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Almut E. D. Veraart

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Almut E. D. Veraart [*]
CREATES
School of Economics and Management
Aarhus University
DK-8000 Aarhus C
Denmark
Email: averaart@creates.au.dk

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Abstract

This paper studies the impact of jumps on volatility estimation and inference based on various realised variation measures such as realised variance, realised multipower variation and truncated realised multipower variation. We review the asymptotic theory of those realised variation measures and present a new estimator for the asymptotic 'variance' of the centered realised variance in the presence of jumps. Next, we compare the finite sample performance of the various estimators by means of detailed Monte Carlo studies where we study the impact of the jump activity, the jump size of the jumps in the price and the presence of additional independent or dependent jumps in the volatility on the finite sample performance of the various estimators. We find that the finite sample performance of realised variance, and in particular of the log–transformed realised variance, is generally good, whereas the jump–robust statistics turn out not to be as jump robust as the asymptotic theory would suggest in the presence of a highly active jump process.

In an empirical study on high frequency data from the Standard & Poor's Depository Receipt (SPY), we investigate the impact of jumps on inference on volatility by realised variance in practice.

Keywords: Realised variance; realised multipower variation; truncated realised variance; inference; stochastic volatility; jumps.
JEL codes: C10, C14, C58, G10.

1 Introduction

Volatility estimation and inference is one of the key problems in finance and has become one of the most active research areas in financial econometrics in the last decade. In particular, the availability of high frequency financial data makes it possible to construct various non–parametric volatility estimators, whose theoretical behaviour is now very well understood, see Andersen, Bollerslev, Diebold & Labys (2001), Barndorff-Nielsen & Shephard (2002, 2004a,b, 2007). Besides the question of estimating volatility, the recent literature has also addressed the problem of testing for jumps in asset price data, see Aït-Sahalia & Jacod (2009b), Barndorff-Nielsen & Shephard (2006), Huang & Tauchen (2005), Lee & Mykland (2008). In this paper, we address a closely related questions, which has not been studied empirically before: How do jumps impact inference on volatility based on various realised variation measures?

In order to answer that question, we will first focus on the most widely used and probably best understood non–parametric stochastic volatility estimator, i.e. on *realised variance*, see Andersen, Bollerslev, Diebold & Ebens (2001), Barndorff-Nielsen & Shephard (2002). Daily realised variance is defined as the sum of the squared intra–day returns. It can be shown, see Protter (2004), that this quantity is a consistent estimator for the quadratic variation of the price process, which is often regarded as a measure of accumulated daily variance. However, in order to make inference on the quadratic variation of the price process, we need a central limit result for the volatility estimator. Recently, Jacod (2008) has established a comprehensive limit theory for realised variance (among other functionals of semimartingales) when the logarithmic asset price is assumed to be an Itô semimartingale; the corresponding *feasible* limit theory has then been derived in Veraart (2010) and we refer to these two papers for details on the underlying theory.

In this paper, we present a new estimator for the jump part of the asymptotic 'variance' of the centered realised variance which is based on the threshold concept.

Next, this paper focuses on practical aspects of the new central limit theory for realised variance when the asset price exhibits jumps. First of all, we study the finite sample performance of the new theory by means of Monte Carlo studies. In particular, we will see that the feasible limit theory makes use of spot volatility estimators. Hence we compare in our simulation experiment the finite sample performance based on two widely used spot volatility estimators: locally average realised bipower variation and locally averaged truncated realised variance. Furthermore, we will see that the asymptotic normality of realised variance in the presence of jumps only holds when the asset price and the volatility do not jump together. In fact, it is now possible to test whether there are co–jumps in price and volatility, see Jacod & Todorov (2010). However, from a practical point of view one might want to ask the question, how big the estimation error is if one uses the standard inference technique even if there are common jumps in the price and the volatility. This is studied in another simulation experiment in this paper.

Since economists are often interested in inference on the continuous part of the quadratic variation, we also study the impact of jumps on realised multipower and truncated realised variation measures, which are known to be robust to jumps. We review the asymptotic theory of these quantities briefly and include them into our Monte Carlo study. While such measures turn out to be robust to rare jumps, they are severely challenged — in finite samples — by the presence of a very active jump process generating many small jumps in addition to some bigger jumps.

Next, we carry out a small empirical study based on high frequency data from the Standard & Poor's Depository Receipt (SPY) over a ten year period from 1998 to 2008. In this study, we investigate the impact of jumps on inference based on realised variance empirically. We estimate the quadratic variation of the asset price by realised variance and compute the corresponding confidence

bounds using the new feasible limit theory. The main aim is to find out how the size of the confidence bounds differs depending on whether we account for jumps or not.

The remaining part of the paper is structured as follows. Section 2 reviews the asymptotic properties of realised variance, realised multipower variation and truncated realised multipower variation in the presence of jumps. Section 3 presents consistent estimators for the asymptotic 'variances' of centered realised variation measures. Here we also present a new estimator of the jump part of the asymptotic 'variance' of centered realised variance. In Section 4, we check the finite sample performance of the asymptotic theory, where we study the impact of jumps in the price and also the impact of co–jumps in the price and in the volatility on inference for realised variance. Next, we study the impact of jumps on the size of the confidence bounds for realised variance empirically in Section 5. Finally, Section 6 concludes. All technical assumptions, proofs and tables from the simulation study are relegated to the Appendix, see Sections A, B and C, respectively.

2 Asymptotic theory for realised variation measures in the presence of jumps

Recent research on nonparametric volatility estimation has focused on the class of Itô semimartingales as a model for the logarithmic asset price. In this section we will review the basic traits of Iô semimartingales and we will present some of the most often used nonparametric volatility estimators.

The focus will be on a non–technical presentation of the underlying theory and we refer to Jacod (2008) and Veraart (2010) for the technical details.

2.1 The Itô semimartingale framework

Let $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq0}, \mathbb{P})$ denoted a filtered probability space, on which we define a real–valued Itô semimartingale $X = (X_t)_{t\geq0}$. Note that an Itô semimartingale is a semimartingale whose characteristics are absolutely continuous with respect to the Lebesgue measure. That means that it can be written as

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + J_t, \tag{1}$$

where $W = (W_t)_{t\geq0}$ is a Brownian motion and $J = (J_t)_{t\geq0}$ is a pure jump process defined by

$$J_t = \int_0^t \int_E \kappa(\delta(s,x))(\mu - \nu)(ds,dx) + \int_0^t \int_E \kappa'(\delta(s,x))\mu(ds,dx), \tag{2}$$

where μ is a Poisson random measure on $\mathbb{R}_+ \times E$ with $(E, \mathcal{E})$ being an auxiliary space on the probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq0}, \mathbb{P})$. Further, ν denotes the predictable compensator of μ and is given by $\nu(ds,dx) = ds \otimes \lambda(dx)$, where λ is a σ–finite measure on $(E, \mathcal{E})$. We denote by κ a truncation function which is bounded, has compact support and $\kappa(x) = x$ in a neighbourhood of 0 and is assumed to be continuous. Then we define $\kappa'(x) = x - \kappa(x)$. The coefficients b, σ and δ are assumed to satisfy some mild integrability conditions, see the Appendix for more details.

Let us now assume that the logarithmic asset price, denoted by $X = (X_t)_{t\geq0}$, is modelled by such an Itô semimartingale. Clearly, this is a very general modelling framework since it allows for stochastic volatility, fairly general jumps and can account for the leverage effect, i.e. the usually negative correlation between asset prices and volatility, see Black (1976), Nelson (1991), since σ and W are not assumed to be independent.

Our aim is now to focus on the *volatility* of the logarithmic asset price X given by (1). A quantity which is closely related to the volatility of the logarithmic asset price, is the *quadratic variation* of X, which is given by

$$[X]_t = \int_0^t \sigma_s^2 ds + \sum_{0 \le s \le t} (\Delta X_s)^2, \tag{3}$$

where $\Delta X_s = X_s - X_{s-}$ denotes the jump of X at time s, see Protter (2004) for more details. We see that the quadratic variation contains two parts: the continuous part given by the integrated squared volatility process σ and the jump part, which consists of the sum of the squared jumps of X. From an economic point of view, the quadratic variation can be interpreted as a measure of the accumulated instantaneous conditional variance of the continuous and the jump part of the price over a time interval $[0, t]$, where t is usually taken to be one day.

2.2 Realised variance

The key question is: How can we estimate this accumulated daily price variation, defined by (3), given high frequency financial data?

Assume that we observe the process X over an interval $[0, T]$ for $T > 0$ at times $i\Delta_n$ for $\Delta_n > 0$ and $i = 0, \dots \lfloor T/\Delta_n \rfloor$. Further, let $0 < t \le T$. Note that we will always assume that $\Delta_n \to 0$ as $n \to \infty$. Typically we have $\Delta_n = 1/n$. So for its discretely observed increments we write

$$\Delta_i^n X = X_{i\Delta_n} - X_{(i-1)\Delta_n}.$$

Now we define the *realised variance* (RV) over the interval $[0, t]$ as the sum of the squared increments, i.e.

$$RV_t^n = \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (\Delta_i^n X)^2.$$

It is well–known, see e.g. Andersen, Bollerslev, Diebold & Ebens (2001), Barndorff-Nielsen & Shephard (2002) and Protter (2004), that the *realised variance* estimates the quadratic variation of the underlying process consistently, i.e.

$$RV_t^n \xrightarrow{ucp} [X]_t, \quad \text{as } n \to \infty,$$

where the convergence is uniform on compacts in probability (ucp), see Protter (2004, p. 57). Realised variance is now a widely used 'volatility' estimator, see e.g. Huang & Tauchen (2005) and Andersen et al. (2007). However, in order to assess the quality of this estimator, one has to construct confidence intervals and, hence, one needs a central limit result for realised variance. From recent work by Jacod (2008, Theorem 2.12 (ii)), we obtain a central limit theory for realised variance in the presence of jumps.

Assume that Assumption (L-2), see Appendix, holds. Then

$$\frac{1}{\sqrt{\Delta_n}} \left(\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (\Delta_i^n X)^2 - [X]_t \right) = \frac{1}{\sqrt{\Delta_n}} \left(\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (\Delta_i^n X)^2 - \left(\int_0^t \sigma_s^2 ds + \sum_{0 \le s \le t} (\Delta X_s)^2 \right) \right)$$

$$\tag{4}$$

converges, as $\Delta_n \to 0$, finite dimensionally stably in law to a random variable which has zero mean and (conditional) asymptotic 'variance' given by

$$V_t := 2I_t + 2D_t, \qquad \text{where} \qquad I_t := \int_0^t \sigma_s^4 ds, \qquad D_t := \sum_{p:T_p \leq t} \left(\sigma_{T_p-}^2 + \sigma_{T_p}^2 \right) (\Delta X_{T_p})^2,$$

where $(T_p)_{p \geq 1}$ denote stopping times which are the enumerations of the jump times of X.

Remark Note that stable convergence in law is a stronger convergence concept than convergence in law, see Rényi (1963) and Jacod (2008) for more details. In particular, it will allow us to replace the asymptotic 'variance' V_t by a consistent estimator $\widehat{V}_t$ without having an impact on the validility of the central limit theorem, see below.

Under the additional assumption that X and σ do not have common jumps, the above result simplifies to

$$\frac{1}{\sqrt{\Delta_n}} \left(\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (\Delta_i^n X)^2 - \left(\int_0^t \sigma_s^2 ds + \sum_{0 \leq s \leq t} (\Delta X_s)^2 \right) \right) \to MN(0, V_t), \qquad (5)$$

finite dimensionally stably in law. In this case, D_t simplifies to $2 \sum_{p:T_p \leq t} \sigma_{T_p}^2 (\Delta X_{T_p})^2$, since the jump times of X are not jump times of σ and, hence, $\sigma_{T_p-}^2 = \sigma_{T_p}^2$ for $p \in \mathbb{N}$ such that $T_p \leq t$. Note that MN stands for *mixed normal*.

Remark When there are no jumps, the limit result is well–known, see e.g. Jacod (1994), Jacod & Protter (1998) and Barndorff-Nielsen & Shephard (2002).

In order to use this limit result in practice, we need a consistent estimator for the asymptotic 'variance' V_t.

Let us assume that $\widehat{V}_t^n$ denotes such a consistent estimator of V_t, then we get

$$\frac{1}{\sqrt{\Delta_n \widehat{V}_t^n}} \left(\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (\Delta_i^n X)^2 - \left(\int_0^t \sigma_s^2 ds + \sum_{0 \leq s \leq t} (\Delta X_s)^2 \right) \right) \to N(0, 1), \qquad (6)$$

where the convergence is in law.

Remark Since the central limit result (5) holds *stably* in law (and not just in law), we can replace V_t by a consistent estimator $\widehat{V}_t^n$ and obtain the *feasible* central limit result (6) without further proof.

2.3 Realised multipower variation

We have seen that the realised variance estimates the entire quadratic variation consistently. However, it is also interesting to study the continuous part and the jump part of the qudratic variation separately. In order to do that, two concepts have been proposed: realised multipower variation, see Barndorff-Nielsen & Shephard (2004*b*, 2006), and truncated realised variation, see Mancini (2001). More recently, a combination of these two concept has also been studied, see Corsi et al. (2010). Here, we will review these concepts briefly.

The concept of *realised multipower variation* (RMPV) has been introduced by Barndorff-Nielsen & Shephard (2004*b*, 2006) as a consistent estimator of the continuous part of the quadratic variation,

which we denote by $[X]_t^c = \int_0^t \sigma_s^2 ds$. Subsequently, it has also been studied by Woerner (2006), Jacod (2006).

In the following, we will denote by $\mu_x = \mathbb{E}\left(|U|^x\right) = \pi^{-1/2} 2^{x/2} \Gamma\left(\frac{x+1}{2}\right)$, for $U \sim N(0,1)$ and $x > 0$ the xth absolute moment of a standard normally distributed random variable. Further, let $I \in \mathbb{N}$ and let $\mathbf{r} = (r_1, \ldots, r_I)$ denote a multi–index where $r_i > 0$ for $i = 1, \ldots, I$. We write

$$|\mathbf{r}| = \sum_{i=1}^{I} r_i, \qquad \mathbf{r}_+ = \max_{i \in 1, \ldots, I} r_i, \qquad \mathbf{r}_- = \min_{i \in 1, \ldots, I} r_i, \qquad \mu_{\mathbf{r}} = \prod_{i=1}^{I} \mu_{r_i}.$$

We define the realised multipower variation (RMPV) by

$$RMPV(\mathbf{r})_t^n = \Delta_n^{1-|\mathbf{r}|/2} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \prod_{j=1}^{I} \left|\Delta_{i+j-1}^n X\right|^{r_j}. \tag{7}$$

From Jacod (2006), we know that under assumption (H), see Appendix, and if $\mathbf{r}_+ < 2$, then

$$RMPV(\mathbf{r})_t^n \overset{ucp}{\to} \mu_{\mathbf{r}} \int_0^t \sigma_s^{|\mathbf{r}|} ds, \qquad \text{as } \Delta_n \to 0. \tag{8}$$

The corresponding central limit result only holds under stricter conditions (mainly on the jump part of X), see Jacod (2006). Under assumption (A), see Appendix, which basically says that the jump part of X is of finite variation and the volatility process σ never vanishes, then for $\mathbf{r}_+ < 1$:

$$\frac{1}{\sqrt{\Delta_n}} \left(RMPV(\mathbf{r})_t^n - \mu_{\mathbf{r}} \int_0^t \sigma_s^{|\mathbf{r}|} ds \right) \to MN\left(0, A(\mathbf{r}) \int_0^t \sigma_s^{2|\mathbf{r}|} ds \right), \tag{9}$$

where the convergence is stably in law and

$$A(\mathbf{r}) = \prod_{i=1}^{I} \mu_{2r_i} - (2I-1) \prod_{i=1}^{I} \mu_{r_i}^2 + 2 \sum_{i=1}^{I-1} \prod_{j=1}^{i} \mu_{r_j} \prod_{j=I-i+1}^{I} \mu_{r_j} \prod_{j=1}^{I-i} \mu_{r_j+r_{j+i}}. \tag{10}$$

Remark Note that, in the presence of jumps, the above central limit theorem does not hold for the realised bipower variation, where $I = 2$ and $r_1 = r_2 = 1$. See Vetter (2010) for a detailed treatment of that case.

The choices $|\mathbf{r}| = 2$ and $|\mathbf{r}| = 4$ are of particular interest in financial econometrics, since they lead to consistent estimators of integrated variance and integrated quarticity, respectively. In many applications, we are interested in the case where $r_i = c/I$ for $c \in \mathbb{N}$ and $i = 1, \ldots, I$. In this case, $A(\mathbf{r})$ simplifies to

$$A((c/I, \ldots, c/I)) = \mu_{2c/I}^I - (2I-1)\mu_{c/I}^{2I} + 2 \sum_{i=1}^{I-1} \mu_{c/I}^{2i} \mu_{2c/I}^{I-i}.$$

If $c = 2, I = 2$, we obtain realised bipower variation and for $c = 2, I = 3$ realised tripower variation. More generally, for estimating the integrated $|\mathbf{r}|$th power of σ, we could use any multipower variation (satisfying the assumptions above) such that $c = |\mathbf{r}|/I$. However, from the formulae above we see that the efficiency of multipower variation decreases as $I \to \infty$, see also Table 1 for some examples.

The main intuition behind the concept of realised multipower variation lies in the fact that the probability of having several jumps in neighbouring intervals is very low, hence the effect of jumps is averaged out compared to realised variance, say.

c, I	$\mu_{c/I}$	$\mu_{\mathbf{r}}^{-1} = \mu_{c/I}^{-I}$	$\mu_{\mathbf{r}}^{-2} A(\mathbf{r})$
$c = 2, I = 2$	$\sqrt{2/\pi} \simeq 0.797$	$\pi/2 \simeq 1.570$	$\frac{1}{4}\pi^2 + \pi - 3 \simeq 2.608$
$c = 2, I = 3$	0.802	1.935	3.061
$c = 4, I = 3$	0.830	1.743	13.649
$c = 4, I = 4$	0.797	2.467	14.916

Table 1: Scaling and efficiency parameters for the realised multipower variation. Here we choose $\mathbf{r} = (r_1, \ldots, r_I)$, where $r_i = c/I$, for $c, I \in \mathbb{N}$, $i = 1, \ldots, I$. Then $\lim_{I \to \infty} \mu_{\mathbf{r}}^{-2} A(\mathbf{r}) = \pi^2 c^2/8$.

Remark In order to improve the finite sample performance of realised multipower, in particular when it is compared to realised variance, one often multiplies the realised multipower variation by the factor $\frac{\lfloor t/\Delta_n \rfloor}{\lfloor t/\Delta_n \rfloor - I + 1}$, which converges to 1 as $n \to \infty$. This factor makes up for the fact that realised variance consists of $\lfloor t/\Delta_n \rfloor$ terms in the sum, whereas RMPV only consists of $\lfloor t/\Delta_n \rfloor - I + 1$ terms.

Another adjusted version of RMPV, which one can often find are staggered versions, see e.g. Huang & Tauchen (2005), which make RMPV robust towards some types of market microstructure effects.

2.4 Truncated realised variance

Besides the concept of realised multipower variation there is the concept of truncated RV or RMVP. The idea of truncating RV to only use 'small' increments and, hence, get rid of jumps has been introduced by Mancini (2001), see also Mancini (2009), and has subsequently been studied in a more general framework by Jacod (2008). The main results are, see Jacod (2008, Theorem 2.4 (iii) and Theorem 2.10), that the truncated realised variance (TRV) defined by

$$TRV(\omega, \alpha)_t^n = \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (\Delta_i^n X)^2 \, \mathbb{I}_{\{|\Delta_i^n X| \le \alpha \Delta_n^\omega\}},$$

for $\alpha > 0$ and $\omega \in (0, 1/2)$ is a consistent estimator of $[X]_t^c$, provided assumption (H), see Appendix, holds. Further, under the stronger assumption (B) (on the activity index of the jumps in X), we have the following central limit theorem:

$$\frac{1}{\sqrt{\Delta_n}} \left(TRV(\omega, \alpha)_t^n - \int_0^t \sigma_s^2 ds \right) \to MN \left(0, 2 \int_0^t \sigma_s^4 ds \right), \qquad \text{as } \Delta_n \to 0.$$

Remark When we compare RMPV($\mathbf{r}$) with $|\mathbf{r}| = 2$ and TRV, we observe that both classes of estimators are consistent and asymptotically unbiased estimators for $[X]^c$. The advantage of the TRV estimator is that it is more efficient then RMPV($\mathbf{r}$). However, the nice feature of the RMPV($\mathbf{r}$) estimator lies in the fact that it does not require to choose a threshold, which can be complicated in practice.

2.5 Truncated/threshold realised multipower variation

Recently, the two concepts of multipower variation and truncation have been combined in order to get an estimator which achieves even better finite sample properties, see Corsi et al. (2010). Truncated

realised multipower variation (TRMPV) is defined by

$$TRMPV(\mathbf{r}, \alpha, \omega)_t^n = \Delta_n^{1-|\mathbf{r}|/2} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \prod_{j=1}^{I} \left| \Delta_{i+j-1}^n X \right|^{r_j} \mathbb{I}_{\{|\Delta_{i+j-1}^n X| \leq \alpha_j \Delta_n^{\omega_j}\}}, \tag{11}$$

where we use the notation as in the RMPV case and, additionally, $\alpha = (\alpha_1, \ldots, \alpha_I)$ and $\omega = (\omega_1, \ldots, \omega_I)$, where all $\alpha_i > 0$ and $\omega_i \in (0, 1/2)$ for $i = 1, \ldots, I$. Regarding the consistency, we get the same result as in (8), where RMPV is replaced by TRMPV. Under the same assumptions as above, we also obtain a central limit theorem of the form (9), where RMPV is replaced by TRMPV. In addition, the central limit theorem also holds for the case that $\mathbf{r}_+ = 1$ and, hence, includes truncated realised bipower variation.

2.6   Difference of RV and RMPV

So far, we have seen that RV can be used for making inference on $[X]$ and RMPV, TRV and TRMPV can be used for inference on $[X^c]$. In order to make inference on the jump part of the quadratic variation, we can use the difference of RV and RMPV, which has been studied in Veraart (2010).

Under assumption (C), see Appendix, we have that for $r_i = 2/I$

$$\frac{1}{\sqrt{\Delta_n}} \left(RV_t^n - \mu_{2/I}^{-I} RMPV(\mathbf{r})_t^n \right) \to MN \left(0, \theta_I I_t + 2D_t \right), \qquad \text{as } \Delta_n \to 0,$$

where $\theta_I = \mu_{2/I}^{-2I} A((2/I, \ldots, 2/I)) - 2$.

2.7   Final remarks regarding the asymptotic theory

It should be noted that the asymptotic theory for the realised measures reviewed above has been derived under the assumption that there is always a Brownian component present, i.e. the price process does not solely consist of a pure jump component. For pure jump processes, the asymptotic theory is much less developed. However, for some subclasses of pure jump processes there are some asymptotic results for realised measures, see e.g. Jacod (2004), Todorov & Tauchen (2010) for some first investigations along those lines. However, the assumption that a Brownian component is always present does not seem to be that unrealistic in the light of recent findings by Cont & Mancini (2007) and Aït-Sahalia & Jacod (2010).

3   Consistent estimators of the asymptotic variance of centered realised variation measures

In order to use the central limit results reviewed above to make inference on $[X]$, $[X^c]$ or on $[X^d]$, we need a consistent estimators for the conditional asymptotic 'variances'. In order to do that, we proceed in two steps: We review the well–known estimators for the continuous part, i.e. for I_t, and then we construct a new estimator for the jump part D_t.

3.1 Estimating the continuous part

From the previous section, we know that we can use RMPV, and in particular realised tripower variation with $r_i = 4/3$, for estimating I_t, i.e. we define

$$\hat{I}_t^n = \mu_{4/3}^{-3} RMPV((4/3, 4/3, 4/3)) = \mu_{4/3}^{-3} \Delta_n^{-1} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor - 2} |\Delta_i^n X|^{4/3} |\Delta_{i+1}^n X|^{4/3} |\Delta_{i+2}^n X|^{4/3}.$$

Then $\hat{I}_t^n$ is a consistent and unbiased estimator of the integrated quarticity even in the presence of jumps in the price process, in particular $\hat{I}_t^n \xrightarrow{ucp} I_t = \int_0^t \sigma_s^4 ds$, as $n \to \infty$. Other choices of RMPV($\mathbf{r}$) with $|\mathbf{r}| = 4$ would be possible, too, but the choice above is the most efficient one from the RMPV($\mathbf{r}$) estimators which are robust towards jumps and are of the type $r_i = 4/I$.

3.2 Estimating the jump part

Next, we propose a new estimator for the jump process $D_t = \sum_{p:T_p \leq t} \left(\sigma_{T_p-}^2 + \sigma_{T_p}^2 \right) \left(\Delta X_{T_p} \right)^2$, which is based on the truncation concept. In particular, we use a localised version of TRV for estimating the spot volatility σ^2 and we use the truncation concept again, this time using the big increments only, for approximating the jumps of D. Altogether, we get the following result.

Proposition 3.1 *Let assumption (G1), see Appendix, be satisfied. Let $(k_n)_{n \in \mathbb{N}} \subseteq \mathbb{N}$ such that $k_n \to \infty$ and $k_n \Delta_n \to 0$ as $n \to \infty$. Further, let*

$$I_{n,t}(i) = \{j \in \mathbb{N} : j \neq i : 1 \leq j \leq \lfloor t/\Delta_n \rfloor, |i - j| \leq k_n\}.$$

For $\alpha > 0$, $\omega \in \left(0, \frac{1}{2}\right)$, we define

$$\widehat{D}_t^{(1),n} := \widehat{D}_t^{(1),n}(\alpha, \omega) := \frac{1}{k_n \Delta_n} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \left(\Delta_i^n X\right)^2 \mathbb{I}_{\{|\Delta_i^n X| > \alpha \Delta_n^\omega\}} \sum_{j \in I_{n,t}(i)} \left(\Delta_j^n X\right)^2 \mathbb{I}_{\{|\Delta_j^n X| \leq \alpha \Delta_n^\omega\}}.$$

Then $\widehat{D}_t^{(1),n} \xrightarrow{\mathbb{P}} D_t$, as $n \to \infty$.

Proof The proof is given in the appendix. $\qquad \square$

An alternative estimator of D_t has recently been proposed in Veraart (2010, Theorem 2). Under assumption (D). Further, we define an index set $I^*(n, i)$ by

$$I_{n,t}^*(i) = \{j \in \mathbb{N} \setminus \{i-1, i, i-1\} : 1 \leq j \leq \lfloor t/\Delta_n \rfloor, |i - j| \leq k_n\}.$$

Then,

$$\widehat{D}_t^{(2),n} = \max \left(\sum_{i=1}^{\lceil t/\Delta_n \rceil} \left(\Delta_i^n X\right)^2 \frac{\mu_1^{-2}}{(k_n - 2)\Delta_n} \sum_{j \in I^*(n,i)} \left|\Delta_j^n X\right| \left|\Delta_{j+1}^n X\right| - 2\hat{I}_t^n, 0 \right) \tag{12}$$

is also a consistent estimator of D_t.

3.3   Putting it together

Recall that, for inference on the realised variance, we need to estimate $V_t = 2I_t + 2D_t$. Hence, we choose the following two estimators:

$$\widehat{V}_t^{(1),n} := \widehat{V}_t^{(1),n}(\alpha, \omega) = 2\hat{I}_t^n + 2\widehat{D}_t^{(1),n}(\alpha, \omega),$$
$$\widehat{V}_t^{(2),n} = 2\hat{I}_t^n + 2\widehat{D}_t^{(2),n}.$$

3.4   Relevant quantities for the simulation study

After we have reviewed the asymptotic theory for (truncated) RV and (truncated) RMPV in the presence of jumps, we study the finite sample properties of these estimators. In order to do that, we define the quantities of interest, which are asymptotically (under the appropriate assumptions) standard normally distributed.

Now we can combine the previous results and obtain a feasible central limit theory for realised variance in the presence of jumps, see (Veraart 2010, Corollary 5).

Let $t = 1$ and $\Delta_n = 1/n$ and assume that $X_0 = 0$. Then $[X]_1 = \int_0^1 \sigma_s^2 ds + \sum_{p:T_p \leq 1} (\Delta X_s)^2$. Let

$$T^{(1)}(k_n) := T^{(1)}(k_n, \alpha, \omega) = \sqrt{n} \left(\sum_{i=1}^n (\Delta_i^n X)^2 - [X]_1 \right) \left(\widehat{V}_1^{(1),n}(\alpha, \omega) \right)^{-1/2},$$

$$T^{(2)}(k_n) = \sqrt{n} \left(\sum_{i=1}^n (\Delta_i^n X)^2 - [X]_1 \right) \left(\widehat{V}_1^{(2),n} \right)^{-1/2}. \tag{13}$$

By applying the delta method for stable convergence, see Dette et al. (2006), to the logarithmic transform of the realised variance, we deduce that

$$\overline{T}^{(1)}(k_n) := \overline{T}^{(1)}(k_n, \alpha, \omega)$$
$$= \sqrt{n} \left(\log \left(\sum_{i=1}^n (\Delta_i^n X)^2 \right) - \log([X]_1) \right) \left(\widehat{V}_1^{(1),n}(\alpha, \omega) \right)^{-1/2} \sum_{i=1}^n (\Delta_i^n X)^2,$$

$$\overline{T}^{(2)}(k_n) = \sqrt{n} \left(\log \left(\sum_{i=1}^n (\Delta_i^n X)^2 \right) - \log([X]_1) \right) \left(\widehat{V}_1^{(2),n} \right)^{-1/2} \sum_{i=1}^n (\Delta_i^n X)^2, \tag{14}$$

are asymptotically standard normally distributed.

In the following simulation study, we will study the finite sample performance of these statistics. Furthermore, we can now construct asymptotic confidence intervals for the quadratic variation of the log–price process. The corresponding upper and lower confidence bounds are hence given by

$$\sum_{i=1}^n (\Delta_i^n X)^2 \pm q_{1-\frac{\alpha}{2}} \sqrt{n^{-1}\widehat{V}_1^{(1),n}(\alpha, \omega)}, \qquad \text{and} \qquad \sum_{i=1}^n (\Delta_i^n X)^2 \pm q_{1-\frac{\alpha}{2}} \sqrt{n^{-1}\widehat{V}_1^{(2),n}},$$

respectively, where $q_{1-\frac{\alpha}{2}}$ corresponds to the $1 - \frac{\alpha}{2}$ quantile of a standard normally distributed random variable. Analogously as above, we construct statistics based on suitably centered and scaled RMPV and TRMPV, see below for more details.

4   Simulation study

After we have reviewed the feasible central limit theory for realised variation measures, we study their finite sample behaviour by carrying out several Monte Carlo experiments. In particular, we simulate data from seven different stochastic volatility models for various different jump specifications, see Section 4.2 for more details. Then we compute 20 different statistics consisting of suitably centered and scaled realised measures. More precisely, we compute the ten quantities defined in Section 4.3 and the corresponding log–transformed quantities (using the delta method for stable convergence).

4.1   Research questions addressed in the simulation study

The simulation is designed such that it addresses the following research questions regarding the finite sample performance of realised variance:

- The estimator $\widehat{D}_t$ of the jump part of the asymptotic 'variance' of centered RV makes use of spot volatility estimators. Here we use locally averaged realised bipower variation and truncated realised variance. Which estimator is more reliable? How shall we choose the local window k_n (which has to satisfy $k_n \to \infty$ and $k_n\Delta_n \to 0$ as $n \to \infty$). We compare the choice of $k_n = \lfloor 50\Delta_n^{-1/4} \rfloor$ as suggested by Aït-Sahalia & Jacod (2009b) with the choice $k_n = \lfloor \Delta_n^{-1/2} \rfloor$ which is more in line with Lee & Mykland (2008) who suggested to choose a relatively small local window so that the computational intensity does not become too high. Note that we use $\Delta_n = 1/n$ for $n \in \{39, 78, \dots, 23400\}$, hence $\lfloor \Delta_n^{-1/2} \rfloor \leq \lfloor 50\Delta_n^{-1/4} \rfloor$.

- Which estimator for D_t is the better one? The one derived in Veraart (2010) or the one presented in this paper?

- How good is the finite sample performance of realised variance in general?

- Does the finite sample performance change when the activity level of the jump process and/or the jump size increase? In the presence of common jumps in the price and the volatility, centered RV is not asymptotically normally distributed. How does this statistic behave in such a framework?

- How do the log–transformed quantities perform compared to the raw statistics?

Regarding realised multipower variation and truncated realised measures, we aim to shed some light on the following questions:

- How robust are RMPV and TRMV towards jumps of finite and infinite activity in finite samples?

- Which jump robust measure performs best in the presence of jumps?

4.2   Simulation design

In order to study the impact of jumps on the performance of realised variation measure, we simulate from a variety of stochastic volatility models with and without jumps.

Throughout the simulation study, we choose one day as basic unit of time for our simulation. We then simulate the diffusion part based on an Euler scheme, where we choose the increment of one second per tick on the Euler clock, where we have 23,400 seconds a day (which corresponds to the equity market being open for 6.5 hours a day).

11

Diffusion (D): The Heston stochastic volatility model serves as a jump–free benchmark model where the choice of parameters is taken from the empirical results in Aït-Sahalia & Kimmel (2007). I.e. $dX_t = \sigma_t dW_t, d\sigma_t^2 = \kappa(\beta - \sigma_t^2)dt + \gamma\sigma_t dB_t, \mathbb{E}(dW_t dB_t) = \rho dt$, where $\beta^{1/2} = 0.4, \gamma = 0.5, \kappa = 5, \rho = -0.5$. See Table 3 for the results.

Jump diffusion (JD): Using the same model specification for the continuous part as in (D), we now add compound Poisson jumps with normally distributed jumps: $dX_t = \sigma_t dW_t + d\sum_{i=1}^{N_t} J_i$, where N_t is a Poisson process with i.i.d. normally distributed jump sizes such that $\sum_{i=1}^{N_t} J_i | N_t \sim N(0, \sigma_J^2)$. In fact, we work with a stratified compound Poisson process having 1, 5 or 10 jumps per day which are uniformly distributed over the day. Also, when we have more jumps, we decrease their size, so that the contribution of the jump 'variance' stays the same throughout the simulation. In order to achieve that, we simulate the jump sizes conditional on $N_t = c$ from $J_i \sim N(0, \frac{\sigma_J^2}{c})$. Note that, working with the stationary σ^2, we have that $\mathbb{E}\left(\int_0^1 \sigma_s^2 ds\right) = \mathbb{E}\left(\sigma_0^2\right) = \beta, \mathbb{E}\left(\sum_{i=1}^c J_i^2\right) = cVar(J_1) = \sigma_J^2$. We choose σ_J^2 such that $p = \frac{\sigma_J^2}{\beta + \sigma_J^2}$, i.e. the percentage of the mean of the jump part of the quadratic variation compared to the mean of the total variation equals 15% or 30%. Note that empirical studies sometimes find a much lower jump contribution, see e.g. Huang & Tauchen (2005) who found up to 7%. See Tables 4, 5, 6 for the results.

Infinite activity jumps in the price (IAP): Now let $dX_t = \sigma_t dW_t + dL_t$, where L is a symmetrical tempered α–stable Lévy process for $0 < \alpha < 1$ and $\lambda > 0$. I.e. the Lévy density of L is given by $\nu(dx) = |x|^{-(\alpha+1)}e^{-\lambda x}dx$. Clearly, $Var(L_1) = 2\Gamma(2 - \alpha)/\lambda^{2-\alpha}$. As before, we want the percentage of the jump part of the variance to be $p = 15\%, 30\%$, where $Var(L_1) = \beta p/(1-p)$. Hence we choose $\lambda = \left(\frac{\beta p}{2(1-p)}\right)^{1/(\alpha-2)}$ for $\alpha = 0.1, 0.5, 0.9$. Note that the simulation of the tempered stable Lévy process is based on the series representation given in Rosinski (2001). See Tables 7, 8, 9 for the results.

Next, we study the case of having jumps in the volatility process.

Independent finite activity jumps in volatility and price (IFAJ): Here we have $dX_t = \sigma_t dW_t + dJ_t^{(1)}, d\sigma_t^2 = \kappa(\beta - \sigma_t^2)dt + \gamma\sigma_t dB_t + dJ_t^{(2)}, \mathbb{E}(dW_t dB_t) = \rho dt$, where $J^{(1)}$ and $J^{(2)}$ being independent stratified compound Poisson processes (as before) with 5 and 10 jumps per day. See e.g. Zhou (2003) and the references therein for the existence of a process σ^2 defined as above. The jump sizes of $J^{(1)}$ are drawn from a double exponential distribution with equal probability of positive and negative jumps and the jump sizes of the independent process $J^{(2)}$ are exponentially distributed. Again, the parameters are chosen such that the jump percentage of the total variation is set to 15%. See Tables 10, 11 for the results.

Common finite activity jumps in volatility and price (CFAJ): The simulation set up is similar to IFAJ, but in order to make the jump processes $J^{(1)}, J^{(2)}$ dependent, we work with the same jump times and, in particular, we set $|J^{(1)}| = J^{(2)}$, where we allow for 5 or 10 jumps a day. See Tables 12, 13 for the results.

BNS model with independent jumps in price and volatility (BNSIJ): Next, we focus on the Barndorff-Nielsen & Shephard (2001, 2002) model, where the volatility process is purely jump–driven, in particular: $dX_t = \sigma_t dW_t + dJ_t, d\sigma_t^2 = -\lambda\sigma_t^2 dt + dL_{\lambda t}$, where L is an inverse Gaussian $(IG(\mu, l))$ subordinator with parameters $\lambda = 1, \mu = 0.16, l = \mu^2$. J is a stratified Poisson

process, independent of L, with normally distributed jumps (as in JD), with 0, 1, 5, 10 jumps a day. Again the percentage of the jump variation is set to 15%. See Tables 14, 15, 16, 17 for the results.

BNS model with common jumps in price and volatility (BNSCJ): Finally, we simulate the classical BNS model with common IG jumps, i.e. $dX_t = \sigma_t dW_t + \rho dL_{\lambda t}$, $d\sigma_t^2 = -\lambda\sigma_t^2 dt + dL_{\lambda t}$, with the parameter choices as in BNSIJ and, further, $\rho \in \{-0.2, -0.4\}$. See Tables 18, 19 and Figure 5 for the results. Note that the parameter ρ in the BNS model is not a correlation parameter as such (since it is not restricted to be in [-1,1]), but it plays its role, see e.g. Veraart & Veraart (2010) for more details.

4.3   Quantities of interest in the simulation study

According to the asymptotic theory, properly centered and scaled (truncated) realised (multi-) power variation converge –under the assumptions stated above– to standard normally distributed random variables. In order to check their finite sample performance, we compute these suitably centered and normalised quantities, where in the tables the column with the heading *R.V.M* specifies, which realised variation measure is studied. Throughout the simulation study, we use the following abbreviations for the realised variance measures:

RV: Centered realised variance, scaled by $\widehat{V}_1^{(1),n}(\alpha, \omega)$, i.e. $T^{(1)}(k_n, \alpha, \omega)$;

RV2: Centered realised variance, scaled by $2\hat{I}_1^n + 2\sum_i(\widehat{\sigma}_{(i-1)\Delta_n}^2 + \widehat{\sigma}_{(i-1)\Delta_n-}^2)(\Delta_i^n X)\mathbb{I}_{\{|\Delta_i^n X| > \alpha\Delta_n^\omega\}}$, where the spot volatility is estimated by locally averaged realised bipower variation;

pRV: Centered realised variance, scaled by $2\hat{I}_1^n + 2\max\{\sum_i(\Delta_i^n X)^2(\widehat{\sigma}_{(i-1)\Delta_n}^2 + \widehat{\sigma}_{(i-1)\Delta_n-}^2) - 2\hat{I}_1^n, 0\}$, where the spot volatility is estimated by locally averaged truncated realised variance;

pRV2: Centered realised variance, scaled by $\widehat{V}_1^{(2),n}$, i.e. $T^{(2)}(k_n)$.

cRV: Centered realised variance, scaled by the estimator of the continuous part of the asymptotic 'variance' only, i.e. by $2\hat{I}_1^n$.

(Truncated) realised multipower variations:

TRV: Centered and scaled truncated realised variance;

TR2V: Centered and scaled truncated realised bipower variation;

R2V: Centered and scaled realised bipower variation;

R3V: Centered and scaled realised tripower variation;

R4V: Centered and scaled realised quadpower variation;

Note that the threshold parameters are chosen to be $\alpha = 2, \omega = 0.47$ following the simulation study in Aït-Sahalia & Jacod (2009*b*) and the asymptotic 'variance' is estimated by $c\hat{I}_1^n$ for suitable constants $c > 0$ as specified above.

4.4   Results from the simulation study

In order to assess the finite sample performance of the various centered and standardised realised measures in the simulation studies, we have computed their biases (which should be 0), their standard deviations (which should be 1) and the empirical rejection rates for the levels 10% and 5%, which are given in the tables in the Appendix. Furthermore, we have studied quantile–quantile plots and density plots of the distributions of the various realised measures, which are not presented here due to space constraints.

4.4.1   Preceding results regarding the spot volatility estimation

Before we evaluate the results from the simulation study for the various realised measures, we briefly focus on spot volatility estimation first since spot volatility estimators are used in the estimator of the jump part of the asymptotic 'variance' of the realised variance.

The are various methods for estimating the squared spot volatility, i.e. σ^2, see e.g. Aït-Sahalia & Jacod (2009*b*), Bandi & Renò (2010), Foster & Nelson (1996), Kristensen (2010), Lee & Mykland (2008).
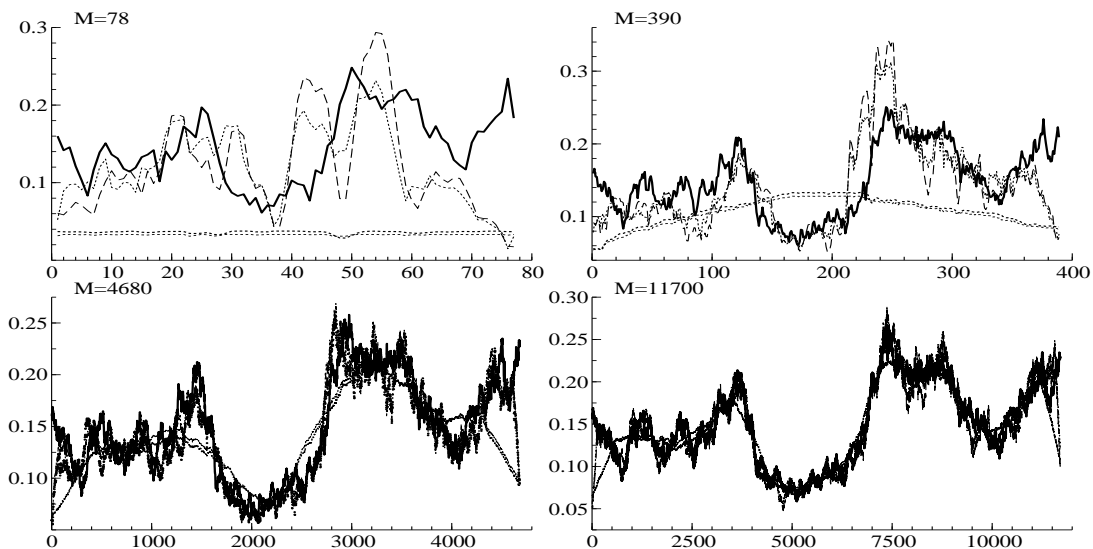
Figure 1: Comparison of various spot volatility estimates for the square–root diffusion model for different numbers of intra–daily observations M.

Here we focus on two approaches: We use locally averaged truncated realised variance and locally averaged realised bipower variation as described above. We compute both estimators for two different choices of the local window size k_n. We work with $\Delta_n = 1/n$ and $k_n \in \{\lfloor \Delta_n^{-1/2} \rfloor, \lfloor 50\Delta_n^{-1/4} \rfloor\}$. Note that both choices satisfy $k_n \to \infty$, $k_n\Delta_n \to 0$. Clearly, $\lfloor \Delta_n^{-1/2} \rfloor \leq \lfloor 50\Delta_n^{-1/4} \rfloor$ for our choices of Δ_n. Note that Aït-Sahalia & Jacod (2009*b*) used the bigger window size whereas Lee & Mykland (2008) suggested to use a relatively small window to ensure that the computing load is not too high. Throughout the various scenarios in the simulation study, we always obtained significantly better results for the smaller window $k_n = \lfloor \Delta_n^{-1/2} \rfloor$, using the root mean square error (RMSE) to compare the various spot estimators. The bigger window size resulted in over–smoothing, which we can clearly
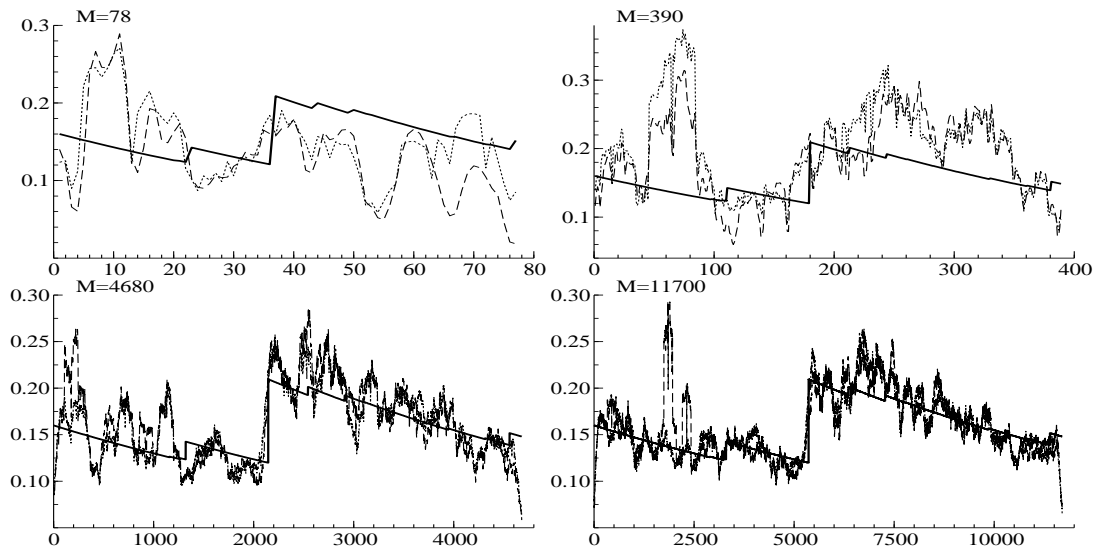
Figure 2: Comparison of various spot volatility estimates for the BNS model, where σ^2 is a non–Gaussian Ornstein Uhlenbeck process with inverse Gaussian innovations for different numbers of intra–daily observations M.

see in Figure 1 and Figure 2. Figure 1 presents a plot of the simulated σ^2 following a square–root diffusion process (represented by the thick solid line). The dashed and dotted lines represent the locally averaged TRV and R2V, where the smoother lines come from the bigger window size. Figure 2 compares the various spot volatility estimators when σ^2 is sampled from a non–Gaussian Ornstein Uhlenbeck process with inverse Gaussian innovations. Note that M denotes the number of intra–daily observations. I.e. the numbers $M = 78, 390, 4680, 11700$ correspond to 5 minute, 1 minute, 5 second and 2 second observations in an equity market which is open 6.5 hours a day. We clearly observe that the spot volatility estimates are rather noisy measures of the underlying volatility. With increasing frequency of the data, however, their performance improves.

Note that we used the simulation design from Aït-Sahalia & Jacod (2009b) for the Brownian–driven stochastic volatility process and, hence, we used their threshold choice for computing TRV. When comparing the performance of the locally averaged TRV and R2V for estimating the spot volatility based on the RMSE, we obtained very similar results throughout the simulation study. But we can nevertheless formulate some tendencies:

- TRV tends to perform better than R2V in the absence of jumps in the price;

- In the presence of (smaller) jumps in the price, R2V tends to perform better for lower frequencies (up to M=390), whereas TRV tends to perform better for higher frequencies;

- In the BNS jump–driven stochastic volatility model, R2V tends to perform better.

- In the presence of rather *big* jumps in the price, TRV tends to perform better.

Clearly, the performance of TRV can be improved by using a data driven, dynamic threshold, see e.g. Aït-Sahalia & Jacod (2010), Corsi et al. (2010), Mancini & Gobbi (2010). However, given that the differences between the two estimators were not that big and if they were, than R2V was in

most cases the better estimator, from a practical point of view, R2V seems to be the winner in that direct comparison given that it does not require to choose a threshold and leads nevertheless to fairly reasonable estimates. Furthermore, we conclude that the choice of the window size k_n seems to be more important than the particular choice of the realised measure. In our simulation experiment, $k_n = \lfloor \Delta_n^{-1/2} \rfloor$ did a fairly good job for the various models studied here.

4.4.2   Results for realised variance

How good is the finite sample performance of realised variance in the presence of jumps? In order to answer that question we have computed the difference of realised variance and the daily quadratic variation of the simulated log–price process and have scaled it suitably by various estimators for the asymptotic 'variance' (and divided it by $\sqrt{\Delta_n}$), which asymptotically should be standard normally distributed. The various statistics are denoted by RV, RV2, pRV, pRV2 and cRV as defined above and the Tables in the Appendix present the bias, the standard deviation and the empirical rejection rate at 10% and 5% level for various numbers $M = \lfloor 1/\Delta_n \rfloor$ of intra–daily observations.

Our general findings are as follows: We obtain good finite sample results (in terms of the bias being close to 0, the standard deviation being close to 1 and the empirical rejection rates being close to the theoretical level of 10% and 5% respectively, for M between 78 and 195 and higher, corresponding to 2-5 min and more frequent returns in an equity market. In general, the results for the log–transformed realised variance were better throughout the simulation study, but we do not report them here and rather focus on the slightly worse results for the raw realised variance since this quantity is more widely used in practice. Note in particular, that the finite sample bias only seems to go away when one studies data at higher frequencies than 1 minute data in a 6.5 hour market, whereas the bias is noticeably smaller for log–transformed realised variances.

Note that throughout the simulation study, we have chosen the model parameters such that the expected integrated variance stays the same. Further, the jump part of the quadratic variation compared to the total variation amounts to 15%. We have rerun the simulations also for the percentage of the jump part being 30%, but got very similar results as before. Our main findings are as follows:

- The finite sample performance becomes good from 5 minute data onwards to more frequent data;

- The finite sample bias goes to zero for data sampled at 1 minute intervals and more frequent data;

- The activity of the jumps in the price process does not seem to have a big impact on the finite sample performance of realised variance, see the results from model IAP, i.e. Tables 7, 8, 9. In fact, it seems like the finite sample performance gets even a bit better for jumps of higher activity.

- In the presence of jumps, realised variance has to be scaled by one of the new estimators derived in this paper and in Veraart (2010) which account for jumps, otherwise the finite sample performance is not good and gets clearly worse when the frequency of the data increases, see the performance of the statistics cRV in the Tables for more details, where the centered realised variance is only standardised by an estimator of the continuous part of the asymptotic 'variance'.

- When comparing the performance of the various standardised realised variances, we observe that pRV tends to perform best for low frequencies (up to M=390), followed closely by pRV2.

However, for higher frequencies, using the new estimator $\widehat{D}^{(1),n}$ leads to better finite sample results, see RV and RV2 in the Tables in the Appendix.

- In the presence of common jumps of the volatility and the price process (model BNSCJ), we still observe a good finite sample performance although the asymptotic theory does not hold in that situation anymore. We have carried out that experiment for different choices of ρ, which scales the jumps in the volatility to a different level for the price jumps. Note that for $\rho = -0.2$, the jump part of the quadratic variation amounts to approximately 4.7% of the total quadratic variation, whereas $\rho = -0.4$ corresponds to 16.7% with our choice of parameters, see Tables 18, 19 and Figure 5.

Note that as before we expect that the performance of the threshold based estimators for the jump part of the asymptotic 'variance' can be improved when choosing a better threshold as discussed in the previous section. However, from an applied point of view, it is encouraging to see that pRV has such a good finite sample performance, where the jump part of the asymptotic 'variance' is estimated based on the difference of generalised realised variance and realised multipower variation and the spot volatility is estimated based on the threshold concept, where a rather crude threshold has been used.

4.4.3   Results for (truncated) realised (multi-) power variation

Besides the question of drawing inference on the entire quadratic variation, it is also of interest to study the continuous and the jump part of the quadratic variation separately. We have computed various jump–robust realised measures denoted by TRV, TR2V, R2V, R3V, R4V as defined above and have analysed their finite sample performance in the same simulation studies as realised variance.

Remark Note that we have included realised bipower variation, R2V, in the simulation experiment even in the presence of jumps although its asymptotic distribution is not Gaussian in that case, see Vetter (2010) for the detailed results. However, truncated realised bipower variation, TR2V, has the corresponding Gaussian limit and, hence, we keep the R2V results to see more clearly the impact of the truncation. So it should be stressed that we cannot expect to get the expected results of standard deviation being equal to 1 and empirical rejection rates being close to 10% and 5% for R2V as soon as there are jumps in the price present. However, we would still expect to find a bias tending to zero as $\Delta_n \to 0$. Further, note that the multipower results only hold theoretically as long as assumption (A) is satisfied.

The main findings for the jump–robust realised measures are as follows. In the absence of jumps in the price, the performance of the various jump–robust measures is very similar and generally good, but we observe that the finite sample bias at low frequencies tend to be fairly big for R3V and R4V see e.g. Table 3 and 14.

In the presence of jumps in the price process, the general trend we observe is as follows:

- If there are rare jumps, the jump robust measures perform fairly well.

- However, if there are (finitely) many small jumps, our estimation results clearly show a (positive) finite sample estimation bias for all jump robust measures. Note here that the sign of the bias for the truncation based methods could potentially change with a different choice of the threshold.

17

- If the price process consists of a Brownian semimartingale and a highly active jump process (e.g. a tempered stable Lévy process), the jump–robust measures show signs of failure. In particular, the more active the jump process gets, the worse results we obtain, see Tables 7, 8, 9.

- Altogether we observe that, in the presence of many jumps, R4V tends to perform best out of the various jump–robust measures studied here at lower frequencies (up to 1 minute data).

- In the BNS modelling framework (BNSCJ), we also obtain reasonable finite sample results for R2V, which is somewhat surprising given that the asymptotic theory does not hold here. That might be due to the fact that the jump activity of the inverse Gaussian subordinator is still rather low.

4.5 Concluding remarks regarding the simulation study

In general, we have observed that the feasible asymptotic theory for realised variance seems to hold also in finite samples when data at a frequency of 2-5 minutes or higher are studied. The differences between the various estimators of the asymptotic 'variance' of realised variance are not that big, but generally, we obtained the best results for pRV at lower frequencies.

When using jump–robust realised measures using the truncation principle, the multipower concept or a combination of both, we found that they perform quite well in the presence of rare jumps. However, as soon as there are many small jumps and in particular, if there is a very active jump process present, then the finite sample performance of the jump–robust measures is not good any longer. This is in line with the findings of Barndorff-Nielsen et al. (2005) and Veraart (2010). In order to get somewhat reliable measures in that situation, higher order multipower variations such as R4V (possibly combined with a dynamically chosen threshold) seem to be the best choice. Further, using log–transformed versions of realised variance rather than the raw realised variance also improves the finite sample performance.

While inference on the entire quadratic variation of the log–price process based on realised variance using the new estimator of the asymptotic 'variance' of RV presented in this paper and the one presented in Veraart (2010) turns out to work well even in finite samples and also in the presence of very active jump processes, the finite sample results for drawing inference on the continuous part or the jump part only are generally not that good when many jumps are present.

5 Empirical study of SPY data

After we have checked the finite sample performance of various feasible estimators of the quadratic variation of an Itô semimartingale, we use our findings now to carry out an empirical study of some high frequency equity data.

5.1 Data description

We use data from the Standard & Poor's Depository Receipt (SPY), which is a very liquid, exchange–traded fund which holds all of the S&P 500 Index stocks. The sample we work with covers a period from 3 August 1998 to 31 July 2008, i.e. 10 years of data. We focus on mid–quote data constructed based on the quote data taken from the TAQ database through the Wharton Research Data Services (WRDS) system. The SPY data were recorded at the AMEX from 1998-2002 and at the PACIF from
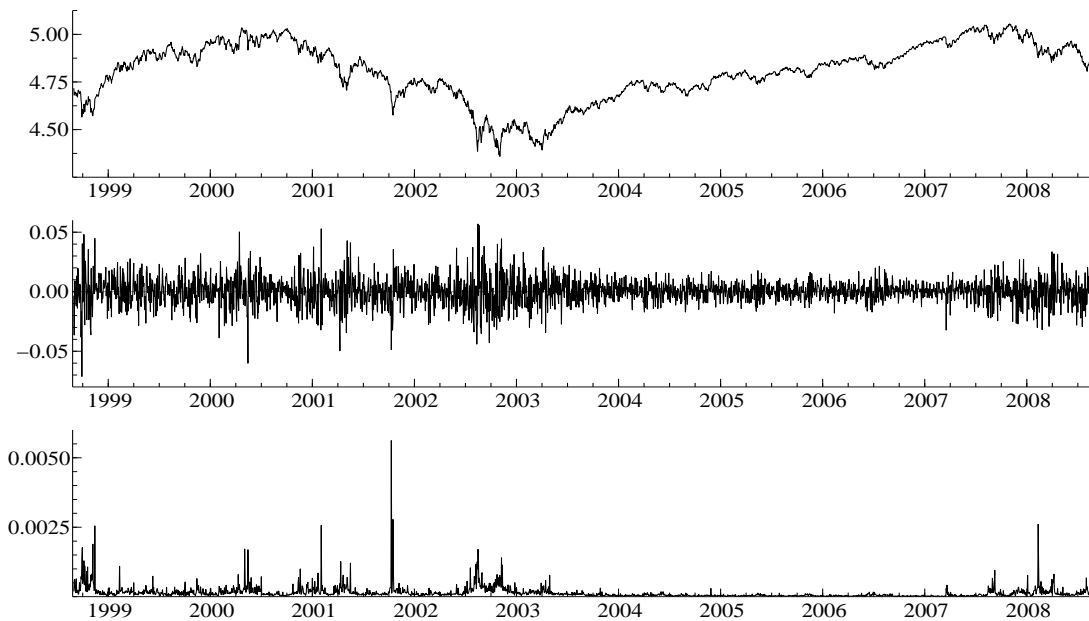
Figure 3: SPY data from 3 August 1998 to 31 July 2008: (a) Cleaned daily log–mid–prices; (b) daily log–mid–price returns; (c) time series of daily realised variances based on five minute returns.

2003-2008. The high frequency data have been kindly supplied to the author by Asger Lunde and have been cleaned using the methods described in Barndorff-Nielsen et al. (2008*b*). We have sampled the data at five minute intervals (using the previous tick method) and have obtained $n = 2515$ days of 78 observations each, i.e. 196,170 data points. A plot of the cleaned SPY log–price data, the returns and the time series of the realised variances is given in Figure 3.

The aim of this study is to analyse the impact of jumps on inference for realised variance. In order to do that, we identify days which have a significant jump contribution and then check how much wider the confidence intervals for realised variance become on those days.

5.2 Testing for jumps and inference on quadratic variation in the presence of jumps

As already mentioned above, there is a wide literature on testing for jumps using high frequency data. In our empirical study, we follow Huang & Tauchen (2005) who found out in detailed Monte Carlo studies that the ratio statistic of the Barndorff-Nielsen & Shephard (2006) test using a maximum adjustment in the denominator has the best size property out of the various test statistics studied, including the properly scaled difference of realised variance and realised bipower (often referred to as *raw* test statistic), the difference of the log–transformed realised variance and realised bipower and the ratio of the difference of realised variance and realised bipower divided by realised variance. When carrying out the test, we get the following results, as reported in Table 2.

Note that Huang & Tauchen (2005) did not find significant differences when the test statistics were scaled by realised tripower or realised quadpower variation, respectively. This is basically in line with what we find here. In our empirical study, we reject the null hypothesis slightly more often when using realised quadpower variation as the scaling factor, which is in line with what we expect: Since realised quadpower variation averages over more returns than realised tripower variation, we

19

Table 2: Empirical results from the maximum adjusted BNS ratio test.

Significance level	M	Number and percentage of rejections	Average contribution of the jump part of the CIs
99%	39	632 (25.12%)	55.61%
	78	931 (37.01%)	64.96%
	130	1136 (45.16%)	58.82%
	195	1328 (52.80%)	53.75%
	390	1779 (70.73%)	40.52%
99.90%	39	390 (15.50%)	57.07%
	78	658 (26.16%)	67.43%
	130	844 (33.55%)	60.59%
	195	1052 (41.82%)	54.63%
	390	1520 (60.43%)	41.38%

expect it to be generally smaller in the presence of jumps, which results in more rejection of the null hypothesis given that the scaling factor appears in the denominator of the ratio statistic.

Our test signals the presence of a significant jump contribution to the total quadratic variation (at 99% significance level) on 25–70% of the days — depending on which data frequency we use for computing the test statistic. In the following we will just refer to those days as *jump days*. Furthermore, we observe that the number of rejections of the null hypothesis increases when the number of intra–daily observations, denoted by M, increases. That is also in line with Huang & Tauchen (2005) findings that when data at lower frequencies are considered, jumps tend to average out and are not necessarily detected by the jump test anymore.

However, in general we can say that the results from the jump test are rather surprising. Rejecting the null hypothesis that frequently is not in line with earlier empirical studies in the literature, see e.g. Huang & Tauchen (2005). However, there are also more recent studies, see e.g. Aït-Sahalia & Jacod (2009a) whose findings point towards the presence of a very active jump component even for highly liquid stocks. If this is true, then we would expect to reject the null hypothesis of no jumps very often and ideally on (almost) all days.

After we have tested for each day in the sample whether there was a significant jump part in the quadratic variation and hence, have identified those jump days, we study how the jumps affect the size of the confidence bounds for realised variance on these jump days. In order to do that, we have computed the average length of the confidence bounds assuming that there were no jumps (i.e. estimating the asymptotic 'variance' by $2\hat{I}$) and the average length of the confidence bounds which accounts for jumps (i.e. estimating the asymptotic 'variance' by $2\hat{I} + 2\hat{D}$). We find that the part of the confidence bounds due to the presence of jumps amounts to approximately 40-60% of the total length, see the last column in Table 2. Further, we observe that the average jump contribution to the total length of the CIs tends to decrease with increasing number of intra–daily observations.

In order to get an additional visual impression on how the confidence bounds for realised variance change in the presence of jumps, we provide a plot of the confidence bounds for RV which account for jumps and add the ones which ignore jumps. In Figure 4, we plot the time series of realised variances with the corresponding 95% confidence bounds for the days when the maximum adjusted BNS ratio test based on ten minute data using realised tripower for the scaling rejects the null hypothesis at the 99.9% level, which was the case on 390 days. The crosses indicated the realised variances, the bars
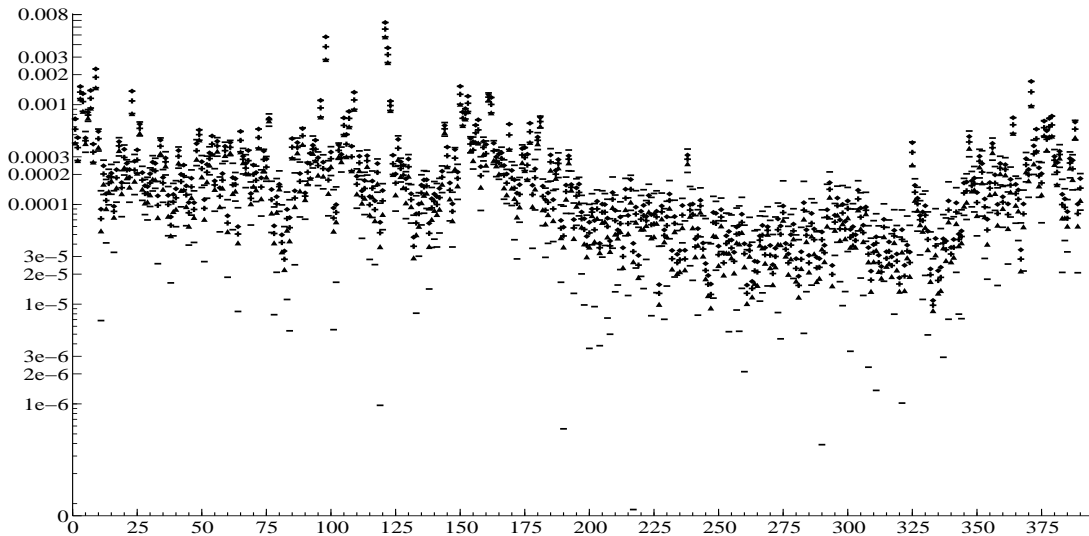
Figure 4: Here we plot the time series of realised variances (using the log–scale) with the corresponding 95% confidence bounds for the days when the maximum adjusted BNS ratio test based on ten minute data using realised tripower for the scaling rejects the null hypothesis at the 99.9% level, which was the case on 390 days.

denote the confidence bounds which account for jumps and the triangles/circles denote the confidence bounds which ignore jumps. We see that the confidence intervals which account for jumps are the wider ones, which is what we expect.

6   Conclusion

This paper has investigated how accurate the asymptotic distribution of realised variation measures is in finite samples when there are finite or infinite activity jumps in the underlying price process. In order to do that, we have proposed a new estimator for the jump part of the asymptotic 'variance' of centered realised variance. Then, we have carried out a detailed Monte Carlo study, where we have compared the finite sample performance of 20 different centered and standardised realised measures. Here we were in particular interested in the performance of realised variance in the presence of jumps. To the best of our knowledge, this is the first paper, which has addressed this important question given the strong evidence for the presence of jumps in financial data, see e.g. Barndorff-Nielsen & Shephard (2004*b*, 2006), Huang & Tauchen (2005), Lee & Mykland (2008), Aït-Sahalia & Jacod (2009*b*). We have studied the finite sample performance of the newly established asymptotic theory for realised variance, see Jacod (2008), Veraart (2010). In our simulation experiment we have compared the results, when the asymptotic 'variance' is estimated based on two different spot volatility estimators: locally averaged realised bipower variation and locally averaged truncated realised variance. In general, we have obtained good results in the Monte Carlo experiments, where the results for the log–transformed realised variance were always slightly better than for the realised variance. Furthermore, both estimators for the jump part of the realised variance provide fairly similar results. The only noticeable differences between the two spot volatility estimators were that truncated realised variance seems to perform slightly better in the stochastic volatility jump diffusion model, whereas

21

realised bipower variation seems to be slightly better in models which allow for jumps in the volatility process.

Regarding the size of the local window k_n, we have obtained good results when choosing $k_n = 1/\sqrt{\Delta_n}$ which furthermore is not that computationally intense. Also, we have clearly seen that the window size k_n has a great impact on the performance of the spot volatility estimator, whereas the particular choice of the spot volatility estimator, i.e. which jump–robust realised quantity is used, does not seem to matter much.

Furthermore, Monte Carlo results suggest that our feasible limit theory might still work even in the case of common jumps of the volatility and the price process as long as the common jump part is rather small.

Regarding the so–called jump robust measures such as realised multipower variation and truncated realised measures, we found that the finite sample performance was not as good as expected. While we have obtained fairly good results for rare jumps in the price, we could clearly see that the performance of the jump robust realised measures studied here worsens dramatically when the jump activity increases.

In addition to the simulation experiment, we have applied the feasible asymptotic theory for realised variance to high frequency SPY data and have computed the confidence intervals for realised variance on the days which were identified as jump days using the maximum adjusted Banrdorff–Nielsen & Shephard ratio test. We have clearly seen that the confidence bounds of realised variance which account for jumps are significantly wider than the ones suggested by Barndorff-Nielsen & Shephard (2002) in the absence of jumps.

In our empirical study, we have worked with rather low frequency data to ensure that the data are not affected by market microstructure noise. However, in future research, it will be interesting to study the impact of jumps on inference on the quadratic variation of the price process if the efficient price is contaminated by market microstructure effects and, hence, not directly observed. In particular, ultra high frequency data (e.g. one second returns) are typically prone to such effects and, hence, volatility estimation and inference in such a framework is carried out using noise robust volatility estimators, see e.g. Zhang et al. (2005), Hansen & Lunde (2006), Zhang (2006), Barndorff-Nielsen & Shephard (2007), Barndorff-Nielsen et al. (2008*a*), and Podolskij & Vetter (2009). Volatility estimators which are robust to market microstructure effects have a slower rate of convergence and do not achieve the square root speed of convergence of realised variance and realised multipower variation. Hence, it will be interesting to study in simulation studies and also empirically which estimators perform better in finite samples in the presence of jumps and market microstructure effects.

A   Technical assumptions

A.1   Assumptions on X and σ

For the central limit theorems of realised variance and realised multipower variation we need some regularity assumptions and a structural assumption for the volatility process which we introduce in the following. All the technical assumptions are taken from Jacod (2008, p. 5–6) and Aït-Sahalia & Jacod (2009*b*, p. 187,204).

We work with a probability space denoted by $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ (or with a slightly enlarged version of the original probability space). Let

$$\phi_r(x) = \begin{cases} 1 \wedge |x|^r, & \text{if } 0 < r < \infty, \\ \mathbb{I}_{\mathbb{R} \setminus \{0\}}(x), & \text{if } r = 0. \end{cases}$$

We focus on Itô semimartingales X which satisfy the following conditions.

Assumption (H): X is an Itô semimartingale as defined in (1) and (2) and the processes $(b_t)_{t\geq 0}$ and $\left(\int \phi_2(x)F_t(dx)\right)_{t\geq 0}$ are locally predictable. Note that $F_t(\omega, dx)$ is the image of the Lebesgue measure on $\mathbb{R}$ by the predictable map $x \mapsto \delta(\omega, t, x)$. Further, the process $(\sigma_t)_{t\geq 0}$ is càdlàg adapted.

Assumption (L-s): (With $s \in [0, 2]$). X is an Itô semimartingale as defined above satisfying assumption (H) and the volatility process σ has the form

$$\sigma_t = \sigma_0 + \int_0^t \widetilde{b}_u du + \int_0^t \widetilde{\sigma}_u dW_u + \int_0^t \widetilde{\sigma}'_u dW'_u + \kappa(\tilde{\delta}) \star (\underline{\mu} - \underline{\nu})_t + \kappa'(\tilde{\delta}) \star \underline{\mu}_t, \quad (15)$$

and

- W' is another Brownian motion on the space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, which is independent of W;

- the process $(\widetilde{b}_t)_{t\geq 0}$ is optional and locally bounded;

- the processes $(b_t)_{t\geq 0}$, $(\widetilde{\sigma}_t)_{t\geq 0}$, $(\widetilde{\sigma}'_t)_{t\geq 0}$ are adapted left–continuous with right limits in t, and locally bounded;

- the functions $\delta, \tilde{\delta} : \Omega \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are defined such that $F_t(\omega, dx)$ is the image of the Lebesgue measure on $\mathbb{R}$ by the map $x \to \delta(\omega, t, x)$; $\delta(\omega, t, x)$ and $\tilde{\delta}(\omega, t, x)$ are predictable, left–continuous with right limits in t. Further $|\delta(\omega, t, x)| \leq \gamma_k(x)$ and $|\delta(\omega, t, x)| \leq \tilde{\gamma}_k(x)$ for all $t \leq T_k(\omega)$ with deterministic functions γ_k and $\hat{\gamma}_k$ on $\mathbb{R}$ with $\int \phi_s \circ \gamma_k(x)dx < \infty$ and $\int \phi_2 \circ \tilde{\gamma}_k(x)dx < \infty$ and $(T_k)_{k\geq 1}$ is a sequence of stopping times increasing to $+\infty$.

Note that assumption (L-s) basically says that the stochastic volatility process σ has to be an Itô semimartingale, too. Further, the parameter s can be regarded as an generalised Blumenthal Getoor index, see Blumenthal & Getoor (1961), measuring the jump activity of the jump part part of X. See Jacod (2008, Remark 2.1) for a wide class of processes which in fact satisfy the above assumptions.

Assumption(H'): Assumption (H) holds and $\sigma_t^2 > 0$ and $\sigma_{t-}^2 > 0$ for all t a.s..

Assumption (G1): Assumptions (L-s) and (H') hold (for some $s \in [0, 2]$) and further

1. All paths $t \mapsto \sup_{x \in E} \frac{|\delta(\omega, t, x)|}{\gamma(x)}$ and $t \mapsto \sup_{x \in E} \frac{|\tilde{\delta}(\omega, t, x)|}{\gamma(x)}$ are locally bounded, where γ is a (nonrandom) nonnegative function satisfying $\int_E (\gamma(x))^2 \wedge 1)\lambda(dx) < \infty$.

2. All paths $t \mapsto \delta'_t(\omega)$ are left–continuous with right limits on the semiopen set $[0, \tau(\omega))$. Here $\delta'_t(\omega) = \int \kappa \circ \delta(\omega, t, x)\lambda(dx)$, if the integral is well–defined, otherwise we set $\delta'_t(\omega) = \infty$.

Assumption (G2): Assumption (G1) holds and $|b_t| + |\sigma_t| + |\widetilde{b}_t| + |\tilde{\sigma}_t| + |\tilde{\sigma}'_t| \leq K$, and $|\delta(t, x)| \leq \gamma(x)$, $|\tilde{\delta}(t, x)| \leq \gamma(x)$ and also $\gamma(x) \leq K$ for some constant K.

Assumption (A): Assumption (L-s) holds for $s < 1$, (H') is satisfied and $\frac{s}{2-s} < \mathbf{r}_- < \mathbf{r}_+ < 1$.

Assumption (B): Assumption (L-s) holds and $s \leq \frac{4\omega-1}{2\omega}$ (hence $\omega \geq 1/4$ and $s < 1$).

Assumption (C): Assumption (L-s) holds for $s < 1$, (H') is satisfied and $I \in \mathbb{N}$ with $2 < I < \frac{2}{s}(2 - s)$. Further, X and σ do not have common jumps.

23

B   Proofs

B.1   Notation and preliminary remarks

Throughout the proofs we use the following notation:

$$\delta_i^n = \sigma_{(i-1)\Delta_n}\Delta_i^n W, \qquad\qquad \theta_i^n = |\Delta_i^n X|\mathbb{I}_{\{|\Delta_i^n X|\leq\alpha\Delta_n^\omega\}},$$

for $\alpha > 0$ and $\omega \in (0, \frac{1}{2})$. Further, we write $\mathbb{E}_{i-1}^n(\cdot) = \mathbb{E}(\cdot|\mathcal{F}_{(i-1)\Delta_n})$ for the conditional expectation.

Furthermore, we write $X = X' + X^d$, where X^c denotes the continuous part and X^d the discontinuous part of X and

$$X_t' = X_0 + \int_0^t b_s ds + X_t^c, \qquad X_t^c = \int_0^t \sigma_s dW_s,$$

$$X^d = \int_0^t \int_E \kappa(\delta(s,x)) \star (\mu - \nu)(ds, dx) + \int_0^t \int_E \kappa'(\delta(s,x)) \star \mu(ds, dx).$$

In the following, we will apply the following inequalities, which are valid under Assumption (G2), see e.g. Aït-Sahalia & Jacod (2009b), Jacod (2008), Vetter (2010), where $K_i > 0$ for $i \in \mathbb{N}$ denote positive constants:

$$\mathbb{E}_{i-1}^n(|\Delta_i^n X^c|) \leq K_1\Delta_n^{1/2}, \qquad \mathbb{E}_{i-1}^n(|\Delta_i^n X'|) \leq K_2\Delta_n^{1/2}, \qquad \mathbb{E}_{i-1}^n(|\Delta_i^n X^d|) \leq K_3\Delta_n, \tag{16}$$
$$\mathbb{E}_{i-1}^n(|\Delta_i^n X|) \leq K_4\Delta_n^{1/2}.$$

Note that in the following, we will denote by $K > 0$ a constant which can change from line to line throughout the proof.

Remark We will prove Proposition 3.1 under the stronger assumption (G2). Using a localisation procedure, it follows from standard arguments, see Jacod (2008) for details, that our result also holds under the weaker assumption (G1).

B.2   Proof

Proof of Proposition 3.1 The proof of the Proposition is a direct consequence of the previous remark and the following two lemmas. $\qquad\square$

Lemma B.1 *Under Assumption (G2) and if*

$$\widehat{D}_t'^n = \frac{1}{k_n\Delta_n} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (\Delta_i^n X)^2 \, \mathbb{I}_{\{|\Delta_i^n X|>\alpha\Delta_n^\omega\}} \sum_{j\in I_{n,t}(i)} (\delta_j^n)^2 \xrightarrow{\mathbb{P}} D_t, \tag{17}$$

then $\widehat{D}_t^n \xrightarrow{\mathbb{P}} D_t$, where

$$\widehat{D}_t^n = \frac{1}{k_n\Delta_n} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (\Delta_i^n X)^2 \, \mathbb{I}_{\{|\Delta_i^n X|>\alpha\Delta_n^\omega\}} \sum_{j\in I_{n,t}(i)} \left(\Delta_j^n X\right)^2 \mathbb{I}_{\{|\Delta_j^n X|\leq\alpha\Delta_n^\omega\}}.$$

Proof The proof goes along the lines of Aït-Sahalia & Jacod (2009*b*, Lemma 1). In particular, we can immediately deduce, see Aït-Sahalia & Jacod (2009*b*, equation (67)), that

$$\frac{1}{\Delta_n}\mathbb{E}\left(\left|(\theta_i^n)^2 - (\delta_i^n)^2\right| \,\big|\, \mathcal{F}_{(i-1)\Delta_n}\right) \le \widehat{\Gamma}_n, \tag{18}$$

for a sequence $(\widehat{\Gamma}_n)_{n\in\mathbb{N}}$ satisfying $\lim_{n\to\infty}\widehat{\Gamma}_n = 0$. Note that the difference $\widehat{D}_t^n - \widehat{D}_t'^n$ consists of a sum with strictly less than $2k_n\lfloor t/\Delta_n\rfloor$ terms, where each term is smaller than

$$\frac{1}{k_n\Delta_n}(\Delta_i^n X)^2\mathbb{I}_{\{|\Delta_i^n X|>\alpha\Delta_n^\omega\}}\left|(\theta_j^n)^2 - (\delta_j^n)^2\right|, \qquad \text{for some } i \ne j.$$

From using the inequalities (16), we get that $\mathbb{E}_{i-1}^n((\Delta_i^n X)^2\mathbb{I}_{\{|\Delta_i^n X|>\alpha\Delta_n^\omega\}}) \le K\Delta_n$. Now, we use two successive conditional expectations and (18) to deduce that each such term is smaller than $\frac{1}{k_n}K\widehat{\Gamma}_n\Delta_n$. So, altogether, we get $\mathbb{E}\left|\widehat{D}_t^n - \widehat{D}_t'^n\right| \le Kt\widehat{\Gamma}_n$ and, hence, the result follows. $\square$

Lemma B.2 *Under Assumption (G2), equation (17) holds.*

Proof In order to prove this result, we adapt the proof given in Aït-Sahalia & Jacod (2009*b*, pp. 216–218) so that it is suitable for the different result, we prove here.

We introduce a continuous function on $[0,1]$ for any $\rho \in (0,1)$ which is defined by $\psi_\rho(x) = \min(1, (2 - |x|/\rho)^+)$, and we define two (increasing) processes by

$$Y(\rho)_t^n = \frac{1}{k_n\Delta_n}\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\psi_\rho(\Delta_i^n X)(\Delta_i^n X)^2\mathbb{I}_{\{|\Delta_i^n X|>\alpha\Delta_n^\omega\}}\sum_{j\in I_{n,t}(i)}(\delta_j^n)^2, \quad Z(\rho)_t^n = \widehat{D}_t'^n - Y(\rho)_t^n.$$

It is sufficient to prove that we have the following three convergence results for a suitable process $Z(\rho)$:

$$\lim_{\rho\to 0}\limsup_n \mathbb{E}(Y(\rho)_t^n) = 0, \tag{19}$$

$$\rho \in (0,1), n \to \infty \Rightarrow Z(\rho)_t^n \xrightarrow{\mathbb{P}} Z(\rho)_t, \tag{20}$$

$$\rho \to 0 \Rightarrow Z(\rho)_t \xrightarrow{\mathbb{P}} D_t. \tag{21}$$

Now we prove (19). Note that $Y(\rho)_t^n$ is a sum of strictly less than $2k_n\lfloor t/\Delta_n\rfloor$ terms, where each term is given by

$$\frac{1}{k_n\Delta_n}\psi_\rho(\Delta_i^n X)(\Delta_i^n X)^2\mathbb{I}_{\{|\Delta_i^n X|>\alpha\Delta_n^\omega\}}(\delta_j^n)^2, \qquad \text{for } i \ne j.$$

From (16), we get that $\mathbb{E}_{j-1}^n((\delta_j^n)^2) \le K\Delta_n$.

Next, we study the term $\psi_\rho(\Delta_i^n X)(\Delta_i^n X)^2\mathbb{I}_{\{|\Delta_i^n X|>\alpha\Delta_n^\omega\}}$. Note that an application of the Markov inequality and the inequalities in (16) leads to $\mathbb{E}_{i-1}^n\left((\Delta_i^n X)^2\mathbb{I}_{\{|\Delta_i^n X|>\alpha\Delta_n^\omega\}}\right) \le K\Delta_n^{1+1/2-\omega}$. Now we take two successive conditional expectations and get $\mathbb{E}(Y(\rho)_t^n) \le Kt\Delta_n^{1/2-\omega}$. So we can deduce (19).

25

Next, we define the process $Z(\rho)$ by

$$Z(\rho)_t = \sum_{q:T_q(\rho)\leq t} (\Delta X_{T_q(\rho)})^2 (1 - \psi_\rho(\Delta X_{T_p(\rho)})) \left(\sigma^2_{T_q(\rho)-} + \sigma^2_{T_q(\rho)}\right),$$

where, for $q = 1, 2, \ldots$, $T_q(\rho)$ denote the (successive) jump times of $\mu([0, t] \times \{x : \gamma(x) > \rho/2\})$. Note that this is the γ from Assumption (G1)! From the same arguments as in Aït-Sahalia & Jacod (2009b), we now obtain (21). Note, in particular, that $1 - \psi_\rho$ converges to the to the indicator function on $\mathbb{R} \setminus \{0\}$.

Finally, we prove (20). Let $\rho \in (0, 1)$ be fixed. For ease of notation, we write $T_q = T_q(\rho)$. Note that

$$Z(\rho)^n_t = \frac{1}{k_n \Delta_n} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (1 - \psi_\rho(\Delta^n_i X))(\Delta^n_i X)^2 \mathbb{I}_{\{|\Delta^n_i X| > \alpha \Delta^\omega_n\}} \sum_{j \in I_{n,t}(i)} (\delta^n_j)^2.$$

From the definition of the Poisson process μ, we see immediately that $|\Delta X_s| \leq \rho/2$ if $s \notin \{T_q, q = 1, 2, \ldots\}$.

So, for each $t \geq 0$ and for each ω and for n large enough, we get: there is no jump time T_q in $(0, k_n \Delta_n]$ and in $(t - (k_n + 1)\Delta_n, t]$. Further, in the interval $((i-1)\Delta_n, i\Delta_n]$ (for $i\Delta_n \leq t$) there is at most one T_q. If there is no jump time in the interval, then $\psi_\rho(\Delta^n_i X) = 1$. So, as soon as n is large enough, we obtain that

$$Z(\rho)^n_t = \sum_{q:k_n\Delta_n < T_q \leq t-(k_n+1)\Delta_n} \zeta^n_q,$$

where $i(n, q) = \inf(i : i\Delta_n \geq T_q)$, $I'(n, q) = \{j : j \neq i(n, q), |j - i(n, q)| \leq k_n\}$ and

$$\zeta^n_q = \frac{1}{k_n \Delta_n}(\Delta^n_{i(n,q)} X)^2 \mathbb{I}_{\{|\Delta^n_{i(n,q)} X| > \alpha \Delta^\omega_n\}}(1 - \psi_\rho(\Delta^n_{i(n,q)} X)) \sum_{j \in I'(n,q)} (\delta^n_j)^2.$$

We have to show that for any q:

$$\zeta^n_q \xrightarrow{\mathbb{P}} (\Delta X_{T_q})^2 (1 - \psi_\rho(\Delta X_{T_q})) \left(\sigma^2_{T_q-} + \sigma^2_{T_q}\right).$$

From Aït-Sahalia & Jacod (2009b), we know that

$$\frac{1}{k_n \Delta_n} \sum_{j \in I'_-(n,q)} (\delta^n_j)^2 \xrightarrow{\mathbb{P}} \sigma^2_{T_q-}, \qquad\qquad \frac{1}{k_n \Delta_n} \sum_{j \in I'_+(n,q)} (\delta^n_j)^2 \xrightarrow{\mathbb{P}} \sigma^2_{T_q},$$

where $I'_-(n, q)$ and $I'_+(n, q)$ are the subsets of $I'(n, q)$ containing all j smaller and bigger, respectively, than $i(n, q)$. Next, from Aït-Sahalia & Jacod (2009b) we also know that

$$(\Delta^n_{i(n,q)} X)^p (1 - \psi_\rho(\Delta^n_{i(n,q)} X)) \to (\Delta X_{T_q})^p (1 - \psi_\rho(\Delta X_{T_q})),$$

pointwise for $p \geq 3$. A straightforward application of the Dubins–Schwarz theorem and Lévy's modulus of continuity leads to

$$(\Delta^n_{i(n,q)} X)^2 \mathbb{I}_{\{|\Delta^n_{i(n,q)} X| > \alpha \Delta^\omega_n\}}(1 - \psi_\rho(\Delta^n_{i(n,q)} X)) \to (\Delta X_{T_q})^2 (1 - \psi_\rho(\Delta X_{T_q})),$$

for $\alpha > 0, 0 < \omega < 1/2$ and, hence, we obtain the result. $\qquad\square$

C   Tables from the simulation study

Table 3: Model (D).

M	R.V.M.	Bias	Std.	0.100	0.050	R.V.M.	Bias	Std.	0.100	0.050
78	RV	-0.31	1.19	0.150	0.100	TRV	-0.31	1.19	0.150	0.100
	RV2	-0.31	1.19	0.150	0.100	TR2V	-0.41	1.18	0.154	0.106
	pRV	-0.28	1.07	0.121	0.078	R2V	-0.41	1.18	0.154	0.106
	pRV2	-0.3	1.15	0.140	0.094	R3V	-0.47	1.17	0.163	0.113
	cRV	-0.31	1.19	0.150	0.100	R4V	-0.52	1.16	0.168	0.115
390	RV	-0.12	1.03	0.105	0.058	TRV	-0.12	1.03	0.105	0.058
	RV2	-0.12	1.03	0.105	0.058	TR2V	-0.17	1.03	0.111	0.064
	pRV	-0.12	0.98	0.092	0.050	R2V	-0.17	1.03	0.111	0.064
	pRV2	-0.12	1.01	0.101	0.056	R3V	-0.2	1.03	0.113	0.067
	cRV	-0.12	1.03	0.105	0.058	R4V	-0.23	1.04	0.116	0.070
780	RV	-0.09	1	0.102	0.050	TRV	-0.09	1	0.101	0.050
	RV2	-0.09	1	0.102	0.050	TR2V	-0.12	1	0.098	0.051
	pRV	-0.09	0.98	0.091	0.044	R2V	-0.12	1	0.098	0.051
	pRV2	-0.09	0.99	0.099	0.048	R3V	-0.14	1	0.102	0.055
	cRV	-0.09	1	0.102	0.050	R4V	-0.16	1	0.104	0.056
4680	RV	-0.07	1.01	0.105	0.054	TRV	-0.07	1.01	0.105	0.054
	RV2	-0.07	1.01	0.105	0.054	TR2V	-0.09	1.01	0.104	0.053
	pRV	-0.07	0.99	0.101	0.054	R2V	-0.09	1.01	0.104	0.053
	pRV2	-0.07	1	0.103	0.054	R3V	-0.09	1.01	0.105	0.054
	cRV	-0.07	1.01	0.105	0.054	R4V	-0.09	1.01	0.106	0.057

Table 4: Model (JD) with one jump a day, where the percentage of the jump part of the quadratic variation of the total quadratic variation is 15%.

M	R.V.M.	Bias	Std.	0.100	0.050	R.V.M.	Bias	Std.	0.100	0.050
78	RV	-0.29	1.21	0.1492	0.0976	TRV	0.2	1.23	0.1752	0.1094
	RV2	-0.29	1.21	0.1502	0.0988	TR2V	-0.17	1.12	0.1218	0.0726
	pRV	-0.25	1.1	0.1228	0.0762	R2V	-0.03	1.14	0.126	0.0726
	pRV2	-0.28	1.19	0.148	0.0948	R3V	-0.19	1.11	0.1204	0.0738
	cRV	-0.29	1.26	0.1686	0.1096	R4V	-0.27	1.1	0.1244	0.078
390	RV	-0.11	1.02	0.1056	0.0532	TRV	0.05	1.02	0.1086	0.0578
	RV2	-0.11	1.02	0.1068	0.0544	TR2V	-0.1	0.98	0.0978	0.053
	pRV	-0.11	1.03	0.109	0.0556	R2V	0.24	1.04	0.1226	0.0678
	pRV2	-0.11	1.06	0.1188	0.0596	R3V	0.04	0.99	0.0976	0.0496
	cRV	-0.11	1.13	0.1392	0.0788	R4V	-0.03	0.98	0.0936	0.0494
780	RV	-0.07	1	0.095	0.0466	TRV	0.01	0.98	0.0972	0.0504
	RV2	-0.07	1.01	0.096	0.0484	TR2V	-0.07	0.97	0.088	0.0464
	pRV	-0.08	1.03	0.1046	0.0544	R2V	0.32	1.04	0.1338	0.0718
	pRV2	-0.08	1.04	0.11	0.0576	R3V	0.1	0.97	0.0964	0.0492
	cRV	-0.08	1.12	0.1372	0.0772	R4V	0.03	0.97	0.0906	0.0478
4680	RV	-0.04	0.98	0.0962	0.0492	TRV	-0.04	0.98	0.0916	0.0526
	RV2	-0.04	0.98	0.0966	0.0498	TR2V	-0.07	0.98	0.0922	0.047
	pRV	-0.04	1.01	0.1026	0.0544	R2V	0.38	1.08	0.1506	0.0896
	pRV2	-0.04	1.01	0.1062	0.0558	R3V	0.11	1	0.1002	0.0494
	cRV	-0.04	1.11	0.1334	0.0784	R4V	0.04	0.99	0.0988	0.0474

Table 5: Model (JD) with five jumps a day, where the percentage of the jump part of the quadratic variation of the total quadratic variation is 15%.

M	R.V.M.	Bias	Std.	0.100	0.050	R.V.M.	Bias	Std.	0.100	0.050
78	RV	-0.3	1.19	0.1492	0.0958	TRV	0.68	1.18	0.2328	0.1436
	RV2	-0.29	1.19	0.1496	0.096	TR2V	0.2	1.06	0.1066	0.0498
	pRV	-0.26	1.05	0.111	0.0682	R2V	0.22	1.06	0.1092	0.0506
	pRV2	-0.28	1.15	0.1388	0.0868	R3V	0.05	1.05	0.0912	0.0488
	cRV	-0.29	1.2	0.1538	0.0988	R4V	-0.03	1.04	0.0898	0.0522
390	RV	-0.11	1.04	0.112	0.0636	TRV	0.96	1.12	0.2852	0.1924
	RV2	-0.11	1.05	0.113	0.0648	TR2V	0.33	0.98	0.12	0.0592
	pRV	-0.1	1.02	0.1082	0.058	R2V	0.64	1	0.183	0.0968
	pRV2	-0.11	1.06	0.1208	0.0664	R3V	0.38	0.97	0.1146	0.0566
	cRV	-0.11	1.11	0.1344	0.0772	R4V	0.27	0.96	0.0982	0.0498
780	RV	-0.07	0.98	0.0906	0.0488	TRV	0.71	1.05	0.21	0.1266
	RV2	-0.07	0.99	0.0914	0.0492	TR2V	0.21	0.94	0.09	0.043
	pRV	-0.06	1.01	0.1002	0.0534	R2V	0.75	0.99	0.203	0.1154
	pRV2	-0.06	1.03	0.1064	0.0584	R3V	0.43	0.94	0.1174	0.0544
	cRV	-0.07	1.08	0.1226	0.0692	R4V	0.31	0.94	0.099	0.045
4680	RV	-0.03	0.96	0.0898	0.0412	TRV	0.15	0.97	0.093	0.0458
	RV2	-0.03	0.96	0.0898	0.0412	TR2V	-0.01	0.95	0.0854	0.0426
	pRV	-0.03	1.01	0.1068	0.055	R2V	0.92	1.05	0.2558	0.1648
	pRV2	-0.03	1.01	0.1064	0.053	R3V	0.44	0.97	0.1198	0.0658
	cRV	-0.03	1.09	0.133	0.0742	R4V	0.3	0.95	0.1008	0.0524

Table 6: Model (JD) with ten jumps a day, where the percentage of the jump part of the quadratic variation of the total quadratic variation is 15%.

M	R.V.M.	Bias	Std.	0.100	0.050	R.V.M.	Bias	Std.	0.100	0.050
78	RV	-0.25	1.17	0.1426	0.0884	TRV	0.77	1.1	0.2272	0.1372
	RV2	-0.25	1.17	0.1432	0.0886	TR2V	0.34	1.02	0.1094	0.049
	pRV	-0.22	1.04	0.1048	0.0636	R2V	0.35	1.02	0.1108	0.05
	pRV2	-0.24	1.13	0.1308	0.0808	R3V	0.18	1.01	0.09	0.0442
	cRV	-0.24	1.17	0.1438	0.0892	R4V	0.09	1	0.087	0.0444
390	RV	-0.14	1.05	0.1192	0.0658	TRV	1.56	1.16	0.4688	0.3574
	RV2	-0.14	1.06	0.119	0.0664	TR2V	0.68	0.97	0.176	0.0892
	pRV	-0.13	0.99	0.101	0.0532	R2V	0.83	0.97	0.2244	0.1256
	pRV2	-0.14	1.05	0.1206	0.0644	R3V	0.56	0.95	0.1456	0.071
	cRV	-0.14	1.08	0.1308	0.0724	R4V	0.45	0.95	0.1212	0.0572
780	RV	-0.1	1.02	0.1088	0.061	TRV	1.5	1.16	0.455	0.3488
	RV2	-0.1	1.03	0.1108	0.0624	TR2V	0.58	0.96	0.152	0.0814
	pRV	-0.09	1.02	0.1066	0.0564	R2V	1	0.98	0.2788	0.167
	pRV2	-0.09	1.05	0.1176	0.0654	R3V	0.65	0.95	0.157	0.0834
	cRV	-0.1	1.09	0.1322	0.077	R4V	0.51	0.94	0.1226	0.066
4680	RV	-0.02	1	0.1026	0.0464	TRV	0.54	1.03	0.1672	0.0984
	RV2	-0.02	1	0.1026	0.0456	TR2V	0.13	0.97	0.0952	0.0464
	pRV	-0.02	1.06	0.125	0.0634	R2V	1.3	1.04	0.382	0.2714
	pRV2	-0.02	1.06	0.1232	0.0626	R3V	0.71	0.97	0.1784	0.1028
	cRV	-0.02	1.12	0.1434	0.0828	R4V	0.52	0.96	0.1342	0.0724

Table 7: Model (IAP) with $\alpha = 0.1$, where the percentage of the jump part of the quadratic variation of the total quadratic variation is 15%.

M	R.V.M.	Bias	Std.	0.100	0.050	R.V.M.	Bias	Std.	0.100	0.050
78	RV	-0.3	1.19	0.1488	0.1004	TRV	0.41	1.22	0.198	0.1256
	RV2	-0.3	1.19	0.1492	0.1012	TR2V	0.03	1.11	0.1182	0.0702
	pRV	-0.26	1.07	0.1172	0.0732	R2V	0.11	1.14	0.1326	0.0776
	pRV2	-0.28	1.15	0.1436	0.0918	R3V	-0.04	1.1	0.1158	0.07
	cRV	-0.29	1.22	0.1588	0.1068	R4V	-0.13	1.09	0.119	0.0722
390	RV	-0.13	1.09	0.1094	0.0616	TRV	0.6	1.11	0.198	0.1178
	RV2	-0.13	1.1	0.111	0.0634	TR2V	0.18	1	0.1072	0.0536
	pRV	-0.12	1.08	0.1052	0.0636	R2V	0.47	1.07	0.1684	0.0954
	pRV2	-0.13	1.12	0.1158	0.069	R3V	0.25	1	0.1134	0.0536
	cRV	-0.13	1.18	0.135	0.0848	R4V	0.15	0.99	0.102	0.046
780	RV	-0.1	1.12	0.1014	0.0518	TRV	0.56	1.07	0.1736	0.1026
	RV2	-0.1	1.12	0.1024	0.052	TR2V	0.18	0.97	0.1008	0.0486
	pRV	-0.1	1.13	0.1054	0.0568	R2V	0.59	1.08	0.189	0.1134
	pRV2	-0.1	1.16	0.1146	0.0608	R3V	0.32	0.99	0.118	0.0546
	cRV	-0.11	1.21	0.1318	0.0758	R4V	0.22	0.97	0.0982	0.0472
4680	RV	-0.06	1.08	0.09	0.044	TRV	0.38	1	0.127	0.0646
	RV2	-0.06	1.09	0.0898	0.044	TR2V	0.1	0.96	0.084	0.0432
	pRV	-0.06	1.15	0.101	0.051	R2V	0.79	1.12	0.2324	0.1522
	pRV2	-0.06	1.15	0.101	0.0518	R3V	0.39	0.98	0.1256	0.0616
	cRV	-0.06	1.24	0.1302	0.073	R4V	0.28	0.97	0.1026	0.0474

Table 8: Model (IAP) with $\alpha = 0.5$, where the percentage of the jump part of the quadratic variation of the total quadratic variation is 15%.

M	R.V.M.	Bias	Std.	0.100	0.050	R.V.M.	Bias	Std.	0.100	0.050
78	RV	-0.27	1.17	0.1476	0.0916	TRV	0.64	1.14	0.2096	0.122
	RV2	-0.27	1.17	0.1476	0.092	TR2V	0.26	1.03	0.111	0.0518
	pRV	-0.25	1.04	0.1092	0.0644	R2V	0.27	1.04	0.1154	0.054
	pRV2	-0.26	1.13	0.136	0.0818	R3V	0.12	1.02	0.0996	0.0516
	cRV	-0.27	1.18	0.1492	0.093	R4V	0.03	1.01	0.0968	0.0522
390	RV	-0.13	1.06	0.1186	0.0658	TRV	1.35	1.13	0.4028	0.2964
	RV2	-0.13	1.06	0.1192	0.0666	TR2V	0.71	0.97	0.1862	0.0994
	pRV	-0.12	1.02	0.1062	0.0576	R2V	0.87	1	0.2338	0.141
	pRV2	-0.13	1.06	0.1192	0.0664	R3V	0.63	0.97	0.1584	0.0802
	cRV	-0.13	1.1	0.1292	0.075	R4V	0.53	0.96	0.1346	0.0664
780	RV	-0.11	1.02	0.1064	0.0602	TRV	1.56	1.14	0.473	0.3588
	RV2	-0.11	1.03	0.1076	0.0612	TR2V	0.8	0.97	0.2006	0.1134
	pRV	-0.1	1.01	0.1038	0.0564	R2V	1.09	1.02	0.3102	0.2046
	pRV2	-0.11	1.04	0.1128	0.066	R3V	0.78	0.97	0.2052	0.1114
	cRV	-0.11	1.08	0.1218	0.0748	R4V	0.65	0.96	0.1612	0.0834
4680	RV	-0.03	0.98	0.1004	0.0474	TRV	1.79	1.11	0.5416	0.4264
	RV2	-0.03	0.98	0.101	0.0474	TR2V	0.86	0.95	0.2096	0.1276
	pRV	-0.03	1.01	0.1134	0.0568	R2V	1.71	1.06	0.5258	0.407
	pRV2	-0.03	1.02	0.117	0.0592	R3V	1.13	0.96	0.3064	0.2018
	cRV	-0.03	1.08	0.1326	0.0734	R4V	0.93	0.94	0.2348	0.1396

Table 9: Model (IAP) with $\alpha = 0.9$, where the percentage of the jump part of the quadratic variation of the total quadratic variation is 15%.

M	R.V.M.	Bias	Std.	0.100	0.050	R.V.M.	Bias	Std.	0.100	0.050
78	RV	-0.29	1.15	0.1384	0.09	TRV	0.68	1.01	0.1822	0.1006
	RV2	-0.29	1.15	0.1384	0.09	TR2V	0.42	0.99	0.1214	0.052
	pRV	-0.27	1.04	0.11	0.0652	R2V	0.42	0.99	0.1216	0.0522
	pRV2	-0.28	1.11	0.129	0.0814	R3V	0.29	0.99	0.1012	0.047
	cRV	-0.29	1.15	0.1386	0.09	R4V	0.21	0.99	0.097	0.0482
390	RV	-0.14	1.04	0.117	0.061	TRV	1.86	1.01	0.5914	0.4686
	RV2	-0.14	1.04	0.1172	0.061	TR2V	1.37	0.92	0.408	0.2762
	pRV	-0.13	0.99	0.099	0.05	R2V	1.38	0.92	0.413	0.2792
	pRV2	-0.13	1.02	0.1108	0.057	R3V	1.18	0.92	0.3302	0.206
	cRV	-0.14	1.04	0.1184	0.0616	R4V	1.07	0.93	0.2874	0.1704
780	RV	-0.09	1.02	0.1072	0.0562	TRV	2.66	1.1	0.8218	0.7366
	RV2	-0.09	1.02	0.1076	0.0562	TR2V	1.89	0.96	0.6158	0.4866
	pRV	-0.09	0.98	0.0928	0.0476	R2V	1.92	0.96	0.6308	0.5006
	pRV2	-0.09	1.01	0.1024	0.0542	R3V	1.65	0.94	0.5164	0.3788
	cRV	-0.09	1.03	0.109	0.0568	R4V	1.5	0.93	0.454	0.3178
4680	RV	-0.03	1.01	0.104	0.0522	TRV	5.41	1.52	0.994	0.9902
	RV2	-0.03	1.01	0.1042	0.053	TR2V	3.33	1.07	0.9354	0.8956
	pRV	-0.03	1	0.0994	0.05	R2V	3.73	1.1	0.9638	0.9418
	pRV2	-0.03	1.03	0.1082	0.054	R3V	3.02	1.02	0.9088	0.85
	cRV	-0.03	1.05	0.1148	0.0596	R4V	2.71	1	0.8556	0.7806

Table 10: Model (IFAJ) with five jumps a day.

M	R.V.M.	Bias	Std.	0.100	0.050	R.V.M.	Bias	Std.	0.100	0.050
78	RV	-0.3	1.17	0.1442	0.0956	TRV	0.46	1.15	0.1798	0.1078
	RV2	-0.3	1.17	0.1456	0.096	TR2V	0.06	1.06	0.098	0.058
	pRV	-0.26	1.06	0.1126	0.0712	R2V	0.16	1.08	0.1088	0.0616
	pRV2	-0.29	1.14	0.1362	0.0898	R3V	0	1.05	0.0994	0.055
	cRV	-0.29	1.21	0.1534	0.103	R4V	-0.08	1.04	0.1048	0.057
390	RV	-0.12	1.03	0.1166	0.0572	TRV	0.49	1.04	0.1608	0.0908
	RV2	-0.12	1.03	0.1178	0.0596	TR2V	0.11	0.98	0.0966	0.0522
	pRV	-0.11	1.03	0.1178	0.0602	R2V	0.53	1.04	0.1634	0.0944
	pRV2	-0.12	1.06	0.1264	0.0682	R3V	0.27	0.99	0.1126	0.053
	cRV	-0.12	1.12	0.1438	0.0818	R4V	0.17	0.98	0.1002	0.0472
780	RV	-0.08	1.02	0.1024	0.0506	TRV	0.36	1.01	0.1264	0.068
	RV2	-0.08	1.02	0.1038	0.0506	TR2V	0.09	0.97	0.0912	0.0446
	pRV	-0.08	1.06	0.1166	0.0596	R2V	0.67	1.05	0.201	0.1144
	pRV2	-0.09	1.07	0.1196	0.0628	R3V	0.37	0.99	0.1208	0.0608
	cRV	-0.09	1.13	0.138	0.0794	R4V	0.27	0.99	0.1084	0.0502
4680	RV	-0.03	1.12	0.0996	0.047	TRV	0.11	0.99	0.1	0.05
	RV2	-0.03	1.12	0.1002	0.0468	TR2V	-0.01	0.97	0.0924	0.0468
	pRV	-0.03	1.19	0.1178	0.0592	R2V	0.83	1.09	0.2438	0.159
	pRV2	-0.03	1.19	0.1178	0.059	R3V	0.39	1	0.124	0.0692
	cRV	-0.03	1.28	0.1442	0.0822	R4V	0.27	0.98	0.1102	0.0536

Table 11: Model (IFAJ) with ten jumps a day.

M	R.V.M.	Bias	Std.	0.100	0.050	R.V.M.	Bias	Std.	0.100	0.050
78	RV	-0.31	1.16	0.1444	0.0972	TRV	0.26	1.12	0.144	0.0838
	RV2	-0.31	1.16	0.1446	0.0974	TR2V	0	1.09	0.1126	0.0636
	pRV	-0.28	1.04	0.115	0.074	R2V	0.01	1.1	0.1136	0.0642
	pRV2	-0.3	1.12	0.1358	0.09	R3V	-0.09	1.09	0.1084	0.0648
	cRV	-0.31	1.17	0.1458	0.098	R4V	-0.16	1.08	0.1146	0.0684
390	RV	-0.16	1.04	0.1144	0.0586	TRV	0.66	1.05	0.2	0.1152
	RV2	-0.16	1.04	0.1152	0.0592	TR2V	0.27	0.98	0.1092	0.0522
	pRV	-0.15	1	0.1026	0.0534	R2V	0.41	1	0.1332	0.0666
	pRV2	-0.16	1.04	0.1146	0.0598	R3V	0.25	0.98	0.1038	0.0472
	cRV	-0.16	1.06	0.1238	0.0654	R4V	0.17	0.98	0.095	0.0446
780	RV	-0.11	1	0.1044	0.0574	TRV	0.69	1.02	0.1918	0.1106
	RV2	-0.11	1.01	0.1052	0.0582	TR2V	0.26	0.96	0.099	0.0472
	pRV	-0.11	1	0.1042	0.0532	R2V	0.53	0.98	0.1442	0.0732
	pRV2	-0.11	1.02	0.1112	0.0604	R3V	0.33	0.97	0.105	0.0522
	cRV	-0.11	1.05	0.1178	0.0676	R4V	0.24	0.97	0.1012	0.0478
4680	RV	-0.04	1	0.0984	0.0514	TRV	0.39	1	0.1296	0.0756
	RV2	-0.04	1	0.0986	0.0512	TR2V	0.1	0.97	0.0918	0.0464
	pRV	-0.04	1.03	0.1108	0.0582	R2V	0.78	1.01	0.214	0.1316
	pRV2	-0.04	1.04	0.1118	0.06	R3V	0.44	0.96	0.1242	0.0622
	cRV	-0.05	1.07	0.122	0.0684	R4V	0.32	0.96	0.1026	0.0566

Table 12: Model (CFAJ) with five jumps a day.

M	R.V.M.	Bias	Std.	0.100	0.050	R.V.M.	Bias	Std.	0.100	0.050
78	RV	-0.31	1.2	0.1468	0.098	TRV	0.38	1.11	0.1578	0.0866
	RV2	-0.31	1.21	0.1474	0.098	TR2V	0.06	1.06	0.0954	0.0524
	pRV	-0.26	1.08	0.1172	0.0746	R2V	0.17	1.08	0.1072	0.0546
	pRV2	-0.29	1.17	0.1412	0.0908	R3V	0.01	1.06	0.092	0.055
	cRV	-0.3	1.24	0.159	0.1058	R4V	-0.07	1.05	0.0938	0.0576
390	RV	-0.14	1.03	0.1082	0.058	TRV	0.45	1.02	0.1446	0.084
	RV2	-0.14	1.04	0.11	0.0604	TR2V	0.12	0.97	0.0924	0.0482
	pRV	-0.13	1.05	0.1164	0.0606	R2V	0.57	1.03	0.1642	0.093
	pRV2	-0.13	1.07	0.126	0.066	R3V	0.32	0.98	0.1126	0.0532
	cRV	-0.13	1.13	0.141	0.081	R4V	0.22	0.97	0.0984	0.0498
780	RV	-0.09	0.99	0.096	0.052	TRV	0.36	0.99	0.1186	0.0646
	RV2	-0.09	0.99	0.0962	0.0522	TR2V	0.08	0.95	0.086	0.0416
	pRV	-0.09	1.03	0.1096	0.063	R2V	0.68	1.01	0.191	0.1056
	pRV2	-0.1	1.04	0.1138	0.0654	R3V	0.39	0.95	0.1118	0.0478
	cRV	-0.1	1.11	0.1344	0.0816	R4V	0.28	0.94	0.0976	0.0422
4680	RV	-0.05	0.96	0.0846	0.0418	TRV	0.09	0.96	0.088	0.0428
	RV2	-0.05	0.96	0.0852	0.0418	TR2V	-0.03	0.95	0.0866	0.0428
	pRV	-0.05	1.02	0.1062	0.0546	R2V	0.85	1.05	0.2396	0.1516
	pRV2	-0.05	1.02	0.1048	0.0536	R3V	0.41	0.97	0.1198	0.0578
	cRV	-0.05	1.11	0.1344	0.074	R4V	0.28	0.96	0.0972	0.0504

Table 13: Model (CFAJ) with ten jumps a day.

M	R.V.M.	Bias	Std.	0.100	0.050	R.V.M.	Bias	Std.	0.100	0.050
78	RV	-0.27	1.15	0.1428	0.088	TRV	0.26	1.08	0.1298	0.0692
	RV2	-0.27	1.16	0.143	0.0882	TR2V	0.02	1.06	0.1014	0.0558
	pRV	-0.24	1.04	0.1114	0.0694	R2V	0.04	1.06	0.1016	0.055
	pRV2	-0.26	1.12	0.1328	0.0814	R3V	-0.06	1.05	0.1028	0.0578
	cRV	-0.26	1.16	0.1454	0.0916	R4V	-0.13	1.05	0.1062	0.063
390	RV	-0.14	1.03	0.112	0.061	TRV	0.64	1.02	0.1788	0.0998
	RV2	-0.14	1.03	0.1118	0.0612	TR2V	0.28	0.97	0.1046	0.0492
	pRV	-0.13	1	0.1012	0.0522	R2V	0.43	0.99	0.1324	0.0644
	pRV2	-0.13	1.04	0.1134	0.0606	R3V	0.28	0.98	0.1036	0.0514
	cRV	-0.14	1.06	0.1216	0.0672	R4V	0.21	0.97	0.0964	0.0488
780	RV	-0.09	1.01	0.103	0.056	TRV	0.67	1.02	0.1886	0.1098
	RV2	-0.09	1.01	0.1038	0.0562	TR2V	0.28	0.96	0.1032	0.0458
	pRV	-0.09	1.01	0.1	0.053	R2V	0.57	0.98	0.1536	0.0782
	pRV2	-0.09	1.03	0.1094	0.0582	R3V	0.37	0.96	0.1118	0.0542
	cRV	-0.09	1.06	0.1182	0.0646	R4V	0.29	0.96	0.104	0.0496
4680	RV	-0.02	1	0.101	0.0512	TRV	0.4	1.01	0.138	0.0738
	RV2	-0.02	1	0.1014	0.0514	TR2V	0.12	0.98	0.1004	0.0492
	pRV	-0.02	1.04	0.1108	0.0586	R2V	0.84	1.02	0.228	0.1464
	pRV2	-0.02	1.04	0.1128	0.0608	R3V	0.49	0.98	0.1366	0.075
	cRV	-0.02	1.08	0.124	0.0676	R4V	0.36	0.97	0.1158	0.0616

Table 14: Model (BNSIJ) without jumps in the price.

M	R.V.M.	Bias	Std.	0.100	0.050	R.V.M.	Bias	Std.	0.100	0.050
78	RV	-0.25	1.18	0.1424	0.0926	TRV	-0.3	1.21	0.1508	0.1
	RV2	-0.26	1.18	0.1426	0.0926	TR2V	-0.4	1.21	0.1514	0.102
	pRV	-0.23	1.08	0.1178	0.0696	R2V	-0.36	1.19	0.1444	0.0948
	pRV2	-0.25	1.15	0.1368	0.0852	R3V	-0.43	1.17	0.1488	0.0968
	cRV	-0.25	1.19	0.1446	0.0936	R4V	-0.47	1.16	0.153	0.0962
390	RV	-0.11	1.03	0.11	0.0622	TRV	-0.19	1.15	0.1256	0.0774
	RV2	-0.11	1.03	0.1098	0.0622	TR2V	-0.21	1.12	0.1264	0.0746
	pRV	-0.1	0.99	0.0998	0.0542	R2V	-0.14	1.03	0.1132	0.0622
	pRV2	-0.11	1.02	0.1068	0.0606	R3V	-0.17	1.03	0.114	0.062
	cRV	-0.11	1.04	0.1108	0.0632	R4V	-0.19	1.03	0.1118	0.0632
780	RV	-0.08	1.01	0.1046	0.054	TRV	-0.19	1.24	0.1252	0.0736
	RV2	-0.08	1	0.1046	0.0534	TR2V	-0.2	1.19	0.123	0.0678
	pRV	-0.08	0.98	0.095	0.0476	R2V	-0.11	1.01	0.1048	0.0506
	pRV2	-0.08	1	0.1018	0.052	R3V	-0.13	1.01	0.1048	0.0556
	cRV	-0.08	1.01	0.1062	0.0552	R4V	-0.15	1	0.1038	0.0534
4680	RV	-0.04	1	0.0996	0.051	TRV	-0.26	1.8	0.1262	0.0768
	RV2	-0.04	1	0.0992	0.0508	TR2V	-0.24	1.65	0.1222	0.0774
	pRV	-0.04	0.99	0.0982	0.049	R2V	-0.05	0.99	0.0982	0.0526
	pRV2	-0.04	1	0.1004	0.051	R3V	-0.05	0.99	0.1008	0.055
	cRV	-0.04	1	0.1012	0.0516	R4V	-0.05	0.99	0.1018	0.053

Table 15: Model (BNSIJ) with one jump in the price, where the percentage of the jump part of the QV is 15%.

M	R.V.M.	Bias	Std.	0.100	0.050	R.V.M.	Bias	Std.	0.100	0.050
78	RV	-0.26	1.2	0.144	0.094	TRV	0.3	1.32	0.2078	0.1382
	RV2	-0.26	1.21	0.1452	0.0956	TR2V	-0.14	1.12	0.1164	0.0674
	pRV	-0.23	1.1	0.1132	0.0688	R2V	0.03	1.12	0.1178	0.0624
	pRV2	-0.25	1.18	0.14	0.0892	R3V	-0.14	1.08	0.1046	0.0606
	cRV	-0.26	1.26	0.1636	0.1094	R4V	-0.23	1.07	0.111	0.0658
390	RV	-0.14	1.12	0.1048	0.058	TRV	0.01	1.17	0.1376	0.0826
	RV2	-0.14	1.13	0.1074	0.059	TR2V	-0.15	1.09	0.1142	0.063
	pRV	-0.13	1.13	0.1094	0.0576	R2V	0.27	1.06	0.1356	0.0716
	pRV2	-0.14	1.16	0.1174	0.0654	R3V	0.05	1.01	0.098	0.0494
	cRV	-0.14	1.25	0.1472	0.0898	R4V	-0.02	1	0.0988	0.0492
780	RV	-0.09	1.05	0.1068	0.0588	TRV	-0.06	1.23	0.1318	0.0744
	RV2	-0.09	1.06	0.1072	0.0602	TR2V	-0.16	1.14	0.1104	0.0646
	pRV	-0.09	1.09	0.1186	0.0656	R2V	0.35	1.06	0.1412	0.0816
	pRV2	-0.09	1.1	0.1234	0.0714	R3V	0.12	0.99	0.102	0.0474
	cRV	-0.1	1.2	0.151	0.0946	R4V	0.04	0.98	0.0932	0.0452
4680	RV	-0.04	1.01	0.0982	0.0502	TRV	-0.24	1.75	0.122	0.073
	RV2	-0.04	1.01	0.0992	0.0506	TR2V	-0.25	1.61	0.118	0.0718
	pRV	-0.04	1.04	0.1114	0.058	R2V	0.42	1.09	0.1588	0.0948
	pRV2	-0.04	1.05	0.112	0.058	R3V	0.12	1	0.1008	0.0504
	cRV	-0.04	1.17	0.1534	0.0888	R4V	0.04	0.99	0.0968	0.0464

Table 16: Model (BNSIJ) with five jumps in the price, where the percentage of the jump part of the QV is 15%.

M	R.V.M.	Bias	Std.	0.100	0.050	R.V.M.	Bias	Std.	0.100	0.050
78	RV	-0.28	1.21	0.1616	0.1062	TRV	0.87	1.34	0.3092	0.2188
	RV2	-0.28	1.21	0.1612	0.1066	TR2V	0.27	1.12	0.1352	0.0708
	pRV	-0.24	1.05	0.1162	0.0674	R2V	0.32	1.08	0.1304	0.065
	pRV2	-0.26	1.16	0.1468	0.0944	R3V	0.13	1.05	0.102	0.05
	cRV	-0.28	1.23	0.1658	0.1096	R4V	0.03	1.04	0.101	0.0512
390	RV	-0.13	1.05	0.1168	0.0648	TRV	1.14	1.41	0.3838	0.2908
	RV2	-0.13	1.06	0.1186	0.0664	TR2V	0.32	1.11	0.1358	0.0754
	pRV	-0.11	1.02	0.1066	0.0578	R2V	0.72	1.02	0.2058	0.1162
	pRV2	-0.12	1.07	0.125	0.068	R3V	0.43	0.97	0.121	0.053
	cRV	-0.13	1.13	0.1466	0.0876	R4V	0.31	0.96	0.0958	0.0468
780	RV	-0.09	1	0.0986	0.0488	TRV	0.8	1.4	0.2886	0.1984
	RV2	-0.09	1.01	0.0998	0.0496	TR2V	0.17	1.18	0.126	0.0678
	pRV	-0.08	1.03	0.104	0.053	R2V	0.85	1.04	0.243	0.1446
	pRV2	-0.09	1.05	0.1142	0.0586	R3V	0.48	0.98	0.137	0.0674
	cRV	-0.09	1.13	0.139	0.078	R4V	0.35	0.97	0.1086	0.0544
4680	RV	-0.03	0.98	0.0902	0.0436	TRV	0.01	1.84	0.1316	0.0786
	RV2	-0.03	0.98	0.09	0.0432	TR2V	-0.17	1.65	0.1086	0.0672
	pRV	-0.03	1.04	0.1146	0.0578	R2V	1.05	1.07	0.2974	0.2
	pRV2	-0.03	1.04	0.1106	0.0556	R3V	0.5	0.97	0.1352	0.0724
	cRV	-0.03	1.16	0.1512	0.0848	R4V	0.34	0.95	0.1072	0.0546

Table 17: Model (BNSIJ) with ten jumps in the price, where the percentage of the jump part of the QV is 15%.

M	R.V.M.	Bias	Std.	0.100	0.050	R.V.M.	Bias	Std.	0.100	0.050
78	RV	-0.28	1.19	0.1532	0.099	TRV	0.88	1.23	0.2958	0.1914
	RV2	-0.28	1.2	0.1532	0.0988	TR2V	0.39	1.1	0.1456	0.0698
	pRV	-0.25	1.06	0.1088	0.0684	R2V	0.43	1.06	0.1386	0.0652
	pRV2	-0.27	1.16	0.1428	0.088	R3V	0.25	1.05	0.107	0.0518
	cRV	-0.28	1.2	0.1562	0.1004	R4V	0.15	1.05	0.1004	0.0512
390	RV	-0.11	1.05	0.1166	0.065	TRV	1.87	1.5	0.5992	0.5086
	RV2	-0.11	1.06	0.1178	0.0658	TR2V	0.74	1.11	0.219	0.1302
	pRV	-0.1	0.98	0.095	0.0464	R2V	0.97	1	0.2656	0.1678
	pRV2	-0.11	1.05	0.1174	0.0618	R3V	0.66	0.95	0.1604	0.0814
	cRV	-0.11	1.1	0.133	0.0764	R4V	0.52	0.93	0.126	0.0618
780	RV	-0.08	1.02	0.104	0.0526	TRV	1.77	1.54	0.588	0.4902
	RV2	-0.08	1.02	0.1038	0.0522	TR2V	0.6	1.15	0.186	0.1104
	pRV	-0.07	1.01	0.1024	0.0492	R2V	1.14	1	0.3338	0.2198
	pRV2	-0.08	1.05	0.1164	0.0574	R3V	0.74	0.94	0.1784	0.0968
	cRV	-0.08	1.1	0.135	0.0728	R4V	0.58	0.93	0.137	0.0672
4680	RV	-0.03	0.98	0.0922	0.0482	TRV	0.46	1.79	0.2178	0.1462
	RV2	-0.03	0.98	0.091	0.0476	TR2V	-0.02	1.55	0.1146	0.0656
	pRV	-0.03	1.05	0.1152	0.0622	R2V	1.41	1.09	0.4264	0.3112
	pRV2	-0.03	1.05	0.1146	0.06	R3V	0.75	0.99	0.1982	0.1152
	cRV	-0.03	1.14	0.1478	0.0822	R4V	0.54	0.97	0.1472	0.0772

Table 18: Model (BNSCJ) with $\rho = -0.2$ corresponding to the jump part of the QV being 4.7%.

M	R.V.M.	Bias	Std.	0.100	0.050	R.V.M.	Bias	Std.	0.100	0.050
78	RV	-0.25	1.18	0.144	0.0944	TRV	-0.27	1.19	0.1494	0.0986
	RV2	-0.25	1.18	0.1444	0.0946	TR2V	-0.38	1.19	0.1466	0.0976
	pRV	-0.23	1.08	0.1184	0.073	R2V	-0.33	1.17	0.138	0.0908
	pRV2	-0.25	1.15	0.1392	0.088	R3V	-0.4	1.16	0.1442	0.0962
	cRV	-0.25	1.19	0.1474	0.0968	R4V	-0.44	1.14	0.147	0.0938
390	RV	-0.1	1.03	0.111	0.063	TRV	-0.16	1.13	0.124	0.0758
	RV2	-0.1	1.03	0.1104	0.0628	TR2V	-0.19	1.11	0.1218	0.071
	pRV	-0.1	1	0.101	0.0554	R2V	-0.1	1.03	0.1096	0.0588
	pRV2	-0.1	1.02	0.109	0.0616	R3V	-0.14	1.03	0.113	0.0604
	cRV	-0.1	1.04	0.1132	0.0654	R4V	-0.17	1.03	0.1092	0.0614
780	RV	-0.07	1.01	0.107	0.055	TRV	-0.15	1.21	0.1248	0.0712
	RV2	-0.07	1.01	0.1066	0.0546	TR2V	-0.18	1.17	0.1214	0.0666
	pRV	-0.07	0.99	0.0974	0.0494	R2V	-0.06	1.01	0.1032	0.0498
	pRV2	-0.07	1	0.1036	0.054	R3V	-0.1	1	0.1014	0.0526
	cRV	-0.07	1.02	0.1084	0.0566	R4V	-0.12	1	0.1002	0.0512
4680	RV	-0.04	1	0.0976	0.0502	TRV	-0.23	1.76	0.1262	0.0762
	RV2	-0.04	0.99	0.097	0.0502	TR2V	-0.21	1.62	0.1218	0.0778
	pRV	-0.04	0.99	0.0974	0.0496	R2V	0	0.99	0.099	0.0536
	pRV2	-0.04	1	0.098	0.0506	R3V	-0.01	0.99	0.0982	0.0546
	cRV	-0.04	1.01	0.1002	0.0522	R4V	-0.02	0.99	0.0984	0.0516

Table 19: Model (BNSCJ) with $\rho = -0.4$ corresponding to the jump part of the QV being 16.7%.

| M	R.V.M.	Bias	Std.	0.100	0.050	R.V.M.	Bias	Std.	0.100	0.050
78	RV	-0.25	1.2	0.1452	0.0958	TRV	-0.23	1.18	0.1446	0.0954
	RV2	-0.25	1.2	0.1444	0.0954	TR2V	-0.35	1.17	0.1384	0.0962
	pRV	-0.23	1.11	0.12	0.0742	R2V	-0.28	1.17	0.1324	0.0882
	pRV2	-0.25	1.18	0.1386	0.0884	R3V	-0.36	1.15	0.1376	0.0908
	cRV	-0.25	1.23	0.1494	0.0996	R4V	-0.41	1.13	0.1428	0.0914
390	RV	-0.1	1.03	0.1126	0.063	TRV	-0.12	1.12	0.123	0.0726
	RV2	-0.1	1.03	0.1118	0.0626	TR2V	-0.17	1.1	0.1194	0.0692
	pRV	-0.09	1.01	0.1052	0.0572	R2V	-0.05	1.04	0.1102	0.0588
	pRV2	-0.1	1.03	0.1114	0.0616	R3V	-0.11	1.03	0.1098	0.0576
	cRV	-0.1	1.06	0.1192	0.0688	R4V	-0.14	1.02	0.1094	0.059
780	RV	-0.07	1.01	0.1082	0.055	TRV	-0.12	1.19	0.1258	0.0718
	RV2	-0.07	1.01	0.1076	0.0536	TR2V	-0.16	1.15	0.1176	0.066
	pRV	-0.07	1	0.1014	0.0514	R2V	-0.01	1.01	0.1044	0.0508
	pRV2	-0.07	1.01	0.1068	0.054	R3V	-0.06	1	0.1004	0.0508
	cRV	-0.07	1.04	0.115	0.0602	R4V	-0.09	0.99	0.0998	0.0496
4680	RV	-0.04	1	0.0968	0.0498	TRV	-0.2	1.72	0.1248	0.076
	RV2	-0.04	0.99	0.0962	0.0496	TR2V	-0.2	1.58	0.1206	0.0766
	pRV	-0.04	1	0.0998	0.0514	R2V	0.07	1.01	0.1064	0.0584
	pRV2	-0.04	1	0.0984	0.0512	R3V	0.01	0.99	0.0982	0.0526
	cRV	-0.04	1.02	0.1058	0.0562	R4V	0	0.98	0.0982	0.0508
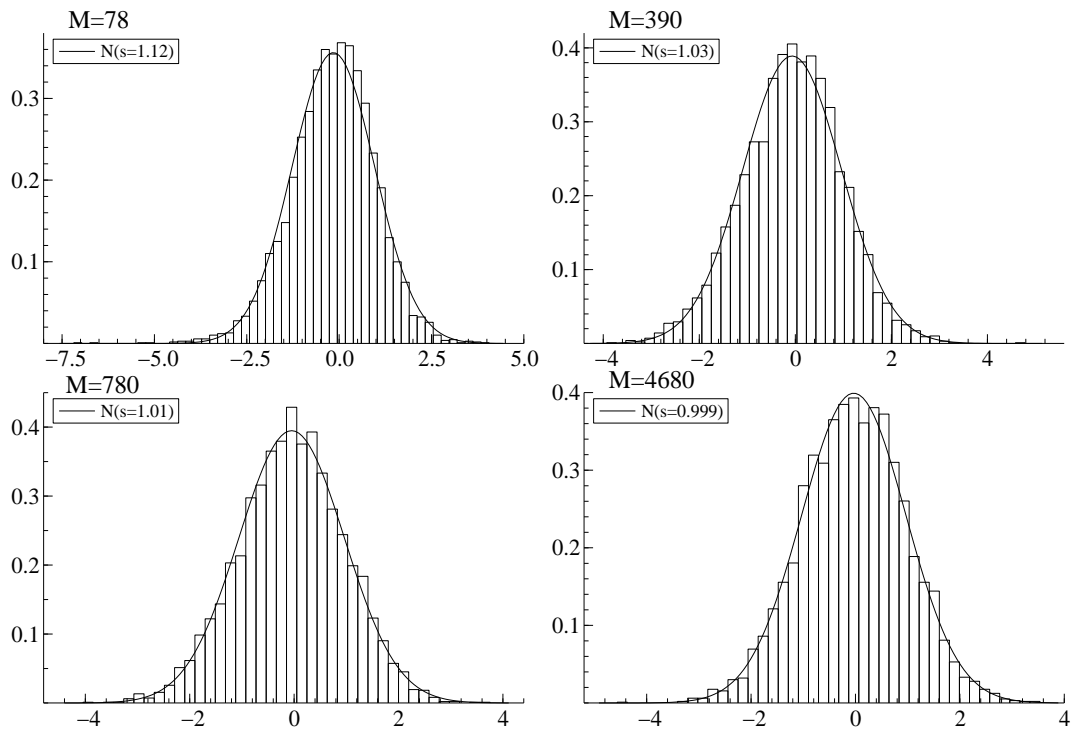
Figure 5: Empirical density of the logarithmic transform of centered and scaled realised variance (denoted by RV in the tables) for the model BNSCJ with $\rho = -0.4$.

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