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# Risk and Return: Long-Run Relationships, Fractional Cointegration, and Return Predictability

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## Abstract

The dynamic dependencies in financial market volatility are generally well described by a long-memory fractionally integrated process. At the same time, the volatility risk premium, defined as the difference between the ex-post realized volatility and the market's ex-ante expectation thereof, tends to be much less persistent and well described by a short-memory process. Using newly available intraday data for the S&P 500 and the VIX volatility index, coupled with frequency domain inference procedures that allow us to focus on specific parts of the spectra, we show that the existing empirical evidence based on daily and coarser sampled data carries over to the high-frequency setting. Guided by these empirical findings, we formulate and estimate a fractionally cointegrated VAR model for the two high-frequency volatility series and the corresponding high-frequency S&P 500 returns. Consistent with the implications from a stylized equilibrium model that directly links the realized and expected volatilities to returns, we show that the equilibrium variance risk premium estimated with the intraday data within the fractionally cointegrated system results in non-trivial return predictability over longer interdaily and monthly horizons. These results in turn suggest that much of the existing literature seeking to establish a risk-return tradeoff relationship between expected returns and expected volatilities may be misguided, and that the variance risk premium provides a much better proxy for the true economic uncertainty that is being rewarded by the market.

**JEL Classification Numbers:** C22, C32, C51, C52, G12, G13, G14

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# 1 Introduction

We develop a unified framework for jointly modeling the dynamic dependencies and interrelatedness in aggregate stock market returns, realized volatilities, and options implied volatilities. Our estimation results rely on newly available high-frequency intraday data for the S&P 500 market portfolio and the corresponding VIX volatility index, along with frequency domain inference procedures that allow us to focus on specific dependencies in the data. Our formal model setup is based on the co-fractional VAR of Johansen (2008, 2009). We show that the longer-run dependencies inherent in the high-frequency data are consistent with the implications from the stylized equilibrium model in Bollerslev, Sizova, and Tauchen (2011) that directly links the dynamics of the two volatility measures and the returns. Further corroborating the qualitative implications from that same theoretical model, we show that the variance risk premium estimated as the long-run equilibrium relationship within the fractionally cointegrated system results in non-trivial return predictability over longer interdaily and monthly return horizons.

An enormous empirical literature has been devoted to characterizing the dynamic dependencies in stock market volatility, and the linkages between volatilities and returns. The most striking empirical regularities to emerge from this burgeon literature are: (i) volatility appears to be highly persistent, with the longer-run dependencies well described by a fractionally integrated process (see, e.g., Ding, Granger, and Engle, 1993; Baillie, Bollerslev, and Mikkelsen, 1996; Andersen and Bollerslev, 1997a; Comte and Renault, 1998); (ii) volatilities implied from options prices typically exceed the corresponding subsequent realized volatilities, implying that the reward for bearing pure volatility risk is negative on average (see, e.g., Bakshi and Kapadia, 2003; Carr and Wu, 2009; Bollerslev, Gibson, and Zhou, 2011); (iii) the volatility risk premium, defined as the difference between options implied and realized volatilities, tends to be much less persistent than the two individual volatility series, pointing to the existence of a fractional cointegration type relationship (see, e.g., Christensen and Nielsen, 2006; Bandi and Perron, 2006); (iv) volatility responds asymmetrically to lagged negative and positive returns, typically referred to as a “leverage effect” (see, e.g., Black, 1976; Nelson, 1991; Bollerslev, Litvinova, and Tauchen, 2006); (v) counter to the implications from a traditional risk-return tradeoff, or “volatility feedback,” type relationship, returns are at best weakly positively related, and sometimes even negatively related, to past

volatilities (see, e.g., French, Schwert, and Stambaugh, 1987; Glosten, Jagannathan, and Runkle, 1993; Campbell and Hentschell, 1992).

The co-fractional VAR for the S&P 500 returns, realized volatilities, and VIX volatility index developed here is generally consistent with *all* of these empirical regularities. In contrast to most of the studies cited above, which are based on daily or coarser sampled data, our use of ultra high-frequency 5-minute observations on returns *and* volatilities allows for much sharper empirical inference concerning the second-order dynamic dependencies and frequency specific linkages between the different variables. This in turn helps us to identify the periodicities in volatility and risk premia that are likely more important economically.

Formally, the spectral density on  $[-\pi, \pi]$  (or on  $[-\pi, \pi]^m$  in the  $m$ -dimensional case) of a discretely sampled continuous time covariance stationary process is well known to be a folded up version of the spectral density of the continuous time process.<sup>1</sup> Aliasing effectively reallocates the spectral mass above the Nyquist frequency to the spectrum at all lower frequencies, but not uniformly so, thereby invariably creating some distortions. Therefore, if we think of 5-minutes as very close to continuous time, such distortions are largely absent. In the same vein, the estimation in the time domain of the co-fractional VAR captures nearly all dynamics with arguably minimal distorting aliasing effects. Moreover, since the coefficients of the co-fractional VAR are defined by second-order moments, which are known to be more precisely estimated the more finely sampled the data so long as the number of parameters to be estimated remains small relative to the size of the data, the efficiency of the inference is also generally enhanced by our use of high-frequency data.

The plan for the rest of the paper is as follows. We begin in the next Section 2 with a description of the high-frequency data underlying our empirical investigations. Section 3 characterizes the long-run dynamic dependencies in the two variance series, including the variance risk premium and the evidence for fractional cointegration. Section 4 details the risk-return relationships inherent in the high-frequency data based on different variance proxies across different frequency bands. These results in turn motivate our empirical implementation of the fractionally cointegrated VAR system discussed in Section 5. Section 6 concludes.

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<sup>1</sup>Technically, the spectral density of a discrete time scalar process may be thought of as identical copies over  $(-\infty, \infty)$  on intervals  $[(2k-1)\pi, (2k+1)\pi]$ ,  $k = 0, \pm 1, \pm 2, \dots$ , with the obvious analogy in the vector case.

## 2 Data

Our analysis is based on high-frequency tick-by-tick observations on the CME futures contract for the S&P 500 aggregate market portfolio and the corresponding CBOE *VIX* volatility index. The data were obtained from Tick Data Inc. and cover the period from September 22, 2003 to December 31, 2008.

Following standard practice in the literature as a way to guard against market-micro structure contaminants (see, e.g., Andersen, Bollerslev, Diebold, and Ebens, 2001), we transform the original tick-by-tick data into equally spaced 5-minute observations using the last price within each 5-minute interval. All-in-all, this leaves us with 78 observations for each of the two series for every trading trading day in the sample.<sup>2</sup>

We denoted the corresponding geometric (log) returns by

$$r_{t+1} = \log(P_{t+1}) - \log(P_t), \quad (1)$$

where the time subscript  $t$  refers to the 77 intraday return observations plus the one overnight return per trading day. Following Andersen and Bollerslev (1998), we estimate the one-month (or 22 trading days) ex-post return variation in a model-free manner by summing all of the within month squared returns,

$$RV_t = \sum_{i=1}^{78 \times 22} r_{t+i}^2. \quad (2)$$

We define the corresponding risk-neutral return variation from the CBOE *VIX* volatility index as<sup>3</sup>

$$VIX_t^2 = \frac{30}{365} (VIX_t^{CBOE})^2. \quad (3)$$

To help stabilize the two variance measures and render them more amenable to time series modeling, we transform both into logarithmic units,

$$rv_t = \log(RV_t), \quad (4)$$

$$vix_t^2 = \log(VIX_t^2). \quad (5)$$

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<sup>2</sup>Out of a total of more than 100,000 5-minute intervals only 447 are missing. We replace these by backfilling. Further details concerning the data are available in Bollerslev, Sizova, and Tauchen (2011), where the same data have been analyzed from a different perspective over a slightly shorter time-span.

<sup>3</sup>The scaling by 30/365 transforms the squared annualized observations on the CBOE index into monthly units.

The overnight returns are purposely included in the definition of the raw return series and the calculation of the realized variances in equations (1) and (2), respectively. The inclusion of the overnight return obviously inflates the unconditional variance of the returns relative to that of an equally spaced intraday 5-minute return series. To alleviate this problem we re-structure all three data series into intraday units by deleting all of the observations corresponding to multiples of  $t = 78$ . With the loss of one month at the end of the sample due to the calculation of  $rv_t$ , our final data set thus covers the period from September 22, 2003 to November 28, 2008, for a total of  $T = 1307 \times 77 = 100,639$  intraday observations.

Standard summary statistics for each of the three series are reported in Table 1. The high-frequency returns are, of course, approximately serially uncorrelated, with a mean indistinguishable from zero. Consistent with the extant literature, the unconditional mean of the realized variance is lower than the mean of the risk-neutral variance, indicative of an on average positive risk premium for bearing volatility risk.<sup>4</sup> At the same time, the risk-neutral variance appears slightly less volatile than the realized variance. Of course, both of the variance series exhibit substantial persistence with extremely slow decay in their autocorrelations. The next section further details these dynamic dependencies in the two variance series.

### 3 Variance Dynamics

#### 3.1 Long-Run Volatility Dependencies

The notion of fractional integration often provides a convenient statistical framework for capturing long-run dependencies in economic time series (see, e.g., the discussion in Baillie, 1996). A stationary time series  $y_t$  is said to be fractionally integrated of order  $d \in (0, 0.5)$ , written  $I(d)$ , if

$$\Delta^d y_t = e_t, \tag{6}$$

where  $e_t$  is an  $I(0)$  process, and  $\Delta^d = (1 - L)^d$  denotes the fractional difference operator,

$$\Delta^d \equiv (1 - L)^d = \sum_{i=0}^{\infty} (-1)^i \binom{d}{i} L^i. \tag{7}$$

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<sup>4</sup>If we split the sample into a *pre-crisis* and a *post-crisis* period, with the start of the crisis defined as February 27, 2007 following the official timeline of the Federal Reserve Bank of St. Louis, the sample means of  $rv_t$  equal 2.210 and 3.593, respectively, compared to 2.718 and 3.824 for the  $vix_t^2$  series. Hence, the unconditional variance risk premium is even larger based on data over the pre-crisis period only.

The spectral density of the process  $y_t$  has a pole of the order  $\omega^{-2d}$  for frequency  $\omega$  near the origin, while the filtered series  $\Delta^d y_t$  has finite spectral density at the origin.

There is ample empirical evidence that financial market volatilities are well described by covariance stationary  $I(d)$  processes with fractional integration parameter fairly close to, but less than, 0.5. in a narrow range between 0.35 and 0.45. For instance, Andersen, Bollerslev, Diebold, and Ebens (2001) report average fractional integration parameters for a set of realized equity return volatilities of approximately 0.35, while the results for foreign exchange rates in Andersen, Bollerslev, Diebold, and Labys (2003) suggest that  $d$  is close to 0.4. Similarly, Christensen and Nielsen (2006) find that daily realized and options implied equity index volatilities are fractionally integrated with  $d$  around 0.4.<sup>5</sup>

Most of the existing literature, however, including the above-cited studies, have relied on daily data or coarser sample data for determining  $d$ . By contrast, both of our volatility series are recorded at a 5-minute sampling frequency. Hence, as a precursor to our more detailed joint empirical analysis, we begin by double checking that the folding of the spectral densities associated with the lower sampling frequency have not distorted the previously reported estimates for  $d$ .

To this end, Figure 1 shows the raw log-log periodogram of the 5-minute  $vi x_t^2$  and  $rv_t$  series at the harmonic frequencies  $\omega_j = \frac{2\pi}{T}j$ ,  $j = 1, 2, \dots, T/2$ . The periodograms of the two variance variables are quite similar, with most of the power concentrated at the low frequencies. At the same time, there appears to be three distinct regions within the frequency domain: a relatively short leftmost low-frequency region up until  $\underline{\omega} \approx 0.0011$ , where the log-log periodograms are linear and nearly flat; an intermediate region between  $\underline{\omega}$  and  $\bar{\omega} \approx 0.0806$ , with more steeply sloped periodograms; and a third rightmost region to the right of  $\bar{\omega}$  corresponding to the within-day variation in the volatilities, where the periodograms are quite erratic.<sup>6</sup>

The approximate linearity of the log-log periodograms for the low frequencies close to zero directly points to long-memory dependencies, or fractional integration. We estimate the fractional integration parameter  $d$  using both the log-periodogram regression of Geweke and Porter-Hudak (1983) and the local-Whittle likelihood procedure of Künsch (1987). For both estimators, we

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<sup>5</sup>One notable exception is Bandi and Perron (2006), who argue that  $d$  is in excess of 0.5, and thus outside the stationary region.

<sup>6</sup>This overall general shape mirrors that of the 5-minute Deutschemark-Dollar absolute returns previously depicted in Andersen and Bollerslev (1997a).

set the required truncation parameter to  $j_{max} = 15$ , corresponding to  $\underline{\omega}$  and periodicities of  $3\frac{1}{2}$  months and longer being used in the estimation.<sup>7</sup> The resulting estimates for  $d$ , with asymptotic standard errors in parentheses, for the  $vi\hat{x}_t^2$  series are 0.376 (0.166) using the log-periodogram and 0.375 (0.129) using the local Whittle approach. For the  $rv_t$  series, the same two estimates are 0.330 (0.166) and 0.338 (0.129), respectively. As such, our findings for the 5-minute volatility series are entirely consistent with the typical estimates for  $d$  reported in the existing literature.

## 3.2 Fractional Cointegration in Variances

The preceding results strongly suggest that each of the two high-frequency variance series are individually long-memory processes. At the same time, one might naturally expect that the two variance series are tied together in the long-run in the form of fractionally cointegrated type relationship.<sup>8</sup>

The simplest case of fractional cointegration occur when the two individual series share the same order of fractional integration, but their difference is integrated of a lower order. In the present context this case naturally corresponds to the ex-post logarithmic variance risk premium,

$$vp_t = vi\hat{x}_t^2 - rv_t. \tag{8}$$

The fact that  $vp_t$  exhibits less persistence than the two individual variance series, has previously been documented with daily and lower frequency data by Christensen and Nielsen (2006), Bandi and Perron (2006), and Chernov (2007), among others.

To establish a similar long-run relationship between our two high-frequency variance series, we begin by testing for equality of the fractional difference parameters using the Wald test of Robinson (1995). The asymptotically distributed  $\chi_1^2$  test statistic equals 0.123 when the  $d$ 's are estimated by the local Whittle procedure, and 0.054 for the  $d$ 's estimated by the log-periodogram estimator. Either way, there seems to be little evidence against the hypothesis that the two variance series are indeed fractionally integrated of the same order.

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<sup>7</sup>Based on visual inspection of Figure 1 this seemingly covers the frequency range where the long-memory behavior hold true; see, also the related discussion in Sowell (1992).

<sup>8</sup>Following Granger (1996), if a linear combination of two fractionally integrated variables is integrated of a lower order than those of the individual series, then the variables are said to be fractionally cointegrated; see also Robinson and Marinucci (2003).



To further explore the possibility of fractional cointegration, we next consider the linear regression,<sup>9</sup>

$$rv_t = \beta_0 + \beta_1 vix_t^2 + \nu_t. \quad (9)$$

The residuals from this regression obviously reduces to the variance risk premium defined in equation (8) above for  $\beta_0 = 0$  and  $\beta_1 = 1$ . However, rather than restricting the relationship between  $rv_t$  and  $vix_t$  to be the same across all frequencies, we estimate the regression using low-pass frequency domain least squares (FDLS) (see, e.g., Robinson, 1994). As before, we truncate the regression at  $j_{max} = 15$ , corresponding to periodicities of  $3\frac{1}{2}$  months and longer. We rely on the local-Whittle approach for estimating  $d$ , together with the techniques developed by Christensen and Nielsen (2006) and Shimotsu and Phillips (2006) for estimating the asymptotic standard errors.

The resulting FDLS estimate for the degree of fractional integration of the residuals equals  $\hat{d}^{(\nu)} = 0.101$  (0.611). As such, the results clearly support the notion that the high-frequency  $vix_t^2$  and  $rv_t$  series are indeed fractionally cointegrated. Moreover, the FDLS estimate of  $\beta_1$  equals 1.268 (0.150), slightly larger than unity but insignificantly so.

### 3.3 Dynamic Dependencies Across Frequencies

In addition to assessing the integration order of  $rv_t$  and  $vix_t^2$ , and the long-run relationship between the two variance measures, the joint distribution of the variance measures may be further illuminated by decomposing each of the variables into their long-run, intermediate, and short-run components. In order to do so, we rely on time-domain band-pass filters for extracting the specific periodicities from the observed series.<sup>10</sup>

The aforementioned Figure 1 reveals a change in the slopes of the periodograms for the realized and risk-neutral variances for frequencies around  $\underline{\omega} \approx 0.0011$ , or periods around  $3\frac{1}{2}$  months.

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<sup>9</sup>This regression is analogous to the Mincer-Zarnowitz style regression commonly used for evaluating the quality of macroeconomic time series forecasts. That is, it evaluates whether  $vix_t^2$  is conditionally unbiased for the ex-post realized variance  $rv_t$ .

<sup>10</sup>Compared to frequency-domain filters (see, e.g., Hassler, Lundvik, Persson, and Soderlind, 1994) the time-domain filters applied here have the advantage that the filtered series are time invariant and do not depend on the length of the sample. Similar filters to the ones used here have previously been applied by Andersen and Bollerslev (1997a) for decomposing high-frequency foreign exchange rates into interday and intradaily components. For additional discussion of band-pass filtering see also Baxter and King (1999), where the same techniques have been used for extracting business cycle components of macroeconomic times series.

Motivated by this observation, we therefore extract a low-frequency, or long-run, component corresponding to dependencies in excess of  $3\frac{1}{2}$  months. That is, for frequency  $\underline{\omega}$  we define a lowpass-filtered series by,

$$y_t^{(low)} = \sum_{i=-k}^k a_i L^i y_t, \quad (10)$$

where

$$a_i = \begin{cases} \frac{\sin(i\underline{\omega})}{i\pi} - \left( \frac{\underline{\omega}}{\pi} + 2 \sum_{j=1}^k \frac{\sin(j\underline{\omega})}{j\pi} - 1 \right) / (2k+1), & \forall i = \pm 1, \pm 2, \dots, \pm k \\ 1 - \sum_{h=-k}^{-1} a_h - \sum_{h=1}^k a_h & \forall i = 0. \end{cases} \quad (11)$$

Intuitively, the  $y_t^{(low)}$  series essentially equates the part of the spectrum in Figure 1 to the right of  $\underline{\omega}$  to zero.

Looking at the two sample periodograms in Figure 1, there is a distinct change in the general patterns for frequencies in excess of the daily frequency  $\bar{\omega} \approx 0.0806$ .<sup>11</sup> To isolate the interdaily component with periodicities of less than  $3\frac{1}{2}$  months from the intraday dynamics, we therefore compute an intermediate-frequency series. Specifically, for frequencies in the band  $\underline{\omega} < \omega < \bar{\omega}$ , we define the bandpass-filtered series by,

$$y_t^{(band)} = \sum_{i=-k}^k (b_i - a_i) L^i y_t, \quad (12)$$

where

$$b_i = \begin{cases} \frac{\sin(i\bar{\omega})}{i\pi} - \left( \frac{\bar{\omega}}{\pi} + 2 \sum_{j=1}^k \frac{\sin(j\bar{\omega})}{j\pi} - 1 \right) / (2k+1) & i = \pm 1, \pm 2, \dots, \pm k \\ 1 - \sum_{h=-k}^{-1} b_h - \sum_{h=1}^k b_h & i = 0. \end{cases} \quad (13)$$

Finally, the highpass-filtered series corresponding to periodicities of a day and shorter is simply computed by,

$$y_t^{(high)} = y_t - \sum_{i=-k}^k b_i L^i y_t. \quad (14)$$

Note that by definition  $y_t \equiv y_t^{(low)} + y_t^{(pass)} + y_t^{(high)}$ .

The higher the value of the truncation parameter  $k$ , the closer the gains of the approximate filters in equations (10), (12), and (14) are to the ideal gains of zero for the frequencies that are

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<sup>11</sup>It is well known that the volatilities of high-frequency returns, or point-in-time volatilities, exhibit strong U-shaped patterns across the trading day; see, e.g., Harris (1986) and Andersen and Bollerslev (1997b). By contrast, the two volatility series analyzed here both measure the variation over a month, and as such even though they behave differently intraday they do not show the same strong almost deterministic intraday patterns.

filtered out and unity for the desired frequency bands. In the implementation reported on below, we set  $k = 77 \times 22$  for all of the three filters, resulting in a loss of one month of observations at the beginning and the end of the sample.

We first focus on the dynamics of the long-run components. Assuming that the effects of market frictions and short-run fluctuations disappear in the long-run, the dynamics of the low-pass filtered series should therefore more clearly reveal the underlying equilibrium relationships between the variables. Figure 2 plots the two filtered variance measures  $rv_t^{(low)}$  and  $vix_t^{2(low)}$  defined by applying the filter in equation (10). Consistent with the results in Section 3.2, the figure reveals strong comovements between the two low-frequency variance components. Indeed, the sample correlation between the two low-pass filtered series equals  $Corr(rv_t^{(low)}, vix_t^{2(low)}) = 0.920$ .

We further detail the relationship between  $vix_t^2$  and  $rv_t$  across all frequencies through measures of their interrelatedness, or coherence.<sup>12</sup> The coherence is analogous to the square of the correlation between two series, taking values from zero (no relation) to one (perfect correlation). In contrast to the standard correlation coefficient, however, the coherence is a function of frequency. The coherence being close to zero for a certain frequency range thus indicates the absence of any relation between the two series across those periodicities. As follows from the first panel in Figure 3 labeled “Total,” the coherence between  $vix_t^2$  and  $rv_t$  is close to one for the lowest frequencies, but decreases to just around 0.2 for the higher frequencies. The almost perfect dynamic relationship between the  $vix_t^2$  and  $rv_t$  series therefore only holds in the long-run.

To get a more nuanced picture of these dependencies, the remaining panels in Figure 3 show the coherence for the high-pass, band-pass, and low-pass filtered variance series. The coherence for the long-run components range between 0.8 and 0.95 across all frequencies. The coherence for the band-pass filter variances is substantially lower and around 0.06, while the coherence for the high-frequency components is practically zero.

In order to interpret these patterns, it is instructive to think about  $vix_t^2$  as the sum of the ex-post realized volatility  $rv_t$ , a premium for bearing volatility risk, say  $\varrho_t$ , along with the corresponding forecast error, say  $\xi_t$ ,

$$vix_t^2 = rv_t + \varrho_t + \xi_t. \tag{15}$$

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<sup>12</sup>Our estimates of the coherence measures are based on the classic Tukey-Hanning method; see, e.g., the discussion in Granger and Hatanaka (1964).

For low frequencies the coherence between  $vix_t^2$  and  $rv_t$  is close to one, implying that the influences of  $\xi_t$  and  $\varrho_t$  are both fairly minor. This is also consistent with the findings that  $rv_t$  is integrated of a higher order than  $vp_t = \xi_t + \varrho_t$ , and that most of the low frequency dynamics in  $vix_t^2$  are due to changes in the volatility. This close coherence is broken at the intermediate and ultra high intraday frequencies, where most of the changes in  $vix_t^2$  stem from changes in the risk premium and/or expectational errors vis-a-vis the future realized volatility. These same arguments also facilitate the interpretation of the empirical risk-return relationships, which we discuss next.

## 4 Risk-Return Relations

A large empirical literature has been devoted to the estimation of risk-return tradeoff relationships in aggregate equity market returns (see, e.g., the discussion in Rossi and Timmermann, 2010, and the many references therein). Much of this research is motivated by simple dynamic CAPM-type reasoning along the lines of,

$$E_t(r_{t+1}) = \gamma \sigma_t^2, \tag{16}$$

where  $\sigma_t^2$  represents the local return variance, and  $\gamma$  is interpreted as a risk aversion parameter.<sup>13</sup> The actual estimation of this equation, of course, necessitates a proxy for  $\sigma_t^2$ . By far the most commonly employed empirical approach relies on the (G)ARCH-M model (Engle, Lilien, and Robins, 1987) for jointly estimating the conditional mean of the returns together with the conditional variance of the returns in place of  $\sigma_t^2 = Var_t(r_{t+1})$ . Instead, by relying on the variance measures analyzed above as directly observable proxies for risk, equation (16) may be estimated directly. The following section explores this idea in our high-frequency data setting, keeping in mind the preceding characterization of the underlying volatility dynamics.<sup>14</sup>

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<sup>13</sup>The Merton (1980) model sometimes used as additional justification for this relationship formally involves the excess return on the market. The requisite Jensen's correction term for the logarithmic returns analyzed here simply adds  $\frac{1}{2}$  to the value of  $\gamma$ . Also, the risk-free rate is essentially zero at the 5-minute level.

<sup>14</sup>The use of high-frequency based realized volatility measures in the estimation of a daily risk-return tradeoff relationship has previously been explored by Bali and Peng (2006).

## 4.1 Return-Variance Regressions

The basic risk-return relation in equation (16) may be conveniently expressed in regression form as,

$$r_{t+1} = \alpha + \beta v_t + u_{t+1}, \quad (17)$$

where  $v_t$  denotes the specific variance proxy used in place of  $\sigma_t^2$ . The first two rows of Table 2 labeled “Raw” show the results of this regression fitted with our high-frequency data using either  $rv_t$  or  $vix_t^2$  in place of  $v_t$ . The results are very weak. This is perhaps not surprising, in view of the difficulty to detect a significant relationship in daily and coarser frequency data reported in the extant literature.

But these raw regressions are also unbalanced and likely not very informative. They involve an essentially white noise  $I(0)$  variable on the left-hand side (the return) and a strongly persistent  $I(d)$  variable on the right-hand side (the realized or risk-neutral variance).<sup>15</sup> Several means to cope with such unbalanced regressions have been suggested in the literature. Maynard, Smallwood, and Wohar (2012), in particular, focus explicitly on the case where the predictor variable is fractionally integrated, and propose an approach to re-balance the regression. A similar approach has also been applied by Christensen and Nielsen (2007), who consider a VAR framework in which the level of returns is predicted by the fractionally filtered variance series.

A simplified version of the return equation in Christensen and Nielsen (2007) is given by,

$$r_{t+1} = \alpha + \beta \Delta^d v_t + u_{t+1}, \quad (18)$$

where  $\Delta^d \equiv (1 - L)^d$  denotes the fractional difference filter.<sup>16</sup> Based on the results reported in the previous section we fix  $d \equiv 0.37$ . The resulting “Long-Memory Adjusted” regressions using  $\Delta^{0.37}rv_t$  or  $\Delta^{0.37}vix_t^2$  as risk proxies are reported in the next two rows of Table 2.<sup>17</sup> Compared to the “Raw” regressions considered above, both of the  $R^2$ s are somewhat higher. The estimated

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<sup>15</sup>This same statistical problem has also previously been discussed in the context of regression-based tests for unbiasedness in the forward foreign exchange market by Baillie and Bollerslev (2000).

<sup>16</sup>More precisely, Christensen and Nielsen (2007) consider a modified version of equation (18), where both  $r_{t+1}$  and  $v_t$  are detrended by their unconditional means and  $\alpha \equiv 0$ . In addition, their VAR structure also allows past returns to affect current returns.

<sup>17</sup>In implementing the fractional filters, we truncate the series expansion for  $(1 - L)^d v_t$  at the beginning of the sample, discarding the first week of filtered observations. All of the regressions reported in the table are based on the identical sample period from September 30, 2003 to November 28, 2008.

coefficient for the realized variance is now also statistically significant, but signal a *negative* risk-return trade-off. By contrast, the fractionally differenced risk-neutral variance results in a positive  $\hat{\beta}$  estimate, albeit statistically insignificant at the usual five-percent level. These mixed results are perhaps again not surprising in view of the existing literature.

As highlighted in Section 3.2 above, another way to render the two variance series stationary, and in turn balance the regression, is to consider their difference, i.e., the variance risk premium  $vp_t$ , as the proxy for risk. The last row in Table 2 labeled “Variance Risk Premium” reports the results from this regression,

$$r_{t+1} = \alpha + \beta vp_t + u_{t+1}. \tag{19}$$

The estimate of  $\beta$  is now positive and statistically significant. The regression  $R^2$  is also much larger than for any of the other regressions reported in the table. Thus, the difference between the two variance variables appears far more informative for the returns than each of the two variance variables in isolation, whether in their raw or fractionally-filtered form. To help further gauge these simple regression-based results, it is informative to decompose the variables into their periodic components.

## 4.2 Risk-Return Relationships Across Frequencies

Following the analysis in Section 3.3, the return  $r_t$  and the variance variables  $rv_t$ ,  $vi x_t^2$ , and  $vp_t$ , are naturally decomposed via band-pass filtering into their short-, intermediate-, and long-run components. Table 3 summarizes the correlations between the resulting components for the different variance variables with the same components of the return. Evidently, there is not much of a relationship at the high and intermediate frequencies. Again, this is not surprising as one might expect the association between risk and return to be more of a long-run than a short-run phenomenon.

Indeed, the low-frequency correlations reported in the rightmost column in the table are all much higher in magnitude. Somewhat paradoxically, however, the correlations are negative for both of the individual variance variables, but positive and larger in magnitude for the variance risk premium.<sup>18</sup> In order to visualize this relationship, Figure 4 shows the low-pass filtered  $vp_t^{(low)}$

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<sup>18</sup>These results are also consistent with the monthly return regressions in Bollerslev, Tauchen, and Zhou (2009),

and  $r_t^{(low)}$  series. The co-movement between the two series is apparent.

In a sum, neither of the two variance series,  $rv_t$  and  $vix_t^2$ , exhibit dynamic co-movements with the returns over any of the three frequency bands that would support the notion of a positive risk-return trade-off relationship. If instead the difference, i.e., the variance risk premium  $vp_t$ , is employed as a proxy for risk, the results are more in line with intuitive expectations regarding a positive low-frequency risk-return tradeoff relationship.

The continuous time model developed by Bollerslev, Sizova, and Tauchen (2011) is useful for interpreting of these results. In the first place, the volatility  $\sigma_t^2$  in the basic dynamic CAPM and equation (16) does not have the actual role of a volatility risk premium per se, but can rather be viewed as a time-varying price on endowment (consumption) risk. The only variable in that simplistic framework that carries a risk premium is consumption, so, despite the intuitive appeal of (16), there really is no reason to expect raw variance variables to relate to returns in any manner, except, perhaps, through some indirect mechanism that could be of either sign.<sup>19</sup> On the other hand, in a generalized long run-risk model with uncertainty about the variability of economic prospects (vol-of-vol), the difference in the variance variables, i.e., the variance risk premium corresponding to  $\varrho_t$  in equation (15), is the key factor connecting returns and variability. The difference is most highly associated with the vol-of-vol and overall economic uncertainty, and that factor commands a substantial risk premium by investors with recursive utility and a strong preference for early resolution of uncertainty. Consequently, the regression in (18) is entirely consistent with more sophisticated versions of the dynamic CAPM.

To fully explore these relations between the return and the two variance measures, and the implications thereof for return predictability, we next turn to the estimation of a joint model for  $r_t$ ,  $rv_t$ , and  $vix_t^2$ , explicitly designed to accommodate the intricate dynamic and cross-variable dependencies highlighted so far.

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which suggest that the return predictability of the variance risk premium is maximized at a four months horizon. Four months lie within the frequency-band classified as low-frequency here.

<sup>19</sup>The sign of the relationship between returns and variances have also previously been called into question on theoretical grounds by Backus and Gregory (1993), among others.

## 5 Co-Fractional System

The co-fractional VAR model of Johansen (2008, 2009) affords a convenient statistical framework to distinguish long-run and short-run effects in a system setting involving fractionally integrated  $I(d)$  variables. Specifically, let  $z_t \equiv (rv_t, vix_t^2, r_t)'$  denote the 5-minute  $3 \times 1$  vector process. Guided by the empirical findings in the previous sections, the simplified versions of the co-fractional VAR model for  $z_t$  estimated here, say  $CFVAR_d(p)$ , takes the form,

$$\Delta^d z_t = \gamma(\delta' (1 - \Delta^d) z_t + \rho') + \sum_{i=1}^p \Gamma_i (1 - \Delta^d)^i \Delta^d z_t + \epsilon_t, \quad (20)$$

where  $\epsilon_t$  denotes a vector white noise process with unconditional covariance matrix  $\Omega$ .<sup>20</sup>

This dynamic CFVAR representation directly parallels the classical error-correction type representation with cointegrated  $I(1)$  variables. The process  $z_t$  contains the fractionally integrated  $I(d)$  variables, analogous to the  $I(1)$  level variables in standard cointegration. The fractional difference operator  $\Delta^d = (1 - L)^d$  thus reduces the left-hand-side of equation (20) from an  $I(d)$  to an  $I(0)$  process, just like the first difference operator for standard cointegrated systems reduces the  $I(1)$  variables to  $I(0)$ .<sup>21</sup> The right-hand side of the equation therefore must also be  $I(0)$ . The first term  $(1 - \Delta^d) z_t$  is what remains after applying the fractional difference operator, and thereby must be  $I(d)$ . The matrix  $\gamma\delta'$  therefore has to be of reduced rank for this to be an  $I(0)$  process. In this situation  $\delta' (1 - \Delta^d) z_t$  has the interpretation of a (fractional) error-correction matrix, with  $\gamma$  the conformable matrix of impact coefficients. The second term on the right-hand side, involving the matrix fractional distributed lag and powers of  $(1 - \Delta^d)$  applied to  $\Delta^d z_t$ , directly mirrors the matrix distributed lag in standard error-correction models, where powers of  $L$  are applied to first differences of the underlying variables. The corresponding  $\Gamma_i$  matrices are essentially nuisance parameters, with  $p$  taken sufficiently large to render the disturbance term  $\epsilon_t$  serially uncorrelated.

The empirical evidence presented in Section 3 suggests that  $rv_t$  and  $vix_t^2$  are both  $I(d)$ , but that they fractionally cointegrate to an  $I(0)$  process, while  $r_t$  is  $I(0)$ . Consequently, the column

<sup>20</sup>Formal regularity conditions for the  $\epsilon_t$  white noise process are spelled out in Johansen (2008, 2009).

<sup>21</sup>The application of the fractional difference operator  $\Delta^d$  to the returns might seemingly result in over-differencing. However, as shown in Appendix A.1 for the specific representation of the CFVAR model adopted here, the resulting return series is still  $I(0)$ .



rank of  $\gamma\delta'$  should be equal to two, with the the natural normalization

$$\delta' = \begin{pmatrix} -\tilde{\delta} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (21)$$

and the corresponding matrix of impact coefficients

$$\gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \\ \gamma_{31} & \gamma_{32} \end{pmatrix}, \quad (22)$$

left unrestricted. The parameter  $\tilde{\delta}$  naturally governs the long-run tie between  $rv_t$  and  $vi x_t^2$ , which in turn defines the variance error correction term as a linear combination of multiple lags of  $(1 - \Delta^d)(vi x_t^2 - \tilde{\delta} rv_t)$ .<sup>22</sup>

The parameters  $\gamma_{11}$  and  $\gamma_{21}$  capture any internal long-run relationships between the variance error-correction term and the variance variables themselves, while  $\gamma_{31}$  captures the relationship between the variance error-correction term and the returns, or the long-run dynamic “volatility feedback” effect implied by the model. The corresponding short-run counterparts are determined by the  $\Gamma_i^{(31)}$  and  $\Gamma_i^{(32)}$  parameters. Similarly, the long-run dynamic “leverage effect” depends on the values of  $\gamma_{12}$  and  $\gamma_{22}$ , with the corresponding short-run effects determined by  $\Gamma_i^{(12)}$  and  $\Gamma_i^{(22)}$ . The overall degree of return predictability implied by the model is jointly determined by the  $\gamma_{3j}$  and  $\Gamma_i^{(3j)}$  parameters.

In sum, by estimating the CFVAR $_d(p)$  model we draw on a richer information set than we did in the previous sections. Importantly, by separately parameterizing the long-run and the short-run dynamics of returns and the variance series, the model is able to accommodate empirically realistic  $I(d)$  long-memory in the realized and the risk-neutral variances and their fractional cointegration, while maintaining that the returns are  $I(0)$ .

## 5.1 Estimation Results

To facilitate the estimation of the CFVAR $_d(p)$  model we begin by fixing the value of the fractional integration parameter  $d$ . The estimation then proceeds in two stages. In the first step, we use the preset fractional differencing parameter  $d$  to construct the vector time series  $\Delta^d z_t$  of filtered realized

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<sup>22</sup>Note, the operator  $(1 - \Delta^d)$  only involves lags  $L^j$ ,  $j \geq 1$ , with no  $L^0$  term.

variances, risk-neutral variances, and returns.<sup>23</sup> In the second stage, we obtain parameter estimates for the co-fractional model by iterated seemingly unrelated regression (SUR). We separately select the number of lags for each of the three equations, allowing for different number of lags of  $rv_t$ ,  $vi x_t^2$ , and  $r_t$  in each of the equations as determined by Schwarz' Bayesian Information Criterion (BIC). Finally, we select the value of  $d$  that maximizes the Gaussian likelihood function subject to the BIC chosen lag specifications.

Turning to the actual CFVAR estimation results reported in Table 4, we find that the likelihood function is maximized at  $d = 0.37$ . This value of  $d$  is directly in line with the semiparametric estimates discussed in Section 3.3. According to the BIC criteria the short-run dynamics of the realized variance depends on four lags of the differenced  $rv_t$  series and one lag of the differenced  $vi x_t^2$  series, while the short-run dynamics of the implied variance depends on two lags of the differenced  $rv_t$  series, three lags of the differenced  $vi x_t^2$  series, and five lags of the differenced  $r_t$  series. Meanwhile, the short-run dynamics of the returns depends on two lags of the differenced  $vi x_t^2$  series and seven lags of the differenced  $r_t$ . We will refer to this particular specification as the CFVAR<sub>0.37(7)</sub> model below. The corresponding standard errors for the parameter estimates reported in the right column of the table are based on 3,000 replications of a moving block bootstrap.<sup>24</sup>

In order to gauge the model fit, Figure 6 compares the spectra of the estimated CFVAR<sub>0.37(7)</sub> model with the sample periodograms for  $rv_t$ ,  $vi x_t^2$ , and  $r_t$ , as well as the variance risk premium  $vp_t = vi x_t^2 - rv_t$ . The match between the model-implied and empirical spectra are exemplary. Particularly noteworthy, the CFVAR model correctly matches the slopes of the spectra near the origin that define the long-run behavior of the two variance series and the less persistent variance risk premium.

The cointegrating vector associated with the realized and risk-neutral variances is estimated to  $(-1.010, 1, 0)$ . This fully parametric CFVAR-based estimate for  $\tilde{\delta}$  is even closer to unity (numerically) than the FDLS estimate discussed in Section 3.2. Of course, a simple  $t$ -test does not

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<sup>23</sup>We again truncate the fractional filter at the beginning of the sample, discarding the first weeks worth of observation, so that the estimation is based on the identical September 30, 2003 to November 28, 2008 sample period underlying the previously reported results in Tables 2 and 3.

<sup>24</sup>Specifically, using the semiparametric estimates for  $d$  we first fractionally filter the two variance series. We then jointly resample blocks of the trivariate  $I(0)$  vector  $(\Delta^d rv_t, \Delta^d vi x_t^2, r_t)'$ , with the length of the blocks set to  $3\frac{1}{2}$  months. We then apply the inverse fractional filter to the resampled variance series and finally reestimate the CFVAR model using the same SUR approach discussed in the text. With the exception of the semiparametric first-step estimation of  $d$ , this bootstrap approach closely mimics that of Davidson (2002).

reject the hypothesis that  $\tilde{\delta} = -1$ , while the absence of fractional cointegration, or  $\tilde{\delta} = 0$ , is strongly rejected by the data. As such, our results are fully supportive of the notion of long-run unbiasedness in variances.<sup>25</sup>

To more clearly illuminate the dynamic dependencies implied by the the CFVAR<sub>0.37</sub>(7) model it is instructive to consider Impulse-Response Functions (IRF) associated with the shocks to the “permanent” and two “transitory” components defined within the model. By definition, the effect of the permanent shock decays hyperbolically and therefore persist over relatively long time periods. The two transitory shocks, on the other hand, both decay at a fast exponential rate. The identification of these shocks does not depend a priori on any known relations between the volatility and return series.<sup>26</sup> However, given the estimates of  $\gamma$  and  $\delta$  in Table 4, the three shocks have clear meanings. The first transitory shock, in particular, drives the wedge between  $vix_t^2$  and  $rv_t$ , and has the interpretation of a shock to the ex-post variance premium. The second transitory shock only affects the returns. The permanent shock is effectively a shock to  $rv_t$  that is unrelated to changes in the variance risk premium. This shock naturally also affects  $vix_t^2$  through expectations of future variances.

The first panel in Figure 7 shows the effect of the permanent shock on  $vix_t^2$  and  $rv_t$ . Initially, both variances are driven up by almost the same amount, so that the permanent shock does not initially affect the size of the variance risk premium. There is some discrepancy in the effect of the shock over the intermediate daily to  $3\frac{1}{2}$  months horizon, but the overlapping nature of the  $rv_t$  series complicates the interpretation of the difference. After  $3\frac{1}{2}$  months the IRFs seem to merge and decay at a common hyperbolic rate. Even at the one-year horizon, however, the “permanent” shock still exercises a non-negligible effect on the  $rv_t$  and  $vix_t^2$  series.

The second panel in Figure 7 shows the IRFs for the first orthogonalized transitory shock associated with the ex-post variance risk premium. This shock fully dissipates after approximately  $3\frac{1}{2}$  months. It essentially reflects the difference between a shock to the ex-ante variance risk premium and an unpredictable shock to the realized variance, or  $\varrho_t$  and  $-\xi_t$ , respectively, in the

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<sup>25</sup>Comparable results based on coarser sampled monthly realized and options implied variances and semiparametric estimates for  $\tilde{\delta} \approx 1$  have previously been reported by Bandi and Perron (2006), among others.

<sup>26</sup>Following Gonzalo and Granger (1995), the permanent and transitory shocks may be constructed mechanically from the  $\epsilon_t$ 's by multiplication with the matrix  $G = [\gamma_{\perp} \ \delta]'$ . The shocks may further be orthogonalized using the method in Gonzalo and Ng (2001).

notation of equation (15) above. The first effect naturally increases  $vi x_t^2$ , while the second effect decreases  $rv_t$ . The first effect is by far the largest over within day horizons, where the overlap in  $rv_t$  invariably temper the effect of any shock.

The third and final panel in the figure shows the effects of the second transitory return shock, from which the simultaneous correlations with the variance and variance risk premium shocks have been removed. This shock has no initial effect on either variance series. However, there is a negative within day impact on the  $vi x^2$  akin to a multi-period dynamic “leverage effect,” as previously discussed by Bollerslev, Litvinova, and Tauchen (2006).

Turning to Figure 8, we show the IRFs for the returns and the two variance shocks. The initial effect both shocks is to decrease the returns. This is entirely consistent with the widely documented “leverage effect” and negative contemporaneous correlations between returns and volatilities. Both of the shocks in turn result in an increase in future returns, as would be expected by a “volatility feedback” type effect, with the variance risk premium shock exercising the slightly larger influence.

Taken as a whole, the IRFs discussed above suggest the potential for non-trivial return predictability through the joint CFVAR modeling of the returns and the two variance measures. The next section explores this idea.

## 5.2 Return Predictability

Until now we have relied on the “forward-looking” monthly  $rv_t$ , or the realized variance over the subsequent month relative to time  $t$ . This ensures that  $rv_t$  is properly aligned with the future expected variance underlying the definition of  $vi x_t^2$ . This also ties in more directly with the theoretical model in Bollerslev, Sizova, and Tauchen (2011) and the forward looking (expected) variance risk premium defined therein.

The “structural”  $CFVAR_{0.37}(7)$  model for  $z_t = (rv_t, vi x_t^2, r_t)'$  estimated above does not easily lend itself to out-of-sample forecasting, however, as  $rv_t$  is not known at time  $t$ . To circumvent this problem, we replace  $rv_t$  with  $\tilde{rv}_t \equiv rv_{t-77 \times 22}$  in the estimation of a CFVAR model for the  $\tilde{z}_t \equiv (\tilde{rv}_t, vi x_t^2, r_t)'$  vector process. The  $CFVAR_{0.37}(7)$  model for  $z_t$  formally implies a  $CFVAR_{0.37}(p)$  model for  $\tilde{z}_t$ . We therefore fix  $d = 0.37$ . We then rely on the same BIC criterion used above for determining the number of lags in each of the three equations in this new  $CFVAR_{0.37}(7)$  model

for  $\tilde{z}_t$ .<sup>27</sup>

In order to convert the dynamic relations implied by the CFVAR model for  $\tilde{z}_t$  into forecasts for the returns and corresponding predictive  $R^2$ s, it is convenient to represent the model in moving average form. In particular, let  $e3' \equiv (0, 0, 1)$  so that  $r_t = e3'\tilde{z}_t$ . The implied infinite moving average representation for the returns may then be expressed as,

$$r_t = e3' \sum_{j=0}^{\infty} \Phi_j \epsilon_{t-j}, \quad (23)$$

where the impulse responses matrices  $\Phi_j$  follow the recursion,

$$\begin{aligned} \Phi_0 &= I \\ \Phi_j &= \sum_{i=0}^{j-1} \Xi_{j-i} \Phi_i, \end{aligned} \quad (24)$$

for,

$$\Xi_i = -I\theta_i^{(0)}(d) - (\gamma\delta')\theta_i^{(0)}(d) + \sum_{j=1}^i (-1)^j \Gamma_j \theta_i^{(j)}(d),$$

and the previously defined parameters of the fractional filter given by  $\theta_i^{(0)}(d) = (-1)^i \binom{d}{i}$  and  $\theta_i^{(j)}(d) = \sum_{l=j-1}^{i-1} \theta_{i-l}^{(0)}(d) \theta_l^{(j-1)}(d)$ ,  $j \geq 1$ , respectively. These expressions readily allow for the calculation of  $h$ -step ahead return forecasts by simply equating all of the values of  $\epsilon_{t+h-j}$  for  $h-j > 0$  to zero in the corresponding expression for  $r_{t+h}$  in equation (23).

In practice, of course, we are typically interested in the cumulative forecasts of the high-frequency returns over longer time-intervals, as opposed to the multi-period forecasts of the high-frequency returns themselves. For illustration, consider the case of one-day returns. With 77 5-minute returns per trading day, the continuously compounded daily return may be written as,

$$r_t^{(\text{day})} = \sum_{j=0}^{77-1} r_{t+j} = e3' \sum_{j=0}^{77-1} \sum_{i=0}^{\infty} \Phi_i \epsilon_{t+j-i}. \quad (25)$$

Going one step further, this expression for the daily return is naturally decomposed into an expected and an unexpected part,

$$r_t^{(\text{day})} = e3' \sum_{j=0}^{77-1} \sum_{i=j+1}^{\infty} \Phi_i \epsilon_{t+j-i} + e3' \sum_{j=0}^{77-1} \sum_{i=0}^j \Phi_i \epsilon_{t+j-i}, \quad (26)$$

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<sup>27</sup>The BIC criterion implies that the short-run dynamics of the backward-looking realized variance depends on five lags of the differenced  $\tilde{r}v_t$  series and one lag of the differenced  $vi x_t^2$  series, while the short-run dynamics of the implied variance depends on three lags of the differenced  $vi x_t^2$  series, and six lags of the differenced returns. The short-run dynamics of the returns depends on two lags of the differenced  $vi x_t^2$  series and seven lags of the differenced returns. Additional details concerning these estimation results are available upon request.

with the first term on the right-hand side corresponding to the former and the second term the latter. Consequently, the  $R^2$  for the daily return implied by the CFVAR model may be conveniently expressed as,

$$R_{\text{day}}^2 = \frac{\sum_{k=1}^{\infty} e3' \left( \sum_{j=0}^{77-1} \Phi_{k+j} \right) \Omega \left( \sum_{j=0}^{77-1} \Phi_{k+j} \right)' e3}{\sum_{k=-(77-1)}^{\infty} e3' \left( \sum_{j=\max(0,-k)}^{77-1} \Phi_{k+j} \right) \Omega \left( \sum_{j=\max(0,-k)}^{77-1} \Phi_{k+j} \right)' e3}. \quad (27)$$

Similar expressions for the  $R^2$ s associated with forecasting 5-minute, hourly, weekly, and monthly returns are readily available by replacing 77 in the equation above with 1, 12, 385, and 1,694, respectively.

The results obtained from evaluating the comparable expression in equation (27) for the 5-minute, hourly, weekly, and monthly forecast horizons at the CFVAR<sub>0.37</sub>(7) model estimates for  $\tilde{z}_t$  are reported in the first row in Table 5.<sup>28</sup> As seen from the table, the CFVAR model for  $\tilde{z}_t$  implies quite substantial  $R^2$ s of 1.987% and 7.091% at the daily and monthly horizons, respectively. As such, these results further corroborate the empirical evidence pertaining to return predictability and a significant risk-return tradeoff relationship discussed above.

To help gauge where this predictability is coming from, we calculate the implied  $R^2$ s for three restricted versions of the CFVAR<sub>0.37</sub>(7) model. In the first two models, we restrict the returns to depend on a single lag of  $\tilde{r}v_t$  and  $vix_t^2$ , respectively, leaving the other dynamic dependencies in the CFVAR<sub>0.37</sub>(7) model intact. The third restricted CFVAR model rules out any “volatility feedback” effects, so that the returns simply follow an autoregressive model. In each case we reestimate the CFVAR model with the relevant restrictions imposed on the return equation, and compute the corresponding VAR representation and implied  $R^2$ s.<sup>29</sup>

The resulting  $R^2$ s are reported in rows two through four in Table 5. The models that only include the lagged  $\tilde{r}v_t$  and  $vix_t^2$  in the return equation result in almost no predictability at the daily horizon, but the  $R^2$ s increase to 4.705% and 4.496%, respectively, at the monthly level. By contrast, a simple autoregressive model for the returns, which does not include any of the variances in the return equation, performs comparatively well in predicting daily and within day returns, but it has essentially no forecasting power at the longer monthly horizon. The general CFVAR<sub>0.37</sub>(7)

<sup>28</sup>In carrying out the numerical calculations, we truncated the infinite sum in (27) at 100,000. Additional robustness checks using longer truncation lags and longer return horizons revealed the same basic conclusions.

<sup>29</sup>Further details concerning the calculation of the  $R^2$ s for the restricted CFVAR models are given in Appendix A.2.

model effectively combines the good forecasting performance of the autoregressive model at high frequencies with that of the risk-based models at lower frequencies, by accounting for the joint dynamic dependencies in the returns and the two variance series and the predictability inherent in the variance risk premium.

To further underscore the gains afforded by jointly modeling the three series, the last three rows in Table 5 report the results from a set of simple univariate “balanced” predictive regressions, in which we regress the 5-minute, hourly, daily, weekly and monthly future returns on  $\Delta^d \tilde{r}_t$ ,  $\Delta^d vix_t^2$ , and  $\tilde{v}p_t = vix_t^2 - \tilde{r}_t$ .<sup>30</sup> As expected, the regressions involving the variance risk premium tend to give rise to the largest  $R^2$ s. However, all of the  $R^2$ s are noticeably lower than those for the CFVAR<sub>0.37</sub>(7) model reported in the first row of the table.

## 6 Conclusion

We provide a detailed characterization of the dynamic dependencies and interrelatedness in aggregate stock market returns and volatilities using newly available high-frequency intraday data on both. The time series of actual realized volatilities and the market’s risk-neutralized expectation thereof are both well described by long-memory fractional integrated processes. At the same time, the two volatility processes appear to be fractionally cointegrated and move in a one-to-one relationship with one another in the long-run. Using frequency domain inference procedures that allow us to focus on specific components of the spectra, we also uncover strong evidence for an otherwise elusive positive risk-return tradeoff relationship in the high-frequency data. Rather than a tradeoff between returns and variances, however, the data clearly point to a tradeoff between returns and the variance risk premium, as defined by the cointegrating relationship between the two variance series. Moreover, we show that the strength of this relationship vary importantly across frequencies, and as a result simple risk-return regressions, as estimated in much of the existing literature, can easily give rise to misleading conclusions. Combining these results, we formulate and estimate a fractionally cointegrated VAR model for the high-frequency returns and two variance series that is able to accommodate all of these dependencies within a coherent joint modeling framework. Going one step further, we show how this high-frequency based multivariate

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<sup>30</sup>These regressions mirror the non-predictive regressions of the contemporaneous returns on the same three predictor variables previously reported in Table 2.

model implies non-trivial return predictability over longer monthly horizons.

The new class of stochastic volatility models proposed by Barndorff-Nielsen and Veraart (2011) provides a direct link between the variance risk premium highlighted here and time-varying volatility-of-volatility. Our quantitative findings are also consistent with the qualitative implications from the stylized equilibrium model in Bollerslev, Sizova, and Tauchen (2011), in which the variance risk premium defined by the long-run cointegrating type relationship deduced from the high-frequency data, is linked to notions of aggregate economic uncertainty and time-varying equity risk premia. It would be interesting to further expand on these models to allow for a more “structural” explanation of the intricate cross-frequency empirical relationships and dynamic dependencies in returns and risk-neutral and realized variances documented here.

## A Appendix

### A.1 CFVAR Model Solution for Returns

Let  $e1' \equiv (1, 0, 0)$ ,  $e2' \equiv (0, 1, 0)$ , and  $e3' \equiv (0, 0, 1)$ , respectively. The preliminary univariate estimates for the two variance series in Section 3 suggest that the first and the second equations of the  $CFVAR_d(p)$ ,  $e1'\Delta^d z_t \equiv \Delta^d r v_t$  and  $e2'\Delta^d z_t \equiv \Delta^d v i x_t^2$ , respectively, are both  $I(0)$ . But the fractional filter  $\Delta^d$  is also applied to the third equation  $e3'\Delta^d z_t$  and the returns. This seemingly may result in over-differencing. However, following Theorem 8 of Johansen (2008), if the conditions for inversion of the  $CFVAR_d(p)$  are satisfied and  $d < \frac{1}{2}$ , then (20) has the solution

$$z_t = D \sum_{i=0}^{\infty} \theta_i^{(0)}(-d) L^i \epsilon_t + Y_t + \mu_t, \quad (\text{A.1})$$

where  $\theta_i^{(0)}(-d) = (-1)^i \binom{-d}{i}$  are the coefficients of the inverse fractional filter,  $\mu_t$  is a function of the restricted constant  $\rho$ ,  $Y_t$  is a stationary  $I(0)$  series, and the parameter matrix  $D$  is defined by,

$$D = \delta_{\perp} \left( \gamma'_{\perp} \left[ I - \sum_{i=1}^p \Gamma_i \right] \delta_{\perp} \right)^{-1} \gamma'_{\perp}, \quad (\text{A.2})$$

where  $\gamma_{\perp}$  and  $\delta_{\perp}$  are  $(3 \times 1)$  vectors such that  $\gamma' \gamma_{\perp} = 0$  and  $\delta' \delta_{\perp} = 0$ , respectively. With  $\gamma$  and  $\delta$  defined by equations (22) and (21), the last row of  $D$  therefore has only zero elements.



Consequently, the solution of the CFVAR model for the third equation and the return process reduces to

$$r_t = e3'Y_t + e3'\mu_t, \quad (\text{A.3})$$

that is an  $I(0)$  process, plus the initial contribution associated with  $\rho$ .

## A.2 CFVAR Restricted $R^2$ s

For illustration, consider a daily forecast horizon. Let

$$z_t = \sum_{j=0}^{\infty} \tilde{\Phi}_j \tilde{\epsilon}_{t-j} \quad (\text{A.4})$$

denote the moving average representation of the CFVAR model for  $z_t$  with the appropriate restrictions on the return equation imposed on the coefficients. Moving average coefficients are calculated using the recursions in equation (24). Define the polynomial

$$\Psi_1 + \Psi_2 L + \Psi_3 L^2 + \dots = \left( \sum_{j=1}^{\infty} \sum_{i=0}^{77-1} \tilde{\Phi}_{j+i} L^j \right) \left( \sum_{j=0}^{\infty} \tilde{\Phi}_j L^j \right)^{-1} \left( \sum_{j=0}^{\infty} \Phi_j L^j \right). \quad (\text{A.5})$$

The daily return forecast implied by the restricted CFVAR model is then given by  $e3' \sum_{j=1}^{\infty} \Psi_j \epsilon_{t-j}$ . The fraction of the cumulative daily returns that can be explained by the restricted model may therefore be computed as follows,

$$\tilde{R}_{\text{day}}^2 = \frac{\sum_{k=1}^{\infty} e3' \Psi_k \Omega \left( \sum_{j=0}^{77-1} \Phi_{k+j} \right)' e3}{\sqrt{\sum_{k=1}^{\infty} e3' \Psi_k \Omega \Psi_k' e3 \times \sum_{k=-(77-1)}^{\infty} e3' \left( \sum_{j=\max(0,-k)}^{77-1} \Phi_{k+j} \right) \Omega \left( \sum_{j=\max(0,-k)}^{77-1} \Phi_{k+j} \right)' e3}}. \quad (\text{A.6})$$

Again, similar expressions for the 5-minute, hourly, weekly and monthly returns are readily available by replacing 77 in the formula above by the integer value corresponding to the relevant forecast horizon.

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Table 1: Summary Statistics

	Mean	St.Dev.	Autocorrelations					
			1	2	3	10	78	78× 22
$r_t$	-0.000	0.142	-0.039	-0.031	-0.005	0.001	-0.030	0.007
$rv_t$	2.704	0.982	0.999	0.999	0.999	0.999	0.996	0.755
$vi\dot{x}_t^2$	3.119	0.785	0.999	0.999	0.999	0.998	0.986	0.787

Note: The table reports standard summary statistics for the returns,  $r_t$ , monthly realized variances,  $rv_t$ , and risk-neutral variances,  $vi\dot{x}_t^2$ . All of the statistics are based on 5-minute observations from September 22, 2003 through November 28, 2008, for a total of 100,639 observations.

Table 2: Univariate Return Regressions

Risk Proxy ( $v_t$ )	$\beta$	SE	$R^2$
Raw			
$rv_t$	-0.00128	0.00092	0.0116%
$vi\dot{x}_t^2$	0.00001	0.00142	0.0000%
Long-Memory Adjusted			
$\Delta^d rv_t$	-0.0373	0.0128	0.0191%
$\Delta^d vi\dot{x}_t^2$	0.0345	0.0190	0.0173%
Variance Risk Premium			
$vp_t = vi\dot{x}_t^2 - rv_t$	0.00445	0.00093	0.0406%

Note: The table reports 5-minute return regressions,  $r_t = \alpha + \beta v_{t-1} + u_t$ , based on data from September 30, 2003 to November 28, 2008. The first two columns report the OLS estimates for  $\beta$  and the corresponding Newey-West standard errors (SE). The last column reports the regression  $R^2$ 's. The fractional difference parameter is fixed at  $d = 0.37$ .

Table 3: Risk-Return Relations Across Frequencies

	$r_{t+1}^{high}$	$r_{t+1}^{band}$	$r_{t+1}^{low}$
$rv_t^{high}$	-0.005		
$rv_t^{band}$		-0.072	
$rv_t^{low}$			-0.644
$vi x_t^{high}$	0.176		
$vi x_t^{band}$		-0.016	
$vi x_t^{low}$			-0.494
$vp_t^{high}$	0.168		
$vp_t^{band}$		0.040	
$vp_t^{low}$			0.666

Note: The table reports the correlations between the short-, medium-, and long-run components of the returns  $r_t$ , the realized variance  $rv_t$ , the risk-neutral variance  $vi x_t^2$ , and the variance risk premium  $vp_t$ . All of the correlations are based on 5-minute observations from September 30, 2003 to November 28, 2008, along with the band-pass filtering procedures discussed in the main text for decomposing the series into the different components.

Table 4: CFVAR Model Estimates

	Estimates	SE
$\rho'$	$\begin{pmatrix} -0.000 & -0.0000 \end{pmatrix}$	$\begin{pmatrix} 0.161 & 0.0002 \end{pmatrix}$
$\gamma'$	$\begin{pmatrix} 0.000269 & -0.00198 & 0.00507 \\ 0.0000 & 0.0310 & -1.953 \end{pmatrix}$	$\begin{pmatrix} 0.000087 & 0.00049 & 0.00166 \\ 0.0006 & 0.0182 & 0.231 \end{pmatrix}$
$\delta'$	$\begin{pmatrix} -1.010 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0.308 & - & - \\ - & - & - \end{pmatrix}$
$\Gamma_1$	$\begin{pmatrix} 1.749 & -0.00646 & 0 \\ 0.050 & 1.579 & -0.058 \\ 0 & 0.511 & 0.904 \end{pmatrix}$	$\begin{pmatrix} 0.014 & 0.00096 & - \\ 0.045 & 0.025 & 0.018 \\ - & 0.119 & 0.230 \end{pmatrix}$
$\Gamma_2$	$\begin{pmatrix} -0.606 & 0 & 0 \\ -0.072 & -0.311 & -0.068 \\ 0 & -0.549 & 0.990 \end{pmatrix}$	$\begin{pmatrix} 0.022 & - & - \\ 0.051 & 0.046 & 0.017 \\ - & 0.128 & 0.239 \end{pmatrix}$
$\Gamma_3$	$\begin{pmatrix} 0.002 & 0 & 0 \\ 0 & -0.258 & -0.059 \\ 0 & 0 & 1.013 \end{pmatrix}$	$\begin{pmatrix} 0.046 & - & - \\ - & 0.027 & 0.021 \\ - & - & 0.243 \end{pmatrix}$
$\Gamma_4$	$\begin{pmatrix} -0.139 & 0 & 0 \\ 0 & 0 & -0.056 \\ 0 & 0 & 0.328 \end{pmatrix}$	$\begin{pmatrix} 0.028 & - & - \\ - & - & 0.048 \\ - & - & 0.368 \end{pmatrix}$
$\Gamma_5$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -0.252 \\ 0 & 0 & 6.909 \end{pmatrix}$	$\begin{pmatrix} - & - & - \\ - & - & 0.074 \\ - & - & 0.950 \end{pmatrix}$
$\Gamma_6$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -10.38 \end{pmatrix}$	$\begin{pmatrix} - & - & - \\ - & - & - \\ - & - & 1.78 \end{pmatrix}$
$\Gamma_7$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 10.64 \end{pmatrix}$	$\begin{pmatrix} - & - & - \\ - & - & - \\ - & - & 2.08 \end{pmatrix}$

Note: The table reports Seemingly Unrelated Regression (SUR) estimates of the CFVAR<sub>0.37</sub>(7) model,

$$\Delta^d z_t = \gamma (\delta' (1 - \Delta^d) z_t + \rho) + \sum_{i=1}^7 \Gamma_i \Delta^d (1 - \Delta^d)^i z_t + \epsilon_t,$$

based on 5-minute observations from September 30, 2003 to November 28, 2008. The fractional difference parameter is fixed at  $d = 0.37$ . The reported standard errors (SE) for the parameter estimates are calculated from the bootstrap procedure discussed in the main text.

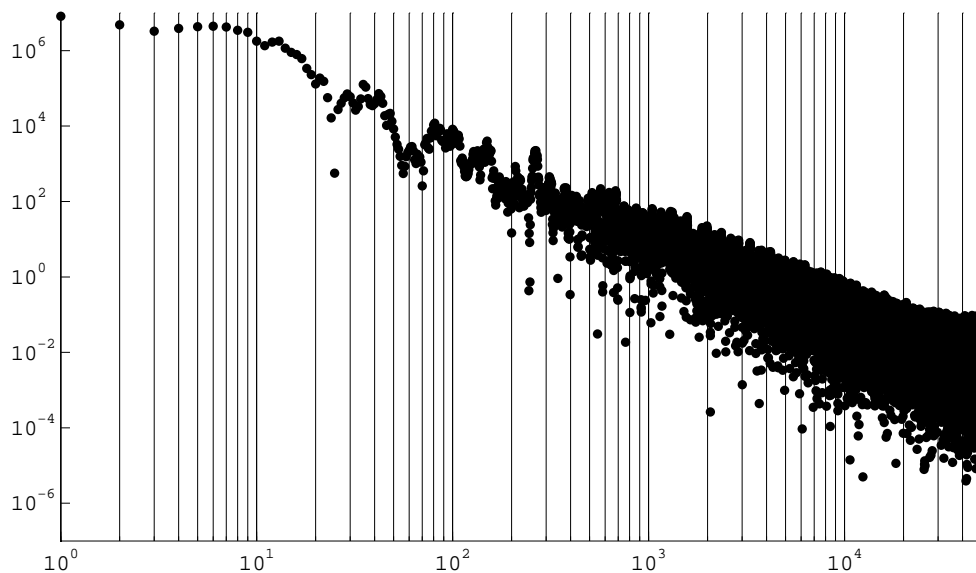


Table 5: Multi-Period Return Predictions

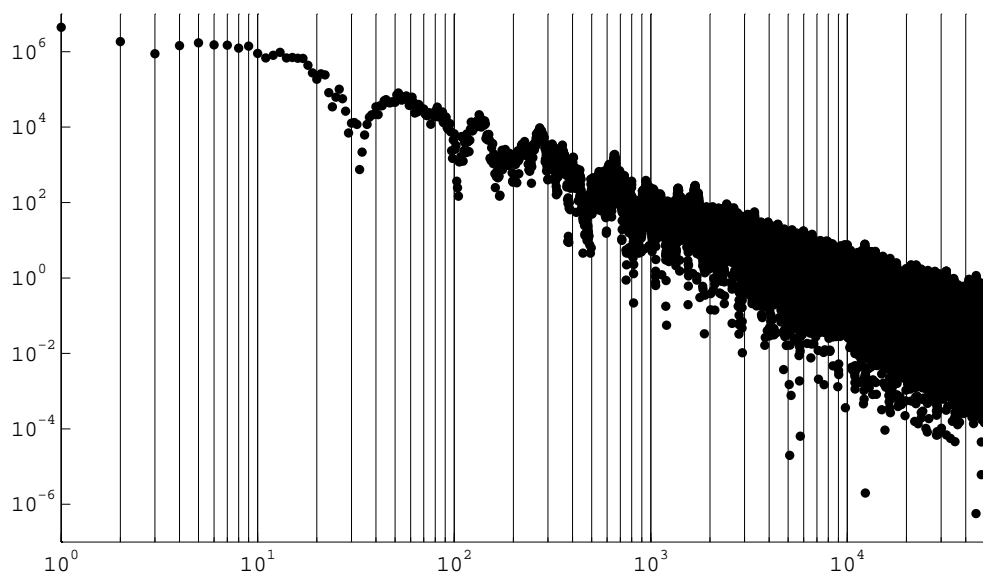
	$r_t^{(5\text{-min})}$	$r_t^{(\text{hour})}$	$r_t^{(\text{day})}$	$r_t^{(\text{week})}$	$r_t^{(\text{month})}$
CFVAR <sub>0.37(7)</sub>	0.256%	0.631%	1.987%	3.786%	7.091%
CFVAR with $\tilde{r}v_t$ only	0.002%	0.020%	0.126%	0.853%	4.705%
CFVAR with $vi x_t^2$ only	0.001%	0.013%	0.090%	0.781%	4.496%
CFVAR with no vol. feedback	0.197%	0.439%	1.345%	1.275%	0.009%
Regression with $\Delta^d \tilde{r}v_t$	0.000%	0.001%	0.007%	0.143%	0.203%
Regression with $\Delta^d vi x_t^2$	0.015%	0.017%	0.070%	0.065%	0.007%
Regression with $\tilde{v}p_t = vi x_t^2 - \tilde{r}v_t$	0.004%	0.032%	0.214%	0.695%	0.021%

Note: The first row reports the predictive  $R^2$ s for 5-minute, hourly, daily, weekly and monthly returns implied by the predictive CFVAR<sub>0.37(7)</sub> model discussed in the main text. Rows two through four gives the predictive  $R^2$ s for the restrictive CFVAR models in which the return equation only includes lagged values of  $\tilde{r}v_t$ ,  $vi x_t^2$ , and the returns, respectively. The final three rows give the  $R^2$ s from simple univariate predictive return regressions. All of the estimates are based on 5-minute observations from September 30, 2003 to November 28, 2008.

Figure 1: Periodograms for Realized and Risk-Neutral Variances



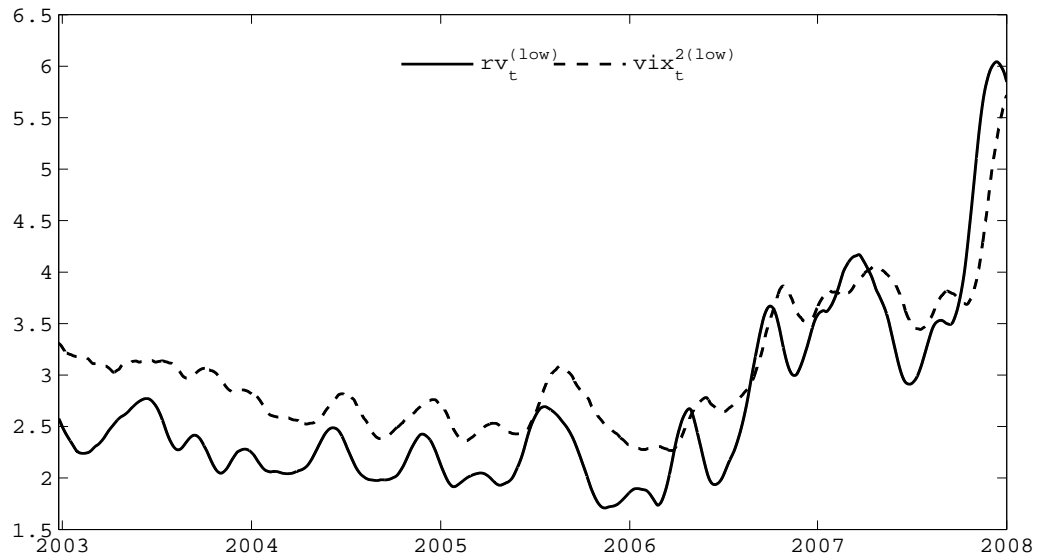
(a) Realized variance



(b) Risk-neutral variance

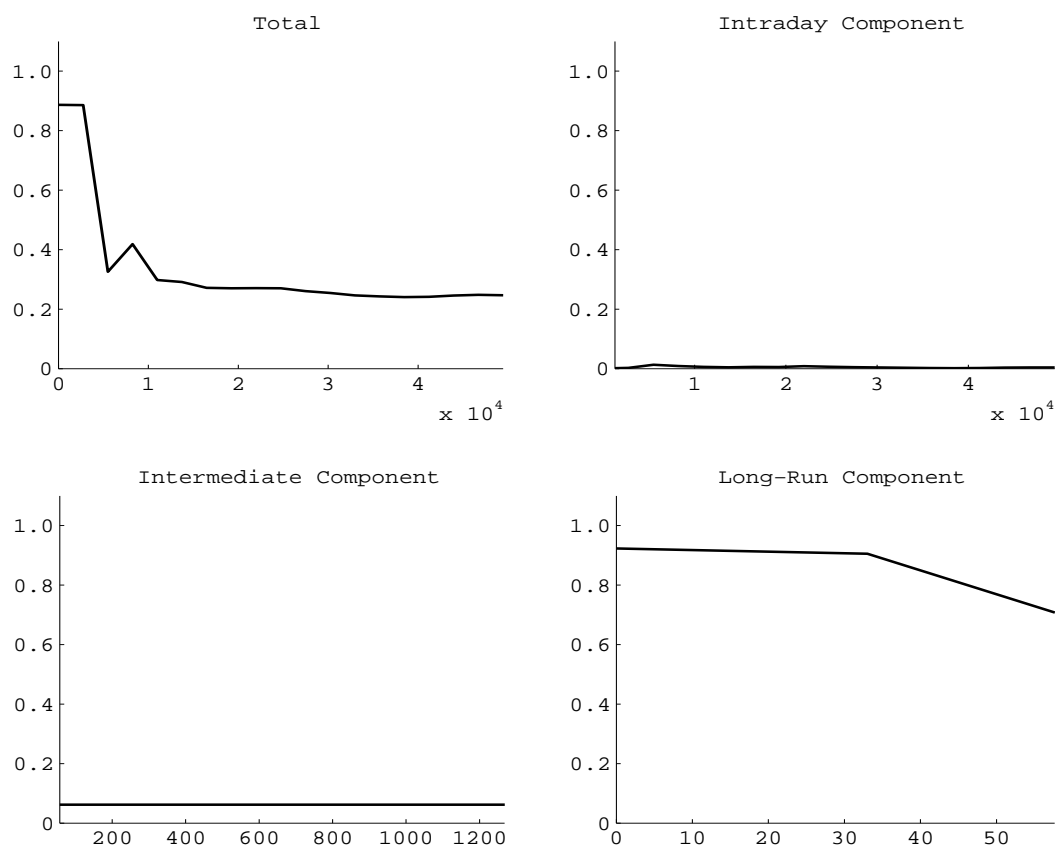
Note: The figure plots the sample periodograms of the realized variance  $rv_t$  (top panel) and the risk-neutral variance  $vi x_t^2$  (bottom panel). The periodograms are plotted on a double logarithmic scale. The estimates are based on 5-minute observations from September 30, 2003 to November 28, 2008.

Figure 2: Low-Pass Filtered Realized and Risk-Neutral Variances



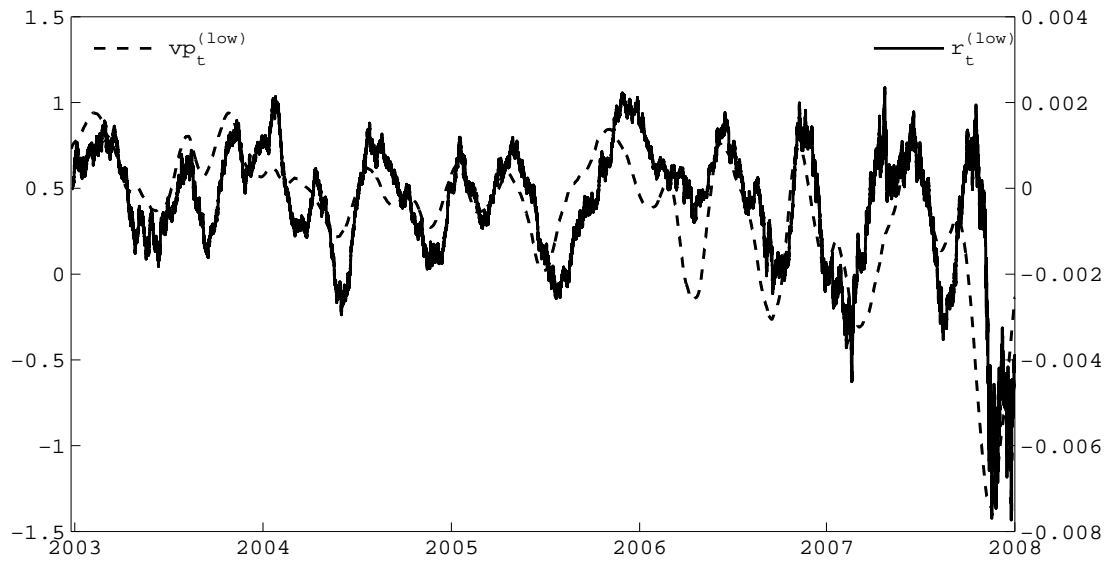
Note: The figure plots the low-pass filtered realized variance  $rv_t$  (solid line) and risk-neutral variance  $vix_t^2$  (dashed line) over the September 30, 2003 to November 28, 2008 sample period.

Figure 3: Coherence between Realized and Risk-Neutral Variances



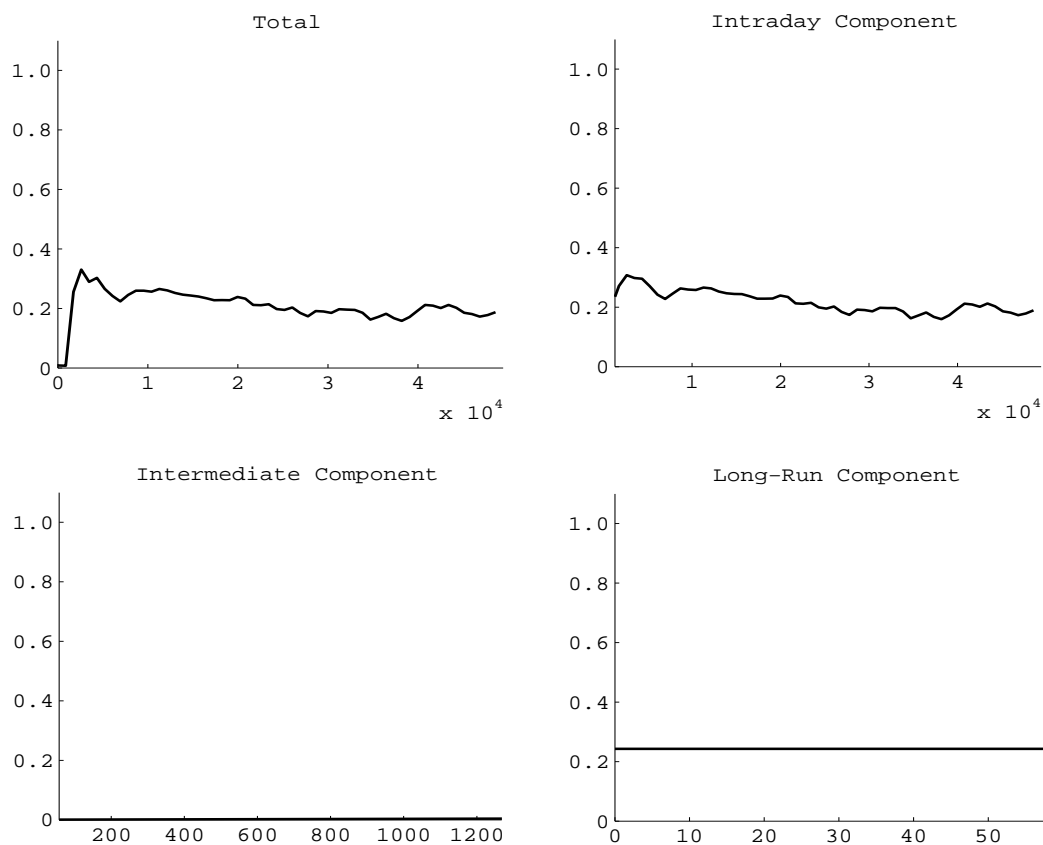
Note: The figure plots the coherence between the realized variance  $rv_t$  and the risk-neutral variance  $vx_t^2$ , as well as their high-pass (less than one day), band-pass (one day to  $3\frac{1}{2}$  months), and low-pass (longer than  $3\frac{1}{2}$  months) filtered counterparts. All of the estimates are based on 5-minute observations from September 30, 2003 to November 28, 2008.

Figure 4: Low-Pass Filtered Variance Risk Premium and Returns



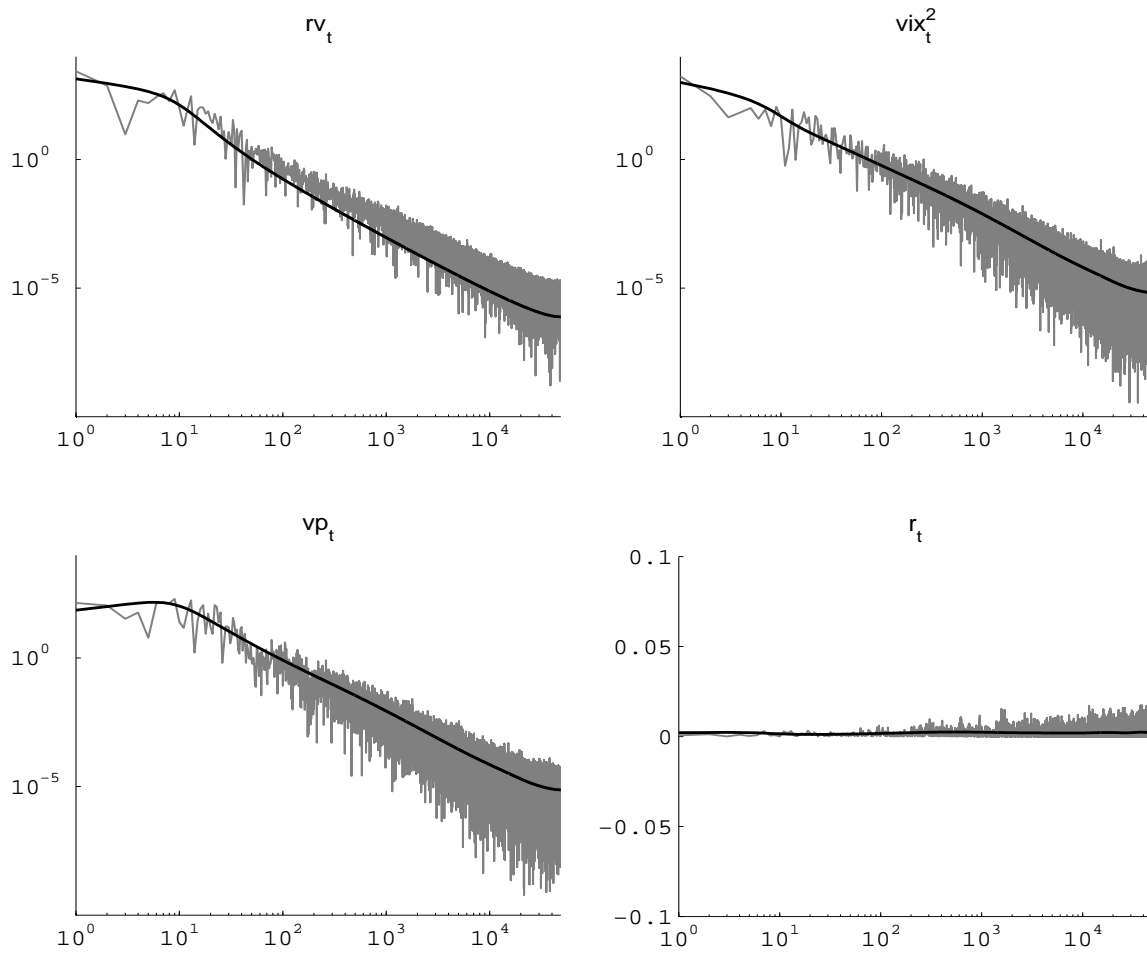
Note: The figure plots the low-pass filtered variance risk premium  $vp_t$  (dashed line) and the returns  $r_t^2$  (solid line) for the September 30, 2003 to November 28, 2008 sample period.

Figure 5: Coherence between Returns and Variance Risk Premium



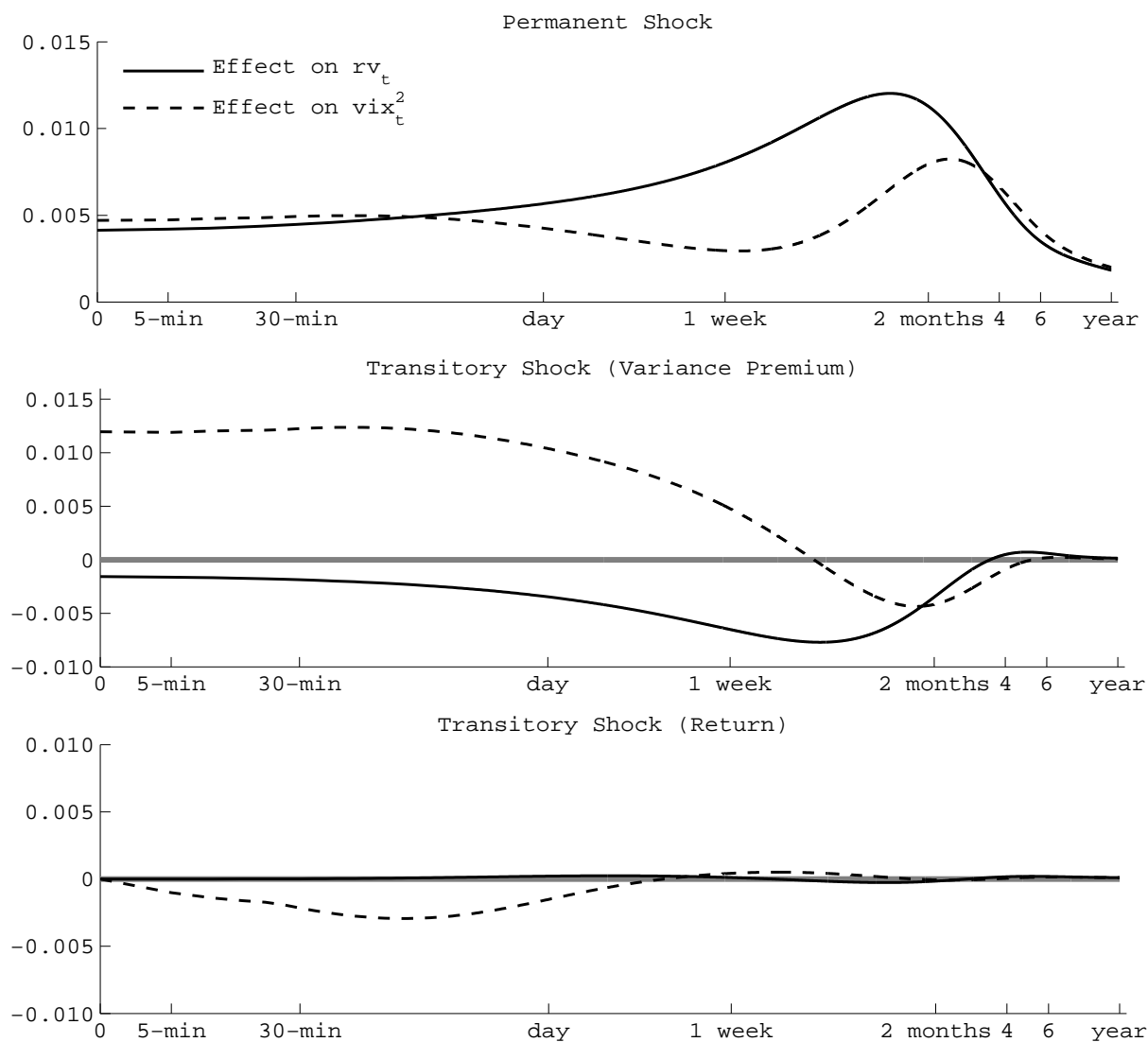
Note: The figure plots the coherence between the returns  $r_t$  and the variance risk premium  $vp_t$ , as well as their high-pass (less than one day), band-pass (one day to  $3\frac{1}{2}$  months), and low-pass (longer than  $3\frac{1}{2}$  months) filtered counterparts. All of the estimates are based on 5-minute observations from September 30, 2003 to November 28, 2008.

Figure 6: CFVAR Model Implied Spectra



Note: The figure plots the CFVAR model implied spectra (solid lines) for the realized variance  $rv_t$ , the risk-neutral variance  $vix_t^2$ , the variance risk premium  $vp_t$ , and the returns  $r_t$ , along with their corresponding sample periodograms (grey lines). All of the estimates are based on 5-minute observations from September 30, 2003 to November 28, 2008.

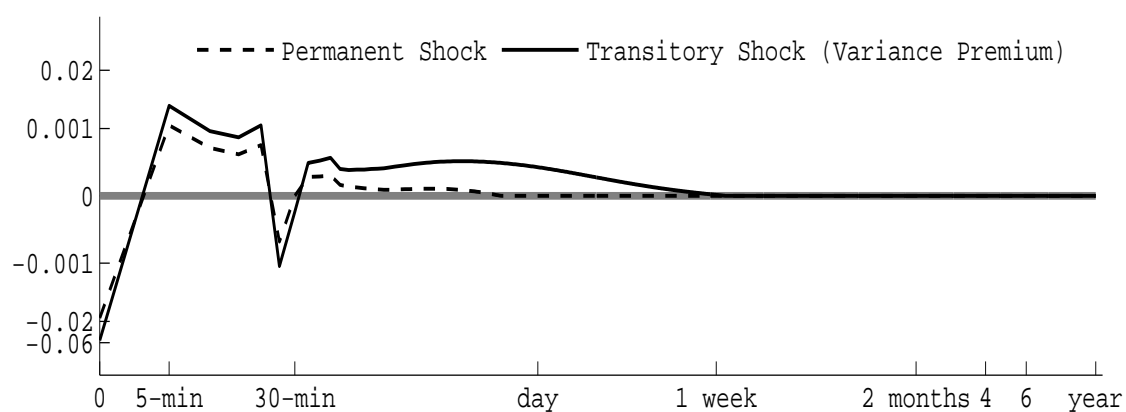
Figure 7: Impulse Response Functions for Realized and Risk-Neutral Variances



Note: The figure plots the CFVAR model implied impulse response functions for the realized variance  $rv_t$  (solid line) and the risk-neutral variance  $vix_t^2$  (dashed line) with respect to the permanent variance shock (top panel), the transitory shock to the variance risk premium (middle panel), and the transitory shock to the returns (bottom panel). All of the estimates are based on 5-minute observations from September 30, 2003 to November 28, 2008.



Figure 8: Impulse Response Functions for Returns



Note: The figure plots the CFVAR model implied impulse response functions for the return  $r_t$  with respect to the permanent variance shock (dashed line) and the transitory shock to the variance risk premium (solid line). All of the estimates are based on 5-minute observations from September 30, 2003 to November 28, 2008.

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