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Nonlinear Kalman Filtering in Affine Term Structure Models*

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Abstract

When the relationship between security prices and state variables in dynamic term structure models is nonlinear, existing studies usually linearize this relationship because nonlinear filtering is computationally demanding. We conduct an extensive investigation of this linearization and analyze the potential of the unscented Kalman filter to properly capture nonlinearities. To illustrate the advantages of the unscented Kalman filter, we analyze the cross section of swap rates, which are relatively simple non-linear instruments, and cap prices, which are highly nonlinear in the states. An extensive Monte Carlo experiment demonstrates that the unscented Kalman filter is much more accurate than its extended counterpart in filtering the states and forecasting swap rates and caps. Our findings suggest that the unscented Kalman filter may prove to be a good approach for a number of other problems in fixed income pricing with nonlinear relationships between the state vector and the observations, such as the estimation of term structure models using coupon bonds and the estimation of quadratic term structure models.

JEL Classification: G12

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1 Introduction

Multifactor affine term structure models (ATSMs) have become the standard in the literature on the valuation of fixed income securities, such as government bonds, corporate bonds, interest rate swaps, credit default swaps, and interest rate derivatives. Even though we have made significant progress in specifying these models, their implementation is still subject to substantial challenges.

One of the challenges is the proper identification of the parameters governing the dynamics of the risk premia (see Dai and Singleton (2002)). It has been recognized in the literature that the use of contracts that are nonlinear in the state variables, such as interest rate derivatives, can potentially help achieve such identification. Nonlinear contracts can also enhance the ability of affine models to capture time variation in excess returns and conditional volatility (see Bikbov and Chernov (2009) and Almeida, Graveline and Joslin (2011)).

Given the potentially valuable information content of nonlinear securities, efficient implementation of ATSMs for these securities is of paramount importance. One of the most popular techniques used in the literature, the extended Kalman filter (EKF), relies on a linearized version of the measurement equation, which links observed security prices to the models' state variables. Our paper is the first to extensively investigate the impact of this linearization. We show that this approximation leads to significant noise and biases in the filtered state variables as well as the forecasts of security prices. These biases are particularly pronounced when using securities that are very nonlinear in the state variables, such as interest rate derivatives. We propose the use of the unscented Kalman filter (UKF), which avoids this linearization, to implement affine term structure models with nonlinear securities, and we extensively analyze the properties of this filter. The main advantage of the unscented Kalman filter is that it accounts for the non-linear relationship between the observed security prices and the underlying state variables.

We use an extensive Monte Carlo experiment that involves a cross-section of LIBOR and swap rates, as well as interest rate caps to investigate the quality of both filters as well as their in- and out-of-sample forecasting ability. The unscented Kalman filter significantly outperforms the extended Kalman filter. First, the UKF outperforms the EKF in filtering the unobserved state variables. Using the root-mean-square-error (RMSE) of the filtered state variables as a gauge for the performance of both filters, we find that the UKF robustly outperforms the EKF across models and securities. In some cases, the median RMSE for the EKF is up to 33 times larger than the median RMSE for the UKF. The outperformance of the unscented Kalman filter is particularly pronounced when interest rate caps are included in the filtering exercise. We also find that the UKF is numerically much more stable than the EKF, exhibiting a much lower dispersion of the RMSE across the Monte Carlo trajectories. Second, the improved precision of the UKF in filtering the state variables translates into more accurate forecasts for LIBOR rates, swap rates, and cap prices. The outperformance of the UKF is particularly pronounced for short horizons. It is also critically important that the superior performance of the UKF comes at a reasonable computational cost. In our applications, the time required for the unscented Kalman filter was about twice the time needed for the extended Kalman filter.

Throughout this paper we keep the structural parameters fixed at their true values. However, the poor results obtained when using the EKF to filter states suggest that parameter estimation based on this technique would be highly unreliable, as the filter is unable to correctly fit rates and prices even when provided with the true model parameters. The dramatic improvements brought by the UKF suggest that it will also improve parameter estimation, especially when derivative prices are used to estimate the parameters, which is of critical importance in the identification of the risk premium parameters.

Even though the use of the unscented Kalman filter has become popular in the engineering literature (see for instance Julier (2000) and Julier and Uhlmann (2004)), it has not been used extensively in the empirical asset pricing literature.[1] Our results suggest that the unscented Kalman filter may prove to be a good approach to tackle a number of problems in fixed income pricing, especially when the relationship between the state vector and the observations is highly nonlinear. This includes for example the estimation of term structure models of credit spreads using a cross section of coupon bonds or credit derivatives, or the estimation of quadratic term structure models.[2]

The paper proceeds as follows. Section 2 briefly discusses the pricing of LIBOR, swaps, and caps in affine term structure models. Section 3 discusses Kalman filtering in ATSMs, including the extended Kalman filter and the unscented Kalman filter. Section 4 reports the results of our Monte Carlo experiments. Section 5 discusses implications for parameter estimation, and Section 6 concludes.

[1] See Carr and Wu (2007) and Bakshi, Carr and Wu (2008) for applications to equity options. van Binsbergen and Koijen (2012) use the unscented Kalman filter to estimate present-value models.

[2] See Fontaine and Garcia (2012) for a recent application of the unscented Kalman filter to the estimation of term structure models for coupon bonds. See Chen, Cheng, Fabozzi, and Liu (2008) for an application of the unscented Kalman filter to the estimation of quadratic term structure models.

2 Affine Term Structure Models

In this section, we define the risk-neutral dynamics in ATSMs, a pricing kernel and the pricing formulas for LIBOR rates, swap rates, and cap prices. We follow the literature on term structure models and assume that the swap and LIBOR contracts as well as the interest rate caps are default-free. See Dai and Singleton (2000), Collin-Dufresne and Solnik (2001), and Feldhutter and Lando (2008) for further discussion.

2.1 Risk-Neutral Dynamics

Affine term structure models (ATSMs) assume that the short rate is given by $r_t = \delta_0 + \delta_1' x_t$, and the state vector x_t follows an affine diffusion under the risk-neutral measure Q

$$dx_t = \widetilde{\kappa}\left(\widetilde{\theta} - x_t\right) dt + \Sigma \sqrt{S_t} d\widetilde{W}_t, \tag{1}$$

where $\widetilde{W}_t$ is a $N-$dimensional vector of independent standard Q-Brownian motions, $\widetilde{\kappa}$ and Σ are $N \times N$ matrices and S_t is a diagonal matrix with a ith diagonal element given by

$$[S_t]_{ii} = \alpha_i + \beta_i' x_t. \tag{2}$$

Following Duffie and Kan (1996), we write

$$\begin{aligned} \psi^Q(u,t,\tau) &= E_t^Q\left[e^{-\int_t^{t+\tau} r_s ds} e^{u' x_t}\right] \\ &= \exp\left\{A_u(\tau) - B_u'(\tau) x_t\right\}, \end{aligned} \tag{3}$$

where τ is the time to maturity, and $A_u(\tau)$ and $B_u(\tau)$ satisfy the following Ricatti ODEs

$$\frac{dA_u(\tau)}{d\tau} = -\widetilde{\theta}'\widetilde{\kappa} B_u(\tau) + \frac{1}{2}\sum_{i=1}^{N} \left[\Sigma B_u(\tau)\right]_i^2 \alpha_i - \delta_0 \tag{4}$$

and

$$\frac{dB_u(\tau)}{d\tau} = -\widetilde{\kappa} B_u(\tau) + \frac{1}{2}\sum_{i=1}^{N} \left[\Sigma B_u(\tau)\right]_i^2 \beta_i + \delta_1. \tag{5}$$

Equations (4) and (5) can be solved numerically with initial conditions $A_u(0) = 0$ and $B_u(0) = -u$. The resulting zero-coupon bond price is exponentially affine in the state vector

$$P(t,\tau) = \psi^Q(0,t,\tau) = \exp\left\{A_0(\tau) - B_0'(\tau) x_t\right\}. \tag{6}$$

2.2 Pricing Kernel

The model is completely specified once the dynamics of the state price are known. The dynamic of the pricing kernel π_t is assumed to be of the form

$$\frac{d\pi_t}{\pi_t} = -r_t dt - \Lambda_t' dW_t, \tag{7}$$

where W_t is a $N-$dimensional vector of independent standard $P-$Brownian motions and Λ_t denotes the market price of risk. The dynamics of the state vector under the actual measure P can be obtained by subtracting $\Sigma\sqrt{S_t}\Lambda_t$ from the drift of equation (1).

The market price of risk Λ_t does not depend on the maturity of the bond and is a function of the current value of the state vector x_t. We study completely affine models which specify the market price of risk as follows

$$\Lambda_t = \sqrt{S_t}\lambda_0. \tag{8}$$

See Cheridito, Filipović and Kimmel (2007), Duffee (2002), and Duarte (2004) for alternative specifications of the market price of risk.

2.3 LIBOR and Swap Rates

In ATSMs, the time-t LIBOR rate on a loan maturing at $t+\tau$ is given by

$$\begin{aligned} L(t,\tau) &= \frac{1-P(t,\tau)}{\tau P(t,\tau)} \\ &= \exp(-A_0(\tau) + B_0'(\tau)x_t) - 1. \end{aligned} \tag{9}$$

while the fair rate at time t on a swap contract with semi-annual payments up to maturity $t+\tau$ can be written as

$$\begin{aligned} SR(t,\tau) &= \frac{1-P(t,\tau)}{0.5\times\sum_{j=1}^{2\tau} P(t,0.5j)} \\ &= \frac{1-\exp(A_0(\tau) - B_0'(\tau)x_t)}{0.5\times\sum_{j=1}^{2\tau}\exp(A_0(0.5j) - B_0'(0.5j)x_t)}. \end{aligned} \tag{10}$$

As mentioned earlier, $A_0(\tau)$ and $B_0(\tau)$ can be obtained numerically from equations (4) and (5).

2.4 Cap Prices

Computing cap prices is more computationally intensive. Given the current latent state x_0, the value of an at-the-money cap C_L on the 3-month LIBOR rate $L(t, 0.25)$ with strike

$\bar{R} = L(0, 0.25)$ and maturity in T years is

$$C_L(0, T, \bar{R}) = \sum_{j=2}^{T/0.25} E^Q \left[e^{-\int_0^{T_j} r_s ds} \, 0.25 \Big(L(T_{j-1}, 0.25) - \bar{R} \Big)^+ \right] = \sum_{j=2}^{T/0.25} c_L \left(0, T_j, \bar{R} \right) \,, \tag{11}$$

where $T_j = 0.25j$. The cap price is thus the sum of the value of caplets $c_L \left(0, T_j, \bar{R}\right)$ with strike $\bar{R}$ and maturity T_j.

The payoff $\Pi_{T_{j-1}}$ of caplet $c_L \left(0, T_j, \bar{R}\right)$ is known at time T_{j-1} but paid at T_j. It is given by

$$\begin{aligned} \Pi_{T_{j-1}} &= 0.25 \left(L(T_{j-1}, 0.25) - \bar{R} \right)^+ \\ &= 0.25 \left(\frac{1 - P(T_{j-1}, 0.25)}{0.25 P(T_{j-1}, 0.25)} - \bar{R} \right)^+ \\ &= \frac{1 + 0.25\bar{R}}{P(T_{j-1}, 0.25)} \left(\frac{1}{1 + 0.25\bar{R}} - P(T_{j-1}, 0.25) \right)^+ \,. \end{aligned} \tag{12}$$

Since the discounted value of the caplet is a martingale under the risk-neutral measure, we have for $K = \frac{1}{1+0.25\bar{R}}$

$$\begin{aligned} c_L \left(0, T_j, \bar{R}\right) &= E^Q \left[e^{-\int_0^{T_j} r_s ds} \, \Pi_{T_{j-1}} \right] \\ &= \frac{1}{K} E^Q \left[e^{-\int_0^{T_{j-1}} r_s ds} \Big(K - P(T_{j-1}, 0.25) \Big)^+ \right] \\ &= \frac{1}{K} \mathcal{P}(0, T_{j-1}, T_j, K) \end{aligned} \tag{13}$$

Equation (13) represents the time-0 value of $1/K$ puts with maturity T_{j-1} and strike K on a zero-coupon bond maturing in T_j years. Duffie, Pan, and Singleton (2000) show that the price of such a put option is given by

$$\begin{aligned} \mathcal{P}(0, T_{j-1}, T_j, K) &= E^Q \left[e^{-\int_0^{T_{j-1}} r_s ds} \Big(K - \exp\left\{ A_0(0.25) - B_0'(0.25) x_{T_{j-1}} \right\} \Big)^+ \right] \\ &= e^{A_0(0.25)} E^Q \left[e^{-\int_0^{T_{j-1}} r_s ds} \Big(e^{-A_0(0.25)} K - \exp\left\{ -B_0'(0.25) x_{T_{j-1}} \right\} \Big)^+ \right] \\ &= e^{A_0(0.25)} \Big[c \, G_{0,d}(\log c, 0, T_{j-1}) - G_{d,d}(\log c, 0, T_{j-1}) \Big], \end{aligned} \tag{14}$$

where $c = e^{-A_0(0.25)} K$, $d = -B_0(0.25)$, and

$$G_{a,b}(y, 0, T_{j-1}) = \frac{1}{2} \psi^Q(a, 0, T_{j-1}) - \frac{1}{\pi} \int_0^\infty \frac{1}{\varkappa} \mathrm{Im} \left[\psi^Q(a + i\varkappa b, 0, T_{j-1}) e^{-i\varkappa y} \right] d\varkappa \tag{15}$$

In general, the integral in (15) can only be solved numerically. Note that this requires solving the Ricatti ODEs for $A_u(\tau)$ and $B_u(\tau)$ in (4) and (5) at each point $u = a + i\varkappa b$.

Empirical studies of cap pricing and hedging can be found in Li and Zhao (2006) and Jarrow, Li and Zhao (2007).

3 Kalman Filtering the State Vector

Consider the following general nonlinear state-space system

$$x_{t+1} = F\left(x_t, \epsilon_{t+1}\right), \tag{16}$$

and

$$y_t = G(x_t) + u_t \tag{17}$$

where y_t is a D-dimensional vector of observables, ϵ_{t+1} is the state noise and u_t is the observation noise that has zero mean and a covariance matrix denoted by R. In term structure applications, the transition function F is specified by the dynamic of the state vector and the measurement function G is specified by the pricing function of the fixed income securities being studied. In our application, the transition function F follows from the affine state vector dynamic in (1), y_t are the LIBOR, swap rates, and cap prices observed weekly for different maturities, and the function G is given by the pricing functions in (9), (10), and (11).

The transition equation (16) reflects the discrete time evolution of the state variables, whereas the measurement equation provides the mapping between the unobserved state vector and the observed variables. If $\{x_t, t \leq T\}$ is an affine diffusion process, a discrete expression of its dynamics is unavailable except for Gaussian processes. When the state vector is not Gaussian, one can obtain an approximate transition equation by exploiting the existence of the two first conditional moments in closed-form and replacing the original state vector with a new Gaussian state vector with identical two first conditional moments. While this approximation results in inconsistent estimates, Monte Carlo evidence shows that its impact is negligible in practice (see Duan and Simonato (1999) and de Jong (2000)).

For the models we are interested in, the conditional expectation of the state vector is an affine function of the state (see Appendix A for explicit expressions of the two first conditional moments). Using (1) and an Euler discretization, the transition equation (16) can therefore be rewritten as follows

$$x_{t+1} = F(x_t, \epsilon_{t+1}) = a + bx_t + \epsilon_{t+1}, \tag{18}$$

where $\epsilon_{t+1|t} \sim \mathcal{N}(0, v\left(x_t\right))$ and $v\left(x_t\right)$ is the conditional covariance matrix of the state vector.

Given that y_t is observed and assuming that it is a Gaussian random variable, the Kalman filter recursively provides the optimal minimum MSE estimate of the state vector. The prediction step consists of

$$x_{t|t-1} = a + bx_{t-1|t-1}, \tag{19}$$

$$P_{xx(t|t-1)} = bP_{xx(t-1|t-1)}b' + v\left(x_{t-1|t-1}\right) \tag{20}$$

$$K_t = P_{xy(t|t-1)} P^{-1}_{yy(t|t-1)}, \tag{21}$$

and

$$y_{t|t-1} = E_{t-1}\left[G(x_t)\right]. \tag{22}$$

The updating is done using

$$x_{t|t} = x_{t|t-1} + K_t\left(y_t - y_{t|t-1}\right), \tag{23}$$

and

$$P_{xx(t|t)} = P_{xx(t|t-1)} - K_t P_{yy(t|t-1)} K_t', \tag{24}$$

When G in (22) is a linear function, e.g. if the observations are zero-coupon yields, then the covariance matrices $P_{xy(t|t-1)}$ and $P_{yy(t|t-1)}$ can be computed exactly and the only approximation is therefore induced by the Gaussian transformation of the state vector used in (18). When the relationship between the state vector and the observation is nonlinear, as is the case when swap contracts, coupon bonds, or interest rate options are used, then $G(x_t)$ needs to be well approximated in order to obtain good estimates of the covariance matrices $P_{xy(t|t-1)}$ and $P_{yy(t|t-1)}$. The approximation of G is different for different implementations of the filter, which is the topic to which we now turn.

3.1 The Extended Kalman Filter

To deal with nonlinearity in the measurement equation, one can apply the extended Kalman filter (EKF), which relies on a first order Taylor expansion of the measurement equation around the predicted state $x_{t|t-1}$.[3] The measurement equation is therefore rewritten as follows

$$y_t = G(x_{t|t-1}) + J_t\left(x_t - x_{t|t-1}\right) + u_t, \tag{25}$$

where

$$J_t = \left.\frac{\partial G}{\partial x_t}\right|_{x_t = x_{t|t-1}}$$

denotes the Jacobian matrix of the nonlinear function $G(x_{t|t-1})$ computed at $x_{t|t-1}$.

The covariance matrices $P_{xy(t|t-1)}$ and $P_{yy(t|t-1)}$ are then computed as

$$P_{xy(t|t-1)} = P_{xx(t|t-1)} J_t, \tag{26}$$

and

$$P_{yy(t|t-1)} = J_t P_{xx(t|t-1)} J_t' + R. \tag{27}$$

[3] For applications of the extended Kalman filter see Chen and Scott (1995), Duan and Simonato (1999), and Duffee (1999).

The estimate of the state vector is then updated using (23), (24), and

$$K_t = P_{xx(t|t-1)} J_t P^{-1}_{yy(t|t-1)}. \tag{28}$$

3.2 The Unscented Kalman Filter

Unlike the extended Kalman filter, the unscented Kalman filter uses the exact nonlinear function $G(x_t)$ and does not linearize the measurement equation. Rather than approximating $G(x_t)$, the unscented Kalman filter approximates the conditional distribution of the x_t using the scaled unscented transformation (see Julier (2000) for more details), which can be defined as a method for computing the statistics of a nonlinear transformation of a random variable. Julier and Uhlmann (2004) prove that such an approximation is accurate to the third order for Gaussian states and to the second order for non-Gaussian states. It must also be noted that the approximation does not require computation of the Jacobian or Hessian matrices and that the computational burden associated with the unscented Kalman filter is not prohibitive compared to that of the extended Kalman filter. In our application below, the computation time for the unscented Kalman filter was on average twice that of the extended Kalman filter.

Consider the random variable x with mean μ_x and covariance matrix P_{xx}, and the nonlinear transformation $y = G(x)$. The basic idea behind the scaled unscented transformation is to generate a set of points, called sigma points, with the first two sample moments equal to μ_x and P_{xx}. The nonlinear transformation is then applied at each sigma point. In particular, the n_x-dimensional random variable is approximated by a set of $2n_x + 1$ weighted points given by

$$\mathcal{X}_0 = \mu_x, \tag{29}$$

$$\mathcal{X}_i = \mu_x + \left(\sqrt{(n_x + \xi)\, P_{xx}}\right)_i, \text{ for } i = 1, \cdots, n_x \tag{30}$$

$$\mathcal{X}_i = \mu_x - \left(\sqrt{(n_x + \xi)\, P_{xx}}\right)_i, \text{ for } i = n_x, \cdots, 2n_x \tag{31}$$

with weights

$$W_0^m = \frac{\xi}{(n_x + \xi)}, \quad W_0^c = \frac{\xi}{(n_x + \xi)} + \left(1 - \rho^2 + \gamma\right) \tag{32}$$

$$W_i^m = W_i^c = \frac{1}{2\,(n_x + \xi)}, \text{ for } i = 1, \cdots, 2n_x, \tag{33}$$

where $\xi = \rho^2\,(n_x + \chi) - n_x$, and where $\left(\sqrt{(n_x + \xi)\, P_{xx}}\right)_i$ is the ith column of the matrix square root of $(n_x + \xi)\, P_{xx}$. The scaling parameter $\rho > 0$ is intended to minimize higher order effects and can be made arbitrary small. The restriction $\chi > 0$ guarantees the positivity of the covariance matrix. The parameter $\gamma \geq 0$ can capture higher order moments of the state distribution; it is equal to two for the Gaussian distribution. The nonlinear transformation is applied to the sigma points (29)-(31)

$$\mathcal{Y}_i = G\left(\mathcal{X}_i\right), \text{ for } i = 0, \cdots, 2n_x.$$

The unscented Kalman filter relies on the unscented transformation to approximate the covariance matrices $P_{xy(t|t-1)}$ and $P_{yy(t|t-1)}$. The state vector is augmented with the state noise ϵ_t and the measurement noise u_t. With N state variables and D observables this results in the $n_a = (2N + D)$ dimensional vector

$$x_t^a = \left[x_t' \;\; \epsilon_t' \;\; u_t'\right]'. \tag{34}$$

The unscented transformation is applied to this augmented vector in order to compute the sigma points.

As shown by equations (30) and (31), the implementation of the unscented Kalman filter requires the computation of the square root of the variance-covariance matrix of the augmented state. There is no guarantee that the variance-covariance matrix will be positive definite. Positive definiteness of the variance-covariance matrix is also not guaranteed with the extended Kalman filter which in turn can affect its numerical stability. In the unscented case, a more stable algorithm is provided by the square-root unscented Kalman filter proposed by van der Merwe and Wan (2002). The basic intuition behind the square-root implementation of the unscented Kalman filter is to propagate and update the square-root of the variance-covariance matrix rather than the variance-covariance matrix itself.[4]

If we denote the square-root matrix of P by S, the square-root implementation of the unscented Kalman filter can be summarized by the following algorithm:

0. Initialize the algorithm with unconditional moments

$$x_{0|0}^a = \left[E\left[x_t\right]' \;\; 0 \;\; 0\right]' \qquad S_{xx(0|0)} = S_{\epsilon\epsilon(0|0)} = \sqrt{var\left[x_t\right]} \qquad S_{uu(0|0)} = \sqrt{R}$$

1. Compute the $2n_a + 1$ sigma points:

$$\mathcal{X}_{t-1|t-1}^a = \left[x_{t-1|t-1}^a \;\; x_{t-1|t-1}^a \pm \sqrt{(n_a + \xi)} S_{t-1|t-1}^a\right],$$

where $S_{t-1|t-1}^a$ is the block diagonal matrix with $S_{xx(t-1|t-1)}$, $S_{\epsilon\epsilon(t-1|t-1)}$, and $S_{uu(t-1|t-1)}$ on its diagonal

[4] While the square-root implementation of the unscented Kalman filter is more stable numerically, its computational complexity is similar to that of the original unscented Kalman.

2. Prediction step:

$$\mathcal{X}^x_{t|t-1} = a + b\mathcal{X}^x_{t-1|t-1} + \mathcal{X}^\epsilon_{t-1|t-1}\,, \qquad x_{t|t-1} = \sum_{i=1}^{2n_a+1} W_i^m \mathcal{X}^x_{i,t|t-1}\,,$$

$$S_{xx(t|t-1)} = \text{CHOLUPDATE}\left\{\text{QR}\left\{\sqrt{W_1^c}\left(\mathcal{X}^x_{2:2n_a+1,t|t-1} - x_{t|t-1}\right)\ \sqrt{v\left(x_{t|t-1}\right)}\right\}, \mathcal{X}^x_{1,t|t-1} - x_{t|t-1}, W_0^c\right\}\,,$$

$$\mathcal{Y}_{t|t-1} = G\left(\mathcal{X}^x_{t|t-1}\right) + \mathcal{X}^u_{t|t-1}\,, \qquad \text{and} \quad y_{t|t-1} = \sum_{i=1}^{2n_a+1} W_i^m \mathcal{Y}_{i,t|t-1},$$

where (i) $\text{QR}\{A\}$ returns the Q matrix from the 'QR' orthogonal-triangular decomposition of A, and (ii) $S_{cu} = \text{CHOLUPDATE}\{S, z, \pm\nu\}$ is a rank-1 update (or downdate) to Cholesky factorization: assuming that S is the Cholesky factor of P, then S_{cu} is the Cholesky factor of $P \pm \nu zz'$. If z is a matrix, the update (or downdate) is performed using the columns of z sequentially.

3. Measurement update:

$$P_{xy(t|t-1)} = \sum_{i=1}^{2n_a+1} W_i^c \left[\mathcal{X}^x_{i,t|t-1} - x_{t|t-1}\right]\left[\mathcal{Y}_{i,t|t-1} - y_{t|t-1}\right]'$$

$$S_{yy(t|t-1)} = \text{CHOLUPDATE}\left\{\text{QR}\left\{\sqrt{W_1^c}\left(\mathcal{Y}_{2:2n_a+1,t|t-1} - y_{t|t-1}\right)\ \sqrt{R}\right\}, \mathcal{Y}_{1,t|t-1} - y_{t|t-1}, W_0^c\right\}$$

where R is the variance of the measurement error. Then

$$K_t = \left(P_{xy(t|t-1)}/S'_{yy(t|t-1)}\right)/S_{yy(t|t-1)}, \qquad x_{t|t} = x_{t|t-1} + K_t\left(y_t - y_{t|t-1}\right)\,,$$

$$S_{xx(t|t)} = \text{CHOLUPDATE}\left\{S_{xx(t|t-1)}, K_t S_{yy(t|t-1)}, -1\right\} \quad \text{and} \quad S_{\epsilon\epsilon(t|t)} = \sqrt{v(x_{t|t})}\,,$$

where '/' denotes the efficient least-squares solution to the problem $Ax = b$. The estimate of the state vector is then updated using equations (23) and (24).

4 Monte Carlo Analysis

The objective of this Monte Carlo study is to evaluate the performance of the UKF and EKF for filtering states as well as for fitting and forecasting fixed income security rates and prices. We want to focus on the numerical stability of each filtering algorithm and the potential biases caused by a linear approximation of LIBOR, the swap rates, and especially the cap prices, which are very nonlinear in the states. By design, the comparison below is not affected by the issue of parameter estimation as we keep the parameters fixed at their true values throughout.

4.1 Monte Carlo Design

Table 1 summarizes the parameter values used in the Monte Carlo experiment. Empty entries represent parameters that have to be set to zero for the purpose of model identification. The grey-shaded entries represent parameters that are set to zero to obtain closed-form solutions for the Ricatti ODEs. Our application involves pricing the cap contracts a large number of times for each model, and pricing each caplet requires solving the Ricatti ODEs at every integration point in (15). Numerically solving the Ricatti ODEs in this setup is therefore prohibitively expensive computationally. We reduce the computational burden by following Ait-Sahalia and Kimmel (2010) and restrict some of the model parameters in order to obtain closed-form solutions for the Ricatti ODEs. These restrictions are such that the models are specified with $N-M$ correlated Gaussian processes that are uncorrelated with M square-root processes, which are uncorrelated among themselves.[5]

With some exceptions, the parameters in Table 1 are from Table 8 in Ait-Sahalia and Kimmel (2010). The exceptions, all motivated by numerical considerations, are as follows. The δ_0 parameter is set to 3% for all models. Ait-Sahalia and Kimmel (2010) find zero values for the κ_{11} parameter in the $A_1(3)$ and $A_2(3)$ models. To enhance numerical stability in the matrix inversions we have shifted these parameter values by one standard error. Ait-Sahalia and Kimmel (2010) also have zero values for the mean θ_j of some of the volatility factors. We set these parameters equal to 5%, which enhances the stability of the simulations. Note that these changes result in parameters estimates that are well inside the estimated confidence intervals in Ait-Sahalia and Kimmel (2010) at conventional significance levels.

Our choice of parameters does not materially affect our findings below. In each case, we have chosen conservative parameterizations that reduce the nonlinearities of bond and cap prices as a function of the state variables vis-a-vis other parameterizations. As a result, for most realistic parameterizations, the benefits of the proposed method will generally be more substantial than what we document.

For each canonical model, we simulate 500 samples of LIBOR rates with maturities of 3 and 6 months, swap rates with maturities of 1, 2, 5, 7, and 10 years, and interest rate caps on 3-month LIBOR with maturities of 1 and 5 years. Each sample contains 260 weekly observations. We use an Euler discretization of the stochastic differential equation of the state vector and divide the week into 100 time steps. Weekly observations are then extracted by taking the 100th observation within each week. We constrain the volatility factors to be above 0.1%, and we constrain the factors to ensure that spot rates do not fall below 25 bps.

[5] Restrictions on the parameters in ATSMs are also imposed in Balduzzi, Das, Foresi and Sundaram (1996), Chen (1996), and Dai and Singleton (2000).

For each simulated sample, the filtered values of the unobserved state variables and the rate or price implied by each filter are compared to the simulated data under various scenarios. The unscented Kalman filter is implemented with the rescaling parameters $\rho = 1$, $\chi = 0$, and $\gamma = 2$.[6]

Figure 1 shows the unconditional term structure of interest rates implied by the parameters in Table 1 for each of the four models. Note that the parameters generate quite different term structures across models, and note in particular that the implied term structure for the $A_3(3)$ model is fairly flat.

4.2 State Vector Extraction

We begin by assessing the ability of each filter to accurately extract the unobserved path of the state variables. For each simulated time series, we compare the filtered state variables implied by each method to the actual state observations. As a gauge for the goodness of fit, we calculate the root mean squared error (RMSE) over 260 weeks for each Monte Carlo sample as follows

$$RMSE_k^F(i) = \sqrt{\frac{1}{260}\sum_{t=1}^{260}\left(x_{i,k,t} - x_{i,k,t}^F\right)^2}$$

where $x_{i,k,t}$ denotes the true but unobserved state variable i in sample k at time t. The filtered state variable is denoted by $x_{i,k,t}^F$ where F is either EKF or UKF. For each model we compute the RMSE for each state variable and on each of 500 Monte Carlo samples.[7]

Tables 2 and 3 provide the mean, median, and standard deviation of the RMSE for each state variable across the 500 simulated time series. For example, for the mean RMSE we compute

$$Mean(RMSE_k^F(i)) = \frac{1}{500}\sum_{k=1}^{500} RMSE_k^F(i).$$

Panel A presents results for the $A_0(3)$ model, Panel B for the $A_1(3)$ model, Panel C for the $A_2(3)$ model, and Panel D for the $A_3(3)$ model. Table 2 provides results for the case where state filtering is done without caps, while Table 3 uses cap prices to extract the state variables. Our prior is that because the cap prices are more nonlinear in the states, the relative performance of the UKF versus the EKF should further improve in Table 3 compared

[6] We choose $\gamma = 2$ because it implies a Gaussian state. This assumption therefore induces a bias in our implementation of the UKF which is identical to the EKF bias. This ensures that our comparison is focused on the implications of nonlinearities in the measurement equation.

[7] To investigate if sample length affects our conclusions, we also conducted two smaller experiments using 520 and 1300 weeks instead of 260. If anything, the UKF outperforms the EKF by an even wider margin.

with Table 2. Tables 2 and 3 include estimates of the interquartile range (IQR) for the true but unobserved state variable. This measure is intended to help gauge the magnitudes of the RMSEs, which of course are a function of the magnitude of the state variables.

Table 2 clearly shows that the UKF is more successful than the EKF in filtering the state variables no matter which metric is used. The ratio of the median RMSE from the UKF to that from the EKF is substantially lower than one in most cases, and is below 10% in several instances. Comparing the RMSEs to the state IQR shows that the differences in RMSE across filters are large in relation to the magnitude of the state variables as well.

The UKF does not outperform the EKF for the first and second factor of the $A_3(3)$ model. This may be due to the fact that the term structure for the $A_3(3)$ model is flat, as seen in Figure 1, or perhaps to the fact that in this model the loading on the second factor is very small, as indicated in Table 1. These features may make the unobserved states harder to identify.

Table 3 provides results for the case where caps are included in filtering. As expected, the superior performance of the UKF compared with EKF is even stronger. Excluding the second factor in the $A_3(3)$ model, the ratio of the median RMSE implied by the UKF to that implied by the EKF now ranges between 3% and 23%. In other words, the median RMSE for the EKF is in some cases 33 times higher than that implied by the UKF.

Looking across Tables 2 and 3, it is also noteworthy that for both cases the standard deviation of the RMSE is an order of magnitude larger for the EKF compared to the UKF. This is a clear indication that the UKF is numerically more stable. Note that the Mean($RMSE$) and Median($RMSE$) values in Tables 2 and 3 are quite close showing that while the EKF is less stable than the UKF the Mean($RMSE$) results are not driven by a few outliers: The Median($RMSE$) ratios across the two filters favor the UKF just as much as the Mean($RMSE$) ratios do.

Figures 2 to 5 provide further insight in these results by showing scatter plots of filtered state variables $x_{i,k,t}^F$ versus actual state variables $x_{i,k,t}$, using all $130,000$ observations across 260 weeks and 500 samples. Each row of panels shows results for a different state variable. The two left-most columns of panels in each figure show the case where caps are excluded from the filtering algorithm, while the two right-side columns are obtained by also including caps in the cross-section of observed securities. The left-side panels clearly indicate the outperformance of the UKF over the EKF. The EKF delivers much less reliable estimates of the state variables; importantly it is also numerically much less stable, as evidenced by the high number of outliers in the scatter plots. The right-side panels of Figures 2-5 also confirm the superiority of the UKF in dealing with securities that are highly nonlinear in the state variables. The EKF is even more unstable compared to the left-side panels where caps are not included in the analysis.

Comparing Tables 2 and 3, it is clear that for the EKF, the state RMSEs are dramatically larger when caps are used in filtering; mean and median RMSEs associated with the EKF in Table 3 are systematically larger than in Table 2. As discussed earlier, the EKF performs a first order Taylor expansion around the predicted state variables. The quality of the EKF filtering thus crucially depend on the numerical gradient used in this first order approximation. The increased RMSEs demonstrate that when the highly nonlinear caps are used, the gradient offers a very poor approximation of the impact that variations in states have on the measurement equation. The UKF does not linearize the measurement equation. As a consequence, the UKF's mean and median RMSEs are systematically smaller in Table 3 than in Table 2.

The results in Tables 2 and 3 are quite striking. The UKF is able to incorporate the additional information contained in caps to extract the underlying states more precisely. The EKF, however, actually suffers from the additional information in caps because of the linearization required.

This initial Monte Carlo exercise leaves little doubt that the UKF is much superior in filtering the state variables than the EKF. The UKF's relative performance versus the EKF further improves when the securities used in the analysis are more nonlinear in the state variables. The UKF's outperformance is of course model-dependent, for the obvious reason that different models and different model parameterizations imply different degrees of nonlinearity. Most notably in our analysis, the chosen parameterization for the $A_3(3)$ model is not very nonlinear, and as a result the UKF does not offer many advantages in this case. We have experimented with other parameterizations of the models, and the results indeed depend on the degree on nonlinearity in the parameterizations chosen.

Based on Tables 2-3 and Figures 2-5 our first main conclusion is that for realistic parameterizations implying sensible amounts of nonlinearity in the state variables and realistic term structures of LIBOR and swap rates, the UKF will improve drastically on the EKF in terms of capturing the dynamics of the state vector.

4.3 Implications for Rates and Prices

In order to assess the economic implications of the two filtering methods we now investigate the filters' ability to match observed LIBOR and swap rates as well as cap prices. We compare the fitted LIBOR, swap rates, and cap prices implied by states from each filter to the actual rates and prices computed from the true states.

Tables 4 and 5 compare the security prices implied by the filtered states to the simulated ones. We provide the RMSEs as well as the Bias based on the true rates and prices and the filtered ones. For each security the RMSE for filter F is computed as

$$RMSE^F = \frac{1}{500}\sum_{k=1}^{500} RMSE_k^F = \frac{1}{500}\sum_{k=1}^{500}\sqrt{\frac{1}{260}\sum_{t=1}^{260}\left(y_{k,t} - y_{k,t}^F\right)^2}$$

where $y_{k,t}$ is the true price or rate in sample k at week t, $y_{k,t}^F$ is the value obtained using a filtered state vector and F is either UKF or EKF. Bias is defined by

$$Bias^F = \frac{1}{130,000}\sum_{k=1}^{500}\sum_{t=1}^{260}\left(y_{k,t} - y_{k,t}^F\right)$$

In order to judge the magnitudes Tables 4 and 5 also report the interquartile range of the observed rates and prices. All estimates are in basis points. Table 4 provides results when cap prices are not used in the filtering step, while Table 5 provides results for the case when cap prices are used to filter the states.

Table 4 indicates that the UKF typically results in a dramatically lower RMSE compared to the EKF. The degree of outperformance is model-dependent and security-dependent. It is generally modest for swap rates, more substantial for LIBOR, and in several cases spectacular in the case of caps. For example, the RMSE on the five-year cap is roughly ten times higher for the EKF compared to the UKF. We conclude that the benefits of the UKF become even more pronounced when pricing nonlinear securities that are not used in state vector filtering. As in Table 2 the improvement of UKF is less dramatic in the $A_3(3)$ model in Table 4 where the nonlinearities are less pronounced.

A large RMSE can arise from either variance or from bias in the filter. Table 4 therefore reports the rate and price bias in addition to the RMSE. Note that bias is generally small for both filters for the LIBOR and swap rates. However, for the cap prices the EKF in many cases contains a strong positive bias meaning that the use of the EKF results in an underpricing of caps.

Table 5 indicates that the superior performance of the UKF versus the EKF remains intact when caps are also included in state filtering. Compared to Table 4, the RMSE implied by the UKF is substantially smaller for cap prices, and slightly larger overall for LIBOR and swap rates. This result is not surprising because the states filtered on all securities represent a compromise between fitting rates and cap prices. Interestingly, the performance of the EKF relative to Table 4 deteriorates for all securities.

In terms of bias, the EKF again tends to underprice caps in Table 5, as was the case in Table 4. But note further that when caps are used in filtering, the underpricing of securities is more widespread. The EKF has a positive bias (underprices) for all securities in the $A_0(3)$, $A_1(3)$, and $A_2(3)$ models. In the much less nonlinear $A_3(3)$ model the bias is less apparent.

Our second main conclusion is that the UKF's improvement over the EKF in extracting states carries over to improvements in securities pricing. Furthermore, while adding nonlinear securities generally improves the performance of the nonlinear UKF filter in state vector extraction, the economic benefits are not evenly distributed across securities. However, the benefits are clear for the pricing of highly nonlinear securities. In contrast, the linearized EKF filter actually performs worse in pricing securities when states have been extracted using nonlinear instruments such as caps.

4.4 Dynamic Implications: Rate and Price Forecasts

Dynamic term structure models are used not only for the valuation of securities at present but also to forecast future rates and prices (see for example Backus, Foresi, Mozumdar and Wu (2001) and Egorov, Hong and Li (2006)). However, the usefulness of the model depends crucially on the accuracy of the state vector filter.

Tables 6-9 summarize the performance of the UKF and EKF for predicting LIBOR, swap rates, and cap prices for various forecasting horizons in each of our four models. For each security we compute the forecast RMSE for each horizon, h, defined by

$$RMSE_k^F(h) = \sqrt{\frac{1}{260-h}\sum_{t=1}^{260-h}\left(y_{k,t+h} - y_{k,t+h|t}^F\right)^2}$$

where $y_{k,t+h|t}^F$ is the price or rate of the security computed using the filter-dependent h-week ahead state vector forecast, $x_{k,t+h|t}^F$. In Tables 6-9, Panel A reports Mean$\left(RMSE_k^F(h)\right)$, Panel B reports Median$\left(RMSE_k^F(h)\right)$, and Panel C reports Stdev$\left(RMSE_k^F(h)\right)$. The moments are computed for $h = 1$, 4 and 12 week horizons across the 500 samples denoted by k. The left-side and right-side panels respectively show results obtained without and with the use of caps in filtering.

Tables 6-9 confirm the conclusions from the contemporaneous fit in Tables 4-5: the UKF significantly outperforms the EKF. The improvement is largest at shorter horizons (1 and 4 weeks). When considering the right-side panels where the states are filtered using LIBOR, swap rates and caps, the relative performance of the EKF deteriorates compared to the left-side panels; this confirms the EKF's problems in dealing with securities that are highly nonlinear in the states. The magnitude of the improvement is smallest in the case of the $A_3(3)$ model. As previously discussed, this is not due to the nature of the $A_3(3)$ model, but rather to the parameterization in Table 1, which determines the extent of nonlinearity in the states for each model.

Figures 6-9 provides more perspective by scatter plotting the 500 individual $RMSE_k^{UKF}(h)$ on the y-axis against the corresponding $RMSE_k^{EKF}(h)$ on the x-axis for the one-week forecast horizon ($h = 1$). The UKF outperforms the EKF when the plots fall below the 45-degree line. Figures 6-9 are quite striking. Note that there are virtually no observations above the 45-degree line. These figures provide a more visual and intuitive assessment of the performance of both filtering methods. The figures confirm that the UKF-implied forecasts substantially outperform the EKF-implied forecasts.

Our third main conclusion is that the UKF generally delivers forecast RMSEs that are much lower than those obtained using the EKF.

4.5 Implications for Long-Maturity Caps

So far we have run two versions of each Monte Carlo experiment for each filter: One where only LIBOR and swap rates are used in filtering, and another one where in addition 1-yr and 5-yr caps are used. In Tables 4-9 we used the models to price the same securities used in filtering.

We now instead consider an application of the term structure models in which 7-yr caps must be priced. These contracts have not been used in any of the filters when extracting states. We restrict attention to contemporaneous pricing just as in Tables 4-5.

Table 10 contains the results. We again compute pricing RMSE and Bias computed from the true rates and prices as well as the extracted rates and prices obtained from each filter. In Panel A we report the results for the EKF and UKF when states are filtered on only LIBOR and swap rates. Note the dramatically lower RMSE for the UKF compared to the EKF for the first three models, even when caps are not used in filtering. For the much less nonlinear $A_3(3)$ model the RMSEs are similar. As in Table 4, the EKF underprices all caps. The bias is dramatic in the case of the $A_2(3)$ model.

In Panel B of Table 10 we report on 7-yr cap pricing when states are filtering using 1-yr and 5-yr caps as well as the LIBOR and swap rates. Not surprisingly, the UKF outperforms the EKF in this case. What is perhaps surprising is the magnitude of the outperformance. The RMSE for the EKF is more than 3 times higher than UKF for the $A_3(3)$ model, 17 times higher for the $A_0(3)$ model, 24 times higher for the $A_1(3)$ model, and 25 times higher for the $A_2(3)$ model. As in Table 5, the EKF systematically underprices caps and the biases can be large.

Our fourth main conclusion is that the UKF outperforms the EKF for the pricing of nonlinear securities even when these have not been used in filtering the underlying state vectors.

5 Discussion: Parameter Estimation

Our Monte Carlo analysis has deliberately kept the structural parameters fixed at their true values. Our analysis of caps makes our Monte Carlo investigation extremely intensive numerically, and we therefore leave the details of parameter estimation for future research, but our analysis does raise important questions on the potential effect of the choice of filter on parameter estimation, and we now provide some general remarks.

The literature contains a large number of empirical methods that can be used to estimate multifactor affine models, including indirect inference, simulated method of moments (SML), and the efficient method of moments (EMM). Most papers use either quasi maximum likelihood or the Kalman filter with a likelihood based criterion.[8] These techniques are popular because they are easier to implement and because Duffee and Stanton (2004) demonstrate in an extensive Monte Carlo experiment that these two methods outperform more complex estimation techniques (EMM and SML).

The QML estimator has the drawback that an ad hoc choice has to be made that the pricing relationship holds without error for certain bonds, which complicates the comparison with the model fit implied by the unscented Kalman filter. Another drawback of QML is that it does not offer any guidance as to how to forecast nonlinear instruments once the state variables are obtained through inversion. Monte Carlo simulation can be used to compute the forecast, but the ensuing computational cost can be significant, especially for multifactor models. See Ait-Sahalia and Kimmel (2010) for a more recent MLE approach.

A nonlinear least squares technique can be used to minimize the following loss function with respect to the parameters of the term structure model

$$MSE = \frac{1}{T}\sum_{t=1}^{T}\left(y_t - y_t^F\right)'\left(y_t - y_t^F\right).$$

where y_t^F is obtained using either the UKF or the EKF as described above. We have found the UKF to be vastly superior to the EKF for filtering purposes and expect the same to be true for estimation purposes.

Following Almeida (2005), principal component analysis can be used to provide intuition for the impact of the EKF on parameter estimation. Denoting the principal components of

[8] See for example Babbs and Nowman (1999), Chen and Scott (1995), Dai and Singleton (2000), Duan and Simonato (1999), Duffee (1999), Duffie and Singleton (1997), and Pearson and Sun (1994). See Thompson (2008) for an alternative approach using Bayesian filtering.

y_t by pc_t, the relationship between the principal components and the state variables x_t is as follows:

$$\Psi pc_t = G(x_t) + \overline{u},$$

where Ψ is the matrix of eigenvectors of the covariance matrix of y_t, and $\overline{u}$ is the sample average of $y_t - G(x_t)$.

Clearly the state variables are related to the principal components via the function G. When this function is linearized as with the EKF, the state vector becomes a linear transformation of the principal components, and therefore the rotation imposed on the state variables changes their statistical properties. Consider for instance the case where the first principal component of the non-linear instrument used in the estimation is very persistent. A linearization of the measurement equation forces the corresponding state variable to inherit the persistence even if the true unobserved state variable is not persistent. This simple analysis also highlights an important difference between linear and non-linear securities. In a linear set-up, the state variables inherit the time series properties from the principal components, hence the labeling of the state variables as level, slope and curvature when zero-coupon yields are used. This is not the case with non-linear securities.

Hence, a potential danger in using the extended Kalman filter is that it can create a significant bias in the parameters that govern the dynamics of the state variables. For instruments that are highly nonlinear in the states variables, like interest rate caps, this problem may be aggravated by poor identification of the latent state variables. Indeed, as highlighted by our results, for highly nonlinear functions G, the Jacobian matrix will provide a poor approximation of the impact of the state variables on the evolution of the observables. Poor estimates of the current state together with biased parameters may therefore cause poor performance of the extended Kalman filter (Julier and Uhlmann (2004)).

While several studies have shown that the approximation of the transition equation for non-Gaussian state variables does not imply large biases, the literature does not contain an assessment of the bias resulting from the use of the extended Kalman filter for nonlinear securities such as swap contracts and interest rate derivatives. To the best of our knowledge, the only paper that addresses the nonlinear mapping between the state variables and the observations in affine term structure models is Lund (1997), who uses the iterative extended Kalman filter (see also Mohinder and Angus, 2001). However, the analysis is limited to the single factor Vasicek (1977) model and no comparison is provided with the traditional extended Kalman filter.

Our Monte Carlo experiments show that the extended Kalman filter is ill-suited to optimally exploit the rich information content of securities that are nonlinear in the state variables. We propose the unscented Kalman filter as an alternative to address the nonlinearity in the measurement equation. Our findings on filtering strongly suggest that the UKF will be superior for the purpose of parameter estimation as well.

6 Conclusion

The extended Kalman filter has become the standard tool to analyze a number of important problems in financial economics, and in term structure modeling in particular. While there is no need to look beyond the extended Kalman filter for some term structure applications, it is not clear how well the method performs for many situations of interest, when the measurement equation is nonlinear in the state variables. Examples include the pricing of fixed income derivatives such as caps, floors and swaptions, as well as modeling the cross section of swap yields. The unscented Kalman filter is moderately more costly from a computational perspective, but better suited to handling these nonlinear securities.

We use an extensive Monte Carlo experiment to investigate the relative performance of the extended and unscented Kalman filter. We study three-factor affine term structure models for LIBOR and swap rates, which are mildly nonlinear in the underlying state variables, and cap prices, which are highly nonlinear. We find that the filtering performance of the unscented Kalman filter is much superior to that of the extended Kalman filter. It filters the states more accurately, which leads to improved security prices and forecasts. These results obtain for cap prices as well as for swap rates, regardless of whether caps are used in estimation.

Our results demonstrate the usefulness of the unscented Kalman filter for problems where the relationship between the state vector and the observations is either mildly nonlinear or highly nonlinear. The results therefore suggest that the UKF may prove to be a good approach for implementing term structure models in a wide variety of applications, including the estimation of term structure models using interest rate derivatives, the estimation of nonlinear term structure models such as quadratic models, and the estimation of models of default risk, such as coupon bonds or credit default swaps. The unscented Kalman filter may also prove useful to estimate other types of term structure models, such as the unspanned stochastic volatility models of Collin-Dufresne and Goldstein (2002), and Collin-Dufresne, Goldstein and Jones (2009).

Appendix: Conditional Moments of the State Vector

We compute explicit expressions for the two first conditional moments following Fackler (2000) who extends the formula provided by Fisher and Gilles (1996).

A.1 Conditional Expectation

The integral form of the stochastic differential equation (1) under the actual probability measure $\mathcal{P}$ is

$$x_{t+\tau} = x_t + \int_t^{t+\tau} \kappa\left(\theta - x_u\right) du + \int_t^{t+\tau} \Sigma\sqrt{S_u} dW_u. \tag{A.1}$$

Applying the Fubini theorem, we get

$$E_t\left[x_{t+\tau}\right] = x_t + \int_t^{t+\tau} \kappa\left(\theta - E_t\left[x_u\right]\right) du.$$

Differentiating with respect to τ implies the following ODE

$$\frac{dE_t\left[x_{t+\tau}\right]}{d\tau} = \kappa\theta - \kappa E_t\left[x_{t+\tau}\right], \tag{A.2}$$

with the initial condition $E_t\left[x_t\right] = x_t$.

The solution to this ODE has the following form

$$E_t\left[x_{t+\tau}\right] = a\left(t,\tau\right) + b\left(t,\tau\right) x_t. \tag{A.3}$$

Using (A.1) for identification yields the following ODEs

$$\frac{\partial a\left(t,\tau\right)}{\partial \tau} = \kappa\theta - \kappa a\left(t,\tau\right) \tag{A.4}$$

and

$$\frac{\partial b\left(t,\tau\right)}{\partial \tau} = -\kappa b\left(t,\tau\right), \tag{A.5}$$

with the initial conditions $a\left(t,\tau\right) = 0$ and $b\left(t,\tau\right) = I_N$.

If the matrix κ is non-singular, the solution of equations (A.4) and (A.5) are

$$a\left(t,\tau\right) = \left(I_N - \exp\left(-\kappa\tau\right)\right)\theta \text{ and } b\left(t,\tau\right) = \exp\left(-\kappa\tau\right),$$

where $\exp\left(-\kappa\left(\tau - t\right)\right)$ is given by the power series

$$\exp\left(-\kappa\tau\right) = I - \tau\kappa + \frac{\tau^2}{2!}\kappa^2 + \cdots.$$

Combining these expressions with (A.3), we get

$$E_t\left[x_{t+\tau}\right] = \left(I_N - \exp\left(-\kappa\tau\right)\right)\theta + \exp\left(-\kappa\tau\right) x_t. \tag{A.6}$$

Notice that if the eigenvalues of the matrix κ are strictly positive, then

$$\lim_{\tau\to\infty} \exp\left(-\kappa\tau\right) = 0,$$

and the unconditional expectation of $x_{t+\tau}$ is given by $E\left[x_t\right] = \theta$.

A.2 Conditional Variance

Applying Itô's lemma to (A.6) yields

$$dE_t\left[x_{t+\tau}\right] = b\left(t,\tau\right)\Sigma\sqrt{S_t}dW_t,$$

or equivalently

$$x_{t+\tau} = E_t\left[x_{t+\tau}\right] + \int_t^{t+\tau} b\left(u, t+\tau-u\right)\Sigma\sqrt{S_u}dW_u.$$

Under some technical conditions (see Neftci (1996))

$$\begin{aligned} var_t\left[x_{t+\tau}\right] &= var_t\left[\int_t^{t+\tau} b\left(u,t+\tau-u\right)\Sigma\sqrt{S_u}dW_u\right] \\ &= E_t\left[\int_t^{t+\tau} b\left(u,t+\tau-u\right)\Sigma S_u \Sigma b\left(u,t+\tau-u\right)^\top du\right] \\ &= \int_t^{t+\tau} b\left(u,t+\tau-u\right)\Sigma diag\left(\alpha+\mathcal{B}E_t\left[x_u\right]\right)\Sigma b\left(u,t+\tau-u\right)^\top du. \end{aligned} \quad \text{(A.7)}$$

Following Fackler (2000), the vectorized version of (A.7) is

$$vec\left(var_t\left[x_{t+\tau}\right]\right) = \int_t^{t+\tau}\left(b\left(u,t+\tau-u\right)\otimes b\left(u,t+\tau-u\right)\right)\left(\Sigma\otimes\Sigma\right)\mathcal{D}\left(\alpha+\mathcal{B}E_t\left[x_u\right]\right)du, \quad \text{(A.8)}$$

where $\otimes$ denotes the Kronecker product operator and $\mathcal{D}$ is a $n^2 \times n$ matrix such that

$$\mathcal{D}_{ij} = \begin{cases} 1 & \text{if } i = \left(j-1\right)n + j, \\ 0 & \text{otherwise.} \end{cases} \quad \text{(A.9)}$$

In the case of a 3-factor model, $\mathcal{D}$ is

$$\mathcal{D} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}'. \quad \text{(A.10)}$$

Using (A.6), expression (A.8) can be rearranged as follows

$$vec\left(var_t\left[x_{t+\tau}\right]\right) = v_0(t,\tau) + v_1(t,\tau)x_t, \quad \text{(A.11)}$$

where

$$v_0(t,\tau) = \int_t^{t+\tau}\left(b\left(u,t+\tau-u\right)\otimes b\left(u,t+\tau-u\right)\right)\left(\Sigma\otimes\Sigma\right)\mathcal{D}\left(\alpha+\mathcal{B}a\left(t,u-t\right)\right)du \quad \text{(A.12)}$$

and

$$v_1(t,\tau) = \int_t^{t+\tau}\left(b\left(u,t+\tau-u\right)\otimes b\left(u,t+\tau-u\right)\right)\left(\Sigma\otimes\Sigma\right)\mathcal{D}\mathcal{B}b\left(t,u-t\right)du. \quad \text{(A.13)}$$

Differentiating (A.12) and (A.13) with respect to τ yields the following ODE's

$$\frac{\partial v_0(t,\tau)}{\partial \tau} = (\Sigma \otimes \Sigma)\,\mathcal{D}\,(\alpha + \mathcal{B}a\,(t,\tau)) - (\kappa \otimes I_N + I_N \otimes \kappa)\,v_0(t,\tau), \tag{A.14}$$

and

$$\frac{\partial v_1(t,\tau)}{\partial \tau} = (\Sigma \otimes \Sigma)\,\mathcal{D}\mathcal{B}b\,(t,\tau) - (\kappa \otimes I_N + I_N \otimes \kappa)\,v_1(t,\tau) \tag{A.15}$$

Combining these ODEs with equations (A.4) and (A.5), we get the following two systems of ODE's

$$\begin{bmatrix} \frac{\partial a(t,\tau)}{\partial \tau} \\ \frac{\partial v_0(t,\tau)}{\partial \tau} \end{bmatrix} = \Theta - \boldsymbol{\kappa} \begin{bmatrix} a\,(t,\tau) \\ v_0(t,\tau) \end{bmatrix}, \tag{A.16}$$

and

$$\begin{bmatrix} \frac{\partial b(t,\tau)}{\partial \tau} \\ \frac{\partial v_1(t,\tau)}{\partial \tau} \end{bmatrix} = -\boldsymbol{\kappa} \begin{bmatrix} b\,(t,\tau) \\ v_1(t,\tau) \end{bmatrix}, \tag{A.17}$$

where

$$\Theta = \begin{bmatrix} \kappa\theta \\ (\Sigma \otimes \Sigma)\,\mathcal{D}\alpha \end{bmatrix} \tag{A.18}$$

and

$$\boldsymbol{\kappa} = \begin{bmatrix} \kappa & 0 \\ -(\Sigma \otimes \Sigma)\,\mathcal{D}\mathcal{B} & (\kappa \otimes I_N + I_N \otimes \kappa) \end{bmatrix}. \tag{A.19}$$

The initial conditions are $a\,(t,\tau) = 0$, $b\,(t,\tau) = I_N$, $v_0(t,0) = 0$ and $v_1(t,0)$. Provided that κ is nonsingular, the solution to these two systems is given by

$$\begin{bmatrix} a\,(t,\tau) \\ v_0(t,\tau) \end{bmatrix} = \left(I_{N(N+1)} - \exp\left(-\boldsymbol{\kappa}\tau\right)\right)\boldsymbol{\kappa}^{-1}\Theta, \tag{A.20}$$

and

$$\begin{bmatrix} b\,(t,\tau) \\ v_1(t,\tau) \end{bmatrix} = \exp\left(-\boldsymbol{\kappa}\tau\right) \begin{bmatrix} I_N \\ 0 \end{bmatrix}, \tag{A.21}$$

where $\exp\left(-\kappa\tau\right)$ is given by the power series

$$\exp\left(-\kappa\tau\right) = I - \tau\kappa + \frac{\tau^2}{2!}\kappa^2 + \cdots. \tag{A.22}$$

Since $\boldsymbol{\kappa}^{-1}$ can be written as

$$\begin{bmatrix} \kappa^{-1} & 0 \\ (\kappa \otimes I_N + I_N \otimes \kappa)^{-1}(\Sigma \otimes \Sigma)\,\mathcal{D}\mathcal{B}\kappa^{-1} & (\kappa \otimes I_N + I_N \otimes \kappa)^{-1} \end{bmatrix}. \tag{A.23}$$

If we assume that the eigenvalues of $\boldsymbol{\kappa}$ are strictly positive, then $\lim_{\tau\to\infty}\exp\left(-\boldsymbol{\kappa}\tau\right) = 0$ and the unconditional vectorized variance is

$$\begin{aligned} vec\,(var\,[x_t]) &= \lim_{\tau\to\infty} v_0(t,\tau) \\ &= (\kappa \otimes I_N + I_N \otimes \kappa)^{-1}(\Sigma \otimes \Sigma)\,\mathcal{D}\,(\mathcal{B}\theta + \alpha)\,. \end{aligned} \tag{A.24}$$

Computing the first two conditional moments involves evaluating the power series (A.22). Several methods for evaluating the exponential of a matrix are provided in the literature, see for example Moler and Van Loan (1978). As pointed out by Fackler (2000), the eigenvalues decomposition, suggested by Fisher and Gilles (1996) and used by Duffee (2002), and the Padé approximation yield good results in this particular context. We use the Padé approximation to compute the conditional expectation and variance.

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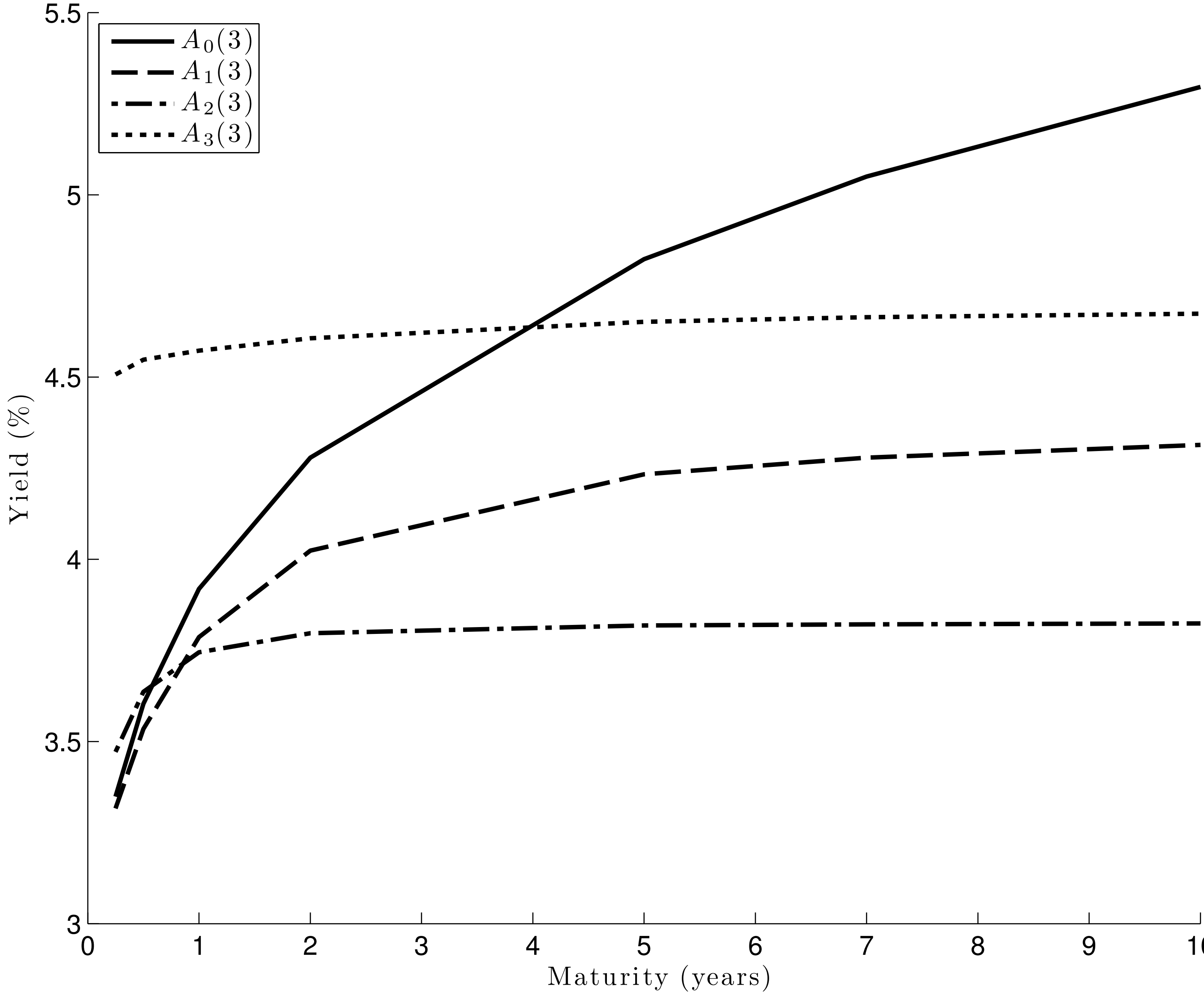
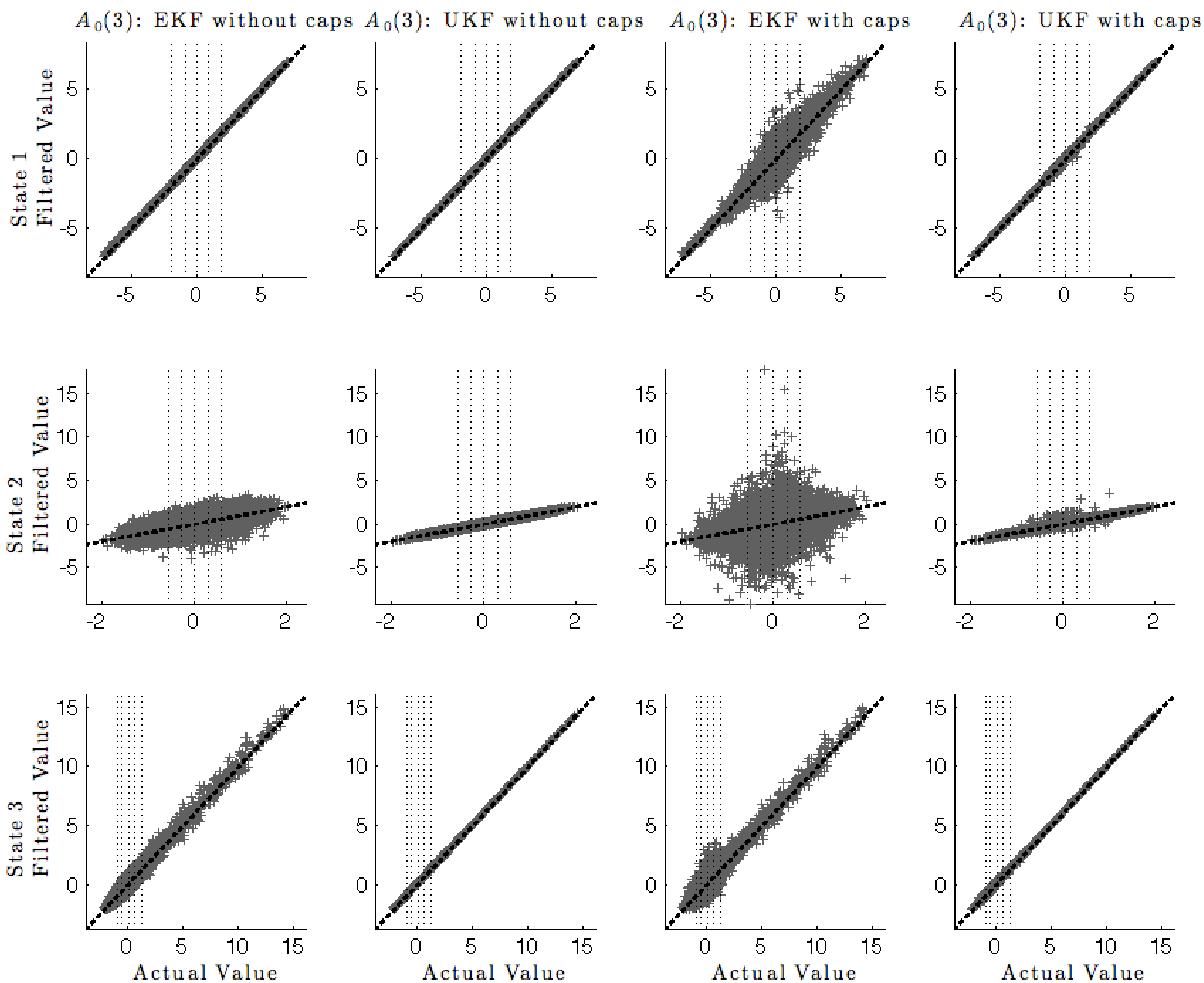
Figure 1: Unconditional Term Structures of Interest Rates. $A_M(3)$ Models



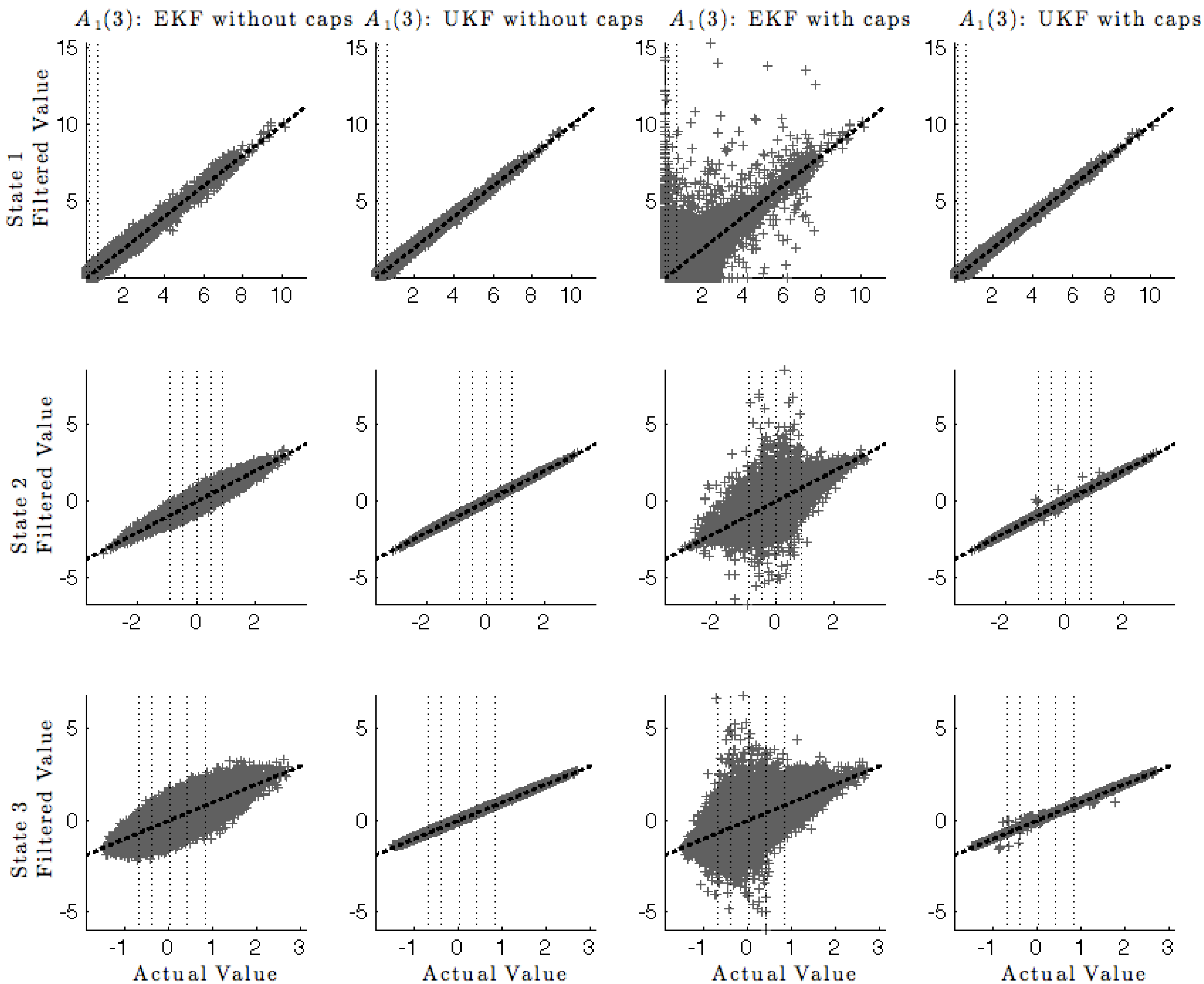
Notes: We display the unconditional term structure of interest rates implied by the four $A_M(3)$ models we consider, using the parameter values in Table 1.

Figure 2: Filtered States versus Actual States. $A_0(3)$ Model



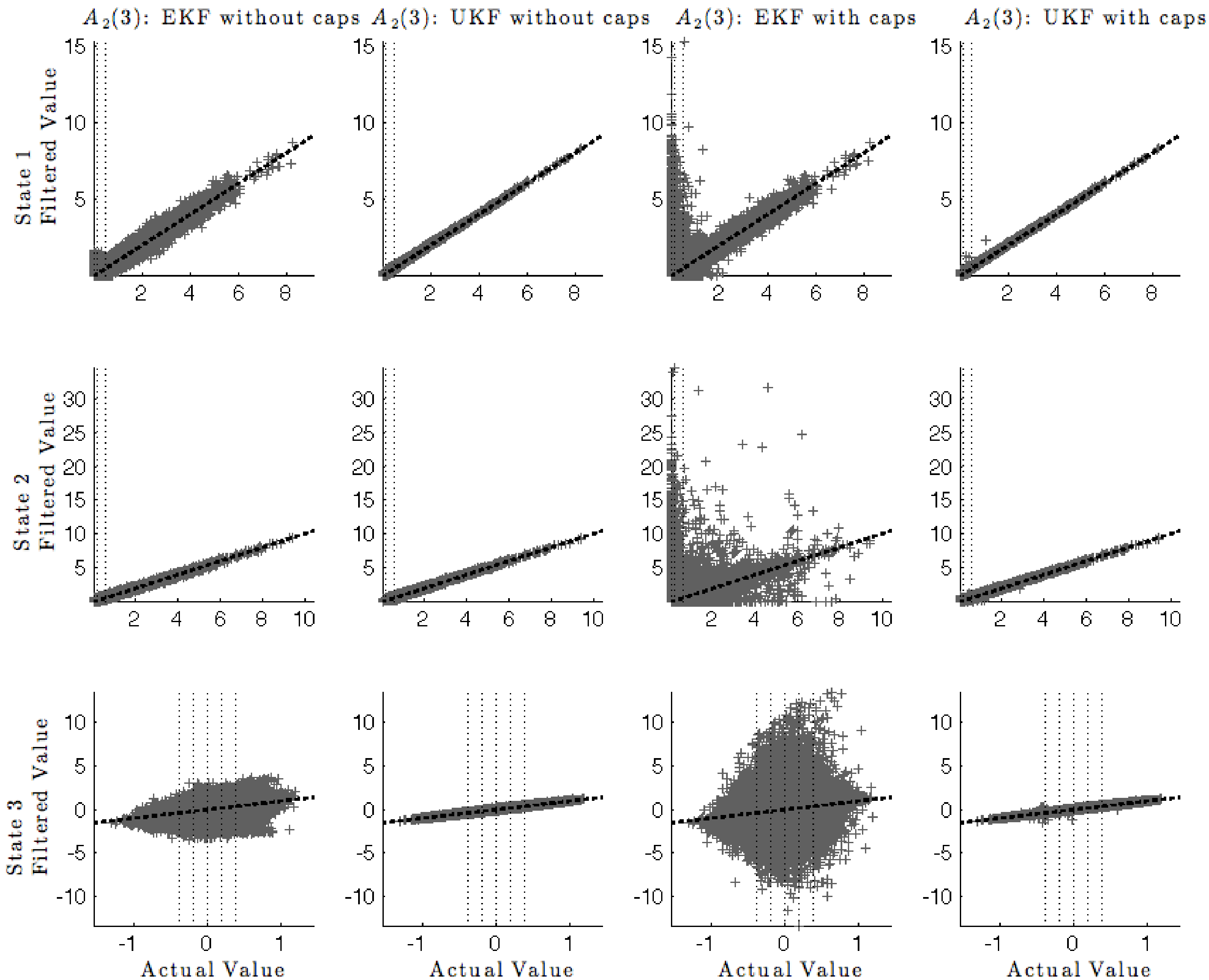
Notes: We scatter plot the filtered states against the actual states for the $A_0(3)$ model. Each row of panels depicts a different state variable. The two left-side columns show states filtered using LIBOR and swap rates only; the two right-side columns show filtered states obtained using the rates as well as the cap prices. Each panel includes the diagonal line (dashes) which would be attained by a perfect filter. The vertical dotted lines denote the 10^{th}, 25^{th}, 50^{th}, 75^{th}, and 90^{th} percentiles of the distribution of the state realizations.

Figure 3: Filtered States versus Actual States. $A_1(3)$ Model



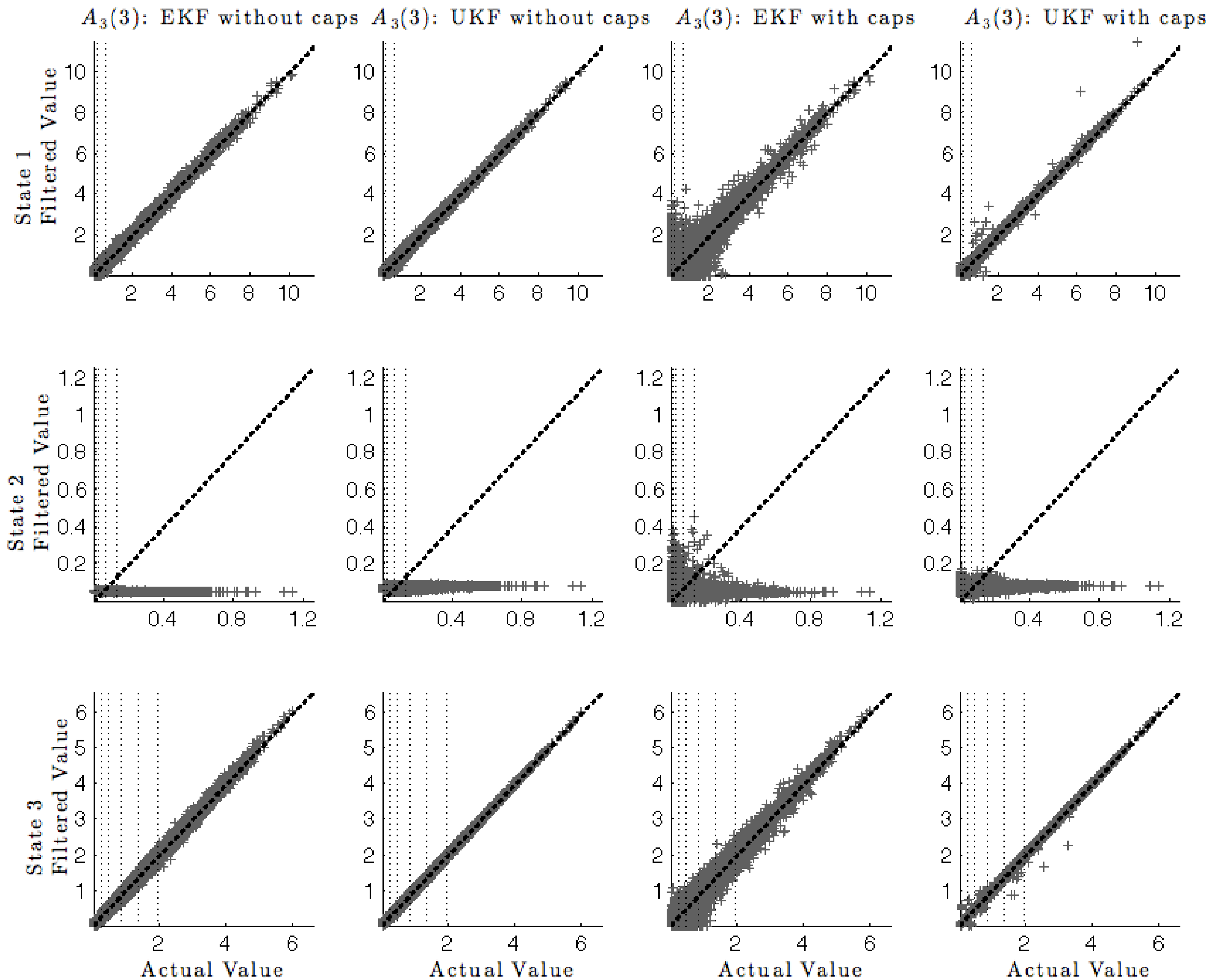
Notes: We scatter plot the filtered states against the actual states for the $A_1(3)$ model. Each row of panels depicts a different state variable. The two left-side columns show states filtered using LIBOR and swap rates only; the two right-side columns show filtered states obtained using the rates as well as the cap prices. Each panel includes the diagonal line (dashes) which would be attained by a perfect filter. The vertical dotted lines denote the 10^{th}, 25^{th}, 50^{th}, 75^{th}, and 90^{th} percentiles of the distribution of the state realizations.

Figure 4: Filtered States versus Actual States. $A_2(3)$ Model



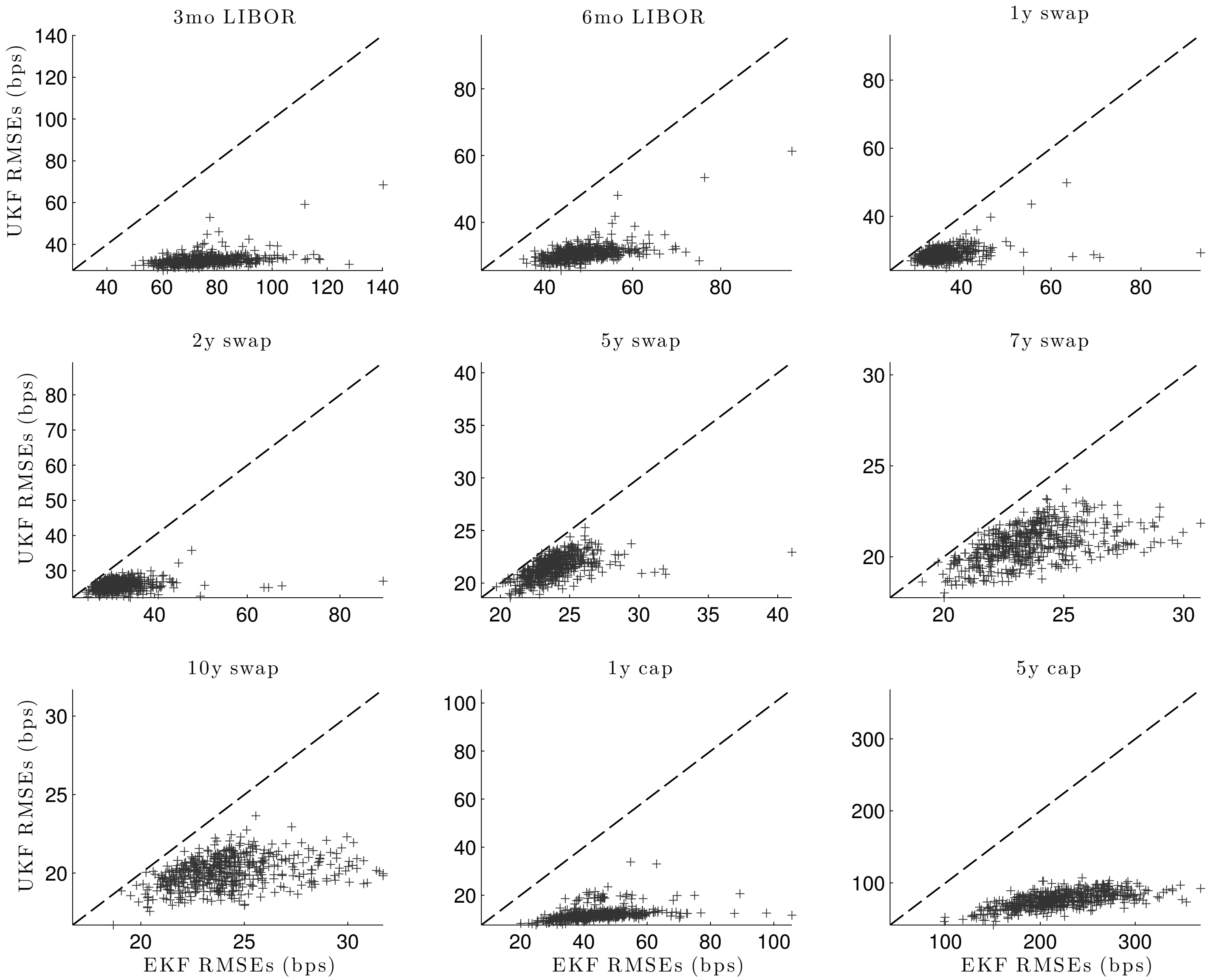
Notes: We scatter plot the filtered states against the actual states for the $A_2(3)$ model. Each row of panels depicts a different state variable. The two left-side columns show states filtered using LIBOR and swap rates only; the two right-side columns show filtered states obtained using the rates as well as the cap prices. Each panel includes the diagonal line (dashes) which would be attained by a perfect filter. The vertical dotted lines denote the 10th, 25th, 50th, 75th, and 90th percentiles of the distribution of the state realizations.

Figure 5: Filtered States versus Actual States. $A_3(3)$ Model



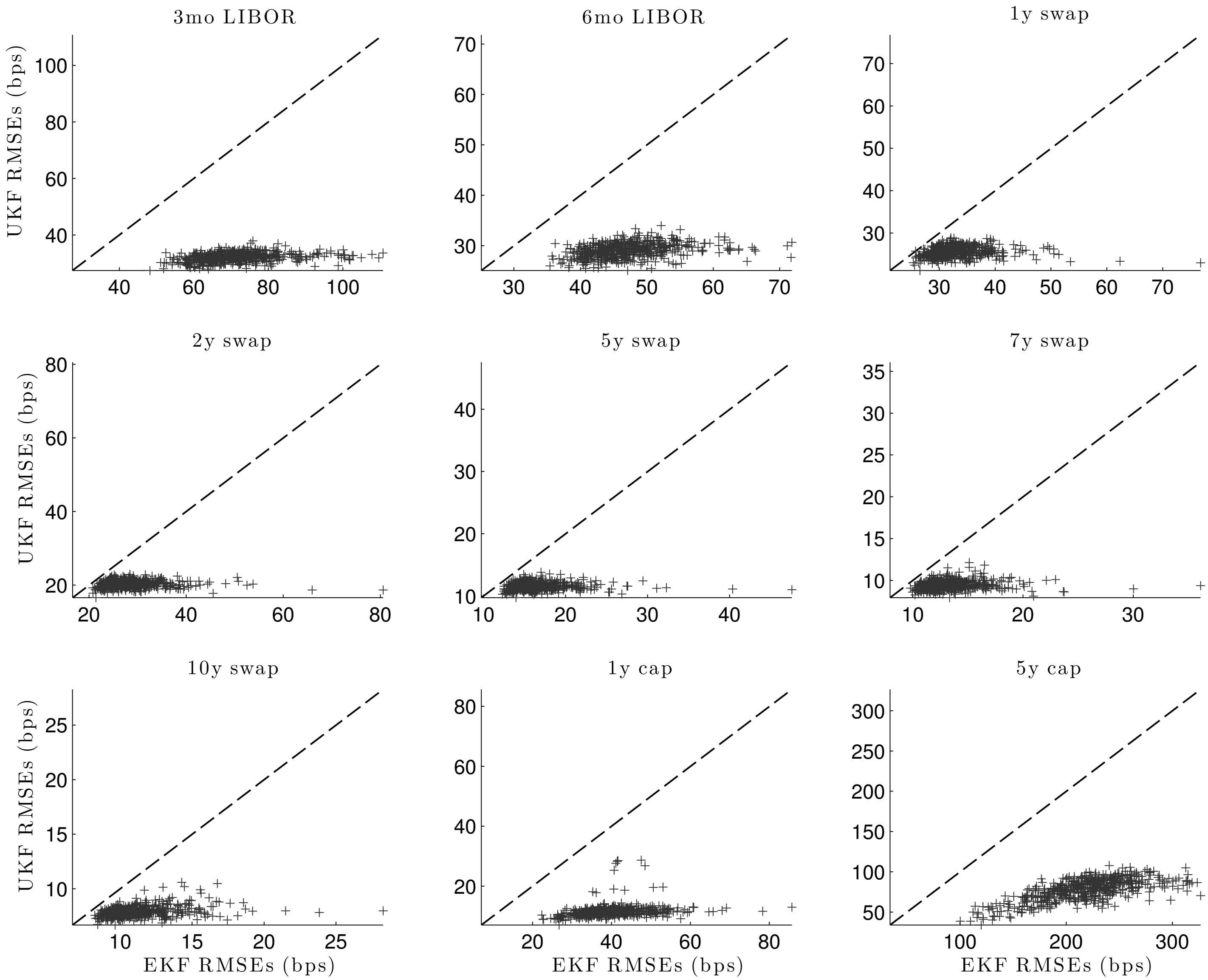
Notes: We scatter plot the filtered states against the actual states for the $A_3(3)$ model. Each row of panels depicts a different state variable. The two left-side columns show states filtered using LIBOR and swap rates only; the two right-side columns show filtered states obtained using the rates as well as the cap prices. Each panel includes the diagonal line (dashes) which would be attained by a perfect filter. The vertical dotted lines denote the 10^{th}, 25^{th}, 50^{th}, 75^{th}, and 90^{th} percentiles of the distribution of the state realizations.

Figure 6: Rate and Price Forecast RMSEs. UKF versus EKF. $A_0(3)$ Model



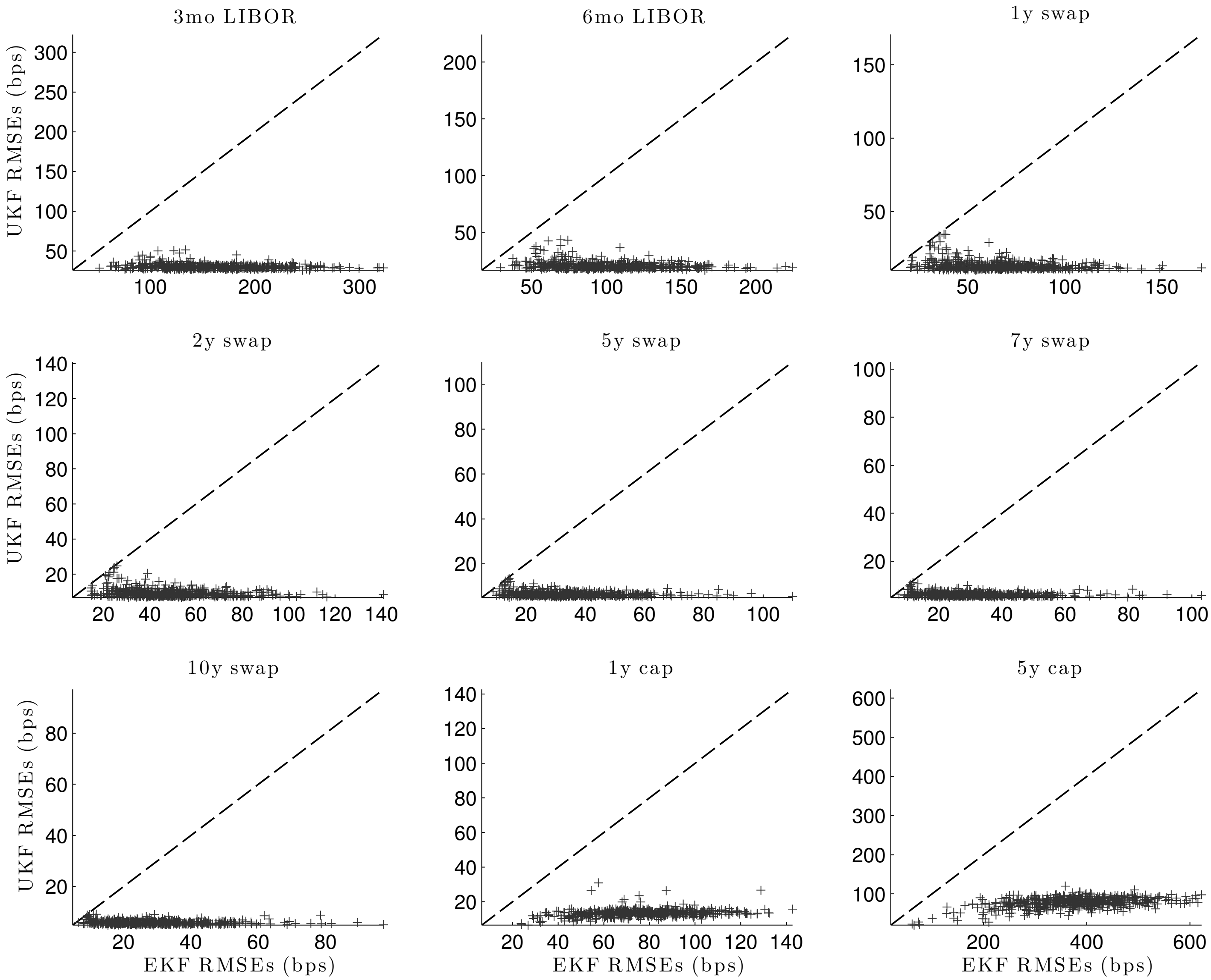
Notes: For each of the nine rates and prices, we scatter the 500 simulated one-week-ahead forecast RMSEs of the UKF model against the corresponding RMSEs for the EKF. The UKF outperforms the EKF when marks fall below the dashed 45-degree line.

Figure 7: Rate and Price Forecast RMSEs. UKF versus EKF. $A_1(3)$ Model



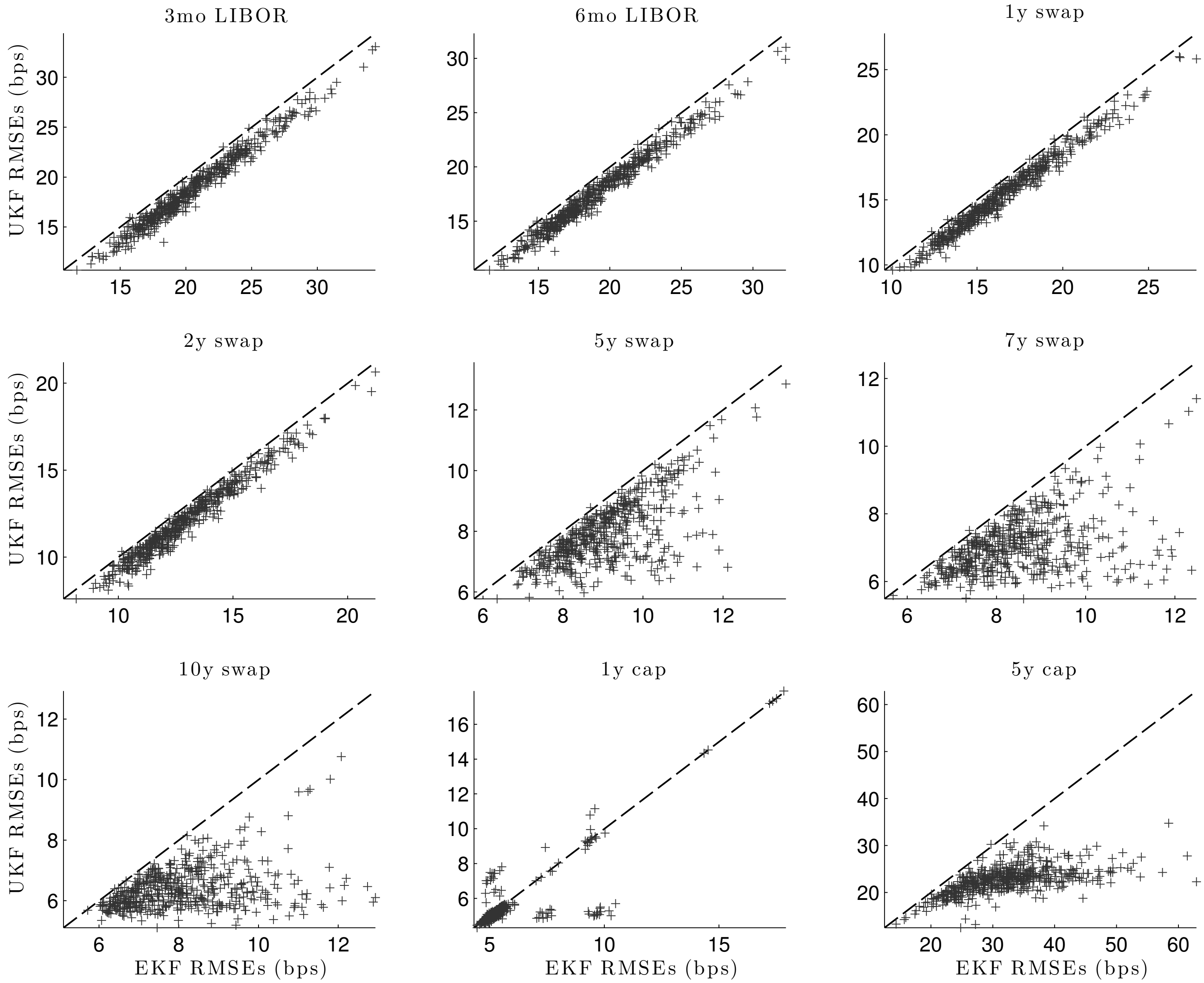
Notes: For each of the nine rates and prices, we scatter the 500 simulated one-week-ahead forecast RMSEs of the UKF model against the corresponding RMSEs for the EKF. The UKF outperforms the EKF when marks fall below the dashed 45-degree line.

Figure 8: Rate and Price Forecast RMSEs. UKF versus EKF. $A_2(3)$ Model



Notes: For each of the nine rates and prices, we scatter the 500 simulated RMSEs of the UKF model against the corresponding RMSEs for the EKF. The UKF outperforms the EKF when marks fall below the dashed 45-degree line.

Figure 9: Rate and Price Forecast RMSEs. UKF versus EKF. $A_3(3)$ Model



Notes: For each of the nine rates and prices, we scatter the 500 simulated one-week-ahead forecast RMSEs of the UKF model against the corresponding RMSEs for the EKF. The UKF outperforms the EKF when marks fall below the dashed 45-degree line.

Table 1: Parameters for the $A_M(3)$ Models

	$A_0(3)$			$A_1(3)$			$A_2(3)$			$A_3(3)$		
Parameter	**Factor 1**	**Factor 2**	**Factor 3**	**Factor 1**	**Factor 2**	**Factor 3**	**Factor 1**	**Factor 2**	**Factor 3**	**Factor 1**	**Factor 2**	**Factor 3**
δ_0	**0.030**	-	-	**0.030**	-	-	**0.030**	-	-	**0.030**	-	-
δ_{1j}	0.0048	-0.0130	0.0241	0.0028	0.0052	0.0281	0.0194	0.0028	0.0391	0.0028	0.0002	0.0145
κ_{1j}	0.0168			***0.0390***			***0.1600***	0		0.0370	0	0
κ_{2j}	0.4000	2.9600		0	0.8800		0	0.0380		0	5.7100	0
κ_{3j}	-0.6400	-2.5600	0.8410	0	-2.3200	2.6900	0	0	5.6500	0	0	0.8100
θ_j				*0.05*			*0.05*	*0.05*		*0.05*	*0.05*	*1.00*
λ_{0j}	-0.190	0.610	-0.970	0.001	-0.120	-0.970	0.680	-0.035	-1.100	-0.034	-0.010	-0.100
α_j	1	1	1		1	1			1			
β_1				1	0	0	1		0	1		
β_2								1	0		1	
β_3												1

Notes: We report the parameter values used in the Monte Carlo simulations for the four $A_M(3)$ models. Empty entries indicate zero parameter values that are implicit to the normalized form of the models or imposed for identification. Grey-shaded 0 entries indicate restrictions placed on the parameters in order to obtain closed-form solutions to the Ricatti equations. With some exceptions, the parameters are from Table 8 in Aït-Sahalia and Kimmel (2010). The exceptions are motivated by numerical considerations in the simulations and filtering. The Monte-Carlo simulations also impose constraints on the volatility factors so that they are at least 0.1%, and on the vector of factors to ensure that spot rates do not fall below 25 bps.

Table 2: State RMSEs from States Filtered without Caps

Panel A: $A_0(3)$ Model

	Factor 1			Factor 2			Factor 3		
	EKF	UKF	UKF/EKF	EKF	UKF	UKF/EKF	EKF	UKF	UKF/EKF
Mean(RMSE)	0.0473	0.0334	0.71	0.5000	0.0615	0.12	0.2367	0.0204	0.09
Median(RMSE)	0.0471	0.0334	0.71	0.4925	0.0615	0.12	0.2346	0.0203	0.09
Stdev(RMSE)	0.0037	0.0015	0.42	0.0631	0.0027	0.04	0.0331	0.0010	0.03
IQR(States)	1.7750			0.5910			1.1409		

Panel B: $A_1(3)$ Model

	Factor 1			Factor 2			Factor 3		
	EKF	UKF	UKF/EKF	EKF	UKF	UKF/EKF	EKF	UKF	UKF/EKF
Mean(RMSE)	0.1223	0.1062	0.87	0.2157	0.0422	0.20	0.4247	0.0329	0.08
Median(RMSE)	0.1123	0.1017	0.91	0.2145	0.0417	0.19	0.4239	0.0329	0.08
Stdev(RMSE)	0.0421	0.0189	0.45	0.0268	0.0033	0.12	0.0525	0.0014	0.03
IQR(States)	0.1577			0.9346			0.8070		

Panel C: $A_2(3)$ Model

	Factor 1			Factor 2			Factor 3		
	EKF	UKF	UKF/EKF	EKF	UKF	UKF/EKF	EKF	UKF	UKF/EKF
Mean(RMSE)	0.1821	0.0398	0.22	0.0746	0.0746	1.00	0.6088	0.0410	0.07
Median(RMSE)	0.1633	0.0400	0.25	0.0693	0.0716	1.03	0.5426	0.0410	0.08
Stdev(RMSE)	0.0752	0.0078	0.10	0.0272	0.0195	0.72	0.2340	0.0051	0.02
IQR(States)	0.1451			0.1636			0.3958		

Panel D: $A_3(3)$ Model

	Factor 1			Factor 2			Factor 3		
	EKF	UKF	UKF/EKF	EKF	UKF	UKF/EKF	EKF	UKF	UKF/EKF
Mean(RMSE)	0.0726	0.0750	1.03	0.0640	0.0714	1.12	0.0545	0.0297	0.54
Median(RMSE)	0.0660	0.0712	1.08	0.0582	0.0659	1.13	0.0530	0.0292	0.55
Stdev(RMSE)	0.0257	0.0154	0.60	0.0228	0.0160	0.70	0.0115	0.0033	0.28
IQR(States)	0.1579			0.0628			0.8767		

Notes: For each model, we report the mean, median, and standard deviation of the state RMSEs from the extended and the unscented Kalman filters using 500 simulated paths. For each statistic, the ratio of the UKF to EKF RMSE is reported in the third column (UKF/EKF). The IQR reports the interquartile range of the distribution of the underlying states (defined as the 75th percentile minus the 25th percentile of the state's distribution). In each of the 500 simulations, 260 weekly LIBOR and swap rates are generated using the parameters from Table 1. States are filtered using LIBOR and swap rates only.

Table 3: State RMSEs from States Filtered with Caps

Panel A: $A_0(3)$ Model

	Factor 1			Factor 2			Factor 3		
	EKF	UKF	UKF/EKF	EKF	UKF	UKF/EKF	EKF	UKF	UKF/EKF
Mean(RMSE)	0.1805	0.0319	0.18	0.6814	0.0543	0.08	0.2543	0.0168	0.07
Median(RMSE)	0.1735	0.0313	0.18	0.6567	0.0509	0.08	0.2490	0.0153	0.06
Stdev(RMSE)	0.0458	0.0031	0.07	0.1304	0.0128	0.10	0.0395	0.0036	0.09
IQR(States)	1.7750			0.5910			1.1409		

Panel B: $A_1(3)$ Model

	Factor 1			Factor 2			Factor 3		
	EKF	UKF	UKF/EKF	EKF	UKF	UKF/EKF	EKF	UKF	UKF/EKF
Mean(RMSE)	0.4530	0.0986	0.22	0.3440	0.0358	0.10	0.4766	0.0191	0.04
Median(RMSE)	0.4128	0.0941	0.23	0.3275	0.0347	0.11	0.4645	0.0183	0.04
Stdev(RMSE)	0.1909	0.0189	0.10	0.0832	0.0052	0.06	0.0774	0.0054	0.07
IQR(States)	0.1577			0.9346			0.8070		

Panel C: $A_2(3)$ Model

	Factor 1			Factor 2			Factor 3		
	EKF	UKF	UKF/EKF	EKF	UKF	UKF/EKF	EKF	UKF	UKF/EKF
Mean(RMSE)	0.5309	0.0361	0.07	0.9187	0.0714	0.08	1.0532	0.0339	0.03
Median(RMSE)	0.5097	0.0359	0.07	0.8409	0.0691	0.08	1.0371	0.0330	0.03
Stdev(RMSE)	0.1949	0.0078	0.04	0.4726	0.0181	0.04	0.2907	0.0070	0.02
IQR(States)	0.1451			0.1636			0.3958		

Panel D: $A_3(3)$ Model

	Factor 1			Factor 2			Factor 3		
	EKF	UKF	UKF/EKF	EKF	UKF	UKF/EKF	EKF	UKF	UKF/EKF
Mean(RMSE)	0.2406	0.0577	0.24	0.0646	0.0709	1.10	0.0970	0.0223	0.23
Median(RMSE)	0.2317	0.0541	0.23	0.0593	0.0652	1.10	0.0944	0.0216	0.23
Stdev(RMSE)	0.0973	0.0173	0.18	0.0226	0.0163	0.72	0.0213	0.0054	0.25
IQR(States)	0.1579			0.0628			0.8767		

Notes: For each model, we report the mean, median, and standard deviation of the state RMSEs from the extended and the unscented Kalman filters using 500 simulated paths. For each statistic, the ratio of the UKF to EKF RMSE is reported in the third column (UKF/EKF). The IQR reports the interquartile range of the distribution of the underlying states (defined as the 75th percentile minus the 25th percentile of the state's distribution). In each of the 500 simulations, 260 weekly LIBOR and swap rates are generated using the parameters from Table 1. States are filtered using LIBOR, swap rates, and caps.

Table 4: Rate and Price Fit of $A_M(3)$ Models. States Filtered without Caps.

		$A_0(3)$		$A_1(3)$		$A_2(3)$		$A_3(3)$	
		EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF
3-mo LIBOR									
	Bias	1.374	0.011	1.191	0.239	1.014	-0.054	-0.033	-0.258
	RMSE	71.230	2.706	66.481	2.793	102.390	2.693	7.476	4.147
	IQR	311.69		247.57		100.00		119.38	
6-mo LIBOR									
	Bias	1.252	-0.016	0.851	-0.192	3.195	0.230	-0.030	-0.195
	RMSE	41.145	4.145	39.677	4.186	57.610	4.451	6.590	4.268
	IQR	308.80		233.33		68.99		111.03	
1-yr Swap									
	Bias	1.083	0.021	0.571	-0.394	4.161	0.486	0.042	-0.004
	RMSE	13.379	3.853	13.105	4.116	25.645	4.672	5.229	4.493
	IQR	297.66		203.91		44.08		95.33	
2-yr Swap									
	Bias	0.765	0.002	0.416	-0.202	3.532	0.550	0.056	0.188
	RMSE	6.725	4.035	5.160	4.053	12.010	4.762	5.104	4.725
	IQR	278.73		154.64		29.10		72.64	
5-yr Swap									
	Bias	0.364	-0.006	0.453	0.436	1.866	0.537	0.079	0.459
	RMSE	5.430	4.328	5.722	4.690	6.787	4.901	4.885	4.877
	IQR	256.30		80.85		18.00		41.15	
7-yr Swap									
	Bias	0.284	0.013	0.485	0.614	1.414	0.545	0.105	0.548
	RMSE	4.564	4.142	5.285	4.828	5.982	4.941	4.824	4.909
	IQR	250.06		61.44		15.57		32.38	
10-yr Swap									
	Bias	0.198	0.002	0.468	0.691	1.036	0.517	0.098	0.581
	RMSE	4.131	3.926	5.014	4.902	5.494	4.953	4.788	4.921
	IQR	244.21		46.53		13.60		25.65	
1-yr Cap									
	Bias	5.286	-0.111	4.742	-0.457	16.517	0.341	0.014	0.029
	RMSE	38.511	7.417	34.745	6.249	53.847	6.117	5.536	5.508
	IQR	46.56		43.69		36.17		7.85	
5-yr Cap									
	Bias	17.012	-0.126	20.483	0.712	86.311	1.923	1.220	1.871
	RMSE	207.900	20.252	200.050	17.935	291.550	17.966	13.316	10.860
	IQR	517.86		543.91		239.95		157.79	

Notes: RMSE and Bias estimates are obtained from 300,000 simulated rates and prices (500 trajectories, 260 weeks), and the corresponding fitted values using the EKF or the UKF. IQR refers to the interquartile range of the true rates and prices. Caps are not used when filtering the states in this table, only LIBOR and swap rates are used for filtering.

Table 5: Rate and Price Fit of $A_M(3)$ Models. States Filtered using Caps.

	$A_0(3)$		$A_1(3)$		$A_2(3)$		$A_3(3)$	
	EKF	**UKF**	**EKF**	**UKF**	**EKF**	**UKF**	**EKF**	**UKF**
3-mo LIBOR								
Bias	1.747	0.002	2.283	0.307	8.524	-0.032	-0.254	0.019
RMSE	77.094	4.515	70.651	4.041	178.250	3.571	9.692	4.417
IQR	311.69		247.57		100.00		119.38	
6-mo LIBOR								
Bias	1.838	-0.007	2.158	-0.099	15.410	0.183	-0.143	0.035
RMSE	43.127	4.492	41.614	4.379	110.160	4.521	8.664	4.485
IQR	308.80		233.33		68.99		111.03	
1-yr Swap								
Bias	1.800	0.044	2.148	-0.300	18.375	0.406	0.111	0.142
RMSE	21.955	4.070	21.306	4.188	67.892	4.724	7.053	4.633
IQR	297.66		203.91		44.08		95.33	
2-yr Swap								
Bias	1.392	0.020	2.251	-0.158	16.585	0.484	0.386	0.214
RMSE	18.655	4.343	19.288	4.251	48.854	4.833	6.309	4.792
IQR	278.73		154.64		29.10		72.64	
5-yr Swap								
Bias	0.658	-0.011	2.455	0.384	11.575	0.515	0.761	0.320
RMSE	10.750	4.508	12.231	4.764	34.162	4.929	6.499	4.895
IQR	256.30		80.85		18.00		41.15	
7-yr Swap								
Bias	0.467	0.003	2.462	0.540	10.112	0.536	0.874	0.366
RMSE	11.811	4.345	10.349	4.868	31.049	4.958	6.775	4.924
IQR	250.06		61.44		15.57		32.38	
10-yr Swap								
Bias	0.291	-0.013	2.363	0.604	8.869	0.517	0.921	0.371
RMSE	13.741	4.161	9.241	4.924	28.667	4.964	6.940	4.936
IQR	244.21		46.53		13.60		25.65	
1-yr Cap								
Bias	7.160	-0.054	6.165	-0.391	30.238	0.234	0.152	0.007
RMSE	46.718	5.730	41.462	5.465	87.167	5.467	5.609	5.618
IQR	46.56		43.69		36.17		7.85	
5-yr Cap								
Bias	20.303	0.006	26.033	0.353	147.660	1.553	4.460	0.693
RMSE	228.350	9.093	222.360	8.943	433.820	12.288	24.986	7.760
IQR	517.86		543.91		239.95		157.79	

Notes: RMSE and Bias estimates are obtained from 300,000 simulated rates and prices (500 trajectories, 260 weeks), and the corresponding fitted values using the EKF or the UKF. IQR refers to the interquartile range of the true rates and prices. Caps as well as LIBOR and swap rates are used when filtering the states in this table.

Table 6: Rate and Price Forecasting Performance. $A_0(3)$ Model

	States Filtered without Caps						States Filtered with Caps					
Panel A: Average Forecast RMSE (bps)												
Forecast horizon	1 week		4 weeks		12 weeks		1 week		4 weeks		12 weeks	
	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF
3-mo LIBOR	72.11	32.64	79.02	63.31	107.85	105.81	76.28	32.55	80.31	63.28	109.61	105.79
6-mo LIBOR	47.49	30.75	65.31	60.50	104.19	103.07	49.06	30.73	66.90	60.49	107.56	103.06
1-yr Swap	30.89	29.03	57.63	57.11	99.05	97.49	36.32	29.03	62.05	57.10	103.25	97.48
2-yr Swap	26.84	25.94	51.66	50.69	87.80	86.18	32.53	25.92	55.38	50.68	90.42	86.17
5-yr Swap	21.92	21.58	42.14	41.82	71.87	71.41	23.93	21.56	43.30	41.80	72.67	71.39
7-yr Swap	20.85	20.74	40.19	40.06	68.74	68.53	23.64	20.70	41.78	40.04	69.79	68.51
10-yr Swap	20.13	20.09	38.67	38.63	66.24	66.16	24.15	20.05	41.00	38.61	67.73	66.15
1-yr Cap	37.25	12.30	34.41	19.33	30.27	25.98	44.46	12.11	38.90	19.29	31.14	26.00
5-yr Cap	203.95	76.89	200.74	140.68	224.71	217.26	220.25	76.26	208.57	140.54	234.34	217.24
Panel B: Median Forecast RMSE (bps)												
Forecast horizon	1 week		4 weeks		12 weeks		1 week		4 weeks		12 weeks	
	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF
3-mo LIBOR	71.07	32.28	77.58	61.55	103.31	102.07	74.85	32.15	78.70	61.56	104.77	102.14
6-mo LIBOR	46.93	30.50	63.89	59.12	100.75	100.40	48.18	30.45	65.57	59.18	103.95	100.23
1-yr Swap	30.73	28.95	56.34	56.18	96.64	95.74	35.28	28.93	60.71	56.12	100.80	95.86
2-yr Swap	26.71	25.99	51.39	50.54	86.72	85.21	31.61	25.96	54.66	50.64	88.83	85.13
5-yr Swap	21.96	21.61	42.04	41.70	71.45	70.95	23.75	21.56	43.15	41.70	72.29	70.95
7-yr Swap	20.85	20.70	40.21	40.13	68.52	68.17	23.44	20.68	41.72	40.01	69.44	68.18
10-yr Swap	20.14	20.08	38.71	38.66	66.01	65.99	23.83	20.05	40.98	38.59	67.09	65.94
1-yr Cap	37.58	11.76	34.49	19.07	30.20	25.57	43.43	11.57	37.82	19.12	30.85	25.62
5-yr Cap	204.04	76.95	200.95	140.00	223.32	215.64	215.62	76.14	206.56	140.00	232.43	215.65
Panel C: Forecast RMSE Standard Deviation (bps)												
Forecast horizon	1 week		4 weeks		12 weeks		1 week		4 weeks		12 weeks	
	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF
3-mo LIBOR	9.94	3.07	11.82	11.39	30.18	27.90	11.66	3.06	12.07	11.36	30.04	27.88
6-mo LIBOR	6.01	2.64	10.57	9.88	27.13	24.77	6.47	2.64	10.76	9.87	26.97	24.76
1-yr Swap	2.53	1.98	8.43	7.36	21.29	19.34	5.39	1.98	9.39	7.36	21.32	19.34
2-yr Swap	1.72	1.39	5.57	4.77	14.45	13.34	5.21	1.39	6.82	4.77	14.66	13.35
5-yr Swap	1.00	1.06	3.12	3.12	9.02	8.97	1.81	1.06	3.26	3.12	9.02	8.98
7-yr Swap	0.98	1.02	3.02	3.06	8.67	8.69	1.96	1.02	3.16	3.05	8.62	8.69
10-yr Swap	0.95	0.97	2.89	2.92	8.46	8.47	2.38	0.98	3.19	2.92	8.39	8.47
1-yr Cap	6.07	2.65	5.09	3.07	4.45	4.39	10.22	2.75	7.75	3.06	4.86	4.39
5-yr Cap	34.96	11.06	28.19	21.30	37.45	38.06	44.57	11.11	33.56	21.23	39.57	38.09

Notes: We forecast rates and cap prices using the EKF and UKF filters. For each of 500 simulations, we compute the forecast RMSE. The mean, median, and standard deviation of these RMSEs is reported for the EKF and the UKF. On the left-hand side, the results were obtained from states filtered without using cap prices. On the right-hand side, caps were used when filtering the states.

Table 7: Rate and Price Forecasting Performance. $A_1(3)$ Model

Panel A: Average Forecast RMSE (bps)

	States Filtered without Caps						States Filtered with Caps					
Forecast horizon	1 week		4 weeks		12 weeks		1 week		4 weeks		12 weeks	
	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF
3-mo LIBOR	68.57	32.19	75.80	60.73	99.05	97.72	71.59	32.07	76.89	60.71	100.41	97.71
6-mo LIBOR	45.77	29.09	60.83	55.79	92.26	92.15	47.35	29.07	62.28	55.79	95.05	92.14
1-yr Swap	27.55	25.63	49.65	49.50	83.24	82.72	33.02	25.61	53.94	49.48	87.44	82.69
2-yr Swap	20.79	20.30	39.47	38.82	65.77	64.58	28.55	20.26	44.68	38.79	69.67	64.55
5-yr Swap	12.24	11.60	21.58	20.99	35.04	34.34	16.57	11.56	24.52	20.97	37.12	34.33
7-yr Swap	9.81	9.45	16.67	16.30	26.74	26.27	13.39	9.42	19.15	16.28	28.47	26.25
10-yr Swap	8.11	7.94	12.95	12.75	20.39	20.10	11.30	7.91	15.26	12.72	22.02	20.09
1-yr Cap	34.07	11.72	32.55	19.05	29.99	25.82	39.99	11.50	36.51	18.85	31.13	25.80
5-yr Cap	199.29	78.37	202.34	143.26	227.87	218.23	217.31	77.57	210.84	143.02	230.46	218.13

Panel B: Median Forecast RMSE (bps)

Forecast horizon	1 week		4 weeks		12 weeks		1 week		4 weeks		12 weeks	
	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF
3-mo LIBOR	68.46	32.26	75.85	60.70	98.70	97.41	70.57	32.13	76.25	60.76	99.98	97.44
6-mo LIBOR	45.78	29.14	60.78	55.67	91.84	91.69	46.57	29.13	62.15	55.68	95.01	91.71
1-yr Swap	27.54	25.63	49.66	49.48	82.90	82.28	32.03	25.61	53.49	49.44	87.50	82.19
2-yr Swap	20.76	20.35	39.32	38.67	65.58	64.37	27.25	20.29	44.00	38.63	69.05	64.43
5-yr Swap	12.24	11.59	21.47	20.87	34.87	34.21	15.73	11.54	24.20	20.83	36.77	34.23
7-yr Swap	9.78	9.41	16.60	16.21	26.60	26.17	12.80	9.38	18.71	16.21	28.23	26.19
10-yr Swap	8.08	7.90	12.87	12.66	20.17	19.89	10.76	7.87	14.79	12.62	21.66	19.86
1-yr Cap	33.88	11.48	32.34	18.75	30.00	25.38	39.29	11.27	35.99	18.67	31.00	25.38
5-yr Cap	199.89	80.02	205.73	145.84	231.63	221.70	219.41	79.22	214.87	145.74	235.37	221.48

Panel C: Forecast RMSE Standard Deviation (bps)

Forecast horizon	1 week		4 weeks		12 weeks		1 week		4 weeks		12 weeks	
	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF
3-mo LIBOR	8.53	1.59	5.59	4.91	12.58	12.95	10.26	1.59	6.26	4.90	12.31	12.95
6-mo LIBOR	5.08	1.43	4.27	4.41	11.95	12.00	5.86	1.43	4.48	4.40	11.76	11.99
1-yr Swap	1.61	1.20	3.68	3.71	10.51	10.42	4.86	1.19	5.27	3.71	10.68	10.41
2-yr Swap	0.96	0.94	2.92	2.84	8.11	8.06	5.71	0.93	5.39	2.83	8.54	8.05
5-yr Swap	0.59	0.58	1.59	1.56	4.35	4.33	3.35	0.58	3.10	1.55	4.60	4.33
7-yr Swap	0.52	0.53	1.28	1.27	3.41	3.40	2.55	0.52	2.45	1.27	3.62	3.41
10-yr Swap	0.47	0.48	1.08	1.08	2.71	2.70	2.18	0.48	2.10	1.08	2.92	2.71
1-yr Cap	4.30	2.19	3.58	2.89	3.93	4.19	7.93	2.21	6.20	2.70	4.49	4.18
5-yr Cap	32.16	13.46	30.38	25.93	43.65	43.65	41.40	13.26	34.56	25.82	43.37	43.61

Notes: We forecast rates and cap prices using the EKF and UKF filters. For each of 500 simulations, we compute the forecast RMSE. The mean, median, and standard deviation of these RMSEs is reported for the EKF and the UKF. On the left-hand side, the results were obtained from states filtered without using cap prices. On the right-hand side, caps were used when filtering the states.

Table 8: Rate and Price Forecasting Performance. $A_2(3)$ Model

Panel A: Average Forecast RMSE (bps)

	States Filtered without Caps						**States Filtered with Caps**					
Forecast horizon	1 week		4 weeks		12 weeks		1 week		4 weeks		12 weeks	
	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF
3-mo LIBOR	94.66	30.40	79.82	52.22	71.73	69.41	160.84	30.32	125.03	52.21	102.93	69.43
6-mo LIBOR	54.05	20.71	48.32	35.61	50.75	48.85	101.84	20.67	87.61	35.60	87.37	48.86
1-yr Swap	25.77	13.47	27.13	22.65	35.19	32.30	66.38	13.42	65.72	22.63	72.92	32.31
2-yr Swap	13.75	9.30	17.08	14.71	24.29	21.34	49.34	9.25	51.24	14.68	56.66	21.34
5-yr Swap	7.93	6.59	9.96	8.94	13.82	12.46	34.47	6.56	35.39	8.93	37.47	12.45
7-yr Swap	7.00	6.17	8.54	7.90	11.55	10.66	31.27	6.14	31.86	7.88	33.19	10.65
10-yr Swap	6.38	5.91	7.53	7.17	9.89	9.36	28.82	5.88	29.16	7.15	29.98	9.35
1-yr Cap	48.92	13.43	38.79	21.18	28.76	26.10	76.97	13.39	53.94	21.16	30.37	26.09
5-yr Cap	262.74	77.90	208.11	130.31	171.30	163.31	374.82	77.54	255.21	130.12	180.92	163.21

Panel B: Median Forecast RMSE (bps)

Forecast horizon	1 week		4 weeks		12 weeks		1 week		4 weeks		12 weeks	
	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF
3-mo LIBOR	83.56	29.76	74.42	51.13	69.25	66.65	156.64	29.66	120.72	51.10	99.22	66.77
6-mo LIBOR	47.32	19.79	45.90	33.61	46.87	44.47	98.85	19.73	84.34	33.65	83.07	44.43
1-yr Swap	22.86	12.50	25.62	20.43	31.04	27.61	65.04	12.41	61.95	20.37	69.63	27.69
2-yr Swap	12.50	8.53	15.78	12.97	21.23	17.77	47.38	8.48	48.18	12.98	52.95	17.82
5-yr Swap	7.48	6.28	9.36	8.14	12.48	10.87	32.13	6.25	33.13	8.12	35.19	10.85
7-yr Swap	6.69	5.95	8.08	7.27	10.51	9.31	28.96	5.93	29.18	7.28	30.72	9.26
10-yr Swap	6.20	5.75	7.15	6.61	8.86	8.21	26.48	5.73	26.63	6.59	27.41	8.21
1-yr Cap	42.68	13.40	35.22	21.19	28.80	25.81	76.32	13.32	53.82	21.18	30.58	25.81
5-yr Cap	232.98	79.19	198.61	132.09	172.26	164.09	377.82	78.75	259.64	132.01	182.54	164.01

Panel C: Forecast RMSE Standard Deviation (bps)

Forecast horizon	1 week		4 weeks		12 weeks		1 week		4 weeks		12 weeks	
	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF
3-mo LIBOR	34.69	3.14	18.64	7.05	15.11	15.75	48.13	3.15	32.59	7.05	23.14	15.74
6-mo LIBOR	19.42	3.42	10.25	7.40	15.72	15.96	32.68	3.43	24.88	7.40	24.73	15.95
1-yr Swap	8.60	3.29	6.78	7.08	14.49	14.60	23.59	3.30	21.69	7.07	23.47	14.59
2-yr Swap	4.00	2.43	5.36	5.45	10.97	11.04	19.18	2.44	18.58	5.44	19.17	11.03
5-yr Swap	1.62	1.10	2.75	2.71	5.71	5.71	15.07	1.10	14.59	2.71	14.06	5.71
7-yr Swap	1.14	0.81	2.12	2.10	4.51	4.51	14.26	0.81	13.87	2.10	13.27	4.51
10-yr Swap	0.83	0.68	1.72	1.73	3.72	3.73	13.62	0.67	13.33	1.72	12.75	3.73
1-yr Cap	19.17	1.93	11.49	3.35	5.07	5.11	20.79	2.15	12.68	3.36	5.02	5.11
5-yr Cap	105.13	13.27	57.91	25.88	39.14	38.14	97.86	13.30	55.34	25.81	40.70	38.02

Notes: We forecast rates and cap prices using the EKF and UKF filters. For each of 500 simulations, we compute the forecast RMSE. The mean, median, and standard deviation of these RMSEs is reported for the EKF and the UKF. On the left-hand side, the results were obtained from states filtered without using cap prices. On the right-hand side, caps were used when filtering the states.

Table 9: Rate and Price Forecasting Performance. $A_3(3)$ Model

	States Filtered without Caps						States Filtered with Caps					
Panel A: Average Forecast RMSE (bps)												
Forecast horizon	1 week		4 weeks		12 weeks		1 week		4 weeks		12 weeks	
	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF
3-mo LIBOR	19.79	19.01	36.04	35.63	57.15	56.93	20.67	18.96	36.42	35.61	57.31	56.92
6-mo LIBOR	18.47	17.78	33.56	33.18	53.22	53.02	19.22	17.74	33.89	33.17	53.36	53.01
1-yr Swap	15.97	15.41	28.77	28.47	45.51	45.36	16.53	15.39	29.02	28.46	45.65	45.36
2-yr Swap	12.47	12.12	21.96	21.77	34.52	34.43	12.97	12.11	22.23	21.77	34.71	34.43
5-yr Swap	8.05	7.97	12.83	12.79	19.61	19.59	9.10	7.94	13.55	12.77	20.14	19.58
7-yr Swap	7.10	7.08	10.58	10.57	15.75	15.74	8.49	7.03	11.61	10.54	16.51	15.73
10-yr Swap	6.45	6.46	8.91	8.92	12.80	12.81	8.10	6.40	10.22	8.88	13.80	12.79
1-yr Cap	5.61	5.46	5.71	5.53	5.88	5.74	5.64	5.45	5.67	5.60	5.95	5.79
5-yr Cap	24.66	23.26	42.26	41.34	66.35	65.41	32.29	22.71	46.41	41.17	68.00	65.40
Panel B: Median Forecast RMSE (bps)												
Forecast horizon	1 week		4 weeks		12 weeks		1 week		4 weeks		12 weeks	
	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF
3-mo LIBOR	19.11	18.43	35.06	34.58	54.67	54.47	20.11	18.35	35.43	34.53	54.80	54.43
6-mo LIBOR	17.99	17.25	32.65	32.28	50.96	50.65	18.66	17.20	32.87	32.27	51.09	50.63
1-yr Swap	15.54	15.02	28.07	27.77	43.65	43.39	16.10	15.00	28.24	27.76	43.74	43.41
2-yr Swap	12.17	11.88	21.35	21.16	33.03	32.84	12.60	11.84	21.53	21.16	33.15	32.82
5-yr Swap	7.87	7.81	12.51	12.44	18.68	18.60	9.02	7.78	13.21	12.44	19.15	18.64
7-yr Swap	6.99	6.97	10.22	10.21	14.95	14.93	8.34	6.94	11.25	10.18	15.61	14.90
10-yr Swap	6.32	6.34	8.54	8.54	12.01	11.96	7.85	6.26	9.99	8.49	13.15	11.99
1-yr Cap	5.14	5.09	5.22	5.19	5.45	5.40	5.21	5.10	5.26	5.21	5.45	5.42
5-yr Cap	24.80	23.37	42.97	42.13	67.26	66.17	31.78	22.80	46.58	41.76	69.16	66.09
Panel C: Forecast RMSE Standard Deviation (bps)												
Forecast horizon	1 week		4 weeks		12 weeks		1 week		4 weeks		12 weeks	
	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF	EKF	UKF
3-mo LIBOR	3.99	3.75	7.94	7.92	14.89	14.98	3.82	3.76	7.86	7.91	14.91	14.96
6-mo LIBOR	3.72	3.49	7.40	7.39	13.90	13.97	3.57	3.50	7.33	7.38	13.90	13.96
1-yr Swap	3.10	2.90	6.27	6.25	11.82	11.85	2.99	2.91	6.21	6.24	11.80	11.85
2-yr Swap	2.26	2.12	4.66	4.63	8.86	8.85	2.12	2.13	4.58	4.63	8.80	8.86
5-yr Swap	1.11	1.04	2.48	2.43	4.92	4.85	1.12	1.06	2.30	2.45	4.75	4.88
7-yr Swap	0.86	0.81	2.01	1.95	4.02	3.95	1.20	0.83	1.94	1.97	3.86	3.98
10-yr Swap	0.72	0.68	1.64	1.59	3.40	3.32	1.36	0.70	1.77	1.61	3.30	3.35
1-yr Cap	1.68	1.58	1.77	1.53	1.65	1.49	1.65	1.53	1.64	1.59	1.77	1.58
5-yr Cap	3.21	3.01	5.15	5.07	9.68	9.27	7.55	2.95	7.52	5.05	10.33	9.35

Notes: We forecast rates and cap prices using the EKF and UKF filters. For each of 500 simulations, we compute the forecast RMSE. The mean, median, and standard deviation of these RMSEs is reported for the EKF and the UKF. On the left-hand side, the results were obtained from states filtered without using cap prices. On the right-hand side, caps were used when filtering the states.

Table 10: Fit of 7-Year Cap Prices. $A_M(3)$ Models.

Panel A: States Filtered without 1-yr and 5-yr Caps

7-yr Cap	**$A_0(3)$ EKF**	**$A_0(3)$ UKF**	**$A_1(3)$ EKF**	**$A_1(3)$ UKF**	**$A_2(3)$ EKF**	**$A_2(3)$ UKF**	**$A_3(3)$ EKF**	**$A_3(3)$ UKF**
Bias	19.078	-0.148	25.377	1.563	114.710	2.497	1.840	2.865
RMSE	271.750	26.803	264.120	21.785	391.090	22.898	18.200	13.986
IQR	757.24		804.06		333.20		246.73	

Panel B: States Filtered with 1-yr and 5-yr Caps

7-yr Cap	**$A_0(3)$ EKF**	**$A_0(3)$ UKF**	**$A_1(3)$ EKF**	**$A_1(3)$ UKF**	**$A_2(3)$ EKF**	**$A_2(3)$ UKF**	**$A_3(3)$ EKF**	**$A_3(3)$ UKF**
Bias	22.024	-0.022	31.852	0.955	193.740	2.094	6.690	1.026
RMSE	295.340	17.346	290.700	11.955	573.900	16.179	36.670	10.581
IQR	757.24		804.06		333.20		246.73	

Notes: Estimates of RMSE and Bias are obtained from 130,000 simulated 7-year cap prices (500 trajectories, 260 weeks), and the corresponding fitted prices using the EKF or the UKF. In Panel A only LIBOR and swap rates are used when filtering the underlying states. In Panel B 1-year and 5-year caps are used in addition when filtering the state. 7-year caps are never used in the filtering step.

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