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Goodness-of-fit testing for fractional diffusions *

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Abstract

This paper presents a goodness-of-fit test for the volatility function of a SDE driven by a Gaussian process with stationary and centered increments. Under rather weak assumptions on the Gaussian process, we provide a procedure for testing whether the unknown volatility function lies in a given linear functional space or not. This testing problem is highly non-trivial, because the volatility function is not identifiable in our model. The underlying fractional diffusion is assumed to be observed at high frequency on a fixed time interval and the test statistic is based on weighted power variations. Our test statistic is consistent against any fixed alternative.

Keywords: central limit theorem, goodness-of-fit tests, high frequency observations, fractional diffusions, stable convergence.

JEL Classification: C10, C13, C14.

1 Introduction

In this paper we consider a stochastic differential equation

$$dX_t = a(t, X_t)dt + \sigma(t, X_t)dG_t, \qquad X_0 = x \in \mathbb{R},$$

driven by a Gaussian process $(G_t)_{t\geq 0}$ with stationary and centered increments. Our aim is to derive a goodness-of-fit test for the volatility function $\sigma^2 : [0,1] \times \mathbb{R} \to \mathbb{R}^+$ on the basis of discrete high frequency observations $X_{i\Delta_n}$, $i = 0, \ldots, [1/\Delta_n]$, of the path $(X_t)_{t\in[0,1]}$. The notion of high frequency refers to the infill asymptotics setting, i.e. $\Delta_n \to 0$. More precisely, we will propose a procedure for testing whether the unknown volatility function σ^2 lies in a linear subspace generated by given volatility functions $\sigma_k^2 : [0,1] \times \mathbb{R} \to \mathbb{R}^+$, $1 \leq k \leq d$.

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This type of goodness-of-fit testing has been studied in the literature for the case of classical diffusions that are driven by a standard Brownian motion (see e.g. [6]; we also refer to [1] for similar testing problems under a different sampling scheme). Compared to the classical case, we will have to deal with additional non-trivial theoretical challenges. First of all, under our mild assumptions on the driving Gaussian process G, the volatility function σ^2 is not identifiable. This fact can be easily seen by observing that a multiplication of the volatility function σ^2 by a constant can not be distinguished from the multiplication of G by the same constant. Secondly, the resulting test statistic should be robust to the presence of the drift process a (i.e. the limit theory should not depend on a), because the drift a can not be estimated on a fixed time interval $[0,1]$. The third theoretical problem is coming from the fact that the (stable) central limit theorems for usual power variation statistics, which will be the main tool for hypothesis testing, only hold for a subrange of smoothness parameters of G, while we would like to obtain a valid testing procedure for all smoothness parameters of G.

Our test statistic is a function of various weighted power variations based on the second order differences of X. Higher order differences is a key instrument for obtaining valid central limit theorems for all smoothness parameters of G, but more importantly they also insure the robustness to the drift process a (in contrast to the first order differences). The second crucial issue of our test statistic is the *self-scaling property*, which makes it independent of the unknown variance of dG_t. This solves the non-identifiability problem that we mentioned above.

Our paper is organised as follows. In section 2 we present the main assumptions on the processes a, σ^2 and G, which ensure the existence of the unique solution of the above SDE and the validity of the asymptotic results. In section 3 we explain the testing problem and construct a distance measure, which enables us to make statistical decisions. In section 4 we derive our test statistic and show its asymptotic mixed normality. Section 5 is devoted to proofs.

2 Model and assumptions

We consider a one-dimensional fractional diffusion model of the form

$$X_t = x + \int_0^t a(s, X_s)ds + \int_0^t \sigma(s, X_s)dG_s, \qquad t \in [0,1], \tag{2.1}$$

defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $x \in \mathbb{R}$. Here the function $a : [0,1] \times \mathbb{R} \to \mathbb{R}$ represents the drift, $\sigma : [0,1] \times \mathbb{R} \to \mathbb{R}$ is the non-vanishing volatility function and $G = (G_t)_{t\in[0,1]}$ is a Gaussian process with stationary and centered increments. To ensure the existence of the second integral in (2.1) (in the Riemann-Stieltjes sense) we need to assume that the process G is Hölder continuous of order bigger than $1/2$. More precisely, we impose structural assumptions on the covariance kernel of the Gaussian driver G. Let R denote the variance function of the increments of G, i.e.

$$R(t) = \mathbb{E}[|G_{t+s} - G_s|^2], \qquad t \geq 0, \tag{2.2}$$

which is independent of s since G has stationary increments. Below, the functions $L_R, L_{R^{(4)}} : \mathbb{R}_{>0} \to \mathbb{R}$ are assumed to be continuous and slowly varying at 0, $f^{(k)}$ denotes the k-th derivative of a function f and $H \in (1/2, 1)$. We assume that the following conditions are satisfied:

(i) $R(t) = t^{2H} L_R(t)$.

(ii) $R^{(4)}(t) = t^{2H-4} L_{R^{(4)}}(t)$.

(iii) There exists a constant $b \in (0, 1)$ such that

$$\limsup_{x \to 0} \sup_{y \in [x, x^b]} \left| \frac{L_{R^{(4)}}(y)}{L_R(x)} \right| < \infty.$$

We remark that condition (i) implies that G is Hölder continuous of all orders smaller than $H \in (1/2, 1)$. Of course, the whole set of conditions is not required to guarantee the existence of a unique solution of the SDE (2.1), but we do require assumptions (i)-(iii) for the central limit theorems that are presented in section 4 (cf. [3]). Notice that condition (i) indicates that the local behaviour of the Gaussian process G is similar to the local behaviour of the fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. This fact will be reflected in the central limit theorems presented in section 4.

Now, the results of [9] imply the existence of a unique solution of the SDE (2.1), where the second integral in (2.1) is defined pathwise in the Riemann-Stieltjes sense, given the following set of conditions is satisfied:

(iv) The function $a(t, \cdot)$ is Lipschitz, uniformly in $t \in [0, 1]$. The function $a(\cdot, x)$ is Hölder continuous of order $\gamma \geq H$, uniformly in $x \in \mathbb{R}$. Furthermore, it satisfies the growth condition

$$|a(t, x)| \leq C|x| + a_0(t)$$

for some $C > 0$ and $a_0 \in L^p([0, 1])$ for some $p > \frac{1}{1-H}$.

(v) The function $\sigma(t, \cdot)$ is Lipschitz, uniformly in $t \in [0, 1]$, and σ is continuously differentiable in x. Furthermore, there exist constants $0 < \alpha, \beta \leq 1$ with $\beta > 1/2$ and $\alpha > \frac{1}{H} - 1$ such that: (a) $\frac{\partial}{\partial x}\sigma(t, \cdot)$ is locally Hölder continuous of order α, uniformly in $t \in [0, 1]$, (b) $\sigma(\cdot, x)$ and $\frac{\partial}{\partial x}\sigma(\cdot, x)$ are Hölder continuous of order β, uniformly in $x \in \mathbb{R}$.

The assumptions (iv) and (v) include some conditions that are required to prove our asymptotic results; hence, they are slightly stronger than considered in [9]. Furthermore, under conditions (iv)-(v), the unique solution of (2.1) satisfies

$$X \in C^{1-\delta}([0, 1]) \quad \text{a.s.,} \tag{2.3}$$

for any $\delta \in (1 - H, \min\{\frac{1}{2}, \alpha, \frac{\beta}{1+\beta}\})$ (see again [9]). Throughout this paper we assume that, for any $k \in \mathbb{N}$ and any $t_1, \ldots, t_k > 0$, the random vector $(X_{t_1}, \ldots, X_{t_k})$ has a Lebesgue density. This technical conditions ensures the invertibility of the matrices Σ and Σ_n defined by (3.6) and (4.3), respectively.

3 The test problem

We assume that the process X defined in (2.1) is observed at time points $t_i = i\Delta_n$, i.e. the high frequency observations

$$X_0, X_{\Delta_n}, X_{2\Delta_n}, \ldots, X_{\Delta_n[1/\Delta_n]}$$

are given and $\Delta_n \to 0$. For $d \geq 1$, let $\sigma_1^2, \ldots, \sigma_d^2 : [0,1] \times \mathbb{R} \to \mathbb{R}$ be known functions that satisfy the assumption (v) from section 2. We assume that the functions $\sigma_1^2, \ldots, \sigma_d^2$ are linearly independent on the sets of the form $[0,1] \times [a,b]$, for all $a < b$, and denote by $\mathbb{V}$ the vector space generated by $\sigma_1^2, \ldots, \sigma_d^2$, i.e.

$$\mathbb{V} = \text{span}\{\sigma_1^2, \ldots, \sigma_d^2\}. \tag{3.1}$$

Our main aim is to decide, on the basis of the underlying observations $X_{i\Delta_n}$, whether the unknown volatility function σ^2 is a linear combination of the known functions $\sigma_1^2, \ldots, \sigma_d^2$. More precisely, the null hypothesis is given by

$$H_0: \ \sigma^2(t, X_t) = \sum_{k=1}^{d} \lambda_k \sigma_k^2(t, X_t) \quad \text{for some } \lambda_k\text{'s and a.s. all } t \in [0,1], \tag{3.2}$$

while the alternative is defined as the complement hypothesis. We remark that any possible test procedure can only give *pathwise* conclusions. For instance, when H_0 holds for the observed path $X(\omega)$ it does not mean that it remains true for a different path $X(\omega')$. Furthermore, H_0 does not imply the identity $\sigma^2(t,x) = \sum_{k=1}^{d} \lambda_k \sigma_k^2(t,x)$ for some $(\lambda_1, \ldots, \lambda_d)$ and a.s. all $(t,x) \in [0,1] \times \mathbb{R}$.

Our test procedure will be based on the estimation of the L^2-distance between the vector space $\mathbb{V}$ and the volatility function σ^2. For this purpose we introduce the following random scalar product: for any functions $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ we define

$$\langle f, g \rangle := \int_0^1 f(s, X_s) g(s, X_s) ds. \tag{3.3}$$

We also set $\|f\| := \sqrt{\langle f, f \rangle}$ and denote by $\|\mathbb{V} - f\|$ the distance between the vector space $\mathbb{V}$ and the function f, i.e.

$$\|\mathbb{V} - f\| := \inf\{\|v - f\| : \ v \in \mathbb{V}\}.$$

Now, we can reformulate the null hypothesis and the alternative as

$$H_0: \ \|\mathbb{V} - \sigma^2\| = 0, \qquad H_1: \ \|\mathbb{V} - \sigma^2\| > 0, \tag{3.4}$$

where the vector space $\mathbb{V}$ is defined in (3.1). The standard arguments from the Hilbert space theory imply the identity

$$\|\mathbb{V} - \sigma^2\|^2 = \|\sigma^2\|^2 - \Big(\langle \sigma_1^2, \sigma^2 \rangle, \ldots, \langle \sigma_d^2, \sigma^2 \rangle\Big) \Sigma^{-1} \Big(\langle \sigma_1^2, \sigma^2 \rangle, \ldots, \langle \sigma_d^2, \sigma^2 \rangle\Big)^\star, \tag{3.5}$$

$$\Sigma = \Big(\langle \sigma_k^2, \sigma_l^2 \rangle\Big)_{1 \leq k,l \leq d}, \tag{3.6}$$

where $v^\star$ denotes the transpose of the vector v. In contrast to the work of [6], who followed a similar approach for the goodness-of-fit testing for classical diffusions (that are driven by a Brownian motion), the distance measure $\|\mathbb{V}-\sigma^2\|^2$ is not the right quantity to estimate in the fractional diffusion case. The main reason lies in the fact that the objects $\|\sigma^2\|$ and $\langle\sigma_k^2,\sigma^2\rangle$, $1\le k\le d$, can not be identified, because our assumptions are not sufficient to fully identify the variance function R. In order to obtain a feasible estimator, we define a self-scaling measure $T\ge 0$ as

$$T=\frac{\|\mathbb{V}-\sigma^2\|^2}{\|\sigma^2\|^2}, \tag{3.7}$$

which turns out to be a crucial transformation of the original distance measure $\|\mathbb{V}-\sigma^2\|$. We will see that the random measure T can be consistently estimated without a precise knowledge of the function R. We also have the appealing property

$$0\le T\le 1,$$

which describes the deviation from the null hypothesis $T=0$ in percent (in contrast, it is not clear when the quantity $\|\mathbb{V}-\sigma^2\|$ can be identified as small).

Remark 3.1 For the simple case $d=1$, $\sigma_1^2=1$, which corresponds to the homoscedasticity testing, we obtain the identity

$$T=1-\frac{\left(\int_0^1\sigma^2(s,X_s)ds\right)^2}{\int_0^1\sigma^4(s,X_s)ds}.$$

□

4 The testing procedure

In this section we construct a consistent, asymptotically mixed normal, estimator T_n of the random variable T defined in (3.7). Our plan is to construct empirical analogues of the quantities $\|\sigma^2\|^2$, $\langle\sigma_k^2,\sigma^2\rangle$, $\langle\sigma_k^2,\sigma_l^2\rangle$, $1\le k,l\le d$, prove their asymptotic mixed normality and apply the delta-method. In the high frequency setting (weighted) power variations based on increments of $X_{i\Delta_n}-X_{(i-1)\Delta_n}$ are known to be consistent estimators of integrated powers of volatility. However, in our framework of fractional diffusions, it makes more sense to use higher order differences. We set

$$\Delta_{i,2}^n X=X_{i\Delta_n}-2X_{(i-1)\Delta_n}+X_{(i-2)\Delta_n},\qquad \tau_n^2=\mathbb{E}[|\Delta_{i,2}^n G|^2].$$

The empirical analogues of $\|\sigma^2\|^2$, $\langle\sigma_k^2,\sigma^2\rangle$, $\langle\sigma_k^2,\sigma_l^2\rangle$, $1\le k,l\le d$, are defined as

$$\|\sigma^2\|_n^2 = \frac{\Delta_n}{3\tau_n^4}\sum_{i=2}^{[1/\Delta_n]} |\Delta_{i,2}^n X|^4, \tag{4.1}$$

$$\langle\sigma_k^2,\sigma^2\rangle_n = \frac{\Delta_n}{\tau_n^2}\sum_{i=2}^{[1/\Delta_n]} \sigma^2_{k,(i-2)\Delta_n}|\Delta_{i,2}^n X|^2,$$

$$\langle\sigma_k^2,\sigma_l^2\rangle_n = \Delta_n\sum_{i=0}^{[1/\Delta_n]} \sigma^2_{k,i\Delta_n}\sigma^2_{l,i\Delta_n},$$

where we use a shorthand notation $\sigma^2_{k,s}=\sigma^2_k(s,X_s)$. In our setting using second order differences has the following crucial advantages: (a) The central limit theorem presented below holds for all $H\in(1/2,1)$ while it would hold only for $H\in(1/2,3/4)$ if we would use the standard increments of X; this effect of higher order differences is known in the literature in the pure Gaussian framework (see e.g. [8]), (b) More importantly, the central limit theorem is not influenced by the presence of the drift function a, which would not be true in the case of the standard increments of X (this is a less known fact).

Finally, we define

$$T_n = \frac{\|\sigma^2\|_n^2 - \Big(\langle\sigma_1^2,\sigma^2\rangle_n,\ldots,\langle\sigma_d^2,\sigma^2\rangle_n\Big)\Sigma_n^{-1}\Big(\langle\sigma_1^2,\sigma^2\rangle_n,\ldots,\langle\sigma_d^2,\sigma^2\rangle_n\Big)^\star}{\|\sigma^2\|_n^2}, \tag{4.2}$$

$$\Sigma_n = \Big(\langle\sigma_k^2,\sigma_l^2\rangle_n\Big)_{1\le k,l\le d}. \tag{4.3}$$

Remark 4.1 The statistics defined in (4.1) are not feasible, because the normalising constant τ_n is unknown! However, it is easy to see that the test statistic T_n is feasible due to the self-scaling property. For instance, when $d=1$ and $\sigma_1^2=1$ we obtain the identity

$$T_n = 1-\frac{3\Big(\Delta_n\sum_{i=2}^{[1/\Delta_n]}|\Delta_{i,2}^n X|^2\Big)^2}{\Delta_n\sum_{i=2}^{[1/\Delta_n]}|\Delta_{i,2}^n X|^4},$$

which obviously does not depend on the unknown constant τ_n. □

As it has been mentioned in e.g. [5], under the assumption (i) of section 2, it holds that

$$\mathrm{corr}(\Delta_{1,2}^n G,\Delta_{1+j,2}^n G)\to\rho(j)=\frac{-(j+2)^{2H}+4(j+1)^{2H}-6j^{2H}+4|j-1|^{2H}-|j-2|^{2H}}{2\Big(4-2^{2H}\Big)},$$

as $n\to\infty$, and $\rho=\rho_H$ is the correlation function of the second order fractional noise $(\Delta_{i,2}^n B^H)_{i\ge1}$ with Hurst parameter H. Notice that $|\rho(j)|\sim j^{2H-4}$ as $j\to\infty$, which implies that $\sum_{j=1}^\infty|\rho(j)|<\infty$ for all $H\in(1/2,1)$.

In the next theorem we demonstrate a multivariate stable central limit theorem. Recall that a sequence of random variables Y_n on $(\Omega, \mathcal{F}, \mathbb{P})$ converges stably in law towards Y ($Y_n \xrightarrow{st} Y$), which is defined on the extension $(\Omega', \mathcal{F}', \mathbb{P}')$ of the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$, iff

$$\lim_{n\to\infty} \mathbb{E}[f(Y_n)Z] = \mathbb{E}'[f(Y)Z]$$

for all bounded random variables Z and all bounded, continuous functions f (see e.g. [2] for more details). We call Y mixed normal with mean 0 and conditional covariance matrix V ($Y = MN(0, V)$) when, conditionally on $\mathcal{F}$, Y has a normal distribution with mean 0 and covariance matrix V. The next results mainly follow from the theory presented in [3], [4] and [5].

Theorem 4.2 *Assume that the conditions (i)-(v) from section 2 hold. Then we obtain the convergence*

$$\|\sigma^2\|_n^2 \xrightarrow{\mathbb{P}} \|\sigma^2\|^2, \quad \langle \sigma_k^2, \sigma^2\rangle_n \xrightarrow{\mathbb{P}} \langle \sigma_k^2, \sigma^2\rangle, \quad \langle \sigma_k^2, \sigma_l^2\rangle_n \xrightarrow{\mathbb{P}} \langle \sigma_k^2, \sigma_l^2\rangle. \tag{4.4}$$

Furthermore, we deduce the stable convergence

$$\Delta_n^{-1/2} \begin{pmatrix} \|\sigma^2\|_n^2 - \|\sigma^2\|^2 \\ \langle \sigma_1^2, \sigma^2\rangle_n - \langle \sigma_1^2, \sigma^2\rangle \\ \vdots \\ \langle \sigma_d^2, \sigma^2\rangle_n - \langle \sigma_d^2, \sigma^2\rangle \end{pmatrix} \xrightarrow{st} MN_{d+1}\Big(0, \int_0^1 V_s ds\Big), \tag{4.5}$$

where $V_s \in \mathbb{R}^{(d+1)\times(d+1)}$ is a symmetric matrix defined as

$$V_s^{11} = \alpha_H \sigma_s^8,$$
$$V_s^{1,k+1} = \beta_H \sigma_{k,s}^2 \sigma_s^6, \qquad 1 \le k \le d,$$
$$V_s^{k+1,l+1} = \gamma_H \sigma_{k,s}^2 \sigma_{l,s}^2 \sigma_s^4, \qquad 1 \le k, l \le d,$$

and the constants α_H, β_H and γ_H are given by

$$\alpha_H = \frac{1}{9}\Big(96 + \sum_{k=1}^{\infty} \{48\rho^4(k) + 144\rho^2(k)\}\Big),$$

$$\beta_H = \frac{1}{3}\Big(12 + 24\sum_{k=1}^{\infty} \rho^2(k)\Big),$$

$$\gamma_H = 2 + 4\sum_{k=1}^{\infty} \rho^2(k).$$

Finally, it holds that

$$\Delta_n^{-1/2}(\Sigma_n - \Sigma) \xrightarrow{\mathbb{P}} 0. \tag{4.6}$$

Now, we want to apply the delta-method. For an invertible matrix $A \in \mathbb{R}^{d\times d}$ and $x \in \mathbb{R}^{d+1}$, we define the function

$$g(x;A) := \frac{x_1 - (x_2,\dots,x_{d+1})A^{-1}(x_2,\dots,x_{d+1})^\star}{x_1}.$$

Then the identity

$$\begin{aligned} T_n - T &= g(\|\sigma^2\|_n^2, \langle\sigma_1^2,\sigma^2\rangle_n,\dots,\langle\sigma_d^2,\sigma^2\rangle_n;\Sigma_n) \\ &- g(\|\sigma^2\|^2, \langle\sigma_1^2,\sigma^2\rangle,\dots,\langle\sigma_d^2,\sigma^2\rangle;\Sigma) \end{aligned}$$

holds. Notice that the matrices Σ and Σ_n are invertible a.s., because the functions $\sigma_1^2,\dots,\sigma_d^2$ are linearly independent on the sets of the form $[0,1]\times[a,b]$ for all $a<b$, and the random vector $(X_{t_1},\dots,X_{t_k})$ has a Lebesgue density for all $k\in\mathbb{N}$ and all $t_1,\dots,t_k>0$. Using (4.6) we deduce that

$$\begin{aligned} T_n - T &= g(\|\sigma^2\|_n^2, \langle\sigma_1^2,\sigma^2\rangle_n,\dots,\langle\sigma_d^2,\sigma^2\rangle_n;\Sigma) \\ &- g(\|\sigma^2\|^2, \langle\sigma_1^2,\sigma^2\rangle,\dots,\langle\sigma_d^2,\sigma^2\rangle;\Sigma) + o_{\mathbb{P}}(\Delta_n^{1/2}). \end{aligned}$$

This decomposition will imply the mixed normality of $\Delta_n^{-1/2}(T_n-T)$ by the delta-method for stable convergence. However, to obtain a feasible test statistic we still need to estimate the conditional covariance matrix $V=\int_0^1 V_s ds$. This consists of two subproblems: (a) Estimation of the parameter $H\in(1/2,1)$ and (b) Estimation of integrated products of various volatility functions. The first estimation problem is solved via a change-of-frequency method

$$H_n := \frac{1}{2}\log_2\left(\frac{\sum_{i=4}^{[1/\Delta_n]}|X_{i\Delta_n} - 2X_{(i-2)\Delta_n} + X_{(i-4)\Delta_n}|^2}{\sum_{i=2}^{[1/\Delta_n]}|X_{i\Delta_n} - 2X_{(i-1)\Delta_n} + X_{(i-2)\Delta_n}|^2}\right),$$

which compares one realised measure at two different frequencies ($\log_2$ denotes the logarithm with basis 2). Indeed, section 4.3 from [5] shows that

$$H_n \xrightarrow{\mathbb{P}} H. \tag{4.7}$$

Notice that the coefficients α_H, β_H and γ_H are continuous in H, which implies that

$$\alpha_{H_n} \xrightarrow{\mathbb{P}} \alpha_H, \quad \beta_{H_n} \xrightarrow{\mathbb{P}} \beta_H, \quad \gamma_{H_n} \xrightarrow{\mathbb{P}} \gamma_H.$$

Now, we are able to construct an empirical (symmetric) analogue of the matrix $V=\int_0^1 V_s ds$:

$$V_n^{11} = \frac{\alpha_{H_n}\Delta_n}{105\tau_n^8}\sum_{i=2}^{[1/\Delta_n]}|\Delta_{i,2}^n X|^8,$$

$$V_n^{1,k+1} = \frac{\beta_{H_n}\Delta_n}{15\tau_n^6}\sum_{i=2}^{[1/\Delta_n]}\sigma^2_{k,(i-2)\Delta_n}|\Delta_{i,2}^n X|^6, \qquad 1\le k\le d,$$

$$V_n^{k+1,l+1} = \frac{\gamma_{H_n}\Delta_n}{3\tau_n^4}\sum_{i=2}^{[1/\Delta_n]}\sigma^2_{k,(i-2)\Delta_n}\sigma^2_{l,(i-2)\Delta_n}|\Delta_{i,2}^n X|^4, \qquad 1\le k,l\le d.$$

Before we present the next theorem, we introduce a shorthand notation

$$\nabla g = \nabla g(\|\sigma^2\|^2, \langle \sigma_1^2, \sigma^2\rangle, \ldots, \langle \sigma_d^2, \sigma^2\rangle; \Sigma),$$

$$\nabla g_n = \nabla g(\|\sigma^2\|_n^2, \langle \sigma_1^2, \sigma^2\rangle_n, \ldots, \langle \sigma_d^2, \sigma^2\rangle_n; \Sigma_n).$$

The next result follows directly from Theorem 4.2 and the delta-method for stable convergence (see e.g. Proposition 2.5 in [10]).

Theorem 4.3 *Assume that the conditions (i)-(v) from section 2 hold. Then we obtain the following results:*

(a) It holds that $\Delta_n^{-1/2}(T_n - T) \xrightarrow{st} MN\Big(0, \nabla g V \nabla g^\star\Big)$ *with* $V = \int_0^1 V_s ds$.

(b) We obtain the convergence in probability $V_n \xrightarrow{\mathbb{P}} V$ *and*

$$\frac{\Delta_n^{-1/2}(T_n - T)}{\sqrt{\nabla g_n V_n \nabla g_n^\star}} \xrightarrow{d} \mathcal{N}(0,1). \tag{4.8}$$

As we have already mentioned, part (a) of Theorem 4.3 follows from Theorem 4.2 and the delta-method for stable convergence. The standard central limit theorem in (4.8) follows again from the properties of stable convergence, part (a) of Theorem 4.3 and $V_n \xrightarrow{\mathbb{P}} V$, $\nabla g_n \xrightarrow{\mathbb{P}} \nabla g$.

Remark 4.4 The statistic $\nabla g_n V_n \nabla g_n^\star$ is indeed feasible, i.e. it does not depend on the normalising constant τ_n. For instance, when $d = 1$ and $\sigma_1^2 = 1$ we deduce the identities

$$\Sigma_n = 1, \qquad \nabla g_n = \left(\frac{\Big(\frac{\Delta_n}{\tau_n^2}\sum_{i=2}^{[1/\Delta_n]} |\Delta_{i,2}^n X|^2\Big)^2}{\Big(\frac{\Delta_n}{3\tau_n^4}\sum_{i=2}^{[1/\Delta_n]} |\Delta_{i,2}^n X|^4\Big)^2}, -\frac{\frac{2\Delta_n}{\tau_n^2}\sum_{i=2}^{[1/\Delta_n]} |\Delta_{i,2}^n X|^2}{\frac{\Delta_n}{3\tau_n^4}\sum_{i=2}^{[1/\Delta_n]} |\Delta_{i,2}^n X|^4} \right),$$

$$V_n = \begin{pmatrix} \frac{\alpha_{H_n}\Delta_n}{105\tau_n^8}\sum_{i=2}^{[1/\Delta_n]} |\Delta_{i,2}^n X|^8 & \frac{\beta_{H_n}\Delta_n}{15\tau_n^6}\sum_{i=2}^{[1/\Delta_n]} |\Delta_{i,2}^n X|^6 \\ \frac{\beta_{H_n}\Delta_n}{15\tau_n^6}\sum_{i=2}^{[1/\Delta_n]} |\Delta_{i,2}^n X|^6 & \frac{\gamma_{H_n}\Delta_n}{3\tau_n^4}\sum_{i=2}^{[1/\Delta_n]} |\Delta_{i,2}^n X|^4 \end{pmatrix},$$

which implies that

$$\nabla g_n V_n \nabla g_n^\star = \left(\frac{9\Big(\sum_{i=2}^{[1/\Delta_n]} |\Delta_{i,2}^n X|^2\Big)^2}{\Big(\sum_{i=2}^{[1/\Delta_n]} |\Delta_{i,2}^n X|^4\Big)^2}, -\frac{6\sum_{i=2}^{[1/\Delta_n]} |\Delta_{i,2}^n X|^2}{\sum_{i=2}^{[1/\Delta_n]} |\Delta_{i,2}^n X|^4} \right)$$

$$\times \begin{pmatrix} \frac{\alpha_{H_n}\Delta_n}{105}\sum_{i=2}^{[1/\Delta_n]} |\Delta_{i,2}^n X|^8 & \frac{\beta_{H_n}\Delta_n}{15}\sum_{i=2}^{[1/\Delta_n]} |\Delta_{i,2}^n X|^6 \\ \frac{\beta_{H_n}\Delta_n}{15}\sum_{i=2}^{[1/\Delta_n]} |\Delta_{i,2}^n X|^6 & \frac{\gamma_{H_n}\Delta_n}{3}\sum_{i=2}^{[1/\Delta_n]} |\Delta_{i,2}^n X|^4 \end{pmatrix}$$

$$\times \left(\frac{9\Big(\sum_{i=2}^{[1/\Delta_n]} |\Delta_{i,2}^n X|^2\Big)^2}{\Big(\sum_{i=2}^{[1/\Delta_n]} |\Delta_{i,2}^n X|^4\Big)^2}, -\frac{6\sum_{i=2}^{[1/\Delta_n]} |\Delta_{i,2}^n X|^2}{\sum_{i=2}^{[1/\Delta_n]} |\Delta_{i,2}^n X|^4} \right)^\star$$

□

Finally, we define our test statistic as

$$S_n = \frac{\Delta_n^{-1/2} T_n}{\sqrt{\nabla g_n V_n \nabla g_n^\star}}. \tag{4.9}$$

The next proposition follows directly from part (b) of Theorem 4.3.

Proposition 4.5 *Assume that the conditions (i)-(v) from section 2 hold. We reject the null hypothesis* $H_0: \ \|\mathbb{V} - \sigma^2\| = 0$ *if*

$$S_n > u_{1-\alpha},$$

where $u_{1-\alpha}$ *denotes the* $(1-\alpha)$ *quantile of* $\mathcal{N}(0,1)$*. Then it holds that*

$$\begin{aligned} \mathbb{P}_{H_0}(S_n > u_{1-\alpha}) &\to \alpha, \\ \mathbb{P}_{H_1}(S_n > u_{1-\alpha}) &\to 1. \end{aligned}$$

In other words, the test statistic S_n *has asymptotic level* α *and it is consistent against any fixed alternative.*

5 Proofs

We concentrate on the proof of Theorem 4.2 as all other theorems follow from this result. The justification of Theorem 4.2 relies on a combination of various methods presented in [3], [4] and [5], which we sketch below. Let us define

$$\overline{X}_t = X_0 + \int_0^t \sigma_s dG_s, \tag{5.1}$$

which corresponds to the process X with $a \equiv 0$ (recall the notation $\sigma_s = \sigma(s, X_s)$).

(a) We start by considering the process $\overline{X}$ instead of the original process X. The theory presented in [3] and [5] implies that the approximation

$$\Delta_{i,2}^n \overline{X} \approx \sigma_{(i-2)\Delta_n} \Delta_{i,2}^n G$$

does not influence the limit behavior. The paper [3] only deals with power variations of the first order differences of the process $\overline{X}$, while [5] considers higher order differences of *Brownian semi-stationary processes*, which do not have the form (5.1). However, the second order differences of both processes are well approximated by the same expression $\sigma_{(i-2)\Delta_n} \Delta_{i,2}^n G$ (we refer to (5.5) and (5.10) in [5] for more details). For this reason we can rely on the methods of [3], [4] and [5] from now on, even though the papers [4] and [5] do not deal with the process $\overline{X}$ directly. □

(b) Using part (a) it can be deduced that

$$\frac{\Delta_n}{\tau_n^p}\sum_{i=2}^{[1/\Delta_n]}|\Delta_{i,2}^n\overline{X}|^p \xrightarrow{\mathbb{P}} m_p\int_0^1|\sigma_s|^p ds, \qquad p>0,$$

with $m_p=\mathbb{E}[|\mathcal{N}(0,1)|^p]$. This result was shown in [3] for the first order differences (see Theorem 2 therein), but the proof remains true for higher order differences (cf. Theorem 3.1 in [5]). Observing the approximation of part (a) and following the lines of the proof in [3] and [4], we see that the above convergence in probability (and also the associated central limit theorem) remains valid if we introduce weight processes σ_k^2 as long as they are Lebesgue integrable. Thus, the convergence in probability stated in (4.4) and the convergence $V_n \xrightarrow{\mathbb{P}} V$ from part (b) of Theorem 4.3 are valid for the process $\overline{X}$. □

(c) It remains to justify the central limit theorem presented in (4.5). Here we will rely on the multivariate central limit theorem for second order differences presented in [5] (see Theorem 3.3 therein). First of all, we need to show that the volatility process $\sigma_s=\sigma(s,X_s)$ is Hölder continuous of order $>1/2$ to apply the aforementioned result. But this is obviously true as

$$|\sigma_s-\sigma_t|\le|\sigma(s,X_s)-\sigma(t,X_s)|+|\sigma(t,X_s)-\sigma(t,X_t)|\le C_1|t-s|^\beta+C_2|t-s|^{1-\delta},$$

for some $C_1,C_2>0$, due to assumption (v) in section 2 and (2.3). Since $\beta>1/2$ and δ can be chosen to satisfy $1-\delta>1/2$ (because $H>1/2$), we deduce that $(\sigma_s)_{s\in[0,1]}$ is Hölder continuous of order $>1/2$.

Now we are able to apply Theorem 3.3 from [5] for the process $\overline{X}$. For simplicity we only present a joint central limit theorem for the vector $\Delta_n^{-1/2}(\langle\sigma_k^2,\sigma^2\rangle_n-\langle\sigma_k^2,\sigma^2\rangle,\langle\sigma_l^2,\sigma^2\rangle_n-\langle\sigma_l^2,\sigma^2\rangle)^\star$ with $1\le k,l\le d$. Theorem 3.3 in [5] (applied to $\overline{X}$) states that

$$\Delta_n^{-1/2}(\langle\sigma_k^2,\sigma^2\rangle_n-\langle\sigma_k^2,\sigma^2\rangle,\langle\sigma_l^2,\sigma^2\rangle_n-\langle\sigma_l^2,\sigma^2\rangle)^\star \xrightarrow{st} MN_2\Big(0,\int_0^1 V_s ds\Big)$$

with

$$V_s^{11}=\mu\sigma_{k,s}^4\sigma_s^4, \qquad V_s^{12}=\mu\sigma_{k,s}^2\sigma_{l,s}^2\sigma_s^4, \qquad V_s^{22}=\mu\sigma_{l,s}^4\sigma_s^4,$$

and

$$\mu=\lim_{n\to\infty}\Delta_n^{-1}\mathrm{Var}\left(\Delta_n\sum_{i=2}^{[1/\Delta_n]}\frac{|\Delta_{i,2}^n B^H|^2}{\mathrm{Var}(|\Delta_{i,2}^n B^H|^2)}\right),$$

where B^H denotes a fractional Brownian motion with Hurst parameter H. A straightforward calculation shows that $\mu=\gamma_H$. Hence, we obtain (4.5) for the process $\overline{X}$. □

(d) Finally, we need to prove that Theorem 4.2 holds for the original process X (rather than only for $\overline{X}$). For simplicity, let us only consider the first estimator $\langle \sigma^2, \sigma^2 \rangle_n$. Since Theorem 4.2 is valid for the process $\overline{X}$, it suffices to show that

$$\frac{\Delta_n^{1/2}}{3\tau_n^4}\Big(\sum_{i=2}^{[1/\Delta_n]} |\Delta_{i,2}^n X|^4 - \sum_{i=2}^{[1/\Delta_n]} |\Delta_{i,2}^n \overline{X}|^4\Big) \xrightarrow{\mathbb{P}} 0.$$

Let us set $A_t = \int_0^t a(s, X_s)ds$. The mean value theorem implies that

$$||\Delta_{i,2}^n X|^4 - |\Delta_{i,2}^n \overline{X}|^4| \leq C|\Delta_{i,2}^n A|(|\Delta_{i,2}^n X|^3 + |\Delta_{i,2}^n \overline{X}|^3)$$

for some $C > 0$. Now, due to assumptions (i), (iv), (v) from section 2 and (2.3), we obtain for some $\varepsilon > 0$ small enough and $K > 0$:

$$\tau_n^{-4} \leq K\Delta_n^{-4H-\varepsilon}, \quad |\Delta_{i,2}^n A| \leq K\Delta_n^{1+H-\varepsilon}, \quad |\Delta_{i,2}^n X| \leq K\Delta_n^{H-\varepsilon}, \quad |\Delta_{i,2}^n \overline{X}| \leq K\Delta_n^{H-\varepsilon}.$$

Thus,

$$\frac{\Delta_n^{1/2}}{3\tau_n^4}\Big(\sum_{i=2}^{[1/\Delta_n]} |\Delta_{i,2}^n X|^4 - \sum_{i=2}^{[1/\Delta_n]} |\Delta_{i,2}^n \overline{X}|^4\Big) \leq K\Delta_n^{1/2-5\varepsilon}.$$

Choosing $\varepsilon > 0$ small enough we deduce the desired result. □

(e) We are left to proving the convergence

$$\Delta_n^{-1/2}(\Sigma_n - \Sigma) \xrightarrow{\mathbb{P}} 0$$

from (4.6). Since the processes $(\sigma_{k,s})_{s\in[0,1]}$, $k = 1, \ldots, d$, are Hölder continuous of order $\eta > 1/2$ (see step (c)), we readily deduce that

$$\Delta_n^{-1/2}|\Sigma_n - \Sigma| \leq \Delta_n^{-1/2}\int_0^1 \Big|\sigma_{k,s}^2\sigma_{l,s}^2 - \sigma_{k,[s/\Delta_n]\Delta_n}^2\sigma_{l,[s/\Delta_n]\Delta_n}^2\Big|ds \xrightarrow{\mathbb{P}} 0$$

for all $1 \leq k, l \leq d$. □

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