

# **GARCH Option Valuation: Theory and Evidence**

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## Abstract

We survey the theory and empirical evidence on GARCH option valuation models. Our treatment includes the range of functional forms available for the volatility dynamic, multifactor models, nonnormal shock distributions as well as style of pricing kernels typically used. Various strategies for empirical implementation are laid out and we also discuss the links between GARCH and stochastic volatility models. In the appendix we provide Matlab computer code for option pricing via Monte Carlo simulation for nonaffine models as well as Fourier inversion for affine models.

JEL code: G13

Key words: GARCH, option valuation.

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# 1 Introduction

This survey presents theory and empirical evidence on GARCH option valuation. We focus on GARCH models for two reasons. First, there is overwhelming empirical evidence that modeling time-varying volatility and volatility clustering is critically important in modeling equity and index returns and equity and index options.<sup>1</sup> We therefore do not discuss in detail the rich literature that models options using homoskedastic models with non-normal innovations, because we believe that from an empirical perspective these models need to be augmented with time-varying volatility to achieve empirical success.<sup>2</sup> Second, we focus on discrete-time GARCH models because they are relatively straightforward to implement. In our opinion option valuation models ought to be tested using large option samples that contain a substantial cross-sectional dimension, as well as an extensive time dimension. Moreover, we will argue below that it is desirable to combine options and return data when estimating model parameters. The resulting optimization problem is computationally demanding, and therefore it is important to use models that capture stylized facts in the data but are computationally less demanding. We believe GARCH constitutes an interesting framework from this perspective.

We emphasize that we do not attempt to provide an overview of the option valuation literature more in general. Several such overviews are available, see for instance Bates (2003). We rather focus more narrowly on GARCH option valuation, because these models are not yet very well understood, and there are many misconceptions regarding the underlying theory and the empirical performance of these models. While at times we discuss related empirical findings in the continuous-time stochastic volatility literature, it is important to keep in mind that available studies do not yet contain a full-fledged comparison of the option pricing performance of these two classes of models, and therefore any differences have to be interpreted very carefully.

The paper proceeds as follows. Section 2 briefly discusses GARCH models. Section 3 discusses risk neutralization for processes with normal innovations, and Section 4 for non-normal innovations. Section 5 discusses several heteroskedastic models with non-normal innovations. Section 6 discusses extensions of the GARCH model to multiple shocks, and also discusses the class of Levy-GARCH processes. Section 7 summarizes valuation techniques for European and American options. Section 8 discusses estimation and filtering. Section 9 provides a brief comparison of GARCH and stochastic volatility models, and Section 10 presents the available empirical evidence. Section 11 concludes.

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<sup>1</sup>For evidence on returns, see for example Andersen, Bollerslev, Christoffersen, and Diebold (2006), Bollerslev (1986), French, Schwert, and Stambaugh (1987) and Schwert (1989). For evidence on index options, see for example Bakshi, Cao, and Chen (1997) and Duan (1996).

<sup>2</sup>For models with non-normal innovations see for example Camara (2003), Schroder (2004), Madan and Seneta (1990), and Heston (1993b, 2004).

## 2 GARCH Processes

We assume that the underlying stock price process follows the conditional distribution  $D$  under the physical measure  $P$ . We write

$$\begin{aligned} R_t &\equiv \ln\left(\frac{S_t}{S_{t-1}}\right) = \mu_t - \gamma_t + \varepsilon_t & \varepsilon_t|F_{t-1} &\sim D(0, h_t) \\ &= \mu_t - \gamma_t + \sqrt{h_t}z_t & z_t|F_{t-1} &\sim D(0, 1) \end{aligned} \quad (2.1)$$

where  $S_t$  is the stock price at time  $t$ , and  $h_t$  is the conditional variance of the log return in period  $t$ . Unless otherwise mentioned, we assume that  $\gamma_t$  is defined from

$$\exp(\gamma_t) \equiv E_{t-1}[\exp(\varepsilon_t)],$$

which ensures that the conditional expected gross rate of return,  $E_{t-1}[S_t/S_{t-1}]$ , equals  $\exp(\mu_t)$ .

$$\begin{aligned} E_{t-1}[S_t/S_{t-1}] &= E_{t-1}[\exp(\mu_t - \gamma_t + \varepsilon_t)] = \exp(\mu_t) \\ &\iff \exp(\gamma_t) = E_{t-1}[\exp(\varepsilon_t)] \end{aligned}$$

Note that the conditional mean  $\mu_t$  and conditional variance  $h_t$  are  $F_{t-1}$  measurable.

The interest in GARCH option valuation models is well-motivated. The literature on GARCH processes originated with the observation of Engle (1982) that many economic and financial time series are conditionally heteroskedastic, meaning that the assumption of constant variance is not appropriate, and that the conditional variance seems to be mean reverting. Option values critically depend on the variance of the underlying asset, and therefore conditional heteroskedasticity and variance mean reversion are of first-order importance for option valuation. Moreover for most financial assets estimating the properties of the conditional variance process is relatively straightforward compared to other conditional moments, including the conditional mean.

Engle's (ARCH) model was extended by Bollerslev (1986) and Taylor (1986), giving rise to the GARCH process. Consider the GARCH(1,1) model

$$\begin{aligned} R_t &\equiv \ln\left(\frac{S_t}{S_{t-1}}\right) = \mu_t - \gamma_t + \varepsilon_t \\ h_t &= w + bh_{t-1} + a\varepsilon_{t-1}^2 \end{aligned} \quad (2.2)$$

A large number of papers have proposed alternatives to the simple GARCH dynamic in (2.2).<sup>3</sup>

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<sup>3</sup>The literature on GARCH processes is voluminous. For overviews see Bollerslev, Chou, and Kroner (1992), Bollerslev, Engle, and Nelson (1994), Diebold and Lopez (1995), and Andersen, Bollerslev, Christoffersen, and Diebold (2006).

We present the evidence on the empirical performance of different models in Section 10. However, we now discuss two issues that are critical to understanding the theory of GARCH option pricing.

First, while the GARCH model in (2.2) is prevalent in the empirical literature on stock returns, interest rates and exchange rates, Heston and Nandi (2000) propose a somewhat different class of GARCH models. The variance dynamic for the Heston-Nandi GARCH(1, 1) model is

$$h_t = w + bh_{t-1} + a(z_{t-1} - c\sqrt{h_{t-1}})^2 \quad (2.3)$$

This variance dynamic is designed to yield a closed-form solution for the price of a European call option, and it is referred to as an affine dynamic. The nature of this closed form solution and the implications of the modeling assumption are discussed in more detail in Sections 7 and 10. At this point note that the motivation underlying (2.3) differs from the motivation underlying (2.2). The variance dynamic in (2.3) is designed to make option valuation analytically convenient.

Second, note that under the simplest assumption in (2.2) that  $\varepsilon_t$  is i.i.d.  $N(0, h_t)$ , the resulting distribution of  $R_t$  is symmetric, while many financial time series are characterized by asymmetric returns. A simple approach to modeling return asymmetry is the NGARCH model of Engle and Ng (1993), which is given by<sup>4</sup>

$$h_t = w + bh_{t-1} + ah_{t-1}(z_{t-1} - c)^2 \quad (2.4)$$

Note that while the constant term and the autoregressive term involving  $b$  are the same in (2.3) and (2.4), the so-called news impact term involving  $a$  is different in the two specifications.

For  $c > 0$ , the distribution of returns is negatively skewed. Negative skewness is often referred to as the leverage effect, owing to a mechanism first emphasized by Black (1976). There is some discussion as to the underlying causes of negative skewness in returns, but this need not concern us here, given the nature of the models under consideration. When allowing for  $c \neq 0$  in fitting returns and option data, we have to keep two stylized facts in mind. When estimating skewness from historical returns, the distribution of index returns seems to be reliably negatively skewed, but estimates for stock returns sometimes indicate positive skewness, and often they yield estimates that are statistically not significantly different from zero (see for instance Campbell, Lo, and Mackinlay (1997)). Second, the risk-neutral distribution of index returns is more negatively skewed than the physical distribution, and the risk-neutral distribution of stock returns is mostly negatively skewed. One of the critical aspects of any option valuation model will therefore be the relationship between the risk-neutral and physical skewness, but one needs to be mindful of the difference between index options and stock options in this regard. The latter issue has not

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<sup>4</sup>For alternative approaches to modeling return asymmetry, see for example Glosten, Jagannathan, and Runkle (1993) and Nelson (1991).

received much attention in the literature, for the simple reason that most existing papers study index options.

### 3 Risk Neutralization for Conditionally Normal Stock Returns

For the purpose of option valuation, it is critical to characterize the risk-neutral dynamic of the underlying return, which enables option valuation using risk-neutral valuation. In this Section we discuss the risk neutralization of the simple GARCH models (2.4) and (2.3) with normal innovations, as well as extensions to GARCH models with multiple volatility components. We discuss the more general case of non-normal innovations in Section 4.

#### 3.1 Risk Neutralization for One-Component Gaussian Models

First consider the case of (2.2) with  $z_t$  i.i.d.  $N(0, 1)$ , but for simplicity assume a constant variance instead of a time varying  $h_t$ , and also assume a constant mean return. The resulting lognormal model may seem similar to the classical Black-Scholes (1973) setup, but the critical difference is that we are in an incomplete markets setup instead of the Black-Scholes complete markets economy, which is set in continuous time. Rubinstein (1976) and Brennan (1979) specify a representative agent economy and establish a set of sufficient conditions on the representative agent's preferences to obtain a risk-neutral valuation relationship (RNVR) for the valuation of European options in this framework.<sup>5</sup> The resulting option price equals the Black-Scholes price. Brennan (1979) interprets his approach as “completing” the market by choice of a utility function of the representative agent.

Duan (1995) provides the first analysis of a RNVR for a GARCH process. His analysis builds on Rubinstein (1976) and Brennan (1979) in the sense that he characterizes sufficient conditions in a representative agent economy that ensure the existence of a RNVR, which he refers to as a local risk-neutral valuation relationship (LRNVR). Duan's result relies on the existence of a representative agent with constant relative risk aversion or constant absolute risk aversion. Amin and Ng (1993) obtain a similar result by characterizing the bivariate distribution of stock returns and the pricing kernel.

We take a somewhat different approach which yields the same results as Duan (1995) in the case of a conditionally normal innovation, but which can easily be generalized to deal with non-normal innovations, which we will discuss in detail in Section 4, and which can also easily be adapted to deal with multiple variance components. This approach does not attempt

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<sup>5</sup>See Camara (2003) for a generalization of Brennan's approach to transformed normal innovations.

to characterize a representative agent economy to obtain an option price. Instead, following Christoffersen, Elkamhi, Feunou, and Jacobs (2010, henceforth CEFJ), we first specify a class of Radon-Nikodym (RN) derivatives, and subsequently derive restrictions that ensure the existence of an equivalent martingale measure (EMM) that makes the discounted stock price process a martingale. This result will allow us later to obtain the distribution of the stock return under the EMM.

Note that with  $z_t$  i.i.d.  $N(0, 1)$ , we get  $\gamma_t = \frac{1}{2}h_t$  since  $\exp(\gamma_t) = E_{t-1}[\exp(\varepsilon_t)]$ . Following CEFJ, it can be shown that

$$\frac{dQ}{dP}\Big|_{F_t} = \exp\left(-\sum_{i=1}^t \left(\frac{\mu_i - r_i}{h_i} \varepsilon_i + \frac{1}{2} \left(\frac{\mu_i - r_i}{h_i}\right)^2 h_i\right)\right) \quad (3.1)$$

defines a Radon-Nikodym (RN) derivative. Moreover, it can also be shown that the probability measure  $Q$  defined by the Radon-Nikodym derivative (3.1) is an EMM, that under the EMM  $Q$  defined by (3.1),  $\varepsilon_t | F_{t-1} \sim N(-(\mu_t - r_t), h_t)$ , and that under the EMM  $Q$ ,

$$\ln\left(\frac{S_t}{S_{t-1}}\right) = r_t - \frac{1}{2}h_t + \varepsilon_t^* \quad (3.2)$$

with  $\varepsilon_t^* | F_{t-1} \sim N(0, h_t)$  and  $E^Q\left[\frac{S_t}{S_{t-1}}\Big|F_{t-1}\right] = \exp(r_t)$ .

Consider now the NGARCH(1,1) process in (2.2) with normal innovations. Using  $\varepsilon_t^* = \varepsilon_t + \mu_t - r_t$ , the volatility process under  $Q$  becomes

$$h_t = w + bh_{t-1} + a(\varepsilon_{t-1}^* - \mu_{t-1} + r_{t-1})^2 \quad \text{with} \quad \varepsilon_t^* | F_{t-1} \sim N(0, h_t)$$

This result coincides with Duan (1995). To see this, note that Duan (1995) specifies the physical GARCH dynamic

$$\begin{aligned} R_t &\equiv \ln\left(\frac{S_t}{S_{t-1}}\right) = r + \lambda\sqrt{h_t} - \frac{1}{2}h_t + \varepsilon_t \\ h_t &= w + bh_{t-1} + a\varepsilon_{t-1}^2 \end{aligned} \quad (3.3)$$

where the price of risk  $\lambda$  is assumed to be constant. To relate this to the notation in (2.1), note that  $r_t = r$  and  $\mu_t = r + \lambda\sqrt{h_t}$ , or  $\lambda = \frac{\mu_t - r}{\sqrt{h_t}}$ . This corresponds to the Radon-Nikodym derivative

$$\frac{dQ}{dP}\Big|_{F_t} = \exp\left(-\sum_{i=1}^t \left(\frac{\varepsilon_i}{\sqrt{h_i}}\lambda + \frac{1}{2}\lambda^2\right)\right) \quad (3.4)$$

and risk neutral innovations  $\varepsilon_t^* = \varepsilon_t + \mu_t - r_t = \varepsilon_t + \lambda\sqrt{h_t}$ . The risk neutral GARCH process is

$$\begin{aligned} R_t &\equiv \ln\left(\frac{S_t}{S_{t-1}}\right) = r - \frac{1}{2}h_t + \varepsilon_t^* \\ h_t &= w + bh_{t-1} + a\left(\varepsilon_{t-1}^* - \lambda\sqrt{h_{t-1}}\right)^2 \end{aligned} \quad (3.5)$$

confirming the result in Duan (1995).

Heston and Nandi (2000) start from the following model

$$\begin{aligned} R_t &\equiv \ln\left(\frac{S_t}{S_{t-1}}\right) = r + \lambda h_t + \varepsilon_t \\ h_t &= w + bh_{t-1} + a(z_{t-1} - c\sqrt{h_{t-1}})^2 \end{aligned} \quad (3.6)$$

Using the notational setup in (2.1), Heston and Nandi therefore assume  $r_t = r$  and  $\mu_t = r + \lambda h_t + 0.5h_t$ . This corresponds to the Radon-Nikodym derivative

$$\frac{dQ}{dP}\Big|_{F_t} = \exp\left(-\sum_{i=1}^t \left(\left(\lambda + \frac{1}{2}\right)\varepsilon_i + \frac{1}{2}\left(\lambda + \frac{1}{2}\right)^2 h_i\right)\right) \quad (3.7)$$

and risk-neutral innovations of the form  $\varepsilon_t^* = \varepsilon_t + \lambda h_t + 0.5h_t$ . The risk-neutral GARCH process is

$$\begin{aligned} R_t &\equiv \ln\left(\frac{S_t}{S_{t-1}}\right) = r - \frac{1}{2}h_t + \varepsilon_t^* \\ h_t &= w + bh_{t-1} + a(z_{t-1}^* - (c + \lambda + \frac{1}{2})\sqrt{h_{t-1}})^2 \end{aligned} \quad (3.8)$$

The Duan (1995) and Heston and Nandi (2000) examples indicate that the risk-neutral volatility dynamic can be obtained without completely characterizing the underlying economy. This will be helpful when we consider return processes with non-normal innovations. The use of the RN derivative in (3.1) and the subsequent characterization of the EMM can be seen as a minimalist approach, which exclusively addresses the valuation problem. The characterization of the economy that gives rise to that particular price is a very interesting question in its own right, which is ignored. This approach is prevalent in the continuous-time option valuation literature. Similar results can of course be obtained by specifying the appropriate pricing kernel.

These examples also illustrate the importance of the specification of the conditional mean. While it may be chosen for analytical convenience, it can actually be specified in a fairly general way. However, it is important to recognize its importance for asymmetries and skewness. Duan's (1995) specification (3.3) starts out with a physical process that is not negatively skewed, whereas



Heston and Nandi's (2000) specification (3.6) implies negative skewness for  $c > 0$ . The risk-neutral dynamic (3.5) exhibits negative skewness, since the price of risk  $\lambda$  has to be positive. The risk neutral asymmetry parameter in (3.8) is larger than the physical asymmetry parameter, again because  $\lambda > 0$ . These mechanics are consistent with stylized facts, because the risk-neutral distribution of returns is more negatively skewed than the physical distribution. Therefore, while we have some flexibility with regard to the specification of the conditional mean, a specification such as (3.3) or (3.6), where returns are a function of return variance or the standard deviation of returns, has desirable implications. Note finally that the flexibility in the specification of the conditional mean obtains because, as discussed above, we are not using a general equilibrium approach.

### 3.2 A Two-Component Gaussian Model

The models (2.3) and (2.4) may not exhibit sufficiently rich volatility dynamics, notably with respect to their autocorrelation functions and volatility term structures. One way proceed is to allow for fractionally integrated GARCH models as in Bollerslev and Mikkelsen (1999), but for the purpose of option valuation this approach may be computationally demanding. An easier approach is to generalize the GARCH(1,1) models by introducing additional variance components.

Multi-component models have substantial advantages. The variance is the main determinant of option prices, and it is likely that one variance component is insufficient to explain the variation in option prices across time, as well as across the maturity and moneyness spectra. In the continuous-time stochastic volatility literature, Bates (2000) and Christoffersen, Heston, and Jacobs (2009) study two-component models. It is straightforward to risk-neutralize GARCH processes with multiple components. Consider the following extension of the NGARCH model with normal innovations.

$$\begin{aligned}
 R_t &\equiv \ln\left(\frac{S_t}{S_{t-1}}\right) = \mu_t - \frac{1}{2}h_t + \varepsilon_t \\
 h_{1,t} &= w_1 + b_1h_{1,t-1} + a_1h_{1,t-1}(z_{1,t-1} - c_1)^2 \\
 h_{2,t} &= w_2 + b_2h_{2,t-1} + a_2h_{2,t-1}(z_{2,t-1} - c_2)^2
 \end{aligned} \tag{3.9}$$

where  $h_t = h_{1,t} + h_{2,t}$ . It is easy to see that using the the RN derivative (3.1), risk neutralization

is very straightforward, yielding

$$\begin{aligned}
R_t &\equiv \ln\left(\frac{S_t}{S_{t-1}}\right) = r_t - \frac{1}{2}h_t + \varepsilon_t \\
h_{1,t} &= w_1 + b_1h_{1,t-1} + a_1\left(\varepsilon_{t-1}^* - \mu_{t-1} + r_{t-1} - \sqrt{h_{1,t-1}}c_1\right)^2 \\
h_{2,t} &= w_2 + b_2h_{2,t-1} + a_2\left(\varepsilon_{t-1}^* - \mu_{t-1} + r_{t-1} - \sqrt{h_{2,t-1}}c_2\right)^2
\end{aligned} \tag{3.10}$$

The researcher has considerable freedom in formulating the conditional mean  $\mu_t$ , either as a function of the total variance  $h_t$ , or as a function of both variance components separately, or in a different way. Unless long samples of returns are available, it is often difficult to reliably choose between different specifications of the conditional mean from returns only. However, the conditional mean specification is of paramount importance when estimating the risk-neutralized version of the model from options, as can be seen from (3.10), because the specification of the conditional mean determines the risk-neutral return skewness and other moments.

It is of course equally straightforward to specify and estimate a two-component model using the Heston-Nandi GARCH dynamic (3.9).

### 3.3 Long-Run and Short Run Volatility Components

Multi-component GARCH models such as (3.10) have to the best of our knowledge not yet been implemented. However, Christoffersen, Jacobs, Ornathanalai, and Wang (2008), and Christoffersen, Dorion, Jacobs, and Wang (2010) study slightly more complex two-component models. Here we consider the affine version of this model studied in Christoffersen, Jacobs, Ornathanalai, and Wang (2008). Note that the unconditional variance for the Heston-Nandi dynamic in (2.3) is given by

$$E[h_t] \equiv \sigma^2 = \frac{w + a}{1 - b - ac^2}$$

The variance process can then be rewritten by substituting out  $w$ ,

$$h_t = \sigma^2 + b(h_{t-1} - \sigma^2) + a\left((z_{t-1} - c\sqrt{h_{t-1}})^2 - (1 + c^2\sigma^2)\right). \tag{3.11}$$

Christoffersen, Jacobs, Ornathanalai, and Wang (2008) generalize (2.3) by replacing  $\sigma^2$  with the time-varying component  $q_t$ , which gives

$$h_t = q_t + \beta(h_{t-1} - q_{t-1}) + \alpha\left((z_{t-1} - \gamma_1\sqrt{h_{t-1}})^2 - (1 + \gamma_1^2q_{t-1})\right). \tag{3.12}$$

This is an affine version of the component model of Engle and Lee (1999), where  $q_t$  is referred to as the long-run component, and  $h_t - q_t$  as the short-run component, which has zero mean. Engle and Lee (1999) find that this model provides a very good fit to return data. This model also has an extensive history in the natural sciences, where natural phenomena are often modeled as a stochastic process with a time-varying long-run mean. The risk-neutralization of this model is more complex than (3.10), and the reader is referred to Christoffersen, Jacobs, Ornathanalai, and Wang (2008) for details.

One final remark is that the two-component models in this Section and Section 3.2 are relatively simple because they are formulated using nonlinear functions of a single innovation  $z_{t-1}$ . This simple structure is one of the advantages of the GARCH framework, as it greatly facilitates estimation which can be done using conventional maximum likelihood techniques. However, the question arises whether generalizations of these models with multiple innovations could perform even better. We will address this question below, but first we consider GARCH models with non-normal innovations.

## 4 Non-Normal Innovations: A General Framework for Risk Neutralization

We now provide results on the risk-neutralization of returns with non-normal innovations. We discuss in detail the empirical evidence on non-normal innovations in Section 10. At this point, it is important to keep the following in mind. First, it is well-known that stock returns are not normally distributed, and there is a long tradition in finance of modeling fat-tailed returns, going back to Mandelbrot (1963). Second, there is a rich tradition in the GARCH literature of modeling non-normal innovations in returns. See for instance Bollerslev (1987) and Nelson (1991) for early examples. Third, even if the one-period conditional returns are assumed to be normally distributed in the GARCH model (2.2), then neither the multiperiod conditional returns, nor the unconditional returns will be normally distributed because of the mixing of dependent normal distributions with different variances.

While the implications of assuming non-normal innovations in GARCH option pricing models may therefore not be immediately evident for long-maturity options, these assumptions are relevant because there is considerable evidence of model misspecification for daily return models that assume normal innovations. Here we provide the general framework for risk-neutralization, and we discuss several examples below in Section 5.

CEFJ (2010) provide a more general valuation result for the processes in (2.2) and its extensions when  $\varepsilon_t|F_{t-1} \sim D(0, h_t)$ . For a given predetermined sequence  $\{\nu_t\}$ , define the following

candidate Radon-Nikodym derivative

$$\frac{dQ}{dP} \Big|_{F_T} = \exp \left( - \sum_{t=1}^T (\nu_t \varepsilon_t + \Psi_t(\nu_t)) \right) \quad (4.1)$$

where  $\Psi_t(u)$  is defined as the natural logarithm of the moment generating function

$$E_{t-1} [\exp(-u\varepsilon_t)] \equiv \exp(\Psi_t(u))$$

The mean correction factor in (2.1) is given by  $\gamma_t = \Psi_t(-1)$ . In the normal case we have  $\Psi_t(u) = \frac{1}{2}h_t u^2$  and  $\gamma_t = \Psi_t(-1) = \frac{1}{2}h_t$ . CEFJ (2010) show that (4.1) is a Radon-Nikodym derivative. For an EMM to exist, the expected return under the risk neutral measure must be equal to the risk-free rate.

$$E_{t-1} \left[ \frac{\frac{dQ_t}{dP_t}}{\frac{dQ_{t-1}}{dP_{t-1}}} R_t \right] = \exp(r_t) \quad (4.2)$$

Using (4.1) this gives

$$E_{t-1} [\exp(-\nu_t \varepsilon_t - \Psi_t(\nu_t)) \exp(\varepsilon_t) \exp(\mu_t - \gamma_t)] = \exp(r_t)$$

Therefore the probability measure  $Q$  defined by the Radon-Nikodym derivative in (4.1) is an EMM if and only if

$$\Psi_t(\nu_t - 1) - \Psi_t(\nu_t) + \mu_t - r_t - \gamma_t = 0. \quad (4.3)$$

The use of the moment generating function in the candidate Radon-Nikodym derivative is merely a convenient mathematical tool to find an analytical expression for risk neutralization in this incomplete markets setup. However, the formulation of the Radon-Nikodym derivative in (4.1) is useful in the sense that it clarifies the relationship between the condition (4.3) and the EMM condition (4.2), and the role of the affine structure of the Radon-Nikodym in assumption (4.1). Alternative choices of the Radon-Nikodym derivative will lead to other EMMs and therefore different derivative prices. CEFJ (2010) discuss such generalizations using a quadratic and even more general EMMs.

Denote the moment generating function as  $MGF_t(\nu_t) = \exp(\Psi_t(\nu_t))$ . Then using the definition of the Esscher (1932) transform

$$ET_t(z; \nu_t) = \frac{MGF_t(z + \nu_t)}{MGF_t(\nu_t)}, \quad (4.4)$$

it can be seen that the problem at hand amounts to finding a  $\nu_t$  such that<sup>6</sup>

$$r_t = \ln(ET_t(1; \nu_t)) \tag{4.5}$$

In our opinion it is more straightforward to use equation (4.1). While it does not require the specification of an underlying economy, it captures the intuition of the intertemporal optimality through (4.2). While the use of (4.4) and (4.5) is algebraically equivalent, it is not very informative on how to analyze more general cases. Moreover, when formulated as (4.5), the role of the specification of the conditional mean of returns is ignored, while it is clear from (4.2) that it critically affects the result. In other words, the formulation in terms of the Esscher transform (4.5) effectively assumes a return process that is distributed with mean zero. It is arguably simpler and more intuitive to formulate the problem using (4.1) and (4.2).

The nonnormal models we discuss in this survey are all parametric in nature: The researcher specifies a particular parametric distribution which is then risk-neutralized and used to compute option prices. Barone-Adesi, Engle, and Mancini (2008) introduce an interesting nonparametric alternative where the empirical return shocks are used in the simulation when computing option prices. This approach requires an assumption that the distribution of the return shocks is the same under the physical and risk neutral measures.

## 5 Non-Normal Innovations: Examples

We now discuss several specific option pricing models that are more general than (2.3) and (2.4) in that the return innovations are non-normal. Section 5.1 introduces the inverse Gaussian model and risk neutralizes it using the framework in Section 4. Section 5.2 analyzes the Generalized Error Distribution using the risk neutralization procedure in Duan (1999). Section 5.3 discusses a particular jump GARCH model which motivates the development of a general class of jump GARCH models in Section 6.

### 5.1 Conditionally Inverse Gaussian Returns

Christoffersen, Heston and Jacobs (2006) analyze a GARCH model with an inverse Gaussian innovation,  $y_t \sim IG(\sigma_t^2/\eta^2)$ . This model can be analyzed using the valuation framework in

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<sup>6</sup>The insurance literature has also suggested the use of the Esscher transform for the purpose of option valuation. See Buhlmann, Delbaen, Embrechts, and Shiryayev (1996,1998), Gerber and Shiu (1994), and Siu, Tong, and Yang (2004).

Section 4. The return dynamic is

$$R_t = r + (\zeta + \eta^{-1}) h_t + \varepsilon_t, \text{ where} \quad (5.1)$$

$$\varepsilon_t = \eta y_t - \eta^{-1} h_t \quad (5.2)$$

and where the conditional return variance,  $h_t$ , is of the GARCH form. The conditional log MGF is

$$\Psi_t(u) = \left( u + \frac{1 - \sqrt{1 + 2u\eta}}{\eta} \right) \frac{h_t}{\eta}$$

The EMM condition

$$\Psi_t(\nu_t - 1) - \Psi_t(\nu_t) - \Psi_t(-1) + \mu_t - r = 0$$

is solved by the constant

$$\nu_t = \nu = \frac{1}{2\eta} \left[ \frac{(2 + \zeta\eta^3)^2}{4\zeta^2\eta^2} - 1 \right], \forall t$$

The EMM is given by

$$\begin{aligned} \frac{dQ}{dP} \Big|_{F_t} &= \exp \left( - \sum_{i=1}^t \left( \nu \varepsilon_i + \left( \nu + \frac{1 - \sqrt{1 + 2\nu\eta}}{\eta} \right) \frac{h_i}{\eta} \right) \right) \\ &= \exp \left( -\nu t \bar{\varepsilon}_t - \delta t \bar{h}_i \right) \end{aligned}$$

where  $\bar{\varepsilon}_t = \frac{1}{t} \sum_{i=1}^t \varepsilon_i$ ,  $\bar{h}_i = \frac{1}{t} \sum_{i=1}^t h_i$ , and  $\delta = \frac{\nu}{\eta} + \frac{1 - \sqrt{1 + 2\nu\eta}}{\eta^2}$ . The risk neutral return dynamic can be written as

$$R_t \equiv \ln \left( \frac{S_t}{S_{t-1}} \right) = r - \Psi_t^{Q^*}(-1) + \varepsilon_t^* = r + (\zeta^* + \eta^{*-1}) \sigma_t^{*2} + \varepsilon_t^*$$

where

$$\zeta^* = \frac{1 - 2\eta^* - \sqrt{1 - 2\eta^*}}{\eta^{*2}} \text{ and } \varepsilon_t^* = \eta^* y_t^* - \eta^{*-1} \sigma_t^{*2}$$

The risk neutral process thus takes the same form as the physical process. Option values can be obtained in closed form using the characteristic function. See Christoffersen, Heston and Jacobs (2006) for more details.

## 5.2 Conditionally Generalized Error Distribution Returns

Duan (1999) introduces a model with innovations that follow the generalized error distribution. This model is also of interest because the approach to risk-neutralization is somewhat different from the approach discussed in Section 4. Stentoft (2008) also uses this approach to obtain a

risk-neutral dynamic for the normal inverse Gaussian model. Consider a GARCH model such as (2.2) or (2.3), and now assume that the *i.i.d.* return shock  $z_t$  follows the Generalized Error Distribution (GED). For notational convenience we denote the innovation by  $\zeta_t$ . Normalizing to obtain zero mean and unit variance, the probability density function is given by

$$g_\varphi(\zeta) = \frac{\varphi}{2^{1+\frac{1}{\varphi}} \theta(\varphi) \Gamma\left(\frac{1}{\varphi}\right)} \exp\left(-\frac{1}{2} \left|\frac{\zeta}{\theta(\varphi)}\right|^\varphi\right) \quad \text{for } 0 < \varphi \leq \infty$$

where  $\Gamma(\cdot)$  is the gamma function and where  $\theta(\varphi) = \left(\frac{2^{-\frac{2}{\varphi}} \Gamma\left(\frac{1}{\varphi}\right)}{\Gamma\left(\frac{3}{\varphi}\right)}\right)^{\frac{1}{2}}$ . The expected return exists provided  $\varphi > 1$ . The GED innovation  $\zeta$  has a skewness of zero and a kurtosis of  $\kappa(\varphi) = \frac{\Gamma\left(\frac{5}{\varphi}\right)\Gamma\left(\frac{1}{\varphi}\right)}{\Gamma\left(\frac{3}{\varphi}\right)^2}$ . It can be verified that  $\kappa(2) = 3$ , and  $g_2(\zeta) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\zeta^2\right)$ . Therefore the standardized GED conveniently nests the standard normal distribution which obtains when  $\varphi = 2$ . For  $\varphi < 2$ , the density function has tails that are fatter than the normal distribution and vice versa. Nelson (1991), Hamilton (1994) and Duan (1999) provide more detail on the properties of the GED distribution.

Duan (1999) extends the risk-neutralization for the normal case in Duan (1995) to the GED or other nonnormal distributions, as follows. Recall that for an *i.i.d.* Normal innovation  $z_t$  we have the following mean shift between the two measures

$$\eta_t = z_t^* - z_t \tag{5.3}$$

For a GED distributed  $\zeta_t$  shock, write instead

$$\eta_t = z_t^* - \Phi^{-1}(G_\varphi(\zeta_t))$$

where  $\Phi^{-1}(\cdot)$  is the standard normal inverse CDF so that  $z_t = \Phi^{-1}(G_\varphi(\zeta_t))$  is normally distributed. We can then rewrite the linear normal mapping in (5.3) as a nonlinear GED mapping given by

$$\zeta_t = G_\varphi^{-1}(\Phi(z_t^* - \eta_t)) \tag{5.4}$$

The derivation of the risk-neutral model requires solving for  $\eta_t$ . This is done by setting the conditionally expected risk-neutral asset return in each period equal to the risk-free rate. In general we can write

$$\exp(r) = E_{t-1}^Q \left[ \exp \left\{ E_{t-1} [R_t] + \sqrt{h_t} G_\varphi^{-1}(\Phi(z_t^* - \eta_t)) \right\} \right]$$

An exact solution for  $\eta_t$  in the GED case involves a prohibitively cumbersome numerical solution for  $\eta_t$  on every day and on every Monte Carlo path. The following approximation from Christoffersen, Dorion, Jacobs, and Wang (CDJW, 2010) can be used. In the normal special case we have  $G_\varphi^{-1}(\Phi(z)) = z$  for all  $z$ . As the Normal and GED are both symmetric we know that  $G_\varphi^{-1}(\Phi(0)) = 0$  for all  $\varphi$ . This suggests the linear approximation

$$G_\varphi^{-1}(\Phi(z)) \approx b_\varphi z$$

where  $b_\varphi$  is easily found for a given value of  $\varphi$  by fitting  $\zeta_i = G_\varphi^{-1}(\Phi(z_i))$  to  $z_i$  for a wide grid of  $z_i$  values. CDJW motivate this approximation by the fact that the probability integral transform,  $\zeta = G_\varphi^{-1}(\Phi(z))$  is very close to linear in  $\varphi$  for their parameter estimates. Using this approximation and taking logs, and solving for  $\eta_t$  yields

$$\eta_t = \left( \frac{\lambda}{b_\varphi} + \frac{1}{2} b_\varphi \right) \sqrt{h_t} \quad (5.5)$$

where the normal case obtains when  $b_\varphi = 1$ . The solution in (5.5) can be used in (5.4), and substituting into the return dynamic yields the risk-neutral processes. Note that while the linear approximation greatly facilitates the computation of  $\eta_t$  in the GED models, the GED option prices still require frequent inversion of the GED cumulative distribution function.

Lehnert (2003) applies an asymmetric version of the GED distribution in an exponential GARCH model and uses it for option valuation of DAX index contracts.

### 5.3 Conditionally Poisson-Normal Jumps

A very interesting approach to generating non-normal innovations in a heteroskedastic model is to use Poisson-normal innovations mixed with normals. This approach is used for option valuation in Duan, Ritchken and Sun (DRS, 2005). DRS (2005) consider the return process

$$R_t = \alpha_t + \varepsilon_t, \quad (5.6)$$

where  $\alpha_t$  is the conditional mean component.<sup>7</sup> The Poisson-normal innovation  $\varepsilon_t$  is defined by

$$\varepsilon_t = \sqrt{h_t} \left( z_t + \sum_{i=1}^{N_t} x_t^i \right)$$

with

$$z_t \sim N(0, 1), \quad x_t^j \sim N(\theta, \delta^2) \text{ for } j = 1, 2, \dots, N_t, \text{ and } N_t \sim \text{Poisson}(\chi)$$

---

<sup>7</sup>Duan, Ritchken, and Sun (2006) consider the continuous-time limit of this model.



The number of jumps  $N_t$  that arrives from time  $t - 1$  to  $t$  is Poisson distributed with intensity  $\chi$ . The jump size is independently drawn from a normal distribution with mean  $\theta$  and variance  $\delta^2$ . The conditional return variance equals  $(1 + \chi(\theta^2 + \delta^2))h_t$ , where  $h_t$  follows a GARCH process. The log return mean  $\mu_t$  can be a function of  $h_t$  as well as the jump and risk premium parameters. The conditional log MGF of a Poisson-Normal innovation  $\varepsilon_t$  is given by

$$\begin{aligned}\Psi_t(u) &= \ln(E_{t-1}[\exp(-u\varepsilon_t)]) \\ &= \frac{1}{2}u^2h_t + \chi \left[ \exp\left(-\theta u\sqrt{h_t} + \frac{1}{2}\delta^2u^2h_t\right) - 1 \right].\end{aligned}$$

In DRS (2005), the mean component  $\alpha_t$  is chosen for tractability in the risk neutralization. Specifically, it is defined as

$$\begin{aligned}\alpha_t &= -\frac{1}{2}h_{t+1} - \nu\sqrt{h_t} + \chi \exp\left(\nu\theta + \frac{1}{2}\delta^2\nu^2\right) (1 - \Delta_t) \quad \text{with} \\ \Delta_t &= \exp\left((\nu\delta^2 + \theta)\sqrt{h_{t+1}} + \frac{1}{2}\delta^2h_t\right).\end{aligned}$$

Using the Radon-Nikodym derivative (4.1) implies that (5.6) is an EMM if and only if

$$\alpha_t = \Psi_t(\nu_t) - \Psi_t(\nu_t - 1) - r = 0.$$

Solving the above restriction shows that  $\nu_t = \nu$ . The risk-neutral distribution is from the same family as the physical distribution. This can be inferred from the risk-neutral log MGF of a Poisson-Normal innovation

$$\begin{aligned}\Psi_t^{Q^*}(u) &= \ln E_{t-1}^Q[\exp(-u\varepsilon_t^*)] \\ &= \frac{1}{2}u^2h_t + \chi_t^* \left[ \exp\left(-\theta_t^*u\sqrt{h_t} + \frac{1}{2}\delta^2u^2h_t\right) - 1 \right]\end{aligned}$$

where

$$\chi_t^* = \chi \exp\left(-\theta\nu\sqrt{h_t} + \frac{1}{2}\delta^2\nu^2h_t\right) \quad \text{and} \quad \theta_t^* = \theta - \delta^2\sqrt{h_t}\nu.$$

In summary, the risk-neutral Poisson-normal innovation  $\varepsilon_t^*$  can be written as

$$\varepsilon_t^* = \sqrt{h_t} \left( z_t^* + \sum_{i=1}^{N_t^*} (x_t^*)^i \right)$$

with

$$z_t^* \sim N(0, 1) \quad ; \quad (x_t^*)^j \sim N(\theta^*, \delta^2) \quad \text{for } j = 1, 2, \dots, N_t^* \quad ; \quad \text{and } N_t^* \sim \text{Poisson}(\chi^*).$$

The DRS (2005) approach is very interesting in that it allows for building various types of non-normal innovations by adding normal and Poisson components. We will next develop a general framework that builds on this idea and allows the multiple components to have separate moment dynamics and risk premia.

## 6 Models with Multiple Shocks

In the standard GARCH literature, asset returns are driven by a univariate innovation. However, empirical evidence from the continuous-time option pricing literature suggests that it may be beneficial to separate returns into two components, a jump part and a diffusive part. Christoffersen, Jacobs, and Ornathanalai (CJO, 2012) study a multiple-shock version of the model with Poisson-Normal innovations in Section 5.3 by separately modeling the compound Poisson jump and the normal component in returns.<sup>8</sup> The advantage of this approach is that it allows for time-varying jump intensity as well as for separate modeling of the jump and normal risk premia. Ornathanalai (2009) develops a more general class of multifactor return models, which he refers to as Lévy GARCH. In this approach, return innovations follow Lévy processes and affine GARCH dynamics are used to generate heteroskedasticity in the models.

We introduce the return processes and the risk neutralization in a multiple-shock setting in Sections 6.1 and 6.2. Section 6.3 discusses the jump model of CJO (2012). Finally, Section 6.4 discusses the Lévy GARCH framework.

### 6.1 Multiple Innovations

CJO (2012) and Ornathanalai (2009) consider a return process that is more general than (2.2) in two ways. First, they allow for multiple shocks in the return process. Second, these shocks can be normal, non-normal, or both. Let  $\mathbf{X}_t$  be a d-dimensional vector of Lévy processes that drive changes in asset price. The return process can be written as

$$R_t = \log \left( \frac{S_t}{S_{t-1}} \right) = r + \mu_t + \boldsymbol{\vartheta}' \mathbf{X}_t - \boldsymbol{\xi}_{\mathbf{X}}(\boldsymbol{\vartheta})' \mathbf{h}_t, \quad (6.1)$$

---

<sup>8</sup>The discussion of CJO (2012) in this survey is based on the 2009 working paper version of the paper.

where  $\boldsymbol{\vartheta}$ , and  $\mathbf{h}_t$  are  $d$ -dimensional vectors. Note that we use bold-faced font to represent a vector. The elements in  $\mathbf{h}_t$  are parameters that control the time-varying dynamics of  $\mathbf{X}_t$ . When  $\mathbf{h}_t$  follows a GARCH dynamic, the model exhibits heteroskedasticity. The setup in (6.1) can be motivated using the time-homogeneity property of Lévy processes. Ornathanalai (2009) shows that in most cases, the log MGF of a scalar Lévy process  $x_t$  can be written as consisting of two parts

$$\Psi_t(u) = \ln(E_{t-1}[\exp(-ux_t)]) = \xi_x(-u)h_t.$$

The first part,  $\xi_x(-u)$ , is time independent and is referred to as the coefficient in the log MGF. The second part,  $h_t$ , can be modeled as time varying. In the case of a normal innovation,  $\Psi_t(u) = \frac{1}{2}u^2h_t$ , thus  $h_t$  is the variance, and  $\xi_x(-u) = \frac{1}{2}u^2$ .

This decomposition can easily be extended to the multiple shock case. If the Lévy processes in  $\mathbf{X}_t$  are independently distributed, then its log MGF can be written as  $\Psi_t(\mathbf{u}) = \boldsymbol{\xi}_{\mathbf{X}}(-\mathbf{u})' \mathbf{h}_t$ . The mean correction factor in (6.1) is given by  $\Psi_t(-\boldsymbol{\vartheta}) = \boldsymbol{\xi}_{\mathbf{X}}(\boldsymbol{\vartheta})' \mathbf{h}_t$ , which serves the same purpose as  $\gamma_t$  in the univariate setup (2.1).

Taking the conditional expectation of the price process in (6.1) shows that the log return in excess of the risk-free rate is equal to

$$\log E_{t-1}[\exp(R_t)] - r = \mu_t.$$

We thus refer to  $\mu_t$  as the conditional equity premium. The conditional variance of the log return is given by  $Var_{t-1}(R_t) = \boldsymbol{\vartheta}' Cov_{t-1}(\mathbf{X}_t) \boldsymbol{\vartheta}$ , where  $Cov_{t-1}(\mathbf{X}_t)$  is the conditional covariance matrix that is diagonal and affine in  $\mathbf{h}_t$ . Consequently, the conditional return variance will also be an affine function of  $\mathbf{h}_t$ .

## 6.2 Risk Neutralization

CJO (2012) and Ornathanalai (2009) provide a general valuation result for the process in (6.1). A candidate for the Radon-Nikodym derivative is

$$\frac{dQ}{dP} \Big|_{F_T} = \exp \left( - \sum_{t=1}^T (\boldsymbol{\nu}_t' \mathbf{X}_t + \Psi_t(\boldsymbol{\nu}_t)) \right), \quad (6.2)$$

where  $\Psi_t(\boldsymbol{\nu}_t)$  is the log of the MGF that is evaluated at a  $d$ -dimensional vector  $\boldsymbol{\nu}_t$  of EMM coefficients. The above Radon-Nikodym derivative can be thought of as a multiple-shock version of (4.1). Applying (6.2) to the return process in (6.1), we can establish the following requirement

for (6.2) to be an EMM

$$\Psi_t(\boldsymbol{\nu}_t - 1) - \Psi_t(\boldsymbol{\nu}_t) - \boldsymbol{\xi}_X(\boldsymbol{\vartheta})' \mathbf{h}_t + \mu_t = 0 \quad (6.3)$$

Multiple solutions  $\boldsymbol{\nu}_t$  exist for these EMM conditions. For tractability, CJO (2012) and Ornathanalai (2009) introduce an affine structure on the conditional equity premium specification,  $\mu_t = \boldsymbol{\lambda}' \mathbf{h}_t$ , where  $\boldsymbol{\lambda}$  is a  $d$ -dimensional vector with each element denoted by  $\lambda_i$ , for  $i = 1..d$ . When  $\mathbf{h}_t$  follows a GARCH dynamic, the model produces time-varying risk premia. CJO (2012) furthermore set  $\boldsymbol{\nu}_t = \boldsymbol{\nu}$  in (6.2). In this case, Ornathanalai (2009) shows that the solution to each element  $\nu_i$  in  $\boldsymbol{\nu}$  is found by solving

$$\lambda_i = \xi_{x_i}(\vartheta_i) + \xi_{x_i}(-\nu_i) - \xi_{x_i}(\vartheta_i - \nu_i). \quad (6.4)$$

For instance, if  $x_t$  is distributed as  $N(0, h_t)$ , the solution to (6.4) is  $\nu = \lambda$ . If  $x_t$  is a Compound Poisson process, an analytical solution to (6.4) does not exist, but the numerical solution is well-behaved. Note that the above EMM condition is very general, and can be applied to any Lévy process. See Ornathanalai (2009) for more details.

### 6.3 Poisson Jump and Normal Innovations

Christoffersen, Jacobs, Ornathanalai, and Wang (2009) and CJO (2012) study option valuation for various two-shock models consisting of a normal and a compound Poisson jump component. Maheu and McCurdy (2004) consider a different discrete-time jump model but do not use it for option valuation. The return process studied by CJO (2012) is

$$R_t \equiv \log\left(\frac{S_t}{S_{t-1}}\right) = r + \left(\lambda_z - \frac{1}{2}\right) h_{z,t} + \left(\lambda_y - \xi_y(1)\right) h_{y,t} + z_t + y_t, \quad (6.5)$$

which is a special case of (6.1) with  $\boldsymbol{\vartheta} = (1, 1)$  and  $\mu_t = \lambda_z h_{z,t} + \lambda_y h_{y,t}$ . This return dynamic does not nest the nonaffine model in DRS (2005) which assumes a constant jump intensity. In (6.5), the return process is a function of a normal innovation  $z_t$ , which is assumed to be distributed  $N(0, h_{z,t})$ , and a pure-jump component  $y_t$ , which is assumed to be contemporaneously independent from  $z_t$ . Therefore, the jump and the normal shocks are modeled separately in the return dynamic (6.5), which is not the case in the DRS (2005) model.

CJO (2012) assume that  $y_t$  is conditionally distributed as compound Poisson with jump intensity  $h_{y,t}$ , mean jump size  $\theta$ , and jump variance  $\delta^2$ . The log MGF of a compound Poisson process is given by

$$\Psi_t(u) = \xi_y(-u) h_{y,t}.$$

Due to the time homogeneity property of a compound Poisson process, its log MGF can be written in two parts, with  $h_{y,t}$  being the time homogeneous parameter and  $\xi_y(-u) = \exp(-u\theta + \frac{u^2\delta^2}{2}) - 1$  being the coefficient. The terms  $\frac{1}{2}h_{z,t}$  and  $\xi_y(1)h_{y,t} \equiv \left(\exp(\theta + \frac{\delta^2}{2}) - 1\right)h_{y,t}$  in (6.5) are mean-adjustment terms for the normal and jump components respectively. The jump component in the period  $t$  return is given by  $y_t = \sum_{j=0}^{n_t} x_t^j$ , where  $x_t^j$ ,  $j = 0, 1, 2, \dots, n_t$ , is an i.i.d. sequence of normally distributed random variables such that  $x_t^j \sim N(\theta, \delta^2)$ . The number of jumps  $n_t$  arriving between times  $t-1$  and  $t$  is a Poisson counting process with intensity  $h_{y,t}$ . The conditional expectation of the number of jumps arriving over the time interval  $(t-1, t)$  is  $E_{t-1}[n_t] = h_{y,t}$ .

The conditional variance and skewness of the return process (6.5) are

$$\text{Var}_{t-1}(R_t) = h_{z,t} + (\delta^2 + \theta^2)h_{y,t} \quad (6.6)$$

$$\text{Skew}_{t-1}(R_t) = \frac{\theta(3\delta^2 + \theta^2)h_{y,t}}{(h_{z,t} + (\delta^2 + \theta^2)h_{y,t})^{3/2}}, \quad (6.7)$$

which show that the model's time-varying dynamics will depend on how  $h_{z,t}$  and  $h_{y,t}$  are specified. In the most general case, CJO (2012) specify that  $h_{z,t}$  and  $h_{y,t}$  are governed by two Heston-Nandi type GARCH(1,1) dynamics

$$\begin{aligned} h_{z,t} &= w_z + b_z h_{z,t-1} + \frac{a_z}{h_{z,t}} (z_{t-1} + y_{t-1} - c_z h_{z,t-1})^2 \\ h_{y,t} &= w_y + b_y h_{y,t-1} + \frac{a_y}{h_{y,t}} (z_{t-1} + y_{t-1} - c_y h_{y,t-1})^2 \end{aligned} \quad (6.8)$$

CJO (2012) empirically study this case, as well as more parsimonious special cases, for instance by letting the jump intensity be a linear function of the variance of the normal component,  $h_{y,t} = kh_{z,t}$ .

The model is risk-neutralized using the multiple-shock Radon-Nikodym derivative (6.2) and setting  $\nu_t = \nu$ . For the two-factor model in CJO (2012), this implies

$$\frac{dQ}{dP} \Big|_{F_T} = \exp \left( - \sum_{t=1}^T \left( \nu_z z_t + \nu_y y_t + \frac{1}{2} \nu_z^2 h_{z,t} + \xi_y(-\nu_y) h_{y,t} \right) \right), \quad (6.9)$$

where

$$\Psi_t(\nu_t) = E_{t-1}[\exp(-\nu_z z_t - \nu_y y_t)] = \frac{1}{2} \nu_z^2 h_{z,t} + \xi_y(-\nu_y) h_{y,t}$$

is the joint log MGF of  $[z_t, y_t]$ . We also recall that  $\xi_y(-\nu_y)$  is the coefficient in the log MGF of a compound Poisson process.

The general solution that makes the above Radon-Nikodym derivative an EMM is given in

(6.4). For the return process studied in CJO (2012), this implies the following restrictions

$$\lambda_y = \left( e^{\frac{\delta^2}{2} + \theta} - 1 \right) + e^{\frac{\nu_y^2 \delta^2}{2} - \nu_y \theta} \left( 1 - e^{(\frac{1}{2} - \nu_y) \delta^2 + \theta} \right) \quad (6.10)$$

$$\lambda_z = \nu_z \quad (6.11)$$

Note that (6.11) is identical to the result implied by Duan's (1995) LRNVR method for a conditionally normal innovation. It is not possible to solve for the second EMM coefficient  $\nu_y$  in (6.10) analytically, but it is well behaved and can easily be solved numerically. Using a change of measure according to (6.9), the stock return process can be written under the risk-neutral measure as

$$\begin{aligned} \log \left( \frac{S_t}{S_{t-1}} \right) &= r - \frac{1}{2} h_{z,t} - \xi_y^*(1) h_{y,t}^* + z_t + y_t^*, \quad (6.12) \\ h_{z,t} &= w_z + b_z h_{z,t-1} + \frac{a_z}{h_{z,t-1}} \left( z_{t-1} + y_{t-1}^* - c_z^* h_{z,t-1} \right)^2 \\ h_{y,t}^* &= w_y^* + b_y h_{y,t-1}^* + \frac{a_y^*}{h_{y,t-1}^*} \left( z_{t-1} + y_{t-1}^* - \nu_z h_{z,t-1} - c_y^* h_{y,t-1}^* \right)^2. \end{aligned}$$

where  $y_t^*$  is compound Poisson with intensity  $h_{y,t}^*$ , mean  $\theta^*$ , and variance  $\delta^2$ , and for  $\Pi = e^{\frac{\nu_y^2 \delta^2}{2} - \nu_y \theta}$  we have

$$\begin{aligned} h_{y,t}^* &= h_{y,t} \Pi, & \xi_y^*(1) &= e^{\frac{\delta^2}{2} + \theta^*} - 1, & w_y^* &= w_y \Pi \\ a_y^* &= \Pi^2 a_y, & c_z^* &= (c_z + \nu_z), & c_y^* &= \frac{c_y}{\Pi} \end{aligned}$$

Due to the fact that the jump innovation does not yield an exponentially affine moment generating function, closed-form option valuation results are not available, but the discrete-time GARCH structure of the model renders option valuation straightforward via Monte-Carlo simulation.

## 6.4 Affine Lévy GARCH Processes

The CJO (2012) model with jump and normal innovations in Section 6.3 is just one example of a model with multiple innovations that can be handled within the structure of the Lévy GARCH framework in Sections 6.1 and 6.2. The class of Lévy processes encompasses most of the distributions used in the finance and economics literature. Besides the normal and the compound Poisson processes, some of the well-known Lévy processes include the Normal Inverse Gaussian (NIG) processes of Barndorff-Nielsen (1998), the Variance Gamma (VG) model of Madan and Milne (1991), and the CGMY process of Carr, Geman, Madan and Yor (2002). Therefore, most existing GARCH processes can be nested within the Lévy framework.

Ornathanalai (2009) studies a class of multifactor models by combining the versatility of Lévy processes with the ease of implementation of GARCH dynamics, and refers to this new class of models as affine Lévy GARCH. For tractability, Ornathanalai (2009) chooses an affine specification for the risk premium. That is, the conditional equity premium in (6.1) is given by

$$\mu_t = \log E_{t-1}^{\mathbb{P}} [\exp (R_t)] - r = \sum_{i=1}^d \lambda_i h_{i,t},$$

where  $\lambda_i$  and  $h_{i,t}$  are the market price of risk and the time-homogenous parameters associated with the  $i^{th}$  Lévy innovation in  $\mathbf{X}_t$ . Using this specification, for most of the well-known Lévy processes, the risk-neutral dynamics obtained using (6.2) are from the same family as the physical distribution. The risk-neutral dynamics of asset prices can therefore be simulated, and derivatives can be priced via Monte Carlo simulation as in Duan (1995). Risk-neutralization results for various Lévy processes are summarized in Ornathanalai (2009).

Ornathanalai (2009) further shows that when combining Lévy processes with certain affine GARCH dynamics, the asset return processes will have analytical solutions for their conditional transform. Consequently, the price of zero-coupon bonds is available in closed-form, and the prices of European-style derivatives can be computed using the Fourier inversion method (see Heston (1993a)). This special sub-class of the Lévy GARCH framework is referred to as affine Lévy GARCH. In this framework, the variance of the normal component  $h_{z,t}$  is governed by a Heston-Nandi type dynamic

$$h_{z,t} = w_z + b_z h_{z,t-1} + \frac{a_z}{h_{z,t}} (z_{t-1} - c_z h_{z,t-1})^2. \quad (6.13)$$

The jump intensity dynamic is given by  $h_{y,t} = k h_{z,t}$ .

It is important to note that the GARCH process (6.13) that is considered in Ornathanalai (2009) differs in an important way from the standard models used in the literature. Unlike standard GARCH models, the total return residual  $z_t + y_t$  does not enter into the GARCH updating dynamic. This has important implications in comparison with the more standard setup in (6.8), where the total return residual  $z_t + y_t$  enters directly into the GARCH dynamic. In the standard case, the filtration of  $h_{z,t}$  and  $h_{y,t}$  in (6.8) is extremely simple and we can directly estimate the model using standard maximum likelihood techniques. Unfortunately, the dynamic of (6.8) does not admit tractable formulae for the cumulant exponent and hence no closed-form solution for European-style derivatives exists. In (6.13) on the other hand, only the normal innovation  $z_{t-1}$  enters the news impact function, which enables the derivation of a closed-form option valuation formula, but on the other hand estimation has to rely on a filtering technique which is not required in standard GARCH models. Ornathanalai (2009) proposes that

the auxiliary particle filtering (APF) technique is used for the filtration of the return innovation. The use of APF has many advantages. It can handle high degrees of nonlinearity, which is an inherent feature of Lévy jump processes. Moreover, the APF builds on a filtering algorithm that facilitates the estimation of the models using the maximum likelihood method, and the implementation of APF does not require that the analytical density of the Lévy processes is known, as long as these processes can be simulated using a robust algorithm. This is clearly advantageous as most of the Lévy processes do not have a closed-form density function. For further details of maximum likelihood estimation using the APF, see Christoffersen, Jacobs and Mimouni (2010) and Ornathanalai (2009).

The added filtering step required in implementing the model in (6.13) means that it can be viewed as a hybrid between the GARCH and SV approaches. It differs from GARCH in that the next period's conditional volatility is not a function of the total return residual. On the other hand, it differs from SV because the next period's conditional volatility is known ex post of the return today. Although the implementation of the model in (6.13) is more involved than that of a standard GARCH, it is simpler to implement than that of the SV approaches. For clarity, we distinguish the mechanic of (6.13) from the SV approaches by comparing it to the stochastic volatilities with jump model (SVJ). For ease of comparison, we cast the continuous-time SVJ model into a discrete-time setting, using the Euler discretization (see for example Johannes and Polson (2006)). The resulting dynamic is given by

$$R_t = r + \mu_t + \sqrt{v_t}z_{1,t} + y_t \tag{6.14}$$

$$v_t = a + bv_{t-1} + \sigma\sqrt{v_{t-1}}\left(\rho z_{1,t} + \sqrt{1-\rho^2}z_{2,t}\right), \tag{6.15}$$

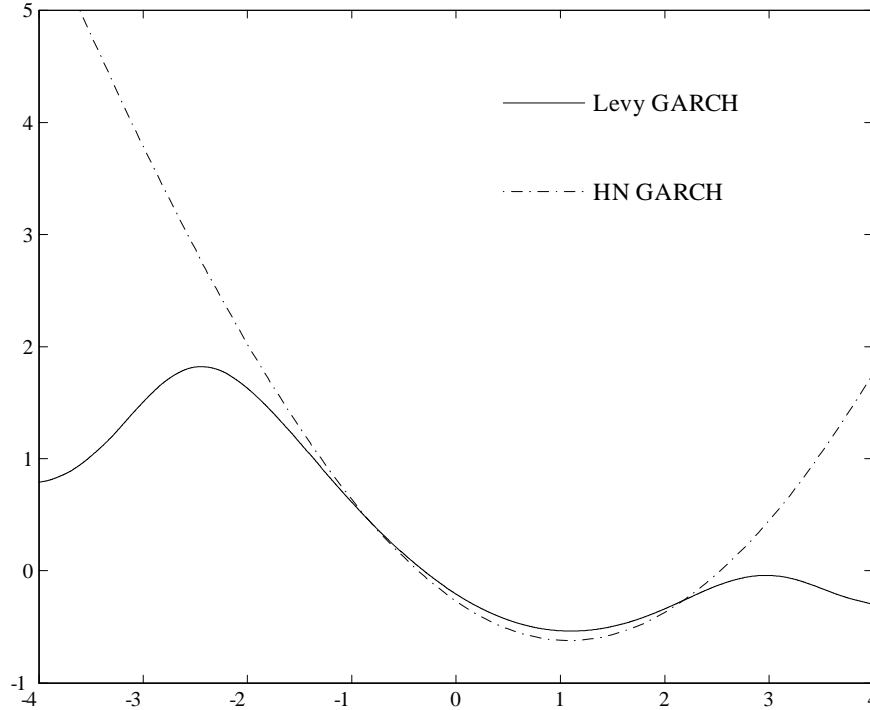
where  $y_t$  represents the compound Poisson jump process, and  $z_{1,t}$  and  $z_{2,t}$  are i.i.d. normal shocks. The contemporaneous correlation between return and variance is captured by  $Corr_t(R_t, v_t) = \rho$ . These dynamics contain three independent shocks. It is possible to decompose the diffusive part  $\sqrt{v_t}z_{1,t}$  from the and the jump part  $y_t$  in (6.14) using a filtering technique. However, unlike the model in (6.13), we cannot determine the next period's conditional volatility given  $z_{1,t}$  because  $v_t$  also depends on the Brownian shock  $z_{2,t}$ , which cannot be inferred from today's return. The filtration of future's volatility in the SVJ model therefore requires an additional step.

The modeling approach in (6.13) differs from (6.8) in another respect. Bates (2006, 2012) argues that standard GARCH models such as NGARCH and the Heston-Nandi model generate excessively large estimates of conditional variance after large stock market movements. As the total return residuals are not always used in the GARCH updating process in (6.13), the models do not suffer from the problem of excessively large estimates of conditional variance after large stock market movements. Figure 1 illustrates this model feature using the news impact curve for



the HN GARCH(1,1) model and the two-factor normal and Lévy jump model (Lévy GARCH) studied in Ornathanalai (2009). The news impact curve illustrates how the current period’s return residuals (news impact) conditionally affect the volatility of returns in the next period. The x-axis represents standardized returns  $R_t/\sqrt{Var_{t-1}(R_t)}$ . The y-axis represents the percentage change in the annualized return volatilities  $(\sqrt{Var_t(R_{t+1})} - \sqrt{Var_{t-1}(R_t)})$ . Current volatility is set to equal the model’s implied long-run volatility level. Figure 1 clearly indicates how the approach in (6.13) can address Bates’ (2006, 2012) concern.

Figure 1: News impact curves



## 7 Valuation

We discuss the valuation of European options when the stock price follows a GARCH process. We first discuss Monte-Carlo valuation. We then discuss how the affine GARCH dynamic (2.3) leads to a (quasi) closed form option price. Subsequently we summarize the literature on approximations to European option prices for GARCH processes. Finally, we discuss the valuation of American options.

### 7.1 Monte-Carlo Valuation of European Options

European call option prices can be computed using Monte Carlo simulation by simulating the risk-neutral return process  $R_t^*$  and computing the sample analogue of the discounted risk neutral

expectation.<sup>9</sup> For a call option,  $C$ , quoted at the close of day  $t$  with maturity on day  $T$ , and with strike price  $X$  we have

$$\begin{aligned} C_{t,T} &= \exp(-r(T-t)) E_t^*[Max(S_T - X, 0)] \\ &\approx \exp(-r(T-t)) \frac{1}{MC} \sum_{i=1}^{MC} \left[ Max \left( S_t \exp \left( \sum_{\tau=1}^{T-t} R_{i,t+\tau}^* \right) - X, 0 \right) \right] \end{aligned}$$

where  $R_{i,t+\tau}^*$  denotes future daily log-return simulated under the risk-neutral measure. The subscript  $i$  refers to the  $i$ th out of a total of  $MC$  simulated paths. It is advisable to use numerical techniques to increase numerical efficiency. Good candidates are stratified random numbers, antithetic variates, control variate techniques, for instance using the Black-Scholes price as the control, and the empirical martingale method of Duan and Simonato (1998). Using these techniques, we have found Monte Carlo simulation to be very reliable. Christoffersen, Dorion, Jacobs, and Wang (2010) provide evidence on the accuracy of Monte Carlo simulation for the Heston and Nandi (2000) model. It stands to reason that Monte Carlo simulation is equally reliable for other models. Appendix A contains matlab code for the computation of Heston-Nandi option prices using Monte Carlo simulation. The code uses Duan and Simonato's empirical martingale technique, and uses a control variate technique with the Black-Scholes price as the control. The code can easily be adapted for other GARCH models.

## 7.2 Closed-Form European Option Valuation

Heston and Nandi's (2000) affine dynamic is specifically designed to yield a closed-form solution. Heston and Nandi (2000) show that a European call option with strike price  $K$  that expires at time  $T$  is worth

$$\begin{aligned} \text{Call Price} &= e^{-r(T-t)} E_t^*[Max(S_T - K, 0)] \tag{7.1} \\ &= \frac{1}{2} S_t + \frac{e^{-r(T-t)}}{\pi} \int_0^\infty \text{Re} \left[ \frac{K^{-i\phi} f^*(t, T; i\phi + 1)}{i\phi} \right] d\phi \\ &\quad - K e^{-r(T-t)} \left( \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left[ \frac{K^{-i\phi} f^*(t, T; i\phi)}{i\phi} \right] d\phi \right) \end{aligned}$$

where  $f^*(t, T; i\phi)$  is the conditional characteristic function of the logarithm of the spot price under the risk neutral measure. For affine GARCH processes, this generating function can be characterized with a set of difference equations. For instance, for the GARCH(1,1) representation

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<sup>9</sup>See for instance Boyle (1977).

in (2.3) we have

$$f^*(t, T; \phi) = S_t^\phi \exp(A_t + B_t h_{t+1})$$

with coefficients

$$\begin{aligned} A_t &= A_{t+1} + \phi r + B_{t+1} w - \frac{1}{2} \ln(1 - 2aB_{t+1}) \\ B_t &= \phi(-0.5 + c^*) - 0.5c^{*2} + \frac{1/2(\phi - c^*)^2}{1 - 2B_{t+1}a} + b_1 B_{t+1} \end{aligned}$$

where  $c^*$  is the risk-neutral leverage parameter in (3.8), which is equal to  $c + \lambda + \frac{1}{2}$ . Note that  $A_t$  and  $B_t$  implicitly are functions of  $T$  and  $\phi$ . This system of difference equations can be solved backwards using the terminal condition  $A_T = B_T = 0$ . Appendix B contains matlab code for the GARCH(1,1) case. Heston and Nandi (2000) provide solutions for the more general GARCH(p,q) case, but their expressions contain some typos. Appendix C contains revised formulas, and also covers the GARCH(2,2) case in order to better motivate the GARCH(p,q) expressions.

Note from (7.1) that even the affine GARCH model does not yield a true closed-form solution, because finding the call price necessitates univariate numerical integration. However, this integration problem is in general well-behaved and the integration is not computer- or time-intensive. Some authors nevertheless prefer to refer to (2.3) as a quasi closed-form solution. A more important question is whether the added convenience of the affine form of the variance dynamic in (2.3) leads to a deterioration in empirical fit. We will address this issue in detail in Section 10.

### 7.3 Approximations for European Option Prices

Duan, Gauthier, and Simonato (1999) provide an analytical approximation for NGARCH European option prices, using the Edgeworth expansions previously used by Jarrow and Rudd (1982). The approximation works very well for short-maturity options. For long-maturity options and very persistent GARCH processes, the approximation is somewhat less reliable. Duan, Gauthier, Sasseville, and Simonato (2006) provide approximations to European option prices under the GJR GARCH of Glosten et al. (1993) and the exponential GARCH of Nelson (1991), and investigate their empirical performance. The results are similar to those for the NGARCH process in Duan, Gauthier, and Simonato (1999). Duan, Gauthier, Sasseville, and Simonato (2006) also provide computation times for the approximations. However, to the best of our knowledge the literature does not contain a detailed comparison of computation times for these approximations and efficiently implemented Monte Carlo prices. Given the trade-off between accuracy and computation time inherent in the use of these approximations, such an exercise would be very valuable.

## 7.4 The Valuation of American Options

Much of the empirical literature on GARCH option valuation is conducted using European-style index options. However, the majority of exchange-traded options are American, and therefore we need techniques that allow for the optimal exercise strategy to be determined as part of the option valuation exercise. For constant volatility models, American options can be valued using Monte Carlo techniques. See for instance Tilley (1993), Barraquand and Martineau (1995), and Broadie, Glasserman, and Jain (1997). Unfortunately, it is not straightforward to implement these Monte Carlo methods for the valuation of American options under GARCH because of the increased complexity.

Instead, three other approaches exist. Duan and Simonato (2001) propose American option valuation for GARCH processes by approximating the underlying asset price process by a finite-state, time-homogeneous Markov chain. They find it provides a very good approximation to the Monte Carlo prices. Ritchken and Trevor (1999) propose a lattice scheme. Stentoft (2005, 2008) analyzes American option valuation for GARCH processes using the least squares Monte Carlo method of Longstaff and Schwartz (2001). Stentoft (2005) investigates NGARCH processes with normal innovations, and finds that the resulting option prices with normal innovations are in most cases very close to the prices obtained using the methods of Duan and Simonato (2001) and Ritchken and Trevor (1999). Stentoft (2008) uses the same approach to value American options for a normal inverse Gaussian (NIG) GARCH model that allows for conditional skewness and kurtosis. Stentoft (2005, 2008) documents that the early exercise premium is substantial and should not be ignored.

## 8 Estimation and Filtering

We first discuss estimation of GARCH processes using return data, and how these estimates can be used for option valuation. We then discuss estimation using option data, and finally we highlight the benefits of estimating and testing GARCH models using a mixture of return and option data.

### 8.1 Estimation Using Returns

It is very straightforward to estimate traditional GARCH processes such as (2.4) or the affine Heston and Nandi (2000) GARCH process (2.3) on return data using maximum likelihood, for a variety of return innovations. See for instance Engle (1982) and Bollerslev (1986). For instance, using the notation for the return dynamic in (2.1), with normal innovations the conditional

density of daily returns is normal, so that

$$f(R(t)|h(t)) = \frac{1}{\sqrt{2\pi h(t)}} \exp\left(-\frac{(R(t) - \mu_t + \gamma_t)^2}{2h(t)}\right).$$

The return log likelihood is therefore

$$\ln L^R \propto -\frac{1}{2} \sum_{t=1}^T \{\ln(h(t)) + (R(t) - \mu_t + \gamma_t)^2 / h(t)\}. \quad (8.1)$$

and we can obtain the parameters of the GARCH process by maximizing the log likelihood. Several studies estimate GARCH parameters using maximum likelihood, and subsequently risk-neutralize the parameters and value options. See for instance Engle and Mustafa (1992), Satchell and Timmermann (1995), Heston and Nandi (2000), Duan and Zhang (2001), Christoffersen and Jacobs (2004), Christoffersen, Heston, and Jacobs (2006), Christoffersen, Jacobs, Ornathanalai and Wang (2008), and Christoffersen, Dorion, Jacobs, and Wang (2010).

It is sometimes argued that unlike the case of option valuation using stochastic volatility models, all parameters needed for option valuation can thus be obtained by estimating the GARCH process under the physical measure using return data. We can risk neutralize the parameters using the arguments in Section 3 or 4, and all parameters needed for option valuation are available. For stochastic volatility models, which contain two sources of uncertainty, the price of volatility risk cannot be estimated from return data, and has to be estimated from option data. See CEFJ (2010) for an extensive discussion. However, Christoffersen, Heston and Jacobs (2010) note that when valuing options using GARCH models, an additional price of volatility risk can be specified which needs to be estimated using option data, just like the case of stochastic volatility models. Effectively this additional risk premium is set to zero in many GARCH option valuation studies. See Christoffersen, Heston and Jacobs (2010) for details.

## 8.2 Estimation using Options

When discussing the evaluation and assessment of GARCH models from option data, it is important to make a number of important distinctions. First, it is possible to estimate GARCH processes on return data as discussed in Section 8.1, to subsequently use these estimates to price options, and to evaluate the models using a variety of loss functions. On the other hand, the GARCH model parameters can also directly be estimated from option prices. Second, when estimating GARCH parameters from option prices, it is critically important whether one estimates a different set of parameters for each cross-section of options, or one set of parameters which is used to price all cross-sections in the sample. The former approach is used for stochastic volatil-

ity models in Bakshi, Cao, and Chen (1997), for example. Duan (1996) estimates a GARCH model using a single cross-section. The latter approach is used in Heston and Nandi (2000) and Christoffersen and Jacobs (2004), for example. A third important distinction is the use of return data when estimating GARCH models on option data. When estimating model parameters using several cross-sections of options, it is in principle necessary to filter volatility from underlying returns, unless one dramatically increases the parameter space by adding a volatility parameter for each cross-section.<sup>10</sup> This is done in most studies that estimate GARCH models from option data, because the required filtering exercise is very simple.<sup>11</sup> The resulting estimates are therefore consistent with returns, in the sense that the volatility used to value options is consistent with the models' risk neutralization and the underlying return data. However, while returns are used in the filtering exercise, the loss function does not explicitly contain a component based on returns. This exercise, which necessitates combining loss functions for options data and returns data, is discussed below in Section 8.3.

We now outline estimation using options data, and we discuss the empirically most relevant case where multiple cross-sections of options are used. This necessitates linking volatility on different dates using the time-series of stock returns. Consider for example the specification of the conditional mean in (3.3). Solving for  $z_t$  we get

$$z_t = \left( R_t - r - \lambda\sqrt{h_t} + \frac{1}{2}h_t \right) / \sqrt{h_t} = \left[ \left( R_t - r + \frac{1}{2}h_t \right) / \sqrt{h_t} \right] - \lambda. \quad (8.2)$$

The updating from  $h_t$  to  $h_{t+1}$  can then be formulated in terms of observables and parameters only by substituting (8.2) in the variance dynamic. Using the NGARCH dynamic in (2.4) for instance, this gives

$$h_t = w + bh_{t-1} + ah_{t-1} \left( \left[ \left( R_{t-1} - r + \frac{1}{2}h_{t-1} \right) / \sqrt{h_{t-1}} \right] - (\lambda + c) \right)^2 \quad (8.3)$$

We can therefore write any loss function of interest as a function of the parameters and observables only. It is important to note that option theory does not suggest the appropriate loss function, as option pricing formulae such as (7.1) do not contain an error term. Loss functions are thus largely chosen out of econometric convenience. For instance, we can minimize the dollar root mean squared option valuation error from a given sample, as in

$$\$RMSE = \sqrt{\frac{1}{N_T} \sum_{t,i} (C_{i,t} - C_{i,t}(h_t(\xi^*)))^2} \quad (8.4)$$

<sup>10</sup>See Bates (2000) and Christoffersen, Heston, and Jacobs (2009) for examples of this approach.

<sup>11</sup>See Heston and Nandi (2000), Christoffersen and Jacobs (2004), Christoffersen, Jacobs, Ornathanalai and Wang (2008), and Christoffersen, Heston, and Jacobs (2006).

where  $N_t$  is the number of option prices present in the sample at time  $t$ ,  $N_T = \sum_{t=1}^T N_t$  and  $T$  is the total number of days included in the sample,  $\xi^*$  is the vector of risk-neutral parameters to be estimated, and  $C_{i,t}(h_t(\xi^*))$  is the model price.

A potential problem with the  $\$RMSE$  loss function is that the weight associated with different option contracts may be highly dependent on moneyness and maturity. Alternatively, relative errors could be used as in

$$\%RMSE = \sqrt{\frac{1}{N_T} \sum_{t,i} \left( \frac{C_{i,t} - C_{i,t}(h_t(\xi^*))}{C_{i,t}} \right)^2} \quad (8.5)$$

A popular approach is to use a loss function based on implied volatilities

$$IVRMSE = \sqrt{\frac{1}{N_T} \sum_{t,i} [\sigma_{i,t} - \sigma_{i,t}(C_{i,t}(h_t(\xi^*)))]^2} \quad (8.6)$$

where  $\sigma_{i,t}$  and  $\sigma_{i,t}(C_{i,t}(h_t(\xi^*)))$  are implied Black-Scholes volatilities from the market price and the model price respectively. This choice of loss function is econometrically convenient because implied volatilities, unlike call option prices, are not dramatically different across maturities and moneyness, thereby avoiding that some options receive much larger weights than others. Unfortunately, the  $IVMSE$  loss function may cause problems when using large option datasets because it is computationally intensive, as it requires inversion of the Black-Scholes formula at each step of the numerical search procedure. To get around this problem, note that we can think of the model option price as an approximation to the market price using a first order approximation around implied Black-Scholes volatility

$$C_{i,t}(h_t(\xi^*)) \approx C_{i,t} + Vega_{i,t} [\sigma_{i,t} - \sigma_{i,t}(C_{i,t}(h_t(\xi^*)))]$$

where  $Vega_{i,t}$  is the Black-Scholes sensitivity of the option price with respect to volatility, computed using  $\sigma_{i,t}$ . Using this approximation, we can assess model fit using the implied volatility root mean squared error ( $IVRMSE$ ) loss function as follows

$$IVRMSE \equiv \sqrt{\frac{1}{N_T} \sum_{i,t} [\sigma_{i,t} - \sigma_{i,t}(C_{i,t}(h_t(\xi^*)))]^2} \approx \sqrt{\frac{1}{N_T} \sum_{i,t} [C_{i,t} - C_{i,t}(h_t(\xi^*))]^2 / Vega_{i,t}^2}$$

The approximation to the implied volatility errors is extremely useful in large scale empirical estimation exercises such as ours, where the Black-Scholes inversion of model prices becomes very costly numerically. See for instance Carr and Wu (2007) and Trolle and Schwartz (2009)

for applications.

A priori we would expect to obtain lower pricing errors when estimating risk-neutral parameters from options, as compared with estimation from return data. Option prices contain forward looking information over and beyond historical returns, and thus using option prices to find parameters can have an important advantage simply from the perspective of the data used. Second, when using maximum likelihood to estimate parameters under the physical measure, it is clear that the loss function is quite different from an out-of-sample loss function which could be something like the mean-squared dollar errors in (8.4) or IVRMSE in (8.6). Following the intuition of Granger (1969), this will lead to larger out-of-sample errors.

### 8.3 Estimation using Returns and Option Data

When estimating model parameters from options as in Section 8.2, GARCH filtering ensures that the estimated parameters are consistent with the return data, as is clear from (8.2) and (8.3). However, because returns are not explicitly weighted in the loss function, it is still possible that we will overfit the option data, and consequently end up with a poor fit to the return data, especially with richly parameterized models. Bates (1996) was the first to notice that this is indeed the case, pointing out that model estimates for a stochastic volatility model obtained from option data were inconsistent with estimates obtained from returns. In recent years increasing importance has been placed on models' ability to reconcile the price of the options and the underlying returns, see for example Pan (2002), Eraker (2004), Santa-Clara and Yan (2010), and Broadie, Chernov, and Johannes (2007).

Despite the consensus that this issue is critically important, most papers continue to estimate option pricing models using either returns or options. Pan (2002) and Chernov and Ghysels (2000) combine returns and options in a loss function based on the method of moments, but they use a univariate time series of an at-the-money option in estimation rather than a time-series of cross-sections. Santa-Clara and Yan use a likelihood-based loss function that combines returns and options, using a more extensive option data sample.

The limited evidence based on loss functions involving returns and options is largely due to the computational complexity associated with most option pricing models. In our opinion, discrete-time models, and GARCH models in particular, have a substantial advantage here, because the computational burden is substantially lower, due to the simple structure of the GARCH filter. In the recent literature, Christoffersen, Jacobs, and Ornathanalai (CJO, 2012), Christoffersen, Heston, and Jacobs (CHJ, 2010), and Ornathanalai (2009) report estimation results using a long time series of results and extensive cross-sections of option data.

In order to combine returns and option data in a single loss function, some error structure



must be assumed on the option data. With  $\sigma_{i,t}$  and  $\sigma_{i,t}(C_{i,t}(h_t(\xi^*)))$  denoting the implied Black-Scholes volatilities from the market price and the model price, CJO and CHJ assume

$$\sigma_{i,t} = \sigma_{i,t}(C_{i,t}(h_t(\xi^*))) + \varepsilon_{i,t} \quad (8.7)$$

and further assume that the implied volatility error  $\varepsilon_{i,t} \sim Normal(0, \sigma_\varepsilon^2)$  is uncorrelated with shocks in returns. Given the normality assumption, the log-likelihood associated with the options data,  $L_{options}$ , is easily constructed, and can be combined with the log-likelihood associated with the return data,  $L_{returns}$ , which is given in (8.1).

While this construction is conceptually straightforward, a practical problem remains. Option prices are available in the time-series as well as in the cross section, thus the number of data points available for computing  $L_{options}$  is significantly larger than the number of data points available for computing  $L_{returns}$ . In order to ensure that the parameter estimates from the joint estimation are not dominated by the option prices, CJO (2012) assign an equal weight to each data point when computing the total log likelihood, as follows

$$L_{joint} = \frac{T + N_T}{2} \frac{L_{returns}}{T} + \frac{T + N_T}{2} \frac{L_{options}}{N}, \quad (8.8)$$

where  $T$  is the number of days in the return sample, and  $N_T$  is the total number of option contracts. Christoffersen, Heston, and Jacobs (2010) try to address this problem by using a return sample in estimation which is considerably longer than the option sample.

## 9 Stochastic Volatility

Most option pricing models are formulated in continuous time. It is therefore useful to investigate the relationship between GARCH models and the benchmark continuous time models. The benchmark model in the continuous time literature is the Heston (1993a) stochastic volatility model. Consider first the one-factor Heston (1993a) model which is one of the most popular models in the option valuation literature. This model specifies the dynamics of the spot price  $S(t)$  as follows

$$\begin{aligned} dS(t) &= (r + \mu v(t))S(t) dt + \sqrt{v(t)}S(t) dz_1(t), \\ dv(t) &= \kappa(\theta - v(t))dt + \sigma\sqrt{v(t)} \left( \rho dz_1(t) + \sqrt{1 - \rho^2} dz_2(t) \right), \end{aligned} \quad (9.1)$$

where  $z_1(t)$  and  $z_2(t)$  are independent Wiener processes. The correlation between the innovations to the stock return and variance is  $\rho$ . A negative  $\rho$  captures negative skewness in the return distribution. Most of the literature follows Heston (1993a) and specifies the following risk-neutral

dynamic

$$\begin{aligned} dS(t) &= rS(t)dt + \sqrt{v(t)}S(t)dz_1^*(t), \\ dv(t) &= (\kappa(\theta - v(t)) - \lambda v(t))dt + \sigma\sqrt{v(t)}(\rho dz_1^*(t) + \sqrt{1 - \rho^2}dz_2^*(t)), \end{aligned} \quad (9.2)$$

where  $z_1^*(t)$  and  $z_2^*(t)$  are independent Wiener processes under the risk-neutral measure. The method underlying this risk neutralization is similar to the one outlined in Section 3 for GARCH processes. The variance process is usually re-written as

$$dv(t) = (\kappa^*(\theta^* - v(t))dt + \sigma\sqrt{v(t)}(\rho dz_1^*(t) + \sqrt{1 - \rho^2}dz_2^*(t))), \quad (9.3)$$

This model yields a quasi closed form option price, because just like the Heston-Nandi GARCH model (2.3), it has an affine structure. It is therefore convenient to use this model to capture the stylized facts of heteroskedasticity and negative skewness, which can be accommodated by a negative value of  $\rho$ , and to add other factors to address the remaining option biases. Most of the literature combines the Heston (1993a) model with Poisson jumps in returns and volatility.<sup>12</sup> Bates (2012), Carr and Wu (2004), and Huang and Wu (2004) investigate infinite-activity Lévy processes. Bates (2000) and Christoffersen, Heston, and Jacobs (2009) investigate stochastic volatility models with multiple diffusive factors.

Several authors have investigated the relationship between GARCH models and (9.1) by considering the limits of the GARCH models as the time interval becomes very small.<sup>13</sup> See for instance Duan (1997) and Heston and Nandi (2000). While this is instructive, these arguments have to be interpreted with care, because a given GARCH model can have several continuous-time limits, and a given continuous-time stochastic volatility model can be the limit of several GARCH models. See for instance Corradi (2000).

It may be more instructive to focus on the more intuitive differences between the GARCH models in (2.3) and (2.4) and their risk-neutralizations in 3 and 4 on the one hand, and the continuous-time stochastic volatility model in (9.1) and (9.2). The Heston model (9.1) is in principle a richer model than a GARCH model because it has two innovations, whereas a GARCH model only has one. The GARCH model uses this one innovation in an ingenious way, by using the lagged squared innovations in the variance dynamic. This makes the model easily amenable to estimation using returns, whereas it is difficult to estimate (9.1) using returns. The absence of a second innovation has distinct implications for option valuation. GARCH models do not

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<sup>12</sup>See for instance Andersen, Benzoni, and Lund (2002), Bakshi, Cao and Chen (1997), Bates (1996b, 2000, 2006), Broadie, Chernov, and Johannes (2007), Chernov and Ghysels (2000), Eraker (2004), Eraker, Johannes, and Polson (2003), Pan (2002), Broadie, Chernov, and Johannes (2007), and Eraker, Johannes, and Polson (2003).

<sup>13</sup>Garcia and Renault (1998) investigate how the GARCH and stochastic volatility models differ in terms of hedging.

contain a separate price for volatility risk, which may affect model fit. Christoffersen, Heston, and Jacobs (2010) demonstrate that it is possible to incorporate a separate price for volatility risk into the GARCH model, even while maintaining a single innovation, and that this improves option pricing performance. We discussed another difference between GARCH models and stochastic volatility models in Section 6.4, and we argued that the model in Ornathanalai (2009) can be seen as a hybrid of both approaches.

Empirical work on the trade-offs between GARCH and stochastic volatility models is very useful, because it is an important empirical question whether the additional flexibility offered by the second innovation in the stochastic volatility model outweighs the increase in computational complexity. Fleming and Kirby (2003) compare the forecasting performance of both classes of models, and find that GARCH performs very well out-of-sample. Lehar, Scheicher, and Schittenkopf (2002) compare the performance of GARCH and continuous-time stochastic volatility models using models estimated over relatively short windows, and find that the GARCH model outperforms the stochastic volatility model out-of-sample. More work along these lines is needed, preferably using loss function containing both option data and long return samples. It may prove interesting to compare the option pricing performance of GARCH models with that of discrete-time stochastic volatility models with two innovations. Such models are formulated by Feunou and Tedongap (2009) and CEFJ (2010).

## 10 Empirical Evidence

We now discuss empirical evidence on GARCH option pricing models. All available evidence indicates that time variation in volatility is critically important, and that GARCH models outperform the Black-Scholes model.<sup>14</sup> We therefore do not discuss this evidence in more detail, but instead organize our discussion around four GARCH modeling features: the leverage effect, differences in fit between affine and non-affine models, the impact of multiple components, and the improvements in fit resulting from non-normal innovations. In some cases we attempt to relate empirical results to the continuous-time stochastic volatility models in Section 9, but it is important to keep in mind the inherent limitations.

As previously mentioned, the presence of the leverage effect, such as in (2.4), is a robust empirical finding. Recall that this finding is partly due to the fact that most of the empirical literature on GARCH option valuation uses index options, and index returns contain substantial negative skewness under the physical as well as risk-neutral measure. When modeling stock options, a somewhat different approach may be warranted, as negative skewness is less prevalent

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<sup>14</sup>See for example Duan (1996), Engle and Mustafa (1992), Heynen, Kemna, and Vorst (1994), and Duan and Zhang (2001).

for stocks under the physical measure.

The modeling of negative skewness and asymmetry in stock returns is the subject of a voluminous literature. See Engle and Ng (1993), Glosten, Jagannathan, and Runkle (1993), Nelson (1991), and Campbell and Hentschel (1992) for important contributions. The modeling of the leverage effect, or equivalently the modeling of negative skewness, is therefore a fortiori important for option pricing, because negative skewness is even more important under the risk-neutral measure. However, the literature on this topic is limited, perhaps because estimation of GARCH models using options is much more time-intensive than when using returns. Christoffersen and Jacobs (2004) investigate the modeling of the leverage effect using the framework provided by Ding, Granger, and Engle (1993) and Hentschel (1995), which nests a lot of existing specifications. Christoffersen and Jacobs (2004) find that more complex models outperform the NGARCH model (2.4) when estimating on return data using maximum likelihood estimation (MLE), consistent with the existing literature. However, this is not the case in- and out-of-sample when estimating the model parameters using option data. Moreover, when valuing options using the risk-neutralized parameters estimated on returns, the simple NGARCH model (2.4) also performs well. Christoffersen and Jacobs (2004) conclude that while it is critically important to model negative skewness for the purpose of option valuation, the simple NGARCH model (2.4) suffices for this task. However, Hardle and Hafner (2000) find that the threshold GARCH model (TGARCH), which has a more flexible news impact function, outperforms the NGARCH model when valuing options. These results are obtained using risk-neutralized MLE estimates from returns, and estimates from option data are not considered.

Another robust finding seems to be that it is rather straightforward to improve on the fit of the affine model (2.3) using non-affine volatility dynamics such as (2.4). Hsieh and Ritchken (2005) compare an affine and a non-affine NGARCH(1,1) model with normal shocks. They estimate the two models using Wednesday data for S&P500 options for 1991-1995, and they estimate the model on each day of the sample, parameterizing the initial volatility. They find that the non-affine model dominates the affine model. Christoffersen, Dorion, Jacobs, and Wang (2010) compare affine and non-affine specifications for several models: models with Gaussian and non-Gaussian innovations, and volatility processes with one or two components. They find that non-affine models dominate affine models both in terms of return fitting and option valuation in all cases. One caveat is that they do not conduct a full-fledged estimation exercise where all model parameters are estimated from option data. Either parameters are estimated using MLE on a long sample of daily returns for 1962-2001, or the MLE parameters are used with a parameterized initial volatility calibrated from options. However, in view of the fact that the results are robust across several models, and in view of the Hsieh and Ritchken (2005) results,

in our opinion the superiority of the non-affine specification is a robust empirical finding.<sup>15</sup>

The superiority of the non-affine specification (2.4) over the affine specification (2.3) should not come as a surprise, and in our opinion it does not mean that the affine specification should be abandoned. The non-affine specification (2.4) was chosen because of its superiority in modeling returns, as well as other financial time series, and its good performance in this respect has been documented by a voluminous literature. The results on option pricing merely confirm these findings. The affine model (2.3) yields a quasi closed form solution, and the convenience offered by this solution makes it a valuable tool. The papers by Hsieh and Ritchken (2005) and Christoffersen, Dorion, Jacobs, and Wang (2010) therefore merely document a trade-off between the convenience offered by the closed form solution and the corresponding deterioration in fit. In other words, the interesting question is not whether the non-affine model fits the data better, but how much better it fits. Moreover, this improvement in fit has to be balanced against the computational cost and the precision of the Monte-Carlo pricing needed for the non-affine model. In our experience, Monte-Carlo pricing using the numerical techniques described in Section 7.1 is very precise and reasonably fast. We therefore believe that the trade-off between affine and non-affine models deserves serious consideration, and needs to be explored further in additional research.

Available research on heteroskedastic non-normal innovations is very limited. In fact, the models presented in Section 5 constitute all available empirical evidence. Christoffersen, Heston, and Jacobs (2006) find that the inverse Gaussian model in Section 5.1 improves upon the Heston-Nandi model in-sample but not out-of-sample for valuing options. Christoffersen, Dorion, Jacobs, and Wang (2010) document that option pricing models with GED innovations as in Section 5.2 also do not fit options better than models with Gaussian innovations, although the models with GED innovations outperform Gaussian models when fitting returns. Lehnert (2003) finds that an EGARCH dynamic with GED innovations outperforms the Heston-Nandi model in- and out-of-sample. This outperformance may of course not be due to the non-normal innovation, but also to the difference in functional form of the volatility dynamic.

In the continuous-time stochastic volatility model, the use of Poisson jumps is the preferred approach to address the shortcomings of the standard affine model. Christoffersen, Jacobs and Ornathanalai (2009) estimate the Poisson-normal model in Section 5.3, as well as a number of special cases. They estimate the models using MLE on daily S&P500 returns for 1985-2004, and find that the Poisson-normal models significantly improve on the fit of the benchmark Heston-Nandi (2000) model. Using the risk-neutralized MLE estimates, they find that models with time-varying jump intensities significantly improve on the Heston-Nandi (2000) benchmark. They

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<sup>15</sup>See also Jones (2003), Ait-Sahalia and Kimmel (2007), and Christoffersen, Jacobs, and Mimouni (2010) for evidence questioning the affine assumption in continuous-time stochastic volatility models.

also confirm the good performance of jump models when estimating parameters using a joint objective function consisting of returns and options, although the relative ranking of the jump models changes. Duan, Ritchken, and Sun (2005) estimate the parameters of their model in Section 5.3 using daily S&P500 returns for 1991-1995. They subsequently risk-neutralize these parameters and value options over the same sample period. They find that the jump model outperforms the NGARCH model (2.4). Ornathanalai (2009) also confirms the importance of jump processes when estimating the model in Section 6.4 and finds that infinite-activity jump processes outperform finite-activity jump processes.

Overall, results for non-normal specifications are mixed. It is important to keep in mind that it is difficult to compare results when the benchmarks are different. For example, in Christoffersen, Jacobs and Ornathanalai (2009) the benchmark is the affine Heston-Nandi (2000) model, whereas in Duan, Ritchken, and Sun (2005) the benchmark is the non-affine model in Duan (1995). Given that the non-affine model significantly outperforms the affine model, the choice of benchmark will influence the conclusion regarding the importance of the jump models. Moreover, different data and objective functions are used in estimation, and in our experience, risk-neutralized parameter estimates obtained from returns sometimes yield surprising results. Estimation using option data, while filtering using returns, may seem preferable, but a potential problem is that one often overfits the benchmark GARCH models such as (2.4) and (2.3). In our opinion models are therefore best compared using an objective function that combines returns and options data. This is the only way to address Bates' (1996) observation that the mapping between the physical and risk-neutral distribution is the most problematic aspect of option pricing models. Not much of this type of evidence is available, even when using normal innovations or continuous-time models.<sup>16</sup> Given the importance of risk premia, as emphasized for example by Pan (2002), Broadie, Chernov, and Johannes (2007), Christoffersen, Ornathanalai, and Jacobs (2009), and Ornathanalai (2009), it is likely that models with non-normal innovations will yield very different results in this type of empirical exercise, because implied risk premia will be very different from the normal case. GARCH models can play an important role here, because of the computational complexity resulting from objective functions including returns and options. The GARCH filter is extremely simple, which makes the resulting optimization problem computationally feasible in many cases. There is a lot of potentially interesting work to be done here, given that many different innovations can be used. See Ornathanalai (2009) for an overview of candidate processes within the Lévy class.

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<sup>16</sup>Chernov and Ghysels (2000) and Pan (2002) use method of moment estimators to combine return data and a time series of option contracts. Jones (2003) and Eraker (2004) use Bayesian methods, and their posteriors are based on both types of data. Christoffersen, Jacobs, and Ornathanalai (2009), Ornathanalai (2009), and Christoffersen, Heston, and Jacobs (2010) present results based on MLE using a likelihood consisting of returns and broad cross-sections of options.

Available empirical evidence on multifactor GARCH models is also rather limited.<sup>17</sup> Christoffersen, Jacobs, Ornathanalai, and Wang (2008) document that the model with long-run and short-run components in (3.12) significantly outperforms the Heston-Nandi (2000) model. Christoffersen, Dorion, Jacobs, and Wang (2010) find that when considering the non-affine versions of these models, the evidence in favor of the model with short-run and long-run components is much less clear-cut.

The limited evidence on multifactor models is perhaps somewhat surprising. In the yield curve literature, which uses models with a mathematical structure similar to those in the option valuation literature, the use of multifactor models of the short rate is widespread. In fact, it is widely accepted in the term structure literature that one factor is not sufficient to capture the time variation and cross-sectional variation in the term structure, and the consensus seems to be that a minimum of three factors are needed. See for instance Litterman and Scheinkman (1991) and Dai and Singleton (2000). Option valuation and term structure modeling have a lot in common: in both cases one faces the demanding task of providing a good empirical fit to the time-series as well as the cross-sectional dimension using tractable, parsimonious models. We therefore believe that the use of multifactor models is as critical for the equity option valuation literature as it is for the term structure literature, and that in the future multifactor models may become as widespread in the option valuation literature as they now are in the term structure literature.<sup>18</sup>

We have even less evidence on the valuation of American options using GARCH processes. Stentoft (2005) compares the performance of several GARCH models with the constant volatility model, using a sample of five individual stocks. He finds that the GARCH models dramatically improve upon the constant volatility model. He documents the importance of the leverage effect, and finds that the EGARCH model performs especially well. Using option data for three underlying stocks, Stentoft (2008) finds that normal inverse Gaussian innovations improve on Gaussian innovations, especially for pricing out-of-the-money options.

We therefore conclude that much work remains to be done. An extensive specification search is needed that involves different types of non-normal innovations and multiple volatility components. The trade-off between affine and non-affine processes in terms of computation time and precision requires special attention. Most importantly, this specification search ought to be implemented using objective functions that combine returns data and options data, in order to address Bates' (1996) observation that the physical and risk-neutral parameters needed to match returns and options data respectively are very different.

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<sup>17</sup>The Poisson-normal models in Duan, Ritchken, and Sun (2005) and Christoffersen, Jacobs, and Ornathanalai (2009) can of course also be considered multifactor models.

<sup>18</sup>See Bates (2000) and Christoffersen, Heston, and Jacobs (2009) for studies of multifactor continuous-time stochastic volatility models.

## 11 Conclusion

The literature on option valuation has expanded considerably over the last three decades. An increasing number of studies investigate option pricing in models that capture the time-varying volatility in returns. It is important to emphasize once more that this development is driven by the stylized facts in the data, as evidenced by the rich literature that estimates GARCH models on return data, following the seminal work of Engle (1982) and Bollerslev (1986), and the existing evidence on option pricing, as articulated forcefully by Bakshi, Cao and Chen (1997).

Even though improvements in computing power make the analysis of data-rich and complex optimization problems ever more feasible, option valuation studies continue to be confronted by computational constraints, due to the size of the data and the complexity of the processes required by the structure of the data. In our opinion, GARCH models therefore have an important role to play in the development of this literature, because of their (relative) ease of implementation.

While the literature has expanded significantly, it seems to us that much remains unknown. Even within the class of models discussed in this survey, our knowledge is limited, and many outstanding questions have not been addressed. For instance, do non-normal innovations or multiple volatility components offer the best chances of improving empirical fit? How many volatility components are needed? Even more importantly, what is the nature of the trade-off between the convenience offered by the affine structure and the resulting deterioration in fit? Can this be addressed by additional volatility components or does the resulting increase in parameters come at too high a price?

While some of these questions have been addressed, most existing work, including our own, suffer in our opinion from two important weaknesses. First, we believe that the Bates (1996) critique has to be taken very seriously, and that consequently the value added of studies that fit models using returns or options separately is limited. Instead, a loss function that combines returns and option data, as in Section 8.3, is indispensable, and clearly the existing evidence using such frameworks is limited. There is a real danger that many existing empirical results are simply caused by overfitting. Second, and somewhat related, a more explicit out-of-sample focus is needed when evaluating option pricing models.

Much additional work is thus needed. Moreover, the question of the economics underlying many of these models has hardly been addressed. The risk-neutralizations used in this survey are analytically convenient, and have the benefit of clarifying the relationship between the risk-neutral and physical measure, but the deeper question of the preferences underlying the construction, as addressed in Brennan (1979), Rubinstein (1976), Duan (1995), and Naik and Lee (1990), are somewhat obscured. Finally, most of the empirical evidence discussed in Section 10 uses index options. More work on stock options is needed. See Elkamhi and Ornthalalai



(2009) for a recent example.

## 12 Appendix A: Matlab Code for Monte Carlo Simulation of European Option Prices

```
% Monte Carlo computation of European option prices for Heston-Nandi (2000) model
% Use Black-Scholes based importance sampling, as well as Duan and Simonato (1998) EMS
% Parameter Values mostly from Heston and Nandi Table 1(a)
r=0; % risk free rate per day continuously compounded
T=90; % days (periods) to maturity
a=1.32e-6; % GARCH MA parameter
b=0.589; % GARCH AR parameter
gam=421.39; % GARCH Asymmetry parameter
om=5.02e-6; % GARCH constant
lam=.205; % risk premium
ht1=(0.15^2)/252; % realistic value for daily unconditional variance
K=100; % strike price
S=100; % stock price
nsim=10000;
xitt=random('norm',0,1,nsim,T);
xt = ones(nsim,1);
xthomo=xt;
ht = ht1*ones(nsim,1);
sigma=sqrt(365*ht1);
bsprice=blsprice(S,K,r*365,T/365,sigma);
for i = 1:T
xitt=xitt(1:nsim,i);
xt = xt.*exp(-0.5*ht+sqrt(ht).*xitt);
xthomo = xthomo.*exp(-0.5*sigma^2/365+sigma/sqrt(365).*xitt);
mxt=mean(xt);
mxthomo=mean(xthomo);
if mxt < 0.5
chgar=-99;
break;
else;
```

```

xt = xt/mxt;
xthomo = xthomo/mxthomo;
ht=om*ones(nsim,1)+b*ht+a*(xit-sqrt(ht)*gamstar).^2;
end;
end;
xta=S*exp(r*T).*xt;
xtahomo=S*exp(r*T).*xthomo;
maxpay=xta-K*ones(nsim,1);
maxpay=max(maxpay,zeros(nsim,1));
maxpayhomo=xtahomo-K*ones(nsim,1);
maxpayhomo=max(maxpayhomo,zeros(nsim,1));
chgarsumhomo=exp(-r*T)*sum(maxpayhomo);
chgarhomo=chgarsumhomo/nsim;
chgarsum=exp(-r*T)*sum(maxpay);
op_price=chgarsum/nsim + bsprice - chgarhomo;

```

## 13 Appendix B: Matlab Code for Heston-Nandi GARCH(1,1) Option Prices

% Matlab code for computation of closed-form GARCH(1,1) option price in Heston and Nandi (2000)

% The first part of the code is called GetHesnanPrice.m

% It sets parameters and computes the integral. It calls the function hntres.m which does the recursion

%-----

% GetHesnanPrice.m

% function [cHN]=GetHesnanPrice(S,T,K,r,a,b,om,gamstar,ht1, resc);

% Parameter Values mostly from Heston and Nandi (2000) Table 1(a)

r=0; % risk free rate per day continuously compounded

T=90; % days (periods) to maturity

a=1.32e-6; % GARCH MA parameter

b=0.589; % GARCH AR parameter

gam=421.39; % GARCH Asymmetry parameter

om=5.02e-6; % GARCH intercept

lam=.205; % risk premium

```

ht1=(0.15^2)/252; % realistic value for daily unconditional variance
K=100; % strike price
S=100; % stock price
integr=quad8('hmintres',.0001,1000,[1e-6 1e-6],[],a,b,om,ht1,-.5,gamstar,T,r,K,S);
% Calculate Heston's European call price
op_price=.5*(S-K*exp(-r*T)) + K/pi*integr
% hmintres.m
% calculates the Heston-Nandi integrand
% note that we are also finding f(i*fi+1) here and not just f(fi)
function fval=hnint(fi,a,b,om,ht1,lam,gam,T,r,K,S);
fip = [i*fi+1; i*fi];
A=fip*r;
B=lam*fip + .5*fip.^2;
% recurse backwards until time t=0
for t=T-1:-1:1;
Ap=A;
Bp=B;
A = Ap + fip*r + Bp*om - .5*log(1-2*a*Bp);
B = fip*(lam+gam)-.5*gam^2 + b*Bp + (.5*(fip-gam).^2)./(1-2*a*Bp);
end;
f1=S.^fip(1,:).*exp(A(1,:)+B(1,:)*ht1);
f2=S.^fip(2,:).*exp(A(2,:)+B(2,:)*ht1);
% first integrand
f1val=real((K.^(-i*fi)).*f1./(i*fi));
% second integrand
f2val=real((K.^(-i*fi)).*f2./(i*fi));
% combined integrand
fval = (f1val/K - f2val)*exp(-r*T);

```

## 14 Appendix C: The Affine GARCH(p,q) Case

This appendix presents the moment generating function (MGF) for the GARCH(p,q) process used in this paper and in Heston and Nandi (2000). We first derive the MGF of a GARCH(2,2)

$$h_{t+1} = w + b_1 h_t + b_2 h_{t-1} + a_1 \left( z_t - c_1 \sqrt{h_t} \right)^2 + a_2 \left( z_{t-1} - c_2 \sqrt{h_{t-1}} \right)^2$$

as an example and then generalize it to the case of the GARCH(p,q). Let  $x_t = \log(S_t)$ . For convenience we will denote the conditional generating function of  $S_t$  (or equivalently the conditional moment generating function (MGF) of  $x_T$ ) by  $f_t$  instead of the more cumbersome  $f(t; T, \phi)$

$$f_t = E_t[\exp(\phi x_T)] \quad (\text{C1})$$

We shall guess that the MGF takes the log-linear form. We again use the more parsimonious notation  $A_t$  to indicate  $A(t; T, \phi)$ .

$$f_t = \exp\left(\phi x_t + A_t + B_{1t}h_{t+1} + B_{2t}h_t + C_t(z_t - c_2\sqrt{h_t})^2\right) \quad (\text{C2})$$

We have

$$f_t = E_t[f_{t+1}] = E_t\left[\exp\left(\phi x_{t+1} + A_{t+1} + B_{1t+1}h_{t+2} + B_{2t+1}h_{t+1} + C_{t+1}(z_{t+1} - c_2\sqrt{h_{t+1}})^2\right)\right] \quad (\text{C3})$$

Since  $x_T$  is known at time T, equations (C2) and (C3) require the terminal condition

$$A_T = B_{iT} = C_T = 0$$

Substituting the dynamics of  $x_t$  into (C3) and rewriting we get

$$f_t = E_t \exp \left( \begin{array}{l} \phi(x_t + r) + (B_{1t+1}a_1 + C_{t+1}) \left( z_{t+1} - (\bar{c}_{t+1} - \frac{\phi}{2(B_{1t+1}a_1 + C_{t+1})}) \sqrt{h_{t+1}} \right)^2 + \\ A_{t+1} + B_{1t+1}w + B_{1t+1}b_2h_t + B_{1t+1}a_2(z_t - c_2\sqrt{h_t})^2 + \\ \left( \phi\lambda + B_{1t+1}b_1 + B_{2t+1} + (\phi\bar{c}_{t+1} - \frac{\phi^2}{4(B_{1t+1}a_1 + C_{t+1})}) \right) + \\ (B_{1t+1}a_1c_1^2 + C_{t+1}c_2^2) - \bar{c}_{t+1}(B_{1t+1}a_1c_1 + C_{t+1}c_2) \end{array} \right) h_{t+1} \quad (\text{C4})$$

where

$$\bar{c}_{t+1} = \frac{B_{1t+1}a_1c_1 + C_{t+1}c_2}{B_{1t+1}a_1 + C_{t+1}}$$

and we have used

$$\begin{aligned} & (B_{1t+1}a_1 + C_{t+1}) \left( z_{t+1} - (\bar{c}_{t+1} - \frac{\phi}{2(B_{1t+1}a_1 + C_{t+1})}) \sqrt{h_{t+1}} \right)^2 \\ &= B_{1t+1}a_1(z_{t+1} - c_1\sqrt{h_{t+1}})^2 + C_{t+1}(z_{t+1} - c_2\sqrt{h_{t+1}})^2 \\ &+ z_{t+1}\phi\sqrt{h_{t+1}} + \left( -\phi\bar{c}_{t+1} + \frac{\phi^2}{4(B_{1t+1}a_1 + C_{t+1})} \right) h_{t+1} \\ &- (B_{1t+1}a_1c_1^2 + C_{t+1}c_2^2 - \bar{c}_{t+1}(B_{1t+1}a_1c_1 + C_{t+1}c_2)) h_{t+1} \end{aligned}$$

Using the result

$$E [\exp(x(z + y)^2)] = \exp\left(-\frac{1}{2} \ln(1 - 2x) + xy^2/(1 - 2x)\right) \quad (\text{C5})$$

in (C4) we get

$$f_t = \exp \left( \begin{array}{l} \phi(x_t + r) + A_{t+1} + B_{1t+1}w - \frac{1}{2} \ln(1 - 2B_{1t+1}a_1 - 2C_{t+1}) + \\ B_{1t+1}b_2h_t + B_{1t+1}a_2(z_t - c_2\sqrt{h_t})^2 + \frac{(B_{1t+1}a_1 + C_{t+1})(\bar{c}_{t+1} - \frac{\phi}{2(B_{1t+1}a_1 + C_{t+1})})^2}{1 - 2B_{1t+1}a_1 - 2C_{t+1}} h_{t+1} + \\ \left( \begin{array}{l} \phi\lambda + B_{1t+1}b_1 + B_{2t+1} + (\phi\bar{c}_{t+1} - \frac{\phi^2}{4(B_{1t+1}a_1 + C_{t+1})}) + \dots \\ (B_{1t+1}a_1c_1^2 + C_{t+1}c_2^2) - \bar{c}_{t+1}(B_{1t+1}a_1c_1 + C_{t+1}c_2) \end{array} \right) h_{t+1} \end{array} \right) \quad (\text{C6})$$

Matching terms on both sides of (C6) and (C2) gives

$$A_t = A_{t+1} + \phi r + B_{1t+1}w - \frac{1}{2} \ln(1 - 2B_{1t+1}a_1 - 2C_{t+1})$$

$$\begin{aligned} B_{1t} &= \phi\lambda + B_{1t+1}b_1 + B_{2t+1} + (B_{1t+1}a_1c_1^2 + C_{t+1}c_2^2) \\ &+ \frac{1/2\phi^2 + 2(B_{1t+1}a_1c_1 + C_{t+1}c_2)(B_{1t+1}a_1c_1 + C_{t+1}c_2 - \phi)}{1 - 2B_{1t+1}a_1 - 2C_{t+1}} \end{aligned}$$

$$B_{2t} = B_{1t+1}b_2$$

$$C_t = B_{1t+1}a_2$$

where we have used the fact that

$$\begin{aligned} &\phi\bar{c}_{t+1} - \frac{\phi^2}{4(B_{1t+1}a_1 + C_{t+1})} - \bar{c}_{t+1}(B_{1t+1}a_1c_1 + C_{t+1}c_2) \\ &+ \frac{(B_{1t+1}a_1 + C_{t+1})(\bar{c}_{t+1} - \frac{\phi}{2(B_{1t+1}a_1 + C_{t+1})})^2}{1 - 2B_{1t+1}a_1 - 2C_{t+1}} \\ &= \frac{1/2\phi^2 + 2(B_{1t+1}a_1c_1 + C_{t+1}c_2)(B_{1t+1}a_1c_1 + C_{t+1}c_2 - \phi)}{1 - 2B_{1t+1}a_1 - 2C_{t+1}} \end{aligned}$$

The case of GARCH(p,q) follows the same logic but is more notation-intensive. Define two

$k \times k$  upper-triangle matrices  $\mathbf{Z}_t = \{Z_{ij,t}\}$  and  $\mathbf{C}_t = \{C_{ij,t}\}$ , where

$$\begin{aligned} Z_{ij,t} &= \left( z_{t-i+1} - c_{j+i} \sqrt{h_{t-i+1}} \right)^2 \text{ for } j+i \leq q \\ &= 0 \text{ for } j+i > q \end{aligned}$$

The moment generating function  $f_t$  is assumed to be of the log-linear form

$$f_t = \exp \left( \phi x_t + A_t + \sum_{i=1}^p B_{it} h_{t+2-i} + I'(\mathbf{C}_t \cdot \times \mathbf{Z}_t) I \right)$$

where  $I$  is a  $k \times 1$  vector of ones and  $k = q - 1$ .  $\cdot \times$  represents element by element multiplication.

After algebra similar to (C1)-(C6), we derive the following results:

$$A_t = A_{t+1} + \phi r + B_{1t+1} w - \frac{1}{2} \ln \left( 1 - 2B_{1t+1} a_1 - 2 \sum_{j=1}^{q-1} C_{1j,t+1} \right)$$

$$\begin{aligned} B_{1t} &= \phi \lambda + B_{1t+1} b_1 + B_{2t+1} + \left( B_{1t+1} a_1 c_1^2 + \sum_{j=1}^{q-1} C_{1j,t+1} c_{j+1}^2 \right) + \\ &\frac{1/2\phi^2 + 2 \left( B_{1t+1} a_1 c_1 + \sum_{j=1}^{q-1} C_{1j,t+1} c_{j+1} \right) \left( B_{1t+1} a_1 c_1 + \sum_{j=1}^{q-1} C_{1j,t+1} c_{j+1} - \phi \right)}{1 - 2B_{1t+1} a_1 - 2 \sum_{j=1}^{q-1} C_{1j,t+1}} \end{aligned}$$

$$B_{it} = B_{1t+1} b_i + B_{i+1t+1}, \text{ for } i = 2 \dots p - 1$$

$$B_{pt} = B_{1t+1} b_p$$

for  $i = 1 \dots k$

$$\begin{aligned} C_{ij,t} &= B_{1t+1} a_{i+1} \text{ for } j = 1 \text{ and } j+i \leq q \\ &= C_{i+1j-1,t+1} \text{ for } j \neq 1 \text{ and } j+i \leq q \\ &= 0 \text{ for } j+i > q \end{aligned}$$

and

$$A_T = B_{iT} = C_{ijT} = 0, \text{ for all } i \text{ and } j.$$

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