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Latent Integrated Stochastic Volatility, Realized Volatility, and Implied Volatility: A State Space Approach

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Abstract

We include simultaneously both realized volatility measures based on high-frequency asset returns and implied volatilities backed out of individual traded at the money option prices in a state space approach to the analysis of true underlying volatility. We model integrated volatility as a latent first order Markov process and show that our model is closely related to the CEV and Barndorff-Nielsen & Shephard (2001) models for local volatility. We show that if measurement noise in the observable volatility proxies is not accounted for, then the estimated autoregressive parameter in the latent process is downward biased. Implied volatility performs better than any of the alternative realized measures when forecasting future integrated volatility. The results are largely similar across the stock market (S&P 500), bond market (30-year U.S. T-bond), and foreign currency exchange market ($/£).

JEL Classifications: C32, G13, G14, G17

Keywords: Autoregression, bipower variation, high-frequency data, implied volatility, integrated volatility, Kalman filter, moving average, option prices, realized volatility, state space model, stochastic volatility

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1 Introduction

The measurement and modelling of volatility using high-frequency return data has attracted great attention in the recent literature. In this realized volatility approach, the sum of squared high-frequency returns over an interval consistently estimates the integrated true but unobserved (latent) volatility process, see, e.g., Barndorff-Nielsen & Shephard (2002*b*) and Andersen, Bollerslev, Diebold & Labys (2001). In a related literature, implied volatility backed out from observed option prices is considered as a forecast of subsequent realized volatility over the life of the option, see, e.g., Christensen & Prabhala (1998) and Bollerslev & Zhou (2006). Thus, realized volatility and implied volatility are both imperfect proxies of true underlying integrated volatility. In this paper, we conduct a state space analysis that explicitly addresses the resulting filtering problem when using simultaneously both types of volatility proxies in studying the stochastic process governing true underlying volatility.

We take integrated volatility as the unobserved state variable process and adopt a first order Markov specification for this.[1] We then introduce data on the available realized and implied volatility measures and extend the specification to a state space model. The measurement equation says that the observed realized and implied volatility measures are noisy proxies for true integrated volatility. Thus, it takes the form

$$\begin{pmatrix} RV_t \\ IV_t \end{pmatrix} \equiv \alpha + \beta \int_{t-1}^{t} \sigma_s^2 ds + error,$$

where RV_t is a realized volatility measure calculated from high-frequency asset returns over the interval from $t-1$ to t, IV_t is an option-implied volatility measure pertaining to the same period, and σ_s^2 is the true but unobserved underlying volatility process. In our empirical implementation, we estimate both the vector α collecting the biases of the available imperfect volatility proxies, and the vector β of sensitivities of the proxies to true volatility. The state equation is the transition equation for integrated volatility. The resulting state space model is linear, so the Kalman filter applies. The log likelihood function is constructed from the prediction error decomposition based on the innovations from the filter. This allows estimating the parameters of both the true underlying volatility process and the measurement error mechanism. We find empirically that a simple specification where latent integrated volatility follows an AR(1) process, and hence measured volatility is AR(1) plus noise, describes the data well. In particular, the additional parameters in higher order specifications for latent integrated volatility turn out insignificant, once measurement error noise is allowed for.

We show how our approach allows the forecasting of future true but unobserved volatility, conditionally on both realized and implied current observable proxies, whereas previous research has focussed on the forecasting of future observable volatility proxies, such as realized volatility

[1] This accomodates general finite order ARMA(p, q) specifications for integrated volatility in the state space framework.

or the standard error of observed returns. In an empirical application, we consider the stock market (S&P 500), bond market (30-year U.S. T-Bond), and foreign exchange (\$/£) market. The results show that both realized and implied volatility measures contribute to the forecasting of future true volatility, and that implied volatility in this context is a much more important component of the optimal volatility forecast than realized volatility measures. This suggests that option prices are more sensitive to, and hence more informative about future volatility than observed returns, even when the latter are measured at high frequency. This makes sense from a market efficiency viewpoint, since option traders have access to the historical return record.

This paper draws on several papers within the econometrics and financial econometrics literature. Based on Kalman filter estimation techniques we investigate the relationship between and forecasting ability of volatility measures such as realized volatility, bipower volatility and implied volatility and as such our paper is related to Canina & Figlewski (1993) who find that implied volatility (IV) has no forecasting ability of future realized volatility. Christensen & Prabhala (1998) shows that using non-overlapping observations at a lower frequency, IV is an important predictor of future realized volatility. Bollerslev & Zhou (2006) is a recent contribution to this literature. Many models have been proposed for the modelling of local volatility, such as the ARCH by Engle (1982), GARCH by Bollerslev (1986) and stochastic volatility models. Our simple AR(1) plus noise specification for realized volatility is closely related to the CEV local volatility model, with the Heston (1993) square-root model and the Nelson (1991) GARCH-diffusion as special cases, and to the Lévy driven Ornstein-Uhlenbeck (OU) model for local volatility proposed by Barndorff-Nielsen & Shephard (2001, 2002a), henceforth referred to as the BNS model. In particular, Granger & Morris (1976) show that an AR(1) plus noise model can always be rewritten as an ARMA(1,1), which is the structure for integrated volatility implied by the CEV and BNS models for local volatility.

Recently, Hansen & Lunde (2010) consider an instrumental variable approach to the measurement error in realized volatility. Instrumental variables and the state space framework are alternative approaches to the problem of measurement error in observable proxies, but neither has previously been applied to the full set of available proxies, including both realized measures and individual option-implied volatility simultaneously, and across different markets. Koopman, Jungbacker & Hol (2005) compare the forecasting power of realized volatility and the model-free implied VIX volatility measure in a state space analysis and conclude that VIX is more informative than realized volatility based on daily returns, but less informative than realized volatility based on high-frequency returns. They do not compare with bipower variation or implied volatility backed out of traded options (VIX may be regarded as a smoothed average of such), and they also do not combine forecasts based on different proxies or consider other markets than the stock market. Since our results show that implied volatility based on individual traded option prices is more informative than realized measures, the indication is that individual Black-Scholes style option-implied volatility forecasts better than model-free VIX, and this is consistent with Andersen & Bondarenko (2007).

The paper is laid out as follows. Section 2 introduces the state space approach, how our AR(1) plus noise model is related to the BNS model and general results for the relationship between AR(1) plus noise models and ARMA(1,1) models. Section 3 discusses volatility measurement and describes our data on the stock market (S&P 500), bond market (30-year U.S. T-Bond), and foreign exchange ($/£) market. The empirical results are presented in Section 4, and Section 5 concludes.

2 The State Space Approach

Let S_t denote the asset price at time t and consider a general continuous time specification of the form

$$dS_t = S_t(\mu_t dt + \sigma_t dW_t^s), \tag{1}$$

where μ_t is the instantaneous expected return, σ_t the local return volatility, and W_t^s a standard Wiener process driving asset returns. In much of finance, the major interest is in the specification of the process for σ_t. In particular, derivative pricing is based on expected cash flow calculations using an equivalent martingale measure where μ_t is replaced by the riskless rate of interest, whereas the volatility process is unaltered and retains critical importance for pricing purposes. The Black & Scholes (1973) and Merton (1973) specification yielding explicit option price formulas is the special case of constant volatility, $\sigma_t \equiv \sigma$, but abounding empirical evidence indicates the need for a specification allowing for time-varying volatility, including volatility clustering, i.e., positive serial dependence in volatility. Of course, volatility is not directly observed, and even asset price observations are not available in continuous time, but only as a discrete time record. Accordingly, empirical volatility models focus on a suitable series of volatility measures covering consecutive discrete time intervals.

We specify a discrete time Markovian model for integrated volatility. In the sequel, both high-frequency return data and observed option prices are used in the estimation of the model. Integrated volatility from $t-1$ to t, defined as

$$\overline{\sigma}_t^2 = \int_{t-1}^{t} \sigma_s^2 ds, \tag{2}$$

is of interest for several reasons. This is the relevant true volatility measure that we have a chance of estimating with discrete time observed proxies. Furthermore, in option pricing this is indeed the relevant volatility measure, see Hull & White (1987).

In our state space approach, we take the state variable to be the latent integrated volatility process,

$$x_t = \overline{\sigma}_t^2 - E\left(\overline{\sigma}_t^2\right). \tag{3}$$

We adopt a first order Markov or autoregressive (AR(1)) specification for the state variable process,

$$x_{t+1} = \gamma x_t + u_{t+1}, \tag{4}$$

and write σ_u^2 for the volatility-of-volatility parameter, the variance of the zero mean, serially uncorrelated shocks u_{t+1} representing the innovations in integrated volatility.

Let y_t be a k-vector of observed volatility proxies. This could contain realized volatility, implied volatility, realized bipower variation, or other variables in the data set. The equation linking volatility proxies and true but latent volatility is

$$y_t = \alpha + \beta x_t + \varepsilon_t, \tag{5}$$

where β is a k-vector of sensitivities of the proxies to true volatility, α allows for biases in the proxies, and ε_t is the k-vector (or a matrix) of measurement errors or noise terms in the volatility proxies.

The model is in state space form. Thus, (4) is the state transition equation, and (5) is the measurement equation. While true volatility is AR(1), the data-generating mechanism for y_t is an AR(1) plus noise model, or, in the case $k > 1$, a VAR(1) plus noise. These models are similar to ARMA(1,1), respectively VARMA(1,1), but there are differences. This may be seen, e.g., from the spectral densities, and will become apparent later. For identification purposes, we normalize one of the coordinates of β to unity, thus equating the scales of latent volatility and the associated observed proxy. The parameters to be estimated are the remaining sensitivities in β, the biases α, the transition parameter γ, the volatility-of-volatility parameter σ_u^2, and the measurement error variances and covariances, say, σ_i^2.and $\rho_{i,j}$, where $var(\varepsilon_t) = \left\{\rho_{i,j}\sigma_i\sigma_j\right\}_{i,j}$.

As usual, the first order Markov specification for the latent state variable is not restrictive, since the state space form easily accommodates higher order specifications through expansion of the state vector. We write X_t for the state vector in the general case, of dimension p. Thus, X_t replaces x_t in the measurement equation (5), and β is $k \times p$, with one coordinate normalized for identification purposes, and zero columns for components of the state vector that do not impact current volatility proxies.

It is useful to briefly review the operation of the general state space approach in the simple cases where integrated volatility x_t, is either AR(2) or ARMA(1,1). The second order Markov or AR(2) model for latent x_t is

$$x_{t+1} = \gamma_1 x_t + \gamma_2 x_{t-1} + u_{t+1}. \tag{6}$$

It is convenient to let $X_t = (x_t, x_{t-1})'$ when latent integrated volatility is AR(2), as in (6).

Thus, $p = 2$, and the transition equation is

$$\begin{pmatrix} x_{t+1} \\ x_t \end{pmatrix} = \begin{pmatrix} \gamma_1 & \gamma_2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_t \\ x_{t-1} \end{pmatrix} + \begin{pmatrix} u_{t+1} \\ 0 \end{pmatrix}. \tag{7}$$

Writing Q in general for the variance-covariance matrix of the error term, say, v_{t+1}, in the state transition, we have in the AR(2) case $v_t = (u_t, 0)$ and Q has σ_u^2 in the upper left corner and is otherwise zero, i.e., there is only one parameter in Q. If the observed proxies y_t, react to true volatility only through its contemporaneous value x_t, not the lagged x_{t-1}, then the second column of β is zero. As an example, with bivariate volatility proxy ($k = p = 2$), e.g., implied and realized volatility, the measurement equation is

$$\begin{pmatrix} y_t^1 \\ y_t^2 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} + \begin{pmatrix} \beta_1 & 0 \\ \beta_2 & 0 \end{pmatrix} \begin{pmatrix} x_{t+1} \\ x_t \end{pmatrix} + \begin{pmatrix} \varepsilon_t^1 \\ \varepsilon_t^2 \end{pmatrix}. \tag{8}$$

Again, either β_1 or β_2 is normalized to unity. The parameters to be estimated are the other component of β, the vectors α and γ, the volatility-of-volatility parameter σ_u^2 in Q, the variances σ_1^2 and σ_2^2 of the measurement errors ε^1 and ε^2, and their correlation $\rho_{1,2}$.

When latent integrated volatility x_t is governed by an autoregressive-moving average or ARMA(1,1) process instead of AR(1) or AR(2),

$$x_{t+1} = \gamma x_t + \theta u_t + u_{t+1}, \tag{9}$$

where θ is the MA-parameter, the state vector may be taken to be $X_t \equiv (x_t, u_t)'$, so again $p = 2$, and the transition equation is

$$X_{t+1} \equiv \begin{pmatrix} \gamma & \theta \\ 0 & 0 \end{pmatrix} X_t + v_{t+1}, \tag{10}$$

where $v_t \equiv (u_t, u_t)'$. The error v_t in the state vector is still serially uncorrelated, since the integrated volatility innovation u_t, is, and in this case,

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sigma_u^2, \tag{11}$$

i.e., again there is only one parameter in Q. Like before, if the volatility proxies y_t react to current latent volatility x_t, then the second column of β is zero in the measurement equation. The state space form is preserved, and the parameters to be estimated now include θ. More generally, the state space framework accomodates any finite order ARMA(p, q) specification for latent integrated volatility.[2]

[2] Our empirical work shows that low order specifications suffice, namely, the AR(1), AR(2), and ARMA(1,1) models shown here.

Due to the assumed linearity of the state space representation, the Kalman filter is the natural empirical approach (see Harvey (1991) and Durbin & Koopman (2001); the appendix provides a brief summary of the Kalman filter recursions). The filter provides estimates of the unobserved x_t conditionally on data through the previous time period $t-1$, the 'predicted' state, $E_{t-1}[x_t]$, and conditionally on the expanded information set including current (time t) data y_t, the 'filtered' state, $E_t[x_t]$. Assuming normal distributions for the measurement errors ε_t and state shocks u_t, these are conditional means, and in general they are minimum mean squared error estimates. This allows calculating the innovations (prediction errors) in the data sequence and the corresponding (prediction error decomposition of the) conditional likelihood function given the initial observation y_0. We maximize this likelihood to obtain our parameter estimates. In addition, we use the filtered states $E_t[x_t]$ as estimates of true but unobserved underlying integrated volatility x_t, and the predicted states $E_{t-1}[x_t]$ as optimal forecasts of future true volatility. This is in contrast to previous literature that has focussed on the forecasting of future observable volatility proxies, such as realized volatility or the standard error of observed returns, and has not included individual option-implied volatility in the information set in the state space representation.

2.1 Specific Local Volatility Models

The model considered so far is based on the assumption that latent integrated volatility follows an AR(1), AR(2) or ARMA(1,1) process. This will likely only be true to a certain degree of approximation in practice. If no specific model for latent local volatility is adopted, it is generally hard to say more about the process for integrated volatility, and it becomes an empirical question whether our specification is useful. Before turning to the empirical analysis, we here consider in addition how to accommodate specific models for latent local volatility into the general framework. The specific models are those studied also by Barndorff-Nielsen & Shephard (2002*a*), namely, the constant elasticity of variance (CEV) process and the Ornstein-Uhlenbeck (OU) or BNS specification of Barndorff-Nielsen & Shephard (2001). Similar steps would be taken in case of alternative models.

The CEV local volatility specification is

$$d\sigma_t^2 = \lambda(\xi - \sigma_t^2)dt + \tau\sigma_t^{2\eta}dW_t, \tag{12}$$

where W_t is a standard Wiener process. This is a mean-reverting specification, with rate of mean reversion λ, and unconditional mean or target for mean reversion of the instantaneous variance σ_t^2 given by ξ. The CEV parameter is $\eta \in [1/2, 1]$, with the special cases $\eta = 1/2$ and 1 corresponding to the Heston (1993) square-root model for volatility and the Nelson (1991) GARCH-diffusion, respectively. The BNS local volatility specification is

$$d\sigma_t^2 = \lambda(\xi - \sigma_t^2)dt + dZ_{\lambda t}, \tag{13}$$

and similarly to the CEV case, the increments dZ to the underlying driving process are mean zero, stationary, and serially independent. The BNS specification is a non-Gaussian OU-process, i.e., Z is not a standard Wiener process. Instead, it is another time-homogeneous Lévy process, i.e., a process with independent and stationary increments, and a special case of a general subordinator, in the sense of Conley, Hansen, Luttmer & Scheinkman (1997). A Wiener process for Z in (13) would imply negative variances σ_t^2 with positive probability. The non-Gaussian specification may have other advantages, such as heavy tails, depending on the exact specification. The subordinator specification implies that Z has positive jumps, and although the drift of σ_t^2 can be negative, it becomes positive when σ_t^2 gets sufficiently small, in such a manner that volatility remains positive, as in the CEV model.[3] By running the subordinator according to time index λt instead of just t (e.g., large λ means running the process faster), the OU structure implies that the unconditional or invariant distribution of σ_t^2 is independent of λ, and only depends on the choice of subordinator process Z. This may be chosen, e.g., such that σ_t^2 has an unconditional Gamma distribution, as in the CIR case, or an inverse Gaussian distribution. In the BNS model without leverage, the inverse Gaussian distribution for σ_t^2 implies a normal-inverse Gaussian (NIG) distribution for returns, which has proved empirically relevant in some cases. For other subordinators, more general return distributions are obtained, all consistent with volatility clustering and non-normal returns, and leverage may be accommodated, too.

Barndorff-Nielsen & Shephard (2002*a*) show that both in the CEV and BNS models, the autocorrelation function for integrated volatility is

$$corr\left(\overline{\sigma}_t^2, \overline{\sigma}_{t+s}^2\right) = d \exp\left(-\lambda\left(s-1\right)\right), \quad s > 0, \tag{14}$$

where

$$d = \frac{\left[1-\exp\left\{-\lambda\right\}\right]^2}{2\left[\exp\left\{-\lambda\right\}-1+\lambda\right]}. \tag{15}$$

This implies that integrated volatility follows a constrained ARMA(1,1) process (see Barndorff-Nielsen & Shephard (2002*a*) and Meddahi (2003)). Similarly, the squared returns have autocorrelation function given by

$$corr\left(r_t^2, r_{t+s}^2\right) = c \exp\left(-\lambda\left(s-1\right)\right), \quad s > 0,$$

where

$$c = \frac{\left[1-\exp\left\{-\lambda\right\}\right]^2}{6\left[\exp\left\{-\lambda\right\}-1+\lambda\right]+2\lambda^2\left(\xi/\omega\right)^2}. \tag{16}$$

[3] We work with a zero mean process, i.e., $Z_t = \widetilde{Z}_t - \xi t$, in terms of the standard subordinator (or background driving Lévy process) $\widetilde{Z}_t$ from BNS, with positive mean ξt. The drift $-\xi dt$ of Z_t is offset by the first term in the drift of σ_t^2, and the second term in the latter, $-\lambda\sigma_t^2 dt$, becomes small with σ_t^2. Note that the unconditional means of $\widetilde{Z}_t$ and σ_t^2 coincide, at ξ, in the BNS model.

Thus, squared returns are also ARMA(1,1), but with $c \leq d$. That is, this specification implies that the latent process is more strongly serially dependent than the observed return based proxy.

Our general state space approach accomodates ARMA(1,1) models for integrated volatility, see (9)-(11). Thus, it applies in many relevant situations, including when latent local volatility is driven by a CEV or BNS model. It also applies outside these cases, and our results suggest that this is empirically relevant.

2.2 The Relation Between AR(1) Plus Noise and ARMA(1,1)

Here, we present the relevant conditions for when AR(1) plus noise and ARMA(1,1) models may be recast in terms of each other, and we show the corresponding relations between parameters of the two models.

Starting with an ARMA(1,1) model of the form

$$y_t = \gamma y_{t-1} + \theta w_{t-1} + w_t, \tag{17}$$

Granger & Morris (1976) (GM) show that this may be recast as an AR(1) plus noise model of the form (4)-(5) if and only if the realizability conditions given by

$$\frac{1}{1+\gamma_{ARMA}^2} > -\frac{\theta}{\left(1+\theta^2\right)\gamma_{ARMA}} \geq 0 \tag{18}$$

are satisfied. They also note that the parameters of the alternative representations satisfy the relations

$$\gamma_{AR} = \gamma_{ARMA}, \tag{19}$$

$$\sigma_\varepsilon^2 = -\frac{\theta}{\gamma_{ARMA}}\sigma_w^2, \tag{20}$$

where $\sigma_w^2 = var(w_t)$. This gives the first two of the three AR(1) plus noise parameters $(\gamma, \sigma_e^2, \sigma_u^2)$ in terms of the three ARMA(1,1) parameters $(\gamma, \theta, \sigma_w^2)$. The third and final AR(1) plus noise parameter is σ_u^2, but GM do not show how this is obtained from the ARMA(1,1) parameters. We show in the appendix that in fact

$$\sigma_u^2 = \left(1+\theta^2\right)\sigma_w^2 + \left(1+\gamma_{ARMA}^2\right)\frac{\theta}{\gamma_{ARMA}}\sigma_w^2, \tag{21}$$

hence completing the reparametrization from ARMA(1,1) to AR(1) plus noise.

The ARMA(1,1) process for integrated volatility implied by the BNS model (see (14)) has both γ_{ARMA} and θ positive (Meddahi (2003)), and so does not satisfy the GM conditions in (18). This may be seen from either (18) or (20), since each requires that the autoregressive and moving average parameters in the ARMA(1,1) representation be of opposite sign.

An AR(1) plus noise model of the form (4)-(5) can always be rewritten as an ARMA(1,1) model. This is stated without proof by GM. We provide a complete proof of this assertion in the appendix. GM also do not show the resulting ARMA(1,1) parameters in terms of the AR(1) parameters. We show in the appendix that they are given as

$$\gamma_{ARMA} = \gamma_{AR}, \tag{22}$$

$$\sigma_w^2 = -\frac{\gamma_{AR}}{\theta}\sigma_\varepsilon^2, \tag{23}$$

$$\theta = \frac{-\sigma_\varepsilon^2 - \gamma_{AR}^2\sigma_\varepsilon^2 - \sigma_u^2 \pm \sqrt{\sigma_u^4 + 2\sigma_u^2\sigma_\varepsilon^2\gamma_{AR}^2 + 2\sigma_u^2\sigma_\varepsilon^2 + \sigma_\varepsilon^4\gamma_{AR}^4 - 2\sigma_\varepsilon^4\gamma_{AR}^2 + \sigma_\varepsilon^4}}{2\gamma_{AR}\sigma_\varepsilon^2}. \tag{24}$$

We also show in the appendix that the natural realizability condition $\sigma_w^2 > 0$ always is satisfied by σ_w^2 from (23) when θ is given by (24) and $|\gamma| \leq 1$.

The constrained ARMA(1,1) models for integrated volatility implied by the CEV and BNS models in the previous subsection have the AR and MA parameters both positive. Our results point to an AR(1) plus noise model for the volatility proxies, and this may be recast as an ARMA(1,1) model with AR and MA parameters of opposite sign. This does not necessarily imply a rejection of the CEV and BNS local volatility models, since it is integrated volatility measured without any noise that are governed by ARMA(1,1) processes with positive parameters in these models. The possibility of measurement noise would suggest an ARMA(1,1) plus noise for the volatility proxies. Ideally, this might provide an encompassing framework including both the CEV, BNS, and our Markovian plus noise specifications. For example, significance of the estimated measurement noise variance in an ARMA(1,1) plus noise would imply rejection of our AR(1) plus noise, since this may be recast as ARMA(1,1) even without noise. The CEV and BNS models would be rejected if the ARMA(1,1) parameters turned out to be of opposite sign. However, when casting the ARMA(1,1) plus noise model in state space form, the parameters are not all identified without further restrictions. A sufficient restriction for identification is that there is no measurement noise, but the upshot is that we cannot use the framework to distinguish between the models. Since our empirical results below show that the AR(1) plus noise specification fits the data well, we interpret the estimates in these terms, while briefly remarking on their implications for the CEV and BNS models.

3 Volatility Measurement and Data

Assume that $M+1$ evenly spaced intra-daily observations for day t are available on the log-price $\log S_{t,j}$. We then denote the M intra-daily continuously compounded returns by

$$r_{t,j} = \log S_{t,j} - \log S_{t,j-1}, \quad j = 1, ..., M, \quad t = 1, ..., T, \tag{25}$$

where T is the number of days in the sample. Realized volatility for day t is given by the sum of squared intra-day returns,

$$RV_t = \sum_{j=1}^{M} r_{t,j}^2, \quad t = 1, ..., T. \tag{26}$$

As shown by Barndorff-Nielsen & Shephard (2002*b*) and Andersen et al. (2001), RV_t converges to quadratic variation, i.e., the integrated volatility plus the sum of squared price jumps during day t. Thus, RV_t is a consistent estimator of integrated volatility if there are no jumps. This is so even in the presence of the leverage effect, i.e., correlation between shocks to volatility and returns, see Barndorff-Nielsen, Hansen, Lunde & Shephard (2006) and Barndorff-Nielsen, Graversen, Jacod & Shephard (2006). An alternative estimator that retains consistency even in the presence of jumps in prices is the staggered (skip-k, with $k \geq 0$) realized bipower variation, defined as

$$SBV_t = \mu_1^{-2} \frac{M}{M-(k+1)} \sum_{j=k+2}^{M} |r_{t,j}| \, |r_{t,j-k-1}| \, , \quad t = 1, ..., T, \tag{27}$$

where $\mu_1 = \sqrt{2/\pi}$. Barndorff-Nielsen & Shephard (2004) show that realized bipower variation, i.e., the non-staggered ($k = 0$) version, BV_t, converges to integrated volatility as observation frequency is increased,

$$BV_t \to_p \int_{t-1}^{t} \sigma_s^2 ds \quad \text{as } M \to \infty.$$

In theory, a higher value of M improves precision of the estimators, but in practice it also makes them more susceptible to market microstructure effects, such as bid-ask bounces, stale prices, measurement errors, etc., introducing artificial (typically negative) serial correlation in returns, see, e.g., Hansen & Lunde (2006) and Barndorff-Nielsen & Shephard (2007). Huang & Tauchen (2005) show that staggering (i.e., setting $k \geq 1$) mitigates the resulting bias in (27), since it avoids the multiplication of the adjacent returns $r_{t,j}$ and $r_{t,j-1}$ that by (25) share the log-price $p_{t,j-1}$ in the non-staggered (i.e. $k = 0$) version of (27). We follow Huang & Tauchen (2005) and also use $k = 1$ in (27) in our empirical work. The choice of k has no impact on asymptotic results, i.e., SBV_t is consistent, too.

An alternative volatility proxy is implied volatility backed out from option prices. For the construction of implied volatility, we let c denote the call option price, X the strike price, τ the time to expiration of the option, F the price of the underlying futures contract with delivery date Δ periods after option expiration, and r the riskless interest rate. We use the futures option pricing formula, see Bates (1996*b*) and Bates (1996*a*),

$$c(F, X, \tau, \Delta, r, \sigma^2) = e^{-r(\tau+\Delta)}[F\Phi(d) - X\Phi(d - \sqrt{\sigma^2\tau})], \quad d = \frac{\ln(F/X) + \frac{1}{2}\sigma^2\tau}{\sqrt{\sigma^2\tau}}, \tag{28}$$

where $\Phi(\cdot)$ is the standard normal c.d.f. and σ the futures return volatility. The case $\Delta = 0$ (no delivery lag) corresponds to the well-known Black (1976) and Garman & Kohlhagen (1983) futures option formula. For general $\Delta > 0$, regarding the futures contract as an asset paying a continuous dividend yield equal to the riskless rate r, the asset price in the standard Black & Scholes (1973) and Merton (1973) formula is replaced by the discounted futures price $e^{-r(\tau+\Delta)}F$. This European style formula is here applied to American style options since early exercise premia are very small for short-term, at-the-money (ATM, $X = F$) futures options, as noted by Jorion (1995), who applied (28) with $\Delta = 0$ to the currency option market, whereas Bates (1996*b*) used delivery lags Δ specific to the Philadelphia Exchange (PHLX) and the Chicago Mercantile Exchange (CME), respectively.

We consider serial \$/£ and S&P 500 futures options with monthly expiration cycle traded at the CME, and equivalent T-Bond futures options traded at the Chicago Board of Trade (CBOT). The contract specifications do not uniquely identify the particular T-Bond serving as underlying asset for the bond futures, requiring merely that it does not mature and is not callable for at least 15 years from the first day of the delivery month of the underlying futures. The delivery month of the underlying futures contracts follows a quarterly (March) cycle, with delivery date on the third Wednesday of the month for currency and bond futures, and the third Friday for stock index futures. The options expire two Fridays prior to the third Wednesday of each month in the currency case, on the last Friday followed by at least two business days in the month in the bond case, and on the third Friday in the stock case, except every third month where it is shifted to the preceding Thursday to avoid "triple witching hour" problems associated with simultaneous maturity of the futures, options, and index options. Upon exercise, the holder of the option receives a position at the strike X in the futures, plus the intrinsic value $F - X$ in cash, on the following trading day, so the delivery lag is $\Delta = 3/365$ (from Friday to Monday), except $\Delta = 1/365$ (Thursday to Friday) every third month in the stock case. Finally, following French (1984), τ is measured in trading days when used with volatilities ($\sigma^2\tau$ in (28)) and in calender days when concerning interest rates (in $r(\tau + \Delta)$).

Given observations on the option price c and the variables F, X, τ, Δ, and r, an implied volatility (IV) estimate can be backed out from (28) by numerical inversion of the nonlinear equation $c = c(F, X, \tau, \Delta, r, IV)$ with respect to IV. Serial futures options with monthly expiration cycle were introduced in January 1987 in the \$/£ market, in October 1990 for T-bonds and October 1987 for S&P 500. Our option price data cover the period from December 1987 through July 2006 for \$/£, from December 1992 trough November 2007 for bonds, and from December 1987 through November 2007 for S&P 500 futures options. We start our series later than the inception times for two reasons. First, from data it seems that trading was relatively thin just after inception and second, initially new options on T-bonds futures where only introduced after expiration of the previous contract and therefore contracts with more than one month to expiration were never available. We use daily closing prices of ATM calls obtained from the Commodity Research Bureau. The US Eurodollar deposit one-month middle rate from Datastream is used for the risk-free rate r. The final samples are time series of length n of annualized IV measures from (28) with at least 6 calender days to expiration and thus, covers overlapping periods ranging from 6 to 37 calendar days, with $n = 4630$ for the currency market,[4] $n = 4996$ for the stock market, and $n = 3622$ for the bond market.

For RV, BV, and SBV, we construct data similarly to Andersen T. G. & Meddahi (2005). Specifically, we use five-minute observations from the CME floor trading on the leading contracts on \$/£ futures exchange rates, S&P 500 futures prices, and T-Bond futures prices. There is a total of 80 high-frequency returns per day ($r_{t,j}$ from (25)) for the currency and bond markets and 93 per day for the stock market. We use the nonparametric procedure from (26) and (27) to construct daily realized volatility measures (in annualized terms). Hence, the RV, BV and SBV measures do not cover the same period as the IV estimates. However, we are forecasting one-day-ahead and since IV is an estimate of expected volatility over the next month we expect it to carry information about the one-day-ahead volatility as well. Since the overnight returns often become very dominating when calculating the volatility measures over one day, we exclude these in the calculation of RV, BV and SBV.

If implied volatility were given by the conditional expectation of future realized volatility, we would expect that RV and IV had equal unconditional means, and RV higher unconditional standard deviation in the time series than IV. Since we exclude overnight returns in our observed non-parametric volatility measures such a comparison cannot be made. Furthermore, in our data we multiply annualized implied volatility by 10 and rescale all three realized volatility measures by a common factor such that the time series averages of implied and realized volatility agree exactly. This facilitates interpretation, since the realized measures leave out overnight and weekend returns that presumably are reflected in implied volatility, to the extent that this is set by traders as a forecast of total volatility through expiration of the option.[5] Of course, matching means in this manner has no impact on information content and the important forecasting properties that we discuss below.

[4] Trading in \$/DM options declined near the introduction of the Euro, and for January 1999 no one-month currency option price is available, even though quarterly contract prices are. An IV estimate is constructed by linear interpolation between IV for December 1998 and February 1999.

[5] One other reason for this rescaling is that the scale of the rescaled variables improves computational efficiency significantly.

Figure 1: Time series plots of monthly volatility measures

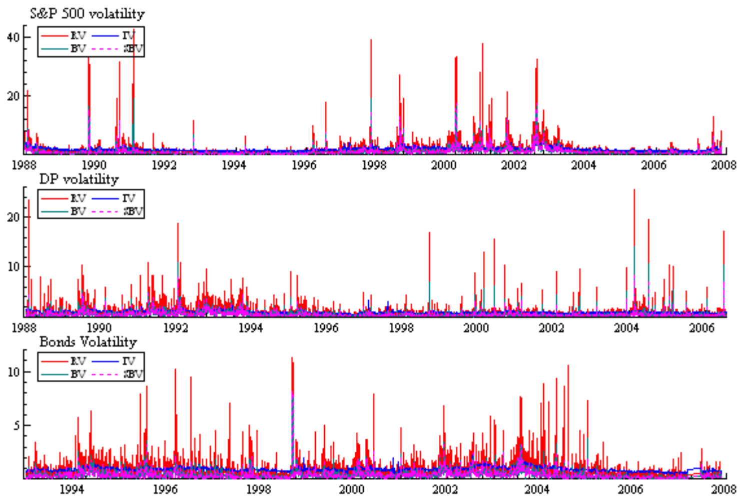

Figure (1) shows time series plots of the four daily volatility measures. For all three markets it is seen that RV exhibits strongest fluctuation, which is expected since this measure is influenced by jumps. Since IV is a measure of expected average volatility over a longer period, it should react less to current jumps. It is also seen from the figure that S&P 500 futures have the highest peak volatilities, and bond futures have the lowest. Further, volatility clustering is clearly present in the data, although the high volatility periods are not identical across the three markets.

Table 1 about here

Summary statistics are shown in Table 1. As noted, the means of RV and IV have been equalized within each market, but BV and the staggered version have lower mean. RV does have higher standard deviation in the time series than IV, and the bipower measures have standard deviations between those of RV and IV. Skewness is positive and excess kurtosis quite large, but both nearly disappear upon log transformation (lower panel of Table 1).

4 Empirical Results

We now apply the state space approach from section 2 to the volatility measures described in section 3. For the simplest specification results appear in Table 2. The first line of the table shows results for a straight autoregression for realized volatility, i.e., RV_t is regressed on its own lagged value. From the table, the autoregressive coefficient is estimated to .60. It is significantly positive, and significantly less than unity. From this estimate, realized volatility is mean-reverting, and not close to a unit root process. The measurement equation intercept, α, is estimated to 1.68, to be compared to the sample average of realized volatility. This AR(1) model for realized volatility is equivalent to setting the variance of the measurement error equal to zero. This way, the estimated AR(1) coefficient may be identified with γ in the state space model.

Table 2 about here

Results of relaxing the assumption of zero measurement error are reported in the second line of Table 2. This is the AR(1) plus noise model. The change in parameter estimates is quite dramatic. Thus, the autoregressive coefficient γ in the latent state variable process is now estimated to .91, much higher than the estimate of .60 in basic regression of realized volatility on its own lag. This difference suggests that the restricted model is misspecified, consistent with the notion that integrated volatility is governed by a process of the assumed autoregressive type, whereas the volatility proxy in the data (here, realized volatility) contains measurement error. The variance of the measurement error is estimated to 2.32 and is significant, reinforcing that the model restricting this parameter to zero is misspecified. Finally, the variance of the noise in the latent state process drops from 4.06 to .71 when introducing measurement noise. Indeed, total noise variance, the sum of measurement and state noise variances, is lower in the unrestricted model than the state noise variance in the restricted model, thus supporting this manner of splitting the noise necessary to explain the data in two.

The results are interesting from a number of perspectives. First, the indication is that measurement noise is part of the process specification for realized volatility. Second, when allowing for this, measured persistence in volatility increases dramatically, by a factor of 1.5. Thus, estimates of persistence based on autocorrelations in proxy series may be misleading, and, indeed, understated. Our results indicate much stronger persistence in true volatility than in measured volatility, and this should matter for the application of volatility models, e.g., to pricing, hedging, forecasting, etc..

The third line of the table reports results where the volatility proxy used is instead implied volatility backed out from option prices. Again, a basic regression of the proxy on its own lag is considered first. In this case, the coefficient estimate is high, at .94, already in the basic regression, and the standard error is even smaller than in the previous models using realized volatility. Together, the results suggest that implied volatility behaves in a manner closer to true volatility than does realized volatility.

The fourth line reports results when using the implied volatility proxy from the third line in the AR(1) plus noise model estimated using the Kalman filter as in the second line. As with realized volatility, the autoregressive coefficient increases when allowing for measurement error, now to .98. In the implied volatility case, both the standard error and the estimated measurement error variance are very small. All noise variances are smaller in the implied volatility case, and as in the realized volatility case, the sum of measurement and state noise variances in the AR(1) plus noise model is less than the state noise in the pure autoregression without measurement error.

The results so far are consistent with the notions that the AR(1) plus noise model is more correct than the pure autoregressive model for realized volatility, that the strength of autocorrelation in true volatility is high, and that option prices carry useful information about the process for volatility.

The next line is for the case $k = 2$, i.e., using both realized volatility and implied volatility simultaneously as proxies in the analysis. As in the previous estimations, it is necessary for identification purposes to restrict one of the β-coefficients from the measurement equation, so as to set the scale of the latent process. We set the coefficient on integrated volatility $\beta_2 = 1$, so that the scale of latent volatility is that of integrated volatility. Again, the autoregressive coefficient in the underlying state process is estimated to .98, as in the AR(1) plus noise model for implied volatility. As expected, the measurement error variance in realized volatility is slightly larger now, at 4.00, than the estimate of 2.32 obtained with realized volatility alone determining the movements of the latent process. The two measurement errors are virtually uncorrelated. The estimated intercepts capture the empirical means of the two data series.

The next three lines show the results when realized volatility is replaced by bipower variation. The previous results are largely confirmed. Bipower variation is slightly more persistent than realized volatility, and with lower state process noise and lower measurement error in the AR(1) plus noise model. This suggests that bipower variation, by removing the jump component of realized volatility, is closer to underlying continuous sample path volatility and that our latent state variable captures the latter. The last three lines of the table shows that these results are confirmed when using staggered bipower variation for robustness against market microstructure noise.

Table 3 about here

Table 3 shows the similar results using log-volatilities throughout. The previous results are largely confirmed. All γ_1 estimates are higher after applying the log-transform to the volatility series, but the estimates still increase sharply when introducing measurement error. Again, state noise drops dramatically when allowing for measurement error, and by a larger amount than the added measurement error noise. In the staggered case, measurement and state noises are significantly negatively correlated, although with a small coefficient. Again, implied volatility data produce a γ_1 estimate in excess of .95, whereas the realized measures when used individually give estimates below .80, and the joint models combining implied and realized measures all yield estimates above .98. The results indicate that there is strong persistence in true volatility, and that this is best reflected in implied volatility data.

Table 4 about here

Table 4 shows results of regressing future volatilities or volatility forecasts on implied and realized volatility measures in the current information set. The first set of regressions in Panel A use future realized volatility measures as dependent variables in specifications of the type

$$RV_t = a + bRV_{t-1} + cIV_{t-1} + \varepsilon_t. \tag{29}$$

The following regressions replace RV by the other realized measures, BV and SBV. From the results in the first line of the table, forecasting next day's realized volatility, both current realized volatility and implied volatility have significant forecasting power. Both the coefficient estimate and the t-statistic are larger for implied than for realized volatility. Similar results are found in the next lines, using log-volatilities, bipower variation instead of realized volatility, and staggered bipower variation. The DW statistics show no indication that the regression is misspecified. The table also reports R^2 statistics, showing that variables in the information set explain a considerable portion of future realized volatility measures.

The future realized volatility measures are only proxies of true future volatility. Using our approach, the future (next day) value of true but unobserved volatility is forecast optimally using the prediction step of the Kalman filter. The resulting forecasts are used as dependent variables in Panel B of the table. The general format is

$$\widehat{x}_{t+1} = a + bRV_t + cIV_t + u_t, \tag{30}$$

where $\widehat{x}_{t+1} = E_t[x_{t+1}]$. Again, later results in the table are for the case where RV is replaced by the other realized measures, BV or SBV. Thus, the regression results show the role played by current realized volatility measures and implied volatility in forming the forecast of future volatility. Again, both realized volatility measures and implied volatility enter significantly into the forecast. The most striking feature of the results is that implied volatility is a much more important forecasting variable relative to any of the realized volatility measures when forecasting true volatility, compared to when forecasting the realized volatility proxies in the first part of the table. This suggests that option prices reflect information about true underlying volatility, whereas realized measures in addition contain considerable noise.

Table 5 about here

Table 6 about here

Table 7 about here

Tables 5 to 7 show the similar results for bonds as those shown in Tables 2 to 4 for stocks. The lower measurement error variance when using staggered instead of raw bipower variation indicates that there is market microstructure noise in the bond market, and that staggering helps alleviate this. From Tables 5 and 6 it is seen that the autoregressive parameter is downward biased if measurement noise is not accounted for, and even more so compared to the results for S&P 500. Comparing the results for measurement noise the results also indicate that measurement noise is as significant in the bond futures market as in the S&P 500 futures market. When considering the results in Panel A of Table 7 for the standard forecasting setup in (29) one might be let to the conclustion that in the bond market IV does not contain incremental information over the realized meassures BV and SBV. On the other hand, the results in Panel B of Table 7 of the forecasting equation (30) shows that IV is outperforming all realized volatility proxies, when forecasting true underlying volatility.

Table 8 about here

Table 9 about here

Table 10 about here

Tables 8 to 10 show the corresponding results for the foreign exchange market. The results for this market are largely similar to those for the stock market in Tables 2 to 4, but there are a few notable differences. The results of Table 8 indicate that if measurement error is ignored in the estimation using realized volatility measures then the autoregressive parameter is going to be even more downward biased than was the case for the S&P 500 and bonds market. As in the S&P 500 market, when forecasting one period ahead volatility in the foreign exchange market it is seen from Table 10 that IV is performing better than all realized measures irrespectively of the forecasting setup.

Table 11 shows results for the case where the AR(1) specification for integrated latent volatility is replaced by an AR(2). This is relevant since volatility is persistent, and misspecification of the dynamic structure could bias the results. The results for all three markets show that once measurement noise is allowed for, the AR(2) coefficient γ_2 becomes insignificant. This indicates that our AR(1) plus noise specification is adequate for these data. Indeed, as noted in Section 2, AR(1) plus noise implies ARMA(1,1), and this flexible structure apparently captures all the empirically relevant higher order features present in the volatility series.

Table 11 about here

We also consider the extended state space form with bivariate VAR(1) state process, thus accommodating ARMA(1,1) integrated volatility, as would be implied by the CEV and BNS models. Estimation results for S&P 500 appear in Table 2.[6] The empirical results from the VAR(1) specification confirm the theoretical results from Section 2, i.e., the ARMA(1,1) parameters are exactly those implied by the estimated AR(1) plus noise model from Table 2, and the likelihood values are the same, too, as they should be. The fact that the AR(1) plus noise estimation yields much higher coefficient than a straight AR(1) matches the comparison $c \leq d$ from (16) and (15), i.e., the latent process is more strongly correlated than the observed proxies. Evidently, measured volatility is not AR(1), but rather ARMA(1,1), or possibly ARMA(1,1) plus noise. The ARMA(1,1) specification is consistent with AR(1) integrated volatility and measurement noise, and also with ARMA(1,1) integrated volatility and no measurement noise. Since the AR and MA parameters are of opposite sign, the ARMA(1,1) for integrated volatility cannot be motivated by the CEV or BNS spot volatility models in this case. These models might be consistent with an ARMA(1,1) plus noise for measured volatility, but not all parameters are identified in the state space representation of this.

[6] We only report results for S&P 500 since similar conclusions are drawn for bonds and foreign exchange.

5 Conclusion

We have introduced implied volatility into the state space approach to the filtering problem that arises when measured volatility is an imperfect proxy of true underlying integrated volatility. Our treatment has relied on the Kalman filter, which is optimal if the state space model is exactly linear. With a more general underlying volatility process, a nonlinear filter would be optimal, but our approach using the Kalman filter would still provide a useful benchmark against which to assess the value-added of generalizations. We adopt a first order Markov specification for integrated volatility and show that the model is closely related to the ARMA(1,1) specification implied by the CEV and BNS models for local volatility. In deed if no measurement error is assumed in the CEV or BNS specification then the parameter estimates in the resulting ARMA(1,1) secification for integrated volatility can be obtained from the estimates obtained in our first order Markov specification.

Our empirical results show that the serial dependence in true integrated volatility is understated if the measurement error in imperfect proxies is ignored, and more so in the bond and foreign exchange marget than the stock market. This result is in general most prominent for RV compared to BV and SBV and always more prominent for the realised high frequency return based measures compared to IV backed out of at the money options. Our results also show that the implied volatility proxy backed out of observed at the money option prices carries considerably more weight in the optimal filtered prediction of true integrated volatility than the various high-frequency return based volatility proxies. This conclusion applies to the stock, bond, and foreign currency exchange markets, alike.

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Table 1: Summary statistics for daily data on RV, IV, BV, SBV for S&P 500, Bonds and FX. S&P 500 data covers the period 1987.12.16-2007.11.8, bonds the period 1992.12.21-2007.11.19 and FX the period 1987.12.23-2006.7.25. IV has been multiplied by 10 and all three realized volatility have been rescaled by a common factor such that the time series averages of implied and realized volatility agree exactly. Descriptive statistics are reported for both level and logs.

	SP 500 (level)				Bonds (level)				FX (level)			
	RV	IV	BV	SBV	RV	IV	BV	SBV	RV	IV	BV	SBV
Mean	1.68	1.68	0.77	0.78	0.94	0.94	0.41	0.40	0.96	0.96	0.44	0.41
Std. dev	2.53	0.61	1.16	1.16	0.95	0.19	0.47	0.46	1.28	0.27	0.59	0.47
Ex. kurt.	73.18	1.38	73.37	58.33	36.97	1.85	113.70	131.87	95.50	3.51	133.63	67.01
Skew.	6.95	1.13	6.86	6.19	4.95	0.69	8.33	9.05	7.68	1.22	8.78	6.20
Min.	0.08	0.43	0.02	0.01	0.08	0.20	0.01	0.01	0.05	0.23	0.02	0.02
Max.	42.62	4.76	19.00	17.32	11.27	2.21	8.18	8.18	25.50	3.27	13.92	8.18
Obs.	4996	4996	4996	4996	3622	3622	3622	3622	4630	4630	4630	4630
	SP 500 (ln)				Bonds (ln)				FX (ln)			
	RV	IV	BV	SBV	RV	IV	BV	SBV	RV	IV	BV	SBV
Mean	0.05	0.46	−0.75	−0.73	−0.34	−0.08	−1.21	−1.23	−0.38	−0.08	−1.16	−1.19
Std. dev	0.89	0.34	0.91	0.92	0.71	0.20	0.77	0.75	0.75	0.27	0.75	0.73
Ex. kurt.	0.36	−0.46	0.36	0.33	0.55	2.38	1.26	0.91	1.14	0.43	1.06	0.79
Skew.	0.49	0.31	0.46	0.44	0.37	−0.34	−0.06	0.04	0.63	0.19	0.59	0.49
Min.	−2.57	−0.85	−3.77	−4.46	−2.48	−1.61	−5.11	−4.50	−2.94	−1.48	−3.88	−3.80
Max.	3.75	1.56	2.94	2.85	2.42	0.79	2.10	2.10	3.24	1.19	2.63	2.10
Obs.	4996	4996	4996	4996	3622	3622	3622	3622	4630	4630	4630	4630

Table 2: Kalman-filter estimation for RV, BV, SBV and/or IV as observable variables for S&P 500 using daily observations for the period 1987.12.16-2007.11.8.

Model (S&P 500)	α_1	α_2	β_1	β_2	γ	θ	σ_1^2	σ_2^2	ρ	u^2	$L(\cdot)$
AR, RV	1.6833 (0.0725)	–	1*	–	0.6048 (0.0591)	–	–	–	–	4.0633 (0.6078)	−10,591
ARn, RV	1.6923 (0.1333)	–	1*	–	0.9093 (0.0453)	–	2.3244 (0.5915)	–	–	0.7088 (0.5214)	−10,405
ARMA, RV	1.6936 (0.1336)	–	1*	–	0.9092 (0.0454)	−0.5604 (0.1526)	–	–	–	3.7702 (0.6021)	−10,405
AR, IV	1.6903 (0.0536)	–	1*	–	0.9438 (0.0089)	–	–	–	–	0.0411 (0.0032)	881
ARn, IV	1.7145 (0.1149)	–	1*	–	0.9849 (0.0044)	–	0.0161 (0.0023)	–	–	0.0113 (0.0020)	1,210
ARMA, IV	1.7149 (0.1150)	–	1*	–	0.9849 (0.0044)	−0.4400 (0.0494)	–	–	–	0.0361 (0.0025)	1,210
ARn, (RV, IV)	1.7611 (0.2812)	1.7116 (0.1074)	2.6026 (0.1127)	1*	0.9832 (0.0048)	–	3.9993 (0.6342)	0.0153 (0.0022)	−0.0133 (0.0222)	0.0125 (0.0022)	−4,813
AR, BV	0.7680 (0.0356)	–	1*	–	0.6428 (0.0546)	–	–	–	–	0.7951 (0.1234)	−6,517
ARn, BV	0.7728 (0.0640)	–	1*	–	0.9097 (0.0389)	–	0.4320 (0.1075)	–	–	0.1595 (0.0931)	−6,339
ARMA, BV	0.7727 (0.0640)	–	1*	–	0.9097 (0.0389)	−0.5307 (0.1275)	–	–	–	0.7405 (0.1217)	−6,339
ARn, (BV, IV)	0.8034 (0.1306)	1.7110 (0.1060)	1.2256 (0.0574)	1*	0.9828 (0.0045)	–	0.8208 (0.1293)	0.0152 (0.0022)	−0.0098 (0.0118)	0.0128 (0.0022)	−877
AR, SBV	0.7821 (0.0383)	–	1*	–	0.6888 (0.0453)	–	–	–	–	0.7017 (0.0897)	−6,205
ARn, SBV	0.7859 (0.0611)	–	1*	–	0.8930 (0.0426)	–	0.3268 (0.0903)	–	–	0.2047 (0.1032)	−6,066
ARMA, SBV	0.7859 (0.0611)	–	1*	–	0.8930 (0.0426)	−0.4397 (0.1311)	–	–	–	0.6638 (0.0890)	−6,066
ARn, (SBV, IV)	0.8183 (0.1339)	1.7111 (0.1063)	1.2528 (0.0572)	1*	0.9829 (0.0049)	–	0.7771 (0.1089)	0.0153 (0.0022)	−0.0068 (0.0109)	0.0127 (0.0023)	−741

Note: The model estimated is either AR(1) (AR) using equations (4) and (5) with $\varepsilon_t = 0$, AR(1) plus noise (ARn) using equations (4) and (5) with $\varepsilon_t \neq 0$ or ARMA(1,1) (ARMA) using equations (9) and (5) with $\varepsilon_t = 0$. ML estimates are reported with robust (sandwich-formula) standard errors in parenthesis. Also reported are $\ln L(\cdot)$, the value of the maximized log-likelihood function. * indicates that the parameter has been fixed to the reported value.

Table 3: Kalman-filter estimation for ln(RV), ln(BV), ln(SBV) and/or ln(IV) as observable variables for S&P 500 using daily observations for the period 1987.12.16-2007.11.8.

Model (S&P 500)	α_1	α_2	β_1	β_2	γ	θ	σ_1^2	σ_2^2	ρ	u^2	$L(\cdot)$
AR, *RV*	0.0568 (0.0360)	–	1*	–	0.7803 (0.0102)	–	–	–	–	0.3122 (0.0083)	−4,181
ARn, *RV*	0.0731 (0.0955)	–	1*	–	0.9722 (0.0048)	–	0.1698 (0.0076)	–	–	0.0349 (0.0053)	−3,730
ARMA, *RV*	0.0731 (0.0955)	–	1*	–	0.9722 (0.0048)	−0.6333 (0.0264)	–	–	–	0.2606 (0.0080)	−3,730
AR, *IV*	0.4651 (0.0323)	–	1*	–	0.9541 (0.0060)	–	–	–	–	0.0105 (0.0008)	4,292
ARn, *IV*	0.4782 (0.0645)	–	1*	–	0.9871 (0.0029)	–	0.0040 (0.0005)	–	–	0.0030 (0.0004)	4,619
ARMA, *IV*	0.4782 (0.0645)	–	1*	–	0.9871 (0.0029)	−0.4315 (0.0443)	–	–	–	0.0092 (0.0006)	4,619
ARn, (*RV*, *IV*)	0.0916 (0.1327)	0.4768 (0.0605)	2.1895 (0.0259)	1*	0.9856 (0.0031)	–	0.2613 (0.0086)	0.0039 (0.0005)	−0.0025 (0.0015)	0.0033 (0.0005)	5,313
AR, *BV*	−0.7454 (0.0368)	–	1*	–	0.7804 (0.0099)	–	–	–	–	0.3256 (0.0085)	−4,287
ARn, *BV*	−0.7288 (0.0962)	–	1*	–	0.9715 (0.0048)	–	0.1774 (0.0079)	–	–	0.0373 (0.0054)	−3,852
ARMA, *BV*	−0.7288 (0.0962)	–	1*	–	0.9715 (0.0048)	−0.6300 (0.0253)	–	–	–	0.2736 (0.0082)	−3,852
ARn, (*BV*, *IV*)	−0.7100 (0.1344)	0.4767 (0.0605)	2.2181 (0.0273)	1*	0.9856 (0.0031)	–	0.2820 (0.0090)	0.0039 (0.0005)	−0.0028 (0.0017)	0.0033 (0.0005)	5,121
AR, *SBV*	−0.7275 (0.0379)	–	1*	–	0.7905 (0.0097)	–	–	–	–	0.3154 (0.0083)	−4,207
ARn, *SBV*	−0.7119 (0.0959)	–	1*	–	0.9707 (0.0050)	–	0.1701 (0.0079)	–	–	0.0391 (0.0057)	−3,797
ARMA, *SBV*	−0.7120 (0.0959)	–	1*	–	0.9707 (0.0050)	−0.6170 (0.0267)	–	–	–	0.2676 (0.0080)	−3,797
ARn, (*SBV*, *IV*)	−0.6918 (0.1361)	0.4768 (0.0604)	2.2450 (0.0031)	1*	0.9855 (0.0031)	–	0.2764 (0.0090)	0.0039 (0.0005)	−0.0032 (0.0016)	0.0033 (0.0024)	5,156

Note: The model estimated is either AR(1) (AR) using equations (4) and (5) with $\varepsilon_t = 0$, AR(1) plus noise (ARn) using equations (4) and (5) with $\varepsilon_t \neq 0$ or ARMA(1,1) (ARMA) using equations (9) and (5) with $\varepsilon_t = 0$. ML estimates are reported with robust (sandwich-formula) standard errors in parenthesis. Also reported are $\ln L(\cdot)$, the value of the maximized log-likelihood function. * indicates that the parameter has been fixed to the reported value.

Table 4: Reported results are for S&P 500 using daily observations for the period 1987.12.16-2007.11.8 Panel A: OLS estimations. Panel B: OLS estimates where the left hand side is the one step ahead forecasted true, but latent, volatility from the indicated AR(1) plus noise model estimated in the previous tables.

Model (S&P500)	Panel A					
	a	b	c	$L(\cdot)$	DW	R^2
RV,IV	−1.5225 (0.0633)	0.3890 (0.1945)	1.5169 (0.1634)	−10, 22	2.05	0.45
$\ln(RV, IV)$	−0.5215 (0.0215)	0.4143 (0.0185)	1.2034 (0.0452)	−3, 64	2.12	0.68
BV,IV	−0.6649 (0.0880)	0.4380 (0.0652)	0.6519 (0.0741)	−6, 18	2.06	0.49
$\ln(BV, IV)$	−0.9614 (0.0503)	0.4349 (0.0131)	1.1720 (0.0350)	−3, 78	2.12	0.68
SBV,IV	−0.6065 (0.0762)	0.4930 (0.0614)	0.5962 (0.0656)	−5, 90	2.06	0.53
$\ln(SBV, IV)$	−0.9317 (0.0338)	0.4501 (0.0181)	1.1532 (0.0454)	−3, 71	2.12	0.69
	Panel B					
	a	b	c	$L(\cdot)$	DW	R^2
ARn, RV	−0.9711 (0.0410)	0.5782 (0.0238)	–	−5, 03	1.25	0.81
ARn, IV	−1.6182 (0.0063)	–	0.9434 (0.0039)	5, 56	1.84	0.98
ARn, (RV, IV)	−1.5864 (0.0080)	0.0121 (0.0017)	0.9141 (0.0062)	6, 01	1.85	0.98
ARn, BV	−0.4802 (0.0167)	0.6196 (0.0225)	–	−1, 20	1.32	0.85
ARn, (BV, IV)	−1.5790 (0.0079)	0.0311 (0.0038)	0.9079 (0.0062)	6, 11	1.82	0.98
ARn, SBV	−0.5307 (0.0116)	0.6743 (0.0156)	–	−242	1.47	0.90
ARn, (SBV, IV)	−1.5797 (0.0082)	0.0294 (0.0038)	0.9087 (0.0064)	6, 07	1.85	0.98
ARn, $\ln(RV)$	−0.0589 (0.0059)	0.7656 (0.0085)	-	−463	1.47	0.87
ARn, $\ln(IV)$	−0.4563 (0.0013)	-	0.9535 (0.0028)	9, 02	1.86	0.99
ARn, $\ln(RV, IV)$	−0.4227 (0.0027)	0.0353 (0.0021)	0.8792 (0.0059)	9, 66	1.87	0.99
ARn, $\ln(BV)$	0.5542 (0.0103)	0.7647 (0.0085)	–	−550	1.47	0.87
ARn, $\ln(BV, IV)$	−0.3964 (0.0041)	0.0339 (0.0020)	0.8813 (0.0057)	9, 66	1.87	0.99
ARn, $\ln(SBV)$	0.5486 (0.0097)	0.7744 (0.0080)	–	−428	1.48	0.88
ARn, $\ln(SBV, IV)$	−0.3932 (0.0041)	0.0359 (0.0019)	0.8762 (0.0057)	9, 70	1.87	0.99

Note: OLS estimates are reported with standard t-test standard errors. Also reported are $\ln L(\cdot)$, the value of the maximized log-likelihood function.

Table 5: Kalman-filter estimation for RV, BV, SBV and/or IV as observable variables for bonds using daily observations for the period 1992.12.21-2007.11.19.

Model (Bonds)	α_1	α_2	β_1	β_2	γ_1	σ_1^2	σ_2^2	ρ	u^2	$L(\cdot)$
AR, RV	0.9387 (0.0229)	−	1*	−	0.3523 (0.0516)	−	−	−	0.7963 (0.0783)	−4,727
ARn, RV	0.9373 (0.0423)	−	1*	−	0.8895 (0.0358)	0.5575 (0.0653)	−	−	0.0734 (0.0325)	−4,593
AR, IV	0.9372 (0.0127)	−	1*	−	0.8849 (0.0220)	−	−	−	0.0079 (0.0011)	3,634
ARn, IV	0.9320 (0.0277)	−	1*	−	0.9757 (0.0099)	0.0038 (0.0011)	−	−	0.0016 (0.0008)	3,844
ARn, (RV, IV)	0.9254 (0.0583)	0.9326 (0.0263)	2.1706 (0.1060)	1*	0.9732 (0.0104)	0.7568 (0.0868)	0.0037 (0.0011)	−0.0102 (0.0029)	0.0017 (0.0008)	2,509
AR, BV	0.4085 (0.0153)	−	1*	−	0.5892 (0.0775)	−	−	−	0.1438 (0.0188)	−1,627
ARn, BV	0.4082 (0.0223)	−	1*	−	0.8389 (0.0740)	0.0652 (0.0136)	−	−	0.0459 (0.0180)	−1,510
ARn, (BV, IV)	0.4024 (0.0284)	0.9330 (0.0254)	1.0960 (0.0893)	1*	0.9714 (0.0107)	0.1816 (0.0358)	0.0038 (0.0011)	−0.0077 (0.0031)	0.0019 (0.0009)	5,056
AR, SBV	0.3950 (0.0162)	−	1*	−	0.6417 (0.0767)	−	−	−	0.1224 (0.0186)	−1,335
ARn, SBV	0.3946 (0.0227)	−	1*	−	0.8409 (0.0762)	0.0490 (0.0124)	−	−	0.0465 (0.0178)	−1,231
ARn, (SBV, IV)	0.3889 (0.0282)	0.9330 (0.0253)	1.0923 (0.1016)	1*	0.9713 (0.0107)	0.1697 (0.0363)	0.0039 (0.0011)	−0.0077 (0.0034)	0.0019 (0.0009)	5,161

Note: The model estimated is either AR(1) (AR) using equations (4) and (5) with $\varepsilon_t = 0$ and AR(1) plus noise (ARn) using equations (4) and (5) with $\varepsilon_t \neq 0$. ML estimates are reported with robust (sandwich-formula) standard errors in parenthesis. Also reported are $\ln L(\cdot)$, the value of the maximized log-likelihood function. * indicates that the parameter has been fixed to the reported value.

Table 6: Kalman-filter estimation for ln(RV), ln(BV), ln(SBV) and/or ln(IV) as observable variables for Bonds using daily observations for the period 1992.12.21-2007.11.19.

Model (Bonds)	α_1	α_2	β_1	β_2	γ_1	σ_1^2	σ_2^2	ρ	u^2	$L(\cdot)$
AR, RV	−0.3405 (0.0180)	−	1*	−	0.4066 (0.0179)	−	−	−	0.4156 (0.0119)	−3,549
ARn, RV	−0.3505 (0.0744)	−	1*	−	0.9812 (0.0057)	0.3065 (0.0115)	−	−	0.0071 (0.0023)	−3,245
AR, IV	−0.0854 (0.0130)	−	1*	−	0.8708 (0.0251)	−	−	−	0.0101 (0.0017)	3,184
ARn, IV	−0.0932 (0.0319)	−	1*	−	0.9785 (0.0092)	0.0052 (0.0016)	−	−	0.0016 (0.0008)	3,427
ARn, (RV, IV)	−0.3587 (0.0634)	−0.0922 (0.0301)	2.0997 (0.0625)	1*	0.9764 (0.0093)	0.3389 (0.0107)	0.0051 (0.0016)	−0.0090 (0.0019)	0.0018 (0.0008)	3,517
AR, BV	−1.2054 (0.0213)	−	1*	−	0.4713 (0.0179)	−	−	−	0.4617 (0.0136)	−3,740
ARn, BV	−1.2126 (0.0751)	−	1*	−	0.9744 (0.0071)	0.3280 (0.0126)	−	−	0.0134 (0.0041)	−3,447
ARn, (BV, IV)	−1.2254 (0.0710)	−0.0920 (0.0298)	2.3693 (0.0735)	1*	0.9760 (0.0093)	0.3916 (0.0132)	0.0052 (0.0016)	−0.0109 (0.0022)	0.0018 (0.0008)	3,235
AR, SBV	−1.2289 (0.0218)	−	1*	−	0.5070 (0.0167)	−	−	−	0.4201 (0.0118)	−3,569
ARn, SBV	−1.2361 (0.0723)	−	1*	−	0.9712 (0.0072)	0.2893 (0.0110)	−	−	0.0157 (0.0041)	−3,271
ARn, (SBV, IV)	−1.2488 (0.0667)	−0.0919 (0.0279)	2.3837 (0.0076)	1*	0.9759 (0.0077)	0.3611 (0.0123)	0.0052 (0.0012)	−0.0106 (0.0020)	0.0018 (0.0040)	3,363

Note: The model estimated is either AR(1) (AR) using equations (4) and (5) with $\varepsilon_t = 0$ or AR(1) plus noise (ARn) using equations (4) and (5) with $\varepsilon_t \neq 0$. ML estimates are reported with robust (sandwich-formula) standard errors in parenthesis. Also reported are $\ln L(\cdot)$, the value of the maximized log-likelihood function. * indicates that the parameter has been fixed to the reported value.

Table 7: Reported results are for Bonds using daily observations for the period 1992.12.21-2007.11.19 Panel A: OLS estimations. Panel B: OLS estimates where the left hand side is the one step ahead forecasted true, but latent, volatility from the indicated AR(1) plus noise model estimated in the previous tables.

Model (Bonds)	Panel A					
	a	b	c	$L(\cdot)$	DW	R^2
RV,IV	−0.6966 (0.1350)	0.2543 (0.1182)	1.4877 (0.2398)	−456	2.06	0.20
$\ln(RV,IV)$	−0.1447 (0.0150)	0.2039 (0.0310)	1.5151 (0.0954)	−319	2.05	0.31
BV,IV	−0.2171 (0.0695)	0.5351 (0.1591)	0.4336 (0.1354)	−1548	2.20	0.38
$\ln(BV,IV)$	−0.7390 (0.0400)	0.2827 (0.0337)	1.5069 (0.1030)	−3432	2.09	0.34
SBV,IV	−0.1762 (0.0629)	0.5946 (0.1440)	0.3583 (0.1206)	−1272	2.23	0.43
$\ln(SBV,IV)$	−0.7186 (0.0410)	0.3192 (0.0335)	1.4147 (0.1004)	−3278	2.10	0.37
	Panel B					
	a	b	c	$L(\cdot)$	DW	R^2
ARn, RV	−0.3218 (0.0227)	0.3440 (0.0271)	–	−105	0.90	0.63
ARn, IV	−0.8140 (0.0125)	–	0.8740 (0.0134)	6584	1.61	0.95
ARn, (RV,IV)	−0.8010 (0.0114)	0.0207 (0.0011)	0.8388 (0.0130)	6956	1.64	0.96
ARn, BV	−0.2410 (0.0119)	0.5905 (0.0330)	–	3472	1.51	0.90
ARn, (BV,IV)	−0.7935 (0.0108)	0.0527 (0.0027)	0.8282 (0.0122)	7121	1.66	0.96
ARn, SBV	−0.2536 (0.0099)	0.6427 (0.0283)	–	3970	1.61	0.93
ARn, (SBV,IV)	−0.7908 (0.0108)	0.0553 (0.0027)	0.8250 (0.0123)	7115	1.66	0.96
ARn, $\ln(RV)$	0.1371 (0.0126)	0.3775 (0.0140)	–	−472	0.76	0.48
ARn, $\ln(IV)$	0.0808 (0.0019)	–	0.8563 (0.0178)	5890	1.56	0.93
ARn, $\ln(RV,IV)$	0.0891 (0.0013)	0.0460 (0.0024)	0.7795 (0.0181)	6408	1.61	0.95
ARn, $\ln(BV)$	0.5318 (0.0201)	0.4362 (0.0134)	–	−738	0.87	0.56
ARn, $\ln(BV,IV)$	0.1265 (0.0017)	0.0447 (0.0023)	0.7716 (0.0182)	6460	1.61	0.95
ARn, $\ln(SBV)$	0.5892 (0.0203)	0.4744 (0.0134)	–	−611	0.94	0.61
ARn, $\ln(BV,IV)$	0.1299 (0.0019)	0.0471 (0.0025)	0.7657 (0.0184)	6480	1.61	0.95

Note: OLS estimates are reported with standard t-test standard errors. Also reported are $\ln L(\cdot)$, the value of the maximized log-likelihood function.

Table 8: Kalman-filter estimation for RV, BV, SBV and/or IV as observable variables for FX using daily observations for the period 1987.12.23-2006.7.25.

Model (FX)	α_1	α_2	β_1	β_2	γ_1	σ_1^2	σ_2^2	ρ	u^2	$L(\cdot)$
AR, RV	0.9593 (0.0232)	–	1*	–	0.2110 (0.0379)	–	–	–	1.5553 (0.2331)	−7, 59
ARn, RV	0.9630 (0.0765)	–	1*	–	0.9765 (0.0063)	1.3299 (0.2135)	–	–	0.0138 (0.0052)	−7, 42
AR, IV	0.9593 (0.0223)	–	1*	–	0.9392 (0.0206)	–	–	–	0.0086 (0.0018)	4, 44
ARn, IV	0.9596 . (0.0354)	–	1*	–	0.9770 (0.0088)	0.0030 (0.0018)	–	–	0.0032 (0.0019)	4, 63
ARn, (*RV, IV*)	0.9600 (0.0684)	0.9597 (0.0354)	1.8473 (0.0883)	1*	0.9769 (0.0087)	1.3893 (0.2281)	0.0030 (0.0018)	0.0007 (0.0032)	0.0032 (0.0019)	1, 55
AR, BV	0.4406 (0.0108)	–	1*	–	0.2182 (0.0394)	–	–	–	0.3308 (0.0590)	−4, 01
ARn, BV	0.4416 (0.0340)	–	1*	–	0.9750 (0.0061)	0.2818 (0.0541)	–	–	0.0032 (0.0012)	−3, 84
ARn, (*BV, IV*)	0.4408 (0.0315)	0.9596 (0.0354)	0.8506 (0.0384)	1*	0.9769 (0.0087)	0.2967 (0.0584)	0.0030 (0.0018)	0.0004 (0.0014)	0.0032 (0.0019)	5, 12
AR, SBV	0.4126 (0.0094)	–	1*	–	0.2998 (0.0346)	–	–	–	0.1990 (0.0262)	−2, 83
ARn, SBV	0.4131 (0.0335)	–	1*	–	0.9764 (0.0062)	0.1576 (0.0231)	–	–	0.0028 (0.0008)	−2, 56
ARn, (*SBV, IV*)	0.4129 (0.0301)	0.9596 (0.0353)	0.8278 (0.0343)	1*	0.9768 (0.0087)	0.1707 (0.0253)	0.0030 (0.0018)	0.0001 (0.0014)	0.0032 (0.0019)	6, 39

Note: The model estimated is either AR(1) (AR) using equations (4) and (5) with $\varepsilon_t = 0$ or AR(1) plus noise (ARn) using equations (4) and (5) with $\varepsilon_t \neq 0$. ML estimates are reported with robust (sandwich-formula) standard errors in parenthesis. Also reported are $\ln L(\cdot)$, the value of the maximized log-likelihood function. * indicates that the parameter has been fixed to the reported value.

Table 9: Kalman-filter estimation for ln(RV), ln(BV), ln(SBV) and/or ln(IV) as observable variables for FX using daily observations for the period 1987.12.23-2006.7.25.

Model (FX)	α_1	α_2	β_1	β_2	γ_1	σ_1^2	σ_2^2	ρ	u^2	$L(\cdot)$
AR, RV	−0.3823 (0.0177)	–	1*	–	0.4429 (0.0155)	–	–	–	0.4498 (0.0131)	−4, 72
ARn, RV	−0.3833 (0.0637)	–	1*	–	0.9755 (0.0048)	0.3241 (0.0115)	–	–	0.0114 (0.0019)	−4, 35
AR, IV	−0.0780 (0.0246)	–	1*	–	0.9506 (0.0096)	–	–	–	0.0069 (0.0008)	4, 95
ARn, IV	−0.0775 (0.0390)	–	1*	–	0.9810 (0.0045)	0.0023 (0.0007)	–	–	0.0026 (0.0008)	5, 15
ARn, (*RV, IV*)	−0.3818 (0.0661)	−0.0776 (0.0387)	1.6772 (0.0375)	1*	0.9807 (0.0044)	0.3646 (0.0121)	0.0023 (0.0007)	−0.0008 (0.0013)	0.0026 (0.0008)	5, 14
AR, BV	−1.1589 (0.0181)	–	1*	–	0.4588 (0.0152)	–	–	–	0.4444 (0.0129)	−4, 69
ARn, BV	−1.1594 (0.0622)	–	1*	–	0.9734 (0.0050)	0.3191 (0.0113)	–	–	0.0128 (0.0022)	−4, 34
ARn, (*BV, IV*)	−1.1583 (0.0668)	−0.0776 (0.0387)	1.6942 (0.0373)	1*	0.9807 (0.0044)	0.3639 (0.0121)	0.0023 (0.0007)	−0.0009 (0.0013)	0.0026 (0.0008)	5, 14
AR, SBV	−1.1905 (0.0182)	–	1*	–	0.4883 (0.0144)	–	–	–	0.4025 (0.0110)	−4, 46
ARn, SBV	−1.1926 (0.0612)	–	1*	–	0.9721 (0.0052)	0.2818 (0.0097)	–	–	0.0135 (0.0022)	−4, 09
ARn, (*SBV, IV*)	−1.1900 (0.0666)	−0.0777 (0.0386)	1.6946 (0.0363)	1*	0.9807 (0.0044)	0.3295 (0.0101)	0.0023 (0.0007)	−0.0013 (0.0013)	0.0027 (0.0007)	5, 36

Note: The model estimated is either AR(1) (AR) using equations (4) and (5) with $\varepsilon_t = 0$ or AR(1) plus noise (ARn) using equations (4) and (5) with $\varepsilon_t \neq 0$. ML estimates are reported with robust (sandwich-formula) standard errors in parenthesis. Also reported are $\ln L(\cdot)$, the value of the maximized log-likelihood function. * indicates that the parameter has been fixed to the reported value.

Table 10: Reported results are for FX using daily observations for the period 1987.12.23-2006.7.25 Panel A: OLS estimations. Panel B: OLS estimates where the left hand side is the one step ahead forecasted true, but latent, volatility from the indicated AR(1) plus noise model estimated in the previous tables.

Model (FX)	a	b	c	$L(\cdot)$	DW	R^2
Panel A						
RV,IV	−0.5692 (0.0901)	0.0913 (0.0315)	1.5020 (0.1059)	−7, 37	2.01	0.13
$\ln(RV,IV)$	−0.2141 (0.0126)	0.1881 (0.0180)	1.2361 (0.0530)	−4, 31	2.04	0.33
BV,IV	−0.2573 (0.0407)	0.1009 (0.0295)	0.6811 (0.0460)	−3, 80	2.01	0.13
$\ln(BV,IV)$:	−0.8241 (0.0213)	0.2071 (0.0177)	1.2166 (0.0538)	−4, 29	2.04	0.34
SBV,IV	−0.2587 (0.0381)	0.1291 (0.0248)	0.6443 (0.0449)	−2, 54	2.02	0.20
$\ln(SBV,IV)$	−0.8270 (0.0222)	0.2276 (0.0181)	1.1868 (0.0530)	−4, 06	2.05	0.36
Panel B						
	a	b	c	$L(\cdot)$	DW	R^2
ARn, RV	−0.1742 (0.0192)	0.1788 (0.0172)	−	−1, 89	0.39	0.28
ARn, IV	−0.8992 (0.0073)	−	0.9371 (0.0076)	9, 55	1.82	0.99
ARn, (RV,IV)	−0.8980 (0.0076)	0.0019 (0.0006)	0.9338 (0.0084)	9, 58	1.83	0.99
ARn, BV	−0.0819 (0.0098)	0.1839 (0.0202)	−	1, 67	0.40	0.29
ARn, (BV,IV)	−0.8980 (0.0076)	0.0038 (0.0013)	0.9341 (0.0084)	9, 58	1.83	0.99
ARn, SBV	−0.1126 (0.0096)	0.2723 (0.0233)	−	1, 91	0.57	0.39
ARn, (SBV,IV)	−0.8964 (0.0078)	0.0084 (0.0022)	0.9306 (0.0090)	9, 62	1.83	0.98
ARn, $\ln(RV)$	0.1581 (0.0130)	0.4103 (0.0142)	−	−792	0.82	0.53
ARn, $\ln(IV)$	0.0736 (0.0005)	−	0.9494 (0.0039)	10, 13	1.87	0.99
ARn, $\ln(RV,IV)$	0.0760 (0.0005)	0.0086 (0.0011)	0.9358 (0.0052)	10, 27	1.89	0.99
ARn, $\ln(BV)$	0.4891 (0.0222)	0.4214 (0.0142)	−	−771	0.84	0.55
ARn, $\ln(BV,IV)$:	0.0830 (0.0011)	0.0089 (0.0011)	0.9351 (0.0052)	10, 28	1.89	0.99
ARn, $\ln(SBV)$	0.5423 (0.0224)	0.4537 (0.0144)	−	−633	0.90	0.59
ARn, $\ln(SBV,IV)$	0.0856 (0.0013)	0.0111 (0.0013)	0.9316 (0.0053)	10, 31	1.89	0.99

Note: OLS estimates are reported with standard t-test standard errors. Also reported are $\ln L(\cdot)$, the value of the maximized log-likelihood function.

Table 11: Kalman-filter estimation for AR(2) specifications.

Model	α_1	α_2	β_1	γ_1	γ_2	σ_1^2	σ_2^2	ρ	u^2	$L(\cdot)$
$AR(2)n$, RV, IV, SP	1.7611 (0.2812)	1.7116 (0.1074)	2.6026 (0.1127)	0.9832 (0.0047)	$3*10^{-7}$ (0.0001)	3.9993 (0.6342)	0.0153 (0.0022)	−0.0133 (0.0222)	0.0125 (0.0022)	−4,813
$AR(2)n$, BV, IV, SP	0.8035 (0.1307)	1.7110 (0.1060)	1.2256 (0.0574)	0.9828 (0.0048)	$1*10^{-7}$ (0.0001)	0.8208 (0.1293)	0.0152 (0.0022)	−0.0098 (0.0118)	0.0128 (0.0022)	−877
$AR(2)n$, SBV, IV, SP	0.8183 (0.1339)	1.7111 (0.1062)	1.2528 (0.0572)	0.9829 (0.0048)	$1.3*10^{-5}$ (0.0001)	0.7771 (0.1089)	0.0153 (0.0022)	−0.0068 (0.0109)	0.0127 (0.0023)	−741
$AR(2)n$, $\ln(RV, IV)$, SP	0.0916 (0.1327)	0.4768 (0.0605)	2.1895 (0.0259)	0.9856 (0.0028)	$2*10^{-6}$ (0.0011)	0.2613 (0.0086)	0.0039 (0.0053)	−0.0025 (0.0015)	0.0033 (0.0004)	5,313
$AR(2)n$, $\ln(BV, IV)$, SP	−0.7100 (0.1344)	0.4768 (0.0605)	2.2181 (0.0273)	0.9856 (0.0028)	$1*10^{-5}$ (0.0011)	0.2820 (0.0090)	0.0039 (0.0005)	−0.0028 (0.0017)	0.0033 (0.0004)	5,121
$AR(2)n$, $\ln(SBV, IV)$, SP	−0.6922 (0.1355)	0.4766 (0.0602)	2.2449 (0.0271)	0.9855 (0.0028)	$2.8*10^{-6}$ (0.0010)	0.2764 (0.0090)	0.0039 (0.0005)	−0.0032 (0.0016)	0.0033 (0.0005)	5,156
$AR(2)$, $ln(RV)$, SP	0.0598 (0.0504)	–	1*	0.5302 (0.0169)	0.3209 (0.0164)	–	–	–	0.2801 (0.0079)	−3,910
$AR(2)n$, $ln(RV)$, SP	0.0731 (0.0955)	–	1*	0.9722 (0.0048)	$7.0*10^{-7}$ ($1.0*10^{-6}$)	0.1698 (0.0076)	–	–	0.0349 (0.0053)	−3,730
$AR(2)$, $ln(IV)$, SP	0.4702 (0.0457)	–	1*	0.6622 (0.0451)	0.3063 (0.0444)	–	–	–	0.0095 (0.0006)	4,538
$AR(2)n$, $ln(IV)$, SP	0.4782 (0.0645)	–	1*	0.9871 (0.0026)	$6.3*10^{-6}$ (0.0016)	0.0040 (0.0005)	–	–	0.0030 (0.0004)	4,619
$AR(2)$, $ln(BV)$, SP	−0.7425 (0.0508)	–	1*	0.5396 (0.0161)	0.3091 (0.0160)	–	–	–	0.2946 (0.0082)	−4,037
$AR(2)n$, $ln(BV)$, SP	−0.7288 (0.0962)	–	1*	0.9715 (0.0048)	$1.4*10^{-5}$ ($1.0*10^{-5}$)	0.1774 (0.0079)	–	–	0.0373 (0.0054)	−3,852
$AR(2)$, $ln(SBV)$, SP	−0.7247 (0.0518)	–	1*	0.5546 (0.0161)	0.2988 (0.0157)	–	–	–	0.2873 (0.0080)	−3,974
$AR(2)n$, $ln(SBV)$, SP	−0.7119 (0.0959)	–	1*	0.9707 (0.0050)	$1.0*10^{-8}$ ($6.3*10^{-6}$)	0.1701 (0.0078)	–	–	0.0391 (0.0057)	−3,797
$AR(2)n$, RV, IV, $Bonds$	0.9258 (0.0583)	0.9328 (0.0263)	2.1700 (0.1060)	0.9735 (0.0059)	−0.0002 (0.0054)	0.7569 (0.0868)	0.0037 (0.0011)	−0.0102 (0.0029)	0.0017 (0.0008)	2,509
$AR(2)n$, BV, IV, $Bonds$	0.4024 (0.0284)	0.9330 (0.0254)	1.0958 (0.0893)	0.9714 (0.0066)	$4.5*10^{-5}$ (0.0049)	0.1816 (0.0358)	0.0038 (0.0011)	−0.0077 (0.0030)	0.0019 (0.0009)	5,056
$AR(2)n$, SBV, IV, $Bonds$	0.3889 (0.0282)	0.9330 (0.0253)	1.0923 (0.1016)	0.9713 (0.0066)	$7.3*10^{-6}$ (0.0049)	0.1697 (0.0363)	0.0039 (0.0011)	−0.0077 (0.0034)	0.0019 (0.0008)	5,161
$AR(2)n$, $\ln(RV, IV)$, $Bonds$	−0.3587 (0.0634)	−0.0922 (0.0301)	2.0997 (0.0625)	0.9763 (0.0053)	$4.9*10^{-6}$ (0.0052)	0.3389 (0.0107)	0.0051 (0.0015)	−0.0090 (0.0019)	0.0018 (0.0008)	3,517
$AR(2)n$, $\ln(BV, IV)$, $Bonds$	−1.2254 (0.0710)	−0.0920 (0.0298)	2.3693 (0.0735)	0.9760 (0.0054)	$2.3*10^{-6}$ (0.0050)	0.3916 (0.0131)	0.0052 (0.0015)	−0.0109 (0.0022)	0.0018 (0.0008)	3,235
$AR(2)n$, $\ln(SBV, IV)$,$Bonds$	−1.2488 (0.0710)	−0.0919 (0.0297)	2.3837 (0.0705)	0.9759 (0.0055)	$6.1*10^{-6}$ (0.0049)	0.3611 (0.0120)	0.0052 (0.0016)	−0.0106 (0.0022)	0.0018 (0.0008)	3,363
$AR(2)$, $ln(RV)$, $Bonds$	−0.3407 (0.0234)	–	1*	0.3037 (0.0185)	0.2531 (0.0165)	–	–	–	0.3889 (0.0113)	−3,429
$AR(2)n$, $ln(RV)$, $Bonds$	−0.3505 (0.0744)	–	1*	0.9812 (0.0041)	$2.5*10^{-6}$ (0.0023)	0.3065 (0.0115)	–	–	0.0071 (0.0023)	−3,245
$AR(2)$, $ln(IV)$, $Bonds$	−0.0867 (0.0166)	–	1*	0.6686 (0.0730)	0.2330 (0.0672)	–	–	–	0.0095 (0.0016)	3,285
$AR(2)n$, $ln(IV)$, $Bonds$	−0.0932 (0.0319)	–	1*	0.9784 (0.0045)	$2.2*10^{-5}$ (0.0072)	0.0052 (0.0016)	–	–	0.0016 (0.0008)	3,427
$AR(2)$, $ln(BV)$, $Bonds$	−1.2057 (0.0280)	–	1*	0.3458 (0.0191)	0.2663 (0.0171)	–	–	–	0.4290 (0.0124)	−3,607
$AR(2)n$, $ln(BV)$, $Bonds$	−1.2125 (0.0751)	–	1*	0.9744 (0.0068)	$-1.4*10^{-6}$ (0.0003)	0.3280 (0.0126)	–	–	0.0134 (0.0041)	−3,447
$AR(2)$, $ln(SBV)$, $Bonds$	−1.2292 (0.0289)	–	1*	0.3686 (0.0184)	0.2729 (0.0168)	–	–	–	0.3888 (0.0108)	−3,429
$AR(2)n$, $ln(SBV)$, $Bond$	−1.2361 (0.0723)	–	1*	0.9712 (0.0070)	$-7.0*10^{-7}$ (0.0002)	0.2893 (0.0110)	–	–	0.0157 (0.0041)	−3,271
$AR(2)n$, RV, IV, DP	0.9598 (0.0683)	0.9596 (0.0354)	1.8472 (0.0883)	0.9769 (0.0063)	$-2.1*10^{-5}$ (0.0033)	1.3893 (0.2281)	0.0030 (0.0018)	0.0007 (0.0032)	0.0032 (0.0019)	1,546
$AR(2)n$, BV, IV, DP	0.4408 (0.0315)	0.9596 (0.0354)	0.8506 (0.0384)	0.9769 (0.0063)	$-8*10^{-7}$ (0.0033)	0.2967 (0.0584)	0.0030 (0.0018)	0.0004 (0.0014)	0.0032 (0.0019)	5,120
$AR(2)n$, SBV, IV, DP	0.4129 (0.0301)	0.9596 (0.0353)	0.8278 (0.0343)	0.9768 (0.0063)	$2.9*10^{-6}$ (0.0033)	0.1707 (0.0253)	0.0030 (0.0018)	0.0001 (0.0014)	0.0032 (0.0019)	6,392
$AR(2)n$, $\ln(RV, IV)$, DP	−0.3818 (0.0661)	−0.0776 (0.0387)	1.6772 (0.0375)	0.9807 (0.0036)	$2*10^{-7}$ (0.0034)	0.3646 (0.0121)	0.0023 (0.0007)	−0.0008 (0.0013)	0.0026 (0.00074)	5,139
$AR(2)n$, $\ln(BV, IV)$, DP	−1.1583 (0.0668)	−0.0776 (0.0387)	1.6942 (0.0373)	0.9807 (0.0036)	$8*10^{-7}$ (0.0034)	0.3639 (0.0121)	0.0023 (0.0007)	−0.0009 (0.0013)	0.0026 (0.0007)	5,141
$AR(2)n$, $\ln(SBV, IV)$, DP	−1.1900 (0.0666)	−0.0777 (0.0386)	1.6946 (0.0363)	0.9807 (0.0036)	$7*10^{-7}$ (0.0033)	0.3295 (0.0101)	0.0023 (0.0007)	−0.0013 (0.0013)	0.0027 (0.0007)	5,364
$AR(2)$, $ln(RV)$, DP	−0.3826 (0.0226)	–	1*	0.3359 (0.0161)	0.2418 (0.0148)	–	–	–	0.4235 (0.0127)	−4,581
$AR(2)n$, $ln(RV)$, DP	−0.3833 (0.0637)	–	1*	0.9755 (0.0045)	$3.4*10^{-6}$ (0.0004)	0.3241 (0.0115)	–	–	0.0114 (0.0019)	−4,347
$AR(2)$, $ln(IV)$, DP	−0.0780 (0.0318)	–	1*	0.7076 (0.0587)	0.2556 (0.0563)	–	–	–	0.0065 (0.0007)	−4,561
$AR(2)n$, $ln(IV)$, DP	−0.0775 (0.0390)	–	1*	0.9810 (0.0036)	$7.0*10^{-7}$ (0.0036)	0.0023 (0.0007)	–	–	0.0026 (0.0007)	5,149
$AR(2)$, $ln(BV)$, DP	−1.1590 (0.0230)	–	1*	0.3511 (0.0160)	0.2346 (0.0147)	–	–	–	0.4199 (0.0125)	−4,561
$AR(2)n$, $ln(BV)$, DP	−1.1594 (0.0622)	–	1*	0.9734 (0.0049)	$2.4*10^{-6}$ (0.0002)	0.3191 (0.0113)	–	–	0.0128 (0.0022)	−4,336
$AR(2)$, $ln(SBV)$, DP	−1.1908 (0.0236)	–	1*	0.3641 (0.0153)	0.2544 (0.0145)	–	–	–	0.3764 (0.0106)	−4,308
$AR(2)n$, $ln(SBV)$, DP	−1.1926 (0.0612)	–	1*	0.9721 (0.0050)	$5.2*10^{-6}$ (0.0002)	0.2818 (0.0097)	–	–	0.0135 (0.0022)	−4,090

Note: The model estimated is either AR(2) (AR) using equations (7) and (5) with $\varepsilon_t = 0$ or AR(2) plus noise (AR(2)n) using equations (7) and (5) with $\varepsilon_t \neq 0$. ML estimates are reported with robust (sandwich-formula) standard errors in parenthesis. Also reported are $\ln L(\cdot)$, the value of the maximized log-likelihood function. * indicates that the parameter has been fixed to the reported value.

6 Appendix

Specification of AR(1) plus noise and ARMA(1,1)

To derive closed form solutions for parameters when going to and from ARMA(1,1) and AR(1) plus noise models we first consider the individual models. From the measurement equation (5) and the state equation (10), the univariate ARMA(1,1) without measurement noise is formulated in the state space as

$$\begin{aligned} \begin{bmatrix} x_{t+1} \\ w_{t+1} \end{bmatrix} &= \begin{bmatrix} \gamma_{ARMA} & \theta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_t \\ w_t \end{bmatrix} + \begin{bmatrix} w_{t+1} \\ w_{t+1} \end{bmatrix}, \\ y_t &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_t \\ w_t \end{bmatrix}. \end{aligned}$$

This can also be written as

$$(1 - \gamma_{ARMA} L)\, y_t = (1 + \theta L)\, w_t. \tag{31}$$

We denote the $j'th$ covariance as η_j and the $j'th$ correlation as ρ_j . The above model implies the correlation function

$$\rho_j = \begin{cases} 1, & j = 0 \\ \frac{\theta^2\gamma + \theta\gamma^2 + \theta + \gamma}{\theta^2 + 2\gamma\theta + 1}, & j = 1 \\ \gamma^{j-1}\rho_1, & j \geq 2 \end{cases}. \tag{32}$$

The variance of the right hand side MA(1) process is

$$\begin{aligned} \eta_0 &= E\left[(w_t + \theta w_{t-1})(w_t + \theta w_{t-1})\right] \\ &= \left(1 + \theta^2\right)\sigma_w^2. \end{aligned} \tag{33}$$

The first auto-covariance of the right hand side MA(1) process is

$$\begin{aligned} \eta_1 &= E\left[(w_t + \theta w_{t-1})(w_{t-1} + \theta w_{t-2})\right] \\ &= \theta\sigma_w^2. \end{aligned} \tag{34}$$

We do not have to calculate higher order auto-covariances since these can be stated in terms of the first order auto-covariance and γ_{ARMA}.

Using the notation in (4) and (5), and thus not quite similar notation as Granger & Morris (1976), the univariate AR(1) plus noise system in state space form is given by

$$\begin{aligned} x_{t+1} &= \gamma_{AR} x_t + u_{t+1}, \\ y_t &= x_t + \varepsilon_t. \end{aligned}$$

Thus we can also write it as

$$(1-\gamma_{AR}L)\,y_t=(1-\gamma_{AR}L)\,\varepsilon_t+u_t. \tag{35}$$

The variance of the MA(1) process on the right hand side is

$$\begin{aligned}\eta_0 &= E\left[(\varepsilon_t-\gamma_{AR}\varepsilon_{t-1}+u_t)\,(\varepsilon_t-\gamma_{AR}\varepsilon_{t-1}+u_t)\right]\\ &= \left(1+\gamma_{AR}^2\right)\sigma_\varepsilon^2+\sigma_u^2. \end{aligned}\tag{36}$$

The first auto-covariance of the right hand side MA(1) process is

$$\begin{aligned}\eta_1 &= E\left[(\varepsilon_t-\gamma_{AR}\varepsilon_{t-1}+u_t)\,(\varepsilon_{t-1}-\gamma_{AR}\varepsilon_{t-2}+u_{t-1})\right]\\ &= -\gamma_{AR}\sigma_\varepsilon^2. \end{aligned}\tag{37}$$

We do not have to calculate higher order auto-covariances since these can be stated in terms of the first order auto-covariance and γ_{AR}.

ARMA(1,1) to AR(1) plus noise:

Knowing θ, σ_w and γ_{ARMA} in the ARMA(1,1) representation we can find expressions for σ_ε, σ_u and γ_{AR} in the corresponding AR(1) plus noise representation. Comparing (35) to (31) it is seen that for the two models to be identical we require

$$\gamma_{AR}=\gamma_{ARMA}. \tag{38}$$

Furthermore the MA(1) term on the right hand side of (35) must match the MA(1) term on the right hand side of (31). This is the case if their auto-covariance functions are identical. Higher order (two and up) auto-covariances are identical if $\gamma_{AR}=\gamma_{ARMA}$ and if the first order auto-covariances are identical. Thus, we require (equating 37 and 34)

$$\eta_1=-\gamma_{AR}\sigma_\varepsilon^2=\theta\sigma_w^2. \tag{39}$$

Furthermore, for variances to be equal (equating 36 and 33)

$$\eta_0=\left(1+\gamma_{AR}^2\right)\sigma_\varepsilon^2+\sigma_u^2=\left(1+\theta^2\right)\sigma_w^2. \tag{40}$$

Parameter solutions:

We already solved for γ_{AR} in terms of γ_{ARMA} in (38). Left is to find solutions for σ_ε and σ_u in terms of θ and σ_w. From (39) and (38)

$$\sigma_\varepsilon^2=-\frac{\theta}{\gamma_{ARMA}}\sigma_w^2. \tag{41}$$

Substituting this into (40) yields

$$\begin{aligned} -\left(1+\gamma_{ARMA}^2\right)\frac{\theta}{\gamma_{ARMA}}\sigma_w^2+\sigma_u^2 &= \left(1+\theta^2\right)\sigma_w^2 \Leftrightarrow \\ \sigma_u^2 &= \left(1+\theta^2\right)\sigma_w^2+\left(1+\gamma_{ARMA}^2\right)\frac{\theta}{\gamma_{ARMA}}\sigma_w^2. \end{aligned}$$

Realizability conditions:

From (41) and $\sigma_\varepsilon^2 > 0$

$$\sigma_\varepsilon^2 = -\frac{\theta}{\gamma_{ARMA}}\sigma_w^2 > 0.$$

Since $\sigma_w^2 > 0$ we require $sign\,(\theta) \neq sign\,(\gamma_{ARMA})$.

From (40), (38) and the requirement of $\sigma_u^2 > 0$

$$\begin{aligned} \sigma_u^2 &= \left(1+\theta^2\right)\sigma_w^2-\left(1+\gamma_{AR}^2\right)\sigma_\varepsilon^2 > 0 \Leftrightarrow \\ \left(1+\theta^2\right)\sigma_w^2 &> \left(1+\gamma_{ARMA}^2\right)\sigma_\varepsilon^2. \end{aligned}$$

Dividing on both sides using (41) and (38) yields

$$\begin{aligned} \frac{\left(1+\theta^2\right)\sigma_w^2}{-\frac{\theta}{\gamma_{ARMA}}\sigma_w^2} &> \frac{\left(1+\gamma_{ARMA}^2\right)\sigma_\varepsilon^2}{\sigma_\varepsilon^2} \Leftrightarrow \\ \frac{1}{1+\gamma_{ARMA}^2} &> -\frac{\theta}{\left(1+\theta^2\right)\gamma_{ARMA}}. \end{aligned} \tag{42}$$

Since $1+\theta^2 > 0$ and $sign\,(\theta) \neq sign\,(\gamma_{ARMA})$ it follows that

$$-\frac{\theta}{\left(1+\theta^2\right)\gamma_{ARMA}} \geq 0, \tag{43}$$

and hence from (42) and (43) the realizability condition becomes

$$\frac{1}{1+\gamma_{ARMA}^2} > -\frac{\theta}{\left(1+\theta^2\right)\gamma_{ARMA}} \geq 0. \tag{44}$$

This result was also shown in less detail and with slightly different model set up in Granger & Morris (1976).

AR(1) plus noise to ARMA(1,1)

In this case we know σ_ε, σ_u and γ_{AR} from the AR(1) plus noise representation and we can find expressions for θ, σ_w and γ_{ARMA} in the corresponding ARMA(1,1) representation. Again, to match the left hand side of (35) to the left hand side of (31) we must have

$$\gamma_{ARMA} = \gamma_{AR}.$$

To match the MA(1) term on the right hand side of (35) to the right hand side of (31) we again require the relations in (40) and (39) to be satisfied. From (39) we have

$$\sigma_w^2 = -\frac{\gamma_{AR}}{\theta}\sigma_\varepsilon^2. \tag{45}$$

Substituting into (40) yields

$$\begin{aligned}
\left(1+\gamma_{AR}^2\right)\sigma_\varepsilon^2 + \sigma_u^2 &= -\left(1+\theta^2\right)\frac{\gamma_{AR}}{\theta}\sigma_\varepsilon^2 \Leftrightarrow \\
\theta^2\gamma_{AR}\sigma_\varepsilon^2 + \theta\left(\sigma_\varepsilon^2 + \gamma_{AR}^2\sigma_\varepsilon^2 + \sigma_u^2\right) + \gamma_{AR}\sigma_\varepsilon^2 &= 0.
\end{aligned}$$

For $\sigma_\varepsilon^2\gamma_{AR} \neq 0$, this implies

$$\theta = \frac{-\sigma_\varepsilon^2 - \gamma_{AR}^2\sigma_\varepsilon^2 - \sigma_u^2 \pm \sqrt{\sigma_u^4 + 2\sigma_u^2\sigma_\varepsilon^2\gamma_{AR}^2 + 2\sigma_u^2\sigma_\varepsilon^2 + \sigma_\varepsilon^4\gamma_{AR}^4 - 2\sigma_\varepsilon^4\gamma_{AR}^2 + \sigma_\varepsilon^4}}{2\gamma_{AR}\sigma_\varepsilon^2}. \tag{46}$$

We have to show that the term inside the square root is non-negative, hence

$$\begin{aligned}
\sigma_u^4 + 2\sigma_u^2\sigma_\varepsilon^2\gamma_{AR}^2 + 2\sigma_u^2\sigma_\varepsilon^2 + \sigma_\varepsilon^4\gamma_{AR}^4 - 2\sigma_\varepsilon^4\gamma_{AR}^2 + \sigma_\varepsilon^4 &\geq 0 \Leftrightarrow \\
\sigma_u^4 + 2\sigma_u^2\sigma_\varepsilon^2\gamma_{AR}^2 + 2\sigma_u^2\sigma_\varepsilon^2 + \sigma_\varepsilon^4\gamma_{AR}^4 + \sigma_\varepsilon^4 &\geq 2\sigma_\varepsilon^4\gamma_{AR}^2.
\end{aligned}$$

All terms on the left hand side are positive and it is thus sufficient to show that

$$\begin{aligned}
\sigma_\varepsilon^4\left(\gamma_{AR}^4 + 1\right) &\geq 2\sigma_\varepsilon^4\gamma_{AR}^2 \Leftrightarrow \\
\gamma_{AR}^4 + 1 &\geq 2\gamma_{AR}^2.
\end{aligned}$$

For a stationary AR parameter, i.e. $|\gamma_{AR}| < 1$ this holds with strict inequality and for a unit root process it holds with equality.

Realizability conditions:

We have now found the equations for the parameters of such a model respecification, but we still need to figure out if there are any realizability conditions. We require $\sigma_w^2 > 0$ and then from (45) we must have

$$sign\left(\theta\right) \neq sign\left(\gamma_{AR}\right).$$

From (46) we see that the denominator is of the same sign as γ_{AR}. Thus if $\gamma_{AR} > 0$ then we require the numerator to be positive and vice versa if $\gamma_{AR} < 0$. We now check that this is satisfied.

Case 1: $\gamma_{AR} > 0$

We require

$$-\sigma_\varepsilon^2 - \gamma_{AR}^2\sigma_\varepsilon^2 - \sigma_u^2 \pm \sqrt{\sigma_u^4 + 2\sigma_u^2\sigma_\varepsilon^2\gamma_{AR}^2 + 2\sigma_u^2\sigma_\varepsilon^2 + \sigma_\varepsilon^4\gamma_{AR}^4 - 2\sigma_\varepsilon^4\gamma_{AR}^2 + \sigma_\varepsilon^4} < 0.$$

It is sufficient to check that

$$\begin{aligned}
-\sigma_\varepsilon^2 - \gamma_{AR}^2\sigma_\varepsilon^2 - \sigma_u^2 + \sqrt{\sigma_u^4 + 2\sigma_u^2\sigma_\varepsilon^2\gamma_{AR}^2 + 2\sigma_u^2\sigma_\varepsilon^2 + \sigma_\varepsilon^4\gamma_{AR}^4 - 2\sigma_\varepsilon^4\gamma_{AR}^2 + \sigma_\varepsilon^4} &< 0 \Leftrightarrow \\
\sigma_u^4 + 2\sigma_u^2\sigma_\varepsilon^2\gamma_{AR}^2 + 2\sigma_u^2\sigma_\varepsilon^2 + \sigma_\varepsilon^4\gamma_{AR}^4 - 2\sigma_\varepsilon^4\gamma_{AR}^2 + \sigma_\varepsilon^4 &< \left(\sigma_\varepsilon^2 + \gamma_{AR}^2\sigma_\varepsilon^2 + \sigma_u^2\right)^2 \Leftrightarrow \\
-2\sigma_\varepsilon^4\gamma_{AR}^2 &< 2\sigma_\varepsilon^4\gamma_{AR}^2.
\end{aligned}$$

Thus, the realizability condition is always satisfied.

Case 2**:** $\gamma_{AR} < 0$

We require

$$-\sigma_\varepsilon^2 - \gamma_{AR}^2\sigma_\varepsilon^2 - \sigma_u^2 \pm \sqrt{\sigma_u^4 + 2\sigma_u^2\sigma_\varepsilon^2\gamma_{AR}^2 + 2\sigma_u^2\sigma_\varepsilon^2 + \sigma_\varepsilon^4\gamma_{AR}^4 - 2\sigma_\varepsilon^4\gamma_{AR}^2 + \sigma_\varepsilon^4} > 0.$$

It is sufficient to check that

$$-\sigma_\varepsilon^2 - \gamma_{AR}^2\sigma_\varepsilon^2 - \sigma_u^2 - \sqrt{\sigma_u^4 + 2\sigma_u^2\sigma_\varepsilon^2\gamma_{AR}^2 + 2\sigma_u^2\sigma_\varepsilon^2 + \sigma_\varepsilon^4\gamma_{AR}^4 - 2\sigma_\varepsilon^4\gamma_{AR}^2 + \sigma_\varepsilon^4} > 0.$$

Following similar calculations as above shows that this is always satisfied. Hence, we can always go from AR(1) plus noise model to an ARMA(1,1) model. This result was postulated in (Granger & Morris (1976)).

We can thus conclude that we can always go from an AR(1) plus noise model to an ARMA(1,1) model, but certain realizability conditions must be satisfied for us to go from an ARMA(1,1) model to an AR(1) plus noise model. Closed form expressions exist for all parameters in terms of the parameters of the alternative model in both cases.

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