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Estimation of long memory in integrated variance

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Estimation of long memory in integrated variance *

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Abstract

A stylized fact is that realized variance has long memory. We show that, when the instantaneous volatility is driven by a fractional Brownian motion, the integrated variance is characterized by long-range dependence. As a consequence, the realized variance inherits this property when prices are observed continuously and without microstructure noise, and the spectral densities of integrated and realized variance coincide. However, prices are not observed continuously, so that the realized variance is affected by a measurement error. Discrete sampling and market microstructure noise induce a finite-sample bias in the fractionally integration semiparametric estimates. A Monte Carlo simulation analysis provides evidence of such a bias for common sampling frequencies.

Keywords: Realized variance, Long memory, fractional Brownian Motion, Measurement error, Whittle estimator.

J.E.L: classification: C10, C22, C80.

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1 Introduction

A well documented stylized fact is that volatility of financial returns is stationary and it is characterized by long-range dependence, or long memory, see, for instance, Baillie (1996), Bollerslev and Mikkelsen (1996), Dacorogna, Muller, Nagler, Olsen, and Pictet (1993), Ding, Granger, and Engle (1993), Granger and Ding (1996). More recently Andersen, Bollerslev, Diebold, and Ebens (2001), Andersen, Bollerslev, Diebold, and Labys (2001), Andersen, Bollerslev, Diebold, and Labys (2003), Martens, Van Dijk, and De Pooter (2009), and Rossi and Santucci de Magistris (2009) report evidence of long memory in the realized variance series. To the best of our knowledge, none of them has explicitly found a link between the long memory properties of the realized volatility, and its theoretical counterpart, the integrated volatility. Furthermore, no theoretical justification for the presence of long memory in integrated variance is given. A recent paper by Lieberman and Phillips (2008) shows that the presence of long memory in the realized variance is due to the aggregation of a finite number of short-memory series and it depends on the choice of the sampling scheme. However, the theory of realized variance, as an estimator of integrated variance (or integrated volatility, IV henceforth) lies in the *continuous time* framework, while Lieberman and Phillips (2008)'s proof is based on the aggregation of discretely sampled squared returns. Hansen and Lunde (2010) suggest an alternative explanation for the presence of long memory in realized variance. In fact, despite integrated volatility could be a close-to-unit-root process, realized volatility, that is characterized by a measurement error, is less persistent and appears as a stationary fractionally integrated process.

In this paper we study the long memory properties of the integrated and the realized variance (or realized volatility, RV) and we investigate the properties of the semiparametric estimation of long memory of IV based on realized measures. We assume that the trajectories of the instantaneous volatility, $\sigma^2(t)$, are generated by a fractional Brownian motion of order d, see Beran (1994) and Comte and Renault (1996, 1998). As shown by Comte and Renault (1998), the fractional Brownian motion represents a parsimonious way to introduce long memory in the volatility series, which encompasses weakly dependent stochastic volatility models. Due to the complexity of the estimation, long memory stochastic volatility models for instantaneous volatility have not found so much widespread use in practice. A notable exception is Casas and Gao (2008), who propose an estimation technique for the long memory stochastic volatility model based on squared daily returns. A number of alternative specifications, both in continuous and discrete time, have been considered in literature in order to generate long range dependence. Among others, Granger and Hyung (2004) point toward the presence of level shifts as causes of long memory in volatility, while Corsi (2009) and Corsi and Reno (2010) suggest that a multi-factor model is able to reproduce the observed high degree of long range dependence.

We first demonstrate that IV has the same fractional integration order of $\sigma^2(t)$, that is d. This result can be interpreted as a consequence of the self similarity feature of the fractional Brownian motion. It is therefore natural that realized measures of volatility have the same integration order of IV in the ideal situation where the price is recorded continuously and without market microstructure noise. In this case, we show that the spectral density of RV converges to the spectral density of IV.

In a more realistic framework, when we consider the presence of market microstructure noise and discrete sampling schemes, see Bandi and Russell (2008), Hansen and Lunde (2006) and for a recent survey McAleer and Medeiros (2008), the realized variance estimator is characterized by a measurement error. The measurement error introduces an additive term in the spectral density of RV. As a consequence, it turns out that the semiparametric estimation of the long memory parameter is biased. We investigate by simulations to what extent the choice of the sampling scheme and the size of the market microstructure noise introduce a finite sample downward bias in the local Whittle estimator of the integration order of IV. The downward bias is an increasing function of the noise-to-signal ratio, namely the ratio between the variance of the measurement error and the variance of IV. We therefore show the dependence of the noise-to-signal ratio on the parameters of the instantaneous volatility, the sampling frequency and the microstructure noise. For moderate values of the variance of the microstructure noise, the noise-to-signal ratio significantly increases as the volatility of volatility decreases.

We also employ the corrected local Whittle estimator by Hurvich, Moulines, and Soulier (2005), which accounts for the presence of the measurement error, in the realized volatility context. We show by means of Monte Carlo simulations its finite-sample features, based on alternative realized measures of volatility for different choices of the sampling frequency. The results highlight the dramatic improvement in terms of bias of the corrected with respect to the uncorrected one. Finally, the *long memory signature plot* of the realized variances of four NYSE assets, namely the plot of the long memory estimates obtained with realized volatility at different sampling frequencies, confirms the simulation results that is the robustness of the corrected estimation to the measurement error, while in contrast the local Whittle estimator is sensitive to the choice of the sampling frequency.

This paper is organized as follows. In Section 2 we show that, when the instantaneous volatility is driven by a fractional Brownian motion, the degree of fractional integration of the integrated volatility process is the same as the instantaneous volatility. Section 3 illustrates the characteristics of the measurement error and the features of the spectrum of the realized volatility process. Section 4 presents the estimation of the long memory parameter. In Section 5 the results of some simulations are illustrated and discussed. Section 7 concludes.

2 Long memory in integrated variance

Let $P(t)$ be the price of an asset, where its logarithm, $p(t)$, follows the stochastic differential equation:

$$\mathrm{d}p(t) = m(t)\mathrm{d}t + \sigma(t)\mathrm{d}W(t) \tag{1}$$

where $W(t)$ is a standard Brownian motion and $m(t)$ is locally bounded and predictable. $\sigma(t)$ is assumed to be independent of $W(t)$ and càdlàg, see Barndorff-Nielsen and Shephard (2002a). The logarithm of the instantaneous volatility is assumed to be driven by a fractional Ornstein-Uhlenbeck process, with zero long run mean, as in Comte and Renault (1998):

$$\mathrm{d}\ln\sigma^2(t) = -k\ln\sigma^2(t)\mathrm{d}t + \gamma\mathrm{d}W_d(t) \tag{2}$$

where $k > 0$ is the drift parameter, while $\gamma > 0$ is the volatility parameter and $W_d(t)$ is the fractional Brownian motion (fBm). The literature on long memory processes in econometrics distinguishes between type I and type II fractional Brownian motion. These processes have been carefully examined and contrasted by Marinucci and Robinson (1999), Davidson and Hashimzade (2008). When considered as real continuous processes on the unit interval, they can be defined respectively by

$$B_d(t) = \frac{1}{\Gamma(1+d)}\int_0^t (t-s)^d\mathrm{d}W(s) + \int_{-\infty}^0 [(t-s)^d - (-s)^d]\mathrm{d}W(s) \tag{3}$$

and

$$W_d(t) = \frac{1}{\Gamma(1+d)}\int_0^t (t-s)^d\mathrm{d}W(s). \tag{4}$$

In the type II case, the second term in (3) is omitted, it is the truncated version of the general fBm, see Comte and Renault (1996, 1998) and Marinucci and Robinson (1999). As shown by Marinucci and Robinson (1999), the increments of (3) are stationary, whereas those of (4) are not. When $d = 0$, both definitions of fBm collapses into the usual Brownian motion. The solution of (2) can be written as $\ln \sigma^2(t) = \int_0^t e^{-k(t-s)}\gamma \, \mathrm{d}W_d(s)$. The process $\ln \sigma^2(t)$ has long memory of order d, if there exists $k_\infty \in \mathbb{R}_+$ so that $\lim_{u\to+\infty} u\widetilde{k}(u) = k_\infty$, where k_∞ is a nonzero finite constant, and $0 < d < 1/2$.

Moreover, Comte (1996) and Comte and Renault (1998) show that the spectral density of the $\ln \sigma^2(t)$ is equal to

$$f(\lambda) = \frac{\gamma^2}{\Gamma(1+d)^2\lambda^{2d}} \frac{1}{\lambda^2 + k^2} \tag{5}$$

so that

$$\lim_{\lambda\to 0} \lambda^{2d} f(\lambda) = \frac{\gamma^2}{\Gamma(1+d)^2 k^2} \tag{6}$$

with the constant term depending on d. It is noteworthy that the spectral density at the origin is proportional to k_∞^2, where $k_\infty = \frac{\gamma d}{k}$ for the process in (2) (see Comte, 1996). This makes clear that the long memory feature of process in (2) is directly linked to the characteristics of the drift term. Indeed, when $k = 0$, e.g., the mean reversion is zero, the condition on k_∞ is no more satisfied. In this case, the process in (2) coincides with the fBm and it is non stationary. Thus the spectral density of $\ln \sigma^2(t)$ turns out to be proportional to $\lambda^{-2(1+d)}$, where $(1 + d)$ is the order of fractional integration of the fBm in (2) (see Flandrin, 1989 and Marinucci and Robinson, 1999). The volatility process $\sigma^2(t)$ is asymptotically equivalent (in quadratic mean) to the stationary process (see Comte and Renault, 1998) [1]:

$$\tilde{\sigma}^2(t) = \exp\left(\int_{-\infty}^t e^{-k(t-u)}\gamma \, \mathrm{d}W_d(u)\right), \quad k > 0 \quad 0 < d < \frac{1}{2}. \tag{7}$$

where the solution is expressed using type I fBm. Comte and Renault (1998) prove that the spectral density, $f_{\tilde{\sigma}^2}(\lambda)$, of the process $\tilde{\sigma}^2(t)$, is equal to $c\lambda^{-2d}$ for $\lambda \to 0$, so that the volatility process inherits the long-memory property induced by the fBm.[2]

Proposition 1 *Given the process in* (2) *for the logarithm of the instantaneous volatility, with* $k > 0$ *, then* $\lim_{\lambda\to 0} \lambda^{2d} f_{IV}(\lambda) = c \in \mathbb{R}_+$ *where* $f_{IV}(\lambda)$ *is the spectral density of*

[1] The volatility process σ_t^2 coincide almost surely with $\tilde{\sigma}^2(t)$.

[2] Note that the integration order of instantaneous and integrated volatility, when $k = 0$, is equal to $1 + d$, since $W_d(t)$ is integrated of order $1 + d$.

$IV = \int_0^1 \tilde{\sigma}^2(u)\mathrm{d}u.$

When the instantaneous volatility is covariance stationary, i.e., $k > 0$, the integrated volatility process has the same degree of fractional integration. This explains the empirical evidence of long memory in the ex-post realized measures of integrated volatility. It is also interesting to note that, when $k = 0$, that is the mean reversion of the log-instantaneous volatility is null, then $\log \sigma^2(t)$ is integrated of order $1 + d$, so that the IV can be supposed to be integrated of order $1 + d$, namely it is nonstationary.

3 The measurement error

In this section we set the notation and characterize the measurement error associated with the RV estimator. The integrated variance is defined as follows

$$IV = \int_0^1 \sigma^2(u)\mathrm{d}u. \tag{8}$$

To simplify the notation we consider an equidistant partition $0 = t_0 < t_1 < \ldots < t_n = 1$, where $t_i = i/n$, and $\Delta = 1/n$, that is the interval is normalized to have unit length. Define $r_{i\Delta,\Delta} = p_{i\Delta,\Delta} - p_{(i-1)\Delta,\Delta}$. Adopting the notation of Hansen and Lunde (2005), the RV at sampling frequency n is

$$RV^{\Delta} = \sum_{i=1}^{n} r^2_{i\Delta,\Delta}. \tag{9}$$

then $RV^{\Delta} \xrightarrow{p} IV$ as $n \to \infty$. Barndorff-Nielsen and Shephard (2002a) and Barndorff-Nielsen and Shephard (2002b) derived a distribution theory for RV^{Δ} when $n \to \infty$,

$$\sqrt{n}(RV^{\Delta} - IV) \xrightarrow{d} N(0, 2IQ)$$

where $IQ = \int_0^1 \sigma^4(u)\mathrm{d}u$ is the *integrated quarticity*. In this paper we focus on the series of nonoverlapping integrated volatilities

$$\int_0^1 \sigma^2(u)\mathrm{d}u, \int_1^2 \sigma^2(u)\mathrm{d}u, \ldots, \int_{T-1}^{T} \sigma^2(u)\mathrm{d}u.$$

where $[0, T]$ represents our sampling period, and given the assumption that each interval has unit length we consider the IV over T intervals, e.g., $IV_1, IV_2, \ldots, IV_T$. Further, the time series of realized variances is composed of $\left\{RV_t^{\Delta}\right\}_{t=1}^{T}$. We are interested in delineating the dynamic features of $\{RV_t^{\Delta}\}_{t=1}^{T}$ compared to those of $\{IV_t\}_{t=1}^{T}$ for different choices of Δ.

3.1 Discretization error

Meddahi (2002) characterizes the discretization error, when realized variance is used to measure integrated variance. He assumes the underlying data generating process is a continuous time, continuous sample-path model. While RV_t^{Δ} converges to IV_t when $\Delta \to 0$, the difference may be not negligible for a given Δ. Following Meddahi (2002) we can decompose the difference between RV and IV, for a given Δ. From (1), the return on the interval $((i-1)\Delta, i\Delta)$

$$r_{t-1+i\Delta,\Delta} = \int_{t-1+(i-1)\Delta}^{t-1+i\Delta} m(u)\mathrm{d}u + \int_{t-1+(i-1)\Delta}^{t-1+i\Delta} \sigma(u)\mathrm{d}W(u)$$

let

$$\mu_{t-1+i\Delta,\Delta} \equiv \int_{t-1+(i-1)\Delta}^{t-1+i\Delta} m(u)\mathrm{d}u \text{ and } \quad \varepsilon_{t-1+i\Delta,\Delta} \equiv \int_{t-1+(i-1)\Delta}^{t-1+i\Delta} \sigma(u)\mathrm{d}W(u)$$

the squared return can be written as:

$$\begin{aligned} r^2_{t-1+i\Delta,\Delta} &= \int_{t-1+(i-1)\Delta}^{t-1+i\Delta} \sigma^2(u)\mathrm{d}u + (\mu_{t-1+i\Delta,\Delta})^2 + 2(\mu_{t-1+i\Delta,\Delta}\varepsilon_{t-1+i\Delta,\Delta}) \\ &\quad + \left((\varepsilon_{t-1+i\Delta,\Delta})^2 - \int_{t-1+(i-1)\Delta}^{t-1+i\Delta} \sigma(u)^2\mathrm{d}u \right) \\ &= \int_{t-1+(i-1)\Delta}^{t-1+i\Delta} \sigma^2(u)\mathrm{d}u + u_{t-1+i\Delta,\Delta}, \end{aligned}$$

Meddahi (2002) shows in Proposition 2.1 that the noise is

$$\begin{aligned} u_{t-1+i\Delta,\Delta} &= (\mu_{t-1+i\Delta,\Delta})^2 + 2(\mu_{t-1+i\Delta,\Delta}\varepsilon_{t-1+i\Delta,\Delta}) \\ &\quad +2\int_{t-1+(i-1)\Delta}^{t-1+i\Delta} \left(\int_{t-1+(i-1)\Delta}^{u} \sigma(s)\mathrm{d}W(s) \right) \sigma(u)\mathrm{d}u \end{aligned} \tag{10}$$

Then the realized variance

$$RV_t^{\Delta} = \sum_{i=1}^{n} \left[\int_{t-1+(i-1)\Delta}^{t-1+i\Delta} \sigma^2(u)\mathrm{d}u \right] + \sum_{i=1}^{n} u_{t-1+i\Delta,\Delta} = IV_t + u_t^{\Delta}. \tag{11}$$

with the discretization error given by

$$u_t^{\Delta} = \sum_{i=1}^{n} u_{t-1+i\Delta,\Delta}. \tag{12}$$

Moreover, Meddahi (2002) proves that u_t^{Δ} has a nonzero mean, when the drift $m(t)$ is nonzero, and is heteroskedastic, since its variance depends on $\sigma(t)$. As pointed out by Barndorff-Nielsen and Shephard (2002b) and Meddahi (2002), the correlation between the integrated variance and the noise term is zero when there is no leverage effect, that is $\mathrm{d}W$ in (1) and $\mathrm{d}W_d$ in (2) are independent. In particular, assuming that the drift in (1) is null and there is no leverage effect, then Barndorff-Nielsen and Shephard (2002a) show that, for finite $\Delta > 0$, the error term can be written as

$$u_{t-1+i\Delta,\Delta} = \sigma^2_{t,i,\Delta}\left(z^2_{t,i} - 1\right) \tag{13}$$

where $z_{t,i}$ is $i.i.d.N(0,1)$ and it is independent of $\sigma^2_{t,i,\Delta} = \int_{t-1+(i-1)\Delta}^{t-1+i\Delta} \sigma^2(s)\mathrm{d}s$. Note that $\sigma^2_{t,i,\Delta}$, is the integrated volatility over the i-th subinterval of length Δ. Therefore, from equation

$$u_t^{\Delta} = \sum_{i=1}^{n} \sigma^2_{t,i,\Delta}\left(z^2_{t,i} - 1\right). \tag{14}$$

It is clear from the result in Proposition 1, that $\sigma^2_{t,i,\Delta}$ has long-memory. Given the representation of the measurement error in equation (13), we are able to characterize the error term, when the integrated volatility has long memory. In particular, from Meddahi (2002) we know that u_t^{Δ} is dynamically uncorrelated and contemporaneously uncorrelated with IV_t, when the leverage effect is absent. Moreover, Barndorff-Nielsen and Shephard (2002a) show that

$$\mathrm{Var}[u_t^{\Delta}] = 2\Delta^{-1} \cdot \left\{\mathrm{Var}[\sigma^2_{t,i,\Delta}] + E\left[\sigma^2_{t,i,\Delta}\right]^2\right\};$$

Barndorff-Nielsen and Shephard (2002a) show that $E\left[\sigma^2_{t,i,\Delta}\right]^2 = \Delta^2 E[\tilde{\sigma}^2(t)]^2$. In our case,

$$\begin{aligned} E[\tilde{\sigma}^2(t)] &= \exp\left(\frac{\omega^2}{2}\right) \\ \mathrm{Var}[\tilde{\sigma}^2(t)] &= [\exp(\omega^2) - 1]\exp(\omega^2) \end{aligned}$$

where $\omega^2 \equiv \mathrm{Var}[\ln \sigma^2(t)] = \frac{\gamma^2 \pi}{\Gamma^2(1+d)k^{1+2d}\cos(d\pi)}$, see Casas and Gao (2008). Therefore, the variance of $\sigma^2_{t,i}$ is given by

$$\mathrm{Var}[\sigma^2_{t,i,\Delta}] = 2\,\mathrm{Var}[\tilde{\sigma}^2(t)] \cdot \int_0^{\Delta}\int_0^{\Delta} \rho(u)\mathrm{d}u\mathrm{d}v \tag{15}$$

where ρ denotes the autocorrelation function of the process $\tilde{\sigma}^2(t)$ see Barndorff-Nielsen and Shephard (2002a). Further, Comte and Renault (1998) prove that the autocovariance function of $\tilde{\sigma}^2(t)$ is equal to

$$\mathrm{Cov}(\tilde{\sigma}^2(t+h), \tilde{\sigma}^2(t)) = \mathrm{Var}[\tilde{\sigma}^2(t)] + \eta h^{2d+1} + o(h^{2d+1}) \text{ for } h \to 0$$

where η is a given constant. Hence the autocorrelation function of $\tilde{\sigma}^2(t)$ decays at a rate that depends on d. When $h \to 0$, the autocorrelation function converges to 1, at rate proportional to $2d+1$. Finally, the $\mathrm{Var}[\sigma^2_{t,i,\Delta}]$ depends on the long memory parameter d, but is constant over time. It is interesting to note that the long memory parameter d affects both the $\mathrm{Var}[\tilde{\sigma}^2(t)]$ and the $\mathrm{Cov}(\tilde{\sigma}^2(t+h), \tilde{\sigma}^2(t))$, and so potentially affects the variance of the discretization error. Nonetheless, it is hard to obtain a closed-form expression for this dependence, we investigate this point by means of simulation in Section 5.

3.2 Microstructure Noise

Suppose now that the observed intradaily return is observed with error, due to the presence of microstructure noise,

$$\widetilde{p}(t) = p(t) + \epsilon(t) \tag{16}$$

where $p(t)$ is the latent true, or efficient, price process that follows (1). The term $\epsilon(t)$ is the noise around the true price, with mean 0 and finite fourth moment. In particular, $\epsilon(t)$ is $i.i.d.$ and it is independent of the efficient price and the true return process.
Over periods of length Δ, we have

$$r_{t,i,\Delta} = (p_{t,i,\Delta} - p_{t,i-1,\Delta}) + (\epsilon_{t,i,\Delta} - \epsilon_{t,i-1,\Delta}) = r_{t,i,\Delta} + \eta_{t,i,\Delta} \tag{17}$$

and $\eta_t = \sum_{i=1}^{n} \eta_{t,i,\Delta}$. With discretization and microstructure noise, the measurement error of realized volatility is given by

$$\xi_t^\Delta = u_t^\Delta + \sum_{i=1}^{n} \eta^2_{t,i,\Delta} + 2\left(\sum_{j=1}^{n} \sigma_{t,i,\Delta} z_{t,i} \eta_{t,i,\Delta}\right). \tag{18}$$

As noted by Bandi and Russell (2006), while the efficient return is of order $O_p(\sqrt{\Delta})$, the microstructure noise is of order $O_p(1)$ over any period of time. This means, that, when $\Delta \to 0$, then the microstructure noise dominates over the true return process, and longer period returns are less contaminated by noise than shorter period returns. Given the properties of $\epsilon(t)$, $E\left(\eta_{t,i,\Delta}\right)=0$, and

Proposition 2 *Let $\Delta>0$, consider the processes $p(t)$, RV^{Δ}, ξ_t^{Δ} defined respectively in* (16), (9), (18)*. Then:*

(i) $E(\xi_t^{\Delta})=\Delta^{-1}\sigma_\eta^2$;

(ii) $\mathrm{Var}(\xi_t^{\Delta})=2\Delta^{-1}E\Big[\Big(\sigma_{t,i,\Delta}^2\Big)^2\Big](1+2\sigma_\eta^2)+\Delta^{-1}\big(\kappa_\eta-\sigma_\eta^4\big)$, *where* $\sigma_\eta^2=\mathrm{Var}\left(\eta_{t,i,\Delta}\right)$ *and* $\kappa_\eta=E[\eta_{t,i,\Delta}^4]$.

(iii) ξ_t^{Δ} *is dynamically uncorrelated, i.e.,* $\mathrm{Cov}(\xi_t^{\Delta},\xi_{t+h}^{\Delta})=0\ \forall h\neq 0$;

(iv) The error term ξ_t^{Δ} is uncorrelated with IV_t;

(v) $\mathrm{Cov}(RV_t^{\Delta},RV_{t-h}^{\Delta})=\mathrm{Cov}(IV_t,IV_{t-h})$, $\forall h\neq 0$.

It is evident that $\mathrm{Var}(\xi_t^{\Delta})\geq\mathrm{Var}(u_t^{\Delta})$, and like in the case of discretization error u_t^{Δ}, the variance of ξ_t^{Δ} depends on the long memory parameter d. Again, as in the previous case, we investigate this feature in Section 5. It is worthy to notice that the impact of the microstructure noise on the variance of ξ_t^{Δ} is of smaller order with respect to the effect on the expected value. However, as we will show later in Section 5, for moderate choices of Δ, i.e. 5-minutes or 10-minutes, the influence of the microstructure noise on the error variance is not negligible.

4 Long memory estimation in the signal-plus-noise framework

Now, we turn our attention to the spectral density of RV since in this study we are interested in the semiparametric estimation of d, for which a local characterization of the spectral density is needed. It is well known that the drawback of global long memory estimators is that they require unnecessary assumptions on the spectral density. Instead, a consistent estimate of d can be obtained simply by specifying the shape of the spectral density at the origin. These methods are referred as local methods. Further, the semiparametric approach has the advantage, over the parametric ones, that it does not require a full specification of the dynamics of the process. This also implies that semiparametric estimation is more robust to the misspecification of the dynamics. In particular, the semiparametric approaches are based on the characterization of the spectrum as $\lambda \to 0$. According to Proposition 1, the spectrum of the IV has a pole as $\lambda \to 0$, that is proportional to λ^{-2d}. However, equation (18) highlights the fact that IV is measured by RV^Δ with an error term, ξ_t^Δ, whose variance depends on the choice of Δ, as shown in equation (13). This is a typical signal-plus-noise problem. Therefore, the quality of the estimate of d based on a realized measure like RV^Δ, can be dramatically affected in finite samples by the measurement error. The spectral density of RV^Δ is defined as,

$$f_{RV^\Delta}(\lambda) = \frac{1}{2\pi}\left\{\mathrm{Var}(RV^\Delta) + 2\sum_{j=1}^{\infty}\left[\mathrm{Cov}(RV_t^\Delta, RV_{t-j}^\Delta)\cos(\lambda j)\right]\right\}, \tag{19}$$

but, given the results in Proposition 2, this can be expressed in terms of the variance of the error ξ_t^Δ and the variance and covariances of IV

$$f_{RV^\Delta}(\lambda) = \frac{1}{2\pi}\left\{\mathrm{Var}(IV^\Delta) + \mathrm{Var}(\xi_t^\Delta) + 2\sum_{j=1}^{\infty}\left[\mathrm{Cov}(IV_t^\Delta, IV_{t-j}^\Delta)\cos(\lambda j)\right]\right\}. \tag{20}$$

This allows to write the spectral density of RV as the sum of two components:

$$f_{RV^\Delta}(\lambda) = f_{IV}(\lambda) + f_{\xi^\Delta}(\lambda) \tag{21}$$

where $f_{\xi^\Delta}(\lambda) = \frac{\mathrm{Var}(\xi_t^\Delta)}{2\pi}$ is the spectral density of the error term and $f_{IV}(\lambda)$ is the spectral of the signal, e.g., the integrated variance. The function $f_{\xi^\Delta}(\lambda)$ is constant with respect to λ because the error term is dynamically uncorrelated. As shown by Barndorff-Nielsen and Shephard (2002a) and Meddahi (2002), when the microstructure noise is absent, the variance of the error term is $\mathrm{Var}(\xi_t^\Delta) = \mathrm{Var}(u_t^\Delta) = 2\Delta^{-1}E\left[(\sigma_{t,i,\Delta}^2)^2\right]$ and it converges to zero as $\Delta \to 0$, so that $f_{\xi^\Delta}(\lambda) \to 0$. In fact, in the ideal situation where prices are recorded continuously and without measurement errors, since $\lim_{\Delta\to 0} RV^\Delta = IV$, it is evident that

$$\lim_{\Delta\to 0} f_{RV^\Delta}(\lambda) = f_{IV}(\lambda)$$

where f_{RV^Δ} is the spectral density of the realized variance. It follows that,

$$\lim_{\lambda\to 0}\left[\lim_{\Delta\to 0}\lambda^{2d} f_{RV^\Delta}(\lambda)\right] = \lim_{\lambda\to 0}\lambda^{2d} f_{IV}(\lambda) \tag{22}$$

so that the realized volatility is characterized by the same degree of long memory of integrated volatility when the instantaneous volatility is generated according to (2). Therefore, if the integrated variance is fractionally integrated of order d, then RV_t will be integrated of order d, since a process which is an $I(d)$ process plus an $I(0)$ process is integrated of order d. Differently from Lieberman and Phillips (2008), we are able to motivate the presence of long memory in realized volatility from a continuous time perspective, where the process of the instantaneous volatility is driven by a fractional Brownian motion of order d. It is also interesting to note that the additive noise term has a spectral density that depends on the variance of $\sigma^2_{t,i,\Delta}$, and on length of the intradaily interval, Δ.

When we consider the presence of the microstructure noise the variance of ξ_t^{Δ} diverges as $\Delta \to 0$ or $n \to \infty$, as noted by Bandi and Russell (2006). So when prices are recorded continuously but with measurement errors, the variance of the noise term, that is $\text{Var}(\xi_t^{\Delta})$, in the spectral density of RV^{Δ} dominates the signal when $\Delta \to 0$. In this case, $\lim_{\Delta\to 0} f_{RV^{\Delta}} = \infty$ for all λ, and thus it is not possible to identify the long memory signal. However, for a given Δ, the $\text{Var}(\xi_t^{\Delta})$ is finite, so that $f_{RV^{\Delta}}$ is finite too. On the other hand the choice of Δ impacts on the variance of ξ_t^{Δ} and through this on the spectral density of RV^{Δ}. If we increase Δ this reduces the variability of ξ_t^{Δ} due to the discretization but increases the microstructure noise component, as shown in Proposition 2, so that the net effect on $\text{Var}(\xi_t^{\Delta})$ is not known a priori. This characterizes a trade-off, which depends on the choice of Δ, that will be studied via simulations in Section 5. A simple and heuristic solution is to sample the returns at intermediate frequencies, say 5 minutes, so that the fourth moment of the microstructure noise is cumulated over finite horizons, and the true volatility signal can be estimated rather precisely, but still with a measurement error.[3]

Therefore, in this section, we discuss the semiparametric estimation of the long memory of the IV, that is robust to the presence of measurement errors. A large literature, see Deo and Hurvich (2001), Hurvich, Moulines, and Soulier (2005) and Haldrup and Nielsen (2007), discusses the properties of the semiparametric long memory estimators, such as the log-periodogram regression and the local Whittle estimator, when the long memory signal is contaminated by a noise term.[4] Deo and Hurvich (2001) show that the Geweke and Porter-

[3]Many alternative realized estimators of volatility have been considered in literature to deal with the problem of the microstructure noise, see among others Zhou (1996), Zhang, Mikland, and Aït-Sahalia (2005), Hansen and Lunde (2006) and Barndorff-Nielsen, Hansen, Lunde, and Shephard (2008).

[4]In this section, we will maintain the assumption that the noise term is dynamically uncorrelated with the signal and it is a white noise. As shown in section 3, this is the relevant case in the realized volatility context when drift in price and leverage are excluded.

Hudak (1984) estimator (GPH) is biased by a constant factor that depends on the variance of the noise term. In order to preserve consistency and asymptotic normality of the GPH estimator, Deo and Hurvich (2001) impose a condition on the growth rate of m, but this relies on the unknown value of d. Sun and Phillips (2003) suggest to introduce an additional term in the log-periodogram regression, $\beta\lambda^{2d}$ to account for the effect of the additive noise term, that is allowed to be weakly dependent. Arteche (2004) suggests that an optimal choice of the bandwidth is important to minimize the influence of the added noise term, since the variance of the measurement error heavily restricts the allowable bandwidth in finite samples. With a larger variance of the noise with respect to the signal, only frequencies very close to the origin contain a valuable information. Hurvich, Moulines, and Soulier (2005) and Arteche (2004) show that, in the signal-plus-noise framework, the local Whittle estimator is consistent for $d \in (0, 1)$ under general assumptions on the noise term. However, in finite samples, the estimates are downward biased. In this paper, we follow the approach suggested by Hurvich, Moulines, and Soulier (2005) in order to show the usefulness of correcting for the measurement error in the realized volatility context.[5] They propose to modify the local Whittle objective function as

$$Q(G, d, \beta)^* = \frac{1}{m} \sum_{j=1}^{m} \left\{ \log \left[G\lambda_j^{-2d}(1 + \beta\lambda_j^{2d}) \right] + \frac{\lambda_j^{2d} I_{RV^\Delta}(\lambda_j)}{G(1 + \beta\lambda_j^{2d})} \right\}, \tag{23}$$

where G is the spectrum at the origin. Concentrating G out, it yields

$$R(d, \beta) = \frac{1}{m} \sum_{j=1}^{m} \log \left(\lambda_j^{-2d}(1 + \beta\lambda_j^{2d}) \right) + \log \left(\frac{1}{m} \sum_{j=1}^{m} \frac{\lambda_j^{2d} I_{RV^\Delta}(\lambda_j)}{(1 + \beta\lambda_j^{2d})} \right), \tag{24}$$

where the local Whittle estimator is obtained setting $\beta = 0$ in the minimization of R. The local Whittle estimates of d and β are

$$(\hat{d}_c, \hat{\beta}) = \arg \min_{(d,\beta) \in \mathbb{D} \times \mathbb{B}} \hat{R}(d, \beta) \tag{25}$$

where $\mathbb{D}$ and $\mathbb{B}$ are the admissible sets of d and β, and m has to tend faster to ∞ than $T^{4d/(4d+1)}$. In the case of RV, $\hat{\beta}$ is interpreted as an estimate of the noise-to-signal ratio, $\frac{\mathrm{Var}(\xi_t^\Delta)}{2\pi f_{IV}(0)}$. The asymptotic variance of $\hat{d}_c$ is $\frac{(1+2d)^2}{16d^2 \cdot m}$, and it is a decreasing function of d. It is interesting to note that, for all the admissible values of d, the asymptotic variance of the bias

[5] Frederiksen, Nielsen, and Nielsen (2008) and Nielsen (2008) suggest to approximate the log-spectrum of the short-memory component of the signal and of the perturbation by means of an even polynomial term.

corrected local Whittle estimator, $\hat{d}_c$, is larger than the corresponding asymptotic variance of the local Whittle estimator, $\hat{d}$, that is $\frac{1}{4\cdot m}$.

5 Simulations

In this section we present the results of a Monte Carlo analysis of the finite sample properties of the long memory estimation of IV based on RV. In particular, we want to evaluate the impact that the measurement error has on the semiparametric long memory estimates, disentangling the contribution of the discretization error from that due to the microstructure noise. The chosen set-up replicates a real situation where the researcher disposes of a fixed number of days, but has alternative choices for Δ. Our purpose is to show how sensitive are the semiparametric estimates of d, in terms of bias and root mean square error (RMSE), to the choice of the sampling frequency (Δ). We also scrutinize the performances of the corrected estimator by Hurvich, Moulines, and Soulier (2005), obtained as in (25), for different Δ. Our simulations are not meant to be an assessment of the relative performances of alternative RV estimators of the integrated variance, like for example in Nielsen and Frederiksen (2008). We focus on the semiparametric estimates of the integration order of the IV based on RV^{Δ}.

For the Monte Carlo simulations we generate the log-price $p(t)$ as:

$$\mathrm{d}p(t) = \sigma(t)\mathrm{d}W(t) \tag{26}$$

and we assume that the log-instantaneous volatility process $\sigma^2(t)$ follows

$$\mathrm{d}\log\sigma^2(t) = k(\beta - \log\sigma^2(t))\mathrm{d}t + \gamma\mathrm{d}W_d(t) \tag{27}$$

In (27) W_d is the fractional Brownian motion of order d independent of $W(t)$. To simulate increments from the fractional Brownian motion we implement the Matlab routine by Yingchun Zhou and Stilian Stoev which is based on the circulant embedding algorithm for the values of interest of the Hurst's exponent, $H = d + \frac{1}{2}$. We simulate from the Euler discrete approximation of (26) and (27). A set of discrete trajectories with a time step of 10 seconds for 6.5 hours per day, which roughly corresponds to the trading period of NYSE. Thus we have a total $6 \times 60 \times 6.5 = 2,160$ log-prices and log-instantaneous volatilities per day for $2,500$ days, that is $Y_j = \{p_j, \log\sigma_j^2\}_{j=1}^{2,160\times 2,500}$. The generated price series are used to compute the RV series, with different Δ. Note that the computational burden is due to the fact that we simulate for each Monte Carlo replication a trajectory of Y_j which has $5,400,000$ observations. The circulant embedding algorithm to simulate a trajectory from a fBm, with 5,400,000 values, for the log-volatilities makes the simulation computationally intensive. The bid-ask bounce is modeled as:

$$\widetilde{p}(t) = p(t) + \frac{\zeta}{2}\mathbb{I}(t) \tag{28}$$

where ζ is the percentage spread, and the order-driven indicator variables $\mathbb{I}(t)$ are independently across p and t and identically distributed with $Pr\{\mathbb{I}(t) = 1\} = Pr\{\mathbb{I}(t) = -1\} = \frac{1}{2}$. This variable takes value 1 when the transaction is buyer-initiated, and -1 when it is seller-initiated. We adopt the simplest bid-ask bounce specification in order to make a comparison with the existing literature. Furthermore it is interesting to note that $\mathrm{d}\widetilde{p}(t)$ exhibits spurious volatility and negative serial correlation, see for instance Nielsen and Frederiksen (2008). A crucial quantity in this framework is the noise-to-signal ratio (nsr), $\frac{\mathrm{Var}(\xi_t^\Delta)}{\mathrm{Var}(IV_t)}$, which clearly depends on the generating process parameters. To figure out this relationship, we estimate the nsr, through Monte Carlo simulations, and plot it as a function of d, γ, ζ, and Δ. In Figure 1(a) it is evident that increasing Δ increases, for each chosen value of d, the nsr, provided that the microstructure noise is absent. Further, the nsr is larger, for any value of Δ, when either $d = 0$ or 0.45, than any other choice. With $d = 0.3$ the nsr is the lowest for all frequencies. When the nsr is plotted for different γ's, see Figure 1(b), the impact of Δ stands out very clearly. All lines are increasing in Δ, starting from 1 minute frequency, while for 10 seconds the microstructure noise dominates. As γ approaches 0, the innovation in the price process becomes the prevailing source of variability, so that the nsr is shifted upwards. This is seen simply noting that:

$$\frac{\mathrm{Var}(\xi_{t,i}^\Delta)}{\mathrm{Var}(\sigma_{t,i,\Delta}^2)} = \frac{\mathrm{Var}(\xi_{t,i}^\Delta)}{E[(\sigma_{t,i,\Delta}^2)^2] - E[(\sigma_{t,i,\Delta}^2)]^2}$$

As $\gamma \to 0$, then $E[(\sigma_{t,i,\Delta}^2)^2] - E[(\sigma_{t,i,\Delta}^2)]^2 \to 0$, so that $\frac{\mathrm{Var}(\xi_t^\Delta)}{\mathrm{Var}(IV_t)} \to \infty$, where $\xi_{t,i}^\Delta$ is defined in (33). In Figure 1(c), the nsr is plotted for different values of ζ, which is the bid-ask spread. It is fairly evident that sampling at 10 seconds, introduces a large microstructure noise such that the variance of the signal is totally dominated by the noise term. For $\zeta = 0.001$ and $\gamma = 0.5$, the nsr is equal to 1.95, when $\Delta = 10$ seconds, and is 1.36 for $\Delta = 30$ minutes.

In the simulations, we use the following set of parameters: $k = 0.9$, $\beta = -9.2$, $\gamma = (0.5, 0.7)$ and $d = 0.4$ that generate a *nsr* of a similar order as in Meddahi (2002), that is between 15% and 20%, in absence of microstructure noise. Second, we explore a situation where the log-instantaneous volatility is characterized by an integration order of $1 + d$. To obtain simulated trajectories of $\log \sigma^2(t)$ that are nonstationary, we set $k = 0$, $\beta = -9.2$, $\gamma = 0.2$ and $d = \{-0.3, -0.4\}$. This implies that the fractional integration order, $\delta = 1 + d$, of the log-instantaneous volatility is 0.7 and 0.6, respectively. We initialize each simulated day with $p(0) = \log(100)$ and $\sigma(0) = \exp(\beta/2)$.

We calculate the RV for $\Delta = 10$ sec, 1 min, 5 min, 10 min, 30 min, and the realized kernel, see Barndorff-Nielsen, Hansen, Lunde, and Shephard (2008), that is robust to the microstructure noise. The estimator is defined as

$$RV_K^{\Delta} = \sum_{h=-H}^{H} k\left(\frac{h}{H+1}\right)\gamma_h \tag{29}$$

where

$$\gamma_h = \sum_{j=|h|+1}^{n} r_{\Delta j,\Delta} r_{\Delta j-|h|,\Delta}$$

and $k(x)$ is a kernel function. We follow the instructions in Barndorff-Nielsen, Hansen, Lunde, and Shephard (2009) to construct the realized kernel. In particular, we focus on the Parzen kernel, since it guarantees non-negative estimates. For the choice of the bandwidth, h, that depends on the estimates of the variance of the noise, we implemented the selection procedure outlined in Barndorff-Nielsen, Hansen, Lunde, and Shephard (2009), Section 2. We then construct realized Kernel measures using 1 minute returns, RV_K^1.

In order to compare the estimators of the long memory parameter d, we calculate for each IV estimator and for each sampling frequency the percentage relative bias from S Monte Carlo simulations:

$$\text{Bias}(\hat{d}) = \frac{100}{d}\left(\frac{1}{S}\sum_{s=1}^{S}(\hat{d}_s - d)\right).$$

The RMSE is calculated as

$$\text{RMSE}(\hat{d}) = \left(\frac{1}{S}\sum_{s=1}^{S}(\hat{d}_s - d)^2\right)^{1/2}.$$

We estimate the long memory parameter of the IV, which is unobserved in practice but known in a simulation study and computed for the day t as $IV_t = \sum_{k=1}^{2,160} \sigma^2_{(t-1)\cdot 2,160+k}$ for

$t = 1, \ldots, 2,500$. The estimates of the long memory parameter based on IV constitutes a natural benchmark for estimates of d based on RV^{Δ}.

Table 1 reports the percentage bias and RMSE of the estimated long memory parameter when $d = 0.4$, obtained with the local Whittle and the corrected Whittle estimators, see equations (25), for different choices of Δ, γ and ζ. In both panels, the estimated d of IV is the closest to the true value, and the percentage bias, smaller than 1%, is due to the Monte Carlo variance. However, in the real world, IV is unobservable and we rely on the realized measures to conduct inference on the degree of long memory. When $\zeta = 0$, the best local Whittle estimates of d are obtained for small choices of Δ, while the largest negative biases are those of RV^{30} and RV_K. This is coherent with the fact that the only source of noise in this case is the discretization error, so that increasing the latter, induces more biased estimates.

When $\zeta = 0.001$ or $\zeta = 0.002$, the largest negative bias is that of RV^{all} (between -24% and -33%), while the bias of RV^{30} is between -13% and -18%. In presence of microstructure noise, the best estimates of d are obtained sampling at 1 and 5 minutes, and the bias is approximately -10%, so that $\hat{d} \approx 0.36$ on average. We also note that the negative bias becomes larger as γ gets smaller, as a result of the increase of the noise-to-signal ratio. Interestingly, correcting for the presence of the measurement errors improves the quality of the estimates, in terms of bias and RMSE, for any choice of Δ, and the relevance of the correction becomes evident as ζ increases. Similar evidence is also confirmed when the IV is long memory but nonstationary ($\log \sigma^2(t)$ simulated with $k = 0$ and $d = -0.3, -0.4$, with $\delta = (1 + d) > 1/2$), that could be the relevant case in practice, see Table 2. First, it is interesting to note that, also in the nonstationary case, IV has the same integration order of $\sigma^2(t)$, as we conjectured above. When $\zeta = 0$ and $\gamma = 0.2$, the variance of the noise becomes dominant on the variance of the signal as Δ increases, see Figure 1(b). Therefore, the impact of the discretization error on the estimates of δ is very large. For example, when $\zeta = 0$ the bias of the Whittle estimator based on RV^{30} is negative and larger than 30% when $d = 0.6$, and larger than 18% when $d = 0.7$. As expected, the negative bias increases, as ζ increases, and we observe extremely large negative biases for $\zeta = 0.002$, so that the estimated d based on the RV can fall in the stationary region, even though the integrated variance is not stationary. For example, RV^5 has a negative bias equal to -28% when $\delta = 0.6$, meaning that $\hat{\delta} \approx 0.43$ on average. It is noteworthy the fact that the corrected Whittle estimator provides unbiased estimates, also in the nonstationary region, for all the choices of Δ.

6 Empirical Analysis

The proposed estimation method is applied to the realized variance series of four stocks traded on NYSE, Caterpillar (C), FedEx (FDX), IBM and JP Morgan, (JPM). The sample period ranges from January 2, 2003 to June 30, 2007, for a total of 1132 trading days. The choice of the sample period is motivated by the idea that the period 2003-2007 has been characterized by low levels of volatility, few jumps and no large level shifts in the volatility mean. Therefore, we avoid the possible upward bias in the semiparametric estimates of d, due to the presence of large shifts as generated by changing bull and bear markets, such as during the 2008-2009 financial crisis.

We base our analysis on realized variances computed with alternative sample frequencies, say 1 minute, 5 minutes, 10 minutes, 15 minutes and 30 minutes. We show, by means of the *long memory signature plot*, displayed in Figure 1, that the proposed correction provides robust estimates of the IV long memory parameter. The Whittle estimator of d on the log-realized variance series (black line) is always in the stationary region, and it is evident the downward trend with respect to Δ. On the other hand, the corrected estimates of d (red line) are always above the Whittle estimates and are constant across different choices of Δ, meaning that the corrected estimator is able to provide estimates of the long memory parameter, that are robust to different choices of the sampling frequency. It is noteworthy the fact that, with the exception of FDX, the corrected estimator, $\hat{d}_c$, falls in the nonstationary region, suggesting that the integrated variance could be a non-stationary process. From this point of view, the fact that the local Whittle estimate of d based on RV^{Δ} turns out to be less than 0.5, namely a stationary long memory process, is mainly due to the role of the measurement error. This also suggests that using a biased long memory estimator leads to wrong conclusions on the stationarity of the integrated and instantaneous volatility processes. Using a similar argument, but in a discrete-time domain framework, Hansen and Lunde (2010) have proposed an instrumental variable estimator of the persistence of the signal when the latter is a unit root process (in the same spirit Rossi and Santucci de Magistris, 2011). At the best of our knowledge, the consequences of a fractional, but non stationary, volatility process are not studied yet in literature and the evidence reported here worths a more detailed investigation of this aspect.

7 Conclusions

A stylized fact is that realized volatility has long memory. In this paper, we investigate the dynamic properties and the source of the long-range dependence of RV. When the instantaneous volatility is driven by a fractional Brownian motion the IV is characterized by long-range dependence. As a consequence, the RV inherits this property in the ideal situation where prices are observed continuously and without microstructure noise. In this case, the spectral densities of IV and RV coincide. In this paper we focus on the dynamic properties of ex-post estimators of IV, such as RV, when we assume that the trajectories of the instantaneous volatility, $\sigma^2(t)$, are generated by a fractional Brownian motion of order d. First, we demonstrate that IV has the same fractional integration order of $\sigma^2(t)$. It is therefore natural that realized measures of volatility have the same integration order of IV in the ideal situation where the price is recorded continuously and without market microstructure noise. We study the dynamic properties of the measurement error associated with the RV, when the efficient price can be directly observed and when instead cannot because contaminated by microstructure noise. The semiparametric estimates of d crucially depends on the use of realized volatility in place of the unobservable IV. In absence of microstructure noise, the RV spectral density converges to the spectral density of IV. On the contrary, when the presence of microstructure noise prevents from using all the available price observations, which would be optimal if only the discretization occurs, the additional component in the spectral density significantly affects the semiparametric estimates of d. We adopt a correction of the local Whittle estimator along the lines of Hurvich, Moulines, and Soulier (2005). A Monte Carlo experiment confirms that the correction of the local Whittle estimator is effective when the microstructure noise is not negligible. Thus the trade-off between discretization error and microstructure noise is neutralized by adopting a corrected version of the local Whittle estimator. Finally, the estimation of the long memory of four NYSE stocks emphasizes the practical importance of considering the measurement error when estimating the degree of long memory of integrated variance. The corrected estimates of d seem to point to the possibility that the IV and the instantaneous volatility can be non-stationary processes. In this study we have not considered the role of jumps in prices and their potential effect on the estimation of long memory in IV. This is left for future research.

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(a) $d = 0.4$ and $\gamma = 0.5$.

	$\zeta = 0.000$				$\zeta = 0.001$				$\zeta = 0.002$			
	Bias $\hat{d}$	RMSE($\hat{d}$)	Bias $\hat{d}_c$	RMSE($\hat{d}_c$)	Bias $\hat{d}$	RMSE($\hat{d}$)	Bias $\hat{d}_c$	RMSE($\hat{d}_c$)	Bias $\hat{d}$	RMSE($\hat{d}$)	Bias $\hat{d}_c$	RMSE($\hat{d}_c$)
IV	-0.67	0.0546	-0.67	0.0565	-0.67	0.0546	-0.67	0.0546	-0.67	0.0546	-0.67	0.0572
RV^{all}	-0.74	0.0546	2.96	0.0571	-5.32	0.0597	-0.96	0.0570	-33.96	0.1508	-5.56	0.1319
RV^{1}	-1.17	0.0546	2.50	0.0465	-3.16	0.0559	0.82	0.0457	-13.22	0.0775	-0.94	0.0810
RV^{5}	-3.09	0.0558	0.73	0.0452	-4.50	0.0574	-0.45	0.0459	-9.37	0.0666	1.14	0.0719
RV^{10}	-5.45	0.0583	-0.25	0.0470	-6.65	0.0601	-0.98	0.0481	-10.39	0.0689	1.26	0.0753
RV^{30}	-13.98	0.0778	-1.28	0.0769	-15.03	0.0809	-1.61	0.0794	-18.08	0.0904	-0.24	0.0913
RV^{1}_{K}	-7.35	0.0616	-0.48	0.0511	-10.48	0.0687	-2.18	0.0584	-14.08	0.0785	-0.71	0.0832

(b) $d = 0.4$ and $\gamma = 0.7$.

	$\zeta = 0.000$				$\zeta = 0.001$				$\zeta = 0.002$			
	Bias $\hat{d}$	RMSE($\hat{d}$)	Bias $\hat{d}_c$	RMSE($\hat{d}_c$)	Bias $\hat{d}$	RMSE($\hat{d}$)	Bias $\hat{d}_c$	RMSE($\hat{d}_c$)	Bias $\hat{d}$	RMSE($\hat{d}$)	Bias $\hat{d}_c$	RMSE($\hat{d}_c$)
IV	-0.94	0.0490	-0.94	0.0490	-0.94	0.0490	-0.94	0.0490	-0.94	0.0490	-0.94	0.0490
RV^{all}	-1.02	0.0490	-0.82	0.0488	-4.10	0.0519	1.48	0.0583	-24.53	0.1172	-4.11	0.1081
RV^{1}	-1.35	0.0491	-0.17	0.0483	-2.54	0.0501	1.25	0.0521	-9.38	0.0638	-0.26	0.0735
RV^{5}	-2.98	0.0501	1.17	0.0525	-3.82	0.0511	1.41	0.0559	-7.22	0.0568	0.39	0.0667
RV^{10}	-4.59	0.0521	1.54	0.0585	-5.37	0.0534	1.31	0.0610	-8.19	0.0586	0.50	0.0692
RV^{30}	-11.29	0.0662	-0.41	0.0750	-11.98	0.0683	-0.66	0.0769	-13.93	0.0743	0.82	0.0813
RV^{1}_{K}	-6.16	0.0542	1.28	0.0632	-8.20	0.0685	0.81	0.0688	-10.81	0.0651	0.03	0.0752

Table 1: Bias and Root mean squared error of Monte Carlo estimates of d. $\hat{d}$ denotes the local Whittle estimator of the long memory parameter, while $\hat{d}_c$ is the corrected local Whittle estimator (see (25)). The term *Bias* is referred to the relative percentage bias, defined in equation (5). The estimates are based on 1,000 samples of 2,500 daily observations from model (26)-(27) with parameter values indicated in table and discretization step set to 10 seconds. The bandwidth used in the estimation of d is $m = T^{0.65}$

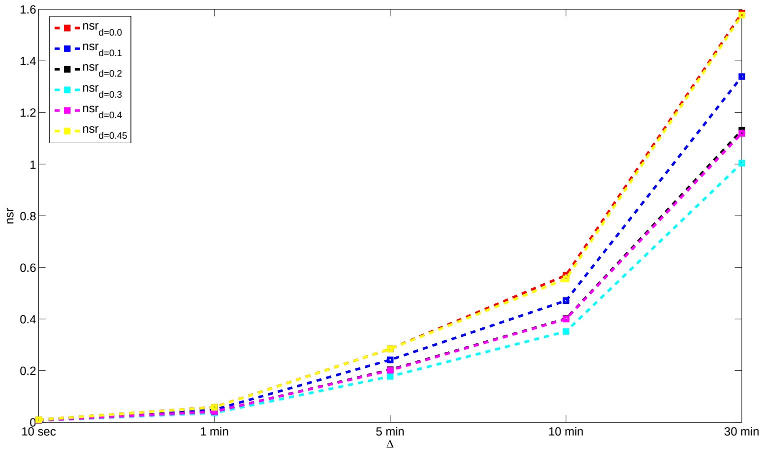
(a) $\delta = 1 + d = 0.6$ and $\gamma = 0.2$.

	$\zeta = 0.000$				$\zeta = 0.001$				$\zeta = 0.002$			
	Bias $\hat{\delta}$	RMSE($\hat{\delta}$)	Bias $\hat{\delta}_c$	RMSE($\hat{\delta}_c$)	Bias $\hat{\delta}$	RMSE($\hat{\delta}$)	Bias $\hat{\delta}_c$	RMSE($\hat{\delta}_c$)	Bias $\hat{\delta}$	RMSE($\hat{\delta}$)	Bias $\hat{\delta}_c$	RMSE($\hat{\delta}_c$)
IV	1.02	0.0596	1.02	0.0596	1.02	0.0596	1.02	0.0596	1.02	0.0596	1.02	0.0596
RV^{all}	0.62	0.0596	2.10	0.0501	-18.84	0.1339	1.78	0.1052	-60.73	0.3741	-2.56	0.1431
RV^1	-1.32	0.0601	3.29	0.0639	-10.55	0.0887	1.79	0.0901	-36.37	0.2316	0.82	0.1313
RV^5	-8.96	0.0797	2.38	0.0881	-14.38	0.1048	1.72	0.0970	-28.15	0.1810	1.55	0.1182
RV^{10}	-16.11	0.1137	1.68	0.1026	-20.10	0.1348	1.62	0.1079	-30.24	0.1927	1.81	0.1216
RV^{30}	-33.70	0.2105	1.23	0.1289	-35.61	0.2216	0.94	0.1317	-40.66	0.2514	0.87	0.1365
RV_K^1	-21.15	0.1402	1.52	0.1111	-28.59	0.1818	1.33	0.1222	-35.04	0.2188	0.81	0.1326

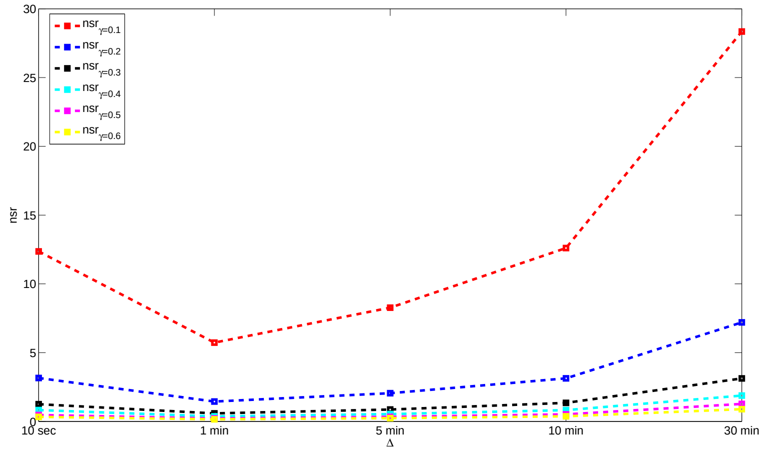
(b) $\delta = 1 + d = 0.7$ and $\gamma = 0.2$.

	$\zeta = 0.000$				$\zeta = 0.001$				$\zeta = 0.002$			
	Bias $\hat{\delta}$	RMSE($\hat{\delta}$)	Bias $\hat{\delta}_c$	RMSE($\hat{\delta}_c$)	Bias $\hat{\delta}$	RMSE($\hat{\delta}$)	Bias $\hat{\delta}_c$	RMSE($\hat{\delta}_c$)	Bias $\hat{\delta}$	RMSE($\hat{\delta}$)	Bias $\hat{\delta}_c$	RMSE($\hat{\delta}_c$)
IV	-0.26	0.0639	-0.26	0.0639	-0.26	0.0639	-0.26	0.0639	-0.26	0.0639	-0.26	0.0639
RV^{all}	-0.41	0.0638	0.07	0.0571	-9.07	0.0999	0.18	0.0911	-37.06	0.2938	-2.28	0.1485
RV^1	-1.12	0.0638	0.20	0.0465	-4.66	0.0720	1.42	0.0827	-18.91	0.1644	-0.99	0.1104
RV^5	-4.40	0.0703	-0.26	0.0452	-6.73	0.0790	0.59	0.0888	-13.90	0.1213	-0.83	0.1015
RV^{10}	-7.70	0.0837	-0.99	0.0470	-9.74	0.0835	0.01	0.0957	-15.32	0.1272	-0.59	0.1073
RV^{30}	-18.57	0.1442	-2.70	0.0769	-19.80	0.1520	-1.02	0.1150	-23.04	0.1739	-1.16	0.1195
RV_K^1	-10.48	0.0970	-1.51	0.0511	-14.42	0.1197	-0.81	0.1044	-18.46	0.1451	-1.11	0.1138

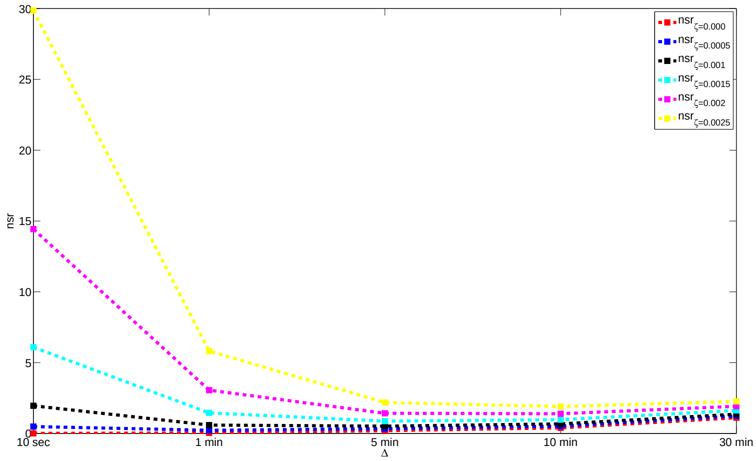
Table 2: Bias and Root mean squared error of Monte Carlo estimates of $\delta = 1 + d$, when $k = 0$. When $d = -0.4$ and $d = -0.3$, then $\delta = 0.6$ and $\delta = 0.7$, respectively. $\hat{\delta}$ denotes the local Whittle estimator, while $\hat{\delta}_c$ is the corrected local Whittle estimator (see (25)). The term *Bias* is referred to the relative percentage bias, defined in equation (5). The estimates are based on 1,000 samples of 2,500 daily observations from model (26)-(27) with parameter values indicated in table and discretization step set to 10 seconds. The bandwidth used in the estimation of δ is $m = T^{0.65}$.



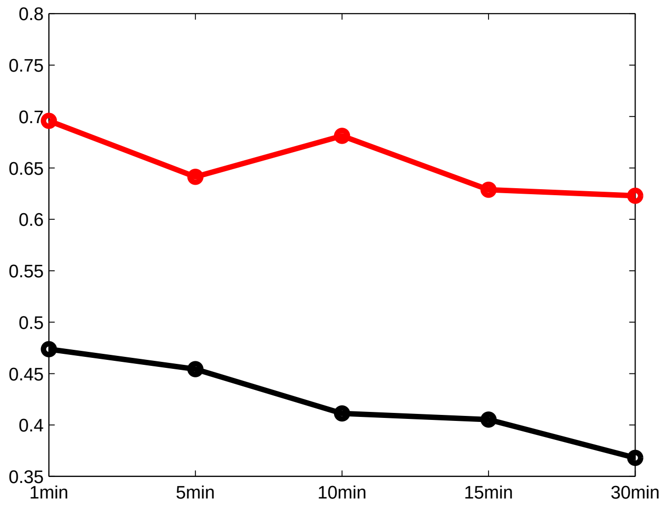
(a) Noise-to-signal ratio, $\mathrm{Var}(\xi_t^{\Delta})/\,\mathrm{Var}(IV_t)$, as a function of $\Delta \in (10sec, 30min)$, with $\gamma = 0.5$ and $\zeta = 0$. Each line corresponds to a different value of d, e.g., $d = \{0, 0.1, 0.2, 0.3, 0.4, 0.45\}$.



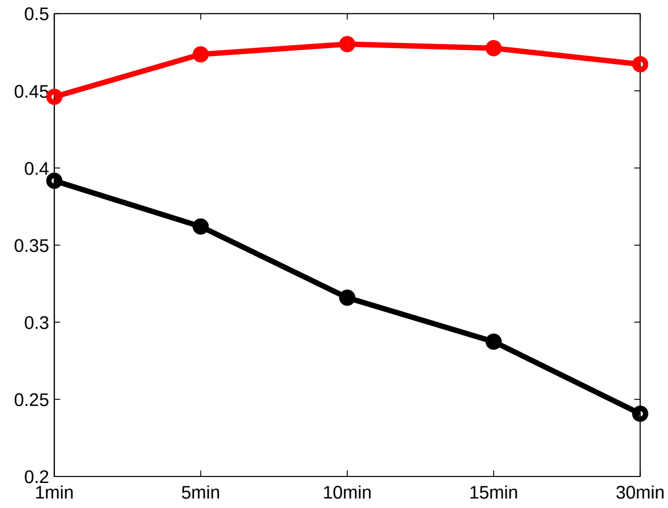
(b) Noise-to-signal ratio, $\mathrm{Var}(\xi_t^{\Delta})/\,\mathrm{Var}(IV_t)$, as a function of $\Delta \in (10sec, 30min)$, with $\zeta = 0.001$ and $d = 0.4$. Each line corresponds to a different value of γ, e.g., $\gamma = \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6\}$.



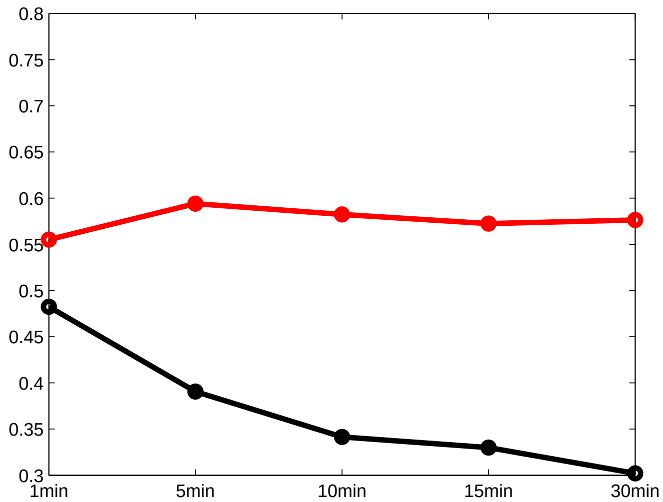
(c) Noise-to-signal ratio, $\mathrm{Var}(\xi_t^{\Delta})/\,\mathrm{Var}(IV_t)$, as function of $\Delta \in (10sec, 30min)$, with $\gamma = 0.5$ and $d = 0.4$. Each line corresponds to a different ζ, e.g., $\zeta = \{0, 0.0005, 0.001, 0.0015, 0.0020, 0.0025\}$.



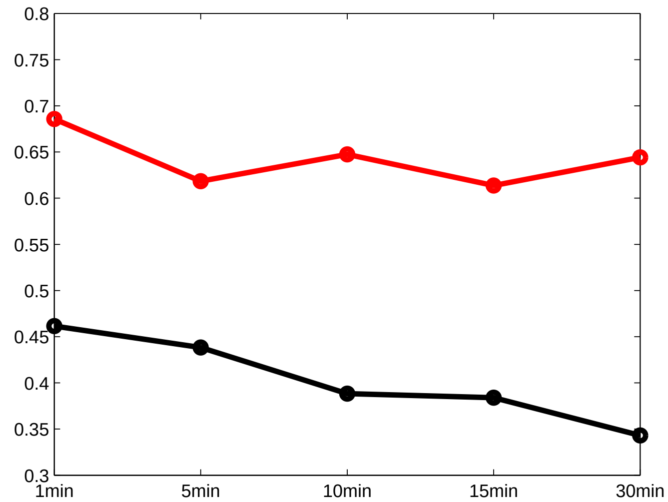
(d) RV of C



(e) RV of FDX



(f) RV of IBM



(g) RV of JPM

Figure 1: *Long memory signature plots*: Long memory parameter estimates for different sampling frequencies (1, 5, 10, 15 and 30 minutes). Black lines represent the local Whittle estimator of the memory parameter (obtained minimizing the function in (24) concentrated with respect to G with $\beta = 0$). Red lines represent the corrected local Whittle estimator (see (25)).

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A Proofs

A.1 Proof of Proposition 1

We know from Comte and Renault (1998) that $\lim_{\lambda\to 0}\lambda^{2d} f_{\tilde{\sigma}^2}(\lambda) = c \in \mathbb{R}_+$. Given that $IV = \int_0^1 \tilde{\sigma}^2(s)ds$. Following Chambers (1996) we express the integral operator in the IV definition as a simple filter that has transfer function

$$T(\lambda) = \int_0^1 e^{-i\lambda u} du = \frac{1}{(-i\lambda)}[e^{-i\lambda} - 1].$$

Therefore the spectral density of IV is given by

$$f_{IV}(\lambda) = |T(\lambda)|^2 f_{\tilde{\sigma}^2}(\lambda). \tag{30}$$

The limit of the spectral density of IV for $\lambda \to 0$ is

$$\lim_{\lambda\to 0} f_{IV}(\lambda) = \lim_{\lambda\to 0}[|T(\lambda)|^2 f_{\tilde{\sigma}^2}(\lambda)] \tag{31}$$

Since $|T(\lambda)|^2 = \frac{2(1-\cos(\lambda))}{|\lambda|^2}$ and $\lim_{\lambda\to 0}(1-\cos(\lambda)) = |\lambda|^2/2$, then $\lim_{\lambda\to 0}|T(\lambda)|^2 = 1$, thus

$$\lim_{\lambda\to 0}\lambda^{2d} f_{IV}(\lambda) = \lim_{\lambda\to 0}\lambda^{2d} f_{\tilde{\sigma}^2}(\lambda) = c \tag{32}$$

that is IV has the same degree of long memory of $\tilde{\sigma}^2(t)$.

A.2 Proof of Proposition 2

(i) $\xi_t^{\Delta} = \sum_{i=1}^{n} \xi_{t,i}^{\Delta}$, where

$$\xi_{t,i}^{\Delta} = \sigma_{t,i,\Delta}^2\left(z_{t,i}^2 - 1\right) + \eta_{t,i,\Delta}^2 + 2\left(\sigma_{t,i,\Delta} z_{t,i}\eta_{t,i,\Delta}\right) \tag{33}$$

Therefore

$$\begin{aligned} E\left(\xi_{t,i}^{\Delta}\right) &= E\left[\sigma_{t,i,\Delta}^2\left(z_{t,i}^2 - 1\right)\right] + E\left(\eta_{t,i,\Delta}^2\right) + 2E\left(\sigma_{t,i,\Delta} z_{t,i}\eta_{t,i,\Delta}\right) \\ &= E\left(\eta_{t,i,\Delta}^2\right) \\ &= \sigma_\eta^2 \end{aligned}$$

$E\left(\sigma_{t,i,\Delta} z_{t,i} \eta_{t,i,\Delta}\right) = E(\sigma_{t,i,\Delta}) \cdot E(z_{t,i}) \cdot E(\eta_{t,i,\Delta}) = 0$, so that $E\left(\sum_{i=1}^{n} \xi_{t,i}^{\Delta}\right) = n\sigma_{\eta}^2$;

(ii) The variance of $\xi_{t,i}^{\Delta}$ is,

$$\begin{aligned}
\operatorname{Var}\left(\xi_{t,i}^{\Delta}\right) &= \operatorname{Var}(u_{t,i}^{\Delta}) + \operatorname{Var}(\eta_{t,i}^2) + 4\operatorname{Var}\left(\sigma_{t,i,\Delta} z_{t,i} \eta_{t,i,\Delta}\right) \\
&= 2E\left[\sigma_{t,i}^2\right] + E(\eta_{i,t}^4) - \sigma_{\eta}^4 + 4\Delta\sigma_{\eta}^2 E\left[\sigma_{t,i}^2\right]
\end{aligned}$$

(iii) The covariance between a generic $\xi_{t,i}^{\Delta}$ and $\xi_{t,j}^{\Delta}$ can be decomposed as

$$\begin{aligned}
\operatorname{Cov}\left(\xi_{t,i}^{\Delta}, \xi_{t,j}^{\Delta}\right) &= E\left[u_{t,i}^{\Delta} u_{t,j}^{\Delta}\right] + E\left[u_{t,i}^{\Delta} \eta_{t,j}^2\right] + 2E\left[u_{t,i}^{\Delta}\left(\sigma_{t,j,\Delta} z_{t,j} \eta_{t,j,\Delta}\right)\right] \\
&\quad + E\left[\eta_{t,i}^2 u_{t,j}^{\Delta}\right] + E\left[\eta_{t,i}^2 \eta_{t,j}^2\right] + 2E\left[\eta_{t,i}^2\left(\sigma_{t,j,\Delta} z_{t,j} \eta_{t,j,\Delta}\right)\right] \\
&\quad + 2E\left[\left(\sigma_{t,i,\Delta} z_{t,i} \eta_{t,i,\Delta}\right) u_{t,j}^{\Delta}\right] + 2E\left[\left(\sigma_{t,i,\Delta} z_{t,i} \eta_{t,i,\Delta}\right) \eta_{t,j}^2\right] \\
&\quad + 4E\left[\left(\sigma_{t,i,\Delta} z_{t,i} \eta_{t,i,\Delta}\right)\left(\sigma_{t,j,\Delta} z_{t,j} \eta_{t,j,\Delta}\right)\right] - \sigma_{\eta}^4 \\
&= \sigma_{\eta}^4 - \sigma_{\eta}^4 = 0 \quad \forall i \neq j
\end{aligned}$$

so that the covariance of ξ_t^{Δ} and ξ_{t+h}^{Δ} is equal to

$$\operatorname{Cov}\left(\sum_{i=1}^{n} \xi_{t,i}^{\Delta}, \sum_{j=1}^{n} \xi_{t+h,j}^{\Delta}\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} \operatorname{Cov}\left(\xi_{t,i}^{\Delta}, \xi_{t+h,j}^{\Delta}\right) = 2n^2 \cdot 0 = 0 \quad \forall h \neq 0. \tag{34}$$

therefore, given that $\operatorname{Cov}\left(\xi_{t,i}^{\Delta}, \xi_{t+h,i}^{\Delta}\right) = 0, \forall h \neq 0$, the variance of ξ_t^{Δ} is

$$\begin{aligned}
\operatorname{Var}\left(\xi_t^{\Delta}\right) &= \operatorname{Var}\left(\sum_{i=1}^{n} \xi_{t,i}^{\Delta}\right) = \sum_{i=1}^{n} \operatorname{Var}\left(\xi_{t,i}^{\Delta}\right) \\
&= 2\Delta^{-1} E\left[\sigma_{t,i}^2\right] + \Delta^{-1} E(\eta_t^4) - n\sigma_{\eta}^4 + 4\sigma_{\eta}^2 \Delta^{-1} E\left[\sigma_{t,i}^2\right]
\end{aligned}$$

(iv) The covariance between the integrated volatility, over the period $(t-1+\Delta(i-1), t-1+\Delta i)$, and the error term is

$$\begin{aligned}
\operatorname{Cov}\left(\sigma_{t,i,\Delta}^2, \xi_{t,i}^{\Delta}\right) &= E\left[\sigma_{t,i,\Delta}^2 \cdot \left(u_{t,i}^{\Delta} + \eta_{t,i,\Delta}^2 + 2\eta_{t,i,\Delta}\sigma_{t,i,\Delta} z_{t,i}\right)\right] - E(\sigma_{t,i,\Delta}^2)\sigma_{\eta}^2 \\
&= E\left(\sigma_{t,i,\Delta}^2 u_{t,i}^{\Delta}\right) + E\left(\sigma_{t,i,\Delta}^2 \eta_{t,i}^2\right) + 2E\left(\sigma_{t,i,\Delta}^2 \eta_{t,i,\Delta}\sigma_{t,i,\Delta} z_{t,i}\right) - E(\sigma_{t,i,\Delta}^2)\sigma_{\eta}^2 \\
&= E(\sigma_{t,i,\Delta}^2)\sigma_{\eta}^2 - E(\sigma_{t,i,\Delta}^2)\sigma_{\eta}^2 = 0
\end{aligned}$$

The same holds for $\operatorname{Cov}\left(\sigma_{t,i,\Delta}^2, \xi_{t,j}^{\Delta}\right)$, for $i \neq j$, which are also equal to zero. Hence, $\operatorname{Cov}\left(IV_t, \xi_t^{\Delta}\right) = 0$.

(v) The autocovariance of realized volatility is

$$
\begin{aligned}
\operatorname{Cov}\left(RV_t^{\Delta}, RV_{t+h}^{\Delta}\right) &= \operatorname{Cov}\left(IV_t+\xi_t^{\Delta}, IV_{t+h}+\xi_{t+h}^{\Delta}\right) \\
&= E\left[\left(IV_t+\xi_t^{\Delta}\right)\left(IV_{t+h}+\xi_{t+h}^{\Delta}\right)\right]-E\left(IV_t+\xi_t^{\Delta}\right)\cdot E\left(IV_{t+h}+\xi_{t+h}^{\Delta}\right) \\
&= E\left(IV_t IV_{t+h}\right)+E\left(\xi_t^{\Delta}\xi_{t+h}^{\Delta}\right)+E\left(IV_t\xi_{t+h}^{\Delta}\right)+E\left(IV_{t+h}^{\Delta}\xi_t^{\Delta}\right) \\
&\quad -E\left(IV_t\right)E\left(IV_{t+h}\right)-E\left(IV_t\right)E\left(\xi_{t+h}^{\Delta}\right) \\
&\quad -E\left(IV_{t+h}\right)E\left(\xi_t^{\Delta}\right)-E\left(\xi_t^{\Delta}\right)E\left(\xi_{t+h}^{\Delta}\right) \\
&= \left[E\left(IV_t IV_{t+h}\right)-E\left(IV_t\right)\cdot E\left(IV_{t+h}\right)\right]+\left[E\left(\xi_t^{\Delta}\xi_{t+h}^{\Delta}\right)-E\left(\xi_t^{\Delta}\right)\cdot E\left(\xi_{t+h}^{\Delta}\right)\right] \\
&\quad +E(IV_t)\sigma_\eta^2+E(IV_{t+h})\sigma_\eta^2-E(IV_t)\sigma_\eta^2-E(IV_{t+h})\sigma_\eta^2 \\
&= \operatorname{Cov}(IV_t, IV_{t+h})+\operatorname{Cov}(\xi_t^{\Delta},\xi_{t+h}^{\Delta})=\operatorname{Cov}(IV_t, IV_{t+h}) \quad \forall h\neq 0
\end{aligned}
$$

because $\operatorname{Cov}(\xi_t^{\Delta},\xi_{t+h}^{\Delta})=0$ as seen in (34).

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