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A Bootstrap Cointegration Rank Test for Panels of VAR Models

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A BOOTSTRAP COINTEGRATION RANK TEST FOR PANELS OF VAR MODELS

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Abstract. This paper proposes a sequential procedure to determine the common cointegration rank of panels of cointegrated VARs. It shows how a panel of cointegrated VARs can be transformed in a set of independent individual models. The likelihood function of the transformed panel is the sum of the likelihood functions of the individual Cointegrated VARs (CVAR) models. A bootstrap based procedure is used to compute empirical distributions of the trace test statistics for these individual models. From these empirical distributions two panel trace test statistics are constructed. The satisfying small sample properties of these tests are documented by means of Monte Carlo. An empirical application illustrates the usefullness of this tests.

Key words: Rank test, Panel data, Cointegration, Bootstrap, Cross section dependence.

JEL classification: C12, C32, C33

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1. Introduction

This paper proposes a panel rank test statistic based on a bootstrap procedure for determination of the cointegration rank in a Panel Cointegrated Vector Autoregressivve (PCVAR) models. The cointegration rank is an important element to understand the dynamics of a system of variables. The sequential likelihood-based procedure for the determination of the cointegration rank in a system of variables which are at most I(1) (see Johansen (1995)) is frequently used in empirical research.

The trace test is a likelihood ratio test for the hypothesis that the true rank of the system (noted r_0 throughout) is equal to r (hereafter $\mathcal{H}(r)$) against the hypothesis that the system has full rank: $\mathcal{H}(p)$ in a system with p endogenous variables. The sequential procedure uses t trace test to first test $\mathcal{H}(r)$ for $r = 0$. If this is rejected r is incremented by 1 and $\mathcal{H}(1)$ is tested, and so on until it is not possible to reject or that $\mathcal{H}(p-1)$ is rejected, in which case the rank is set to p. The poor small sample performances of this procedure have been documented (see among others Reinsel and Ahn (1992) and Johansen (2002b)). Johansen (2002b) proposes a small sample Bartlett correction based on finding the expectation of the likelihood ratio test statistics to correct it. However this correction doesn't always produce sizes close to their nominal values.

Bootstrap methods are increasingly used to compute empirical test statistic distributions that are more accurate than the asymptotic distribution, thus yielding tests with small sample sizes closer to their nominal values. Swensen (2006) proposes a bootstrap algorithm to estimate an empirical distribution of the trace test statistics. He shows that this algorithm is valid only for a restricted class of data generating processes (DGP). Cavaliere, Rahbek, and Taylor (2010) (hereafter CRT) proposes a different bootstrap algorithm valid for any DGP with variables at most I(1). They obtain sizes very close to their nominal level even for samples of only 100 observations. The main aim of this paper is to extend the procedure of CRT to multivariate panel models.

Panels are increasingly used in empirical economics to analyze data set composed of many countries, regions, industries or markets. One of their advantages is that they can theoretically be used to improve inference on the parameters of a model by exploiting the information contained in several series representing the same quantities for different individuals.

Two difficulties arise when working with panels.

(1) Panels where the number of N is small relative to T can be estimated as a single model. However the number of parameters in such a model grows quadratically with the number of individuals. This is often referred to as the curse of dimensionality.

In order to estimates panels with large N and T, one has to control the number of parameters.

(2) Many economical series exhibit common patterns across individuals. When ignored in the modelling of the panel, this common patterns translate in cross section dependence of the residuals, leading to biased inference.

Breitung and Pesaran (2008) review the literature on cointegration and rank test in panels. Only a handful of procedures exist to test for multiple cointegration (i.e. a rank potentially greater than one). Larsson et al. (2001) adapts the Johansen (1995) likelihood based framework to panels VAR, using a standardized rank test statistics to obtain a normally distributed panel rank test statistic. The procedure by Larsson et al. (2001) has three major drawbacks. First, it ignores cross section dependence and consequently is potentially biased. Second, it requires the asymptotic distribution of the trace test statistic to be homogeneous across individuals. Third, it requires the two first moments of the asymptotic distribution of the trace test statistic to be simulated.

Pesaran (2006) and Kapetanios et al. (2009) propose a method to overcome the curse of dimensionality and model cross section dependence in order to obtain residuals uncorrelated through space and time. It is applied by Dees et al. (2005) to construct the Global VAR which has spawned a large litterature on multicountry macroeconomic and financial models.This method is based on the use of weighted cross section averages of the data to construct proxy for the unobserved common factors responsible for cross section dependence.

In the next section I introduce the PCVAR model and show how it can be transformed into set of independent individual models using results by Dees et al. (2005). I then introduce the bootstrap algorithm and the two panel trace test statistics, the properties of which are investigated by means of a Monte Carlo experiment. In the last section an empiricall application demonastrate the potential of this method. Table 1 summarizes the notation used throughout this paper.

2. The Model

The basic model is the panel vector autoregression. Its structure isn't restricted, it is identified by the subscript U:

$$Y_t = \sum_{l=1}^{k+1} \Psi_l Y_{t-l} + \epsilon_t \tag{1}$$

This can be reformulated in its error correction form:

$$\Delta Y_t = \Pi Y_{t-1} + \sum_{l=1}^{k} \Gamma_l \Delta Y_{t-l} + \epsilon_t \tag{2}$$

TABLE 1. Notation

Symbol	Description	Symbol	Description
N	Total number of individual	i	Individual index, from 1 to N
T	Number of observations	t	Observation index
B	Total number of bootstrap iterations	b	Iteration index for the bootstrap
p	Number of variables per individual	p^*	Number of local average variables per individual
r,r_0	Cointegration ranks	k_i	Lag order of the model for individual i
$\xrightarrow{d}$	Convergence in distribution	$\xrightarrow{p}$	Convergence in probability
$\xrightarrow{w.p.}$	Weak convergence in probability[1]	$\xrightarrow{q.m.}$	Convergence in quadratic mean
$\underset{N}{\rightarrow}$	with $N \to \infty$	$\underset{T}{\rightarrow}$	with $T \to \infty$
$\underset{N,T}{\longrightarrow}$	with $N \to \infty$ and $T \to \infty$ in no specific order	$\underset{seq(T,N)}{\longrightarrow}$	$T \to \infty$ followed by $N \to \infty$
$.^*$	Local averages	$.^\dagger$	Bootstrap variable or statistic
$\bar{.}$	Arithmetic average	$\|\|.\|\|$	quadratic norm
$\mathtt{0_p}$	A $p \times p$ matrix of zeros	$._\perp$	The ortogonal complement[2]
$\mathtt{I_p}$	The $p \times p$ identity matrix		

[1] see Hansen (1996); Giné and Zinn (1990)
[2] Defined such that for a matrix A: $A'_\perp A = 0$

The endogenous variables are stacked $Y_t := (Y'_{1,t}, \ldots, Y'_{N,t})'$ in an $[Np \times 1]$ vector. $Y_{i,t}$ is the $p \times 1$ vector of endogenous variables for unit i. The parameter matrices Γ_l and Π are $Np \times Np$. ϵ_t is a $Np \times 1$ vector of unobserved Gaussian shocks to the system, with mean 0 and variance-covariance matrix Ω_U.

$$\Omega_U = \begin{bmatrix} \Omega^U_{11} & \ldots & \Omega^U_{1N} \\ \vdots & \ddots & \vdots \\ \Omega^U_{N1} & \ldots & \Omega^U_{NN} \end{bmatrix}$$

The log-likelihood function for the model given in equation 2 under the assumption that the innovations are normaly distributed reads (see

Johansen (1995)):

$$LL_U = -\frac{T}{2}\ln\left(\det\left(\Omega_U\right)\right) + K \tag{3}$$

where K is some constant term.

Even for a system with a moderate number of individuals the number of parameters to be estimated in the model becomes quickly unmanageable for conventional sizes of T. This section will focus on showing how the model can be transformed in a set of independent individual models. Under this transformation the likelihood function of the panel is the sum of the likelihoods of the individual models.

As in Dees et al. (2005) I assume the following data generating process (DGP) for the variables of individual i:

Assumption 1. *The DGP of Y_i is given by the common factor model:*

$$Y_{i,t} = \delta_{i0} + \delta_{i1}t + \gamma_{f,i}f_t + \gamma_{0,i}Y_{0,t} + \Psi_{i,t} + \upsilon_{i,t} \tag{4}$$

Where f_t is a $m_f \times 1$ vector of common unobserved factors, with $\gamma_{f,i}$ the associated $p \times m_f$ matrix of individual loadings. $Y_{0,t}$ is a $m_0 \times 1$ vector of observed common effects with $\gamma_{0,i}$ the associated matrix of individual loadings. $\Psi_{i,t}$ is the vector of unit specific effects containing past value of $Y_{i,t}$ as well as unit specific deterministic components. $\upsilon_{i,t}$ is a vector of idiosyncratic shocks. $\upsilon_{i,t}$ has a variance covariance matrix equal to Ω_{ii}. υ_t, the stacked vector of individual idiosyncratic shocks, has covariance matrix:

$$\Omega = \begin{bmatrix} \Omega_{11} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \Omega_{NN} \end{bmatrix}$$

Under assumption 1, dependence between the units in the panel is only generated by the common factors. The shocks $\upsilon_{i,t}$ are by assumption normaly distributed and uncorrelated across individuals. In the model given by equation 2 the unobserved common factors are not accounted for. Hence there is dependence among the residual. Ω_U is in general not block diagonal. Hence, noting LL_i the log likelihood of the individual model:

$$\begin{aligned} LL_U &= -\frac{T}{2}\ln\left(\det\left(\Omega_U\right)\right) + K \\ &\neq -\frac{T}{2}\sum_{i=1}^{N}\ln\left(\det\left(\Omega_{ii}^U\right)\right) + K \\ &\neq \sum_{i=1}^{N} LL_i \end{aligned}$$

Dees et al. (2005) and Pesaran (2006) show how to construct a proxy for the unobserved common factors. Define:

$$Y_{i,t}^* = w_{i,t} Y_t = \sum_{j=1}^{N} w_{i,j,t} Y_{j,t}$$

as the local average for individual i. The local average is a weighted sum of every individual with

$$\begin{aligned} w_{i,i,t} &= 0 \\ w_{i,j,t} &\in]0,1[\quad \forall i \neq j \\ \sum_{j=1}^{N} w_{i,j,t} &= 1 \end{aligned}$$

The weights must also fulfil the atomistic condition formally defined in Lemma A.1 in Pesaran (2006). These conditions ensure that the local average is not dominated by a single individual. Using local averages on equation 4, we get:

$$Y_{i,t}^* = \delta_{i0}^* + \delta_{i1}^* t + \gamma_{f,i}^* f_t + \gamma_{0,i} Y_{0,t} + \Psi_{i,t}^* + \upsilon_{i,t}^*$$

Assuming that the $p \times m_f$ matrix of average factor loadings coefficients $\gamma_{f,i}^*$ has full column rank and $p \geq m_f$, Dees et al. (2005) shows that

$$f_t \xrightarrow[N]{q.m.} \left(\gamma_{f,i}^{\prime *} \gamma_{f,i}^*\right)^{-1} \gamma_{f,i}^* \left(Y_{i,t}^* - \delta_{i0}^* - \delta_{i1}^* t - \gamma_{0,i}^* Y_{0,t} - \Psi_{i,t}^*\right) \tag{5}$$

They also show that:

$$\Phi_i(L, k_i) \left(Y_{i,t} - \delta_{i0} - \delta_{i1} t - \gamma_{f,i} f_t - \gamma_{0,i} Y_{0,t}\right) \approx \upsilon_{i,t} \tag{6}$$

Where $\Phi_i(L, k_i)$ is a lag polynomial of order k_i. By inserting the result of equation 5 in equation 6 it follows that:

$$\Phi_i(L, k_i) \left(Y_{i,t} - \widetilde{\delta}_{i0} - \widetilde{\delta}_{i1} t - \widetilde{\gamma}_{f,i} Y_{i,t}^* - \widetilde{\gamma}_{0,i} Y_{0,t}\right) \approx \upsilon_{i,t} \tag{7}$$

Where the $\widetilde{\ }$ parameters are defined as combinations of the original individual and starred (*) parameters. Using these results, I can rewrite the model given by equation 7 as:

$$\begin{aligned} \Delta Y_{i,t} = \widetilde{\alpha}_i \widetilde{\beta}_i' \left(Y_{i,t-1}, Y_{i,t-1}^*, Y_{0,t-1}\right) &+ \Lambda_{i,0} \Delta \left(Y_{i,t}^*, Y_{0,t}\right) \\ &+ \sum_{l=1}^{k} \widetilde{\Gamma}_{i,l} \Delta \left(Y_{i,t-l}, Y_{i,t-l}^*, Y_{0,t-l}\right) + \tilde{\epsilon}_{i,t} \end{aligned} \tag{8}$$

Thus $\hat{\widetilde{\epsilon}}_{it} \to \upsilon_{i,t}$. Since by assumption $E[\upsilon_{i,t} \upsilon_{j,t}'] = 0$ for $i \neq j$, we have:

$$\mathrm{Cov}(\epsilon_{it} \epsilon_{jt}') \to 0$$

Remark Using the local averages, the full panel now reads:

$$\Delta Y_t = \widetilde{\alpha}\widetilde{\beta}'(Z_{t-1}, Y_{0,t}) + \Lambda_0 \Delta\left(Y^*_{i,t}, Y_{0,t}\right) + \sum_{l=1}^{k} \tilde{\Gamma}_l \Delta(Z_{t-l}, Y_{0,t-l}) + \tilde{\epsilon}_t \quad (9)$$

Where, by construction:

$$\alpha = \begin{bmatrix} \alpha_1 & \dots & 0_p \\ \vdots & \alpha_i & \vdots \\ 0_p & \dots & \alpha_N \end{bmatrix} \qquad \beta = [\beta_1, \cdots, \beta_N]$$

and

$$Z_{it} = \left(Y'_{i,t-1}, Y'^*_{i,t-1}\right)' = W_{i,t} Y_t$$

with

$$W_{i,t} = \begin{bmatrix} 0_p & \dots & 0_p & I_p & 0_p & \dots & 0_p \\ w_{i,1,t} & \dots & w_{i,i-1,t} & 0_p & w_{i,i+1,t} & \dots & w_{i,N,t} \end{bmatrix}$$

This formulation of the model stress the importance of the $W_{i,t}$ matrix as the link between every unit in the model.

The variance covariance matrix for the full panel is block diagonal:

$$\begin{aligned} \widehat{\widetilde{\Omega}} &= E[\hat{\tilde{\epsilon}}_t \hat{\tilde{\epsilon}}'_t] \\ &\to \begin{bmatrix} \widetilde{\Omega}_{11} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \widetilde{\Omega}_{NN} \end{bmatrix} \end{aligned} \quad (10)$$

Hence the log likelihood of the full panel given in equation 9 is:

$$LL = -\frac{T}{2} \ln \det(\hat{\widetilde{\Omega}}) + K \quad (11)$$

$$= -\frac{T}{2} \sum_{i=1}^{N} \ln \det(\hat{\widetilde{\Omega}}_{ii}) + K = \sum_{i=1}^{N} LL_i \quad (12)$$

Remark I use the property that the determinant of a block diagonal matrix is equal to the product of the determinants of its blocks.

The full panel can now be estimated by maximizing the log likelihood function of each individual model instead of maximizing the likelihood function of the full panel. The dimension of the parameter matrices of the individual models given by equation 8 are independent of N. Hence the transformed panel given in equation 9 is not subject to cross section dependence of the residuals nor to the curse of dimensionality. This is done at the price of an increase of the dimension of the the individual model with the inclusion of the weakly exogenous local averages.

3. Bootstrap Panel Rank Test

The cointegration rank of a VAR is an important element to understand the dynamics of a system of integrated variables. The sequential procedure by Johansen (1995) for determination of the rank of a system of I(1) variables in the individual VAR is used extensively in applied economic research. There are two main motivations for using a bootstrap based procedure to construct a panel cointegration rank test:

(1) Bootstrap procedures are known to potentially yield sizeable improvements in finite sample performances of statistical tests. A bootstrap procedure can be used to compute critical values of test in finite samples, thus producing tests with sizes closer to their nominal levels. The finite sample performance of the bootstrap sequential rank test procedure for a single individual CVAR reported in Cavaliere, Rahbek, and Taylor (2010) show a significant improvement over the testing procedure based on asymptotic p-values.
(2) The trace test statistic's asymptotic distribution depends on the deterministic terms in the models as well as on the number of endogenous and weakly exogenous variables. This distribution needs to be simulated for every specific model. The bootstrap procedure doesn't require simulating the asymptotic distribution of the trace test statistic for the model under investigation.

Swensen (2006) proposes an algorithm for constructing a sequential rank test using the bootstrap. As CRT shows, this procedure has a major caveat: the model used to generate the bootstrap samples is a mix of parameters estimated under the null and under the alternative hypothesis. The short run parameters are estimated under the alternative hypothesis of full rank $\mathcal{H}(p)$ while the rest of the model is estimated under $\mathcal{H}(r)$. In this case the bootstrap is in general not valid.

The procedure by CRT has one major difference with Swensen's: the short run parameters used to generate the pseudo-data are estimated under the null hypothesis $\mathcal{H}(r)$. This ensures that the DGP used is non explosive and that the bootstrap procedure is valid even when the pseudo-data is generated under an invalid rank choice.

Before stating the algorithm and the panel bootstrap rank test statistics, some assumptions on the dynamics of the model have to be made.

Assumption 2. *The innovations follow an* i.i.d. *sequence* $\{\epsilon_{it}\}$ *satisfying:*

(1) $E(\epsilon_{it})=0$
(2) $E(\epsilon_{it}\epsilon_{it}') = \Omega_i$
(3) $E||\epsilon_{it}||^4 \leq K < \infty$
(4) $E(\epsilon_{it}\epsilon_{jt}') = \mathtt{0_p}$ *for* $i \neq j$.

The 3 first points of the assumption are standard and ensure well behaved residuals. The 4^{th} is the block diagonal structure of the variance covariance matrix.

Assumption 3. *Some assumptions on the dynamics of the individual model:*

(1) *All the characteristic roots associated with the model described by equation 8 are on or outside the unit circle.*
(2) α_i *and* β_i *have full column rank* r
(3) $\det(\alpha'_{i,\perp}(I_p - \sum_l \Gamma_{i,l})\beta_{i,\perp}) \neq 0$

Assumption 4. *Distinct eigenvalues: The* r_0 *non-zero eigenvalues that solve the eigenvalue problem (see appendix A) are distinct.*

Remark To control for cross section dependence, weak exogenous variables are included in the model. To create the bootstrap sample it is necessary to condition on these variables. Harbo et al. (1998) establish that the sequential rank test procedure is consistent when conditioning on weak exogenous variables and only estimating a partial system.

I can now write the panel bootstrap rank test algorithm.

Algorithm 1. *Initialize with r=0.*

(1) *Under* $\mathcal{H}(r)$*, estimate the parameter matrices of equation 8 for each individual model by Gaussian quasi-maximum likelihood estimation. Save the resulting residuals* $\hat{\epsilon}_{r,i,t}$*. Compute the individual trace test statistic* $Q_{r,i}$ *for* $\mathcal{H}_0 : r_0 = r$ *against* $\mathcal{H}_A : r_0 = p$*.*
(2) *Recenter the residuals* $\hat{\epsilon}^c_{r,i,t} = \hat{\epsilon}_{r,i,t} - \frac{1}{T}\sum_t \hat{\epsilon}_{r,i,t}$*. Using the centered residuals construct a bootstrap matrix of residuals. This can be done by resampling or wild bootstrap. It is also possible to draw residuals from a multivariate normal distribution* $\mathcal{N}(0, \hat{\Omega}_i)$*. Define the bootstrap residuals as* $\hat{\epsilon}^\dagger_{r,i,t}$
(3) *Using the parameter matrices estimated under* $\mathcal{H}(r)$ *and the bootstrap residuals, generate for each individual B bootstrap samples using the model:*

$$\begin{aligned}\Delta X^\dagger_{i,t} &= \widehat{\alpha}^r_i \widehat{\beta}^r_i (X^\dagger_{i,t-1}, Y^*_{i,t-1}) + \widehat{\lambda}^r_{i,0} \Delta Y^*_{i,t-1} \\ &+ \sum_l \widehat{\Gamma}^r_{i,l} \Delta (X^\dagger_{i,t-l}, Y^*_{i,t-1}) + \widehat{\epsilon}^{\dagger,r}_{i,t}\end{aligned}$$

(4) *For each individual bootstrap sample compute the bootstrapped trace test statistics* $Q^{\dagger,b}_{r,i}$ *where b denotes the realization index of the bootstrap.*
(5) *For each individual compute the bootstrap p-value of the trace test statistic:*

$$p^\dagger_{i,r} = \frac{1}{B}\sum_{b=1}^{B} \mathbb{1}_{\{Q_{i,r} < Q^{\dagger,b}_{i,r}\}}$$

and compute the average of the standardized individual p-values:

$$\bar{P}_r = \frac{\sum_{i=1}^{N}\left(p_{i,r}^{\dagger} - \frac{1}{2}\right)}{\frac{\sqrt{N}}{12}}$$

If the rank tested is the true rank ($r = r_0$):

$$\bar{P}_r \xrightarrow[seq(T,N)]{d} \mathcal{N}(0,1)$$

For the proof, see appendix B. Let $p\left(\bar{P}_r\right)$ be the p-value associated to the statistic $\bar{P}_r$. If $p\left(\bar{P}_r\right) \geq \eta$ where η is the selected significance level, then the rank of the panel is set to r. Otherwise, if $r+1 < p$ restart from step 1 with $r = r+1$ or if $r+1 = p$, the rank of the panel is p.

Remark I do not describe the details of the computation of the trace test statistics. This is done using the standard method exposed, for example, in Johansen (1995) and briefly described in appendix A.

Remark The $\bar{P}_r$ is a one sided test and the associated p-value should be computed using critical values from the appropriate quantile of the left tail of a standard normal distribution.

I propose an other test statistic based on a pulling of demeaned individual trace test statistics. The algorithm is identical to algorithm 1 until step 5. Hence I will not repeat every step of the algorithm but only state the alternative ending:

Algorithm 2. *Initialize with r=0. Use steps 1 to 4 of algorithm 1.*

(5) *For each individual compute the mean of the bootstrap test statistics $\bar{Q}_{i,r}^{\dagger} = \frac{1}{B}\sum_{b=1}^{B} Q_{i,r}^{\dagger,b}$. Note:*

$$\begin{aligned} NQ_r &= \frac{1}{\sqrt{N}}\left(\sum_{i=1}^{N}\left(Q_{i,r} - \bar{Q}_{i,r}^{\dagger}\right)\right) \\ NQ_r^{\dagger,b} &= \frac{1}{\sqrt{N}}\left(\sum_{i=1}^{N}\left(Q_{i,r}^{\dagger,b} - \bar{Q}_{i,r}^{\dagger}\right)\right) \end{aligned}$$

NQ_r and $NQ_r^{\dagger,b}$ both follow the same normal distribution with mean zero and unknown, but identical, variance. For the proof, see appendix B.

Let $p\left(NQ_r\right)$ be the p-value associated to the statistic NQ_r.

$$p\left(NQ_r\right) = \frac{1}{B}\sum_{b=1}^{B}\mathbf{1}_{\{NQ_r < NQ_r^{\dagger,b}\}}$$

If $p(NQ_r) \geq \eta$ where η is the selected significance level, then the rank of the panel is set to r. Otherwise, if $r+1 < p$ restart from step 1 with $r = r+1$ or if $r+1 = p$, the rank of the panel is p.

Theorem 3.1. *Consistency of the $\bar{P}_r$ and NQ_r statistics: let $\hat{r}$ be the rank selected by algorithm 1 or 2. Then under assumptions 2, 3 and 4:*

$$\begin{aligned} \lim_{T\to\infty} P(\hat{r} = r) &= 0, \forall r < r_0 \\ \lim_{T\to\infty} P(\hat{r} = r_0) &= \begin{cases} 1-\eta \text{ if } r_0 < p \\ 1 \text{ if } r_0 = p \end{cases} \\ \lim_{T\to\infty} \sup_{r\in\{r_0+1,\cdots,p\}} P(\hat{r} = r) &\leq \eta \text{ for } r > r_0 \end{aligned}$$

For the proof see appendix B.

The $\bar{P}_r$ panel test statistic proposed in this paper is based on pooled p-values associated to the individual bootstrap trace test statistics. When the deterministic structure is different for individual i and j, the asymptotic distributions of the trace test statistics are different: $tr(Q_{r,\infty,i}) \neq tr(Q_{r,\infty,j})$. Thus pooling the test statistics is impossible thus the NQ_r isn't valid in this case. The p-values generated by the bootstrap sequential procedure of CRT converges to the uniform distribution when $r = r_0$ and to 0 when $r < r_0$ regardless of the asymptotic distribution of the underlying individual trace test statistic. Thus the p-values can be pooled to generate a panel test statistic. In general this test statistic allows for a greater degree of heterogeneity across individual models than the normalized test statistic discussed previously.

4. Simulation Results

Swensen (2006) and Cavaliere, Rahbek, and Taylor (2010) use a similar DGP to evaluate the size properties of their sequential bootstrap rank test procedure. This DGP is also used by Johansen (2002a) to estimate the small sample performances of the Bartlett corrected rank test. I will use this DGP as the benchmark case for this simulation study and progressively introduce modifications to this benchmark. The model in VECM form is:

$$\Delta Y_{i,t} = \alpha_i \beta_i' Y_{i,t-1} + \epsilon_{i,t}$$

where $\alpha_i' = [-0.4, -0.4, 0, 0, 0]$ and $\beta_i' = [1, 0, 0, 0, 0]$. $\epsilon_{i,t} \sim \mathcal{N}(0, \Omega_i)$ where $\Omega_i = \mathtt{I}_5$. This DGP (noted DGP1 henceforth) is very simple, with no lagged first differences, only one cointegration vector an no correlation between the innovations. This DGP, and the following ones, are estimated with a restricted constant. The simulation results are reported in table 2.

DGP2 is a small modification of the first one. The number of endogenous variables is reduced to 3, with $\alpha_i' = [-0.4, -0.4, 0]$ and $\beta_i' = [1, 0, 0]$. The innovations are still uncorrelated within the individual model and across them. Local averages are constructed with uniform weights ($w_{ijt} = \frac{1}{N-1} \quad \forall i, j, t$) and are included as weak exogenous variable. No common factor is included. Thus the rank of the resulting system is still 1. The system has now 3 endogenous variables and 3 weakly exogenous variables. Results are reported in table 3

DGP3a introduces two modification from DGP 2. A I(1) common factor is included in the DGP. The common factor is generated by adding the same sequence of $\mathcal{N}(0, 1)$ distributed innovations to every Y_{it}. This results in the introduction of a random walk in the DGP common to every individual. The model becomes:

$$\Delta Y_{i,t} = \alpha_i \beta_i' Y_{i,t-1} + \lambda_i F_t + \epsilon_{i,t}$$

Where $\alpha_i' = [-0.4, -0.4, 0.4]$ and $\beta_i' = [1, 1, -1]$. The common factor only impact the first variable: $\lambda_i' = [0.5, 0, 0]$. In the estimated DGP, local averages based on uniform weights are introduced to proxy for this common factor. Since the endogenous variables and the local averages share a common stochastic trend, the rank of this system is 2. Simulation results are reported in table 4.

DGP3b is identical except for the loadings of the common factor which becomes: $\lambda_i' = [1, 0, 0]$. Simulation results are reported in table 5.

It appears that the individual rank test performs similarly, and quite poorly, for DGP3a and DGP3b. The panel rank test outperforms the individual one in both cases, getting close to the nominal size when N and T become large. The panel rank test seems to perform better when the loading of the common factor is stronger. The $\bar{P}_r$ performs very well in samples with small T but gains relatively little power from the extension of the sample length.

TABLE 2. Bootstrap[1] procedure for selecting individual and panel rank.

DGP1, true rank is 1[2]

Individual rank test[3]							
N	T	$r=0$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$
5	100	0.1180	**0.8140**	0.0500	0.0080	0.0080	0.0020
5	200	0.0000	**0.9380**	0.0500	0.0040	0.0000	0.0080
5	500	0.0000	**0.9440**	0.0480	0.0020	0.0020	0.0040
10	100	0.1080	**0.8500**	0.0340	0.0040	0.0010	0.0030
10	200	0.0000	**0.9460**	0.0430	0.0060	0.0010	0.0040
10	500	0.0000	**0.9480**	0.0360	0.0100	0.0010	0.0050
20	100	0.1325	**0.8085**	0.0465	0.0075	0.0030	0.0020
20	200	0.0000	**0.9505**	0.0365	0.0085	0.0015	0.0030
20	500	0.0000	**0.9385**	0.0530	0.0045	0.0020	0.0020
50	100	0.1246	**0.8312**	0.0338	0.0066	0.0024	0.0014
50	200	0.0000	**0.9444**	0.0456	0.0058	0.0020	0.0022
50	500	0.0000	**0.9456**	0.0410	0.0082	0.0024	0.0028
Panel rank test[3]							
N	T	$r=0$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$
5	100	0.0000	**0.9100**	0.0900	0.0000	0.0000	0.0000
5	200	0.0000	**0.9300**	0.0700	0.0000	0.0000	0.0000
5	500	0.0000	**0.9500**	0.0500	0.0000	0.0000	0.0000
10	100	0.0000	**0.9900**	0.0100	0.0000	0.0000	0.0000
10	200	0.0000	**0.9500**	0.0500	0.0000	0.0000	0.0000
10	500	0.0000	**0.9800**	0.0200	0.0000	0.0000	0.0000
20	100	0.0000	**0.9600**	0.0400	0.0000	0.0000	0.0000
20	200	0.0000	**0.9500**	0.0500	0.0000	0.0000	0.0000
20	500	0.0000	**0.9100**	0.0900	0.0000	0.0000	0.0000
50	100	0.0000	**0.9900**	0.0100	0.0000	0.0000	0.0000
50	200	0.0000	**0.9700**	0.0300	0.0000	0.0000	0.0000
50	500	0.0000	**0.9300**	0.0700	0.0000	0.0000	0.0000
P_r panel rank test[3]							
N	T	$r=0$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$
5	100	0.0000	**0.6100**	0.3300	0.0500	0.0000	0.0100
5	200	0.0000	**0.6400**	0.3200	0.0200	0.0100	0.0100
5	500	0.0000	**0.7200**	0.2600	0.0100	0.0000	0.0100
10	100	0.0000	**0.7300**	0.2700	0.0000	0.0000	0.0000
10	200	0.0000	**0.6800**	0.3100	0.0100	0.0000	0.0000
10	500	0.0000	**0.6800**	0.3200	0.0000	0.0000	0.0000
20	100	0.0000	**0.6700**	0.3300	0.0000	0.0000	0.0000
20	200	0.0000	**0.6500**	0.3500	0.0000	0.0000	0.0000
20	500	0.0000	**0.6700**	0.3300	0.0000	0.0000	0.0000
50	100	0.0000	**0.7200**	0.2800	0.0000	0.0000	0.0000
50	200	0.0000	**0.6100**	0.3900	0.0000	0.0000	0.0000
50	500	0.0000	**0.7100**	0.2900	0.0000	0.0000	0.0000

[1] Results based on 199 bootstrap and 100 Monte Carlo replications.
[2] Bootstrap residuals constructed by resampling of the centered residuals.
[3] Frequency of rank selection.

TABLE 3. Bootstrap[1] procedure for selecting individual and panel rank.

DGP2, true rank is 1[2]

Individual rank test[3]					
N	T	$r=0$	$r=1$	$r=2$	$r=3$
5	100	0.0000	**0.9440**	0.0400	0.0160
5	200	0.0000	**0.9580**	0.0400	0.0020
5	500	0.0000	**0.9260**	0.0560	0.0180
10	100	0.0000	**0.9370**	0.0410	0.0220
10	200	0.0000	**0.9350**	0.0510	0.0140
10	500	0.0000	**0.9330**	0.0520	0.0150
20	100	0.0000	**0.9510**	0.0350	0.0140
20	200	0.0000	**0.9430**	0.0440	0.0130
20	500	0.0000	**0.9440**	0.0430	0.0130
50	100	0.0000	**0.9486**	0.0388	0.0126
50	200	0.0000	**0.9510**	0.0374	0.0116
50	500	0.0000	**0.9464**	0.0396	0.0140
Panel rank test[3]					
N	T	$r=0$	$r=1$	$r=2$	$r=3$
5	100	0.0000	**0.9600**	0.0400	0.0000
5	200	0.0000	**0.9600**	0.0400	0.0000
5	500	0.0000	**0.9300**	0.0500	0.0200
10	100	0.0000	**0.9300**	0.0700	0.0000
10	200	0.0000	**0.9100**	0.0900	0.0000
10	500	0.0000	**0.9600**	0.0400	0.0000
20	100	0.0000	**0.9500**	0.0500	0.0000
20	200	0.0000	**0.9500**	0.0400	0.0100
20	500	0.0000	**0.9500**	0.0500	0.0000
50	100	0.0000	**0.9500**	0.0500	0.0000
50	200	0.0000	**0.9500**	0.0500	0.0000
50	500	0.0000	**0.9600**	0.0400	0.0000
P_r panel rank test[3]					
N	T	$r=0$	$r=1$	$r=2$	$r=3$
5	100	0.0000	**0.7600**	0.1200	0.1200
5	200	0.0000	**0.5600**	0.3000	0.1400
5	500	0.0000	**0.6100**	0.2700	0.1200
10	100	0.0000	**0.6900**	0.2000	0.1100
10	200	0.0000	**0.6600**	0.2100	0.1300
10	500	0.0000	**0.7000**	0.2200	0.0800
20	100	0.0000	**0.6100**	0.3500	0.0400
20	200	0.0000	**0.7200**	0.1900	0.0900
20	500	0.0000	**0.6200**	0.3300	0.0500
50	100	0.0000	**0.6900**	0.2700	0.0400
50	200	0.0000	**0.6700**	0.3100	0.0200
50	500	0.0000	**0.6700**	0.3000	0.0300

[1] Results based on 199 bootstrap and 100 Monte Carlo replications.
[2] Bootstrap residuals constructed by resampling of the centered residuals.
[3] Frequency of rank selection.

TABLE 4. Bootstrap[1] procedure for selecting individual and panel rank.

DGP3a, $\lambda_i' = [0.5, 0, 0]$, true rank is 2[2]

		Individual rank test[3]			
N	T	$r = 0$	$r = 1$	$r = 2$	$r = 3$
5	100	0.0000	0.9360	**0.0460**	0.0180
5	200	0.0000	0.9160	**0.0640**	0.0200
5	500	0.0000	0.7980	**0.1600**	0.0420
10	100	0.0000	0.9150	**0.0680**	0.0170
10	200	0.0000	0.8900	**0.0850**	0.0250
10	500	0.0000	0.8050	**0.1700**	0.0250
20	100	0.0000	0.9340	**0.0470**	0.0190
20	200	0.0000	0.9070	**0.0675**	0.0255
20	500	0.0000	0.8285	**0.1460**	0.0255
50	100	0.0000	0.9342	**0.0500**	0.0158
50	200	0.0000	0.9282	**0.0512**	0.0206
50	500	0.0000	0.8862	**0.0904**	0.0234
		Panel rank test[3]			
N	T	$r = 0$	$r = 1$	$r = 2$	$r = 3$
5	100	0.0000	0.9300	**0.0700**	0.0000
5	200	0.0000	0.8900	**0.1100**	0.0000
5	500	0.0000	0.5000	**0.5000**	0.0000
10	100	0.0000	0.8900	**0.1100**	0.0000
10	200	0.0000	0.8100	**0.1900**	0.0000
10	500	0.0000	0.2900	**0.7000**	0.0100
20	100	0.0000	0.8800	**0.1200**	0.0000
20	200	0.0000	0.7000	**0.3000**	0.0000
20	500	0.0000	0.1600	**0.8400**	0.0000
50	100	0.0000	0.9300	**0.0700**	0.0000
50	200	0.0000	0.8000	**0.2000**	0.0000
50	500	0.0000	0.2400	**0.7600**	0.0000
		P_r panel rank test[3]			
N	T	$r = 0$	$r = 1$	$r = 2$	$r = 3$
5	100	0.0000	0.5400	**0.3700**	0.0900
5	200	0.0000	0.4000	**0.5100**	0.0900
5	500	0.0000	0.1100	**0.7000**	0.1900
10	100	0.0000	0.5900	**0.3400**	0.0700
10	200	0.0000	0.3400	**0.5700**	0.0900
10	500	0.0000	0.0900	**0.7400**	0.1700
20	100	0.0000	0.6500	**0.3200**	0.0300
20	200	0.0000	0.3100	**0.6400**	0.0500
20	500	0.0000	0.0000	**0.9400**	0.0600
50	100	0.0000	0.5100	**0.4900**	0.0000
50	200	0.0000	0.3400	**0.6600**	0.0000
50	500	0.0000	0.0100	**0.9700**	0.0200

[1] Results based on 199 bootstrap and 100 Monte Carlo replications.
[2] Bootstrap residuals constructed by resampling of the centered residuals.
[3] Frequency of rank selection.

TABLE 5. Bootstrap[1] procedure for selecting individual and panel rank.

DGP3b, $\lambda_i' = [1, 0, 0]$, true rank is 2[2]					
Individual rank test[3]					
N	T	$r = 0$	$r = 1$	$r = 2$	$r = 3$
5	100	0.0000	0.85600	**0.10800**	0.036000
5	200	0.0000	0.75400	**0.21600**	0.030000
5	500	0.0000	0.39400	**0.56000**	0.046000
10	100	0.0000	0.89900	**0.080000**	0.021000
10	200	0.0000	0.84000	**0.13200**	0.028000
10	500	0.0000	0.55600	**0.40500**	0.039000
20	100	0.0000	0.92500	**0.056500**	0.018500
20	200	0.0000	0.88800	**0.091000**	0.021000
20	500	0.0000	0.74100	**0.21900**	0.040000
50	100	0.0000	0.93780	**0.046600**	0.015600
50	200	0.0000	0.92140	**0.059200**	0.019400
50	500	0.0000	0.87140	**0.10060**	0.028000
Panel rank test[3]					
N	T	$r = 0$	$r = 1$	$r = 2$	$r = 3$
5	100	0.0000	0.72000	**0.26000**	0.020000
5	200	0.0000	0.39000	**0.59000**	0.020000
5	500	0.0000	0.0000	**0.99000**	0.010000
10	100	0.0000	0.75000	**0.25000**	0.0000
10	200	0.0000	0.42000	**0.57000**	0.010000
10	500	0.0000	0.0000	**0.98000**	0.020000
20	100	0.0000	0.86000	**0.14000**	0.0000
20	200	0.0000	0.47000	**0.53000**	0.0000
20	500	0.0000	0.010000	**0.99000**	0.0000
50	100	0.0000	0.89000	**0.11000**	0.0000
50	200	0.0000	0.77000	**0.23000**	0.0000
50	500	0.0000	0.080000	**0.92000**	0.0000
P_r panel rank test[3]					
N	T	$r = 0$	$r = 1$	$r = 2$	$r = 3$
5	100	0.0000	0.24000	**0.62000**	0.14000
5	200	0.0000	0.11000	**0.81000**	0.080000
5	500	0.0000	0.0000	**0.85000**	0.15000
10	100	0.0000	0.35000	**0.59000**	0.060000
10	200	0.0000	0.10000	**0.82000**	0.080000
10	500	0.0000	0.0000	**0.89000**	0.11000
20	100	0.0000	0.40000	**0.58000**	0.020000
20	200	0.0000	0.11000	**0.85000**	0.040000
20	500	0.0000	0.0000	**0.95000**	0.050000
50	100	0.0000	0.46000	**0.54000**	0.0000
50	200	0.0000	0.30000	**0.70000**	0.0000
50	500	0.0000	0.010000	**0.98000**	0.010000

[1] Results based on 199 bootstrap and 100 Monte Carlo replications.
[2] Bootstrap residuals constructed by resampling of the centered residuals.
[3] Frequency of rank selection.

5. Application

To illustrate a possible use of this test, I propose a small empirical application. I test the cointegration rank between the log US Dollar nominal exchange rate ($XR_{i,t}$) and log prices ($CPI_{i,t}$) for a set of 9 currencies. I start by constructing local averages of the variables using weights based on *ppp* GDP. The resultsing variables are noted $XR^*_{i,t}$ and $CPI^*_{i,t}$. $XR^*_{i,t}$ is the weighted average of the exchanges rates between every country but i and the US Dollar. By a simple transformation of the data, I define:

$$EXR_{i,t} = XR_{i,t} - XR^*_{i,t} \tag{13}$$

which is the effective exchange rate (the weighted average of bilateral exchange rate) for individual i. Thus the model estimated is:

$$\Delta \begin{bmatrix} EXR_{i,t} \\ CPI_{i,t} \end{bmatrix} = \Pi_i \begin{bmatrix} EXR_{i,t-1} \\ CPI_{i,t-1} \\ CPI^*_{i,t-1} \\ D_{t-1} \end{bmatrix} + \lambda_{0,i} CPI^*_{i,t} + \Gamma_{i,1} \Delta \begin{bmatrix} EXR_{i,t-1} \\ CPI_{i,t-1} \\ CPI^*_{i,t-1} \end{bmatrix}$$

The VAR order is 2. The model is estimated with a restricted constant noted D_t. All the results below are obtained with 499 bootstrap replications.

Table 6. Individual bootstrap rank test statistics

	rank	trace 0	trace 1	share 0	share 1
Canada	0	12.37	1.40	0.27	0.49
Denmark	0	9.17	0.78	0.63	0.93
Japan	1	24.78	0.48	0.02	0.86
Mexico	1	38.67	2.47	0.00	0.47
Sweden	0	9.51	1.58	0.69	0.68
Switzerland	1	37.03	9.22	0.01	0.17
United Kingdom	2	36.97	9.32	0.00	0.02
United States	1	25.92	4.20	0.01	0.05
Euro Zone	0	11.48	0.67	0.24	0.76

Table 6 reports the results of the individual bootstrap rank test. In four cases the individual test cannot reject the hypothesis of no cointegration (*i.e.* the rank is 0). In four cases the selected rank is 1, and in one case the selected rank is 2. The panel bootstrap test statistics reported in table 7 both select a rank of 1 at any conventional significance level, in accordance with the economic theory predicting a stationary relation between nominal exchange rates and prices (*ppp*).

I then extend the sample with a measure of interest rates, IR. The system has now three endogenous variables and 2 weakly exogenous

TABLE 7. Panel bootstrap rank test statistics

	$r = 0$	$r = 1$
NQ_r	297.54	-5.96
$p(NQ_r)$	0.00	0.562
$\bar{P}_r$	-10.504	-0.224
$p\left(\bar{P}_r\right)$	0.000	0.411

ones, the vector of variables $Z_{t,i}$ becomes:

$$Z_{i,t} = \left[EXR_{i,t}, CPI_{i,t}, IR_{i,t}, CPI^*_{i,t}, IR^*_{i,t}, D_t\right]$$

TABLE 8. Individual bootstrap rank test statistics

	rank	trace 0	trace 1	trace 2	share 0	share 1	share 2
Canada	0	33.68	13.35	0.83	0.17	0.23	0.83
Denmark	0	45.67	17.02	1.31	0.07	0.31	0.89
Japan	0	34.26	8.42	0.27	0.19	0.70	0.98
Mexico	1	49.12	12.57	1.41	0.02	0.59	0.84
Sweden	0	39.51	17.99	5.65	0.24	0.51	0.27
Switzerland	1	67.93	23.31	5.67	0.01	0.13	0.25
United Kingdom	1	50.97	19.08	2.82	0.05	0.12	0.38
United States	2	57.08	27.72	3.84	0.00	0.01	0.07
Euro Zone	0	32.23	14.99	1.77	0.14	0.12	0.64

TABLE 9. Panel bootstrap rank test statistics

	$r = 0$	$r = 1$	$r = 2$
NQ_r	409.95	67.552	3.576
$p(NQ_r)$	0.000	0.092	0.368
$\bar{P}_r$	-14.448	-6.688	2.720
$p\left(\bar{P}_r\right)$	0.000	0.000	0.996

For the larger system, the results of the individual bootstrap test are reported in table 8. The panel bootstrap test statistics results are reported in table 9. The $\bar{P}_r$ test selects a rank of 2, consistent with the theory predicting a stationary relation between nominal exchange rate and prices (*ppp*), and another one between the exchange rate and the interest rate (Interest rate parity). The NQ_r returns a rank of 1 at a 5% significance level and a rank of 2 at 10%.

The results are robust to different measures of the interest rate (3-month, 10-years), of the prices level (CPI, CPI minus food and energy), the use of different weighting matrices (*ppp*-GDP, equal weights) and changes in the selected sample.

6. Conclusion

The aim of this paper was to adapt the bootstrap sequential rank test procedure by Cavaliere, Rahbek, and Taylor (2010) to panels of cointegrated VARs. The major difficulties when working with panels are to control the cross section dependence and overcome the curse of dimenstionality. Using results from Dees et al. (2005) I have shown that these two difficulties can be circumvented by transforming the PC-VAR in a set of independent individual models. I then introduce two bootstrap test statistics, the NQ_r and $\bar{P}_r$ statistics. The $\bar{P}_r$ statistics allows a greater amount of heterogeneity across models as it does not impose homogeneity of the individual asympotic trace test distribution. Their finite sample performances are assessed using Monte Carlo experiments. Both tests show satisfactory finite sample performances which decreases when the complexity of the models increase for a given T and N.

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Appendix A. Likelihood-based inference in CVAR

The purpose of this appendix is to recall the main elements of the likelihood framework for inference in the single unit cointegrated VAR. It is also used to define some notations used throughout the body of this paper. It draws heavily on Juselius (2006)

Note the VAR model in its error correction form:

$$\Delta X_t = \Pi X_{t-1} + \sum_{l=1}^{k} \Gamma \Delta X_{t-l} + \Phi D_t + \epsilon_t \tag{14}$$

Where X_t is a $p \times 1$ vector of I(1) or I(0) endogenous variables. Π is the so-called level matrix and D_t is a vector of deterministic components. $\Pi = \alpha\beta'$ where α and β are $p \times r$ matrices.
Estimation. Define:

$$\begin{aligned} Z_{0t} &= \Delta X_t \\ Z_{1t} &= X_{t-1} \\ Z_{2t} &= [\Delta X_{t-1}, \cdots, \Delta X_{t-k}, D_t] \end{aligned}$$

Model 14 can be written in a more compact form as:

$$Z_{0t} = \alpha\beta' Z_{1t} + \Psi Z_{2t} + \epsilon_t$$

Using the Frisch-Waugh theorem we define the two auxiliary regressions:

$$\begin{aligned} Z_{0t} &= \hat{B}_0 Z_{2t} + R_{0t} \\ Z_{1t} &= \hat{B}_1 Z_{2t} + R_{1t} \end{aligned}$$

and get:

$$R_{0t} = \alpha\beta' R_{1t} + \epsilon_t$$

with $\epsilon \sim \mathcal{N}(0, \Omega)$. It can be shown that the log likelihood function of a VAR is:

$$\ln L_{max} = -\frac{T}{2} \ln |\Omega| + \text{constant terms}$$

Define $S_{ij} = T^{-1} \sum_t R_{it} R'_{jt}$ It can be shown that

$$|\hat{\Omega}(\hat{\beta})| = |S_{00}| \prod_{i=1}^{p} (1 - \lambda_i)$$

where λ_i are the p eigenvalues, ordered such that $\hat{\lambda_1}, \cdots, \hat{\lambda_p} \geq 0$, solution of the eigenvalue problem:

$$|(1 - \rho) S_{11} - S_{10} S_{00}^{-1} S_{01}| = 0$$

The p normalized eigenvectors corresponding to the eigenvalues will be noted β_i.

Deterministics. Deterministics in the cointegration space and its orthogonal space:

TABLE 10. Deterministics

Case	Name	ΦD_t [1]	$R_{1,t}$	$Z_{2,t}$
1	No deterministics	0	-	-
2	Restricted Constant	$\alpha\beta^0$	1	-
3	Restricted Trend	$\alpha\beta^1 t$	t	1

[1] β^0 and β^1 Are the loadings on the restricted constant and trend.

Rank testing. The rank r of a cointergated VAR gives the number of stationary linear combination between the variables (cointegration relations). p-r is then the number of unit roots in the system. Note $\mathcal{H}_p$ the hypothesis that rank=p, i.e. there are no unit roots in the system; x_t is stationary. Note $\mathcal{H}_r$ the hypothesis rank=r.

The trace test is found as:

$$-2\ln \mathcal{Q}(\mathcal{H}_r/\mathcal{H}_p) = T\ln\left(\prod_{j=r+1}^{p}(1-\hat{\lambda}_j)\right)$$

Which is a test for rank=r against rank=p. Therefore if $\mathcal{H}_r$ cannot be rejected we conclude that there are at least p-r unit roots in the model. It follows that the rank can be consistently estimated by using a sequential testing procedure. First, test $\mathcal{H}_0$ against $\mathcal{H}_p$. If this cannot be rejected than the rank is set to zero. Otherwise test $\mathcal{H}_1$ against $\mathcal{H}_p$. Again if this cannot be rejected the rank is set to 1, otherwise continue with $\mathcal{H}_2$. If $\mathcal{H}_{p-1}$ is rejected the rank is then set ot p.

Appendix B. Proofs

Before proving theorem 3.1, some results have to be recalled.

Lemma B.1. *Consistency if the asymptotic trace test. Johansen (1995) proves some results on the convergence of the trace test distribution:*

$$\begin{cases} Q_{r,i} \underset{T}{\rightarrow} \infty & \text{for } r < r_0 \\ Q_{r_0,i} \underset{T}{\overset{w}{\rightarrow}} tr\left(Q_{r_0,i,\infty}\right) \end{cases}$$

Where $tr\left(Q_{r_0,i,\infty}\right)$ is the asymptotic distribution of the trace test statistic for $r = r_0$. This implies for the p-values associated to the trace test statistics:

$$\begin{aligned} p_{r,i} &\underset{T}{\overset{p}{\rightarrow}} 0 \text{ for } r < r_0 \\ p_{r_0,i} &\underset{T}{\overset{w}{\rightarrow}} \mathcal{U}[0,1] \end{aligned}$$

Lemma B.2. *Consistency of the bootstrap sequential algorithm for the individual CVAR: Let $\hat{r}$ be the estimated rank selected by the bootstrap algorithm in CRT. Then*

$$\begin{aligned} Q_{r,i}^{\dagger,b} &\underset{T}{\overset{w.p.}{\rightarrow}} tr(Q_{r,i,\infty}) \text{ for } r \leq r_0 \\ p_{r,i}^{\dagger} &\underset{T}{\overset{p}{\rightarrow}} 0 \text{ for } r < r_0 \\ p_{r_0,i}^{\dagger} &\underset{T}{\overset{w}{\rightarrow}} U[0,1] \end{aligned}$$

Where $p_{r,i}^{\dagger}$ is the p-value associated with the bootstrap rank test of $\mathcal{H}(r)$ against $\mathcal{H}(p)$. It implies:

$$\begin{aligned} \lim_{T\to\infty} P(\hat{r} = r) &= 0, \forall r < r_0 \\ \lim_{T\to\infty} P(\hat{r} = r_0) &= \begin{cases} 1-\eta \text{ if } r_0 < p \\ 1 \text{ if } r_0 = p \end{cases} \\ \lim_{T\to\infty} \sup_{r\in\{r_0+1,\cdots,p\}} P(\hat{r} = r) &\leq \eta \end{aligned}$$

Remark This result requires assumption 2, 3 and 4 to hold and relies on lemma B.1. The proof can be found in CRT.

I can now prove theorem 3.1:

Consistency of the $\bar{P}_r$ procedure: Note $Q_{i,r}$ the trace test statistic of unit i under $\mathcal{H}(r)$ and $Q_{i,r}^{\dagger,b}$ the b^{th} iteration of the corresponding bootstrap trace statistic and $tr(Q_{r,i,\infty})$ the asymptotic distribution of the trace test statistic. Let

$$p_{i,r}^{\dagger} = \frac{1}{B}\sum_{b=1}^{B} \mathbb{1}_{\{Q_{i,r} < Q_{i,r}^{\dagger,b}\}}$$

and

$$\lim_{T\to\infty} p^{\dagger}_{i,r} = p^{\dagger}_{i,r,\infty}$$

From lemma B.2 if $r = r_0$:

$$p^{\dagger}_{i,r,\infty} \sim \mathcal{U}[0,1]$$

$p^{\dagger}_{i,r,\infty}$ are independent draws from a uniform distribution, thus:

$$\begin{aligned}\bar{P}_r &\xrightarrow[seq(T,N)]{d} \lim_{N\to\infty}\left(\lim_{T\to\infty}\frac{\sum_{i=1}^{N}\left(p^{\dagger}_{i,r}-\frac{1}{2}\right)}{\frac{\sqrt{N}}{12}}\right)\\ &\xrightarrow[seq(T,N)]{d} \lim_{N\to\infty}\left(\frac{\sum_{i=1}^{N}\left(p^{\dagger}_{i,r,\infty}-\frac{1}{2}\right)}{\frac{\sqrt{N}}{12}}\right)\sim\mathcal{N}(0,1)\end{aligned}$$

$\bar{P}_r$ tends sequentially with T and N (in that order) in distribution to a standard normal distribution when $r = r_0$. Since when $r < r_0$, $p^{\dagger}_{r,i} \xrightarrow[T]{p} 0$ the statistic $\bar{P}_r$ should be compared critical values from the left tail of the standard normal distribution.

□

Assumption 5. *$tr(Q_{r,i,\infty}) = tr(Q_{r,j,\infty})\forall i,j$: The asymptotic distribution of the trace test statistic $tr(Q_{r,i,\infty})$ is the same for every individual. This requires that the models used for each individuals are the same. The deterministics, dummies and number of weak exogenous variables must be the same for every individual for the asymptotic distribution to be uniform across individuals (Johansen (1995)).*

Consistency of the NQ_r procedure: By lemma B.2:

$$\frac{1}{B}\sum Q^{\dagger,b}_{r,i} \underset{B}{\to} E[Q_{r,i}]$$

Define:

$$NQ_r = \frac{1}{\sqrt{N}}\left(\sum_{i=1}^{N}\left(Q_{r,i}-\frac{1}{B}\sum Q^{\dagger,b}_{r,i}\right)\right) \tag{15}$$

$$NQ^{\dagger,b}_r = \frac{1}{\sqrt{N}}\left(\sum_{i=1}^{N}\left(Q^{\dagger,b}_{r,i}-\frac{1}{B}\sum Q^{\dagger,b}_{r,i}\right)\right) \tag{16}$$

Then, using lemma B.1:

$$\begin{aligned} NQ_r &\underset{N}{\rightarrow} \infty \\ NQ_r &\underset{N}{\xrightarrow{d}} \mathcal{N}(0,\omega) \\ NQ_r^{\dagger,b} &\underset{N}{\xrightarrow{d}} \mathcal{N}(0,\omega) \end{aligned}$$

Define:

$$p(NQ_r) = \frac{1}{B}\mathbb{1}_{\{NQ_r < NQ_r^{\dagger b}\}}$$

it follows from the results above that:

$$\begin{aligned} p(NQ_r) &\underset{T}{\xrightarrow{p}} 0 \quad \text{for } r < r_0 \\ p(NQ_r) &\underset{T}{\xrightarrow{d}} \mathcal{U}[0,1] \quad \text{for } r = r_0 \end{aligned}$$

□

Theorem 3.1 is proven.

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