CREATES Research Paper 2010-18

Modelling energy spot prices by Lévy semistationary processes

Ole E. Barndorff-Nielsen, Fred Espen Benth
and Almut E. D. Veraart

Modelling energy spot prices by Lévy semistationary processes

Ole E. Barndorff–Nielsen
Thiele Center, Department of Mathematical Sciences
and
CREATES, School of Economics and Management
Aarhus University
Ny Munkegade 118
DK–8000 Aarhus C, Denmark
oebn@imf.au.dk

Fred Espen Benth
Centre of Mathematics for Applications
University of Oslo
P.O. Box 1053, Blindern
N–0316 Oslo, Norway
and
Faculty of Economics
University of Agder
Serviceboks 422
N–4604 Kristiansand, Norway
fredb@math.uio.no

Almut E. D. Veraart*
CREATES, School of Economics and Management
Aarhus University
DK–8000 Aarhus, C, Denmark
averaart@creates.au.dk

Abstract

This paper introduces a new modelling framework for energy spot prices based on Lévy semistationary processes. Lévy semistationary processes are special cases of the general class of ambit processes. We provide a detailed analysis of the probabilistic properties of such models and we show how they are able to capture many of the stylised facts observed in energy markets. Furthermore, we derive forward prices based on our spot price model. As it turns out, many of the classical spot models can be embedded into our novel modelling framework.

Keywords: Energy markets; forward price; Lévy semistationary process; stochastic integration; spot price.

JEL codes: C0, C1, C5, G1.

1   Introduction

This article introduces the concept of Lévy semistationary processes and, based on this, we define a new class of energy spot price models. This class is analytically tractable and encompasses many classical models such as those based on the Schwartz one-factor mean-reversion model, see Schwartz (1997), and the wider class of continuous-time autoregressive moving-average processes. Our main innovation lies in the fact that we model the spot price directly by a stationary process, whereas the traditional approach focuses on modelling the dynamics of the spot price. We will discuss in detail the advantages of such a modelling approach. Furthermore, we will show that we can construct both geometric and arithmetic spot models based on Lévy semistationary processes. We will also give a detailed account of the theoretical properties of the class of Lévy semistationary processes, which generalises that of Brownian semistationary processes and are special types of ambit processes. This framework enables us to model both the spot and the forward price (which we compute based on our new spot model) in a consistent way.

Our novel modelling framework is motivated, in particular, by the fact that it naturally incorporates many of the stylised facts observed in energy markets. Although we aim at modelling energy prices in general (including oil, gas, electricity and coal), we will mainly focus on electricity prices in this paper.

There is a clear economical justification for *mean-reversion* in commodity markets in general and in power prices in particular. This is due to the fact that supply and demand will determine the spot price. It is natural to see a downward push in prices when they tend to be large since demand will naturally decrease, and vice-versa when prices are cheap because consumption will most likely increase. Such effects are usually modeled by applying Ornstein-Uhlenbeck processes, where mean-reversion means that up to an additive noise term the *speed* of the (log-)price at each moment is proportional to the negative of the current (log-)price, i.e. (in a heuristic sense)

$$\frac{dY_t}{dt} = -\alpha Y_t + \frac{dN_t}{dt},$$

for a noise process N. The speed of reversion is measured by α. In an integral form, this means that

$$Y_t = Y_0 + \int_0^t g(t-s)\,dN_s\,,$$

where $g(x) = \exp(-\alpha x)$. A much weaker concept of 'mean-reversion' is to assume *stationarity* of the log-price process. With the classical interpretation of mean-reversion, the prices will be physically pushed back to some mean–level, whereas stationarity implies a push of prices back to a mean level in a probabilistic sense. Starting out with this, we can relax the choice of g above to a much wider class of functions, still preserving stationarity and thereby modelling 'mean-reversion'. This is one of our modelling ideas for the spot.

Also, we believe that the price of the spot (and the forward price) is influenced by a *stochastic volatility* process ω, see e.g. Hikspoors & Jaimungal (2008) and Benth (2009), which, based on empirical evidence, describes the volatility of the spot market as a whole. More precisely, we will assume that the volatility of the spot depends on previous states of the volatility.

Finally, we wish to account for the so-called *Samuelson effect*, see Samuelson (1965), which is another stylised fact of electricity markets related to the relationship between the volatility of the spot and the volatility of forward prices. Usually the volatility of the forward price is smaller than the volatility of the spot and decreases with time to maturity. The Samuelson effect refers to the finding that, when the time to maturity approaches zero, the volatility of the forward starts increasing and

converges to the volatility of the spot eventually. We will show how our new modelling framework allows for such effects in a natural way.

The remaining part of the paper is structured as follows. We start by introducing the class of Lévy semistationary processes in Section 2. Section 3 develops two new types of spot price models based on Lévy semistationary processes: a geometric and an arithmetic model. Furthermore we describe how our new models embed many of the traditional models used in the recent literature on modelling electricity spot prices. We proceed by discussing some of the probabilistic properties of our new models in Section 4; and in Section 5, we derive the forward price dynamics of the models and consider questions like affinity of the forward price with respect to the underlying spot. Finally, Section 6 concludes and the appendix, see Section A, contains the proofs of the main results.

2 Lévy semistationary processes

Throughout this paper, we suppose that we have given a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}_{t \in \mathbb{R}}$ satisfying the 'natural conditions', see Karatzas & Shreve (2005).

Now we introduce the class of Lévy semistationary (from now on $\mathcal{LSS}$) processes, which we will use as a building block for our new models for electricity (or, more generally, commodity) spot prices. A $\mathcal{LSS}$ process $Y = \{Y_t\}_{t \in \mathbb{R}}$ is given by

$$Y_t = \mu + \int_{-\infty}^{t} g(t-s)\omega_s \mathrm{d}L_s + \int_{-\infty}^{t} q(t-s)a_s \mathrm{d}s, \tag{1}$$

where μ is a constant, L is a Lévy process, g and q are nonnegative deterministic functions on $\mathbb{R}$, with $g(t) = q(t) = 0$ for $t \leq 0$, and ω and a are càdlàg processes. Note that the name Lévy semistationary processes has been derived from the fact that the process Y is stationary as soon as ω and a are stationary. As an abbreviated version of the above formula we write

$$Y = \mu + g * \omega \bullet L + q * a \bullet Leb, \tag{2}$$

where Leb denotes Lebesgue measure.

The stochastic integral in (1) could be defined in a weak sense as described in Rajput & Rosinski (1989). However, in order to have more powerful tools for analysing such processes, we will work with a general integration concept developed in Basse-O'Connor et al. (2010) which specialises to the concept of Rajput & Rosinski (1989) for deterministic integrands and to Itô integration, see Protter (2005), when we are in the semimartingale framework. Note that the stochastic processes defined in (1) are defined on the whole real line in contrast to the standard stochastic integrals which are defined on compact time intervals. We summarise the main points of the general integration concept in Section 4.

Clearly, the term $g * \omega \bullet L$ in Y is a special case of a volatility modulated Lévy-driven Volterra (VMVP) process, see Barndorff-Nielsen & Schmiegel (2008), which has the form

$$\int_{-\infty}^{t} g(t,s)\omega_s \, dL_s, \tag{3}$$

where L is a Lévy process and g is a nonnegative real-valued measurable function on $\mathbb{R}^2$, such that the integral with respect to L exists. Throughout the paper we shall also be concerned with such VMVP processes.

Remark. A special type of $\mathcal{LSS}$ processes is given by the class of Brownian semistationary ($\mathcal{BSS}$) processes, where the driving Lévy process $L = W$, for a standard Brownian motion $W = (W_t)_{t \in \mathbb{R}}$.

Such processes have recently been introduced by Barndorff-Nielsen & Schmiegel (2009) in the context of modelling turbulence in physics. The problem of drawing inference on the volatility – by means of generalised realised multipower variations – in such general models has been studied in Barndorff-Nielsen, Corcuera & Podolskij (2009, 2010).

Note that the class of $\mathcal{BSS}$ processes can be considered as the natural analogue for (semi-) stationary processes of Brownian semimartingales ($\mathcal{BSM}$) , given by

$$Y_t = \int_0^t \omega_s \mathrm{d}B_s + \int_0^t a_s \mathrm{d}s.$$

The class of $\mathcal{LSS}$ processes can be embedded into the class of ambit processes, which are very general tempo–spatial stochastic processes. I.e. let $Y = \{Y_t(x)\}$ denote a stochastic field in space-time $\mathcal{X} \times \mathbb{R}$ and let $\tau(\theta) = (x(\theta), t(\theta))$ denote a curve in $\mathcal{X} \times \mathbb{R}$. We define

$$Y_t(x) = \mu + \int_{A_t(x)} g(\xi, s; x, t)\omega_s(\xi)L(d\xi, ds) + \int_{D_t(x)} q(\xi, s; x, t)a_s(\xi)d\xi ds, \tag{4}$$

where L denotes a Lévy basis, g, q are deterministic kernel functions, $A_t(x), D_t(x) \subset \mathcal{X} \times \mathbb{R}_t$, and ω, a denote stochastic *volatility* fields. Clearly, $X = (X_\theta)$ with $X_\theta = Y_{t(\theta)}(x(\theta))$ denotes a stochastic process. We call $(Y_t(x))$ an *ambit field* and (X_θ) and *ambit process*, see Barndorff-Nielsen & Schmiegel (2007), Barndorff-Nielsen, Benth & Veraart (2009).

When we consider the null–spatial case in the context of ambit processes, i.e. when the space $\mathcal{X}$ consists of a single point (or we just consider $Y_t(x)$ of (4) in its dependence on t keeping x fixed), the concept of ambit processes specialises to that of Lévy semistationary processes.

The direct link between $\mathcal{LSS}$ and ambit processes makes it possible to model both electricity spot and forward prices in a consistent framework. When we model spot prices based on $\mathcal{LSS}$ processes and forward prices based on ambit fields, one can show, see Barndorff-Nielsen, Benth & Veraart (2010), that these two models can be linked to each other in a very natural way.

3   The new model for the electricity spot price

3.1   Model building

Based on the new class of Lévy semistationary processes, we can now start developing a new class of models for electricity spot prices.

First of all, we wish to point out why we think that the class of $\mathcal{LSS}$ processes is particularly suitable as a building block for modelling commodity markets. We choose a Lévy process $L = (L_t)_{t \in \mathbb{R}}$ as the driving process of the electricity spot price to reflect the fact that there are periods of relative smooth prices followed by usually shorter periods of price spikes and jumps. Moreover, as already mentioned, we wish to allow for stochastic volatility denoted by $\omega = (\omega_t)_{t \in \mathbb{R}}$. This is done by scaling of the driving Lévy process, i.e. we work with a stochastic integral of ω with respect to L. Furthermore, we multiply the stochastic volatility component by a deterministic kernel function g, say, to get more flexibility in order to model the Samuelson effect in a realistic way. I.e. the building block for the electricity spot price $Y = (Y_t)_{t \in \mathbb{R}}$ will be a special case of a $\mathcal{LSS}$ process, which is given by

$$Y_t = \int_{-\infty}^t g(t-s)\omega_s dL_s, \tag{5}$$

for a damping function $g : \mathbb{R}_+ \to \mathbb{R}_+$, and a càdlàg, positive, stationary process ω which is independent of the two–sided Lévy process L. In the definition of Y in (5), we could add a drift term, as in (1). However, in order to keep matters as simple as possible, we restrain from doing so.

3.2 A geometric and an arithmetic model

After having presented the main building block of our new model, the question is now how exactly we wish to model the electricity spot price. Basically, we can follow two paths: We can construct a geometric or an arithmetic model.

3.2.1 Geometric model

In a geometric set up, we define the spot price $S^g = (S^g_t)_{t \in \mathbb{R}}$ by

$$S^g_t = \Lambda(t) \exp(Y_t), \tag{6}$$

where $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$ denotes a bounded and measurable deterministic seasonal function.

In such a modelling framework, the deseasonalised, logarithmic spot price is given by a driftless $\mathcal{LSS}$ process. Special cases of such a model include the CARMA–based model and the classical Schwartz model, as discussed later.

3.2.2 Arithmetic model

Alternatively, one can construct a spot price model which is of arithmetic type. Since we wish to ensure that the prices can only take positive values (although electricity prices can sometimes turn negative, when the supply highly exceeds the demand), we formulate the following condition which is sufficient for price positivity.

Assumption (P): Let L be a Lévy subordinator and let the kernel function g in (5) be positive.

If assumption (P) is satisfied, we define the electricity spot price $S^a = (S^a_t)_{t \geq 0}$ by

$$S^a_t = \Lambda(t)Y_t, \tag{7}$$

where $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$ again denotes a bounded and measurable deterministic seasonal function and Y is defined as in (5).

Note that a process Y defined by (5) and satisfying condition (P) generalises the class of convoluted subordinators defined in Bender & Marquardt (2009) to allow for stochastic volatility.

We remark that we do not assume that the dynamics of the spot model S^g_t or S^a_t starts at the current spot price S^g_0 or S^a_0, respectively, as is customary in most spot price models for financial asset prices in the literature. We rather *assume* that the observed spot price today is an *observation* of the random variable $S^g_0 = \Lambda(0) \exp(Y_0)$ and $S^a_0 = \Lambda(0)Y_0$. The reason being that we in fact model a deseasonalised price series which is in stationarity. Most models for the spot price of commodities, are modelled by stochastic processes which tend to stationary processes as time $t \to \infty$. In fact, this means that at every time instance, it will take some time before the stochastic price dynamics are in stationarity. Therefore it is in fact much more reasonable to model directly under stationarity than using a model where we condition on the observation today.

3.3 Model discussion and further extensions

After having presented our new models we would like to give some further justification why we think they are particularly suitable for modelling electricity spot prices and we will also indicate some further extensions which can be easily included in our new modelling framework.

There are several key features which make our new models both theoretically interesting and practically relevant compared to the traditional models.

First and foremost, we model the deseasonalised, (logarithmic) spot price Y *directly*, rather than its stochastic dynamics. By doing so, we can introduce a general kernel function g, which adds much more flexibility in modelling the mean-reversion of the price process. For instance, as we shall see in the section on forward prices, we can achieve greater flexibility in modelling the shape of the Samuelson effect observed in forward prices, including the hyperbolic one suggested by Bjerksund et al. (2000) as a reasonable volatility feature in power markets.

Note that our $\mathcal{LSS}$–based models are able to produce various types of autocorrelation functions depending on the choice of the kernel function g, see Section 4.4 for more details.

Furthermore, we account for stochastic volatility ω since this is clearly an issue in energy markets, see e.g. Hikspoors & Jaimungal (2008), Trolle & Schwartz (2009), Benth (2009) and Benth & Vos (2009). A very general model for the volatility process would be that we model it itself as a Lévy Volterra process, i.e. $\omega_t^2 = Z_t$ and

$$Z_t = \int_{-\infty}^t i(t, s) \, dU_s \,, \tag{8}$$

where i denotes a deterministic kernel function. In fact, if we want to ensure that the volatility Z is stationary, we can work with a kernel function of the form $i(t, s) = i^*(t - s)$, for a deterministic function i^* which is integrable w.r.t. U. In order to have Z positive, we assume i to be a positive function and U a subordinator.

A straightforward extension of our model is to study a superposition of $\mathcal{LSS}$ processes for the spot price dynamics. I.e. we could replace the process Y by a superposition of $J \in \mathbb{N}$ factors:

$$\sum_{i=1}^J w_i Y_t^{(i)}, \quad \text{where } w_1, \dots, w_J \geq 0, \ \sum_{i=1}^J w_i = 1,$$

and where all $Y_t^{(i)}$ are defined as in (5) for independent Lévy processes $L^{(i)}$ and independent stochastic volatility processes $\omega^{(i)}$, in both the geometric and the arithmetic model. Such models include the Benth, Kallsen & Meyer-Brandis (2007) model as a special case. A superposition of factors $Y^{(i)}$ opens for the separate modelling of spikes and other effects. For instance, one could let the first factor account for the spikes, using a Lévy process with big jumps at low frequency, while the function g forces the jumps back at a high speed. The next factor(s) could model the "normal" variations of the market, where one observes a slower force of mean-reversion, and high frequent Brownian-like noise. Note that all the results we derive in this paper based on the one factor model can be easily generalised to accommodate for the multifactor framework. For notational simplicity, in the following, we focus on the one-factor case.

Also, according to Burger et al. (2003, 2007) it might be necessary to allow for non-stationary effects. Such effects can be easily modelled by a second factor which is added to our spot model, e.g. as a drifted Brownian motion say. However, such models generally do not belong to the class of Lévy semistationary models and thus we disregard them here since the scope of this paper is to study suitable models for the stationary part of the price process.

Finally, note that an alternative model specification could be based on a stochastic time change $\int_{-\infty}^t g(t - s) dL_{\omega_s^{2+}}$, where $\omega_s^{2+} = \int_0^t \omega_u^2 du$. However, since models based on (5) are more straightforward to generalise to a multivariate framework, we stick to the former class. Note that extensions to a multivariate framework can be considered along the lines of Barndorff-Nielsen & Stelzer (2009, 2010).

3.4   Traditional spot price models as $\mathcal{LSS}$–based models

One of the main advantages of the new model class besides its great generality is that it nests most of the standard models which have been used in the literature on modelling electricity prices in recent years.

Thus our new model nests the stationary version of the classical one-factor Schwartz model, see Schwartz (1997), studied for oil prices. In order to see that, one just has to choose $g(t - s) = \exp(-\alpha(t - s))$ for a positive constant α, the volatility being a positive constant ω, and L a standard Brownian motion. By letting L be a Lévy process with the pure-jump part given as a compound Poisson, Cartea & Figueroa (2005) succesfully fitted the Schwartz model to electricity spot prices in the UK market. Benth & Šaltytė Benth (2004) used a normal inverse Gaussian Lévy process L to model UK spot gas and Brent crude oil spot prices.

Another example which is nested by the class of Lévy semistationary processes is a model studied in Benth (2009) in the context of gas markets, where the deseasonalised logarithmic spot price dynamics is assumed to follow a one-factor Schwartz process with stochastic volatility. More precisely, the function g is chosen to be $g(t - s) = \exp(-\alpha(t - s))$, whereas the volatility $\omega_t^2 = Z_t$ is defined by

$$Z_t = \int_{-\infty}^t e^{-\lambda(t-s)} \, dU_{\lambda t} \,, \tag{9}$$

with U being a subordinator process to ensure positivity of Z. This model we recall as the BNS stochastic volatility model, see Barndorff-Nielsen & Shephard (2002).

A more general class of models which we nest is the class of so-called CARMA-processes, which has been successfully used in temperature modelling and weather derivatives pricing, see Benth, Šaltytė Benth & Koekebakker (2007), Benth et al. (2009) and López Cabrera & Härdle (2009), and more recently for electricity prices by Bernhardt et al. (2008). A CARMA process is the continuous-time analogue of an ARMA time series, see Brockwell (2001a,b) for definition and details. More precisely, suppose that for $p > q$

$$Y_t = \mathbf{b}'\mathbf{V}_t \,,$$

where $\mathbf{b} \in \mathbb{R}^p$ and $\mathbf{V}_t$ is a p dimensional OU process of the form

$$d\mathbf{V}_t = A\mathbf{V}_t dt + \mathbf{e}_p dL_t, \tag{10}$$

with

$$A = \left[\begin{array}{cc} 0 & I_{p-1} \\ -\alpha_p & -\alpha_{p-1} \cdots -\alpha_1 \end{array} \right] \,.$$

Here we use the notation I_n for the $n \times n$-identity matrix, $\mathbf{e}_p$ the pth coordinate vector and $\mathbf{b}' = [b_0, b_1, \ldots, b_{p-1}]$ is the transpose of $\mathbf{b}$, with $b_q = 1$ and $b_j = 0$ for $q < j < p$. In Brockwell (2004), it is shown that if all the eigenvalues of A have negative real parts, then $\mathbf{V}_t$ defined as

$$\mathbf{V}_t = \int_{-\infty}^t e^{A(t-s)} \mathbf{e}_p \, dL(s) \,,$$

is the (strictly) stationary solution of (10). Moreover,

$$Y_t = \mathbf{b}'V_t = \int_{-\infty}^t \mathbf{b}'e^{A(t-s)} \mathbf{e}_p \, dL(s) \,, \tag{11}$$

is a CARMA(p, q) process. Hence, specifying $g(x) = \mathbf{b}' \exp(Ax)\mathbf{e}_p$ in (11), the log-spot price dynamics will be a $\mathcal{LSS}$-process (without stochastic volatility). Bernhardt et al. (2008) argue for

CARMA(2,1) dynamics as an appropriate class of models for the deseasonalised log-spot price at the Singapore New Electricity Market. The innovation process L is chosen to be in the class of stable processes. From Benth, Šaltytė Benth & Koekebakker (2007), CARMA(3,0) models seem appropriate for modelling daily average temperatures, and are applied for temperature derivatives pricing, including forward price dynamics of various contracts. More recently, the dynamics of wind speeds have been modelled by a CARMA(4,0) model, and applied to wind derivatives pricing. See Benth & Šaltytė Benth (2009) for more details.

Finally note that the arithmetic model based on a superposition of $\mathcal{LSS}$ processes nests the non–Gaussian Ornstein–Uhlenbeck model which has recently been proposed for modelling electricity spot prices, see Benth, Kallsen & Meyer-Brandis (2007). In order to see that, we just have to choose $g^{(i)}(x) = \exp(-\lambda^{(i)}x)$ for $\lambda^{(i)} > 0$ and $i = 1, \ldots, J$ and $J \in \mathbb{N}$, and, also, to specialise the stochastic volatility processes $\omega^{(i)}$ to positive, bounded, *deterministic* functions.

4   Probabilistic properties of the model

In this section we discuss the conditions under which the process Y_t is well-defined and describe in which sense the stochastic integration in (2) is to be understood.

Furthermore, we study the second–order characteristics of a $\mathcal{LSS}$ process. In particular, the covariance function of Y_t is of practical importance since the empirical covariance structure is a statistical measure applied to fix models. We provide analytical expressions for the autocovariance function for Y_t in many cases, and discuss other probabilistic properties as well.

4.1   Stochastic integration and integrability conditions

There are various ways in which the stochastic integral in (2) can be defined. One possibility would be to use the weak integration concept with respect to a random measure, which has been introduced in Rajput & Rosinski (1989). However, such integrals have only been studied for deterministic integrands and, furthermore, we wish to have a stronger integration concept, which specialises to classical Itô integrals when we are in the semimartingale framework.

In this paper we use the stochastic integration concept described in Basse-O'Connor et al. (2010) where a stochastic integration theory on $\mathbb{R}$, rather than on compact intervals as in the classical framework, is presented. In the following, we briefly review this general integration theory. Recall, that we work with a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, \mathbb{P})$, where the filtration is assumed to be right–continuous and complete. The aim is now to define integrals of the type

$$\int_{\mathbb{R}} \phi_s dZ_s, \tag{12}$$

where ϕ is a predictable stochastic process and Z is a $(\mathcal{F}_t)_{t \in \mathbb{R}}$ (increment) semimartingale, see Basse-O'Connor et al. (2009). A special case of such an integral is given by

$$X_t = \int_{\mathbb{R}} \phi_{t-s} dZ_s, \tag{13}$$

for fixed t. In particular, for

$$\phi_{t-s} = g(t-s)\omega_s \mathbb{I}_{[0,\infty)}(t-s),$$

(13) specialises to a $\mathcal{LSS}$ process.

Note that $Z = (Z_t)_{t \in \mathbb{R}}$ has $\{\mathcal{F}_t\}$–adapted increments if for all $s \leq t$ the difference $X_t - X_s$ is $\mathcal{F}_t$–measurable. We denote by $\mathcal{J}$ the set of $\{\mathcal{F}_t\}$–stopping times, see Basse-O'Connor et al. (2010), which take at most finitely many values and by $\mathcal{A}$ the ring on $\mathcal{R} \times \Omega$ which is generated by the stochastic intervals $(S, T]$ for stopping times $S, T \in \mathcal{J}$ with $S \leq T$. One can now define an additive set function on $\mathcal{A}$ which is given by

$$m_Z((S, t]) = Z_T - Z_S, \quad \text{for } S, T \in \mathcal{J}, S \leq T.$$

In a next step, Basse-O'Connor et al. (2010) introduce the space $\mathcal{V}$ as the vector lattice of step functions over $\mathcal{A}$ which is equipped the Schwartz topology, i.e. the inductive uniform topology corresponding to uniform convergence on compact intervals. A simple integral $I_Z : \mathcal{V} \to L^0$ can then be defined for simple functions $\phi \in \mathcal{V}$ with $\phi = \sum_{i=1}^n r_i \mathbb{I}_{(S_i, T_i]}$, where $n \in \mathbb{N}$, $S_i, T_i \in \mathcal{J}, S_i \leq T_i$ and $r_1, \ldots, r_n \in \mathbb{R}$ by

$$I_Z(\phi) = \sum_{i=1}^n r_i (Z_{T_i} - Z_{S_i}).$$

The general integral (12) is then defined based on the limit of simple functions satisfying a dominated convergence theorem.

Basse-O'Connor et al. (2010) formulate general integrability conditions which ensure that the integral (12) is well defined. In the case of a $\mathcal{LSS}$ process, i.e. $Y_t = \int_{-\infty}^t g(t - s)\omega_s dL_s$ those integrability conditions specialise as follows. Let (γ, σ^2, ℓ) denote the Lévy triplet of L associated with a truncation function h.

According to Basse-O'Connor et al. (2010), the process $g \star \omega$ is integrable with respect to L if and only if the following conditions hold almost surely:

$$\int_{-\infty}^t g^2(t - s)\omega_s^2 \sigma^2 < \infty, \qquad \int_{-\infty}^t \int_{\mathbb{R}} \left(1 \wedge |g(t - s)\omega_s z|^2\right) \ell(dz)ds < \infty$$
$$\int_{-\infty}^t \left| g(t - s)\omega_s \gamma + \int_{\mathbb{R}} (h(zg(t - s)\omega_s) - g(t - s)\omega_s h(z)) \, \ell(dz) \right| ds < \infty. \tag{14}$$

Example 1. *In the case of a Gaussian Ornstein–Uhlenbeck process, i.e. when $g(t - s) = \exp(-\alpha(t - s))$ for $\alpha > 0$ and $\omega \equiv 1$, then the integrability conditions above are clearly satisfied, since we have*

$$\int_{-\infty}^t \exp(-2\alpha(t - s))ds\sigma^2 = \frac{1}{2\alpha}\sigma^2 < \infty.$$

For many financial applications it is natural to restrict the attention to models where the variance is finite, and we focus therefore on Lévy processes L with finite second moment. Note that the integrability conditions above do not ensure square-integrability of Y_t. But substitute the first condition in (14) with the stronger condition

$$\int_{-\infty}^t g^2(t - s)\mathbb{E}[\omega_s^2] \, ds < \infty \,, \tag{15}$$

then $\int_{-\infty}^t g(t - s)\omega_s d(L_s - \mathbb{E}(L_s))$ is square integrable. Clearly, $\mathbb{E}[\omega_s^2]$ is constant in case of stationarity.

For the Lebesgue integral part, we need

$$\mathbb{E}\left[\left(\int_{-\infty}^t g(t - s)\omega_s \, ds\right)^2\right] < \infty \,. \tag{16}$$

Appealing to the Cauchy–Schwarz inequality, we find

$$\mathbb{E}\left[\left(\int_{-\infty}^{t} g(t-s)\omega_s\,ds\right)^2\right] \leq \int_{0}^{\infty} g^{2a}(x)\,dx \int_{-\infty}^{t} g^{2(1-a)}(t-s)\mathbb{E}[\omega_s^2]\,ds\,,$$

for a constant $a \in (0,1)$. Thus, a sufficient condition for (16) to hold is that there exists an $a \in (0,1)$ such that

$$\int_{0}^{\infty} g^{2a}(x)\,dx < \infty\,, \qquad\qquad \int_{-\infty}^{t} g^{2(1-a)}(t-s)\mathbb{E}[\omega_s^2]\,ds < \infty\,. \tag{17}$$

Given a model for ω_s and g, these conditions are simple to verify. Let us consider an example.

Example 2. *Recall that the Schwartz model specified the function g as $g(x) = \exp(-\alpha x)$ for $\alpha > 0$. In case of constant volatility, we find from a straightforward calculation that the conditions (14) are satisfied. Next, suppose that ω_s is defined by the BNS stochastic volatility model, that is (9), where $i(t,s) = \exp(-\lambda(t-s))$ and U_s a subordinator. Suppose now that U_t has cumulant function $\int_0^\infty (\exp(\mathrm{i}\theta z) - 1)\,\ell_U(dz)$ for a Lévy measure ℓ_U supported on the positive real axis, and U_t has finite expectation. In this case we have that*

$$\mathbb{E}\left[\omega_s^2\right] = \mathbb{E}[Z_s] = \frac{1}{\lambda}\int_{0}^{\infty} z\,\ell_U(dz) < \infty\,.$$

Thus, both (15) and (16) are satisfied (the latter seen after using the sufficient conditions), and we find that Y_t is a square-integrable stochastic process.

4.2   Wold–Karhunen representation of stationary processes

Using an $\mathcal{LSS}$ process as a building block for the spot price is in fact a very general modelling choice. Indeed, due to the continuous time Wold-Karhunen decomposition any second order stationary stochastic process, possibly complex valued, of mean 0 and continuous in quadratic mean can be represented as

$$Y_t = \int_{-\infty}^{t} \phi(t-s)\,\mathrm{d}\Xi_s + V_t, \tag{18}$$

where the deterministic function ϕ is an, in general complex, deterministic square integrable function, the process Ξ has orthogonal increments with $\mathbb{E}\left\{|\mathrm{d}\Xi_t|^2\right\} = \varpi dt$ for some constant $\varpi > 0$ and the process V is nonregular (i.e. its future values can be predicted, in the L^2 sense, by linear operations on past values without error). Under the further condition that $\cap_{t\in\mathbb{R}}\overline{\mathrm{sp}}\left\{Y_s : s \leq t\right\} = \{0\}$, the function ϕ is real and uniquely determined up to a real constant of proportionality; and the same is therefore true of Ξ (up to an additive constant). In particular, if $\mathrm{d}\Xi_s = \omega_s\mathrm{d}L_s$, with ω and L as in (2) then Ξ is of the above type with $\varpi = \mathbb{E}\left\{\omega_0^2\right\}$.

4.3   Absence of arbitrage

A natural question to ask is whether our model is prone to arbitrage opportunities. Clearly, an $\mathcal{LSS}$ process is in general not a semimartingale. Hence, at first sight, one might think that this fact can give rise to arbitrage opportunities. However, the standard semimartingale assumption in mathematical finance is only valid for *tradeable* assets in the sense of assets which can be held in a portfolio. The electricity spot is naturally not storable, and thus cannot be part of any financial portfolio. The requirement of being a martingale under some equivalent measure Q is therefore not necessary.

Remark. In order to use $\mathcal{LSS}$ processes in other applications than the ones studied in this paper, it will be necessary to derive appropriate semimartingale conditions for $\mathcal{LSS}$ processes. Such conditions can be derived along the lines of Barndorff-Nielsen & Schmiegel (2009) and Barndorff-Nielsen & Basse-O'Connor (2009) and will be studied in future research.

Furthermore, we would like to stress that we work with stochastic processes defined on the entire real line. The standard theory of mathematical arbitrage, see Delbaen & Schachermayer (2008), is defined for stochastic processes starting from 0 and not from $-\infty$ as in our model. Our new modelling framework makes it therefore necessary to define a more general concept of mathematical arbitrage as it has traditionally been used.

If absence of arbitrage is defined in the classical set up, then Guasoni et al. (2008) have pointed out that, while in frictionless markets martingale measures play a key role, this is not the case anymore in the presence of market imperfections. In fact, in markets with transaction costs, *consistent price systems* as introduced in Schachermayer (2004) are essential. In such a setup, even processes which are not semimartingales can ensure that we have *no free lunch with vanishing risk* in the sense of Delbaen & Schachermayer (1994). It turns out that if a continuous price process has *conditional full support*, then it admits consistent price systems for arbitrarily small transaction costs, see Guasoni et al. (2008). It has recently been shown by Pakkanen (2010), that under certain conditions, see below, a $\mathcal{BSS}$ process has conditional full support. This means that such processes can be used in financial applications without giving rise to arbitrage opportunities. We briefly review the main result proved in Pakkanen (2010). The distributional property of *conditional full support* is defined (for continuous processes) in the following way: We define for any $x \in \mathbb{R}$ the set of functions $f \in C([u,v])$ for $u, v \in \mathbb{R}$, $u \leq v$ such that $f(u) = v$, which we denote by $C_x([u,v])$. The function spaces are endowed with the uniform norm topology. For a fixed time horizon $T \in (0, \infty)$, a continuous process $(X_t)_{t \in [0,T]}$ is said to have *conditional full support* (CFS), if for every $\tau \in [0, T)$ and a.e. $\omega \in \Omega$,

$$\text{supp}(\text{Law}[(X_t)_{t \in [\tau,T]}|\mathcal{F}_\tau](\omega)) = C_{X_\tau(\omega)}([\tau, T]),$$

where supp denotes the support and where $\text{Law}[(X_t)_{t \in [\tau,T]}|\mathcal{F}_\tau]$ denotes the $\mathcal{F}_\tau$–conditional distribution of the $C([\tau, T])$–random variable $X|_{[\tau,T]}$. For the $\mathcal{BSS}$ processes, we get the following result, see Pakkanen (2010, Theorem 1.3). Let $\Xi = (\Xi_t)_{t \in [0,T]}$ be a continuous process, let $(\omega_t)_{t \in (-\infty,T]}$ be cádlág satisfying $\sup_{t \in (-\infty,T]} \mathbb{E}(\omega_t^2) < \infty$ and $\lambda(\{t \in [0,T] : \omega_t = 0\}) = 0$ a.s.. Also, let $(W_t)_{t \in (-\infty,T]}$ be a Brownian motion and let $g : (0, \infty) \to \mathbb{R}$ be a function satisfying $g \in L^2((0, \infty))$ and there exists $\alpha, C > 0$ such that

$$\int_0^\infty g^2(s)ds - \int_0^\infty g(t+s)g(s)ds \leq Ct^\alpha, \quad \text{for all } t \in [0, T],$$

and, furthermore, $\int_0^\epsilon |g(s)|ds > 0$ for all $\epsilon > 0$. If (Ξ, ω) is independent of B, then the process

$$\Psi_t = \Xi_t + \int_{-\infty}^t g(t-s)\omega_s dB_s, \quad t \in [0, T],$$

has conditional full support.

4.4   Second order structure of Lévy–driven semistationary models

We round off this section on the probabilistic properties of Lévy semistationary processes by focusing on their second order structure. In fact, we will state the results for the more general volatility

modulated Volterra process $\overline{Y} = (\overline{Y}_t)_{t\in\mathbb{R}}$, where

$$\overline{Y}_t = \int_{-\infty}^t G(t,s)\omega_s dL_s, \tag{19}$$

with a deterministic function G such that the integral in (19) exists. Clearly, for $G(t,s) = g(t-s)$ we get that $\overline{Y}_t = Y_t$ with Y as defined in (5). Let $\kappa_1 = \mathbb{E}(L_1)$ and $\kappa_2 = Var(L_1)$.

Under the additional assumption that the stochastic volatility ω is independent of the driving Lévy process, we can compute the conditional second order structure, which we do in the following.

Theorem 1. *Let L and ω be independent. The conditional second order structure of $\overline{Y}$ is given by*

$$\mathbb{E}(\overline{Y}_t|\omega) = \kappa_1 \int_{-\infty}^t G(t,s)\omega_s ds, \qquad Var(\overline{Y}_t|\omega) = \kappa_2 \int_{-\infty}^t G(t,s)^2\omega_s^2 ds,$$

and

$$Cov((\overline{Y}_{t+h},\overline{Y}_t)|\omega) = \kappa_2 \int_{-\infty}^t G(t+h,s)G(t,s)\omega_s^2 ds.$$

Corollary 1. *Let L and ω be independent. The conditional second order structure of Y is given by*

$$\mathbb{E}(Y_t|\omega) = \kappa_1 \int_0^\infty g(x)\omega_{t-x}dx, \qquad Var(Y_t|\omega) = \kappa_2 \int_0^\infty g(x)^2\omega_{t-x}^2 dx,$$

and

$$Cov((Y_{t+h},Y_t)|\omega) = \kappa_2 \int_0^\infty g(x+h)g(x)\omega_{t-x}^2 dx.$$

The unconditional second order structure of $\overline{Y}$ is then given as follows.

Theorem 2. *The second order structure of $\overline{Y}$ for stationary ω is given by*

$$\mathbb{E}(\overline{Y}_t) = \kappa_1\mathbb{E}(\omega_0) \int_{-\infty}^t G(t,s)ds,$$

$$Var(\overline{Y}_t) = \kappa_2\mathbb{E}\left(\omega_0^2\right) \int_{-\infty}^t G(t,s)^2 ds + \kappa_1^2 \int_{-\infty}^t \int_{-\infty}^t G(t,s)G(t,u)\gamma(|s-u|)dsdu,$$

$$Cov(\overline{Y}_{t+h},\overline{Y}_t) = \kappa_2\mathbb{E}\left(\omega_0^2\right) \int_{-\infty}^t G(t+h,s)G(t,s)ds$$

$$+ \kappa_1^2 \int_{-\infty}^{t+h} \int_{-\infty}^t G(t+h,s)G(t,u)\gamma(|s-u|)dsdu,$$

where $\gamma(x) = Cov(\omega_{t+x},\omega_t)$ denotes the autocovariance function of ω.

The unconditional second order structure of Y is then given as follows.

Corollary 2. *The second order structure of Y for stationary ω is given by*

$$\mathbb{E}(Y_t) = \kappa_1\mathbb{E}(\omega_0) \int_0^\infty g(x)dx,$$

$$Var(Y_t) = \kappa_2\mathbb{E}\left(\omega_0^2\right) \int_0^\infty g(x)^2 dx + \kappa_1^2 \int_0^\infty \int_0^\infty g(x)g(y)\gamma(|x-y|)dxdy,$$

$$Cov(Y_{t+h},Y_t) = \kappa_2\mathbb{E}\left(\omega_0^2\right) \int_0^\infty g(x+h)g(x)dx + \kappa_1^2 \int_0^\infty \int_0^\infty g(x+h)g(y)\gamma(|x-y|)dxdy,$$

where $\gamma(x) = Cov(\omega_{t+x}, \omega_t)$ denotes the autocovariance function of ω. Hence, we have

$$Cor(Y_{t+h}, Y_t) = \frac{\kappa_2 \mathbb{E}\left(\omega_0^2\right) \int_0^\infty g(x+h)g(x)dx + \kappa_1^2 \int_0^\infty \int_0^\infty g(x+h)g(y)\gamma(|x-y|)dxdy}{\kappa_2 \mathbb{E}\left(\omega_0^2\right) \int_0^\infty g(x)^2 dx + \kappa_1^2 \int_0^\infty \int_0^\infty g(x)g(y)\gamma(|x-y|)dxdy}.$$

Corollary 3. *If $\kappa_1 = 0$ or if ω has zero autocorrelation, then*

$$Cor(Y_{t+h}, Y_t) = \frac{\int_0^\infty g(x+h)g(x)dx}{\int_0^\infty g(x)^2 dx}.$$

The last Corollary shows that we get the same autocorrelation function as in the $\mathcal{BSS}$ model. From the results above, we clearly see the influence of the general damping function g on the correlation structure. A particular choice of g, which is interesting in the energy context is studied in the next example.

Example 3. *Consider the case $g(x) = \frac{\sigma}{x+b}$, for $\sigma, b > 0$, which is motivated from the model of Bjerksund et al. (2000), as we shall return to in the next section when we deal with forward pricing. We have that $\int_0^\infty g^2(x)\,dx = \frac{\sigma^2}{b}$. This ensures integrability of $g(t-s)$ over $(-\infty, t)$ with respect to any square integrable martingale Lévy process L_t. Furthermore, $\int_0^\infty g(x+h)g(x)\,dx = \frac{\sigma^2}{h}\ln\left(1 + \frac{h}{b}\right)$. Thus,*

$$Cor(Y_{t+h}, Y_t) = \frac{b}{h}\ln\left(1 + \frac{h}{b}\right).$$

Observe that since g can be written as

$$g(x) = \frac{\sigma}{x+b} = \int_0^x \frac{-\sigma\,ds}{(s+b)^2} + \frac{\sigma}{b},$$

it follows that the process $Y(t) = \int_{-\infty}^t g(t-s)\,dB_s$ is a semimartingale according to the Knight condition, see Knight (1992) and also Basse (2008), Basse & Pedersen (2009), Basse-O'Connor et al. (2010).

5 Pricing of forward contracts

In this subsection we are concerned with the calculation of the forward price $F_t(T)$ at time t for contracts maturing at time $T \geq t$. We denote by $T^* < \infty$ a finite time horizon for the forward market, meaning that all contracts of interest mature *before* this date.

Recall that $S = (S)_{t \in \mathbb{R}}$ denotes the electricity spot price, being either of arithmetic or geometric kind as defined in (7) and (6), respectively. As in Section 4.4, we consider a general Lévy-driven Volterra process

$$\overline{Y}_t = \int_{-\infty}^t G(t,s)\omega_s\,dL_s,$$

with ω_s being the stochastic volatility as defined in (19). We consider the general case of $G(t,s)$ rather than the stationary situation with $G(t,s) = g(t-s)$ since this leads to some interesting modelling issues in the forward market.

Let $F_t(T)$ denote the forward price at time t of a contract delivering the underlying commodity (electricity) at time $T \geq t$. We use the conventional definition of a forward price in incomplete markets, see Duffie (1992), ensuring the martingale property of $t \mapsto F_t(T)$,

$$F_t(T) = \mathbb{E}_Q\left[S_T \mid \mathcal{F}_t\right], \tag{20}$$

with Q being an equivalent probability to P. Here, we suppose that $S_T \in L^1(Q)$, the space of integrable random variables. In a moment we shall introduce sufficient conditions for this.

Usually in finance one talks of equivalent martingale measures Q, meaning that the equivalent probability Q should turn the discounted price dynamics of the underlying asset into a (local) Q-martingale. However, as we have already discussed, this restriction is not relevant in electricity markets since the spot is not tradeable. Thus, we may choose any equivalent probability Q as pricing measure. In practice, however, one restricts to a parametric class of equivalent probabilities, and the standard choice seems to be given by the Esscher transform, see Benth et al. (2008), Shiryaev (1999). The Esscher transform naturally extends the Girsanov transform to Lévy processes.

To this end, consider Q_L^θ defined as the Esscher transform of L for a parameter $\theta(t)$ being a Borel measurable function. Following Shiryaev (1999) (or Benth et al. (2008), Barndorff-Nielsen & Shiryaev (2010)), Q_L^θ is defined via the Radon-Nikodym density process

$$\frac{dQ_L^\theta}{dP}\Big|_{\mathcal{F}_t} = \exp\left(\int_{-\infty}^t \theta(s)\,dL_s - \int_{-\infty}^t \phi_L(\theta(s))\,ds\right),$$

for $\theta(s)$ being integrable with respect to the Lévy process on $(-\infty, t]$ for every $t \le T^*$, and $\phi_L(x)$ being the log-moment generating function of L_1.

A special choice is the 'constant' measure change, that is, letting

$$\theta(t) = \theta \mathbf{1}_{(0,\infty)}(t). \tag{21}$$

In this case, if L has characteristic triplet (d, b, ℓ_L), where d is the drift, b is the volatility of the continuous martingale part and ℓ_L is the Lévy measure in the Lévy-Kintchine representation, see Shiryaev (1999), a fairly straightforward calculation shows that, see Shiryaev (1999) again, the Esscher transform preserves the Lévy property of L_t, $t > 0$, and the characteristic triplet becomes $(d_\theta, b, \exp(\theta \cdot) \ell_L)$, where

$$d_\theta = d + b\theta + \int_{|z|<1} z(\mathrm{e}^{\theta z} - 1)\,\ell_L(dz).$$

This comes from the simple fact that the log-moment generating function of L_t under Q_L^θ is

$$\phi_L^\theta(x) \triangleq \phi_L(x + \theta) - \phi_L(\theta). \tag{22}$$

Note that the choice of $\theta(t)$ as in (21) forces us to choose a starting time since the function will not be integrable with respect to L_t on $(-\infty, t)$. Starting at zero is convenient since $L_0 = 0$, however, it is also practically reasonable since this can be considered as the current time. With such a choice we do not introduce any risk premium for $t < 0$. In the general case, with a time-dependent parameter function $\theta(t)$, the characteristic triplet of L_t under Q_L^θ will become time-dependent, and hence the Lévy process property is lost. Instead, L_t will be an independent increment process (sometimes called an additive process). We remark that the choice (21) of θ also gives an independent increment process L_t when considered as a process over all times t. Note that if $L = B$, a Brownian motion, the Esscher transform is simply a Girsanov change of measure where $dB_t = \theta(t)\,dt + dW_t$ for a Q_L^θ-Brownian motion W.

Similarly, we do an Esscher transform of U, the subordinator driving the stochastic volatility model, see (8). We define Q_U^η having the Radon-Nikodym density process

$$\frac{dQ_U^\eta}{dP}\Big|_{\mathcal{F}_t} = \exp\left(\int_{-\infty}^t \eta(s)\,dU_s - \int_{-\infty}^t \phi_U(\eta(s))\,ds\right),$$

for $\eta(t) \in \mathbb{R}$ being some measurable function which is integrable with respect to U on $(-\infty, t)$ for all $t \leq T^*$, and $\phi_U(x)$ being the log-moment generating function of U_1. Since U is a subordinator, we can write the Lévy-Kintchine representation of it as

$$\phi_U(x) = \widetilde{d}x + \int_\mathbb{R} \left(\mathrm{e}^{xz} - 1 \right) \ell_U(dz) \,.$$

Choosing $\eta(t) = \eta \mathbf{1}_{(0,\infty)}(t)$, with η a constant, an Esscher transform will give a characteristic triplet $(\widetilde{d}, 0, \exp(\eta \cdot) \ell_U)$, which thus preserves the subordinator property of U_t, $t > 0$, under Q_U^η. For the general case, the process U will be a time-inhomogeneous subordinator (independent increment process with positive jumps). The log-moment generating function is denoted $\phi_U^\eta(x)$.

To ensure the existence of the Esscher transforms, we need some conditions. We need that there exists a constant $c > 0$ such that $\sup |\theta(s)| \leq c$, and where $\int_{|z|>1} \exp(cz)\ell(dz) < \infty$. (Similarly we must have the same for the stochastic volatility, ℓ_U). Also, we must require that exponential moments of L_1 and U_1 exist. More precisely, we suppose that the Esscher transform parameter functions $\theta(t)$ and $\eta(t)$ are such that

$$\int_{-\infty}^{T^*} \int_{|z|>1} \mathrm{e}^{|\theta(s)|z} \, \ell_L(dz) \, ds < \infty \,, \qquad \int_{-\infty}^{T^*} \int_1^\infty \mathrm{e}^{|\eta(s)|z} \, \ell_U(dz) \, ds < \infty \,. \tag{23}$$

The exponential integrability conditions of the Lévy measures of L and U imply the existence of exponential moments, and thus that the Esscher transforms Q_L^θ and Q_U^η are well-defined.

We define the probability $Q^{\theta,\eta} \triangleq Q_L^\theta \times Q_U^\eta$ as the class of pricing measures for deriving forward prices. In this respect, $\theta(t)$ may be referred to as the market price of risk, whereas $\eta(t)$ is the market price of volatility risk. We note that a choice $\theta > 0$ will put more weight to the positive jumps in the price dynamics, and less on the negative, increasing the "risk" for big upward movements in the prices under $Q^{\theta,\eta}$.

Let us denote by $\mathbb{E}_{\theta,\eta}$ the expectation operator with respect to $Q^{\theta,\eta}$, and by $\mathbb{E}_\eta$ the expectation with respect to Q_U^η.

5.1   Geometric case

Suppose that the spot price is defined by the geometric model

$$S_t = \Lambda(t) \exp(\overline{Y}_t) \,,$$

where $\overline{Y}$ is defined as in (19). In order to have the forward price $F_t(T)$ well-defined, we need to ensure that the spot price is integrable with respect to the chosen pricing measure $Q^{\theta,\eta}$. We discuss this issue in some more detail.

We know that ω_t is positive and in general not bounded since it is defined via a subordinator. Thus, $G(t, s)\omega_s + \theta(s)$ is unbounded as well. Supposing that L has exponential moments of all orders, we can calculate as follows using iterated expectations conditioning on the filtration $\mathcal{G}_t$ generated by the paths of $\omega_s, s \leq t$:

$$\mathbb{E}_{\theta,\eta} \left[S_T \right] = \Lambda(T) \mathbb{E}_{\theta,\eta} \left[\mathbb{E}_{\theta,\eta} \left[\exp \left(\int_{-\infty}^T G(T, s)\omega_s \, dL_s \right) \mid \mathcal{G}_T \right] \right]$$

$$= \Lambda(T) \mathbb{E}_\eta \left[\exp \left(\int_{-\infty}^T \phi_L^\theta(G(T, s)\omega_s) \, ds \right) \right] \,.$$

To have that $S_T \in L^1(Q^{\theta,\eta})$, the last integral must be finite. This puts additional restrictions on the choice of η and the specifications of $G(t, s)$ and $i(t, s)$. We note that when applying the Esscher

15

transform, we must require that L has exponential moments of all orders, a very strong restriction on the possible class of driving Lévy processes.

We are now ready to price forwards under the Esscher transform. For general volatility modulated Volterra processes (without a 'drift term') it holds:

Proposition 1. *Suppose that $S_T \in L^1(Q^{\theta,\eta})$. Then, the forward price is given as*

$$F_t(T) = \Lambda(T) \exp \left(\int_{-\infty}^t G(T,s)\omega_s \, dL_s \right) \mathbb{E}_\eta \left[\exp \left(\int_t^T \phi_L^\theta(G(T,s)\omega_s) \, ds \right) \bigg| \mathcal{F}_t \right].$$

As a special case, consider $L = B$. In this case we apply the Girsanov transform rather than Esscher, and it turns out that a rescaling of the transform parameter function $\theta(t)$ by the volatility ω_t is convenient for pricing of forwards. To this end, consider the Girsanov transform

$$dB_t = dW_t + \frac{\theta(t)}{\omega_t} \, dt. \tag{24}$$

Supposing that the Novikov condition

$$\mathbb{E} \left[\exp \left(\frac{1}{2} \int_{-\infty}^{T^*} \frac{\theta^2(s)}{\omega_s^2} \, ds \right) \right] < \infty,$$

holds, we know that W_t is a Brownian motion for $t \leq T^*$ under a probability Q_B^θ having density process

$$\frac{dQ_B^\theta}{dP} \Big|_{\mathcal{F}_t} = \exp \left(- \int_{-\infty}^t \frac{\theta(s)}{\omega_s} \, dB_s - \frac{1}{2} \int_{-\infty}^t \frac{\theta^2(s)}{\omega_s^2} \, ds \right).$$

Suppose that there exists a measurable function $j(t)$ such that

$$j(t) \geq \frac{i(t,s)}{i(0,s)}, \tag{25}$$

for all $s \leq t \leq T^*$, with

$$\int_{-\infty}^{T^*} \frac{\theta^2(s)}{j(s)} \, ds < \infty.$$

Furthermore, suppose that ω_0^{-2} has exponential moments up to a constant C_U. Then, for all $\theta(t)$ such that $\int_{-\infty}^{T^*} \theta^2(s)/j(s) \, ds/2 \leq C_U$, the Novikov condition is satisfied, since by the subordinator property of U_t (restricting our attention to $t \geq 0$)

$$\omega_t^2 = \int_{-\infty}^t i(t,s) \, dU_s \geq \int_{-\infty}^0 i(t,s) \, dU_s \geq j(t) \int_{-\infty}^0 i(0,s) \, dU_s = j(t)w_0^2,$$

and therefore

$$\mathbb{E} \left[\exp \left(\frac{1}{2} \int_{-\infty}^{T^*} \frac{\theta^2(s)}{\omega_s^2} \, ds \right) \right] \leq \mathbb{E} \left[\exp \left(\frac{1}{2} \int_{-\infty}^{T^*} \frac{\theta^2(s)}{j(s)} \, ds \, \omega_0^{-2} \right) \right] < \infty.$$

Specifying $i(t,s) = \exp(-\lambda(t-s))$, we have that $i(t,s)/i(0,s) = \exp(-\lambda t) = j(t)$, and condition (25) holds with equality.

We discuss the integrability of S_T with respect to $Q^{\theta,\eta} \triangleq Q^\theta_B \times Q^\eta_U$. By double conditioning with respect to the filtration generated by the paths of ω_t, we find

$$\mathbb{E}_{\theta,\eta}[S_T] = \exp\left(\int_{-\infty}^T G(T,s)\theta(s)\,ds\right) \mathbb{E}_{\theta,\eta}\left[\mathbb{E}_{\theta,\eta}\left[\exp\left(\int_{-\infty}^T G(T,s)\omega_s\,dW_s\right) \mid \mathcal{G}_T\right]\right]$$

$$= \exp\left(\int_{-\infty}^T G(T,s)\theta(s)\,ds\right) \mathbb{E}_\eta\left[\exp\left(\frac{1}{2}\int_{-\infty}^T G^2(T,s)\omega_s^2\,ds\right)\right].$$

Collecting together the conditions on G, i, θ and η for verifying all the steps above, we find that if $s \mapsto G(T,s)\theta(s)$ is integrable on $(-\infty, T)$ and $s \mapsto G^2(T,s)i(s,v)$ is integrable on (v, T) for all $-\infty < v \leq T$, then $S_T \in L^1(Q^{\theta,\eta})$ as long as

$$\int_{-\infty}^T \int_1^\infty \exp\left(z\left\{\frac{1}{2}\int_v^T G^2(T,s)i(s,v)\,ds + \eta(v)\right\}\right)\ell_U(dz)\,dv < \infty. \qquad (26)$$

We assume these conditions to hold.

We state the forward price for the case $L = B$ and the Girsanov change of measure discussed above.

Proposition 2. *Suppose that $L = B$ and that Q^θ is defined by the Girsanov transform in* (24). *Then, for $t \leq T \leq T^*$,*

$$F_t(T) = \Lambda(T)\exp\left(\int_{-\infty}^t G(T,s)\omega_s\,dW_s + \frac{1}{2}\int_{-\infty}^t\int_t^T G^2(T,v)i(v,s)\,dv\,dU_s\right.$$

$$\left. + \int_{-\infty}^T G(T,s)\theta(s)\,ds + \int_t^T \phi_U^\eta\left(\frac{1}{2}\int_s^T G^2(T,v)i(v,s)\,dv\right)ds\right).$$

Let us consider an example.

Example 4. *In the BNS stochastic volatility model, we have $i(t,s) = \exp(-\lambda(t-s))$. Hence, from Proposition 2*

$$\int_t^T G^2(T,v)e^{-\lambda(v-s)}\,dv = e^{-\lambda(t-s)}\int_t^T G^2(T,v)e^{\lambda(t-v)}\,dv$$

which yields,

$$\int_{-\infty}^t\int_t^T G^2(T,v)i(v,s)\,dv\,dU_s = Z_t\int_t^T G^2(T,v)e^{\lambda(t-v)}\,dv.$$

This implies from Proposition 2 that the forward price is affine in Z, the (square of the) stochastic volatility. The stochastic volatility model studied in Benth (2009) is recovered by choosing $G(t,s) = g(t-s)$, $g(x) = \exp(-\alpha x)$.

5.1.1 The case of constant volatility

Suppose for a moment that the stochastic volatility process ω_t is identical to one (*i.e.*, that we do not have any stochastic volatility in the model). In this case the forward price becomes

$$F_t(T) = \Lambda(T)\exp\left(\int_{-\infty}^t G(T,s)\,dW_s + \int_{-\infty}^T G(T,s)\theta(s)\,ds\right)$$

$$= \Lambda(T)\exp\left(\int_{-\infty}^t G(T,s)\,dB_s + \int_t^T G(T,s)\theta(s)\,ds\right).$$

Hence, the logarithmic forward (log-forward) price is

$$\ln F_t(T) = \Lambda(T) + \int_t^T g(T, s)\theta(s)\, ds + M_t(T)\,,$$

with

$$M_t(T) = \int_{-\infty}^t g(T, s)\, dB_s\,,$$

for $t \leq T$. Note that $t \mapsto M_t(T)$ for $t > 0$ is a P-martingale with the property

$$M_t(t) = Y_t = \ln S_t - \ln \Lambda(t)\,.$$

In the classical Ornstein-Uhlenbeck case, with $G(t, s) = g(t - s)$, $g(x) = \exp(\alpha x)$, we easily compute that

$$M_t(T) = e^{-\alpha(T-t)} Y_t\,,$$

and the forward price is explicitly dependent on the current spot price. In the general case, this does not hold true. We have that $M_T(T) = Y_T$, not unexpectedly, since the forward price converges to the spot at maturity. However, apart from the special time point $t = T$, the forward price will in general *not* be a function of the current spot, but a function of the process $M_t(T)$. Thus, at time t, the forward price will depend on

$$M_t(T) = \int_{-\infty}^t G(T, s)\, dB_s\,,$$

whereas the spot is given by

$$Y_t = \int_{-\infty}^t G(t, s)\, dB_s\,.$$

Both Y_t and $M_t(T)$ are generated by integrating over the same paths of a Brownian motion, since the two stochastic integrals can be pathwise interpreted (they are both Wiener integrals since the integrands are deterministic functions). However, the paths are scaled by two different functions $G(T, s)$ and $G(t, s)$. This allows for an additional degree of flexibility when creating forward curves compared to affine structures.

In the classical Ornstein-Uhlenbeck case, the forward curve as a function of time to maturity $T - t$ will simply be a discounting of today's spot price, discounted by the speed of mean reversion of the spot (in addition comes deterministic scaling by the seasonality and market price of risk). To highlight the additional flexibility in our modelling framework of semistationary processes, suppose for the sake of illustration that $G(t, s) = g_1(t)g_2(s)$. Then

$$M_t(T) = \frac{g_1(T)}{g_1(t)} Y_t\,.$$

If furthermore $\lim_{T\to\infty} g_1(T) := g_1(\infty) \neq 0$, we are in a situation where the long-end (that is, T large) of the forward curve is not a constant. In fact, we find

$$\lim_{T\to\infty}\left(\ln F_t(T) - g_1(t)\int_t^T g_2(s)\theta(s)\, ds - \ln \Lambda(T)\right) = (\ln S_t - \ln \Lambda(t))\frac{g_1(\infty)}{g_1(t)}\,.$$

Since $\ln S_t$ is random, we will have a randomly fluctuating long end of the forward curve. In fact, the long end will be distributed as the stationary distribution of the deseasonalized log-spot scaled by $g_1(\infty)/g_1(t)$. This is very different from the situation with a classical mean-reverting spot dynamics, which implies a deterministic forward price in the long end (dependent on the seasonality and market

18

price of risk only). Various shapes of the forward curve $T \mapsto F_t(T)$ can also be modelled via different specifications of G. For instance, if $g_1(T)$ is a decreasing function, we obtain the contango and backwardation situations depending on the spot price being above or below the mean. If $T \mapsto g_1(T)$ has a hump, we will also observe a hump in the forward curve. For general specifications of G we can have a high degree of flexibility in matching desirable shapes of the forward curve.

Observe that the time-dynamics of the forward price can be considered as correlated with the spot rather than directly depending on the spot. In the Ornstein-Uhlenbeck situation, the log-forward price can be considered as a linear regression on the current spot price, with time-dependent coefficients. This is not the case for general specifications. However, we have that $M_t(T)$ and Y_t are both normally distributed random variables (recall that we are still restricting our attention to $L = B$), and the correlation between the two is

$$Cor(M_t(T), Y_s) = \frac{\int_{-\infty}^{t} G(T,s)G(t,s)\,ds}{\sqrt{\int_{-\infty}^{t} G^2(T,s)\,ds \int_{-\infty}^{t} G^2(t,s)\,ds}}\,.$$

Obviously, for $G(t,s) = g(t-s) = \exp(-\alpha(t-s))$, the correlation is 1. In conclusion, we can obtain a weaker stochastic dependency between the spot and forward price than in the classical mean-reversion case by a different specification of the "mean-reversion" function G.

5.1.2 Affine structure of the forward price

In the discussion above we saw that the choice $G(t,s) = g_1(t)g_2(s)$ yielded a forward price expressible in terms of Y_t. In the next Proposition we prove that this is the only choice of G yielding an affine structure. The result is slightly generalising the analysis of Carverhill (2003).

Proposition 3. *The forward price in Proposition 2 is affine in Y_t and Z_t if there exist functions g_1, g_2, i_1 and i_2 such that $G(t,s) = g_1(t)g_2(s)$ and $i(t,s) = i_1(t)i_2(s)$. Opposite, if the forward price is affine in Y_t and Z_t, and G and i are strictly positive and continuously differentiable in the first argument, then there exists functions g_1, g_2, i_1 and i_2 such that $G(t,s) = g_1(t)g_2(s)$ and $i(t,s) = i_1(t)i_2(s)$.*

Obviously, the choice of G and i coming from OU-models,

$$G(t,s) = g(t-s) = \exp(-\alpha(t-s))\,, \qquad i(t,s) = \exp(-\lambda(t-s))\,,$$

satisfy the conditions in the Proposition above. In fact, appealing to similar arguments as in the proof of Proposition 3 above, one can show that this is the *only* choice (modulo multiplication by a constant) which is stationary and gives an affine structure in the spot and volatility for the forward price dynamics. In particular, the specification $g(x) = \sigma/(x+b)$ considered in Example 3 gives a stationary spot price dynamics, but not an affine structure in the spot for the forward price.

5.1.3 Risk–neutral dynamics of the forward price

We next turn our attention to the risk-neutral dynamics of the forward price.

Proposition 4. *The risk-neutral dynamics of the forward price $F_t(T)$ in Proposition 2 is*

$$\frac{dF_t(T)}{F_{t-}(T)} = G(T,t)\omega_t\,dW_t + \frac{1}{2}\int_t^T G^2(T,s)i(s,t)\,ds\,d\widetilde{U}_t^{\eta}\,,$$

where $\widetilde{U}_t^{\eta} = U_t - \frac{d}{dx}\phi_U^{\eta}(x)|_{x=0}\,t$, $t > 0$ is a Q^{η}-martingale.

We observe that the dynamics will jump according to the changes in volatility given by the process U_t. Thus, although the spot dynamics has continuous paths, the forward price will jump. As expected, the integrand in the jump expression tends to zero when $T - t \to 0$, since the forward price must converge to the spot when time to maturity goes to zero.

The forward dynamics will have a stochastic volatility given by $G(T, t)\omega_t$. Whenever $\lim_{t \uparrow T} G(T, t)$ exists, and $G(T, T) = 1$, we have $a.s.$,

$$\lim_{t \uparrow T} G(T, t)\omega_t = \omega_T.$$

When passing to the limit, we have implicitly supposed that we work with the version of ω_t having left-continuous paths with right-limits. By the definition of our integral in $\overline{Y}_t$, where the integrand is supposed predictable, this can be done. Thus, we find that the forward volatility converges to the spot volatility as time to maturity tends to zero, which is known as the Samuelson effect. Contrary to the classical situation where this convergence goes exponentially, we may have many different shapes of the volatility term structure resulting from our general modelling framework.

In Bjerksund et al. (2000), a forward price dynamics for electricity contracts is proposed to follow

$$\frac{dF_t(T)}{F_t(T)} = \left\{ a + \frac{\sigma}{T - t + b} \right\} dW_t, \tag{27}$$

where a, b and σ are positive constants. They argue that in electricity markets, the Samuelson effect is stronger close to maturity than what is observed in other commodity markets, and they suggest to capture this by letting it increase by the rate $1/(T - t + b)$ close to maturity of the contracts. This is in contrast to the common choice of volatility being $\sigma \exp(-\alpha(T - t))$, resulting from using the Schwartz model for the spot price dynamics. There is no reference to any spot model in the Bjerksund et al. (2000) model. The constant a comes from a non-stationary behaviour, which we have not taken into account in our modelling framework. However, for $a = 0$ we see that we can model the spot price by the $\mathcal{BSS}$ process

$$Y_t = \int_{-\infty}^{t} g(t - s) \, dB_s$$

with

$$g(x) = \frac{\sigma}{x + b}.$$

Thus, after doing a Girsanov transform, we recover the risk-neutral forward dynamics of Bjerksund et al. (2000). The general case with $a \neq 0$ is easily included by adding an independent Brownian motion term to the logarithmic spot price dynamics. It is interesting to note that with this spot price dynamics, the forward dynamics is not affine in the spot. Hence, the Bjerksund et al. model is an example of a non-affine forward dynamics. Whenever $\sigma \neq b$, we do not have that $g(t, t) = 1$, and thus the Bjerksund model does not satisfy the Samuelson effect, either.

5.1.4 Option pricing

We end this section with a discussion of option pricing. Let us assume that we have given an option with exercise time τ on a forward with maturity at time $T \geq \tau$. The option pays $f(F_\tau(T))$, and we are interested in finding the price at time $t \leq \tau$, denoted $C(t)$. From arbitrage theory, it holds that

$$C(t) = e^{-r(\tau - t)}\mathbb{E}_Q\left[f(F_\tau(T)) \mid \mathcal{F}_t\right], \tag{28}$$

where Q is the risk-neutral probability. Choosing Q as coming from the Esscher transform above, we can derive option prices explicitly in terms of the characteristic function of U by Fourier transform.

Moreover, we observe from Proposition 4 that we can state the forward price as

$$F_\tau(T) = H(\tau, T) \exp\left(\int_{-\infty}^{t} G(T, s)\omega_s \, dW_s + \int_{-\infty}^{t} h(T, s) \, dU_s \right),$$

for suitably defined functions H and h. Let now $p(x) = f(\exp(x))$, and suppose that $p \in L^1(\mathbb{R})$. By applying the definitions of Fourier transforms and the inverse in Folland (1984), we have that

$$p(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{p}(y) \mathrm{e}^{\mathrm{i}xy} \, dy,$$

with $\widehat{p}(y)$ is the Fourier transform of $p(x)$ defined by

$$\widehat{p}(y) = \int_{\mathbb{R}} p(x) \mathrm{e}^{-\mathrm{i}xy} \, dx.$$

Hence, we find

$$f(F_\tau(T)) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{p}(y) \mathrm{e}^{\mathrm{i}y \ln H(\tau, T)} \exp\left(\mathrm{i}y \left(\int_{-\infty}^{\tau} G(T, s)\omega_s \, dW_s + \int_{-\infty}^{\tau} h(T, s) \, dU_s \right) \right) dy. \tag{29}$$

Next, by commuting integration and expectation using dominated convergence and applying double conditioning and the stochastic Fubini theorem, it holds that

$$C_t = \mathrm{e}^{-r(\tau-t)} \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{p}(y) \exp\left(\mathrm{i}y \left(\ln H(\tau, T) + \int_{-\infty}^{t} G(T, s)\omega_s \, dW_s + \int_{-\infty}^{t} h(T, s) \, dU_s \right) \right)$$
$$\times \mathbb{E}_\eta\left[\exp\left(\int_t^\tau \mathrm{i}yh(T, v) - \frac{1}{2} \int_v^\tau i(s, v)G^2(T, s) \, ds \, dU_v \right) \mid \mathcal{F}_t \right] dy.$$

The last equality holds by the stochastic Fubini Theorem. Using the independent increment property of U, we reach

$$C_t = \mathrm{e}^{-r(\tau-t)} \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{p}(y) \exp\left(\mathrm{i}y \left(\ln H(\tau, T) + \int_{-\infty}^{t} G(T, s)\omega_s \, dW_s + \int_{-\infty}^{t} h(T, s) \, dU_s \right) \right)$$
$$\times \exp\left(\int_t^\tau \phi_U^\eta\left(\mathrm{i}yh(T, v) - \frac{1}{2} \int_v^\tau i(s, v)G^2(T, s) \, ds \right) dv \right) dy.$$

One can calculate option prices by applying the fast Fourier transform as long as the characteristic function of U, ϕ_U, is known. If p is not integrable (as is the case for a call option), one may introduce a damping function to regularize it, see Carr & Madan (1998) for details.

5.2   Arithmetic case

Let us consider the arithmetic spot price model,

$$S_t = \Lambda(t)\overline{Y}_t,$$

where $\overline{Y}$ is defined as in (19). We analyse the forward price for this case, and discuss the affinity. The results and discussions are reasonably parallel to the geometric case, and we refrain from going into details but focus on some main results.

Under a natural integrability condition of the spot price with respect to the Esscher transform measure $Q^{\theta,\eta}$, we find the following forward price for the arithmetic model:

Proposition 5. *Suppose that $S_T \in L^1(Q^{\theta,\eta})$. Then, the forward price is given as*

$$F_t(T) = \Lambda(T) \left\{ \int_{-\infty}^{t} G(T,s)\omega_s \, dL_s + \mathbb{E}_\theta[L_1] \int_t^T G(T,s)\mathbb{E}_\eta[\omega_s \,|\, \mathcal{F}_t] \, ds \right\}$$

The price is reasonably explicit, except of the conditional expectation of the stochastic volatility ω_s, which in general is very hard to compute. By the same arguments as in Proposition 3, the forward price becomes affine in the spot (or in $\overline{Y}_t$) if and only if $G(t,s) = g_1(t)g_2(s)$ for sufficiently regular functions g_1 and g_2.

In the case $L = B$, we can obtain an explicit forward price when using the Girsanov transform as in (24). We easily compute that the forward price becomes

$$F_t(T) = \Lambda(T) \left\{ \int_{-\infty}^{t} G(T,s)\omega_s \, dW_s + \int_{-\infty}^{T} G(T,s)\theta(s) \, ds \right\} . \tag{30}$$

We note that there is no explicit dependence of the spot volatility ω_s except indirectly in the stochastic integral. This is in contrast to the Lévy case with Esscher transform. The dynamics of the forward price becomes

$$dF_t(T) = \Lambda(T)G(T,t)\omega_t \, dW_t . \tag{31}$$

It is interesting to notice that the volatility of the forward price in the arithmetic case depends on the seasonality function directly. We refer to Benth et al. (2008) for a discussion of the seasonality effects in the term structure of volatility.

If we furthermore let $G(t,s) = g_1(t)g_2(s)$ for some sufficiently regular functions g_1 and g_2, we find that

$$F_t(T) = \frac{\Lambda(T)g_1(T)}{\Lambda(t)g_1(t)} S_t + \Lambda(T) \int_t^T G(T,s)\theta(s) \, ds . \tag{32}$$

Hence, the forward curve moves stochastically as the spot price, whereas the shape of the curve is deterministically given by $\Lambda(T)g_1(T)/\Lambda(t)g_1(t)$. This shape is scaled stochastically by the spot price. In addition, there is a deterministic term which is derived from the market price of risk θ.

We finally remark that also in the arithmetic case one may derive expressions for the prices of options which are computable by fast Fourier techniques.

6   Conclusion

This paper has introduced a new class of models for energy spot prices, which is based on Lévy semistationary processes. We have discussed the probabilistic properties of such models such as suitable integrability conditions, absence of arbitrage and the second order structure. Due to their very general structure, Lévy semistationary processes can account for most of the stylised facts observed in energy prices, including mean–reversion, stationarity, the presence of jumps and spikes, volatility clusters and the Samuelson effect. Furthermore, our new class of models embeds most of the traditional models used in energy finance, such as the Schwartz model, CARMA models and models based on non–Gaussian Ornstein Uhlenbeck processes. We have derived explicit formulae for the electricity forward prices based on our new spot price models.

In future research it will be interesting to investigate how the new models can be estimated and how one can draw inference on the stochastic volatility process. In $\mathcal{BSS}$ models, the latter question has been studied in Barndorff-Nielsen, Corcuera & Podolskij (2009, 2010). Extending such results to more general $\mathcal{LSS}$ processes is subject to future research.

Furthermore, we have mentioned in Section 4 that in order to ensure that there are no arbitrage opportunities in a market with transaction costs, when non–semimartingales are used for modelling

asset prices, a distributional property called *conditional full support* plays a key role in the context of continuous stochastic process. So a natural question to study is: what is the analogue of such a distributional property for jump processes in general and for Lévy semistationary processes in particular?

A   Proofs

Proof of Proposition 1. First, write

$$\int_{-\infty}^{T} G(T,s)\omega_s \, dL_s = \int_{-\infty}^{t} G(T,s)\omega_s \, dL_s + \int_{t}^{T} G(T,s)\omega_s \, dL_s$$

and observe that the first integral on the right-hand side is $\mathcal{F}_t$-measurable. The result follows by using double conditioning, first with respect to the σ-algebra $\mathcal{G}_T$ generated by the paths of $\omega_s, s \leq T$ and $\mathcal{F}_t$, and next with respect to $\mathcal{F}_t$. $\square$

Proof of Proposition 2. By the Girsanov change of measure we have

$$\int_{-\infty}^{T} G(T,s)\omega_s \, dB_s = \int_{-\infty}^{T} G(T,s)\theta(s) \, ds + \int_{-\infty}^{T} G(T,s)\omega_s \, dW_s \,.$$

By following the argumentation in the proof of Proposition 1, we are led to calculate the expectation

$$\mathbb{E}_\eta \left[\exp\left(\frac{1}{2} \int_{t}^{T} G^2(T,s)\omega_s^2 \, ds \right) \mid \mathcal{F}_t \right] \,.$$

But, by the stochastic Fubini Theorem, see e.g. Barndorff-Nielsen & Basse-O'Connor (2009),

$$\int_{t}^{T} G^2(T,s) \int_{-\infty}^{s} i(s,v) \, dU_v \, ds$$
$$= \int_{t}^{T} \int_{-\infty}^{t} G^2(T,s)i(s,v) \, dU_v \, ds + \int_{t}^{T} \int_{t}^{s} G^2(T,s)i(s,v) \, dU_v \, ds$$
$$= \int_{-\infty}^{t} \int_{t}^{T} G^2(T,s)i(s,v) \, ds \, dU_v + \int_{t}^{T} \int_{v}^{T} G^2(T,s)i(s,v) \, ds \, dU_v \,.$$

Using the adaptedness to $\mathcal{F}_t$ of the first integral and the independence from $\mathcal{F}_t$ of the second, we find the desired result. $\square$

Proof of Proposition 3. If $G(t,s) = g_1(t)g_2(s)$ it holds that

$$\int_{-\infty}^{T} G(T,s)\omega_s \, dW_s = \frac{g_1(T)}{g_1(t)} \int_{-\infty}^{t} G(t,s)\omega_s \, dW_s = \frac{g_1(T)}{g_1(t)} Y_t \,.$$

Similarly, of $i(t,s) = i_1(t)i_2(s)$,

$$\int_{-\infty}^{t} \int_{t}^{T} G^2(T,v)i(v,s) \, dv \, dU_s = i_1^{-1}(t) \int_{t}^{T} G^2(T,v)i_1(v) \, dv \int_{-\infty}^{t} i(t,s) \, dU_s$$
$$= i_1^{-1}(t) \int_{t}^{T} G^2(T,v)i_1(v) \, dv \, Z_t \,,$$

and affinity holds in both the volatility and the spot price.

23

Opposite, to have affinity in Y_t we must have that

$$\int_{-\infty}^{t} G(T,s)\omega_s \, dW_s = \xi(T,t) \int_{-\infty}^{t} G(t,s)\omega_s \, dW_s \,,$$

for some function $\xi(T,t)$, which means that the ratio $\xi(T,t) = G(T,s)/G(t,s)$ is independent of s. $\xi(T,t)$ is differentiable in T as long as g is. Furthermore, $\xi(T,T) = 1$ by definition. Thus, by first differentiating ξ with respect to T and next letting $T = t$, it holds that

$$G_T(t,s) = \xi_T(t,t)G(t,s) \,,$$

where we use the notation $G_T = \partial G/\partial T$, the derivative with respect to the first argument. Hence, we must have that

$$G(t,s) = G(s,s) \exp\left(\int_s^t \xi_T(u,u) \, du \right) \,,$$

and the separation property holds.

Likewise, to have affinity in the volatility $Z(t)$, we must have that $\int_t^T G^2(T,v)i(v,s) \, dv/i(t,s)$ must be independent of s. Denote the ratio by $\xi(T,t)$, and differentiate with respect to T to obtain

$$G^2(T,T)i(T,s) + 2\int_t^T G(T,v)G_T(T,v)i(v,s) \, dv = \xi_T(T,t)i(t,s) \,.$$

Hence

$$i(T,s) = -\int_t^T I(T,v)i(v,s) \, dv + J(T,t)i(t,s) \,,$$

for $I(T,t) = 2G^{-2}(T,T)G(T,v)G_T(T,v)$ and $J(T,t) = G^{-2}(T,T)\xi(T,t)$. Differentiating with respect to T, and next letting $T = t$ gives

$$i_T(t,s) = i(t,s) \left(J_T(t,t) - I(t,t) \right) \,.$$

Whence,

$$i(t,s) = i(s,s) \exp\left(\int_s^t J_T(v,v) - I(v,v) \, dv \right) \,,$$

and the separation property holds for i. The Proposition is proved. $\qquad\square$

Proof of Proposition 4. Let

$$H_T(t,s) = \int_t^T G^2(T,v)i(v,s) \, dv \,,$$

and from Proposition 2 we have that

$$F_t(T) = \Theta(t,T) \exp\left(\int_{-\infty}^t G(T,s)\omega_s \, dW_s + \frac{1}{2} \int_{-\infty}^t H_T(t,s) \, d\widetilde{U}_s^\eta \right) \,,$$

for some deterministic function $\Theta(t,T)$. Note that the process

$$M_T(t) \triangleq \int_{-\infty}^t G(T,s)\omega_s \, dW_s$$

is a (local) $Q^{\theta,\eta}$-martingale for $t \le T$. Moreover, from the stochastic Fubini Theorem it holds that

$$\int_{-\infty}^t H_T(t,s) \, d\widetilde{U}_s^\eta = \int_{-\infty}^t H_T(s,s) \, d\widetilde{U}^\eta(s) + \int_{-\infty}^t \int_{-\infty}^u \frac{\partial H_T}{\partial t}(t,s) \, d\widetilde{U}_s^\eta \, du \,,$$

24

where we note that

$$\frac{\partial H_T}{\partial t}(t,s) = -G^2(T,t)i(t,s)\,.$$

Hence, the result follows by the Itô Formula for jump processes. $\qquad\square$

Proof of Proposition 5. Observe that

$$\mathbb{E}_{\theta,\eta}\left[\int_{-\infty}^{T} G(T,s)\omega_s\, dL_s \,|\, \mathcal{F}_t\right] = (-\mathrm{i})\frac{d}{dx}\mathbb{E}_{\theta,\eta}\left[\exp\left(\mathrm{i}x\int_{-\infty}^{T} G(T,s)\omega_s\, dL_s\right)\,|\, \mathcal{F}_t\right]_{x=0}\,.$$

We then proceed as in the proof of Proposition 1, and finally we perform the differentiation and let $x = 0$. $\qquad\square$

Acknowledgement

References

Barndorff-Nielsen, O. E. & Basse-O'Connor, A. (2009), Quasi Ornstein–Uhlenbeck Processes. Aarhus University, Thiele Centre Research Report No. 15.

Barndorff-Nielsen, O. E., Benth, F. E. & Veraart, A. E. D. (2009), Ambit processes and stochastic partial differential equations. Preprint.

Barndorff-Nielsen, O. E., Benth, F. E. & Veraart, A. E. D. (2010), Modelling electricity forward markets by ambit fields. In preparation.

Barndorff-Nielsen, O. E., Corcuera, J. & Podolskij, M. (2009), Multipower variation for brownian semistationary processes. Working paper.

Barndorff-Nielsen, O. E., Corcuera, J. & Podolskij, M. (2010), Second order difference multipower variations for bss processes. Working paper.

Barndorff-Nielsen, O. E. & Schmiegel, J. (2007), Ambit processes; with applications to turbulence and cancer growth, *in* F. Benth, G. Di Nunno, T. Lindstrøm, B. Øksendal & T. Zhang, eds, 'Stochastic Analysis and Applications: The Abel Symposium 2005', Springer, Heidelberg, pp. 93–124.

Barndorff-Nielsen, O. E. & Schmiegel, J. (2008), Time change, volatility and turbulence, *in* A. Sarychev, A. Shiryaev, M. Guerra & M. Grossinho, eds, 'Proceedings of the Workshop on Mathematical Control Theory and Finance, Lisbon 2007', Springer, pp. 29–53.

Barndorff-Nielsen, O. E. & Schmiegel, J. (2009), Brownian semistationary processes and volatility/intermittency, *in* H. Albrecher, W. Rungaldier & W. Schachermeyer, eds, 'Advanced Financial Modelling', Radon Series on Computational andApplied Mathematics 8, W. de Gruyter, Berlin, pp. 1–26.

Barndorff-Nielsen, O. E. & Shephard, N. (2002), 'Econometric analysis of realised volatility and its use in estimating stochastic volatility models', *Journal of the Royal Statistical Society B* **64**, 253–280.

Barndorff-Nielsen, O. E. & Shiryaev, A. (2010), *Change of time and change of measure*, World Scientific. Forthcoming.

Barndorff-Nielsen, O. E. & Stelzer, R. (2009), The multivariate SupOU stochastic volatility model. Preprint.

Barndorff-Nielsen, O. E. & Stelzer, R. (2010), 'Multivariate SupOU processes', *Annals of Applied Probability* . Forthcoming.

Basse, A. (2008), 'Gaussian moving averages and semimartingales', *Electronic Journal of Probability* **13**, 1140–1165.

Basse, A. & Pedersen, J. (2009), 'Lévy driven moving averages and semimartingales', *Stochastic Processes and Their Applications* **119**(9), 2970–2991.

Basse-O'Connor, A., Graversen, S.-E. & Pedersen, J. (2009), Martingale–type processes indexed by $\mathbb{R}$. Preprint.

Basse-O'Connor, A., Graversen, S.-E. & Pedersen, J. (2010), Stochastic integration on the real line. Manuscript in preparation.

Bender, C. & Marquardt, T. (2009), 'Integrating volatility clustering into exponential Lévy models', *Journal of Applied Probability* **46**(3), 609–628.

Benth, F. (2009), 'The stochastic volatility model of Barndorff–Nielsen and Shephard in commodity markets', *Mathematical Finance* . Forthcoming.

Benth, F., Kallsen, J. & Meyer-Brandis, T. (2007), 'A non–gaussian ornstein–uhlenbeck process for electricity spot price modelling and derivatives pricing', *Applied Mathematical Finance* **14**(2), 153–169.

Benth, F., López Cabrera, B. & Härdle, W. (2009), Pricing asian temperature risk. Discussion paper, SFB 649, Humboldt-Universität zu Berlin.

Benth, F. & Vos, L. (2009), A multivariate non–gaussian stochastic volatility model with leverage for energy markets. E-print No. 20, October, Department of Mathematics, University of Oslo, Norway.

Benth, F. & Šaltytė Benth, J. (2004), 'The normal inverse gaussian distribution and spot price modelling in energy markets', *International Journal of Theoretical and Applied Finance* **7**(2), 177–192.

Benth, F. & Šaltytė Benth, J. (2009), 'Dynamic pricing of wind futures', *Energy Economics* **31**(1), 16–24.

Benth, F., Šaltytė Benth, J. & Koekebakker, S. (2007), 'Putting a price on temperature', *Scandinavian Journal of Statistics* **34**, 746–767.

Benth, F., Šaltytė Benth, J. & Koekebakker, S. (2008), *Stochastic Modelling of Electricity and Related Markets*, Vol. 11 of *Advanced Series on Statistical Science and Applied Probability*, World Scientific.

Bernhardt, C., Klüppelberg, C. & Meyer-Brandis, T. (2008), 'Estimating high quantiles for electricity prices by stable linear models', *Journal of Energy Markets* **1**(1), 3–19.

Bjerksund, P., Rasmussen, H. & Stensland, G. (2000), Valuation and risk management in the nordic electricity market. Working paper, Institute of Finance and Management Science, Norwegian School of Economics and Business Administration.

Brockwell, P. (2001*a*), Continuous–time ARMA processes, *in* D. Shanbhag & C. Rao, eds, 'Handbook of Statistics', Vol. 19 of *Stochastic Processes: Theory and Methods*, Elsevier, Amsterdam, pp. 249–275.

Brockwell, P. (2001*b*), 'Lévy–driven CARMA processes', *Annals of the Institute of Statistical Mathematics* **53**, 113–124.

Brockwell, P. (2004), 'Representations of continuous-time arma processes', *Journal of Applied Probability* **41**(A), 375–382.

Burger, M., Graeber, B. & Schindlmayr, G. (2007), *Managing Energy Risk: An Integrated View on Power and Other Energy Markets*, Wiley Finance.

Burger, M., Klar, B., Mueller, A. & Schindlmayr, G. (2003), 'A spot market model for pricing derivatives in electricity markets', *Quantitative Finance* **4**(1), 109–122.

Carr, P. & Madan, D. (1998), 'Option valuation using the fast fourier transform', *Journal of Computational Finance* **2**, 61–73.

Cartea, A. & Figueroa, M. (2005), 'Pricing in electricity markets: a mean reverting jump-diffusion model with seasonality', *Applied Mathematical Finance* **12**, 313–335.

Carverhill, A. (2003), An affine model of long maturity forward rates, with predictable risk premium. School of Business, University of Hong Kong, Working Paper.

Delbaen, F. & Schachermayer, W. (1994), 'A general version of the fundamental theorem of asset pricing', *Mathematische Annalen* **300**, 463–520.

Delbaen, F. & Schachermayer, W. (2008), *The Mathematics of Arbitrage*, 2. edn, Springer Finance.

Duffie, D. (1992), *Dynamic Asset Pricing Theory*, Princeton University Press.

Folland, G. (1984), *Real Analysis – Modern Techniques and their Applications*, John Wiley & Sons.

Guasoni, P., Rásonyi, M. & Schachermayer, W. (2008), 'Consistent price systems and face–lifting pricing under transaction costs', *Annals of Applied Probability* **18**(2), 491–520.

Hikspoors, S. & Jaimungal, S. (2008), 'Asymptotic pricing of commodity derivatives for stochastic volatility spot models', *Applied Mathematical Finance* **15**(5&6), 449–467.

Karatzas, I. & Shreve, S. (2005), *Brownian Motion and Stochastic Calculus*, Vol. 113 of *Graduate Texts in Mathematics*, 2nd edn, Springer. Corr. 8th printing.

Knight, F. (1992), *Foundations of the Prediction Process*, Clarendon Press.

López Cabrera, B. & Härdle, W. (2009), Infering the market price of weather risk. Discussion paper, SFB 649, Humboldt-Universität zu Berlin.

Pakkanen, M. (2010), Brownian semistationary processes and conditional full support. Preprint: arXiv:1002.4774v1.

Protter, P. (2005), *Stochastic Integration and Differential Equations*, Springer. Version 2.1Springer Series: Stochastic Modelling and Applied Probability , Vol. 21,2nd ed. 2003. Corr. 3rd printing.

Rajput, B. & Rosinski, J. (1989), 'Spectral representation of infinitely divisible distributions', *Probability Theory and Related Fields* **82**, 451–487.

Samuelson, P. (1965), 'Proof that properly anticipated prices fluctuate randomly', *Industrial Management Review* **6**, 41–44.

Schachermayer, W. (2004), 'The fundamental theorem of asset pricing under proportional transaction costs in finite discrete time', *Mathematical Finance* **14**, 19–48.

Schwartz, E. (1997), 'The stochastic behavior of commodity prices: Implications for valuation and hedging', *The Journal of Finance* **52**(3), 923–973.

Shiryaev, A. (1999), *Essentials of Stochastic Finance: Facts, Models, Theory*, Vol. 3 of *Advanced Series on Statistical Science and Applied Probability*, World Scientific.

Trolle, A. & Schwartz, E. (2009), 'Unspanned stochastic volatility and the pricing of commodity derivatives', *Review of Financial Studies* **22**(11), 4423–4461.

Research Papers
2010

2010-05:   Martin M. Andreasen: Non-linear DSGE Models and The Optimized
            Particle Filter

2010-06:   Søren Johansen and Bent Nielsen: Discussion of The Forward Search:
            Theory and Data Analysis by Anthony C. Atkinson, Marco Riani, and
            Andrea Ceroli

2010-07:   Giuseppe Cavaliere, Anders Rahbek and A.M.Robert Taylor: Bootstrap
            Sequential Determination of the Co-integration Rank in VAR Models

2010-08:   Peter R. Hansen and Asger Lunde: Estimating the Persistence and the
            Autocorrelation Function of a Time Series that is Measured with Error

2010-09:   Tom Engsted, Thomas Q. Pedersen and Carsten Tanggaard: Pitfalls in
            VAR based return decompositions: A clarification

2010-10:   Torben G. Andersen and Luca Benzoni: Stochastic Volatility

2010-11:   Torben B. Rasmussen: Affine Bond Pricing with a Mixture Distribution
            for Interest Rate Time-Series Dynamics

2010-12:   Martin M. Andreasen and Bent Jesper Christensen: The SR Approach:
            a new Estimation Method for Non-Linear and Non-Gaussian Dynamic
            Term Structure Models

2010-13:   Peter Reinhard Hansen, Zhuo (Albert) Huang and Howard Howan
            Shek: Realized GARCH: Complete Model of Returns and Realized
            Measures of Volatility

2010-14:   Bent Jesper Christensen and Michel van der Wel: An Asset Pricing
            Approach to Testing General Term Structure Models including Heath-
            Jarrow-Morton Specifications and Affine Subclasses

2010-15:   Nektarios Aslanidis and Charlotte Christiansen: Smooth Transition
            Patterns in the Realized Stock Bond Correlation

2010-16:   Tim Bollerslev and Viktor Todorov: Estimation of Jump Tails

2010-17:   Ole E. Barndorff-Nielsen, Fred Espen Benth and Almut E. D. Veraart:
            Ambit processes and stochastic partial differential equations

2010-18:   Ole E. Barndorff-Nielsen, Fred Espen Benth and Almut E. D. Veraart:
            Modelling energy spot prices by Lévy semistationary processes