

Continuous Trading Dynamically Effectively Complete Market with Heterogeneous Beliefs

Zhenjiang Qin

CREATES Research Paper 2012-24

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ZHENJIANG QIN[†]

Department of Economics and Business and CREATES

Aarhus University, Denmark

15 April 2012

Abstract

In a framework of heterogeneous beliefs, I investigate a two-date consumption model with continuous trading over the interval $[0, T]$, in which information on the aggregate consumption at time T is revealed by an Ornstein-Uhlenbeck Bridge. This information structure allows investors to speculate on the heterogeneous posterior variance of dividend throughout $[0, T]$. The market populated with many time-additive exponential-utility investors is dynamically effectively complete, if investors are allowed to trade in only two long-lived securities continuously. The underlying mechanism is that these assumptions imply that the Pareto efficient individual consumption plans are measurable with respect to the aggregate consumption. Hence, I may not need a dynamically complete market to facilitate a Pareto efficient allocation of consumption, the securities only have to facilitate an allocation which is measurable with respect to the aggregate consumption. With normally distributed dividend, the equilibrium stock price is endogenized in a Radner equilibrium as a precision weighted average of the investors' posterior mean minus a risk premium determined by the average posterior precision. The stock price is also a sufficient statistic for computation of the price of redundant dividend derivative and the equilibrium portfolios. The investors form their Pareto optimal trading strategies as if they intend to dynamically endogenously replicate the value of the dividend derivative.

*Financial support from the Center for Research in the Econometric Analysis of Time Series (CREATES) funded by the Danish National Research Foundation is gratefully acknowledged.

[†]Address correspondence: Department of Economics and Business, Building 1322, Bartholins Allé 10, Aarhus University, DK-8000 Aarhus C, Denmark; zqin@econ.au.dk, +45 8716 5287.

Keywords: Heterogeneous Beliefs; Continuous Trading; Dynamically Effectively Complete Market; Asset Pricing

JEL classification: G1, G11, G12

1 Introduction

In a homogeneous belief setting, Duffie and Huang (1985) and Duffie and Zame (1989) demonstrate that continuous trading can play a role as a compensation of long lived securities to dynamically complete the financial market, with finite number of securities. In other words, a continuous trading Radner equilibrium can implement the same Arrow-Debreu consumption allocations. These are two of the very few papers to address the issue of welfare consequences of continuous-trading opportunities in a few long-lived securities (Sundaresan 2000).

However, Duffie and Zame (1989) model the endowment by Arithmetic Brownian Motion, and their information structure cannot be generalized to the heterogeneous beliefs case with heterogeneous volatility. Since it follows from Girsanov's Theorem that the instantaneous volatility of the endowment is identical across investors under both individual perceived dynamics and risk-neutral dynamics. Therefore, to investigate the effects of speculation on the heterogeneity of perceived dividend variance, another information structure which allows difference in perceived variance is in order. Furthermore, in Duffie and Huang (1985), information structure is modeled by a complete probability space which constitutes all possible states of the world that could exist at a terminal date. The investors can receive information which is not relevant to the Pareto optimal consumption. However, in the real world, investors usually receive only part of the information in the economy, for instance, information on the aggregate consumption. Naturally, questions arise: How to generalize Duffie and Huang (1985) to a case with heterogeneous beliefs and information only relevant to aggregate consumption? Can the Pareto consumption allocations in the Arrow-Debreu equilibrium with heterogeneous beliefs in Christensen and Qin (2012) be implemented by some counterpart continuous trading model? In this paper, I show that, fortunately, dynamically complete market can be substituted by additional assumptions about preferences, in order to yield effectively identical results. I find with more restrictive assumptions of utility, i.e., with time-additive negative-exponential utility, continuous trading can dynamically *effectively* complete the financial market with heterogeneous beliefs. This result can be considered to be a consequence of continuous-time Pareto efficient side-betting based on the heterogeneously perceived variance of dividend.

This work is a heterogeneous-belief extension of Duffie and Huang (1985). I summarize the main results in the following three aspects. First, an information structure is constructed to allow heterogeneity in perceived variance of the terminal dividend. The investors can speculate on the variance of the dividend throughout the interval $[0, T)$. Second, with continuous-time Pareto efficient side-betting on confidence among investors, the market is

dynamically *effectively* complete, if investors are allowed to trade in only two long-lived securities continuously. Third, I provide an example to endogenously replicate the value of the redundant derivative in a Radner equilibrium.

I investigate a two-date consumption model in which information on the aggregate consumption at the terminal date T is revealed by an Ornstein-Uhlenbeck Bridge, which is driven by a Standard Brownian Motion. The investors update their beliefs on the normally distributed dividend in a Bayesian fashion. The continuous speculative trade motivated by the heterogeneous beliefs can dynamically effectively complete the markets. The underlying mechanism is that the assumptions of heterogeneous beliefs and time-additive exponential utility imply that the Pareto efficient individual consumption plans are measurable with respect to aggregate consumption. Hence, I may not need a dynamically complete market to facilitate the Pareto efficient allocation of consumption, the securities only have to facilitate an allocation which is measurable with respect to the aggregate consumption. The dynamically effectively complete market in the model has the property that any consumption plan, which is measurable with respect to aggregate consumption, can be implemented, despite the fact that investors may not be able to implement any financially feasible consumption plan as they are in a dynamically complete market.

The model in this paper is a counterpart extreme case of that in Christensen and Qin (2012). Based on the idea of Wilson (1968), they introduce a dividend derivative which pays off the square of dividend at the terminal date to facilitate side-betting and achieve a Pareto efficient equilibrium. This extreme case gets rid of the need for dynamic trading based on public signals. In contrast, in this paper, the investors trade dynamically based on continuous signals. The payoff of the "Dividend Square Security" can be endogenously replicated by trading a riskless bond and a risky asset continuously. Specifically, under the assumptions of exponential utility and the normally distributed dividend, I derive analytical expressions of the equilibrium security prices and the equilibrium portfolios in the Radner equilibrium to implement the Arrow-Debreu equilibrium in Christensen and Qin (2012). The equilibrium stock price in the Radner equilibrium is given as a precision weighted average of the investors' posterior mean minus a risk premium determined by the average posterior precision. The stock price is driven by the heterogeneously updated posterior beliefs and, thus, driven by the prior beliefs and the public signals (as functions of the Brownian motion). Moreover, the stock price is a sufficient statistic for computation of the optimal portfolios, the optimal wealth processes, and the price of the redundant dividend derivative. The investors form their Pareto optimal trading strategies by investing as if they intend to dynamically replicate the value of the dividend derivative. Therefore, continuous trading can be viewed as a replacement of the convexity in the payoff of the derivatives needed for Pareto efficient

side-betting, and implement the Pareto efficient consumption allocations.

Review of the Literature

Current work is closely related to several continuous-time models in different directions. Duffie and Huang (1985) study a dynamically complete market with homogeneous belief. Compare to their model, this paper's information structure allows me to explicitly study the effects of heterogeneous updated posterior beliefs on asset pricing properties such as equilibrium stock price and equilibrium portfolios. Besides, the dynamically effectively complete market in this paper cannot facilitate all kinds of consumption plans as the dynamically complete market can.

With homogeneous prior belief, Brennan and Cao (1996) assume investors receive signals with different precision. New exogenous supply shocks are needed to generate the trading volume in stock to achieve Pareto efficient consumption allocations, which can also be obtained by trading in a quadratic option. In contrast, in this paper, the trading on stock is endogenized in the equilibrium as a result of speculation on heterogeneous variances. In a framework of homogeneous belief, Christensen, Graversen, and Miltersen (2000) show that continuous trading of long-lived contingent claims on aggregate consumption can substitute the need for an infinite number of primitive securities in a dynamically effectively complete market. Zuasti (2008) extends the above literature on continuous trading, by formally including insurance as a non-tradable asset and studying its price and demand. He provides a framework of heterogeneous von Neumann–Morgenstern preferences to study of the interaction between insurance and dynamic financial market which is effectively complete with homogeneous belief. Anderson and Raimondo (2008) provide a non-degeneracy condition on the terminal security dividends to insure completeness in equilibrium with homogeneous belief.

Beyond the classical two-consumption date economy, many other works are contributed to study the infinite-consumption model with heterogeneous beliefs. Buraschi and Jiltsov (2006) and David (2008) investigate continuous-time model in which the power utility investors update beliefs with the dividend process following a geometric Brownian motion, with heterogeneous prior beliefs. The market in their model is effectively incomplete. Moreover, their information structure does not allow heterogeneity in the dividend volatility. Thus, investors conduct no speculative activity with respect to the volatility of dividend and signals. Beker and Espino (2011), in a discrete-time framework, analyze the dynamic properties of portfolios that sustain dynamically complete markets equilibria when investors have heterogeneous priors.

Although both the Black-Scholes model and this paper involve the notion of replicating

the value of redundant derivative, the method in this paper fundamentally differs from that in the Black-Scholes model. First, the Radner equilibrium endogenously replicates the payoff of the redundant asset, in which both the price process of the underlying asset and the replicating strategies are endogenized in the Radner equilibrium. In contrast, in the Black and Scholes (1973) option pricing model, the underlying price process is exogenously given. Furthermore, Black and Scholes (1973) cannot be generalized to the case with heterogeneously perceived volatility, since under individual equivalent probability measure, the volatility has to be identical across investors.

This paper is organized as follows. The primitives of the economy and the learning mechanism in continuous time are established in Section 2. Section 3 establishes a continuous trading Radner equilibrium to implement the same Arrow-Debreu consumption allocations with heterogeneous beliefs in Christensen and Qin (2012). Propositions present the expressions of the equilibrium stock price, the equilibrium portfolios, and the price of the dividend derivative as functions of posterior beliefs, revealing the impacts of public signals. Section 4 concludes the paper. Proofs of lemmas are provided in Appendix A, and the proofs of propositions and the methods to implement the allocation in the Arrow-Debreu equilibrium with heterogeneous beliefs are provided in Appendix B.

2 The Model

I examine an economy with two consumption dates and investors can trade continuously in between with heterogeneous beliefs on the terminal dividend. The model extends the model in Duffie and Huang (1985) to a heterogeneous-belief framework with information only contingent on the terminal aggregate consumption. The investors implement their Pareto optimal consumption plans in a continuous-time and continuous state-space economy.

2.1 The Investors' Beliefs and Preferences

Uncertainty in the economy is represented by an individual-specific product probability space¹ $(\Omega_{D_T} \times \Omega, \mathcal{F}_{D_T} \times \mathcal{F}, P_{D_T}^i \times P)$, where Ω_{D_T} and Ω are independent. The sample

¹For a brief introduction of product probability space, see, e.g. Grigoriu (2002). Consider two probability spaces $(\Omega_1, \mathcal{F}_1, P_1)$, and $(\Omega_2, \mathcal{F}_2, P_2)$, describing two experiments. These two experiments can be characterized jointly by the product probability space $(\Omega_0, \mathcal{F}_0, P_0)$ with the following components.

(i) Product sample space:

$$\Omega_0 = \Omega_1 \times \Omega_2 = \{(\omega_1, \omega_2) : \omega_k \in \Omega_k, k = 1, 2\}.$$

(ii) Product σ -field: $\mathcal{F}_0 = \mathcal{F}_1 \times \mathcal{F}_2 = \sigma(\mathfrak{R})$, where measurable rectangles (ii) Product σ -field:

space Ω_{D_T} contains all possible terminal dividend, and Ω on which is defined an Ornstein-Uhlenbeck Bridge, y , driven by a one-dimensional Brownian motion W . Let $\{\mathcal{F}_t\}$ denotes the augmented filtration generated by $y(t)$, and \mathcal{F}_{D_T} is a σ -field independent of \mathcal{F} . The field \mathcal{F}_{D_T} , whose role is to allow for heterogeneity in investors' priors, consists of all possible initial beliefs. The complete information filtration is the augmentation of the filtration $\mathcal{F}_{D_T} \times \{\mathcal{F}_t\}$.

There are two consumption dates, $t = 0$ and $t = T$, and there are I investors who are endowed at $t = 0$ with a portfolio of marketed securities. The investors potentially receive public information which is revealed by a Standard Brownian Motion continuously at $t \in [0, T]$, and receive terminal normally-distributed dividends at $t = T$. The trading of the marketed securities takes place at $t \in [0, T]$. There are two marketed securities: a zero-coupon bond which pays one unit of consumption at $t = T$ and is in zero net supply, the shares of a single risky firm which have net supplies Z at $t \in [0, T]$. The assumptions of endowment in this paper are identical to that in Christensen and Qin (2012). The investors are endowed with $\bar{\gamma}_i$ units of the $t = T$ zero-coupon bond and \bar{z}_i shares of the risky asset, $i = 1, 2, \dots, I$. In addition, the investors are endowed with $\bar{\kappa}_i$ units of a zero-coupon bond, also in zero net-supply, paying one unit of consumption at $t = 0$. Let x_{it} and γ_{it} present investor i 's portfolio of share and units had of the zero-coupon bond after trading at date t , respectively. Hence, the market clearing conditions at date t are

$$\sum_{i=1}^I \gamma_{it} = 0, \quad \sum_{i=1}^I x_{it} = Z \equiv \sum_{i=1}^I \bar{z}_i, \quad t \in [0, T]. \quad (1)$$

The investor i 's consumption at date t is denoted c_{it} and they have time-additive utility. The common period-specific utility is negative exponential utility with respect to consumption, i.e., $u_{i0}(c_{i0}) = -\exp[-rc_{i0}]$ and $u_{iT}(c_{iT}) = -\exp[-\delta] \exp[-rc_{iT}]$, where $r > 0$ is the investors' common constant absolute risk aversion parameter. Moreover, the investor i 's has common utility discount rate, δ , for date $t = T$ consumption.

$$\mathfrak{R} = \{A_1 \times A_2 : A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2\}.$$

(iii) Product probability measure $P_0 = P_1 \times P_2$, on the measurable space (Ω, \mathcal{F}) : The probability P_0 is unique and has the property

$$P_0(A_1 \times A_2) = P_1(A_1)P_2(A_2) : A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2.$$

2.2 Learning Mechanism

In this section, I construct an information structure which is a continuous-time extension of the two-period learning model in Christensen and Qin (2012). This information structure satisfies that: (1) Signals gradually reveal the information about the terminal dividend; (2) Investors hold heterogeneous prior beliefs, and continuously update their beliefs according to the Bayes' rule. They know the terminal dividend perfectly at $t = T$; (3) The information structure should allow heterogeneity in perceived variance of the terminal dividend and, thus, the investors can speculate on the variance of the dividend throughout the interval $[0, T]$.

A share of the risky asset pays a dividend d_0 at date $t = 0$ and a dividend d_T at date $t = T$. At $t = 0$, the investor i views d_T as a normally distributed variable, with mean m_{i0} and variance σ_{i0}^2 . The investors observe a continuous signal process, y_t , follows the differential form according to Eq. (3) in the following Lemma.

Lemma 1 (*Ornstein-Uhlenbeck Bridge*) *Given deterministic functions $A(t) > 0$ and $B(s)$ satisfying that*

$$\lim_{t \rightarrow T} A(t) = 0 \text{ and } \lim_{s \rightarrow T} B(s)^2 e^{\int_0^s A(u)^{-1} du} = 0, \quad (2)$$

then there exists a unique stochastic signal process $y = (y_t)$ solving the following SDE²

$$dy_t = A(t)^{-1} (d_T - y_t) dt + B(t) dW_t, \quad t \in (0, T), \quad y_0 \in R, \quad (3)$$

and $y_t \rightarrow d_T$, P -a.s., as $t \rightarrow T$, where d_T is the terminal dividend at $t = T$.

Proof. See the Appendix.

Note the terminal dividend d_T and Brownian motion in the signal process is independent, and the signal y_t is measurable to $\Omega_{D_T} \times \Omega$. Particularly, the prior beliefs about d_T at date $t = 0$ is \mathcal{F}_{D_T} -measurable. This assumption is consistent with the discrete-time information structure in Christensen and Qin (2012)³.

Given the signal process defined above, the posterior mean and posterior variance of each investor can be derived by employing the standard filtering theorem in Liptser and

²Although the investors perceive the terminal dividend d_T as a random variable, the nature determines the terminal dividend. Thus, the terminal dividend in the SDE of the public signal can be viewed as an exogenous parameter. As a result, the public signal process is adapted to the filtration $\{\mathcal{F}_t\}$. Moreover, with filtration \mathcal{F}_t the investors can observe the signal y_t , but cannot observe the Brownian motion W .

³The information structure in this paper is different from the Kyle-Back model of "insider trading" (see Kyle 1985 and Back 1992) or the dynamic Markov bridges motivated by models of insider trading (see Campi, Cetin, and Danilova 2011), in which a gradually informed insider observes a signal process (unknown to the market), and the signal process converges to a terminal value which is stochastic and not known in advance.

Shiryayev (1977). To ensure the signal process converges to d_T , the coefficients of the signal process have to meet condition (2). Moreover, according to Liptser and Shiryayev (1977), the following conditions are required to achieve a Bayesian learning:

$$\int_0^T A(t)^{-2} dt < \infty, \quad \int_0^T B(t)^2 dt < \infty. \quad (4)$$

I specify that

$$A = \alpha^{-1} (T - t)^{-\frac{k}{2}}, \quad -1 < k < 0, \quad \alpha > 0, \quad (5)$$

and

$$B = \beta (T - t)^q, \quad q > 0, \quad \beta \in R, \quad (6)$$

and proof that the specified coefficients A and B meet all the requirements in (2) and (4). See proofs in the Appendix.

With the specified coefficients, the signal process which converges to the terminal dividend can be stated as

$$dy_t = \alpha (T - t)^{\frac{k}{2}} (d_T - y_t) dt + \beta (T - t)^q dW_t, \quad t \in (0, T), \quad y_0 \in R. \quad (7)$$

Note there is a linear dependence of the observable component y_t in the drift coefficient of the signal process. Denote the expectation and variance conditional on observed signals up to date t by m_{it} and σ_{it}^2 . I use $h_{it} \equiv 1/\sigma_{it}^2$ throughout to denote precisions for the associated variances. According to Theorem 10.3 in Liptser and Shiryayev (1977), the dynamics of the conditional expectations and conditional variances are given by (also see Lemma 2 in the Appendix, i.e., a benchmark case of Theorem 10.3 in Liptser and Shiryayev (1977))

$$dm_{it} = \frac{\alpha \sigma_{it}^2}{\beta^2} (T - t)^{\frac{k}{2} - 2q} \left[dy_t - \alpha (T - t)^{\frac{k}{2}} (m_{it} - y_t) dt \right], \quad (8)$$

and

$$d\sigma_{it}^2 = -\sigma_{it}^4 \frac{\alpha^2}{\beta^2} (T - t)^{k - 2q} dt.$$

The posterior variance follows an ODE, solve for the ODE, I obtain,

$$\sigma_{it}^2 = \frac{(k - 2q + 1) \beta^2 \sigma_{i0}^2}{\alpha^2 \sigma_{i0}^2 T^{k - 2q + 1} - \alpha^2 \sigma_{i0}^2 (T - t)^{k - 2q + 1} + \beta^2 (k - 2q + 1)}. \quad (9)$$

When $t \rightarrow 0$, $\sigma_{it}^2 \rightarrow \sigma_{i0}^2$, and when $t \rightarrow T$, if $k - 2q + 1 < 0$, then $(T - t)^{k - 2q + 1} \rightarrow \infty$, thus, $\sigma_{it}^2 \rightarrow 0$. Note $k < 0 < 2q - 1$, thus, when $q > \frac{1}{2}$, the posterior variance decreases from the prior variance to zero, and all the investors know the terminal dividend at $t = T$ perfectly.

The heterogeneously updated beliefs give basis for side-betting over the interval $[0, T)$.

With normally distributed terminal dividend, Christensen and Qin (2012) employ a Bayesian learning model in discrete time to model the information structure. As a counterpart information structure in continuous time, the information is revealed and the posterior mean is driven by the Standard Brownian Motion with multiplicity one⁴. This mathematical characteristic has important asset pricing implications, as shown in following sections.

3 Equilibrium with Heterogeneity in Beliefs and Information on Terminal Aggregate Consumption

In this section, I show with heterogeneous beliefs, how the trading strategies and the asset prices in a continuous trading Radner (1972) equilibrium implement the identical Arrow-Debreu consumption allocations in Christensen and Qin (2012), with only two long-lived securities.

3.1 Arrow-Debreu Equilibrium

Under the assumptions of exponential utility investors with heterogeneous beliefs and normally distributed dividend, Christensen and Qin (2012) show that effectively complete market can be achieved if allow the investors trade in only three assets, i.e., a zero coupon bond, a stock, and a derivative which pays off the square of the dividend at the terminal date. In such a Arrow-Debreu equilibrium, the Pareto efficient consumption is a linear function of the aggregate consumption plus a state-dependent term. In other words, the investors share risk (side bet) linearly with heterogeneous beliefs.

3.2 Radner Equilibrium

It is a standard result that the assumptions of heterogeneous beliefs and time-additive preferences represented by exponential utility imply that the Pareto efficient individual consumption plans are measurable with respect to the aggregate consumption (see, e.g., Christensen and Feltham (2003), Chapter 4). Hence, under such a framework, to facilitate the Pareto

⁴A brief introduction of martingale multiplicity is given as follows. The space of square-integrable martingales on (Ω, \mathcal{F}, P) which are null at zero is denoted M_P^2 . Two martingales \tilde{X} and \tilde{Y} are said to be orthogonal if the product $\tilde{X}\tilde{Y}$ is a martingale. Defined an orthogonal 2-basis for M_P^2 as a minimal set of mutually orthogonal elements of M_P^2 with the representation property. Then, the number of elements of a 2-basis, whether countably infinite or some positive integer, is called the multiplicity of M_P^2 , denoted $\mathbf{M}(M_P^2)$. Refer to the Appendix in Duffie and Huang (1985) for a detailed description of the martingale multiplicity.

efficient allocation of consumption, the securities only have to facilitate allocations which are measurable with respect to aggregate consumption. The notion of complete market which needs infinite securities is not necessary anymore.

3.2.1 Implementing Pareto Efficient Consumption Allocation

Duffie and Huang (1985) provide a procedure to implement the Pareto consumption plan in dynamically complete market. Each consumption allocation including those not measurable to the aggregate consumption can be implemented by the price-contingent portfolio. However, in this paper, investors only receive the information which is about the aggregate consumption, thus, I can use their procedure to implement consumption plan, which is measurable to the aggregate consumption.

Assume Q as the martingale measure, the space of square-integrable martingales under Q , denoted M_Q^2 ; its multiplicity, denoted $\mathbf{M}(M_Q^2)$. To employ their procedure, I first have to specify an orthogonal 2-basis for M_Q^2 . Since the multiplicity, $\mathbf{M}(M_Q^2)$, determines how many securities are needed to dynamically effectively complete the market.

According the information structure in the previous section, a Standard Brownian Motion W reveals the information on the aggregate consumption. It is a well known result that the underlying Brownian Motion W is a 2-basis for M_P^2 . Assuming $Q \approx P$, the process

$$\Lambda(t) = \mathbb{E} \left[\frac{dQ}{dP} \mid \mathcal{F}_t \right], \quad t \in [0, T],$$

is a square-integrable martingale on (Ω, \mathcal{F}, P) , with $\mathbb{E}[\Lambda(T)] = 1$.

Note in this paper, individual Pareto optimal consumption allocation is measurable with respect to the aggregation consumption, hence, applying Theorem 4.1 in Duffie and Huang (1985), there exists a trading strategy contingent on the information of the aggregate consumption to implement the individual Pareto optimal consumption plan. In other words, there exists some $\varrho \in L_P^2[W]$ giving the representation

$$\Lambda(t) = 1 + \int_0^t \varrho(s) dW(s),$$

where $\Lambda(t)$ is measurable to the aggregate consumption. It follows from Ito's Lemma that, defining the process $\eta(t) = \varrho(t)/\Lambda(t)$, yields the alternative representation

$$\Lambda(t) = \exp \left(\int_0^t \eta(s) dW(s) - \frac{1}{2} \int_0^t [\eta(s)]^2 ds \right).$$

From this representation, the new process

$$W^*(t) = W(t) - \int_0^t \eta(s) ds,$$

defines a Standard Brownian Motion on (Ω, \mathcal{F}, Q) by Girsanov's Fundamental Theorem (Liptser and Shiriyayev 1977, p.232). It remains to show that W^* is itself a 2-basis for M_Q^2 , but this is immediate from Theorem 5.18 of Liptser and Shiriyayev (1977), using the uniform absolute continuity of P and Q .

With the orthogonal 2-basis for M_Q^2 , W^* , I can apply the procedure in Duffie and Huang (1985) which includes four steps to implement all the allocations which are measurable with respect to aggregate consumption by trading in only two long-lived securities: (See details in the Appendix)

(1) Specify a set of long-lived securities: Since the multiplicity M_Q^2 is 1, according Proposition 5.1 in Duffie and Huang (1985), I only need $M_Q^2 + 1$, i.e., two securities to dynamically effectively complete the markets: A riskless bond and a risky asset.

(2) Announce a price for $t = 0$ consumption and price processes for the long-lived securities: Duffie and Huang (1985) point out that the valid price processes for the riskless bond and stock should be 1 and the orthogonal 2-basis for M_Q^2 on Q , $W^*(t)$, respectively.

(3) Allocate a trading strategy to each investor which generates that investor's Arrow-Debreu allocation and which, collectively, clears markets.

(4) Prove that no investor has any incentive to deviate from the allocated trading strategy.

Essentially, by marketing only two long-lived securities, one paying $W^*(T)$ in date T consumption, the other paying one unit of date T consumption with certainty, and announcing their price processes as $W^*(t)$ and 1 (for all t), a Radner equilibrium in dynamically effectively complete markets is achieved. Summarize the above results, I achieve the following proposition.

Proposition 1 *Assume the exponential-utility investors rationally update their posterior beliefs of the terminal dividend according to the optimal filtering equations (8) and (9), given the observed realizations of the signal.*

- (a) *The market in which the investors can trade in one stock in addition to a riskless bond continuously is dynamically effectively complete.*
- (b) *The Radner equilibrium implements the Pareto optimal consumptions in the corresponding Arrow-Debreu equilibrium. Both the stock price and the trading strategies in the Radner equilibrium are contingent on the information of the aggregate consumption.*

Proof: See Appendix.

Proposition 1 notes the stock price and the trading strategies are contingent on the information of the aggregate consumption, however, it does not provide the explicit expressions for the stock price and portfolios to show how they exactly depend on the public signals. Next subsection presents concrete expressions of the equilibrium stock price and the equilibrium portfolios in the Radner equilibrium, under specific assumptions of utility and dividend structure.

To sum up, the market in this paper is called dynamically effectively complete markets. This type of market is sufficient to ensure the existence of an equilibrium with Pareto optimal consumption allocations, despite the fact that investors may not be able to implement any financially feasible consumption plan as they are in a dynamically complete market.

3.2.2 Equilibrium Security Prices with Impacts from Posterior Beliefs and Public Signals

In general, it is technically difficult to derive the candidate prices to implement the Arrow-Debreu equilibrium, since these prices are usually given by conditional expectations which cannot be computed explicitly. However, under the assumptions of exponential utility and the normally distributed dividend, in this paper, I can derive the explicit expressions of the equilibrium security prices and equilibrium portfolios in the Radner equilibrium to implement the Arrow-Debreu equilibrium in Christensen and Qin (2012).

First of all, I assume the investors trade in the riskless bond with price 1 all the time. In the dynamically effectively complete market, I can employ the standard theory of martingale approach⁵ to solve for the individual-specific state-price deflator, and thus, the security price.

The individual decision problem for the investor i is

$$\max u_{iT}(c_{iT}) \quad s.t. \quad E^i [\zeta_{iT}c_{iT} | \mathcal{F}_t] \leq w_{i0},$$

where ζ_{iT} is the individual-specific state-price deflator at the terminal date T , and w_{i0} is the wealth of investor i at date 0. Let the Lagrangian multiplier be λ_{i0} , thus, the First-Order Condition gives

$$u'_{iT}(c_{iT}^*) = \lambda_{i0}\zeta_{iT}^*,$$

thus, the individual-specific state-price deflator at the terminal date T , ζ_{iT} , is given as a

⁵For an introduction of the standard theory of martingale approach, see, e.g., the Chapter 9 in (?), or a more recent paper, (?) .

function of the optimal terminal consumption,

$$\zeta_{iT}^* = \frac{u'_{iT}(c_{iT}^*)}{\lambda_{i0}} = \frac{r}{\lambda_{i0}} \exp(-rc_{iT}^*).$$

Note the individual-specific state-price deflator is proportional to the marginal utility of the optimal terminal consumption. Christensen and Qin (2012) derive the Pareto optimal terminal consumption in the Arrow-Debreu equilibrium, $c_{iT}^* = \theta_{i0}^\dagger d_T^2 + x_{i0}^\dagger d_T + \gamma_{i0}^\dagger$, where θ_{i0}^\dagger , x_{i0}^\dagger and γ_{i0}^\dagger are the $t = 0$ equilibrium portfolios of the dividend derivative, the stock, and the riskless bond in the effectively complete market, respectively. Specifically,

$$x_{i0}^\dagger = \rho [h_{i0}m_{i0} - \overline{hm}] + Z/I, \quad \rho \equiv \frac{1}{r}, \quad \overline{hm} \equiv \frac{1}{I} \sum_{i=1}^I h_{i0}m_{i0}, \quad i = 1, \dots, I,$$

$$\theta_{i0}^\dagger = \frac{1}{2}\rho [\bar{h} - h_{i0}], \quad \bar{h} \equiv \frac{1}{I} \sum_{i=1}^I h_{i0}, \quad i = 1, \dots, I.$$

Given the portfolios θ_{i0}^\dagger , x_{i0}^\dagger and γ_{i0}^\dagger all known, and from the perspective of the investor i at date t , $d_T \sim N(m_{it}, \sigma_{it}^2)$, the conditional expectation of the individual-specific state-price deflator at the terminal date T , ζ_{iT}^* , can be calculated. Also note that due to the fact that the riskless interest rate is constantly zero, the state-price density process, $\zeta_i^* = (\zeta_{it}^*)$, for each investor i , is a martingale, i.e.,

$$\zeta_{it}^*(y_t) = \mathbb{E}^i[\zeta_{iT}^* | \mathcal{F}_t],$$

hence, the stock price at date t can be computed according to

$$p_t(y_t) = \frac{\mathbb{E}^i[\zeta_{iT}^* d_T | \mathcal{F}_t]}{\zeta_{it}^*}. \quad (10)$$

Calculate the conditional expectation, the stock price at date t is simplified as a function of the priors and the posterior beliefs,

$$p_t(y_t) = \frac{\overline{hm} - h_{i0}m_i + h_{it}m_{it}(y_t) - rZ/I}{\bar{h} - h_{i0} + h_{it}}.$$

At the first glimpse, the stock price looks individual specific. However, by substituting for the posterior mean $m_{it}(y_t)$ and the posterior precision h_{it} (in terms of the priors and signal y_t), I show the price is common for all the investors in the following proposition.

Proposition 2 *Assume the economy is populated by exponential-utility investors who per-*

ceive normally distributed dividend with heterogeneous beliefs. The equilibrium stock price in the Radner equilibrium is given as a precision weighted average of the investors' posterior mean minus a risk premium determined by the average posterior precision, i.e.,

$$p_t(y_t) = \bar{m}_t^h(y_t) - r\bar{\sigma}_t^2 Z/I,$$

where \bar{m}_t^h is the precision weighted average of the investors' posterior means, i.e.,

$$\bar{m}_t^h(y_t) \equiv \frac{1}{I} \sum_{i=1}^I \frac{h_{it}}{\bar{h}_t} m_{it}(y_t), \quad \bar{h}_t \equiv \frac{1}{I} \sum_{i=1}^I h_{it},$$

and $\bar{\sigma}_t^2$ is the inverse of the average posterior precision, i.e., $\bar{\sigma}_t^2 \equiv 1/\bar{h}_t$.

Proof. See Appendix.

Note the expression of the stock price is consistent to that in Christensen and Qin (2012), in a sense that the stock price is given as the discounted expected risk-adjusted dividend. In this paper, the riskless discount factor is a constant one, hence, the stock price is immediately equal to the expected risk-adjusted dividend, i.e., $p_t(y_t) = E^Q[d_T|\mathcal{F}_t]$ ⁶. Moreover, define a representative investor holding a consensus belief of the terminal dividend as $d_T \sim N(\bar{m}_t^h, \bar{\sigma}_t^2)$, thus, a homogeneous-belief model with the representative investor generates the same equilibrium prices as in Proposition 2. Obviously, the stock price is driven by the heterogeneously updated posterior beliefs and, thus, driven by the prior beliefs and the public signals (as functions of the Brownian motion). Aggregation of heterogeneous beliefs is also discussed in Chiarella, Dieci, and He (2006) and Jouini and Napp (2007).

3.2.3 Equilibrium Portfolios with Impacts from Posterior Beliefs and Public Signals

With the concrete expression of equilibrium stock price, I can derive the self-financing optimal trading strategies. Note in most stochastic optimization problems, posed in the general financial market models, investors ascertain only the existence of the associated portfolio strategies, since the Martingale Presentation Theorem does not provide explicit relating integrand. However, following the pioneer work on explicit descriptions of the integrand by

⁶Similar to Christensen and Qin (2012), I define the risk-adjusted probability measure Q explicitly such that conditional on the information at date t under Q , the terminal dividend is normally distributed as $d \sim N(\bar{m}_t^h - r\bar{\sigma}_t^2 Z/I, \bar{\sigma}_t^2)$. Note while the expected dividend under Q is uniquely determined in equilibrium, the variance of the dividend under Q is not uniquely determined due to the market incompleteness and, thus, I just take it be $\bar{\sigma}_t^2$. Fortunately, the lack of the uniqueness of the variance has no consequences in the subsequent analysis.

Clark (1970), Ocone and Karatzast (1991) generalize the Clark-Ocone formula. Using the tool from Malliavin calculus, they derive a general representation formula for the optimal portfolios⁷. Their formulae provide very explicit expressions for the optimal portfolios in feedback form on the current level of wealth, in the market with deterministic riskless interest rate and deterministic market price of risk. Note rewriting the form of the stock price, Eq. (B11) shows that the market price of risk implied in stock price under P -measure is deterministic, and the riskless interest rate is constantly zero. Thus, by using the generalized Clark-Ocone formula in Proposition 2.5 and formula of Eq. (3.11) in Ocone and Karatzast (1991), I calculate the self-financing optimal trading strategies, and obtain the following proposition.

Proposition 3 *Assume the economy is populated by exponential-utility investors who perceive normally distributed dividend with heterogeneous beliefs. In the Radner equilibrium, the equilibrium portfolios in the risky asset and riskless bond are respectively given as*

$$x_{it}^*(y_t) = 2\theta_{i0}^\dagger p_t(y_t) + x_{i0}^\dagger, \quad \gamma_{it}^*(y_t) = \theta_{i0}^\dagger \bar{\sigma}_t^2 + \gamma_{i0}^\dagger - \theta_{i0}^\dagger p_t^2(y_t).$$

Proof. See Appendix.

As shown by the expressions of the equilibrium portfolios, intuitively, to achieve the Pareto optimal consumption c_{iT}^* , the investor i hold constantly x_{i0}^\dagger share of the stock, and γ_{i0}^\dagger share of the riskless bond, then he achieves the part of $x_{i0}^\dagger d_T + \gamma_{i0}^\dagger$ in the optimal consumption. In the following subsection, I show that the part of $\theta_{i0}^\dagger d_T^2$ in the optimal consumption is achieved by holding $2\theta_{i0}^\dagger p_t(y_t)$ share of the stock, and $\theta_{i0}^\dagger (\bar{\sigma}_t^2 - p_t^2(y_t))$ share of the riskless bond. Furthermore, the stock price $p_t(y_t)$ is a sufficient statistic for computation of the optimal portfolios and thus, the optimal wealth processes.

3.2.4 Replicate Payoff of Derivative Paying off the Square of Terminal Dividend

Christensen and Qin (2012) introduce a dividend derivative which pays off the square of the dividend at the terminal date to effectively complete the market in a Arrow-Debreu Equilibrium. The "Dividend-Square Security" with ideal convexity in its payoff profile facilitates Pareto efficient side-betting. Interestingly, similar to Black and Scholes (1973), in which the value of derivative (option) can be replicated by trading underlying asset continuously, here in my model, the payment of the "Dividend-Square Security" in Christensen and Qin (2012)

⁷Other particularly significant works on explicit descriptions of the integrand are Haussmann (1979), Ocone (1984), and Karatzast, Ocone, and Li (1991), among others. Davis (2005) survey the problems of martingale representation, especially those martingales with finite multiplicity. More recently, Renaud and Remillard (2007) apply the Clark-Ocone formula to option pricing and obtain explicit trading portfolios.

can be replicated by trading two securities continuously. Since any function of the dividend is measurable with respect to aggregate consumption, thus, the payoff of "Dividend-Square Security" is measurable with respect to aggregate consumption. Hence, there exists a trade strategy to replicate the payoff of the "Dividend-Square Security". This fact shows that from a welfare perspective, continuous trading is a replacement of the convexity in the payoff of the derivative, which can be attained by using Gamma trading strategies.

Specifically, the dividend derivative in the Radner equilibrium is a redundant asset. The price of "Dividend-Square Security" and the replicating trading strategies are given by the following proposition.

Proposition 4 *Assume the economy is populated by exponential-utility investors who perceive normally distributed dividend with heterogeneous beliefs. The price of the redundant dividend derivative in the Radner equilibrium is given as*

$$\pi_t(y_t) = \frac{\mathbb{E}^i[\zeta_{iT}^* d_T^2 | \mathcal{F}_t]}{\zeta_{it}^*} = \bar{\sigma}_t^2 + p_t^2(y_t),$$

and the value of θ_{i0}^\dagger share of the dividend derivative can be replicated by investing in $2\theta_{i0}^\dagger p_t(y_t)$ share of the stock, and $\theta_{i0}^\dagger (\bar{\sigma}_t^2 - p_t^2(y_t))$ share of the riskless bond continuously.

Proof. See Appendix.

Note the riskless discount factor is a constant one, hence, the price of the dividend derivative is immediately equal to the expected risk-adjusted payment of square of the dividend, i.e., $\pi_t(y_t) = \mathbb{E}^Q[d_T^2 | \mathcal{F}_t]$. Intuitively, the risk-adjusted expectation of the derivative payoff is affected by the posterior beliefs and, thus, affected by the prior beliefs and public signals. Moreover, the expression of the price of the dividend derivative shows that the investors form their Pareto optimal trading strategies by investing as if they intend to dynamically replicate the value of the dividend derivative. This result intuitively demonstrates how the dynamically effectively complete market in this paper is equivalent to the effectively complete market in Christensen and Qin (2012).

Although the interpretation of the equilibrium involves the notion of replication, the method in this paper fundamentally differs from that in the Black-Scholes model. First, the Radner equilibrium endogenously replicate the payoff of the redundant asset, in which both the price process of the underlying asset and the replicating strategies are endogenized in the Radner equilibrium. In contrast, in the Black and Scholes (1973) option pricing model, the underlying price process is exogenously given. Furthermore, Black and Scholes (1973) cannot be generalized to the heterogeneous beliefs case with heterogeneous volatility, since

under individual equivalent probability measure, the volatility has to be identical across investors. However, information structure in this paper allows heterogeneity in perceived variance of the terminal dividend. It is the investors' speculations on the variance of the dividend throughout the interval $[0, T]$ dynamically effectively complete the market.

4 Conclusion

The assumptions of heterogeneous beliefs and the information on aggregate consumption have substantial influence in continuous-time financial models. Comparing to the benchmark homogeneous belief model, I achieve a less strong but effectively equivalent result, i.e., continuous trading can effectively dynamically complete the financial market with heterogeneous beliefs. The investors in such an economy can deal with all the risk on the aggregate consumption and attain their Pareto optimal allocations by trading in a few securities.

The assumptions of negative exponential utility and normally distributed dividends enable me to achieve explicit expressions of the equilibrium security prices and equilibrium portfolios in the Radner equilibrium to implement the Arrow-Debreu equilibrium. More realistic assumptions of preferences and non-normal distributed dividend may not lead to these analytical equilibrium properties. However, adding jumps into the information structure in this paper may maintain some nice properties, I leave this for future research.

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A Appendix: Proof of Lemmas

A.1 Proof of Lemma 1

The proof is similar to that in the Appendix of Christensen, Larsen, and Munk (2011). However, the structure of the drift and volatility is different, and I introduce a new condition, i.e., Eq. (2) to ensure the convergence to the terminal dividend.

I define the deterministic function $b(t) \equiv -A(t)^{-1}$ and note that $b(t) \rightarrow -\infty$ as $t \rightarrow T$. A direct application of Ito’s product rule gives that the stochastic process

$$\bar{X}_s = e^{\int_0^s b(u)du} \left(\bar{X}_0 + \int_0^s e^{-\int_0^t b(u)du} (d_T A(t)^{-1} dt + B(t) dW_t) \right), \quad s \in [0, T],$$

satisfies the SDE (3). Furthermore, L'Hopital's rule gives

$$\lim_{s \rightarrow T} \frac{\int_0^s e^{-\int_0^t b(u)du} (d_T A(t)^{-1}) dt}{e^{\int_0^s -b(u)du}} = \lim_{s \rightarrow T} \frac{d_T A(s)^{-1}}{-b(s)} = d_T.$$

The proof can therefore be concluded by showing

$$e^{\int_0^s b(u)du} M_s = e^{\int_0^s b(u)du} \int_0^s e^{-\int_0^t b(u)du} B(s) dW_t \rightarrow 0, \quad \text{P-a.s.},$$

as $s \rightarrow T$. The quadratic variation of M is given by

$$\langle M \rangle_s = \int_0^s e^{-2\int_0^t b(u)du} \frac{1}{B(t)^{-2}} dt, \quad s \in [0, T].$$

If $\langle M \rangle_T < \infty$, I trivially have that M is a continuous martingale on the interval $[0, T]$ and, in particular, M_T is a real valued random variable and the claim follows. If $\langle M \rangle_T = \infty$, I can use Exercise II.15 in Protter (2004) to see that

$$\lim_{s \rightarrow T} \frac{M_t}{\langle M \rangle_t} = 0, \quad \text{P-almost surely.}$$

L'Hopital's rule gives

$$\lim_{s \rightarrow T} \langle M \rangle_s e^{\int_0^s b(u)du} = \lim_{s \rightarrow T} \frac{\int_0^s e^{-2\int_0^t b(u)du} \frac{1}{B(t)^{-2}} dt}{e^{-\int_0^s b(u)du}} = \lim_{s \rightarrow T} \frac{e^{-\int_0^s b(u)du} dt}{-b(s) B(s)^{-2}}, \quad (\text{A1})$$

and the condition (2) suffices to ensure the above limit is zero. This completes the proof.

A.2 Proof the Specified Coefficients Meet the Requirements of Signal Process and Filtering Equation

Now I proof the coefficients

$$A = \frac{1}{\alpha (T-t)^{\frac{k}{2}}}, \quad -1 < k < 0, \quad \alpha > 0,$$

and

$$B = \beta (T-t)^q, \quad q > 0, \quad \beta \in R,$$

meet following conditions⁸

$$\lim_{t \rightarrow T} A(t) = 0, \quad \lim_{s \rightarrow T} B(s)^2 e^{\int_0^s A(u)^{-1} du} = 0, \quad \int_0^T A(t)^{-2} dt < \infty, \quad \int_0^T B(t)^2 dt < \infty.$$

Since $k < 0$, so that $A \rightarrow 0$ as $t \rightarrow T$, and $B \rightarrow 0$ as $t \rightarrow T$. Moreover, to proof

$$\lim_{s \rightarrow T} B(s)^2 e^{\int_0^s A(u)^{-1} du} = 0,$$

I only need to proof

$$\lim_{s \rightarrow T} e^{\int_0^s A(u)^{-1} du} < \infty.$$

Note let $v = T - u$,

$$\begin{aligned} \lim_{s \rightarrow T} e^{\alpha \int_0^s (T-u)^{\frac{k}{2}} du} &= \lim_{s \rightarrow T} e^{-\alpha \int_0^s (T-u)^{\frac{k}{2}} d(T-u)} = \lim_{s \rightarrow T} e^{-\alpha \int_T^{T-s} v^{\frac{k}{2}} dv} \\ &= \lim_{s \rightarrow T} e^{\alpha \left(-\frac{v^{\frac{k}{2}+1}}{\frac{k}{2}+1} \Big|_T^{T-s} \right)} = \lim_{s \rightarrow T} e^{\alpha \left(-\frac{(T-s)^{\frac{k}{2}+1}}{\frac{k}{2}+1} + \frac{T^{\frac{k}{2}+1}}{\frac{k}{2}+1} \right)} \\ &= \lim_{s \rightarrow T} e^{\alpha \left(\frac{T^{\frac{k}{2}+1} - (T-s)^{\frac{k}{2}+1}}{\frac{k}{2}+1} \right)} = \lim_{s \rightarrow T} e^{\frac{2\alpha}{k+2} \left(T^{\frac{k}{2}+1} - (T-s)^{\frac{k}{2}+1} \right)} \\ &= \lim_{s \rightarrow T} e^{\left(\frac{2\alpha}{k+2} T^{\frac{k}{2}+1} - \frac{2\alpha}{k+2} (T-s)^{\frac{k}{2}+1} \right)} = \lim_{s \rightarrow T} e^{\frac{2\alpha}{k+2} T^{\frac{k}{2}+1}} e^{-\frac{2\alpha}{k+2} (T-s)^{\frac{k}{2}+1}} \\ &= e^{\frac{2\alpha}{k+2} T^{\frac{k}{2}+1}} \lim_{s \rightarrow T} e^{-\frac{2\alpha}{k+2} (T-s)^{\frac{k}{2}+1}}, \end{aligned}$$

since $k > -2$, thus, $\frac{k}{2} + 1 > 0$, and

$$\lim_{s \rightarrow T} e^{-\frac{2\alpha}{k+2} (T-s)^{\frac{k}{2}+1}} = 1,$$

⁸Note a general Brownian bridge converging to d_T

$$dy_t = \frac{d_T - y_t}{T - t} \cdot dt + dW_t,$$

does not meet the condition required by the filtering equation in Liptser and Shiryaev (1977), since

$$\int_0^T \left(\frac{1}{T-t} \right)^2 dt = - \int_0^T (T-t)^{-2} d(T-t),$$

let $T - t = u$,

$$\int_0^T \left(\frac{1}{T-t} \right)^2 dt = - \int_T^0 u^{-2} du = - \left(\frac{u^{-1}}{-1} \Big|_T^0 \right) = \frac{1}{u} \Big|_T^0 = \infty.$$

thus,

$$\lim_{s \rightarrow T} \beta^2 (T - s)^{2q} e^{\alpha \int_0^s (T-u)^{\frac{k}{2}} du} = 0.$$

Furthermore, given $0 > k > -1$,

$$\int_0^T A(t)^{-2} dt = \alpha^2 \int_0^T (T-t)^k dt,$$

thus, as $t \rightarrow T$, $(T-t)^k \rightarrow \infty$, I now have to check the convergence of improper integrals of the second kind with singularity at $t = T$. Note

$$\begin{aligned} \int_0^T A(t)^{-2} dt &= \alpha^2 \int_0^T (T-t)^k dt \\ &= -\alpha^2 \int_0^T (T-t)^k d(-t) \stackrel{u=-t}{=} -\alpha^2 \int_0^{-T} (u+T)^k du \\ &= -\alpha^2 \int_0^{-T} (u-(-T))^k du = \alpha^2 \int_{-T}^0 (u-(-T))^k du \\ &= \alpha^2 \left[\frac{1}{1+k} (u-(-T))^{1+k} \right]_{-T}^0 = \frac{\alpha^2 T^{1+k}}{1+k}, \end{aligned}$$

and

$$\begin{aligned} \int_0^T B(t)^2 dt &= \beta^2 \int_0^T (T-t)^{2q} dt = -\beta^2 \int_0^T (T-t)^{2q} d(T-t) \\ \stackrel{u=T-t}{=} -\beta^2 \int_T^0 u^{2q} du &= \beta^2 \int_0^T u^{2q} du = \frac{\beta^2}{2q+1} u^{2q+1} \Big|_0^T = \frac{\beta^2 T^{2q+1}}{2q+1}. \end{aligned}$$

thus, the specified coefficients meet the requirements of signal process and filtering equation. With the specified coefficients, I can apply the following lemma, which is a simple specific case of the Theorem 10.3 in Liptser and Shiriyayev (1977).

Lemma 2 (Filtering Equation) *Let ξ_t be the observable process, and the coefficients of equations in*

$$d\xi_t = (A_1(t) \varsigma_t + A_2(t) \xi_t) dt + B_1(t) dW_t, \quad (\text{A2})$$

satisfy the conditions of

$$\int_0^T [A_1(t)^2 + A_2(t)^2] dt < \infty, \text{ and } \int_0^T B_1(t)^2 dt < \infty,$$

then the vector $m_t = M\left(\varsigma_t | \mathcal{F}_t^\xi\right)$ and the $\gamma_t = M\left((\varsigma_t - m_t)^2 | \mathcal{F}_t^\xi\right)$ are solutions of the system of equations

$$dm_t = \gamma_t A_1(t) B_1(t)^{-2} [d\xi_t - (A_1(t) m_t + A_2(t) \xi_t) dt],$$

and

$$d\gamma_t = -\gamma_t^2 A_1^2(t) B_1(t)^{-2} dt,$$

with the initial conditions $m_0 = M(\theta_0 | \xi_0)$, and $\gamma_0 = M[(\theta_0 - m_0)^2]$.

A.3 Learning Model with Specific Coefficients

With the specified coefficients (5) and (6), the signal process can be stated as

$$dy_t = \alpha (T - t)^{\frac{k}{2}} (d_T - y_t) dt + \beta (T - t)^q dW_t, \quad t \in (0, T), \quad y_0 \in R.$$

According to Lemma 2, the updating equations of the posterior mean and posterior variance for investor i is

$$dm_{it} = \frac{\alpha \sigma_{it}^2}{\beta^2} (T - t)^{\frac{k}{2} - 2q} \left[dy_t - \alpha (T - t)^{\frac{k}{2}} (m_{it} - y_t) dt \right],$$

and

$$d\sigma_{it}^2 = -\sigma_{it}^4 \frac{\alpha^2}{\beta^2} (T - t)^{k-2q} dt.$$

The posterior variance follows an ODE, solve for the ODE,

$$\begin{aligned} -\frac{1}{\sigma_{it}^4} d\sigma_{it}^2 &= \frac{\alpha^2}{\beta^2} (T - t)^{k-2q} dt \\ &\rightarrow \int -\frac{1}{\sigma_{it}^4} d\sigma_{it}^2 = \frac{\alpha^2}{\beta^2} \int (T - t)^{k-2q} dt + C_0 \\ &\rightarrow \int -\frac{1}{\sigma_{it}^4} d\sigma_{it}^2 = -\frac{\alpha^2}{\beta^2} \int (T - t)^{k-2q} d(T - t) + C_0 \\ &\rightarrow \frac{1}{\sigma_{it}^2} = -\frac{\alpha^2 (T - t)^{k-2q+1}}{\beta^2 (k - 2q + 1)} + C_0, \end{aligned}$$

when $t = 0$, the constant $C_0 = \frac{1}{\sigma_{i0}^2} + \frac{\alpha^2 T^{k-2q+1}}{\beta^2(k-2q+1)}$, thus,

$$\begin{aligned} \frac{1}{\sigma_{it}^2} &= -\frac{\alpha^2 (T-t)^{k-2q+1}}{\beta^2(k-2q+1)} + \frac{1}{\sigma_{i0}^2} + \frac{\alpha^2 T^{k-2q+1}}{\beta^2(k-2q+1)} \\ &\rightarrow \frac{1}{\sigma_{it}^2} = \frac{\alpha^2 \sigma_{i0}^2 T^{k-2q+1} - \alpha^2 \sigma_{i0}^2 (T-t)^{k-2q+1} + \beta^2(k-2q+1)}{(k-2q+1)\beta^2 \sigma_{i0}^2} \\ &\rightarrow \sigma_{it}^2 = \frac{(k-2q+1)\beta^2 \sigma_{i0}^2}{\alpha^2 \sigma_{i0}^2 T^{k-2q+1} - \alpha^2 \sigma_{i0}^2 (T-t)^{k-2q+1} + \beta^2(k-2q+1)}. \end{aligned}$$

This completes the derivation of the dynamics of the posterior mean and posterior variance.

A.4 Proof of Lemma 2

This proof is a special case when setting $a_0 = a_1 = a_2 = b_1 = b_2 = A_0 = B_2 = 0$ in Theorem 10.3 of Liptser and Shirayev (1977). The basic idea of the proof is that using a transformation to get rid of the linear dependence of the observable component ξ_t , and thus, I can obtain the filtering equation by applying the Theorem 10.1. In other words, the model reduces to the benchmark case in Brennan (1998).

Linear dependence of the observable component ξ_t is introduced into the coefficients of transfer in (A2). To prove Lemma 2, I shall need the following Lemma 3.

Lemma 3 *Let the matrix process $D = (D_t, \mathcal{F}_t)$, be such that for almost all t , $0 \leq t \leq T$, ($P - a.s.$)*

$$B_{1t}^2 = D_t^2,$$

then there is a Wiener process \bar{W}_s , such that for each t , $0 \leq t \leq T$, ($P - a.s.$)

$$\int_0^t B_{1s} dW_s = \int_0^t D_s d\bar{W}_s.$$

Note Lemma 3 is an one-dimension benchmark case of Lemma 10.4 in Liptser and Shirayev (1977), which is on a multidimensional Wiener process. By Lemma 3, for the system of equations in (A2), there is also the representation

$$d\xi_t = (A_1(t) \varsigma_t + A_2(t) \xi_t) dt + D(t) d\bar{W}_t, \quad (\text{A3})$$

let

$$\bar{\varsigma}_t = \varsigma_t, \bar{\xi}_t = \xi_t - \int_0^t A_2(s) \xi_s ds, \quad (\text{A4})$$

by (A3) I have

$$\bar{W}_2(t) = \int_0^t D_2^{-1}(s) [d\xi_s - (A_1(s)\varsigma_s + A_2(s)\xi_t)] ds. \quad (A5)$$

From (A3), (A4), and (A5), I have

$$d\bar{\xi}_t = A_1(t)\bar{\theta}_t dt + D_2(t) d\bar{W}_2(t).$$

From the construction of the process $\bar{\xi} = (\bar{\xi}_t), 0 \leq t \leq T$, it follows that $\mathcal{F}_t^\xi \supseteq \mathcal{F}_t^{\bar{\xi}}$. It will be shown that actually the σ -algebras \mathcal{F}_t^ξ and $\mathcal{F}_t^{\bar{\xi}}$ coincide for all $t, 0 \leq t \leq T$.

For the proof I shall consider the linear system of equations

$$d\xi_t = A_2(t)\xi_t dt + d\bar{\xi}_t, \quad \xi_0 = \bar{\xi}_0,$$

obtained from (A4).

This linear system of equations has a unique, strong solution (see Theorem 4.10 in Liptser and Shiriyayev (1977) and the note to it) which implies $\mathcal{F}_t^{\bar{\xi}} \supseteq \mathcal{F}_t^\xi, 0 \leq t \leq T$, i.e., $\mathcal{F}_t^{\bar{\xi}} = \mathcal{F}_t^\xi$ and

$$\bar{m}_t = M\left(\bar{\varsigma}_t | \mathcal{F}_t^{\bar{\xi}}\right) = M\left(\bar{\varsigma}_t | \mathcal{F}_t^\xi\right),$$

hence,

$$m_t = M\left(\varsigma_t | \mathcal{F}_t^\xi\right) = M\left(\bar{\varsigma}_t | \mathcal{F}_t^\xi\right) = \bar{m}_t,$$

and

$$\bar{\varsigma}_t - \bar{m}_t = \varsigma_t - m_t.$$

From this,

$$\gamma_t = \bar{\gamma}_t.$$

According to Theorem 10.1 in Liptser and Shiriyayev (1977), I have

$$d\bar{m}_t = \bar{\gamma}_t A_1(t) D(t)^{-2} [d\bar{\xi}_t - A_1(t)\bar{m}_t dt],$$

and

$$d\bar{\gamma}_t = -\bar{\gamma}_t^2 A_1^2(t) D(t)^{-2} dt.$$

From this, taking into account that $m_t = \bar{m}_t$ and $\gamma_t = \bar{\gamma}_t$, after some simple transformations I arrive at for m_t and γ_t .

B Proof of Propositions

B.1 Proof of Proposition 1 and Algorithm to Implement Pareto Efficient Allocation

The proof and the algorithm is a special case of that in Duffie and Huang (1985). Specifically in this paper, there are only two securities that are traded continuously, and the signals only reveal the information contingent on the aggregate consumption.

Christensen and Qin (2012) provide a specific Arrow-Debreu Equilibrium with specific securities and dividend structure. In general, the economy can be summarized in the usual way by the collection

$$\bar{E} = (V_i, v_i, \succeq; i = 1, \dots, I),$$

where the individual-specific consumption space is thus formalized as $V_i \subset V = R \times L^2(P)$, where $L^2(P)$ is the space of (equivalence classes) of square-integrable random-variables on (Ω, \mathcal{F}, P) . The consumption pair $\bar{v}_i = (\bar{\omega}_{i0}, \bar{\omega}_{iT}) \in V_i$ is the i th investor's endowment. The Arrow-Debreu equilibrium for an economy \bar{E} can be defined as a nonzero linear (price) functional $\Psi : V \rightarrow R$ and a set of allocations $(v_i^* = (\omega_{i0}, \omega_{iT}) \in V_i; i = 1, \dots, I)$ satisfying, for all i ,

$$\begin{aligned} \Psi(v_i^*) &\leq \Psi(\bar{v}_i), \\ v &\succ_i v_i^* \rightarrow \Psi(v) > \Psi(v_i^*), \quad \forall v \in V_i, \\ \sum_{i=1}^I v_i^* &= \sum_{i=1}^I \bar{v}_i. \end{aligned}$$

Duffie and Huang (1985) proof that $\Psi(\omega_{i0}, \omega_{iT}) = \bar{a}\omega_{i0} + E^Q(\omega_{iT}), \forall (\omega_{i0}, \omega_{iT}) \in V_i$, where $\bar{a} \in R_+$ is a price for date zero consumption. They note this relationship hold for both homogeneous belief case and heterogeneous beliefs case.

The Radner equilibrium to implement the Pareto optimal consumption allocations is comprised of: (1) a set of long-lived securities claiming, i.e., the riskless bond and the stock, with corresponding price processes 1 and W^* , and $S \equiv [1, W^*]'$; (2) a set of trading strategies $\theta_i = [\gamma_i, x_i]'$, (omit the subscription t) for each investor $i = 1, \dots, I$; and (3) a price $\bar{a} \in R_+$ for date zero consumption. All of these satisfy budget constrained optimality: for each investor i ,

$$\bar{\omega}_{i0} - \frac{\theta'_i(0) S(0)}{\bar{a}}, \quad \bar{\omega}_{iT} + \theta'_i(T) S(T),$$

is \succeq_i -maximal in the budget set, and market clearing.

With the orthogonal 2-basis for M_Q^2 , I can apply the procedure in Duffie and Huang (1985) to implement all the allocations which are measurable with respect to the aggregate consumption by trading in only two long-lived securities as follows.

For any investor i , for $1 \leq i \leq I - 1$, let $e_i = c_{iT}^* - \bar{\omega}_{iT}$, and c_{iT}^* is the Pareto optimal consumption at $t = T$, then the process

$$\check{X}_i(t) = \mathbb{E}^Q(e_i | \mathcal{F}_t) - \mathbb{E}^Q(e_i), \quad t \in [0, T],$$

is an element of M_Q^2 , given $Q \approx P$, which can be reconstructed via Theorem 4.1 in Duffie and Huang (1985) as

$$\check{X}_i(t) = \int_0^t x_i(s) dW^*(t), \quad \forall t \in [0, T] \text{ a.s.}, \quad (\text{B1})$$

for some $x_i(s) \in L_P^2[W^*]$.

In order to meet the accounting restriction

$$\theta'_i(t) S(t) = \theta'_i(0) S(0) + \int_0^t \theta'_i(s) dS(s), \quad \forall t \in [0, T] \text{ a.s.}, \quad (\text{B2})$$

I set the following trading process for the "store-of-value" security:

$$\gamma'_i(t) = \mathbb{E}^Q(e_i) + \int_0^t x_i(s) dW^*(t) - x_i(s) W^*(t), \quad t \in [0, T]. \quad (\text{B3})$$

Substituting Eq. (B3) into Eq. (B1), noting that $W^*(0) = 0, \forall n$, yields and confirms the accounting restriction, Eq. (B2). This yields the final requirement for claiming the trading strategy $\theta_i = (\gamma_i, x_i)'$ is admissible. Evaluating Eq. (B2) at times T and 0 , using the definitions of \check{X}_i yields:

$$\theta'_i(T) S(T) + \bar{\omega}_{iT} = c_{iT}^* \quad \text{a.s.},$$

and

$$\theta'_i(0) S(0) = \mathbb{E}^Q(c_{iT}^* - \bar{\omega}_{iT}) = \Psi(0, c_{iT}^*) - \Psi(0, \bar{\omega}_{iT}) = (\bar{\omega}_{i0} - c_{i0}^*) \bar{a},$$

where c_{i0}^* is the Pareto optimal consumption at $t = 0$, and the last line making use of the budget constraint on the Arrow-Debreu allocation for investor i . Thus by adopting the trading strategy θ_i , and faced with the date-zero consumption price of \bar{a} , investor i can consume precisely $(c_{i0}^*, c_{iT}^*) = v_i^*$.

The above construction applies for investors 1 through $I - 1$. For the last investor, investor I , let $\theta_I = -\sum_{i=1}^{I-1} \theta_i$, thus market clearing is obviously met by construction. To complete

this step it remains to show that θ_I generates the consumption allocation $(c_{i0}^*, c_{iT}^*) = v_i^*$, but this is immediate from the linearity of stochastic integrals and market clearing in the Arrow-Debreu equilibrium.

Now prove that no investor has any incentive to deviate from the allocated trading strategy by contradiction. Suppose some investor j can obtain a strictly preferred allocation $(c_0, c_T) \succ_j (c_{j0}^*, c_{jT}^*)$ by adopting a different trading strategy θ . Then the Arrow-Debreu price of (c_0, c_T) must be strictly higher than that of (c_{j0}^*, c_{jT}^*) , or

$$\bar{a}c_0 + \mathbb{E}^Q(c_T) > \bar{a}c_{j0}^* + \mathbb{E}^Q(c_{jT}^*).$$

Substituting the Radner budget constraint for c_0 and c_T ,

$$\bar{a}\bar{\omega}_{j0} - \theta'(0)S(0) + \mathbb{E}^Q \left[\bar{\omega}_{jT} + \theta'(0)S(0) + \int_0^T \theta'(t)dS(t) \right] > \bar{a}c_{j0}^* + \mathbb{E}^Q(c_{jT}^*),$$

or

$$\bar{a}\bar{\omega}_{j0} + \mathbb{E}^Q(\bar{\omega}_{jT}) > \bar{a}c_{j0}^* + \mathbb{E}^Q(c_{jT}^*). \quad (\text{B4})$$

The last line uses the fact that $\mathbb{E}^Q[\int_0^t \theta(s)dS(t)] = 0$ since $\int \theta dS$ is a Q -martingale. But Eq. (B4) contradicts the Arrow-Debreu budget-constrained optimality of (c_{j0}^*, c_{jT}^*) . This establishes the whole implement process. Moreover, the market is dynamically effectively complete, since no investor has any incentive to deviate from the self-financing trading strategy, and all the markets clear. This completes the proof.

B.2 Proof of Proposition 2

To obtain the price of the risky asset, I first derive the state-price deflator and the expected risk-adjusted dividend, and then calculate the price of the risky asset according to Eq. (10).

B.2.1 State-Price Deflator at the Terminal Date

Note the state-price deflator at the terminal date T is given as

$$\zeta_{iT}^* = \frac{u'_i(c_{iT}^*)}{\lambda_{i0}} = \frac{r \exp(-rc_{iT}^*)}{\lambda_{i0}},$$

where the Pareto optimal consumption at the terminal date

$$c_{iT}^* = \frac{1}{2}\rho [\bar{h} - h_{i0}] d_T^2 + \rho [h_{i0}m_i - \overline{hm}] d_T + (Z/I) d_T + \gamma_{i0}^\dagger,$$

thus, the state-price deflator is

$$\begin{aligned}\zeta_{iT}^* &= \frac{r \exp\left(-r \left[\frac{1}{2}\rho [\bar{h} - h_{i0}] d_T^2 + \rho [h_{i0}m_i - \bar{h}m] d_T + (Z/I) d_T + \gamma_{i0}^\dagger\right]\right)}{\lambda_{i0}} \\ &= \frac{\exp\left(-\frac{1}{2} [\bar{h} - h_{i0}] d_T^2 - [h_{i0}m_i - \bar{h}m] d_T - r (Z/I) d_T\right)}{\psi_{i0}},\end{aligned}$$

where $\psi_{i0} = \frac{\lambda_{i0}}{r \exp(-r\gamma_{i0}^\dagger)}$.

B.2.2 Expected Risk-Adjusted Dividend

Note let $\bar{\psi}_{i0} = 2\sqrt{2\pi\sigma_{it}^2}\psi_{i0}$, thus the expected risk-adjusted dividend follows

$$\begin{aligned}\mathbb{E}^i [\zeta_{iT}^* d_T | \mathcal{F}_t] &= \frac{1}{\psi_{i0}} \mathbb{E}^i \left[\exp\left(-\frac{1}{2} [\bar{h} - h_{i0}] d_T^2 - [h_{i0}m_i - \bar{h}m] d_T - r (Z/I) d_T\right) d_T | \mathcal{F}_t \right] \\ &= \frac{1}{\bar{\psi}_{i0}} \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2} [\bar{h} - h_{i0}] d_T^2 - [h_{i0}m_i - \bar{h}m] d_T - r (Z/I) d_T - \frac{1}{2} \left(\frac{d_T - m_{it}}{\sigma_{it}}\right)^2\right) d d_T^2.\end{aligned}$$

To simplify, let

$$\begin{aligned}\alpha_0 &\equiv \frac{1}{2} [\bar{h} - h_{i0}] + \frac{1}{2} h_{it}, \\ \alpha_1 &\equiv \frac{h_{i0}m_i - \bar{h}m - h_{it}m_{it} + r (Z/I)}{\bar{h} - h_{i0} + h_{it}}, \\ \alpha_2 &\equiv \frac{(h_{i0}m_i - \bar{h}m - h_{it}m_{it} + r (Z/I))^2}{\bar{h} - h_{i0} + h_{it}} - \frac{1}{2} h_{it}m_{it}^2,\end{aligned}$$

hence,

$$\begin{aligned}\mathbb{E}^i [\zeta_{iT}^* d_T | \mathcal{F}_t] &= \frac{1}{\bar{\psi}_{i0}} \int_{-\infty}^{+\infty} \exp\left(-\alpha_0 [d_T^2 + 2\alpha_1 d_T] - \frac{1}{2} h_{it}m_{it}^2\right) d d_T^2 \\ &= \frac{1}{\bar{\psi}_{i0}} \int_{-\infty}^{+\infty} \exp\left(-\alpha_0 [(d_T + \alpha_1)^2 - \alpha_1^2] - \frac{1}{2} h_{it}m_{it}^2\right) d d_T^2 \\ &= \frac{1}{\bar{\psi}_{i0}} \int_{-\infty}^{+\infty} \exp\left(-\alpha_0 (d_T + \alpha_1)^2 + \alpha_2\right) d d_T^2.\end{aligned}$$

Since $h_{it} \geq h_{i0}$, for $0 < t \leq T$, hence, $\alpha_0 > 0$. Let $a_1 = \sqrt{\alpha_0}$, and $b_1 = a_1\alpha_1$, thus,

$$\begin{aligned}
& \int_{-\infty}^{+\infty} \exp - [\alpha_0 [d_T + \alpha_1]^2 + \alpha_2] d d_T^2 = \int_{-\infty}^{+\infty} \exp - [(a_1 d_T + b_1)^2 + \alpha_2] d d_T^2 \\
& = \exp [-\alpha_2] \int_{-\infty}^{+\infty} \exp - [(a_1 d_T + b_1)^2] d d_T^2 \\
& = \frac{\exp [-\alpha_2]}{a_1^2} \int_{-\infty}^{+\infty} \exp - [(a_1 d_T + b_1)^2] d [(a_1 d_T + b_1)^2 - 2a_1 b_1 d_T - b_1^2] \\
& = \frac{\exp [-\alpha_2]}{a_1^2} \int_{-\infty}^{+\infty} \exp - [(a_1 d_T + b_1)^2] d [(a_1 d_T + b_1)^2 - 2b_1 (a_1 d_T - b_1)],
\end{aligned}$$

let $u = a_1 d_T - b_1$, note $u \rightarrow +\infty$, as $d_T \rightarrow +\infty$, $u \rightarrow -\infty$, as $d_T \rightarrow -\infty$, thus,

$$\begin{aligned}
& \int_{-\infty}^{+\infty} \exp - [\alpha_0 [d_T + \alpha_1]^2 + \alpha_2] d d_T^2 \\
& = \frac{\exp [-\alpha_2]}{a_1^2} \int_{-\infty}^{+\infty} \exp [-u^2] d [u^2 - 2b_1 u] \\
& = \frac{\exp [-\alpha_2]}{a_1^2} \left[\int_{-\infty}^{+\infty} \exp [-u^2] du^2 - 2b_1 \int_{-\infty}^{+\infty} \exp [-u^2] du \right],
\end{aligned}$$

let $v = u^2$, note $v \rightarrow +\infty$, as $u \rightarrow +\infty$, $v \rightarrow +\infty$, as $u \rightarrow -\infty$, thus,

$$\begin{aligned}
& \int_{-\infty}^{+\infty} \exp - [\alpha_0 [d_T + \alpha_1]^2 + \alpha_2] d d_T^2 \\
& = \frac{\exp [-\alpha_2]}{a_1^2} \left[\int_{+\infty}^{+\infty} \exp [-v] dv - 2b_1 \int_{-\infty}^{+\infty} \exp [-u^2] du \right] \\
& = \frac{\exp [-\alpha_2]}{a_1^2} \left[0 - 2b_1 \int_{-\infty}^{+\infty} \exp [-u^2] du \right] \\
& = \frac{\exp [-\alpha_2]}{a_1^2} [-2b_1 \sqrt{\pi}] = -\frac{2b_1 \sqrt{\pi}}{a_1^2} \exp [-\alpha_2].
\end{aligned}$$

Thus, the expected risk-adjusted dividend

$$E^i [\zeta_{iT}^* d_T | \mathcal{F}_t] = -\frac{2b_1 \sqrt{\pi}}{a_1^2 \psi_{i0}} \exp [-\alpha_2].$$

B.2.3 State-Price Deflator at Date t

The state-price deflator at date t follows

$$\begin{aligned}
\zeta_{it}^* &= \mathbb{E}^i [\zeta_{iT}^* | \mathcal{F}_t] \\
&= \frac{1}{\psi_{i0}} \int_{-\infty}^{+\infty} \exp \left(-\frac{1}{2} [\bar{h} - h_{i0}] d_T^2 - [h_{i0} m_i - \bar{h} m] d_T - r (Z/I) d_T - \frac{1}{2} \left(\frac{d_T - m_{it}}{\sigma_{it}} \right)^2 \right) d d_T \\
&= \frac{1}{\psi_{i0}} \int_{-\infty}^{+\infty} \exp - [\alpha_0 [d_T + \alpha_1]^2 + \alpha_2] d d_T \\
&= \frac{1}{\psi_{i0}} \sqrt{2\pi \frac{1}{2\alpha_0}} \exp [-\alpha_2] \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi \frac{1}{2\alpha_0}}} \exp \left[\frac{-[d_T - (-\alpha_1)]^2}{2 \frac{1}{2\alpha_0}} \right] d d_T \\
&= \frac{1}{\psi_{i0}} \sqrt{\frac{\pi}{\alpha_0}} \exp [-\alpha_2].
\end{aligned}$$

B.2.4 Stock Price

Therefore, the stock price at date t is given as

$$\begin{aligned}
p_t &= \frac{\mathbb{E}^i [\zeta_{iT}^* d_T | \mathcal{F}_t]}{\zeta_{it}^*} = \frac{-\frac{2b_1 \sqrt{\pi}}{a_1^2 \psi_{i0}} \exp [-\alpha_2]}{\frac{1}{\psi_{i0}} \sqrt{\frac{\pi}{\alpha_0}} \exp [-\alpha_2]} = \frac{-\frac{b_1}{a_1^2}}{\sqrt{\frac{1}{\alpha_0}}} = -\frac{b_1}{a_1} \\
&= -\alpha_1 = \frac{\bar{h} m - h_{i0} m_i + h_{it} m_{it} - r Z/I}{\bar{h} - h_{i0} + h_{it}}.
\end{aligned} \tag{B5}$$

At the first glimpse, the stock price looks individual specific. However, by substituting for the posterior mean m_{it} and the posterior precision h_{it} (in terms of the priors and signal y_t), I show the price is common for all the investors in the following subsection.

B.2.5 Express the Stock Price as a Function of Priors

I first express the posterior beliefs as functions of priors. By Eq. (9), I obtain the posterior precision

$$h_{it} = h_{i0} + \frac{\alpha^2 T^{k-2q+1} - \alpha^2 (T-t)^{k-2q+1}}{(k-2q+1) \beta^2} = h_{i0} + J_0(t), \tag{B6}$$

where

$$J_0(t) \equiv \frac{\alpha^2 T^{k-2q+1} - \alpha^2 (T-t)^{k-2q+1}}{(k-2q+1) \beta^2},$$

is common for all the investors.

Moreover, note the posterior mean is driven by the signal y , substitute Eq. (7) into Eq.

(8), I have

$$\begin{aligned}
dm_{it} &= \frac{\alpha\sigma_{it}^2}{\beta^2} (T-t)^{\frac{k}{2}-2q} \left[\alpha (T-t)^{\frac{k}{2}} d_T dt + \beta (T-t)^q dW_t - \alpha (T-t)^{\frac{k}{2}} m_{it} dt \right] \\
&= \frac{\alpha\sigma_{it}^2}{\beta^2} (T-t)^{\frac{k}{2}-2q} \left[\alpha (T-t)^{\frac{k}{2}} (d_T - m_{it}) dt + \beta (T-t)^q dW_t \right],
\end{aligned}$$

thus, posterior mean m_{it} follows a SDE, solve for the SDE I obtain that

$$\begin{aligned}
m_{it} &= \Phi(t) \left(m_{i0} + \int_0^t \Phi^{-1}(s) \Theta(s) ds + \int_0^t \Phi^{-1}(s) G(s) dW_s \right) \\
&= \Phi(t) m_{i0} + \Phi(t) \int_0^t \Phi^{-1}(s) \Theta(s) ds + \Phi(t) \int_0^t \Phi^{-1}(s) G(s) dW_s, \tag{B7}
\end{aligned}$$

where

$$\Phi(t) = \frac{\beta^2 (k-2q+1)}{\alpha^2 \sigma_{i0}^2 T^{k-2q+1} + \beta^2 (k-2q+1) - \alpha^2 \sigma_{i0}^2 (T-t)^{k-2q+1}},$$

follows the ODE

$$\dot{\Phi}(t) = \bar{A}(t) \Phi(t), \quad \Phi(0) = 1, \tag{B8}$$

and,

$$\begin{aligned}
\bar{A}(t) &\equiv -\frac{\alpha^2 \sigma_{it}^2}{\beta^2} (T-t)^{k-2q}, \\
G(t) &\equiv \frac{\alpha \sigma_{it}^2}{\beta} (T-t)^{\frac{k}{2}-q}, \\
\Theta(t) &\equiv -\frac{\alpha^2 \sigma_{it}^2 d_T}{\beta^2} (T-t)^{k-2q}.
\end{aligned}$$

See subsection B.2.7 for the derivation of solving for ODE (B8).

Thus,

$$\begin{aligned}
&\int_0^t \Phi^{-1}(s) \Theta(s) ds \\
&= -\int_0^t \frac{T^{k-2q+1} + \frac{\beta^2}{\alpha^2 \sigma_{i0}^2} (k-2q+1) - (T-s)^{k-2q+1}}{\frac{\beta^2}{\alpha^2 \sigma_{i0}^2} (k-2q+1)} \frac{\alpha^2 \sigma_{it}^2 d_T}{\beta^2} (T-s)^{k-2q} ds \\
&= -\int_0^t \frac{\sigma_{i0}^2 \alpha^2 d_T}{\beta^2} (T-s)^{k-2q} ds = \int_0^t \frac{\sigma_{i0}^2 \alpha^2 d_T}{\beta^2} (T-s)^{k-2q} d(T-s) \\
&= \frac{\sigma_{i0}^2 \alpha^2 d_T}{\beta^2} \frac{1}{k-2q+1} \left((T-t)^{k-2q+1} - T^{k-2q+1} \right),
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^t \Phi^{-1}(s) G(s) dW_s \\
= & \int_0^t \frac{\alpha^2 \sigma_{i0}^2 T^{k-2q+1} + \beta^2 (k-2q+1) - \alpha^2 \sigma_{i0}^2 (T-s)^{k-2q+1}}{\beta^2 (k-2q+1)} \frac{\alpha \sigma_{it}^2}{\beta} (T-s)^{\frac{k}{2}-q} dW_s \\
= & \frac{\alpha \sigma_{i0}^2}{\beta} \int_0^t (T-s)^{\frac{k}{2}-q} dW_s.
\end{aligned}$$

Note $h_{it}\Phi(t) = h_{i0}$, thus I have

$$\begin{aligned}
h_{it}m_{it} &= h_{it}\Phi(t)m_{i0} + h_{it}\Phi(t) \int_0^t \Phi^{-1}(s) \Theta(s) ds + h_{it}\Phi(t) \int_0^t \Phi^{-1}(s) G(s) dW_s \\
&= h_{i0}m_i + h_{i0} \int_0^t \Phi^{-1}(s) \Theta(s) ds + h_{i0} \int_0^t \Phi^{-1}(s) G(s) dW_s \\
&= h_{i0}m_i + \frac{\alpha^2 d_T}{\beta^2} \frac{1}{k-2q+1} \left((T-t)^{k-2q+1} - T^{k-2q+1} \right) + \frac{\alpha}{\beta} \int_0^t (T-s)^{\frac{k}{2}-q} dW_s \\
&= h_{i0}m_i + J_1(t, W), \tag{B9}
\end{aligned}$$

where

$$J_1(t, W) \equiv \frac{\alpha^2 d_T}{\beta^2} \frac{1}{k-2q+1} \left((T-t)^{k-2q+1} - T^{k-2q+1} \right) + \frac{\alpha}{\beta} \int_0^t (T-s)^{\frac{k}{2}-q} dW_s,$$

is also common for all the investors.

Hence, substitute Eq. (B6) and Eq. (B9) into Eq. (B5), I obtain

$$p_t = \frac{\overline{hm} + J_1(t, W) - rZ/I}{\overline{h} + J_0(t)}, \tag{B10}$$

from which, it is easy to see the stock price is common for all the investors.

B.2.6 Proof the Stock Price $p_t = \overline{m}_t^h - r\overline{\sigma}_t^2 Z/I$

Note

$$\begin{aligned}
\overline{m}_t^h - r\overline{\sigma}_t^2 Z/I &= \frac{\frac{1}{I} \sum_{i=1}^I h_{it}m_{it} - rZ/I}{\overline{h}_t} = \frac{\frac{1}{I} \sum_{i=1}^I h_{it}m_{it} - rZ/I}{\frac{1}{I} \sum_{i=1}^I h_{it}} \\
&= \frac{\frac{1}{I} \sum_{i=1}^I (h_{i0}m_i + J_1(t, W)) - rZ/I}{\frac{1}{I} \sum_{i=1}^I \left(\frac{\alpha^2 T^{k-2q+1} - \alpha^2 (T-t)^{k-2q+1}}{(k-2q+1)\beta^2} + h_{i0} \right)} = \frac{\overline{hm} + J_1(t, W) - rZ/I}{\overline{h} + J_0(t)} = p_t.
\end{aligned}$$

This completes the proof of Proposition 1.

B.2.7 Solve for the ODE (B8)

Solve for the ODE (B8), I obtain

$$\dot{\Phi}(t) = \bar{A}(t) \Phi(t) \iff d \ln \Phi(t) = \bar{A}(t) dt \iff d \ln \Phi(t) = -\frac{\alpha^2 \sigma_{i0}^2}{\beta^2} (T-t)^{k-2q} dt,$$

thus, substitute for the posterior variance, I obtain

$$\begin{aligned} d \ln \Phi(t) &= -\frac{(k-2q+1) \sigma_{i0}^2}{\sigma_{i0}^2 T^{k-2q+1} - \sigma_{i0}^2 (T-t)^{k-2q+1} + \frac{\beta^2}{\alpha^2} (k-2q+1)} (T-t)^{k-2q} dt \\ &= \frac{(k-2q+1) \sigma_{i0}^2}{\sigma_{i0}^2 T^{k-2q+1} - \sigma_{i0}^2 (T-t)^{k-2q+1} + \frac{\beta^2}{\alpha^2} (k-2q+1)} \frac{1}{k-2q+1} d(T-t)^{k-2q+1} \\ &= \frac{1}{T^{k-2q+1} + \frac{\beta^2}{\alpha^2 \sigma_{i0}^2} (k-2q+1) - (T-t)^{k-2q+1}} d(T-t)^{k-2q+1} \\ &= -\frac{1}{T^{k-2q+1} + \frac{\beta^2}{\alpha^2 \sigma_{i0}^2} (k-2q+1) - (T-t)^{k-2q+1}} \\ &\quad \times d \left[\left(T^{k-2q+1} + \frac{\beta^2}{\alpha^2 \sigma_{i0}^2} (k-2q+1) \right) - (T-t)^{k-2q+1} \right] \\ &= d \left[-\ln \left[\left(T^{k-2q+1} + \frac{\beta^2}{\alpha^2 \sigma_{i0}^2} (k-2q+1) \right) - (T-t)^{k-2q+1} \right] \right]. \end{aligned}$$

Therefore, I have a general solution for the ODE

$$\begin{aligned} \ln \Phi(t) &= \ln \left[\left(T^{k-2q+1} + \frac{\beta^2}{\alpha^2 \sigma_{i0}^2} (k-2q+1) \right) - (T-t)^{k-2q+1} \right]^{-1} + C_1 \\ \iff \Phi(t) &= \left[\left(T^{k-2q+1} + \frac{\beta^2}{\alpha^2 \sigma_{i0}^2} (k-2q+1) \right) - (T-t)^{k-2q+1} \right]^{-1} \exp C_1, \end{aligned}$$

where C_1 is a constant determined by the boundary condition.

Substitute the boundary condition at $t = 0$, I get

$$\begin{aligned} \Phi(0) &= \left[\frac{\beta^2}{\alpha^2 \sigma_{i0}^2} (k-2q+1) \right]^{-1} \exp C_1 = 1 \\ \iff C_1 &= \ln \frac{\beta^2}{\alpha^2 \sigma_{i0}^2} (k-2q+1), \end{aligned}$$

thus, I obtain the expression of $\Phi(t)$ as

$$\Phi(t) = \frac{\beta^2(k-2q+1)}{\alpha^2\sigma_{i0}^2 T^{k-2q+1} + \beta^2(k-2q+1) - \alpha^2\sigma_{i0}^2(T-t)^{k-2q+1}}.$$

B.3 Proof of Proposition 3

The Pareto optimal portfolios should be self-financing, and satisfy both optimal wealth processes and the market clearing conditions. It is proofed as follows.

By Eq. (B10), the differential form of the stock price follows

$$\begin{aligned} dp_t &= \frac{1}{\bar{h} + J_0(t)} dJ_1(t, W) \\ &= \frac{1}{\bar{h} + J_0(t)} d \left(- \int_0^t \frac{\alpha^2 d_T}{\beta^2} (T-s)^{p-2q} ds + \frac{\alpha}{\beta} \int_0^t (T-s)^{\frac{p}{2}-q} dW_s \right) \\ &= \frac{1}{\bar{h} + J_0(t)} \left(- \frac{\alpha^2 d_T}{\beta^2} (T-t)^{p-2q} dt + \frac{\alpha}{\beta} (T-t)^{\frac{p}{2}-q} dW_t \right), \end{aligned} \quad (\text{B11})$$

thus, the SDE of the stock price under P -measure gives rise to the implied market price of risk

$$\phi_t = - \frac{\alpha d_T}{\beta} (T-t)^{\frac{p}{2}-q}. \quad (\text{B12})$$

Note the Brownian motion under Q -measure is

$$dW_t^* = dW_t + \phi_t dt, \quad (\text{B13})$$

substitute the Eq. (B13) into (B11), yields dynamics of the stock price under Q -measure,

$$dp_t^* = \sigma_t^* dW_t^*, \quad \sigma_t^* \equiv \frac{\alpha (T-t)^{\frac{p}{2}-q}}{\beta (\bar{h} + J_0(t))}. \quad (\text{B14})$$

In the spirit of Eq. (B1), I have

$$\begin{aligned} \check{X}_i(t) &= \mathbf{E}^Q(e_i | \mathcal{F}_t) - \mathbf{E}^Q(e_i), \quad t \in [0, T], \\ &= \mathbf{E}^Q(c_{iT}^* - \bar{\omega}_{iT} | \mathcal{F}_t) - \mathbf{E}^Q(c_{iT}^* - \bar{\omega}_{iT}) \\ &= \mathbf{E}^Q(c_{iT}^* | \mathcal{F}_t) - \mathbf{E}^Q(\bar{\omega}_{iT} | \mathcal{F}_t) - \mathbf{E}^Q(c_{iT}^*) + \mathbf{E}^Q(\bar{\omega}_{iT}) \\ &= \mathbf{E}^Q(c_{iT}^* | \mathcal{F}_t) - \mathbf{E}^Q(c_{iT}^*), \end{aligned}$$

thus, according to the Martingale Presentation Theorem, there exists some process $x_i(t)$, allows the optimal wealth process for the investor i to have a stochastic integral representation

$$\mathbb{E}^Q(c_{iT}^* | \mathcal{F}_t) - \mathbb{E}^Q(c_{iT}^*) = \int_0^t x_i(s) dp_t^* = \int_0^t x_i(s) \sigma_t^* dW_t^*, \quad \forall t \in [0, T] \text{ a.s.}$$

Moreover, since $\mathbb{E}^Q(c_{iT}^*)$ is the initial wealth of the investor i , thus,

$$\mathbb{E}^Q(c_{iT}^*) = \frac{\mathbb{E}(\zeta_{iT}^* c_{iT}^*)}{\zeta_{i0}^*} = \mathbb{E}(\zeta_{iT}^* c_{iT}^*), \quad \zeta_{i0}^* = 1,$$

thus yields

$$\mathbb{E}^Q[c_{iT}^* | \mathcal{F}_t] = \mathbb{E}(\zeta_{iT}^* c_{iT}^*) + \int_0^t x_i(s) \sigma_t^* dW_t^*, \quad (\text{B15})$$

which indicates the martingale under Q -measure has a stochastic integral representation. Note Ocone and Karatzast (1991) generalize the Clark-Ocone formula and provide explicit expression of the integrand in the martingale representation formulae under Q -measure.

Before showing their results, I first demonstrate the definition of the Malliavin derivative⁹ as follows. Let $\tilde{W} = \int_0^T \tilde{h}(s) dW_s^*$ be defined for $\tilde{h}(s) \in L^2([0, T])$. For a smooth Brownian functional \tilde{F} , i.e., a random variable of the form

$$\tilde{F} = \tilde{f}\left(\tilde{W}\left(\tilde{h}_1\right), \dots, \tilde{W}\left(\tilde{h}_n\right)\right),$$

where \tilde{f} is a smooth bounded function with bounded derivatives of all orders, the Malliavin derivatives is defined by

$$\partial_t \tilde{F} = \sum_{i=1}^n \partial_i \tilde{f}\left(\tilde{W}\left(\tilde{h}_1\right), \dots, \tilde{W}\left(\tilde{h}_n\right)\right) \tilde{h}_i(t),$$

where ∂_i stands for the i^{th} partial derivative. Note that $\partial_t \left(\int_0^T \tilde{h}(s) dW_s^*\right) = \tilde{h}(t)$ and in particular $\partial_s(W_t^*) = 1$ for $s \leq t$.

Employing the tool of Malliavin derivatives, I summarize the results of Proposition 2.5, Corollary 2.6, and Eq. (3.11) in Ocone and Karatzast (1991) in the following Lemma.

Lemma 4 *Consider the setting in which the market price of risk $\{\phi(t); 0 \leq t \leq T\}$ is a deterministic bounded process, and the riskless interest rate is constantly 0.*

(a) *The process of market price of risk¹⁰ $\phi(t) \in L_{1,1}^a$ and there exist some $\tau > 1, \mu > 1$*

⁹See Oksendal (1996) for a concise introduction to Malliavin calculus. A more detailed and general one can be found in the book of Nualart (1995).

¹⁰Note for notations, norm $\|\cdot\|$ denotes the $L^2([0, T])$ norm, and $D_{1,1}$ is the Banach space and the closure

and some ϵ such that $(1/\epsilon) + (1/\tau) < 1$, s.t., for a random variable Ξ ,

$$\mathbb{E} \left[\int_0^T \|\partial\phi(s)\|^2 ds \right]^{\tau/2} < \infty, \quad \mathbb{E} [\|\partial\Xi\|^\mu] < \infty, \quad \Xi \in L^\epsilon(P), \text{ and } \Xi \in D_{1,1}.$$

(b) Product $\Xi \cdot \Lambda(T) \in D_{1,1}$, where

$$\Lambda(T) = \exp \left(- \int_0^T \phi(t) dW(t) - \frac{1}{2} \int_0^T [\phi(t)]^2 dt \right).$$

(c) Conditions (a) and (b) are sufficient for the establishment of the stochastic integral representation formula

$$\Xi = \mathbb{E}(\Xi\Lambda(T)) + \int_0^T \mathbb{E}^Q(\partial_t\Xi|\mathcal{F}_t) dW_t^*,$$

and it follows also that

$$\mathbb{E}^Q(\Xi|\mathcal{F}_t) = \mathbb{E}(\Xi\Lambda(T)) + \int_0^t \mathbb{E}^Q(\partial_s\Xi|\mathcal{F}_s) dW_s^*, \quad 0 \leq t \leq T. \quad (\text{B16})$$

Note in this paper, both the market price of risk $\phi(t)$, and the riskless interest rate are deterministic, thus, I can apply the stochastic integral representation formula in Lemma 4. Compare Eq. (B16) to Eq. (B15), note $\Lambda(T) = \zeta_{iT}^*$, hence, I obtain the optimal portfolio in the stock market as

$$x_{it}^* = (\sigma_t^*)^{-1} \mathbb{E}^Q(\partial_t(c_{iT}^*)|\mathcal{F}_t), \quad (\text{B17})$$

where $\partial_t(c_{iT}^*)$ is the Malliavin derivative of the optimal terminal consumption c_{iT}^* . This result enable me to achieve very explicit expressions for the optimal portfolios.

Note at the terminal date when the dividend is perfectly known, $p_T^* = d_T$, and by Eq.

of the class of smooth functionals under the norm

$$\|\Xi\|_{1,1} = \mathbb{E} \left[|\Xi| + \left(\|\partial\Xi\|^2 \right)^{\frac{1}{2}} \right],$$

where Ξ is a random variable, and $|\cdot|$ denote the Euclidean norm on \mathbb{R} . Moreover, let $L_{1,1}^a$ denote the set of \mathbb{R} -valued progressively measurable processes $\{\check{u}(s, \tilde{\omega}); 0 \leq s \leq T, \tilde{\omega} \in \Omega\}$ such that

- i) For a.e. $s \in [0, T]$, $\check{u}(s, \cdot) \in (D_{1,1})^1$,
- ii) $(s, \tilde{\omega}) \mapsto \partial\check{u}(s, \tilde{\omega}) \in (L^2([0, T]))^1$ admits a progressively measurable version; and
- iii) $\|\check{u}\|_{1,1}^a = \mathbb{E} \left[\left(\int_0^T |\check{u}(s)|^2 ds \right)^{\frac{1}{2}} + \left(\int_0^T \|\partial\check{u}(s)\|^2 ds \right)^{\frac{1}{2}} \right] < \infty$.

(B14), the stock price under Q -measure is given as

$$p_T^* = p_0^* + \int_0^T \sigma_t^* dW_t^*.$$

To apply the Malliavin derivative, let

$$\tilde{h}(t) = \sigma_t^*, \quad \tilde{W}(\tilde{h}) = \int_0^T \sigma_t^* dW_t^* = \int_0^T \tilde{h}(t) dW_t^*,$$

hence, the stock price is

$$p_T^* = p_0^* + \tilde{W}(\tilde{h}).$$

Moreover, to express c_{iT}^* as a function of $\tilde{W}(\tilde{h})$, let

$$\tilde{F} = \tilde{f}(\tilde{W}(\tilde{h})) = \theta_{i0}^\dagger (p_0^* + \tilde{W}(\tilde{h}))^2 + x_{i0}^\dagger (p_0^* + \tilde{W}(\tilde{h})) + \gamma_{i0}^\dagger,$$

thus, according to the definition of the Malliavin derivative, I have

$$\begin{aligned} \partial_t(c_{iT}^*) &= \partial_t(\theta_{i0}^\dagger (d_T)^2 + x_{i0}^\dagger d_T + \gamma_{i0}^\dagger) = \partial_t(\theta_{i0}^\dagger (p_T^*)^2 + x_{i0}^\dagger p_T^* + \gamma_{i0}^\dagger) \\ &= \partial_t(\theta_{i0}^\dagger (p_0^* + \tilde{W}(\tilde{h}))^2 + x_{i0}^\dagger (p_0^* + \tilde{W}(\tilde{h})) + \gamma_{i0}^\dagger) = \partial_t \tilde{F} \\ &= \partial \tilde{f}(\tilde{W}(\tilde{h})) \cdot \tilde{h} = (2\theta_{i0}^\dagger (p_0^* + \tilde{W}(\tilde{h})) + x_{i0}^\dagger) \cdot \tilde{h} = (2\theta_{i0}^\dagger p_T^* + x_{i0}^\dagger) \sigma_t^*, \end{aligned}$$

note the volatility of the stock price under Q -measure, σ_t^* , is a deterministic function of date t , hence, the optimal portfolio in stock for investor i is

$$\begin{aligned} x_{it}^* &= (\sigma_t^*)^{-1} \mathbb{E}^Q \left[(2\theta_{i0}^\dagger p_T^* + x_{i0}^\dagger) \sigma_t^* | \mathcal{F}_t \right] \\ &= \mathbb{E}^Q \left[2\theta_{i0}^\dagger p_T^* + x_{i0}^\dagger | \mathcal{F}_t \right] = \mathbb{E}^Q \left[2\theta_{i0}^\dagger d_T + x_{i0}^\dagger | \mathcal{F}_t \right] = 2\theta_{i0}^\dagger p_t + x_{i0}^\dagger. \end{aligned}$$

Now turn to the portfolio in the riskless bond. Note the proof of Proposition 3 shows that the price of payment of d_T^2 , i.e.,

$$\frac{\mathbb{E}^i[\zeta_{iT}^* d_T^2 | \mathcal{F}_t]}{\zeta_{it}^*} = \bar{\sigma}_t^2 + p_t^2,$$

hence, the optimal wealth at date t follows

$$\begin{aligned} \mathbb{E}^Q [c_{iT}^* | \mathcal{F}_t] &= \frac{1}{\zeta_{it}^*} \mathbb{E}^i [\zeta_{iT}^* c_{iT}^* | \mathcal{F}_t] = \frac{1}{\zeta_{it}^*} \mathbb{E}^i \left[\zeta_{iT}^* \left(\theta_{i0}^\dagger d_T^2 + x_{i0}^\dagger d_T + \gamma_{i0}^\dagger \right) | \mathcal{F}_t \right] \\ &= \theta_{i0}^\dagger (\bar{\sigma}_t^2 + p_t^2) + x_{i0}^\dagger p_t + \gamma_{i0}^\dagger, \end{aligned}$$

given the price of the bond is always 1. Thus, the optimal portfolio in the riskless bond is

$$\gamma_{it}^* = \mathbb{E}^Q [c_{iT}^* | \mathcal{F}_t] - x_{it} p_t = \theta_{i0}^\dagger \bar{\sigma}_t^2 + \gamma_{i0}^\dagger - \theta_{i0}^\dagger p_t^2.$$

Therefore, similar to the general proof for Proposition 1 (compare with Eq. (B1) and Eq. (B3)), the optimal portfolios here in a special case with exponential-utility investors and normally distributed dividend are self-financing and are able to construct the optimal wealth process. I now verify that the Pareto optimal portfolios satisfy the market clearing condition, i.e., (1). Note that

$$\begin{aligned} \sum_{i=1}^I x_{it}^* &= 2p_t \sum_{i=1}^I \theta_{i0}^\dagger + \sum_{i=1}^I x_{i0}^\dagger = Z, \\ \sum_{i=1}^I \gamma_{it}^* &= (\bar{\sigma}_t^2 - p_t^2) \sum_{i=1}^I \theta_{i0}^\dagger + \sum_{i=1}^I \gamma_{i0}^\dagger = 0, \end{aligned}$$

thus, the market clearing conditions establish. This completes the proof.

B.4 Proof of Proposition 4

The price of the dividend derivative is given as

$$\pi_t = \frac{\mathbb{E}^i [\zeta_{iT}^* d_T^2 | \mathcal{F}_t]}{\zeta_{it}^*},$$

I obtain state-price deflator at date t in the previous subsection, thus I only need to calculate the expected risk-adjusted payment of the dividend derivative.

Use the notation in the previous subsection, I have

$$\begin{aligned}
& \mathbb{E}^i [\zeta_{iT}^* d_T^2 | \mathcal{F}_t] \\
&= \frac{1}{\psi_{i0}} \int_{-\infty}^{+\infty} d_T^2 \exp - [\alpha_0 [d_T + \alpha_1]^2 + \alpha_2] d d_T \\
&= \exp [-\alpha_2] \int_{-\infty}^{+\infty} d_T \exp - [(a_1 d_T + b_1)^2] d d_T^2 \\
&= \frac{\exp [-\alpha_2]}{a_1^2} \int_{-\infty}^{+\infty} d_T \exp - [(a_1 d_T + b_1)^2] d [(a_1 d_T + b_1)^2 - 2a_1 b_1 d_T - b_1^2] \\
&= \frac{\exp [-\alpha_2]}{a_1^2} \int_{-\infty}^{+\infty} d_T \exp - [(a_1 d_T + b_1)^2] d [(a_1 d_T + b_1)^2 - 2b_1 (a_1 d_T + b_1)],
\end{aligned}$$

let $u = a_1 d_T + b_1$, and $d_T = \frac{u-b_1}{a_1}$ note $u \rightarrow +\infty$, as $d_T \rightarrow +\infty$, $u \rightarrow -\infty$, as $d_T \rightarrow -\infty$, thus,

$$\begin{aligned}
\mathbb{E}^i [\zeta_{iT}^* d_T^2 | \mathcal{F}_t] &= \frac{\exp [-\alpha_2]}{a_1^2} \int_{-\infty}^{+\infty} \frac{u-b_1}{a_1} \exp [-u^2] d [u^2 - 2b_1 u] \\
&= \frac{\exp [-\alpha_2]}{a_1^3} \left(\int_{-\infty}^{+\infty} u \exp [-u^2] d [u^2 - 2b_1 u] - b_1 \int_{-\infty}^{+\infty} \exp [-u^2] d [u^2 - 2b_1 u] \right) \quad (\text{B18})
\end{aligned}$$

Note the first integral in (B18)

$$\begin{aligned}
& \int_{-\infty}^{+\infty} u \exp [-u^2] d [u^2 - 2b_1 u] \\
&= \int_{-\infty}^{+\infty} u \exp [-u^2] du^2 - 2b_1 \int_{-\infty}^{+\infty} u \exp [-u^2] du \\
&= \int_0^{+\infty} u \exp [-u^2] du^2 + \int_{-\infty}^0 u \exp [-u^2] du^2 - b_1 \int_{-\infty}^{+\infty} \exp [-u^2] du^2 \\
&= \int_0^{+\infty} u \exp [-u^2] du^2 - \int_{-\infty}^0 -u \exp [-(-u)^2] d(-u)^2 - b_1 \int_{-\infty}^{+\infty} \exp [-u^2] du^2 \\
&= 2 \int_0^{+\infty} u \exp [-u^2] du^2 - b_1 \int_{-\infty}^{+\infty} \exp [-u^2] du^2,
\end{aligned}$$

let $v = u^2$, note $v \rightarrow +\infty$, as $u \rightarrow +\infty$, $v \rightarrow +\infty$, as $u \rightarrow -\infty$, thus,

$$\begin{aligned}
& \int_{-\infty}^{+\infty} u \exp[-u^2] d[u^2 - 2b_1u] \\
&= 2 \int_0^{+\infty} \sqrt{v} \exp[-v] dv - b_1 \int_{+\infty}^{+\infty} \exp[-v] dv \\
&= 2 \int_0^{+\infty} \sqrt{v} \exp[-v] dv - 0 \stackrel{v=\frac{1}{2}x}{=} 2 \int_0^{+\infty} \sqrt{\frac{1}{2}x} \exp\left[-\frac{1}{2}x\right] d\frac{1}{2}x \\
&= \sqrt{\frac{1}{2}} \int_0^{+\infty} \sqrt{x} \exp\left[-\frac{1}{2}x\right] dx.
\end{aligned}$$

Note

$$\begin{aligned}
& \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} x^{\frac{1}{2}} \exp\left(-\frac{1}{2}x\right) dx \\
&= \frac{1}{\sqrt{2\pi}} \left[-x^{\frac{1}{2}} 2 \exp\left(-\frac{1}{2}x\right) dx \Big|_0^{+\infty} + \int_0^{+\infty} \frac{1}{2} x^{-\frac{1}{2}} 2 \exp\left(-\frac{1}{2}x\right) dx \right] \\
&= \frac{1}{\sqrt{2\pi}} \left[(0 - 0) + \int_0^{+\infty} x^{-\frac{1}{2}} \exp\left(-\frac{1}{2}x\right) dx \right] \\
&= \int_0^{+\infty} \frac{1}{\sqrt{2\pi}} x^{-\frac{1}{2}} \exp\left(-\frac{1}{2}x\right) dx = \int_0^{+\infty} f_X(x) dx = 1,
\end{aligned}$$

where $f_X(x)$ is the probability density function of X which is a Chi-square random variable with 1 degrees of freedom.

Therefore,

$$\int_{-\infty}^{+\infty} u \exp[-u^2] d[u^2 - 2b_1u] = \sqrt{\frac{1}{2}} \times \sqrt{2\pi} = \sqrt{\pi}.$$

Note the second integral in (B18) is derived in previous subsection, i.e.,

$$\int_{-\infty}^{+\infty} \exp[-u^2] d[u^2 - 2b_1u] = -2b_1\sqrt{\pi},$$

hence,

$$E^i [\zeta_{iT}^* d_T^2 | \mathcal{F}_t] = \frac{\exp[-\alpha_2]}{a_1^3 \bar{\psi}_{i0}} (\sqrt{\pi} + 2b_1^2 \sqrt{\pi}).$$

Therefore, the price for the dividend derivative

$$\begin{aligned}
\pi_t &= \frac{\mathbb{E}^i [\zeta_{iT}^* d_T^2 | \mathcal{F}_t]}{\zeta_{it}^*} = \frac{\frac{\exp[-\alpha_2]}{a_1^3 \psi_{i0}} (\sqrt{\pi} + 2b_1^2 \sqrt{\pi})}{\frac{1}{\psi_{i0}} \sqrt{\frac{\pi}{\alpha_0}} \exp[-\alpha_2]} = \frac{\frac{1}{2a_1^3} (1 + 2b_1^2)}{\sqrt{\frac{1}{\alpha_0}}} \\
&= \frac{1}{2a_1^2} (1 + 2b_1^2) = \frac{1}{2a_1^2} + \frac{b_1^2}{a_1^2} = \frac{1}{2\alpha_0} + p_t^2 = \bar{\sigma}_t^2 + p_t^2.
\end{aligned}$$

Moreover, by investing in $2\theta_{i0}^\dagger p_t$ share of the stock, and $\theta_{i0}^\dagger (\bar{\sigma}_t^2 - p_t^2)$ share of the riskless bond continuously, the value of the portfolios is $\theta_{i0}^\dagger (p_t^2 + \bar{\sigma}_t^2) = \theta_{i0}^\dagger \pi_t$ which is the value of θ_{i0}^\dagger share of the dividend derivative at date t . This completes the proof.

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