\min\{\eta,1\}})$$

uniformly in j , whence

$$[A_\delta]_t^{(p)} = O_{\text{a.s.}}(\delta^{p \min\{\eta,1\}-1}).$$

Checking the sufficiency of the asserted conditions is now a straightforward task. \square

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