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Asymptotic normality of the QMLE in the level-effect ARCH model*

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Abstract

In this paper consistency and asymptotic normality of the quasi maximum likelihood estimator in the level-effect ARCH model of Chan, Karolyi, Longstaff and Sanders (1992) is established. We consider explicitly the case where the parameters of the conditional heteroskedastic process are in the stationary region and discuss carefully how the results can be extended to the region where the conditional heteroskedastic process is nonstationary. The results illustrate that Jensen and Rahbek's (2004a, 2004b) approach can be extended further than to traditional ARCH and GARCH models.

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1 Introduction

After Engle (1982) initiated the literature on autoregressive conditional heteroskedasticity (ARCH) and the model proved itself to be very useful in empirical applications, an immense amount of research has been directed towards extending Engle's original ideas empirically as well as theoretically. Recently, Jensen and Rahbek (2004a) have provided a very significant and pathbreaking theoretical contribution in the context of the simple ARCH(1) process. In short, they provide a simple proof-technique on establishing the asymptotic normality of the quasi-maximum likelihood (QML) estimator both in the stationary and the nonstationary regions of the parameter space. In this paper, we show that Jensen and Rahbek's (2004a, 2004b) methodology can be applied to the important level-effect ARCH model first suggested in the influential paper by Chan, Karolyi, Longstaff and Sanders (1992). In their level-effect ARCH model they introduce the lagged level of the spot interest rate in the conditional variance equation. This model has subsequently been successfully used and extended by Brenner et al (1996), Andersen and Lund (1997), Ball and Torous (1999) among others. However, despite of its empirical success, the asymptotic behavior of the QML estimator associated with the level-effect ARCH model has, to the best of our knowledge, not been formally established yet. Actually and perhaps surprisingly, most papers on conditional heteroskedastic time series (see e.g. Berkes and Horváth (2004), Ling (2004), Straumann and Mikosch (2006)) do not allow for the introduction of the level of a series, such as the interest rate, in the conditional variance equation. In the following sections we will present the model and provide a simple proof of asymptotic normality and consistency of the QML estimator within the traditional level-effect ARCH(1) setting.

2 The level-effect ARCH model

The model considered is partially specified by (see for example equations (5) and (6) in Andersen and Lund (1997, page 350))

$$y_t^* = \sigma_t y_{t-1}^{\gamma} z_t, \tag{1}$$

$$\sigma_t^2 = w + \alpha \left(\frac{y_{t-1}^*}{y_{t-2}^{\gamma}} \right)^2, \tag{2}$$

for $t = 1, ..., T$. Let us denote $\theta = (\gamma, w, \alpha)'$ and let the true parameter values be given by $\theta_0 = (\gamma_0, w_0, \alpha_0)'$. Further, in many applications y_t^* is chosen as a transformation of current and lagged values of y_t, i.e., $y_t^* = y_t^*(y_t, y_{t-1}, ...)$. For example, y_t^* may equal $y_t - E(y_t)$ or $\Delta y_t - E(\Delta y_t)$. Similarly, we could set $y_t^* = y_t^*(y_t, y_{t-1}, y_{t-2}, ...; \delta)$ where δ is a vector of parameters. One such specification could be $y_t^* = \Delta y_t - (a + by_{t-1})$, where $\delta = (a, b)'$. In practice, δ can be pre-estimated in a first stage. This pre-estimation approach is very common in empirical research, particularly, when modelling spot interest rates, see, for example, Ball and Torous (1999, page 2349). However, to avoid additional complexity of the proofs, we assume throughout this paper that δ is known, see also Remark 2 below. Typically, y_t^* has been assumed to be a strictly stationary process (as in Theorem 1) but, as it has been shown for the regular GARCH model, we will argue that this assumption can be relaxed such that the asymptotics can be extended to the nonstationary region. It should be noted that (1)-(2) is a generalization of Frydman (1994), who consider a discrete-time process, but where $\alpha_0 = 0$. Brooze, Scaillet and Zakoian (1995) have also analyzed the regular level effect model (without the ARCH component) without assuming stationarity of the data generating process. However, their setting is more limited than ours due to more restrictions placed on the parameter space of γ_0. Moreover, for the case where γ_0 is known, (1)-(2) is the standard linear model for which a complete characterization of the estimation theory has been developed Jensen and Rahbek (2004a, 2004b) and in Kristensen and Rahbek (2005, 2008). The quasi log likelihood function associated with (1)-(2) is given as

$$l_T(\theta) = \sum_{t=1}^{T} l_t(\theta) = -\frac{1}{2} \sum_{t=1}^{T} \ln \left[y_{t-1}^{2\gamma} \left(w + \alpha \left(\frac{y_{t-1}^*}{y_{t-2}^{\gamma}} \right)^2 \right) \right] - \frac{1}{2} \sum_{t=1}^{T} \frac{(y_t^*)^2}{y_{t-1}^{2\gamma} \left(w + \alpha \left(\frac{y_{t-1}^*}{y_{t-2}^{\gamma}} \right)^2 \right)}. \tag{3}$$

We will proceed under the following set of maintained assumptions:

Assumption A

A1 $z_t \sim i.i.d.\,(0,1)$,

A2 $\infty > w_0 > 0$, $\infty > |\gamma_0|$, $\infty > \alpha_0 > 0$,

A3 $E\left(\left(1-z_t^2\right)^2\right) = \zeta \in (0,\infty)$,

A4 y_t only takes positive values, $E\left[\left(\frac{y_{t-1}}{y_{t-2}^i}\right)^{\varphi} |\ln(y_{t-1})|^3\right] < \infty$, $E\left[\left(\frac{y_{t-1}}{y_{t-2}^i}\right)^{\varphi} (\ln(y_{t-1}))^2 |\ln(y_{t-2})|\right] < \infty$, $E\left[\left(\frac{y_{t-1}}{y_{t-2}^i}\right)^{\varphi} (\ln(y_{t-2}))^2 |\ln(y_{t-1})|\right] < \infty$, for both $\varphi = 0$ and for some $\varphi > 0$ for $i = 1, 2$,

A5 (y_t, y_t^*) is ergodic and strictly stationary.

Assumption B

B1 $E\ln\left(\alpha_0 z_{t-1}^2\right) < 0$,

Assumptions A1-A3 and B1 are very common in the traditional ARCH literature (see e.g. Jensen and Rahbek (2004a, 2004b, page 1205)). Note also, that γ_0 is required to be bounded. A4 is an assumption very specific for the level-effect ARCH model, where the volatility process is well defined only for strictly and uniformly positive values of y_t. This restriction is probably one of the reasons why the level-effect ARCH specification has been most successfully applied to interest rates (see, e.g. Andersen and Lund (1997)). By allowing for nonstationarity, one could potentially use the level-effect ARCH specification to represent security prices in general. Assumption A5 too, is common in the literature, see, e.g., Kristensen and Linton (2006, page 327). As stated in Andersen and Lund (1997, page 350), interest rates, apart from taking positive values, usually exhibit a high degree of persistence but they are in most applications not expected to violate assumptions A4 and A5. The condition for strict stationarity of σ_t^2 is given by the following Lemma:

Lemma 1 Let Assumption A hold. A necessary and sufficient condition for strict stationarity of σ_t^2 as generated by (1)-(2) is given by

$$E\ln\left(\alpha_0 z_{t-1}^2\right) < 0.$$

Proof of Lemma 1 Given in Appendix 1.

The direct implication of Lemma 1 is that Assumptions A and B jointly ensure strictly stationarity of σ_t^2. Next, the main result of the paper regarding the limiting distribution of the QML estimator in the level-effects ARCH model can be established.

Theorem 1 Define $u_{1t}(\theta_0) = \left(\ln y_{t-1} - \ln y_{t-2} + w_0 (\ln y_{t-2}) \left(\frac{1}{\sigma_t^2(\theta_0)}\right)\right)$, $u_{2t}(\theta_0) = \left(\frac{1}{\sigma_t^2(\theta_0)}\right)$ and $u_{3t}(\theta_0) = \left(\frac{w_0}{\alpha_0}\right)\left(\frac{1}{w_0} - \frac{1}{\sigma_t^2(\theta_0)}\right)$ and let Assumptions A and B hold. Consider the quasi log likelihood function given by (3). Then, there exists a fixed open neighborhood $U = U(\theta_0)$ of θ_0 such that with probability tending to one as $T \longrightarrow \infty$, $l_T(\theta)$ has a unique maximum point $\widehat{\theta}$ in U. In addition, the QML estimator $\widehat{\theta}$ is consistent and asymptotically normal

$$\sqrt{T}\left[\widehat{\theta} - \theta_0\right]' \xrightarrow{d} N\left(0, (2\zeta)^{-2}\Lambda\right),$$

where

$$\Lambda = \zeta \begin{bmatrix} \overline{m}_{11} & \frac{1}{2}\overline{m}_{12} & \frac{1}{2}\overline{m}_{13} \\ \frac{1}{2}\overline{m}_{12} & \frac{1}{4}\overline{m}_{22} & \frac{1}{4}\overline{m}_{23} \\ \frac{1}{2}\overline{m}_{13} & \frac{1}{4}\overline{m}_{23} & \frac{1}{4}\overline{m}_{33} \end{bmatrix} > 0,$$

and $\overline{m}_{ij} = E\left(u_{it}(\theta_0)\, u_{jt}(\theta_0)\right)$ for $i = 1,2,3$ and $j = 1,2,3$.

Proof of Theorem 1 The proof of Theorem 1 is given in Appendix 2.

Importantly, Theorem 1 applies to QMLE of the stationary level-effect ARCH(1) process. However, if γ_0 is known under Assumption A but Assumption B fails then the asymptotics for the QML estimator of α can still be established. To see this, simply notice that the model (1)-(2) in this case can be rewritten as

$$\begin{aligned} \widetilde{y}_t &= \sigma_t z_t \\ \sigma_t^2 &= \omega + \alpha \widetilde{y}_{t-1}^2 \end{aligned}$$

for $\widetilde{y}_t \equiv \frac{y_t^*}{y_{t-1}^{\gamma_0}}$. This representation of the process $\widetilde{y}_t$ is exactly identical to the model given by equation (1) in Jensen and Rahbek (2004a). Consequently, when $E \ln\left(\alpha_0 z_{t-1}^2\right) \geq 0$ (Assumption B fails) then $\widetilde{y}_t^2 \xrightarrow{a.s.} \infty$ from Lemma 1 as $\sigma_t^2 \xrightarrow{a.s.} \infty$ and the asymptotic results follows directly from Jensen and Rahbek (2004a, Lemmas 1-5).[1] Three remarks should be added:

[1] We let $\xrightarrow{a.s.}$ denote convergence "almost surely" as T$\rightarrow \infty$.

Remark 1 It is well known, that the stationary level-effect ARCH model, can be estimated by non-parametric techniques, since the variance function is smooth and only depends on y_{t-1}. In the nonstationary case, Han and Zhang (2009) consider ARCH models by applying the results of Wang and Phillips (2009a, b), although they do not allow for a level-effect .

Remark 2 In model (1)-(2), we are assuming that $y_t^* = y_t^*(y_t, y_{t-1}, y_{t-2}, ...; \delta)$ where A5 holds. If y_t is stationary and different from y_t^*, A5 can be verified to hold when

$$\begin{aligned} y_t^* &= y_t - E(y_t), \\ y_t^* &= \Delta y_t - E(\Delta y_t), \\ y_t^* &= \Delta y_t - (a + b y_{t-1}) = y_t - a - (1+b)\, y_{t-1}, \end{aligned}$$

where $\delta = (a, b)'$, $b < 0$. Also, if we let y_t^* equal δy_{t-1}, we need the restriction that $|\delta| < 1$ in order for A5 to hold. In practice, δ can be pre-estimated in a first stage. If δ and θ are estimated jointly in mean and variance equation (contrary to Ball and Torous (1999)), then our proof will need to be extended to account for the joint estimation. At this stage we do not know if this would impose stronger assumptions than those in Assumptions A and B.

Remark 3 The generalization of the asymptotic results when going from ARCH(1) to GARCH(1,1) can most likely be provided in a similar fashion as the extension from Jensen and Rahbek (2004a) to Jensen and Rahbek (2004b), with the added complexity in the proofs.

3 Conclusion

In this paper we establish consistency and asymptotic normality of the QML estimator in the popular level-effect ARCH model. We allow the parameters of the conditional heteroskedastic process to be in the region where the process is stationary and discuss how the results carry over into the region of the parameter space where the conditional heteroskedastic process is nonstationary.

Appendix 1

Proof of Lemma 1 Let Assumption A hold and write the process as

$$\sigma_t^2 = w_0 + \left(\alpha_0 z_{t-1}^2\right) \sigma_{t-1}^2 = B_t + A_t \sigma_{t-1}^2,$$

where $A_t = \left(\alpha_0 z_{t-1}^2\right)$ and $B_t = w_0$. Then, applying Theorem 1.1 of Bougerol and Picard (1992a, page 1715), we verify the conditions that $E\left(\log\left(\max\left\{|1, A_0|\right\}\right)\right) < \infty$, $E\left(\log\left(\max\left\{|1, B_0|\right\}\right)\right) < \infty$ (since by Assumption A $w_0, \alpha_0 > 0$), and also σ_t^2 is strictly stationary if the Lyapunov exponent τ

$$\tau = \inf\left\{E\left(\frac{1}{T+1}\ln|A_0 \cdots A_T|\right)\right\} < 0.$$

In the case of one-dimensional recurrence equations

$$\frac{1}{T+1}E\left(\ln|A_0 \cdots A_T|\right) = \frac{1}{T+1}\sum_{i=0}^{T} E\ln|A_i| = E\ln|A_0| < 0.$$

Therefore, σ_t^2 is strictly stationary if

$$E\ln|A_0| = E\ln\left(\alpha_0 z_{t-1}^2\right) < 0.$$

This provides the conditions under which σ_t^2 is strictly stationary. Following Bougerol and Picard (1992b, pages 120-121), we can show the "if" and "only if" part of the Lemma. Therefore, under assumption A, the pair $\left(y_t^*, \sigma_t^2\right)' = \left(\sigma_t y_{t-1}^{\gamma_0} z_t, \sigma_t^2\right)'$ is strictly stationary. We note that this is the same sufficient and necessary condition when we have and i.i.d. process in the innovation term in a regular ARCH(1). The proof presented here is similar to Nelson (1990) and Bougerol and Picard (1992b).■

Appendix 2

We provide now three important propositions that we need in order to prove Theorem 1. The proof technique for the QMLE utilizes the classic Cramér type conditions for consistency and asymptotic normality (central limit theorem for the score, convergence of the Hessian and uniformly bounded third-order derivatives); see e.g. Lehmann (1999).

Proposition 1 Let $u_{jt}(\theta_0)$ be defined as in Theorem 1. Under Assumptions A and B, the joint distribution of the score functions evaluated at $\theta = \theta_0$ are asymptotically Gaussian,

$$\frac{1}{\sqrt{T}} \frac{\partial}{\partial \theta} l_T(\theta_0) \xrightarrow{d} N(0, \Lambda),$$

where

$$\Lambda = \zeta \begin{bmatrix} \overline{m}_{11} & \frac{1}{2}\overline{m}_{12} & \frac{1}{2}\overline{m}_{13} \\ \frac{1}{2}\overline{m}_{12} & \frac{1}{4}\overline{m}_{22} & \frac{1}{4}\overline{m}_{23} \\ \frac{1}{2}\overline{m}_{13} & \frac{1}{4}\overline{m}_{23} & \frac{1}{4}\overline{m}_{33} \end{bmatrix} > 0,$$

and $\overline{m}_{ij} = E(u_{it}(\theta_0) u_{jt}(\theta_0))$ for $i = 1, 2, 3$ and $j = 1, 2, 3$.

Proof of Proposition 1 Given in a technical appendix available upon request from any of the authors.■

Proposition 2 Let $u_{jt}(\theta_0)$ be defined as in Theorem 1. Under Assumptions A and B, the observed information evaluated at $\theta = \theta_0$ converges in probability, i.e.,

$$-\frac{1}{T} \frac{\partial^2}{\partial \theta \partial \theta'} l_T(\theta_0) \xrightarrow{p} \Omega,$$

where

$$\Omega = \begin{bmatrix} 2\overline{m}_{11} & \overline{m}_{12} & \overline{m}_{13} \\ \overline{m}_{12} & \frac{1}{2}\overline{m}_{22} & \frac{1}{2}\overline{m}_{23} \\ \overline{m}_{13} & \frac{1}{2}\overline{m}_{23} & \frac{1}{2}\overline{m}_{33} \end{bmatrix} > 0,$$

and $\overline{m}_{ij} = E(u_{it}(\theta_0) u_{jt}(\theta_0))$ for $i = 1, 2, 3$ and $j = 1, 2, 3$.

Proof of Proposition 2 Given in a technical appendix available upon request from any of the authors.■

Proposition 3 Define the lower and upper values for each parameter in θ_0 as $\gamma_L < \gamma_0 < \gamma_U, w_L < w_0 < w_U$, and $\alpha_L < \alpha_0 < \alpha_U$, respectively and the neighborhood $N(\theta_0)$ around θ_0 as

$$N(\theta_0) = \{\theta \backslash \gamma_L \leq \gamma \leq \gamma_U, w_L \leq w \leq w_U, \text{ and } \alpha_L \leq \alpha \leq \alpha_U\}.$$

Under Assumptions A and B, there exists a neighborhood $N(\theta_0)$ for which for $i, j, k = 1, 2, 3$

$$\sup_{\theta \in N(\theta_0)} \left| \frac{1}{T} \frac{\partial^3}{\partial \theta_i \partial \theta_j \partial \theta_k} l_T(\theta) \right| \leq \frac{1}{T} \sum_{t=1}^{T} w_{ijkt},$$

where w_{ijkt} is stationary. Furthermore $\frac{1}{T} \sum_{t=1}^{T} w_{ijkt} \xrightarrow{a.s.} E(w_{ijkt}) < \infty$ for $\forall ijk$.

Proof of Proposition 3 Given in a technical appendix available upon request from any of the authors.■

Proof of Theorem 1 Given the conditions provided by Propositions 1 - 3, Theorem 1 follows from Lumsdaine (1996, pages 593-595, Theorem 3), the ergodic theorem and Lemma 1, page 1206 in Jensen and Rahbek (2004b).■

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Supplementary Technical Appendix for "Asymptotic normality of the QMLE in the level-effect ARCH model", by Christian M. Dahl and Emma M. Iglesias.

In this Appendix, we provide proofs of Propositions 1, 2 and 3 in the paper: "Asymptotic normality of the QMLE in the level-effect ARCH model", by Christian M. Dahl and Emma M. Iglesias. Note that the analytical expressions for the first and second order derivatives are provided initially, but that the third order derivatives of the quasi log likelihood function needed in the proof of Proposition 3 are provided in the last part of the appendix.

Result 1

The first order derivatives are given by

$$\begin{aligned}
\frac{\partial}{\partial\gamma} l_T(\theta) &= \sum_{t=1}^{T} s_{1t}(\theta), \\
&= -\sum_{t=1}^{T}\left(1-\frac{(y_t^*)^2}{y_{t-1}^{2\gamma}\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^{\gamma}}\right)^2\right)}\right)\left(\ln y_{t-1}-\ln y_{t-2}+w\left(\ln y_{t-2}\right)\left(\frac{1}{\sigma_t^2}\right)\right), \qquad (1)\\
\frac{\partial}{\partial w} l_T(\theta) &= \sum_{t=1}^{T} s_{2t}(\theta), \\
&= -\sum_{t=1}^{T}\frac{1}{2}\left(1-\frac{(y_t^*)^2}{y_{t-1}^{2\gamma}\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^{\gamma}}\right)^2\right)}\right)\frac{1}{\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^{\gamma}}\right)^2\right)}, \qquad (2)\\
\frac{\partial}{\partial\alpha} l_T(\theta) &= \sum_{t=1}^{T} s_{3t}(\theta), \\
&= -\sum_{t=1}^{T}\frac{1}{2}\left(1-\frac{(y_t^*)^2}{y_{t-1}^{2\gamma}\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^{\gamma}}\right)^2\right)}\right)\frac{\left(\frac{y_{t-1}^*}{y_{t-2}^{\gamma}}\right)^2}{\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^{\gamma}}\right)^2\right)}. \qquad (3)
\end{aligned}$$

In particular,

$$\begin{aligned}
s_{1t}(\theta_0) &= -\left(1-z_t^2\right)\left(\ln y_{t-1}-\ln y_{t-2}+w_0\left(\ln y_{t-2}\right)\left(\frac{1}{\sigma_t^2(\theta_0)}\right)\right), \\
s_{2t}(\theta_0) &= -\frac{1}{2}\left(1-z_t^2\right)\frac{1}{\sigma_t^2(\theta_0)}, \\
s_{3t}(\theta_0) &= -\frac{1}{2}\left(1-z_t^2\right)\left(\frac{w_0}{\alpha_0}\right)\left(\frac{1}{w_0}-\frac{1}{\sigma_t^2(\theta_0)}\right).
\end{aligned}$$

Result 2

The second order derivatives evaluated at evaluated at $\theta = \theta_0$ are given by

$$
\begin{aligned}
\frac{\partial^2}{\partial \gamma^2} l_T(\theta_0) &= -2 \sum_{t=1}^{T} z_t^2 \left(\ln y_{t-1} - \ln y_{t-2} + w_0 (\ln y_{t-2}) \left(\frac{1}{\sigma_t^2(\theta_0)} \right) \right)^2 \\
&\quad - \sum_{t=1}^{T} 4 \left(1 - z_t^2\right) \alpha_0 \left(\ln^2 y_{t-2}\right) w_0^2 \left(\frac{1}{w_0} - \frac{1}{\sigma_t^2(\theta_0)} \right), \qquad (4) \\
\frac{\partial^2}{\partial w^2} l_T(\theta_0) &= \frac{1}{2} \sum_{t=1}^{T} \left(1 - 2z_t^2\right) \left(\frac{1}{\sigma_t^2(\theta_0)} \right)^2, \qquad (5) \\
\frac{\partial^2}{\partial \alpha^2} l_T(\theta_0) &= \frac{1}{2} \sum_{t=1}^{T} \left(1 - 2z_t^2\right) \left(\frac{w_0}{\alpha_0} \right)^2 \left(\frac{1}{w_0} - \frac{1}{\sigma_t^2(\theta_0)} \right)^2, \qquad (6) \\
\frac{\partial^2}{(\partial \gamma \partial w)} l_T(\theta_0) &= -\sum_{t=1}^{T} z_t^2 \left(\frac{1}{\sigma_t^2(\theta_0)} \right) \left(\ln y_{t-1} - \ln y_{t-2} + w_0 (\ln y_{t-2}) \left(\frac{1}{\sigma_t^2(\theta_0)} \right) \right) \\
&\quad - \sum_{t=1}^{T} \left(1 - z_t^2\right) \left((\ln y_{t-2}) \left(\frac{w_0}{\sigma_t^2(\theta_0)} \right) \left(\frac{1}{w_0} - \frac{1}{\sigma_t^2(\theta_0)} \right) \right), \qquad (7) \\
\frac{\partial^2}{(\partial \gamma \partial \alpha)} l_T(\theta_0) &= -\sum_{t=1}^{T} z_t^2 \left(\frac{w_0}{\alpha_0} \right) \left(\frac{1}{w_0} - \frac{1}{\sigma_t^2(\theta_0)} \right) \left(\left(\ln y_{t-1} - \ln y_{t-2} + w_0 (\ln y_{t-2}) \left(\frac{1}{\sigma_t^2(\theta_0)} \right) \right) \right) \\
&\quad + 2 \sum_{t=1}^{T} \left(1 - z_t^2\right)_0 (\ln y_{t-2}) \left(\frac{w_0}{\alpha_0} \right) \left(\frac{1}{w_0} - \frac{1}{\sigma_t^2(\theta_0)} \right), \qquad (8) \\
\frac{\partial^2}{\partial w \partial \alpha} l_T(\theta_0) &= \frac{1}{2} \sum_{t=1}^{T} \left(1 - 2z_t^2\right) \left(\frac{1}{\sigma_t^2(\theta_0)} \right) \left(\frac{w_0}{\alpha_0} \right) \left(\frac{1}{w_0} - \frac{1}{\sigma_t^2(\theta_0)} \right). \qquad (9)
\end{aligned}
$$

Proofs of Propositions 1, 2 and 3

For the proof of Proposition 1, we need first the following 2 Lemmas.

Lemma A Let Assumptions A and B hold and define $u_{1t}(\theta_0) = \left(\ln y_{t-1} - \ln y_{t-2} + w_0 (\ln y_{t-2}) \left(\frac{1}{\sigma_t^2(\theta_0)}\right)\right)$, $u_{2t}(\theta_0) = \left(\frac{1}{\sigma_t^2(\theta_0)}\right)$ and $u_{3t}(\theta_0) = \left(\frac{w_0}{\alpha_0}\right)\left(\frac{1}{w_0} - \frac{1}{\sigma_t^2(\theta_0)}\right)$. Then $u_{it}(\theta_0)$ is a stationary and ergodic sequence. In addition $\frac{1}{T}\sum_{t=1}^{T} u_{it}(\theta_0) \xrightarrow{p} E(u_{it}(\theta_0)) \equiv \overline{u}_i$ and $\frac{1}{T}\sum_{t=1}^{T} u_{it}^2(\theta_0) \xrightarrow{p} E\left(u_{it}^2(\theta_0)\right) \equiv \overline{m}_{ii}$ for $i = 1, 2, 3$.

Proof of Lemma A Define $I_t = \{y_t, z_t, y_{t-1}, z_{t-1}, y_{t-2}, z_{t-2}, ...\}$. Note first that

$$
\begin{aligned}
|u_{1t}(\theta_0)| &\leq |\ln y_{t-1}| + |\ln y_{t-2}| + w_0 |\ln y_{t-2}| \left(\frac{1}{\sigma_t^2(\theta_0)}\right) \\
&\leq |\ln y_{t-1}| + 2|\ln y_{t-2}|,
\end{aligned}
$$

hence

$$
E|u_{1t}(\theta_0)| \leq 3E((\ln y_t)) < \infty,
$$

where we have used that y_t (hence $\ln(y_t)$) is stationary (A5) and where the last inequality follows from A4 where the first two moments of y_t are assumed to be bounded. Hence we can write

$$
u_{1t}(\theta_0) \equiv g_1\left(y_{t-1}, y_{t-2}, \sigma_t^2(\theta_0)\right),
$$

where g_1 is a I_t-measurable function and where all arguments y_{t-1}, y_{t-2} and $\sigma_t^2(\theta_0)$ are stationary and ergodic as a consequence of A5 and Lemma 1. This implies that $u_{1t}(\theta_0)$ is stationary and ergodic by Theorem 3.35 in White (1984). Consequently $\frac{1}{T}\sum_{t=1}^{T} u_{1t}(\theta_0) \xrightarrow{p} E(u_{1t}(\theta_0))$ follows by the Ergodic Theorem. Similarly, it follows straightforwardly that $E|u_{2t}(\theta_0)| \leq \left(\frac{1}{w_0}\right)$ and $E|u_{3t}(\theta_0)| \leq \left(\frac{2}{\alpha_0}\right)$. We can write $u_{2t}(\theta_0) \equiv g_2\left(\sigma_t^2(\theta_0)\right)$ and $u_{3t}(\theta_0) \equiv g_3\left(\sigma_t^2(\theta_0)\right)$ and as above conclude that $(u_{2t}(\theta_0), u_{3t}(\theta_0))$ is stationary and ergodic, and hence $\frac{1}{T}\sum_{t=1}^{T} u_{it}(\theta_0) \xrightarrow{p} E(u_{it}(\theta_0))$ for $i = 2, 3$. Second, notice that

$$
\begin{aligned}
\left|u_{1t}^2(\theta_0)\right| &= |\ln^2 y_{t-1} - 2\ln y_{t-1}\ln y_{t-2} + \ln^2 y_{t-2} - 2\frac{w_0}{\sigma_t^2(\theta_0)}\ln^2 y_{t-2} + \frac{w_0^2}{\sigma_t^4(\theta_0)}\ln^2 y_{t-2} + 2\frac{w_0}{\sigma_t^2(\theta_0)}\ln y_{t-1}\ln y_{t-2}| \\
&\leq \ln^2 y_{t-1} + 2|\ln y_{t-1}\ln y_{t-2}| + \ln^2 y_{t-2} + 2\frac{w_0}{\sigma_t^2(\theta_0)}\ln^2 y_{t-2} + \frac{w_0^2}{\sigma_t^4(\theta_0)}\ln^2 y_{t-2} + 2\frac{w_0}{\sigma_t^2(\theta_0)}|\ln y_{t-1}\ln y_{t-2}| \\
&\leq \ln^2 y_{t-1} + 4\ln^2 y_{t-2} + 4|\ln y_{t-1}\ln y_{t-2}|,
\end{aligned}
$$

such that

$$
E\left|u_{1t}^2(\theta_0)\right| \leq 5E\left((\ln y_t)^2\right) + 4E|\ln y_t \ln y_{t-1}| < \infty.
$$

On the right hand side of the first inequality we have used A5 (stationarity) and the second inequality follows from A4 (existence of second order moments). In addition, $E\left|u_{2t}^2(\theta_0)\right| \leq \left(\frac{1}{w_0^2}\right)$ and $E\left|u_{3t}^2(\theta_0)\right| \leq \left(\frac{4}{\alpha_0^2}\right)$. We can therefore conclude, by Theorem 3.35 in White (1984), that since $u_{it}(\theta_0)$ is stationary and ergodic then so is $u_{it}^2(\theta_0)$ for $i = 1, 2, 3$. Furthermore as $E|u_{it}^2(\theta_0)|$ is bounded then $\frac{1}{T}\sum_{t=1}^{T} u_{1t}^2(\theta_0) \xrightarrow{p} E\left(u_{1t}^2(\theta_0)\right)$ for $i = 1, 2, 3$ follows from the ergodicity theorem. This completes the proof of Lemma A.■

Lemma B Under Assumptions A and B, the marginal distributions of the score functions given by equations (1)-(3) evaluated at $\theta = \theta_0$ are asymptotically Gaussian,

$$\begin{aligned}
\frac{1}{\sqrt{T}}\frac{\partial}{\partial\gamma}l_T(\theta_0) &= \frac{-1}{\sqrt{T}}\sum_{t=1}^{T}\left(1-z_t^2\right)u_{1t}(\theta_0) \xrightarrow{d} N\left(0,\zeta\overline{m}_{11}\right), &(10)\\
\frac{1}{\sqrt{T}}\frac{\partial}{\partial w}l_T(\theta_0) &= \frac{-1}{\sqrt{T}}\sum_{t=1}^{T}\frac{1}{2}\left(1-z_t^2\right)u_{2t}(\theta_0) \xrightarrow{d} N\left(0,\zeta\overline{m}_{22}\right), &(11)\\
\frac{1}{\sqrt{T}}\frac{\partial}{\partial\alpha}l_T(\theta_0) &= \frac{-1}{\sqrt{T}}\sum_{t=1}^{T}\frac{1}{2}\left(1-z_t^2\right)u_{3t}(\theta_0) \xrightarrow{d} N\left(0,\zeta\overline{m}_{33}\right), &(12)
\end{aligned}$$

where $\overline{m}_{ii}$, $i = 1,2,3$ and ζ are defined by Lemma A and A3 respectively.

Proof of Lemma B We will prove (10) in detail. The results in (11) and (12) hold by identical arguments. Define again $I_t = \{y_t, z_t, y_{t-1}, z_{t-1}, y_{t-2}, z_{t-2}, ...\}$ and recall from Result 1 that

$$s_{1t}(\theta_0) = -\left(1-z_t^2\right)u_{1t}(\theta_0).$$

Consequently

$$\begin{aligned}
E\left(s_{1t}|I_{t-1}\right) &= -E\left(\left(1-z_t^2\right)u_{1t}(\theta_0)|I_{t-1}\right)\\
&= -E\left(\left(1-z_t^2\right)\right)u_{1t}(\theta_0)\\
&= 0. &(13)
\end{aligned}$$

Since $\{s_{1t}, I_t\}$ is an adapted stochastic sequence the result in (13) implies that $\{s_{1t}, I_t\}$ is a martingale difference sequence according to Definition 3.75 in White (1984). Further, notice that

$$\begin{aligned}
V_{1T}^2(\theta_0) &= \sum_{t=1}^{T}E\left(s_{1t}^2(\theta_0)|I_{t-1}\right)\\
&= \sum_{t=1}^{T}E\left(\left(1-z_t^2\right)^2\right)u_{1t}^2(\theta_0)\\
&= \zeta\sum_{t=1}^{T}u_{1t}^2(\theta_0).
\end{aligned}$$

Hence,

$$\begin{aligned}
E(V_{1T}^2(\theta_0)) &= \zeta\sum_{t=1}^{T}E\left(u_{1t}^2(\theta_0)\right)\\
&= T\zeta\overline{m}_{11}.
\end{aligned}$$

Furthermore, according to Lemma A we have that

$$\frac{1}{T}\sum_{t=1}^{T}u_{1t}^2(\theta_0) \xrightarrow{p} \overline{m}_{11},$$

implying that

$$\frac{1}{T}V_{1T}^{2}\left(\theta_0\right)\xrightarrow{p}\zeta\overline{m}_{11}.$$

From this we see that

$$\left(V_{1T}^{2}\left(\theta_0\right)\right)\left(E(V_{1T}^{2}\left(\theta_0\right))\right)^{-1}\xrightarrow{p}1. \tag{14}$$

Importantly, the result given by equation (14) corresponds to Condition (1), page 60 in Brown (1971).

Finally, we need to prove that the Lindeberg type condition, which is Condition (2) in Brown (1971). In particular, we need to show that

$$\left(E(V_{1T}^{2}\left(\theta_0\right))\right)^{-1}\sum_{t=1}^{T}E\left(s_{1t}^{2}\left(\theta_0\right)1\left\{\left|s_{1t}\left(\theta_0\right)\right|>\epsilon\sqrt{E(V_{1T}^{2}\left(\theta_0\right))}\right\}\right)\xrightarrow{p}0,$$

for all $\epsilon>0$. By inserting the expression for s_{1t}^{2} and $E(V_{1T}^{2}\left(\theta_0\right))$ we get

$$\begin{aligned}
& \lim_{T\to\infty}\frac{1}{T\zeta\overline{m}_{11}}\sum_{t=1}^{T}E\left(s_{1t}^{2}\left(\theta_0\right)1\left\{\left|s_{1t}\left(\theta_0\right)\right|>\epsilon\sqrt{T\zeta\overline{m}_{11}}\right\}\right) \\
= & \lim_{T\to\infty}\frac{1}{\zeta\overline{m}_{1}}E\left(\left(\left(1-z_t^2\right)^2u_{1t}^{2}\left(\theta_0\right)\right)1\left\{\left|\left(1-z_t^2\right)^2u_{1t}^{2}\left(\theta_0\right)\right|>\sqrt{T\zeta\overline{m}_{11}}\right\}\right) \\
\to & \ 0,
\end{aligned}$$

for all $\zeta\overline{m}_{11}$ because, from Lemma A and A3, $u_{1t}^{2}\left(\theta_0\right)$ and z_t^2 have finite moments and are stationary and ergodic. Consequently, the Lindeberg condition holds.

According to Theorem 2, page 60, in Brown (1971) we can therefore conclude that

$$\frac{1}{\sqrt{T\zeta\overline{m}_{11}}}\sum_{t=1}^{T}s_{1t}\left(\theta_0\right)\xrightarrow{d}N(0,1),$$

which completes the proof.

Along the same lines

$$\frac{1}{T}\sum_{t=1}^{T}E\left(s_{2t}^{2}\mid I_{t-1}\right) = \frac{1}{T}\sum_{t=1}^{T}\frac{\zeta}{4}\frac{1}{\left(w_0+\alpha_0\left(\frac{y_{t-1}^{*}}{y_{t-2}^{\gamma}}\right)^2\right)}\xrightarrow{p}\frac{\zeta}{4w_0^2}>0, \tag{15}$$

$$\frac{1}{T}\sum_{t=1}^{T}E\left(s_{3t}^{2}\mid I_{t-1}\right) = \frac{1}{T}\sum_{t=1}^{T}\frac{\zeta}{4}\frac{\left(\frac{y_{t-1}^{*}}{y_{t-2}^{\gamma}}\right)^2}{\left(w_0+\alpha_0\left(\frac{y_{t-1}^{*}}{y_{t-2}^{\gamma}}\right)^2\right)}\xrightarrow{p}\frac{\zeta}{4\alpha_0^2}>0. \tag{16}$$

and

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T}E\left(s_{2t}^{2}1\left\{\left|s_{2t}\right|>\sqrt{T}\delta\right\}\right) &\le E\left(\left(\frac{\left(1-z_t^2\right)^2}{4w_0^2}\right)1\left\{\left|\frac{\left(1-z_t^2\right)}{2w_0}\right|>\sqrt{T}\delta\right\}\right)\to 0, \\
\frac{1}{T}\sum_{t=1}^{T}E\left(s_{3t}^{2}1\left\{\left|s_{3t}\right|>\sqrt{T}\delta\right\}\right) &\le E\left(\left(\frac{\left(1-z_t^2\right)^2}{4\alpha_0^2}\right)1\left\{\left|\frac{\left(1-z_t^2\right)}{2\alpha_0}\right|>\sqrt{T}\delta\right\}\right)\to 0,
\end{aligned}$$

for some $\delta>0$ and as T tends to ∞.■

Proof of Proposition 1 In order to fully characterize the asymptotic distribution we need to determine the off-diagonal elements of the variance covariance matrix of the score vectors given by Λ. In particular, because $u_{1t}(\theta_0), u_{2t}(\theta_0)$ and $u_{3t}(\theta_0)$ are all stationary and ergodic with finite first moments (from Lemma A) it follows straightforwardly that

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T} s_{1t}(\theta_0)\, s_{2t}(\theta_0) &= \frac{1}{T}\sum_{t=1}^{T}\left(1-z_t^2\right)^2 u_{1t}(\theta_0)\, u_{2t}(\theta_0) \xrightarrow{p} \frac{1}{2}\zeta\overline{m}_{12},\\
\frac{1}{T}\sum_{t=1}^{T} s_{1t}(\theta_0)\, s_{3t}(\theta_0) &= \frac{1}{T}\sum_{t=1}^{T}\frac{1}{2}\left(1-z_t^2\right)^2 u_{1t}(\theta_0)\, u_{3t}(\theta_0) \xrightarrow{p} \frac{1}{2}\zeta\overline{m}_{13},\\
\frac{1}{T}\sum_{t=1}^{T} s_{2t}(\theta_0)\, s_{3t}(\theta_0) &= \frac{1}{T}\sum_{t=1}^{T}\frac{1}{4}\left(1-z_t^2\right)^2 u_{2t}(\theta_0)\, u_{3t}(\theta_0) \xrightarrow{p} \frac{1}{4}\zeta\overline{m}_{23}.
\end{aligned}$$

Since all the elements in the score vector are asymptotically normal (see Lemma B), the result follows directly from application of the Cramer-Wold device, see for example Proposition 5.1 in White (1984), which completes the proof.■

Proof of Proposition 2 Recall from Result 2 that

$$\begin{aligned}
-\frac{1}{T}\frac{\partial^2}{\partial\gamma^2} l_T(\theta_0) &= 2\frac{1}{T}\sum_{t=1}^{T} z_t^2 u_{1t}^2(\theta_0)\\
&\quad+\frac{1}{T}\sum_{t=1}^{T} 4\left(1-z_t^2\right)\alpha_0\left(\ln^2 y_{t-2}\right) w_0^2\left(\frac{1}{w_0}-\frac{1}{\sigma_t^2(\theta_0)}\right).
\end{aligned}$$

Since z_t^2 and $u_{1t}^2(\theta_0)$ are independent, the first term on the right hand side converges to $2\overline{m}_{11}$ by Lemma A. Furthermore, since $4\alpha_0\left(\ln^2 y_{t-2}\right) w_0^2\left(\frac{1}{w_0}-\frac{1}{\sigma_t^2(\theta_0)}\right)$ has bounded moments, it is ergodic and stationary and since $E\left(1-z_t^2\right)=0$, it follows from the ergodic theorem that the last term on the right hand side converges in probability to zero. Therefore, the result follows. Using identical arguments we find

$$\begin{aligned}
-\frac{1}{T}\frac{\partial^2}{\partial w^2} l_T(\theta_0) &= -\frac{1}{2}\frac{1}{T}\sum_{t=1}^{T}\left(1-2z_t^2\right) u_{2t}^2(\theta_0) \xrightarrow{p} \frac{1}{2}\overline{m}_{22},\\
-\frac{1}{T}\frac{\partial^2}{\partial\alpha^2} l_T(\theta_0) &= -\frac{1}{2}\frac{1}{T}\sum_{t=1}^{T}\left(1-2z_t^2\right) u_{3t}^2(\theta_0) \xrightarrow{p} \frac{1}{2}\overline{m}_{33},\\
-\frac{1}{T}\frac{\partial^2}{(\partial\gamma\partial w)} l_T(\theta_0) &= \frac{1}{T}\sum_{t=1}^{T} z_t^2 u_{1t}(\theta_0)\, u_{2t}(\theta_0) + \frac{1}{T}\sum_{t=1}^{T}\left(1-z_t^2\right)\left(\left(\ln y_{t-2}\right)\left(\frac{w_0}{\sigma_t^2(\theta_0)}\right)\left(\frac{1}{w_0}-\frac{1}{\sigma_t^2(\theta_0)}\right)\right) \xrightarrow{p} \overline{m}_{12},\\
-\frac{1}{T}\frac{\partial^2}{(\partial\gamma\partial\alpha)} l_T(\theta_0) &= \frac{1}{T}\sum_{t=1}^{T} z_t^2 u_{1t}(\theta_0)\, u_{3t}(\theta_0) - \frac{1}{T}\sum_{t=1}^{T}\left(1-z_t^2\right) w_0\left(\ln y_{t-2}\right)\left(\frac{w_0}{\alpha_0}\right)\left(\frac{1}{w_0}-\frac{1}{\sigma_t^2(\theta_0)}\right) \xrightarrow{p} \overline{m}_{13},\\
-\frac{1}{T}\frac{\partial^2 l_t(\theta_0)}{\partial w\partial\alpha} &= -\frac{1}{2}\frac{1}{T}\sum_{t=1}^{T}\left(1-2z_t^2\right) u_{2t}(\theta_0)\, u_{3t}(\theta_0) \xrightarrow{p} \frac{1}{2}\overline{m}_{23}.
\end{aligned}$$

Finally notice that since $\Omega = 2\Lambda\zeta^{-1}$, then $\Omega > 0$. This completes the proof of Proposition 2.■

Proof of Proposition 3 Let us start from the components of $\left|\frac{1}{T}\frac{\partial^3}{\partial\gamma^3}l_T(\theta)\right|$ defined in Result 3 below. Part I (which is also defined in Result 3 below) can be written as

$$\left|\frac{1}{T}4\sum_{t=1}^{T}\frac{\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2(\ln(y_{t-2}))^2(\ln(y_{t-1})-\ln(y_{t-2}))}{\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)^3}\left(1+\frac{(y_t^*)^2}{y_{t-1}^{2\gamma}\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)}\right)\right|$$

$$\leq \left|\frac{4}{T}\sum_{t=1}^{T}\frac{\left(\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)-w\right)^3(\ln(y_{t-2}))^2(\ln(y_{t-1})-\ln(y_{t-2}))}{\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)^3}\left(\frac{(y_t^*)^2}{y_{t-1}^{2\gamma}\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)}+1\right)\right|$$

$$\leq \left|\frac{4}{T}\sum_{t=1}^{T}\left(\frac{(y_t^*)^2}{y_{t-1}^{2\gamma}\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)}+1\right)(\ln(y_{t-2}))^2\left|(\ln(y_{t-1})-\ln(y_{t-2}))\right|\right|$$

$$\leq \left|\frac{4}{T}\sum_{t=1}^{T}\left(\left(\frac{y_{t-1}^{2\gamma_0}\left(w_0+\alpha_0\left(\frac{y_{t-1}^*}{y_{t-2}^{\gamma_0}}\right)^2\right)}{y_{t-1}^{2\gamma}\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)}\right)z_t^2+1\right)(\ln(y_{t-2}))^2\left|(\ln(y_{t-1})-\ln(y_{t-2}))\right|\right|$$

$$\leq \left|\frac{4}{T}\sum_{t=1}^{T}\left(\left(\frac{w_0}{w}y_{t-1}^{2(\gamma_0-\gamma)}+\frac{\alpha_0}{\alpha}\left(\frac{y_{t-1}}{y_{t-2}}\right)^{2(\gamma_0-\gamma)}\right)z_t^2+1\right)(\ln(y_{t-2}))^2\left|(\ln(y_{t-1})-\ln(y_{t-2}))\right|\right|$$

$$\leq \left|\frac{4}{T}\sum_{t=1}^{T}\left(\left\{\frac{w_U}{w_L}\Lambda_{t-1}+\frac{\alpha_U}{\alpha_L}\Lambda_{t-2}\right\}z_t^2+1\right)(\ln(y_{t-2}))^2\left|(\ln(y_{t-1})-\ln(y_{t-2}))\right|\right|$$

$$\leq \left|\frac{4}{T}\sum_{t=1}^{T}\left(\left\{\frac{w_U}{w_L}\Lambda_{t-1}+\frac{\alpha_U}{\alpha_L}\Lambda_{t-2}\right\}z_t^2+1\right)(\ln(y_{t-2}))^2\left|(\ln(y_{t-1})-\ln(y_{t-2}))\right|\right|$$

$$\leq \left|\frac{4}{T}\sum_{t=1}^{T}\left(\left\{\frac{w_U}{w_L}\Lambda_{t-1}+\frac{\alpha_U}{\alpha_L}\Lambda_{t-2}\right\}z_t^2+1\right)(\ln(y_{t-2}))^2\left|(\ln(y_{t-1})-\ln(y_{t-2}))\right|\right|,$$

where we can define the lower bound for all t, $y_L \leq y_{t-1}$, $y_L \leq y_{t-2}$, $\Lambda_{t-1} = \max\left\{y_L^{2|\gamma_U-\gamma_L|}, y_{t-1}^{2|\gamma_U-\gamma_L|}\right\}$, $\Lambda_{t-2} = \max\left\{1, \left(\frac{y_{t-1}}{y_{t-2}}\right)^{2|\gamma_U-\gamma_L|}\right\}$ and the result follows by setting $2|\gamma_U-\gamma_L| = \varphi$, Assumptions A and B and the law of large numbers (see Jensen and Rahbek (2004a), Lemma 5). Parts II and III follow the same argument. Part IV requires also assumption A4 since

$$\left|\frac{1}{T}6\sum_{t=1}^{T}\frac{\alpha^3\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^6(\ln(y_{t-2}))^3}{\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)^3}\left(1-\frac{4(y_t^*)^2}{y_{t-1}^{2\gamma}\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)}\right)\right|$$

$$\leq \left|\frac{6}{T}\sum_{t=1}^{T}\left(\frac{4(y_t^*)^2}{y_{t-1}^{2\gamma}\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)}-1\right)[\ln(y_{t-2})]^3\right| \leq \left|\frac{6}{T}\sum_{t=1}^{T}\left(4\left\{\frac{w_U}{w_L}\Lambda_{t-1}+\frac{\alpha_U}{\alpha_L}\Lambda_{t-2}\right\}z_t^2+1\right)[\ln(y_{t-2})]^3\right|.$$

Parts V and VI follow the same argument.

Along the same lines for $\left|\frac{1}{T}\frac{\partial^3}{\partial\alpha^3}l_T\left(\theta\right)\right|$

$$\left|\frac{1}{T}\frac{\partial^3}{\partial\alpha^3}l_T\left(\theta\right)\right| = \left|\frac{1}{T}\sum_{t=1}^{T}\left(3\frac{\left(y_t^*\right)^2}{y_{t-1}^{2\gamma}\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^{\gamma}}\right)^2\right)}-1\right)\frac{\left(\frac{y_{t-1}^*}{y_{t-2}^{\gamma}}\right)^6}{\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^{\gamma}}\right)^2\right)^3}\right|$$

$$\leq \left|\frac{1}{T}\sum_{t=1}^{T}\left(3\frac{\left(y_t^*\right)^2}{y_{t-1}^{2\gamma}\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^{\gamma}}\right)^2\right)}-1\right)\right|\frac{1}{\alpha_L^3} \leq \frac{1}{T}\sum_{t=1}^{T}\left(3\left\{\frac{w_U}{w_L}\Lambda_{t-1}+\frac{\alpha_U}{\alpha_L}\Lambda_{t-2}\right\}z_t^2+1\right)\frac{1}{\alpha_L^3}.$$

The rest of the cases follow directly using the same argument. This completes the proof of Proposition 3.■

Result 3

The third order derivatives are given by

$$\frac{\partial^3}{\partial w^3} l_T(\theta) = -\sum_{t=1}^{T} \left(1 - 3\frac{(y_t^*)^2}{y_{t-1}^{2\gamma}\left(w + \alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)} \right) \frac{1}{\left(w + \alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)^3},$$

$$\frac{\partial^3}{\partial \alpha^3} l_T(\theta) = -\sum_{t=1}^{T} \left(1 - 3\frac{(y_t^*)^2}{y_{t-1}^{2\gamma}\left(w + \alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)} \right) \frac{\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^6}{\left(w + \alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)^3},$$

$$\begin{aligned}
\frac{\partial^3}{\partial \gamma \partial w^2} l_T(\theta) &= \frac{1}{2}\sum_{t=1}^{T} \frac{-2\left(w + \alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)\left(-2\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2 \ln(y_{t-2})\right)}{\left(w + \alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)^4} \\
&\quad - \sum_{t=1}^{T} \frac{-(y_t^*)^2 y_{t-1}^{2\gamma} 3\left(w + \alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)^2 \left(-2\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2 \ln(y_{t-2})\right)}{y_{t-1}^{4\gamma}\left(w + \alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)^6} \\
&= 2\sum_{t=1}^{T} \left(1 - \frac{3(y_t^*)^2}{y_{t-1}^{2\gamma}\left(w + \alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)}\right) \frac{\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2 \ln(y_{t-2})}{\left(w + \alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)^3},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^3}{\partial \gamma \partial \alpha^2} l_T(\theta) &= -\sum_{t=1}^{T} \frac{2\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^4 \ln(y_{t-2})}{\left(w + \alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)^2} \left(1 - \frac{2(y_t^*)^2}{y_{t-1}^{2\gamma}\left(w + \alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)}\right) \\
&\quad + \sum_{t=1}^{T} \frac{2\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^6 \ln(y_{t-2})}{\left(w + \alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)^3} \left(1 - \frac{3(y_t^*)^2}{y_{t-1}^{2\gamma}\left(w + \alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)}\right),
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^3}{\partial\gamma^2\partial w} l_T(\theta) &= -2\sum_{t=1}^{T} \frac{\left(\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2 \ln(y_{t-2})\left(\ln(y_{t-1})-\ln(y_{t-2})\right)\right)}{\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)^2} \left(1+\frac{2\left(y_t^*\right)^2}{y_{t-1}^{2\gamma}\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)}\right) \\
&\quad -2\sum_{t=1}^{T}\left(\frac{\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2 \ln(y_{t-1})\ln(y_{t-2})}{\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)^2} - \frac{2\alpha^2\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^4 \ln(y_{t-2})\left(\ln(y_{t-1})-\ln(y_{t-2})\right)}{\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)^3}\right) \\
&\quad \times\left(1-\frac{3\left(y_t^*\right)^2}{y_{t-1}^{2\gamma}\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)}\right) \\
&\quad +2\sum_{t=1}^{T}\frac{\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2 \ln(y_{t-2})\ln(y_{t-1})\, w}{\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)^3}\left(2-\frac{9\left(y_t^*\right)^2}{y_{t-1}^{2\gamma}\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)}\right) \\
&\quad +\sum_{t=1}^{T}\frac{\left(y_t^*\right)^2}{y_{t-1}^{2\gamma}\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)}\left(\frac{-4\left(\ln(y_{t-1})\right)^2}{\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)} + \frac{6\left(w+2\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)\left(\ln(y_{t-1})\right)^2 w}{\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)^3}\right. \\
&\quad \left. +\frac{6\alpha^2\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^4 \ln(y_{t-1})\left(\ln(y_{t-1})-\ln(y_{t-2})\right)}{\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)^3}\right),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^3}{\partial\gamma^3} l_T(\theta) = & -4\sum_{t=1}^{T} \frac{\alpha\left(\frac{y^*_{t-1}}{y^\gamma_{t-2}}\right)^2 (\ln(y_{t-2}))^2 (\ln(y_{t-1}) - \ln(y_{t-2}))}{\left(w+\alpha\left(\frac{y^*_{t-1}}{y^\gamma_{t-2}}\right)^2\right)} \left(1 + \frac{(y^*_t)^2}{y^{2\gamma}_{t-1}\left(w+\alpha\left(\frac{y^*_{t-1}}{y^\gamma_{t-2}}\right)^2\right)}\right) \\
& +8\sum_{t=1}^{T} \frac{\alpha^2\left(\frac{y^*_{t-1}}{y^\gamma_{t-2}}\right)^4 (\ln(y_{t-2}))^2 (\ln(y_{t-1}) - \ln(y_{t-2}))}{\left(w+\alpha\left(\frac{y^*_{t-1}}{y^\gamma_{t-2}}\right)^2\right)^2} \left(1 - \frac{(y^*_t)^2}{y^{2\gamma}_{t-1}\left(w+\alpha\left(\frac{y^*_{t-1}}{y^\gamma_{t-2}}\right)^2\right)}\right) \\
& +4\sum_{t=1}^{T} \frac{\alpha w\left(\frac{y^*_{t-1}}{y^\gamma_{t-2}}\right)^2 (\ln(y_{t-2}))^2 \ln(y_{t-1})}{\left(w+\alpha\left(\frac{y^*_{t-1}}{y^\gamma_{t-2}}\right)^2\right)^2} \left(1 - \frac{3(y^*_t)^2}{y^{2\gamma}_{t-1}\left(w+\alpha\left(\frac{y^*_{t-1}}{y^\gamma_{t-2}}\right)^2\right)}\right) \\
& +6\sum_{t=1}^{T} \frac{\alpha^3\left(\frac{y^*_{t-1}}{y^\gamma_{t-2}}\right)^6 (\ln(y_{t-2}))^3}{\left(w+\alpha\left(\frac{y^*_{t-1}}{y^\gamma_{t-2}}\right)^2\right)^3} \left(1 - \frac{4(y^*_t)^2}{y^{2\gamma}_{t-1}\left(w+\alpha\left(\frac{y^*_{t-1}}{y^\gamma_{t-2}}\right)^2\right)}\right) \\
& -6\sum_{t=1}^{T} \left(\frac{\alpha^2 w\left(\frac{y^*_{t-1}}{y^\gamma_{t-2}}\right)^4 (\ln(y_{t-2}))^2 \ln(y_{t-1})}{\left(w+\alpha\left(\frac{y^*_{t-1}}{y^\gamma_{t-2}}\right)^2\right)^3} + \frac{\alpha^3\left(\frac{y^*_{t-1}}{y^\gamma_{t-2}}\right)^6 \ln(y_{t-1}) (\ln(y_{t-2}))^2}{\left(w+\alpha\left(\frac{y^*_{t-1}}{y^\gamma_{t-2}}\right)^2\right)^3}\right) \\
& \times\left(1 - \frac{6(y^*_t)^2}{y^{2\gamma}_{t-1}\left(w+\alpha\left(\frac{y^*_{t-1}}{y^\gamma_{t-2}}\right)^2\right)}\right) \\
& -4\sum_{t=1}^{T} \frac{(y^*_t)^2}{y^{2\gamma}_{t-1}\left(w+\alpha\left(\frac{y^*_{t-1}}{y^\gamma_{t-2}}\right)^2\right)} \left(\frac{-w(\ln(y_{t-1}))^2\left(\ln(y_{t-1})w + \alpha\left(\frac{y^*_{t-1}}{y^\gamma_{t-2}}\right)^2 (2\ln(y_{t-1}) - \ln(y_{t-2}))\right)}{\left(w+\alpha\left(\frac{y^*_{t-1}}{y^\gamma_{t-2}}\right)^2\right)}\right. \\
& + \frac{\alpha^2\left(\frac{y^*_{t-1}}{y^\gamma_{t-2}}\right)^4\left((\ln(y_{t-1}))^2\ln(y_{t-2}) - (\ln(y_{t-1}))^3\right) + \alpha\left(\frac{y^*_{t-1}}{y^\gamma_{t-2}}\right)^2 \ln(y_{t-1})\ln(y_{t-2})(\ln(y_{t-1}) - \ln(y_{t-2}))}{\left(w+\alpha\left(\frac{y^*_{t-1}}{y^\gamma_{t-2}}\right)^2\right)^2} \\
& \left. +3\frac{\left(w^2\alpha\left(\frac{y^*_{t-1}}{y^\gamma_{t-2}}\right)^2 + 2w\alpha^2\left(\frac{y^*_{t-1}}{y^\gamma_{t-2}}\right)^4 + \alpha^3\left(\frac{y^*_{t-1}}{y^\gamma_{t-2}}\right)^6\right)\ln(y_{t-2})(\ln(y_{t-1}))^2}{\left(w+\alpha\left(\frac{y^*_{t-1}}{y^\gamma_{t-2}}\right)^2\right)^3}\right) \\
& = I + II + III + IV + V + VI,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^3}{\partial\gamma^2\partial\alpha} l_T(\theta) &= 2\sum_{t=1}^{T} \frac{\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2 \ln(y_{t-2})\left(\ln(y_{t-1})-\ln(y_{t-2})\right)}{\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)} \left(1+\frac{(y_t^*)^2}{y_{t-1}^{2\gamma}\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)}\right) \\
&\quad -2\sum_{t=1}^{T} \frac{\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^4 \ln(y_{t-2})\left(\ln(y_{t-1})-\ln(y_{t-2})\right)}{\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)^2} \left(1+\frac{2(y_t^*)^2}{y_{t-1}^{2\gamma}\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)}\right) \\
&\quad +4\sum_{t=1}^{T} \frac{\alpha^2\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^6 \ln(y_{t-2})\left(\ln(y_{t-1})-\ln(y_{t-2})\right)}{\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)^3} \left(1-\frac{3(y_t^*)^2}{y_{t-1}^{2\gamma}\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)}\right) \\
&\quad +2\sum_{t=1}^{T} \frac{\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^4 \ln(y_{t-2})\ln(y_{t-1})\,w}{\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)^3} \left(2-\frac{9(y_t^*)^2}{y_{t-1}^{2\gamma}\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)}\right) \\
&\quad -4\sum_{t=1}^{T} \frac{\alpha^2\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^4 \ln(y_{t-2})\left(\ln(y_{t-1})-\ln(y_{t-2})\right)}{\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)^3} - 2\sum_{t=1}^{T} \frac{w\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2 \ln(y_{t-1})\ln(y_{t-2})}{\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)^3} \\
&\quad +\sum_{t=1}^{T} \frac{(y_t^*)^2}{y_{t-1}^{2\gamma}\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)} \left(-\frac{2w\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2 \ln(y_{t-1})\left(2\ln(y_{t-1})-3\ln(y_{t-2})\right)}{\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)^2}\right. \\
&\quad +\frac{12\alpha w\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^4 \left(\ln(y_{t-1})\right)^2}{\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)^3} - \frac{4\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^4 \left(\ln(y_{t-1})-2\ln(y_{t-2})\right)\left(\ln(y_{t-1})-\ln(y_{t-2})\right)}{\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)^2} \\
&\quad \left.+\frac{6\alpha^2\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^6 \ln(y_{t-1})\left(\ln(y_{t-1})-\ln(y_{t-2})\right)+6w^2\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\left(\ln(y_{t-1})\right)^2}{\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)^3}\right),
\end{aligned}
$$

$$
\frac{\partial^3}{\partial\alpha\partial w^2} l_T(\theta) = -\sum_{t=1}^{T}\left(1-\frac{3(y_t^*)^2}{y_{t-1}^{2\gamma}\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)}\right)\frac{\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2}{\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)^3},
$$

$$
\frac{\partial^3}{\partial w\partial\alpha^2} l_T(\theta) = -\sum_{t=1}^{T}\left(1-\frac{3(y_t^*)^2}{y_{t-1}^{2\gamma}\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)}\right)\frac{\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^4}{\left(w+\alpha\left(\frac{y_{t-1}^*}{y_{t-2}^\gamma}\right)^2\right)^3}\ ,
$$

$$
\begin{aligned}
\frac{\partial^3}{\partial w \partial \alpha \partial \gamma} l_T(\theta) &= -\sum_{t=1}^{T} \frac{\left(\frac{y^*_{t-1}}{y^{\gamma}_{t-2}}\right)^2 \ln(y_{t-2})}{\left(w + \alpha \left(\frac{y^*_{t-1}}{y^{\gamma}_{t-2}}\right)^2\right)^2} \left(1 - \frac{2\left(y^*_t\right)^2}{y^{2\gamma}_{t-1}\left(w + \alpha \left(\frac{y^*_{t-1}}{y^{\gamma}_{t-2}}\right)^2\right)}\right) \\
&\quad + \sum_{t=1}^{T} \frac{2\alpha \left(\frac{y^*_{t-1}}{y^{\gamma}_{t-2}}\right)^4 \ln(y_{t-2})}{\left(w + \alpha \left(\frac{y^*_{t-1}}{y^{\gamma}_{t-2}}\right)^2\right)^3} \left(1 - \frac{3\left(y^*_t\right)^2}{y^{2\gamma}_{t-1}\left(w + \alpha \left(\frac{y^*_{t-1}}{y^{\gamma}_{t-2}}\right)^2\right)}\right).
\end{aligned}
$$

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