SCHOOL OF ECONOMICS AND MANAGEMENT
FACULTY OF SOCIAL SCIENCES
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Center for Research in Econometric Analysis of Time Series

CREATES Research Paper 2010-17

Ambit processes and stochastic partial differential equations

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Abstract

Ambit processes are general stochastic processes based on stochastic integrals with respect to Lévy bases. Due to their flexible structure, they have great potential for providing realistic models for various applications such as in turbulence and finance. This papers studies the connection between ambit processes and solutions to stochastic partial differential equations. We investigate this relationship from two angles: from the Walsh theory of martingale measures and from the viewpoint of the Lévy noise analysis.

Keywords: Ambit processes, stochastic partial differential equations, Lévy bases, Lévy noise, Walsh theory of martingale measures, turbulence, finance.

JEL codes: C0, C1, C5.

1 Introduction

In physics, partial differential equations (PDEs) give a dynamic way to describe how phenomena in nature evolve over time and space. For instance, the classical heat equation of Einstein gives a dynamic model for how heat diffuses in a medium. Stochastic partial differential equations (SPDEs) add randomness to such evolution equations, where the noise source may come from uncertainties in measurements, non–explainable effects and turbulent phenomena. The noise is usually modelled as a random field in time and space, also called white noise or, more generally, Lévy noise. We shall be mostly concerned with parabolic PDEs in this paper.

Ambit processes have been proposed and introduced by Barndorff–Nielsen & Schmiegel and have thereafter been applied in various areas such as turbulence modelling (see e.g. [6; 13]), in medical context in form of describing tumor growth ([12]), and more recently for modelling energy markets ([4; 5]).

The solution of a parabolic differential equation is often represented as an integral over a Green's function (the fundamental solution of the PDE) convoluted with some initial condition. Such representations look very similar to the definition of stationary ambit processes of [13]. The Green's function representation is an explicit solution as long as the Green's function is known, where the deterministic space–time dynamics of the phenomena in question is encapsulated in the form of this function. It is closely linked to density functions of stochastic diffusion processes.

Introducing noise leads to complications of interpreting in what sense we have a solution. This requires a theory for stochastic integration in time and space, such as proposed in Walsh [46]. It turns out, that solutions of parabolic equations with an additive source of noise can be represented as the stochastic convolution of the Green's function and the initial value, where the integration is with respect to the random field. We present the theory of Walsh [46] and link it to ambit processes.

When having a stochastic source term, one may have solutions being singular. This is the starting point for applying white noise analysis (WNA) or, more generally, Lévy noise analysis (LNA) to analyse SPDEs. We discuss the theory of LNA and link it to ambit processes. Here we will also include discussions of SPDEs and how they are related to ambit processes.

Note that ambit processes may provide a statistical approach to model physical processes in nature far simpler than SPDEs, since they provide a way to specify directly the model based on a probabilistic understanding of the phenomena in question. They also give a framework for extending the solutions of SPDEs. In order to have a solution in the sense of Walsh, often strong integrability conditions are imposed. The ambit processes are well–defined under very weak conditions of integrability, and thereby we may extend the solutions of certain equations to include far more general initial conditions, say, or more general types of noise.

The main issue of this paper is to relate the use of the building stone in ambit processes, Lévy bases, to the language of Walsh and the theory of LNA. The latter talks about processes being the derivatives of Levy processes, while Walsh talks about random measures and their derivatives.

The outline for the remaining part of the paper is as follows. In Section 2 the concepts of ambit fields and processes are outlined, and the important special case of spatial dimension 0 is treated in some detail; in that case the ambit processes are referred to as Lévy semistationary ($\mathcal{LSS}$) processes or, in the Gaussian case, as Brownian semistationary ($\mathcal{BSS}$) processes. In particular, an indication of the theory and use of multipower variations for inference on the volatility process is given. Section 2 concludes by a brief discussion of some applications to turbulence and energy markets. Section 3 connects the idea of Lévy bases to the theory of random fields due to Walsh. We show how, subject to an L^2 restriction and based on the theory of Hilbert space random fields, it is possible to define Lévy noise for Lévy bases, and the associated integration theory is discussed. Finally, some applications to

SPDEs and their relation to ambit processes are considered. Section 4 links the theory of Lévy noise analysis for Lévy processes, as developed in Holden, Øksendal, Ubøe and Zhang [31], to that of Lévy bases and ambit processes, and discusses SPDEs in that context. The concluding Section 5 briefly brings the various strands together.

2 Ambit processes

2.1 Background

The general background setting for the concept of ambit processes consists of a stochastic field $Y = \{Y_t(x)\}$ in space–time $\mathcal{X} \times \mathbb{R}$, a curve $\tau(\theta) = (x(\theta), t(\theta))$ in $\mathcal{X} \times \mathbb{R}$, and the values $X_\theta = Y_{t(\theta)}(x(\theta))$ of the field along the curve, the focus being on the dynamic properties of the stochastic process $X = \{X_\theta\}$. Here the space $\mathcal{X}$ is often, but not necessarily, taken as $\mathbb{R}^d$ for $d = 1, 2$ or 3. The stochastic field is supposed to be generated by innovations in space–time and the values $Y_t(x)$ are assumed to depend only on innovations that occur prior to or at time t. More precisely, at each point (x, t) only the innovations in some subset $A_t(x)$ of $\mathcal{X} \times \mathbb{R}_t$ (where $\mathbb{R}_t = (-\infty, t]$) are influencing the value of $Y_t(x)$, and we refer to $A_t(x)$ as the *ambit set*, associated to (x, t), and to Y and X as an *ambit field* and an *ambit process*, respectively; see Figure 1 for an illustration.

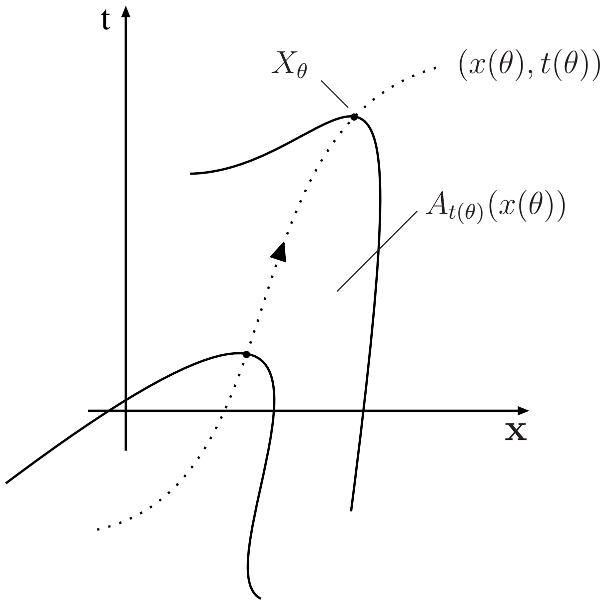

Figure 1: Example of an ambit process X_θ along the curve $(x(\theta), t(\theta))$, where the ambit set is given by $A_{t(\theta)}(x(\theta))$.

Obviously, without further structure nothing interesting can be said about the field Y and the process X, and we shall specify such structure in mathematical detail in a moment. But in verbal terms, $Y_t(x)$ will be defined in the form of a stochastic integral plus a smooth term, and the integrand in the stochastic integral will consist of a deterministic kernel times a positive random variate which is taken to embody the *volatility* or *intermittency* of the field Y. We shall mostly consider specifications under which $Y_t(x)$ is stationary in time for each fixed x.

The volatility field, denoted by σ, is given also as an ambit field, and a central issue is what can be learned about σ from observation of Y or X.

Note that, in general, ambit processes are not semimartingales. Many of the standard tools from semimartingale theory are therefore not applicable and alternative methods are required.

The more precise mathematical specification of what is meant generally by ambit fields and processes is given in Section 2.2. In Sections 2.3, 2.4 and 2.5 we focus on the null–spatial case, i.e. where $\mathcal{X}$ consists of a single point. There the concept of ambit processes specialises to that of Lévy and Brownian semistationary processes ($\mathcal{LSS}$ and $\mathcal{BSS}$ processes). Already in that setting there are many interesting questions of a nonstandard character. These have important analogues in the genuinely tempo–spatial case.

As for semimartingales, the questions of existence and properties of quadratic variations, and more generally multipower variations, are of central importance in the study of ambit fields and processes, in particular as these objects relate to the volatility/intermittency. We will review the main results in that context in Section 2.6 and refer to [17], [8] and [9] for more details.

Section 2.7 contains some applications of ambit processes to turbulence (Section 2.7.1) and energy finance (Section 2.7.2), respectively.

2.2 Ambit fields and processes

Generally we think of ambit fields as being of the form

$$Y_t(x) \quad = \quad \mu \;+\; \int_{A_t(x)} g(\xi, s; x, t)\, \sigma_s(\xi)\, L(\mathrm{d}\xi, \mathrm{d}s) \;+\; \int_{D_t(x)} q(\xi, s; x, t)\, a_s(\xi)\, \mathrm{d}\xi \mathrm{d}s. \quad (1)$$

where $A_t(x)$, and $D_t(x)$ are ambit sets, g and q are deterministic function, $\sigma \geq 0$ is a stochastic field referred to as the *intermittency* or *volatility*, and L is a *Lévy basis*, defined as follows (see [20], [36]): Let $\mathcal{B}(\mathbb{R}^k)$ be the Borel sets of $\mathbb{R}^k$ and denote $\mathcal{B}_b(S)$ the bounded Borel sets of $S \in \mathcal{B}(\mathbb{R}^k)$.

Definition 1. *A family $\{\Lambda(A) : A \in \mathcal{B}_b(S)\}$ of random vectors in $\mathbb{R}^d$ is called an $\mathbb{R}^d$–valued Lévy basis on S if the following three properties are satisfied:*

1. *The law of $\Lambda(A)$ is infinitely divisible for all $A \in \mathcal{B}_b(S)$.*
2. *If $A_1, \ldots, A_n$ are disjoint subsets in $\mathcal{B}_b(S)$, then $\Lambda(A_1), \ldots, \Lambda(A_n)$ are independent.*
3. *If $A_1, A_2, \ldots$ are disjoint subsets in $\mathcal{B}_b(S)$ with $\bigcup_{i=1}^{\infty} A_i \in \mathcal{B}_b(S)$, then*

$$\Lambda\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \Lambda(A_i)\,, a.s.,$$

where the convergence on the right hand side is $a.s.$.

Conditions (2) and (3) define an independently scattered random measure. Note that we use Λ when we refer to a general Lévy basis, and when we have separated out time as one dimension, we talk of Lévy bases defined on $S = \mathcal{X} \times \mathbb{R}$ and we indicate integration with respect to such bases by $L(\mathrm{d}\xi, \mathrm{d}s)$.

Inference on the volatility/intermittency field σ is a focal point for the study of ambit processes and fields. Often the volatility field (or the logarithmic volatility field) will itself be defined as an ambit field through

$$\sigma_t^2(x) = \int_{C_t(x)} h(\xi, s; x, t)\, L(\mathrm{d}\xi, \mathrm{d}s), \tag{2}$$

with h a positive function, $C_t(x)$ some ambit set and where L is a nonnegative non–Gaussian Lévy basis.

At the present level of generality we take the integrals in (1) to be defined in the sense of independently scattered random measures, cf. [38], assuming that g, σ, q and a are sufficiently regular for the integrals to exist. However, in more concrete cases it is often of interest to establish whether the definition of the integrals can be sharpened to a more dynamical version, for instance in the sense of Itô–type integrals. We return to this question later, see in particular Sections 3.4 and 4.

Of particular interest are ambit processes that are stationary in time and nonanticipative. More specifically, they may be derived from ambit fields Y of the form

$$Y_t(x) \;=\; \mu + \int_{A_t(x)} g(\xi, t-s; x)\, \sigma_s(\xi)\, L(\mathrm{d}\xi, \mathrm{d}s) + \int_{D_t(x)} q(\xi, t-s; x)\, a_s(\xi)\, \mathrm{d}\xi \mathrm{d}s. \tag{3}$$

Here the ambit sets $A_t(x)$, and $D_t(x)$ are taken to be *homogeneous* and *nonanticipative*, i.e. $A_t(x)$ is of the form $A_t(x) = A + (x, t)$ where A only involves negative time coordinates, and similarly for $D_t(x)$. Further, we assume that $g(\xi, \tau; x) = 0$ and $q(\xi, \tau; x) = 0$ for all $\tau < 0$.

Remark Recall from [12; 36] that every Lévy basis L exhibits a Lévy–Itô decomposition. Let N denote the Poisson basis associated with the Levy basis L through such a decomposition and let ν denote the intensity measure of N. Clearly, we have $\mathbb{E}(N(dx; d\xi, ds)) = \nu(dx; d\xi, ds)$. In the following, we are interested in *homogeneous* Lévy bases, i.e. Lévy bases which satisfy $\nu(dx; d\xi, ds) = \widetilde{\nu}(dx; d\xi)ds$ for a measure $\widetilde{\nu}$.

Remark Many prominent tempo–spatial models are constructed from an ordinary, partial or fractional differential equation by adding a noise term, for instance in the form of white noise, to the equation. The solution to the equation then being often representable as an integral with respect to the noise of the Green's function of the original deterministic differential equation (see [3; 24]). Thus the solution is taking the form of an ambit process. For some examples with discussion, see Sections 3.5 and 4.2.

Note that, in general, ambit processes involve time varying ambit sets and allow for a stochastic volatility factor. Such stochastic volatility is important in many areas in science, not only in the contexts of turbulence and finance which are in focus in this paper.

For understanding the nature of ambit processes $X_\theta = Y_{t(\theta)}(x(\theta))$, and as a step towards handling questions of inference on σ, it is useful to discuss the cores of Y and X. With the ambit field given by (1), the *cores* $Y_\circ$ and $X_\circ$ of Y and X are defined, respectively, by

$$Y_{\circ t}(x) = \int_{A_t(x)} g(\xi, s; x, t)\, L(\mathrm{d}\xi, \mathrm{d}s),$$

and

$$X_{\circ\theta} = \int_{A(\theta)} g(\xi, s; \tau(\theta))\, L(\mathrm{d}\xi, \mathrm{d}s),$$

where, as above, $\tau(\theta) = (x(\theta), t(\theta))$ and where we have used $A(\theta)$ as a shorthand for $A_{t(\theta)}(x(\theta))$. In case the Lévy basis L is the Wiener basis W we speak of a *Gaussian core*.

Remark A class of processes having some properties common with one–dimensional ambit processes is studied in [44] under the name *mixed moving averages*. More precisely the authors study processes $X = (X_t)_{t\in\mathbb{R}}$ of the form

$$X_t = \int_{\mathcal{X}\times\mathbb{R}} f(x, t-s)\,\Lambda(dx, ds), \tag{4}$$

where $\mathcal{X}$ is a non–empty set and Λ is a symmetric α–stable (SαS) random measure on $\mathcal{X}\times\mathbb{R}$ with Lévy measure $\nu \times leb$, where leb is the Lebesgue measure and ν is a σ–finite measure on $\mathcal{X}$. Note that such processes are always stationary. In the SαS non–Gaussian case, they show that this is the smallest class containing all superpositions and weak limits of ordinary SαS moving averages. Furthermore, Rosinski [39] has obtained a Wold–Karhunen type decomposition of stationary SαS non–Gaussian processes in which mixed moving averages play a role similar to ordinary moving averages in the Gaussian case. And in [40] this type of result is extended to a broad range of non–Gaussian infinitely divisible processes.

2.3 Null–spatial case: Lévy Semistationary Processes ($\mathcal{LSS}$)

When the space $\mathcal{X}$ consists of a single point (or we just consider $Y_t(x)$ of (1) in its dependence on t keeping x fixed) the concept of ambit processes specialises to that of *Lévy Semistationary Processes* ($\mathcal{LSS}$), introduced in [5], which are processes $Y = \{Y_t\}_{t\in\mathbb{R}}$ of the form

$$Y_t = \mu + \int_{-\infty}^{t} g(t-s)\sigma_s \mathrm{d}L_s + \int_{-\infty}^{t} q(t-s)a_s \mathrm{d}s, \tag{5}$$

where μ is a constant, L is a Lévy process, g and q are nonnegative deterministic functions on $\mathbb{R}$, with $g(t) = q(t) = 0$ for $t \leq 0$, and σ and a are càdlàg processes. When σ and a are stationary, as we will require henceforth, then so is Y. Hence the name Lévy semistationary processes. It is convenient to indicate the formula for Y as

$$Y = \mu + g * \sigma \bullet L + q * a \bullet leb, \tag{6}$$

where leb denotes Lebesgue measure.

Generally we have taken the stochastic integrals as defined in the sense of [38]. However, in the present case, of $\mathcal{LSS}$ processes, one may define the integrals in the Itô sense, relative to the filtration $\mathcal{F}^L$ generated by the increments $L_t - L_s$, $-\infty < s \leq t < \infty$. Here we adopt the latter definition, noting that the two versions agree with respect to all finite dimensional distributions.

When $L = B$ in formula (5) for a standard Brownian motion B, then Y specialises to a *Brownian Semistationary Process* ($\mathcal{BSS}$), introduced in [17]. The Gaussian core of a $\mathcal{BSS}$ process is

$$Y_{\circ t} = \int_{-\infty}^{t} g(t-s)\mathrm{d}B_s. \tag{7}$$

We consider the $\mathcal{BSS}$ processes to be the natural analogue, for stationarity related processes, of the class $\mathcal{BSM}$ of Brownian semimartingales

$$Y_t = \int_0^t \sigma_s \mathrm{d}B_s + \int_0^t a_s \mathrm{d}s.$$

Already in this null–spatial case the question of drawing inference on σ^2 is highly nontrivial. The main tool is multipower variation, see [8] and [9].

2.4 Key example for a $\mathcal{BSS}$ process

An example of particular interest in the context of $\mathcal{BSS}$ processes is where

$$g(t) = t^{\nu-1} e^{-\lambda t}, \quad \text{for} \quad t \in (0, \infty), \tag{8}$$

for some $\lambda > 0$ and with $\nu > \frac{1}{2}$. The latter condition is needed to ensure the existence of the stochastic integral in (7).

Remark For the key example (8) the derivative g' of g is not square integrable if $\frac{1}{2} < \nu < 1$ or $1 < \nu \leq \frac{3}{2}$; hence, in these cases Y is not a semimartingale. For $\frac{1}{2} < \nu < 1$ we have $g(0+) = \infty$ while $g(0+) = 0$ when $1 < \nu \leq \frac{3}{2}$. These two cases are radically different in nature. Of course, for $\nu = 1$ the process $Y = \int_{-\infty}^{\cdot} g(\cdot - s)\sigma_s B(\mathrm{d}s)$ is simply a modulated version of the Gaussian Ornstein–Uhlenbeck process, and in particular, a semimartingale. Note also that when $\nu > \frac{3}{2}$ then Y is of finite variation and hence, trivially, a semimartingale. To summarise, the nonsemimartingale cases are $\nu \in \left(\frac{1}{2}, 1\right) \cup \left(1, \frac{3}{2}\right]$.

2.5 Generality of $\mathcal{BSS}$

As a modelling framework for continuous time stationary processes the specification (6) is quite general. In fact, the continuous time Wold–Karhunen decomposition says that any second order stationary stochastic process, possibly complex valued, of mean 0 and continuous in quadratic mean can be represented as

$$Z_t = \int_{-\infty}^{t} \phi(t-s)\, \mathrm{d}\Xi_s + V_t, \tag{9}$$

where the deterministic function ϕ is an in general complex, deterministic square integrable function, the process Ξ has orthogonal increments with $E\left\{|\mathrm{d}\Xi_t|^2\right\} = \varpi dt$ for some constant $\varpi > 0$ and the process V is nonregular (i.e. its future values can be predicted, in the L^2 sense, by linear operations on past values without error).

Under the further condition that $\cap_{t\in\mathbb{R}} \overline{\mathrm{sp}}\{Z_s : s \leq t\} = \{0\}$, the function ϕ is real and uniquely determined up to a real constant of proportionality; and the same is therefore true of Ξ (up to an additive constant).

In particular, if $\mathrm{d}\Xi_s = \sigma_s \mathrm{d}B_s$ with σ and B as in (6), then Ξ is of the above type with $\varpi = \mathrm{E}\left\{\sigma_0^2\right\}$.

2.6 Multipower variations

One of the interesting aspects in the context of $\mathcal{BSS}$ models is the question on how to estimate the stochastic volatility σ and how to make inference on it. A key tool for tackling this question is a statistic called *realised variance* and, more generally, *realised multipower variation*.

A realised multipower variation of a stochastic process X is an object of the type

$$\sum_{i=1}^{[nt]-k+1} \prod_{j=1}^{k} |\Delta_{i+j-1}^n X|^{p_j}, \tag{10}$$

where $\Delta_i^n X = X_{\frac{i}{n}} - X_{\frac{i-1}{n}}$ and $p_1, \ldots, p_k \geq 0$. I.e. it is assumed that the process $X = (X_t)_{t\geq 0}$ is observed at times $i\delta$, where $\delta = \frac{1}{n}$ and $i = 0, 1, \ldots, [nt]$. These concepts have been developed in the context of financial times series, see e.g. [10; 11; 18; 19; 21] for results in a framework based on Brownian semimartingales. In the presence of jumps, these quantities have been studied by [32; 33] and [45]. A detailed survey on this aspect is also given by [2]. However, in the non–semimartingale set up the underlying theory is much more involved. We just sketch the main results here briefly and refer to [17], [8] and [9] for more details.

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, assuming the existence thereon of a $\mathcal{BSS}$ process Y defined as in (5), where $L = B$ is a standard Brownian motion. Let G denote the Gaussian core of Y as defined in (7), i.e.

$$G_t = Y_{\circ t} = \int_{-\infty}^{t} g(t-s)dB_s,$$

and let $\mathcal{G}$ be the σ–algebra generated by G. The correlation function of the increments of G is given by

$$r_n(j) = \text{cov}\left(\frac{\Delta_1^n G}{\tau_n}, \frac{\Delta_{1+j}^n G}{\tau_n}\right) = \frac{\bar{R}(\frac{j+1}{n}) - 2\bar{R}(\frac{j}{n}) + \bar{R}(\frac{j-1}{n})}{2\tau_n^2}.$$

Next, we introduce a class of measures that is crucial for establishing an asymptotic theory for realised multipower variations. We define

$$\pi_\delta(A) = \frac{\int_A (g(x-\delta) - g(x))^2 \mathrm{d}x}{\int_0^\infty (g(x-\delta) - g(x))^2 \mathrm{d}x}\,, \qquad y \geq 0,$$

and we further set $\overline{\pi_\delta}(x) = \pi_\delta(\{y :\ y > x\})$. Note that π_δ is a probability measure on $\mathbb{R}_+$.

We are interested in the asymptotic behaviour of the *normalised multipower variations*

$$\bar{V}(Y, p_1, \ldots, p_k)_t^n = \frac{1}{n\tau_n^{p_+}} \sum_{i=1}^{[nt]-k+1} \prod_{j=1}^{k} |\Delta_{i+j-1}^n Y|^{p_j},$$

where $p_+ = \sum_{j=1}^k p_j$ and $\tau_n^2 = \bar{R}(1/n)$ with $\bar{R}(t) = E[|G_{s+t} - G_s|^2]$, $t \geq 0$.

In order to establish a weak law of large numbers, one needs the following assumption.
(LLN): There exists a sequence $r(j)$ with

$$r_n^2(j) \leq r(j)\,, \qquad \frac{1}{n}\sum_{j=1}^{n-1} r(j) \to 0.$$

Moreover, it holds that

$$\lim_{n\to\infty} \overline{\pi_\delta}(\varepsilon) = 0,$$

for any $\varepsilon > 0$.

Then the law of large numbers is given by the following proposition.

Proposition 1. *Assume that the condition **(LLN)** holds for $Y = g * \sigma \bullet W + q * a \bullet leb$. Define*

$$\rho_{p_1,\ldots,p_k}^{(n)} = E\left[\left|\frac{\Delta_1^n G}{\tau_n}\right|^{p_1} \cdots \left|\frac{\Delta_k^n G}{\tau_n}\right|^{p_k}\right].$$

Then we have

$$\bar{V}(Y, p_1, \ldots, p_k)_t^n - \rho_{p_1,\ldots,p_k}^{(n)} \int_0^t |\sigma_s|^{p_+} \mathrm{d}s \stackrel{ucp}{\longrightarrow} 0,$$

where the convergence is uniform on compacts in probability (ucp).

Furthermore, for a central limit theorem, one needs the following assumption.
(CLT): Assumption **(LLN)** holds, and

$$r_n(j) \to \rho(j)\ , \qquad j \geq 0,$$

where $\rho(j)$ is the correlation function of some stationary centered discrete time Gaussian process $(Q_i)_{i\geq 1}$ with $E[Q_i^2] = 1$ (as before). Moreover, for any $j, n \geq 1$, there exists a sequence $r(j)$ with

$$r_n^2(j) \leq r(j)\ , \qquad \sum_{j=1}^{\infty} r(j) < \infty.$$

Finally, the tail mass function $\overline{\pi}^n$ is assumed to satisfy an additional mild condition.

Now, we can formulate a joint central limit theorem for a family $(\bar{V}(Y, p_1^j, \ldots, p_k^j)_t^n)_{1\leq j\leq d}$ of multipower variations as follows.

Proposition 2. *Assume that the process σ is $\mathcal{G}$–measurable and the condition **(CLT)** holds. Then we obtain the stable convergence*

$$\sqrt{n}\Big(\bar{V}(Y, p_1^j, \ldots, p_k^j)_t^n - \rho^{(n)}_{p_1^j,\ldots,p_k^j} \int_0^t |\sigma_s|^{p_+^j} \mathrm{d}s\Big)_{1\leq j\leq d} \stackrel{\mathcal{G}-st}{\longrightarrow} \int_0^t Z_s^{1/2} \mathrm{d}B_s,$$

where B is a d–dimensional Brownian motion that is defined on an extension of the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ and is independent of $\mathcal{F}$, and Z is a $d\times d$–dimensional process given by

$$Z_s^{ij} = \beta_{ij} |\sigma_s|^{p_+^i + p_+^j}\ , \qquad 1 \leq i, j \leq d\ ,$$

where the $d \times d$ matrix β is defined as in [8].

Note that in order to obtain an asymptotic limit theory for a wide range of multipower variations, one is forced to consider also multipower variations of second order differences. (For Brownian semimartingales passing to second order differences would make no essential change in the limit theory.) Multipower variations based on second order differences are quantities having the same form as (10) but using

$$\diamond_j^n X = X_{j\delta} - 2X_{(j-1)\delta} + X_{(j-2)\delta},$$

instead of $\Delta_j^n X$. However, we shall not dwell on this aspect here, but refer to [7; 9] for discussions, detailed results and applications.

2.7 Applications to turbulence and finance

After having introduced the concept of ambit fields and ambit processes, we turn our attention to applications of such processes in turbulence and in finance.

2.7.1 Tempo–spatial settings in turbulence

The idea of ambit processes arose out of a project aimed at establishing realistic stochastic models of the velocity fields in stationary turbulent regimes (cf. [6; 12] and also [13–17]). In turbulence the basic notion of *intermittency* refers to the fact that the energy in a turbulent field is unevenly distributed in space and time, and the paper [12] introduced stochastic models for turbulent intermittency (also referred to as *energy dissipation*) fields, in the form of ambit fields. The later paper [13] proposed a class of ambit processes for the description of the velocity field, in the form

$$\begin{aligned} Y_t(x) &= \mu + \int_{A_t(x)} g(\xi - x, t - s)\, \sigma_s(\xi)\, W(\mathrm{d}\xi, \mathrm{d}s) \\ &\quad + \int_{D_t(x)} q(\xi - x, t - s)\, \sigma_s^2(\xi)\, \mathrm{d}\xi \mathrm{d}s, \end{aligned} \tag{11}$$

for a Gaussian Lévy basis W with associated intermittency (or energy–dissipation) field

$$\sigma_t^2(x) = \int_{C_t(x)} h(\xi - x, t - s)\, L(\mathrm{d}\xi, \mathrm{d}s), \tag{12}$$

where L is a nonnegative Lévy basis. An alternative way of modelling σ is by defining $\log \sigma^2$ as

$$\log \sigma_t^2(x) = \int_{C_t(x)} h(\xi - x, t - s)\, L(\mathrm{d}\xi, \mathrm{d}s). \tag{13}$$

This latter specification has the advantage of allowing coupling to cascade theories in turbulence, see [43].

Clearly, the choice of the ambit sets $A_t(x), D_t(x), C_t(x)$ influences the behaviour of an ambit process. Therefore, it is important to investigate what shape of the ambit set reflects the empirical facts best.

In order to illustrate how such ambit sets may look, we provide a plot of a particular type of ambit set, the shape of which is rooted in turbulence (see [12]).

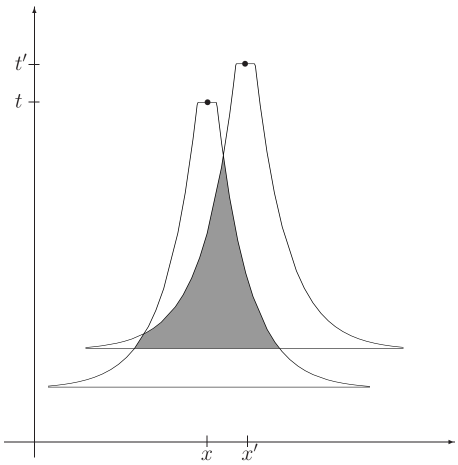

Figure 2: Example of the choice of an ambit set $A_t(x)$ for turbulence modelling, see [12].

Note that the mathematics of turbulence is inherently linked to stochastic partial differential equations (see [24]), as will be discussed in Sections 3 and 4.

2.7.2 Modelling energy markets by ambit fields

Following the success in describing turbulence, it transpires that ambit fields have also great potential in financial applications. In particular, recent research, see [4; 5], has focused on using ambit fields for modelling energy markets. Due to the general structure of ambit fields, these new models are able to capture many stylised facts of energy markets in general, and electricity prices in particular. Special features of those markets are e.g. strong seasonal patterns, very pronounced volatility clusters, high spikes/jumps, the existence of the so called Samuelson effect, i.e. the fact that the volatilities of the forward price are generally smaller than the ones of the underlying spot price and converge, when time to maturity tends to zero, to the volatilities of the spot at a fast rate. Furthermore, there are strong correlations between forward contracts which are close in maturity. In the following we will describe how the structure of ambit processes can be exploited to account for these stylised facts.

2.7.3 Spot price

We start with the the question of how to model the electricity spot price. A natural choice of processes taken from the ambit world is the class of $\mathcal{LSS}$ processes as previously described. In [5], we propose to model the electricity spot price $S = (S_t)_{t\in\mathbb{R}}$ by

$$S_t = \Lambda(t) \exp(Y_t) \,, \tag{14}$$

where $\Lambda : \mathbb{R} \to \mathbb{R}_+$ denotes a deterministic seasonal function and

$$Y_t = \int_{-\infty}^{t} g(t-s)\omega_s dL_s, \tag{15}$$

for a deterministic damping function $g : \mathbb{R} \to \mathbb{R}_+$ with $g(t) = 0$ for $t < 0$ and a càdlàg, positive, stationary process $\omega = (\omega_t)_{t\in\mathbb{R}}$ which is independent of the two–sided Lévy process $L = (L_t)_{t\in\mathbb{R}}$.

There are several key features which make a model for the electricity spot price which is based on a $\mathcal{LSS}$ process both theoretically interesting and practically relevant compared to the traditional models. First and foremost, the deseasonalised, logarithmic spot price Y is modelled *directly* rather than its stochastic dynamics. By doing so, one can introduce a general damping function g, which adds much more flexibility in modelling the mean–reversion of the price process and in accounting for the well–known Samuelson effect ([41]).

Furthermore, we account for stochastic volatility ω since this is clearly an issue in energy markets (see e.g. Hikspoors and Jaimungal [30] and Benth [23]). A very general model for the volatility process would be that we model it itself as a Lévy Volterra process, i.e. $\omega_t^2 = Z_t$ and $Z_t = \int_{-\infty}^{t} h(t,s)\, d\widetilde{L}_s$, where $\widetilde{L} = (\widetilde{L})_{t\in\mathbb{R}}$ is another Lévy process. The function h is assumed to satisfy the same conditions as g.

For further details on $\mathcal{LSS}$–based models for electricity spot prices we refer to [5] and turn our attention now to models for electricity forward contracts based on ambit fields. In the context of forward modelling, we do not stick to the zero spatial case of ambit fields, but rather allow for both a temporal and a spatial component to reflect the fact that the forward price does not only depend on the current time, but also on the time to maturity.

2.7.4 Forward price

In [4], we propose to use an ambit field given by

$$f_t(x) = \int_{A_t(x)} k(\xi, t-s; x)\sigma_s(\xi)L(d\xi, ds), \tag{16}$$

for modelling the forward price of electricity. Here, $t \geq 0$ denotes the current time, $T > 0$ denotes the time of maturity of the forward contract and $x = T - t$ the corresponding time to maturity.

Clearly, in order to specify the model completely, we have to specify the ambit set $A_t(x)$, the damping or weight function k and the stochastic volatility field $\sigma_s(\xi)$. It is important to note that in modelling terms we can vary the choice of the ambit set, the weight function h and the volatility field σ and can still achieve the particular dependence structure we are aiming for. As such there is generally not a unique choice of the ambit set or the weight function or the volatility field to achieve a particular type of dependence structure and the choice will be based on market intuition and considerations of mathematical/statistical tractability.

We assume that the volatility $\sigma_s(\xi) > 0$ is a stochastic field on $\mathbb{R}_+ \times \mathbb{R}$, which is stationary in the time domain, i.e. with respect to s, and which expresses the volatility on the forwards market as a whole, and L is a Lévy basis (integration in the sense of [38]) and k is a damping function. For analytical tractability, we assume that σ is independent of L, and in order to ensure that $f_t(x)$ is stationary in time t, we take the ambit sets to be of the form $A_t(x) = A_0(x) + (0, t)$. Regarding the choice of ambit sets, we just illustrate, in Figure 3, two possibilities of interest.

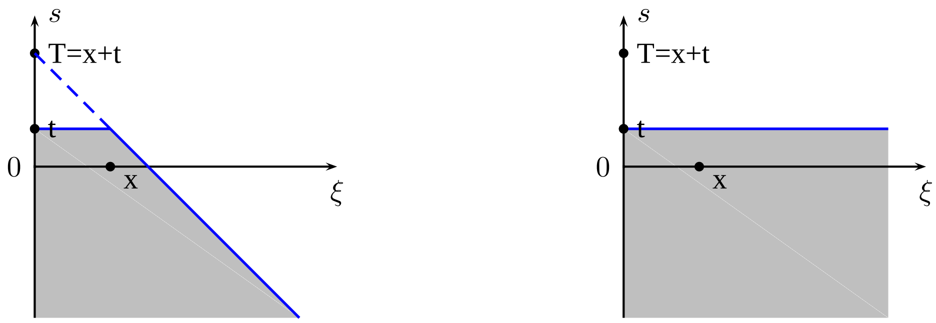

Figure 3: Two relevant choices of the ambit set $A_t(x)$ in the context of modelling electricity forward prices.

Furthermore, we suggest to model the volatility field by

$$\sigma_t^2(x) = \int_{C_t(x)} q(\xi, t-s; x)\widetilde{L}(d\xi, ds),$$

for a nonnegative Lévy basis $\widetilde{L}$, a deterministic damping function q (with $q(\xi, \tau; x) = 0$ for $\tau < 0$) and an ambit set $C_t(x) = C_0(x) + (0, t)$. In order to have that forward contracts close in maturity dates are strongly correlated with each other (as indicated by empirical studies), we could choose the Lévy kernel q such that

$$Cor(\sigma_t^2(x), \sigma_t^2(\bar{x}))$$

is high for values of x and $\bar{x}$ which are close to 0 (i.e. closeness to maturity).

3 Lévy bases and the theory of Walsh

In this section we connect the notion of a Lévy basis to the theory of *white noise* random fields of Walsh [46]. Further, we show how to define the *noise* of sufficiently regular Lévy bases based on the theory of Hilbert space random fields. We summarise the stochastic integration theory of Walsh [46] and present some applications to stochastic partial differential equations in view of ambit processes.

3.1 Brief account on the stochastic integration theory of Walsh

In this subsection we briefly present the approach of Walsh [46] to define stochastic integration with respect to random fields. We keep the discussion on a heuristic level, focusing on the ideas only, since we in any case will introduce the concepts of Walsh in detail below.

The purpose of Walsh [46] is to study stochastic partial differential equations rigorously. The equations are of parabolic type, meaning that the solutions are functions of time and space where their derivative in time is equal to some elliptic operator in space. The partial differential equations are perturbed by random fields, that is, stochastic processes in both time and space (or rather, derivatives of such, called the noise), and in order to make sense out of such equations, one must have available a theory for stochastic integration with respect to such processes.

The key question is how to make sense out of stochastic integrals of the form

$$\int_0^t \int_B X(s,x)\, M(dx,ds)\,,$$

where B is some measurable subset of $\mathbb{R}^d$, and X is some random field in space and time. The M integrator comes from the ”noise” driving the stochastic partial differential equation, and heuristically we may think of this as the time–space derivative of a random field, that is, $M(dx,ds) = \dot{M}(s,x)\,ds\,dx$. However, as is the case for classical Itô integration with respect to a Brownian motion, the time–derivative may not be well–defined.

In the setting of Walsh [46], the approach is to separate the role of time and space, and introduce a class of so–called *martingale measures* $M_t(A)$ for A being a suitable class of measurable subsets of $\mathbb{R}^d$. The martingale measures are so that for each time $t \geq 0$, M_t is a measure–valued square–integrable random variable, and for each set A, the process $t \mapsto M_t(A)$ is a martingale (with respect to a given filtration). In addition, the *covariance functional*

$$\overline{Q}_t(A,B) =< M(A), M(B) >_t$$

plays a crucial role in the construction. Under some technical assumptions on $\overline{Q}$, Walsh [46] constructs the stochastic integral following the scheme of Itô. He shows that for elementary integrands, the stochastic integral is a martingale measure, and by limiting procedures, the definition can be extended to predictable integrands X satisfying some quadratic integrability condition (yielding an extension of the Itô isometry). In fact, the stochastic integral will become a martingale measure.

As it turns out, when studying the relation between Lévy bases and the Walsh theory, so–called *orthogonal* martingale measures are the crucial objects. A martingale measure is called orthogonal if, for two disjoint sets A and B, the processes $M_t(A)$ and $M_t(B)$ are orthogonal. Orthogonal martingale measures satisfy the additional assumptions on the covariance functional, and it is moreover sufficient to study the *covariance measure*

$$Q([0,t] \times A) =< M(A) >_t\,,$$

instead when defining the stochastic integral. In fact, the integrands will be predictable and square integrable with respect to Q. Noteworthy is that the measure Q is closely linked to the control measure of a Lévy basis.

We now go on with a rigorous study of Lévy bases, white noise and stochastic integration in the sense of Walsh, where many of the above concepts will be introduced and discussed in mathematical detail.

3.2 Lévy bases and white noise

In order to relate Lévy bases Λ to the white noise random fields introduced by Walsh [46], it is convenient to slightly reformulate the definition of a Lévy basis given in Definition 1.

We first show that a Lévy basis Λ is countably additive since its law is infinitely divisible:

Lemma 1. *A Lévy basis Λ is countably additive, that is, for a sequence of sets $\{A_n\} \subset \mathcal{B}_b(S)$ where $A_n \downarrow \emptyset$ it holds that*

$$\lim_{n\to\infty} P(|\Lambda(A_n)| \geq \varepsilon) = 0\,, \tag{17}$$

for every $\varepsilon > 0$.

Proof. From the general theory of infinitely divisible laws, there exists a characteristic triplet such that the law of $\Lambda(A)$ has the triplet $(\Sigma_A, \gamma_A, \nu_A)$. One can show (see Pedersen [36, p. 3]) that $A \mapsto \gamma_A^i, \Sigma_A^{ij}$ are signed measures for $i \neq j$, and $A \mapsto \nu_A(B), \Sigma_A^{ii}$ are measures for every i and $B \in \mathcal{B}(\mathbb{R}^d)$. Hence, if $A_n \downarrow \emptyset$ is a sequence of bounded Borel sets, then by standard properties of measures it holds that $(\Sigma_{A_n}, \gamma_{A_n}, \nu_{A_n}) \to (0,0,0)$, and thus the law of $\Lambda(A_n)$ converges to δ_0. Hence, in probability and $a.s.$ it holds that $\Lambda(A_n)$ converges to zero. The countable additivity in (17) follows. □

The following Lemma follows from countable additivity of Λ:

Lemma 2. *Condition (3) in Definition 1 is equivalent to the condition: For each pair of disjoint sets A and B, it holds $a.s.$ that*

$$\Lambda(A \cup B) = \Lambda(A) + \Lambda(B)\,.$$

Proof. Consider $C_N = \bigcup_{i=1}^N A_i$ and $D_N = \bigcup_{i=N+1}^\infty A_i$, and use that C_N and D_N are disjoint to find that

$$\Lambda\left(\bigcup_{i=1}^\infty A_i\right) = \sum_{i=1}^N \Lambda(A_i) + \Lambda(D_N).$$

Since $D_N \downarrow \emptyset$, and by the countable additivity of Λ, we can use Chebyshev's inequality to find

$$P\left(\left|\Lambda\left(\bigcup_{i=1}^\infty A_i\right) - \sum_{i=1}^N \Lambda(A_i)\right| \geq \varepsilon\right) = P\left(|\Lambda(D_N)| \geq \varepsilon\right) \leq \frac{1}{\varepsilon^2}\mathbb{E}[\Lambda(D_N)^2],$$

and the right hand side tends to zero by countable additivity. This gives us convergence in probability of the series $\sum_{i=1}^N \Lambda(A_i)$ when $N \to \infty$. But since the $\Lambda(A_i)$'s are independent random variables, we get convergence P–a.s. by the Itô–Nisio Theorem. □

Recall Condition (2) of independence for Lévy bases Λ in Definition 1. We note that it is equivalent to assume this condition for $n = 2$ only. To see this, let $A_1, A_2, \ldots, A_n$ be n disjoint subsets in $\mathcal{B}_b(S)$. Then, $\Lambda(A_i)$ and $\Lambda(A_j)$ are independent for any combination $i \neq j$, $i, j = 1, \ldots, n$. But then $\Lambda(A_1), \ldots, \Lambda(A_n)$ are independent.

We may give an equivalent definition of a Lévy basis Λ as follows:

Definition 2. *A family $\{\Lambda(A) \,:\, A \in \mathcal{B}_b(S)\}$ of random vectors in $\mathbb{R}^d$ is called an $\mathbb{R}^d$–valued Lévy basis on S if the following three properties are satisfied:*

1. *The law of $\Lambda(A)$ is infinitely divisible for all $A \in \mathcal{B}_b(S)$.*
2. *If A and B are disjoint subsets in $\mathcal{B}_b(S)$, then $\Lambda(A)$ and $\Lambda(B)$ are independent.*

3. *If A and B are disjoint subsets in $\mathcal{B}_b(S)$, then*

$$\Lambda(A \cup B) = \Lambda(A) + \Lambda(B)\,, a.s..$$

The above definition of a Lévy basis provides a natural generalisation of the object defined as *white noise* in Walsh [46]. A white noise is a random set function W on a σ–finite space $(E, \mathcal{E}, \nu)$ defined as follows:

Definition 3. *A white noise W is a random set function on $\mathcal{E}_b$, the sets $A \in \mathcal{E}$ where $\nu(A) < \infty$, such that*

1. *$W(A)$ is normally distributed with zero mean and variance $\nu(A)$;*
2. *$W(A)$ and $W(B)$ are independent as long as A and B are disjoint;*
3. *$W(A \cup B) = W(A) + W(B)$ as long as A and B are disjoint.*

We observe that in the case $E = \mathbb{R}^d$, this white noise concept is a very particular example of a homogeneous Lévy basis (and the definition of Lévy bases, as given in the Appendix, could easily be extended to more general spaces E). Hence, homogeneous Lévy bases provide a generalisation of white noise to *Lévy noise*.

As a note in passing, Walsh [46] concentrates on random measures which have finite variance, in the sense that for each $A \in \mathcal{B}_b(S)$, $\Lambda(A) \in L^2(P)$. Further, the following stronger countable additivity condition is introduced: Λ is said to be *countably additive* if for a sequence of sets $\{A_n\} \subset \mathcal{B}_b(S)$ where $A_n \downarrow \emptyset$ it holds that

$$\lim_{n\to\infty} \mathbb{E}[\Lambda(A_n)^2] = 0\,. \tag{18}$$

This is stronger than the condition (17), which only holds in probability and does not require any finite variance of the random measure. However, the strong condition of Walsh [46] is suitable when defining a theory of stochastic integration which we will consider in Section 3.4.

Walsh [46] also introduces a concept of σ–finiteness of the random measures Λ. To this end, suppose there exists an increasing sequence of sets $\{S_n\}_n \subset \mathcal{B}(S)$ such that $\cup_{n=1}^{\infty} S_n = S$, and for all n, it holds that $\mathcal{B}(S)|S_n \subset \mathcal{B}_b(S)$ and

$$\sup_{A\in\mathcal{B}(S)|S_n} \mathbb{E}[\Lambda(A)^2] < \infty\,.$$

If this is true, we say that Λ is *σ–finite*. If Λ is σ–finite, then Λ is countably additive on $\mathcal{B}(S)|S_n$ if and only if for any sequence of sets $A_n \downarrow \emptyset$ with $A_n \in \mathcal{B}(S)|S_n$ we have $\lim_{n\to\infty} \mathbb{E}[\Lambda(A_n)^2] = 0$. Walsh [46] makes this extension since for such Λ one may extend their domain of definition to include some new sets $A \in \mathcal{B}(S)$: If $A \in \mathcal{B}(S)$, we define

$$\Lambda(A) := \lim_{n\to\infty} \Lambda(A \cap S_n)\,,$$

if the limit exists in $L^2(P)$, and consider $\Lambda(A)$ undefined otherwise. This leaves Λ unchanged on each $\mathcal{B}(S)|S_n$, but may change its value for sets $A \in \mathcal{B}(S)$ which are not in any $\mathcal{B}(S)|S_n$. In Walsh [46], Λ extended in this way is called a σ–finite L^2–valued random measure. Note that we can make this extension for all Lévy bases Λ trivially whenever S is bounded. For S unbounded, the σ–finiteness follows whenever Λ has mean zero. To see this, we make the following computation:

$$\begin{aligned}\mathbb{E}[\Lambda^2(S_n)] &= \mathbb{E}[\Lambda^2(S_n \setminus A)] + 2\mathbb{E}[\Lambda(A)]\mathbb{E}[\Lambda(S_n \setminus A)] + \mathbb{E}[\Lambda^2(A)] \\ &\geq \mathbb{E}[\Lambda^2(A)]\,.\end{aligned}$$

Thus, the variance of $\Lambda(A)$ is bounded by the variance of $\Lambda(S_n)$, which is finite, and σ–finiteness follows.

3.3 Lévy bases and random variables in a Hilbert space

For certain types of Lévy bases Λ, we introduce the mapping $x \mapsto \dot{\Lambda}(x)$ for $x \in S$, being the *noise* of Λ. For this purpose, it will be convenient to interpret the Lévy bases in terms of Hilbert space valued random variables.

To this end, let S be a bounded Borel set in $\mathbb{R}^k$, and introduce the measure space $(S, \mathcal{S}, leb)$, with leb being the Lebesgue measure and $\mathcal{S}$ the Borel sets on S. Assume that S is such that $L^2(S, \mathcal{S}, leb)$ is separable and denote by $\{e_k\}_{k \in \mathbb{N}}$ a complete orthonormal system in the Hilbert space $H = L^2(S, \mathcal{S}, leb)$. We suppose in addition that for all $A \in \mathcal{S}$ with $leb(A) = 0$ we have $\Lambda(A) = 0$ a.s.. Finally, we assume that Λ has *nuclear covariance*[1], that is,

$$\sum_{k=1}^{\infty} \mathbb{E}\left[\left(\int_S e_k(x)\Lambda(dx)\right)^2\right] < \infty\,, \tag{19}$$

where the integration of e_k with respect to $\Lambda(dx)$ is understood in the sense of Rajput and Rosinski as reviewed in Section A.3. We note that in Walsh [46], it is supposed that the integrals with respect to $\Lambda(dx)$ is in the sense of Bochner ([25] and also Chapter III in [27]), which is a stronger concept defined by convergence in variance.

The nuclear covariance condition (19) implies that $\Lambda(A)$ has finite variance, as the following Lemma shows.

Lemma 3. *For every $A \in \mathcal{S}$, $\Lambda(A) \in L^2(P)$.*

Proof. Let $A \in \mathcal{S}$. Since obviously $1_A(x) \in L^2(S, leb)$, we have that

$$1_A(x) = \sum_{k=1}^{\infty} \int_A e_k(y)dy\, e_k(x),$$

and therefore

$$\Lambda(A) = \int_A \Lambda(dx) = \sum_{k=1}^{\infty} \int_A e_k(y)\, dy \int_S e_k(x)\Lambda(dx)\,.$$

But by the Cauchy–Schwarz inequality for sums, we find

$$\begin{aligned}\mathbb{E}[\Lambda(A)^2] &\le \sum_{k=1}^{\infty}\left(\int_A e_k(y)\,dy\right)^2 \times \sum_{k=1}^{\infty}\mathbb{E}\left[\left(\int_S e_k(x)\Lambda(dx)\right)^2\right] \\ &= |1_A|_2^2 \sum_{k=1}^{\infty}\mathbb{E}\left[\left(\int_S e_k(x)\Lambda(dx)\right)^2\right] < \infty\,.\end{aligned}$$

□

For every $\phi \in L^2(S, \mathcal{S}, leb)$, let us introduce the following functional on $L^2(S, \mathcal{S}, leb)$:

$$\phi \mapsto \Lambda(\phi) := \int_S \phi(x)\Lambda(dx)\,. \tag{20}$$

Lemma 4. *The mapping $\phi \mapsto \Lambda(\phi)$ defined in* (20) *is a linear functional on $L^2(S, \mathcal{S}, leb)$.*

[1] This is in accordance with the definition of Walsh [46, p. 288].

Proof. We show that the operator is bounded. We have that $\phi = \sum_{k=1}^{\infty} \phi_k e_k$ and thus

$$\int_S \phi(x)\Lambda(dx) = \sum_{k=1}^{\infty} \phi_k \int_S e_k(x)\Lambda(dx)\,.$$

The Cauchy–Schwarz inequality for sums now yields

$$\mathbb{E}\left[\left|\int_S \phi(x)\Lambda(dx)\right|^2\right] \leq \sum_{k=1}^{\infty} \phi_k^2 \times \sum_{k=1}^{\infty} \mathbb{E}\left[\int_S e_k(x)\Lambda(dx)^2\right] < \infty\,,$$

and hence, the integral is finite $a.s.$. Obviously, $\phi \mapsto \Lambda(\phi)$ is linear, and it therefore defines a linear functional on $L^2(S, \mathcal{S}, leb)$. □

We are now ready to show that Λ has a Radon–Nikodym derivative with respect to the Lebesgue measure.

Proposition 3. *There exists a function $\dot{\Lambda} \in L^2(S, \mathcal{S}, leb)$ such that*

$$\Lambda(\phi) = \int_S \dot{\Lambda}(x)\phi(x)dx\,. \tag{21}$$

Thus $\dot{\Lambda}$ is the Radon–Nikodym derivative of Λ with respect to the Lebesgue measure on $(S, \mathcal{S})$.

Proof. Since any linear functional on a Hilbert space may be represented via the inner product with some element of the Hilbert space, we are ensured the existence of a function $\dot{\Lambda} \in L^2(S, \mathcal{S}, leb)$ such that (21) holds. Note that since $1_A(x)$ is a function in $L^2(S, \mathcal{S}, leb)$ for all $A \in \mathcal{S}$, we have

$$\Lambda(A) = \int_A \dot{\Lambda}(x)dx\,. \tag{22}$$

□

Moreover,

$$\Lambda(A)\Lambda(B) = \int_{A\times B} \dot{\Lambda}(x)\dot{\Lambda}(y)dxdy\,. \tag{23}$$

Note that

$$\dot{\Lambda}(x) = \sum_{k=1}^{\infty} \int_S e_k(y)\Lambda(dy)\, e_k(x)\,.$$

Introduce

$$Q(A \times B) = \mathbb{E}\left[\Lambda(A)\Lambda(B)\right]. \tag{24}$$

Then we have that

$$Q(A \times B) = \int_{A\times B} \mathbb{E}\left[\dot{\Lambda}(x)\dot{\Lambda}(y)\right]\, dx\, dy.$$

We call the signed measure Q the *covariance measure* of the Lévy basis.

Define now the linear operator $\widetilde{Q}$ as

$$\widetilde{Q}f(x) = \int_S q(x,y)f(y)\,dy\,, \tag{25}$$

with $q(x,y) = \mathbb{E}[\dot{\Lambda}(x)\dot{\Lambda}(y)]$. We prove that $\widetilde{Q}$ is a non–negative, nuclear operator from $L^2(S, \mathcal{S}, leb)$ into itself.

Proposition 4. *The linear operator* $\widetilde{Q}$ *defined in* (25) *maps* $L^2(S,\mathcal{S},leb)$ *into itself. The operator is non–negative and nuclear.*

Proof. By the Minkowski and Cauchy–Schwarz inequalities, we have

$$\begin{aligned}\left|\int_S q(\cdot,y)f(y)dy\right|_2 &\leq \int_S |q(\cdot,y)f(y)|_2 dy \\ &= \int_S \left(\int_S q^2(x,y)dx\right)^{1/2} |f(y)|dy \\ &\leq \left(\int_S\int_S q^2(x,y)dxdy\right)^{1/2} |f|_2 \\ &= \left(\int_S\int_S \mathbb{E}[\dot\Lambda(x)\dot\Lambda(y)]^2 dxdy\right)^{1/2} |f|_2 \\ &\leq \left(\int_S\int_S \mathbb{E}[\dot\Lambda^2(x)]\mathbb{E}[\dot\Lambda^2(y)]dxdy\right)^{1/2} |f|_2 \\ &= \mathbb{E}\left[\left|\dot\Lambda\right|_2^2\right]|f|_2 \,.\end{aligned}$$

However, by Parseval's identity and the nuclear covariance condition (19), we have that

$$\mathbb{E}\left[\left|\dot\Lambda\right|_2^2\right] = \sum_{k=1}^{\infty}\mathbb{E}\left[\left(\int_S e_k(x)\Lambda(dx)\right)^2\right] < \infty\,,$$

and hence $\widetilde{Q}f$ is in $L^2(S,\mathcal{S},leb)$. Furthermore, we have that the operator is non–negative in the sense that $(\widetilde{Q}f,f)_2 \geq 0$ for all $f \in L^2(S,leb)$. This follows since

$$(\widetilde{Q}f,f)_2 = \mathbb{E}\left[\left(f,\dot\Lambda\right)_2^2\right] \geq 0\,.$$

We check whether the operator is nuclear. By using the series representation of $\dot\Lambda(y)$, we find

$$\begin{aligned}\widetilde{Q}f(x) &= \int_S q(x,y)f(y)\,dy \\ &= \sum_{k=1}^{\infty}\int_S \mathbb{E}\left[\dot\Lambda(x)\int_S e_k(z)\,\Lambda(dz)\right]e_k(y)f(y)\,dy \\ &= \sum_{k=1}^{\infty}(e_k,f)_2\mathbb{E}\left[\dot\Lambda(x)\int_S \mathrm{e}_k(y)\,\Lambda(dy)\right]\,.\end{aligned}$$

This is the representation in Definition A.1 in Peszat and Zabczyk [37] of nuclear operators, where we identify $a_k(x) = e_k(x)$ and
$b_k(x) = E[\dot\Lambda(x)\int_S \mathrm{e}_k(y)\,\Lambda(dy)]$. Now, $\widetilde{Q}$ is nuclear if $\sum_{k=1}^{\infty}|a_k|_2|b_k|_2 < \infty$. But this is equivalent to

$$\sum_{k=1}^{\infty}\int_S\left(\mathbb{E}\left[\dot\Lambda(x)\int_S e_k(y)\,\Lambda(dy)\right]\right)^2 dx < \infty,$$

since e_k is an orthonormal basis. But, by the Cauchy–Schwarz inequality, we find

$$\begin{aligned}
&\sum_{k=1}^{\infty} \int_S \left(\mathbb{E}\left[\dot{\Lambda}(x) \int_S e_k(y)\, \Lambda(dy) \right] \right)^2 dx \\
&\leq \sum_{k=1}^{\infty} \int_S \mathbb{E}\left[\dot{\Lambda}^2(x) \right] \, dx \, \mathbb{E}\left[\left(\int_S e_k(y)\, \Lambda(dy) \right)^2 \right] \\
&= \mathbb{E}\left[\left| \dot{\Lambda} \right|_2^2 \right] \sum_{k=1}^{\infty} \mathbb{E}\left[\left(\int_S e_k(y)\, \Lambda(dy) \right)^2 \right] .
\end{aligned}$$

And this is finite by the nuclear covariance condition (19). □

We conclude that $\widetilde{Q}$ is a covariance operator in the sense of Peszat and Zabczyk [37, p. 30], where it is defined for Gaussian random variables with values in a Hilbert space. This links the Lévy bases to the theory of square–integrable Hilbert space valued random variables. We note that the nuclear covariance condition (19) makes the Lévy basis sufficiently regular to create random fields with values in a Hilbert space, where we can define covariance operators as the crucial object to understand the covariance structure. Tracing back, we see that the covariance measure of the Lévy basis Λ can be represented by the covariance operator of $\dot{\Lambda}$ as

$$Q(A \times B) = (\widetilde{Q} 1_A, 1_B)_2 \,. \tag{26}$$

Thus, the covariance measure is representable via an integral kernel.

3.4 Extension of the stochastic integration theory of Walsh

Let us consider a Lévy basis Λ on $[0,T] \times S \in \mathcal{B}(\mathbb{R}^{k+1})$, that is, a Lévy basis where we have separated out the first variable to denote time.

We introduce the following measure–valued process

$$M_t(A) := \Lambda((0,t] \times A) \,, \tag{27}$$

for any $A \in \mathcal{B}_b(S)$. The following properties are inherited from the Lévy basis for a fixed set $A \in \mathcal{B}_b(S)$:

Proposition 5. *The measure valued process $M_t(A)$ for $A \in \mathcal{B}_b(S)$ defined in* (27) *is an additive process*[2]*, i.e. it satisfies the following properties:*

1. *The law of $M_t(A)$ is infinitely divisible for each t.*
2. *The increments of $M_t(A)$ are independent.*
3. *The process $M_t(A)$ is stochastically continuous.*
4. *The process $M_t(A)$ is right–continuous, with $M_0(A) = 0$, $a.s.$.*

[2] More precisely, we have that $M_t(A)$ is an *additive process in law*, see Definition 1.6 in Sato [42].

Proof. The first property follows from the fact that the Lévy basis Λ is infinitely divisible. To see the second property, we observe from the additivity of Λ that

$$\Lambda((0,t] \times A) = \Lambda(\{(0,s] \times A\} \cup \{(s,t] \times A\}) = \Lambda((0,s] \times A) + \Lambda((s,t] \times A)\,.$$

From the independence property of Λ, it holds that $\Lambda((s,t] \times A)$ is independent of $\Lambda((0,\tau] \times A)$ for all sets $(0,\tau] \times A$ where $\tau \leq s$. Hence, $M_t(A) - M_s(A)$ is independent of $M_s(A)$. We continue with proving property (3). Observe that

$$P(|M_t(A) - M_s(A)| > \varepsilon) = P(|\Lambda((s,t] \times A)| > \varepsilon),$$

and when $t \downarrow s$ we have that $(s,t] \times A \downarrow \emptyset$. Hence, from the countable additivity in probability, which holds for Lévy bases, it follows that

$$\lim_{t \downarrow s} P(|M_t(A) - M_s(A)| > \varepsilon) = 0\,.$$

This proves property (3). In particular, we find

$$\lim_{t \downarrow 0} P(|M_t(A)| > \varepsilon) = 0,$$

and therefore $M_t(A)$ converges in probability to zero, which implies convergence in law to δ_0. This gives that $\lim_{t\downarrow 0} M_t(A) = 0, a.s.$, and we have that $M_0(A) = \lim_{t\downarrow 0} M_t(A) = 0, a.s.$. Moreover, following the same argument as above, we see that for $s > t$, (using independence of Λ)

$$\Lambda((0,s] \times A) = \Lambda((0,t] \times A) + \Lambda((t,s] \times A)\,.$$

The countable additivity of Λ yields that

$$\Lambda((t,s] \times A) \to 0\,,$$

in probability when $s \downarrow t$ since $(t,s] \times A \downarrow \emptyset$, and therefore $\Lambda((t,s] \times A)$ converges in law to δ_0. Hence,

$$\Lambda((0,s] \times A) \to \Lambda((0,t] \times A)\,,$$

and it follows that $M_t(A)$ is right–continuous. Hence, we have shown the last property. $\square$

Remark To obtain a Lévy process, we would need to have stationarity of increments, i.e. the law of the increment $M_{s+t}(A) - M_s(A)$, $s, t > 0$ should be independent of s. But

$$M_{s+t}(A) - M_s(A) = \Lambda((s, s+t] \times A),$$

and the characteristic triplet for the law is thus $(\Sigma_{(s,s+t]\times A}, \gamma_{(s,s+t]\times A}, \nu_{(s,s+t]\times A})$. If there exist measures $\widetilde{\Sigma}_A$ and $\widetilde{\nu}_A$, and a signed measure $\widetilde{\gamma}_A$ such that $\Sigma_{\tau\times A} = leb(\tau)\widetilde{\Sigma}_A$, $\gamma_{\tau\times A} = leb(\tau)\widetilde{\gamma}_A$ and $\nu_{\tau\times A} = leb(\tau)\widetilde{\nu}_A$, for τ a bounded Borel subset of the positive real line, we would have stationarity. Such a separation property of the characteristic triplet would imply that $M_t(A)$ is a Lévy process.

We want to use $M_t(A)$ as integrators like in Walsh [46], where the Itô integration approach is used. We conveniently suppose that for each A, $M_t(A) \in L^2(\Omega, \mathcal{F}, P)$. Furthermore, we define the filtration $\mathcal{F}_t$ by $\mathcal{F}_t = \cap_{n=1}^{\infty} \mathcal{F}^0_{t+1/n}$, where

$$\mathcal{F}^0_t = \sigma\{M_s(A) : A \in \mathcal{B}_b(S), 0 < s \leq t\} \vee \mathcal{N},$$

and where $\mathcal{N}$ denotes the P–null sets of $\mathcal{F}$. Then, $\mathcal{F}_t$ is right–continuous by construction. Finally, we suppose that the expected value of the Lévy basis Λ is equal to zero, that is, $\mathbb{E}[M_t(A)] = 0$. If this is not the case, we can always redefine the Lévy basis by subtracting its mean value in order to obtain a mean–zero process.

It turns out that $M_t(A)$ is a square–integrable martingale satisfying an orthogonality property:

Proposition 6. *Under the assumption of square–integrability and mean zero of $M_t(A)$, the following two properties hold*

1. *For each A, $t \mapsto M_t(A)$ is a (square–integrable) martingale with respect to the filtration $\mathcal{F}_t$.*
2. *If A and B are two disjoint sets in $\mathcal{B}_b(S)$, then $M_t(A)$ and $M_t(B)$ are independent.*

Proof. The second property holds trivially from the independence property of the Lévy basis. To see the first property, let $s \leq t$. We have by the independence property of the Lévy basis that

$$\Lambda((0,t] \times A) = \Lambda((0,s] \times A \cup (s,t] \times A) = \Lambda((0,s] \times A) + \Lambda((s,t] \times A),$$

and therefore

$$M_t(A) = M_s(A) + \Lambda((s,t] \times A)\,.$$

Furthermore, we have that $\Lambda((s,t] \times A)$ is independent of $\mathcal{F}_s$ since any sets $[0,s_i] \times B$ will be disjoint with $(s,t] \times A$ as long as $s_i \leq s$. Therefore

$$\mathbb{E}[M_t(A)\,|\,\mathcal{F}_s] = \mathbb{E}[M_s(A)\,|\,\mathcal{F}_s] + \mathbb{E}[\Lambda((s,t] \times A)] = M_s(A)\,.$$

The last equality is obtained by the zero–mean assumption on the Lévy basis and the measurability of $M_s(A)$ to $\mathcal{F}_s$. □

These two properties together with the fact that $M_0(A) = 0$ $a.s.$, are essentially defining what is called an *orthogonal martingale measure* in Walsh [46]. Walsh [46] adds a further regularity condition on $A \mapsto M_t(A)$ which he calls σ–finiteness to make up the definition of an orthogonal martingale measure. As we have seen earlier, σ–finiteness follows for Lévy bases with mean zero, which is what is supposed here.

As is shown in Walsh [46] (see also [34] for a survey), for orthogonal martingale measures we may introduce a *covariance measure* Q as

$$Q([0,t] \times A) =< M(A) >_t\,, \tag{28}$$

for $A \in \mathcal{B}_b(S)$. The covariance measure Q is positive, and is used as the control measure in the Walsh sense when defining stochastic integration with respect to M. We now describe the integration procedure followed by Walsh [46], which is essentially the Itô approach to stochastic integration. To make matters slightly simpler, we suppose that S is a bounded Borel set, and we recall the notation $\mathcal{S}$ for the Borel subsets of S. Furthermore, we treat only integration up to a finite time T. Note that extensions to unbounded S and infinite time interval follow by standard arguments (see [46, p. 289]).

First, we say that a random field $f(s,x)$ is *elementary* if it has the form

$$f(s,x,\omega) = X(\omega) 1_{(a,b]}(s) 1_A(x)\,, \tag{29}$$

where $0 \leq a < t$, X is bounded and $\mathcal{F}_a$–measurable and $A \in \mathcal{S}$. For elementary functions we can define stochastic integration as

$$\int_0^t \int_B f(s,x)\, M(dx,ds) := X\left(M_{t\wedge a}(A\cap B) - M_{t\wedge b}(A\cap B)\right)\,, \tag{30}$$

for every $B \in \mathcal{S}$. In fact, the stochastic integral becomes a martingale measure as discussed earlier. The extension of stochastic integration to finite linear combinations of elementary random fields is obvious. A finite linear combinations of elementary random fields is called a *simple* random field, and the set of simple random fields is denoted $\mathcal{T}$. The *predictable* σ–algebra $\mathcal{P}$ is the σ–algebra generated by $\mathcal{T}$, and a random field is called *predictable* as long as it is $\mathcal{P}$–measurable. A norm $\|\cdot\|_M$ is defined on the predictable random fields f by

$$\|f\|_M^2 := \mathbb{E}\left[\int_{[0,T]\times S} f^2(s,x)\, Q(dx,ds)\right], \tag{31}$$

which determines the Hilbert space $\mathcal{P}_M := L^2(\Omega \times [0,T] \times S, \mathcal{P}, Q)$. In Walsh [46] it is proved that $\mathcal{T}$ is dense in $\mathcal{P}_M$. To define the stochastic integral of $f \in \mathcal{P}_M$, we choose an approximating sequence $\{f_n\}_n \subset \mathcal{T}$ such that $\|f - f_n\|_M \to 0$ as $n \to \infty$. It is easy to see that for each $A \in \mathcal{S}$, $\int_{[0,t]\times A} f_n(s,x)\, M(dxds)$ is a Cauchy sequence in $L^2(\Omega, \mathcal{F}, P)$, and thus there exists a limit which we define as the stochastic integral of f. It turns out that this stochastic integral is again a martingale measure, and that the “Itô isometry” holds;

$$\mathbb{E}\left[\left(\int_{[0,t]\times A} f(s,x) M(dx,ds)\right)^2\right] = \|f\|_M^2 . \tag{32}$$

See Walsh [46], Theorem 2.5 for the complete result and proof.[3]

The weak integration of Rajput and Rosinski ([38]) extends this definition of stochastic integration in the following sense. For any sequence $\{f_n\}_n \subset \mathcal{T}$ of deterministic functions converging to f in $\mathcal{P}_M$, there exists a subsequence $\{f_{n'}\}_{n'} \subset \mathcal{T}$ converging to f Q–a.e., and for this sequence the stochastic integrals converge in probability since they converge in variance by definition. Hence, for $f \in \mathcal{P}_M$, the definition of weak integration according to Rajput and Rosinski presented in Section A.3 in the Appendix extends that of Walsh as long as the control measure λ of the Lévy basis Λ is absolutely continuous with respect to Q. (See Section A.2 in the Appendix). However, as the following computation shows, Q and λ are equivalent: Since we have assumed that the Lévy basis Λ has zero mean, it follows from the characteristic exponent in formula (48) of the Appendix that

$$Q([0,t] \times A) = \int_{[0,t]\times A} \left(\sigma^2(x,s) + \int_{\mathbb{R}} z^2\, \rho(x,s,dz)\right)\, \lambda(dx,ds) .$$

Therefore we conclude that the weak integration concept of Rajput and Rosinski is a true generalisation of that due to Walsh as long as deterministic integrands are considered. We remark in passing that the integration theory of Rajput and Rosinski is not restricted to square–integrable Lévy bases, as is the Walsh integration concept we have presented here.

Remark Note that we do not know if we have disintegration with the theory of Walsh. However, we know that the integral is a martingale process in time, which adds important dynamics which gives us a big advantage compared to the weaker form of integration available from Rajput and Rosinski ([38]).

[3] Note that in Walsh [46], the argument is made for so–called *worthy* martingale measures. As argued in Walsh [46], an orthogonal martingale measure is worthy, and moreover the *control measure* used to define stochastic integrals sits in this case on the diagonal of $S \times S$. We have chosen to present that particular case.

Note also that in the definition of weak integration in the appendix only deterministic integrands are used. The general definition of ambit processes involves stochastic integrands. This can be accommodated by further extension of the Walsh theory. Such extension is currently under development in collaboration with Andreas Basse–O'Connor, Svend Erik Graversen and Jan Pedersen, see e.g. [22].

3.5 Stochastic partial differential equations and ambit processes

In this subsection we consider a class of parabolic stochastic partial differential equations (SPDE) analysed in detail by Walsh [46]. The motivation with our presentation here is to relate the solutions of such SPDEs to ambit processes, and discuss possible extensions based on these.

Letting $\dot{W}$ be a white noise in the sense of Walsh, we introduce the following non–linear parabolic SPDE

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} - v + f(t,v)\dot{W}\,, & t>0, 0<x<K, \\ \frac{\partial v}{\partial x}(t,0) = \frac{\partial v}{\partial x}(t,K) = 0, & t>0, \\ v(0,x) = v_0(x)\,, & 0<x<K\,, \end{cases} \tag{33}$$

where $K>0$ is some constant and f is a Lipschitz continuous function in x of at most linear growth. Furthermore, it is supposed that v_0 is $\mathcal{F}_0$–measurable and $\mathbb{E}[v_0^2(x)]$ is bounded. Since white noise is too rough to expect smooth solutions of the parabolic SPDE, Walsh [46] introduces a *weak solution* concept. We say that v is a *weak solution* of (33) if for every $\phi \in C^\infty([0,K])$ with $\phi'(0)=\phi'(K)=0$ it holds that

$$\int_0^K (v(t,x)-v_0(x))\phi(x)\,dx = \int_0^t\int_0^K v(s,x)(\phi''(x)-\phi(x))\,dxds + \int_0^t\int_0^K f(s,v(s,x))\phi(x)\,W(dx,ds)\,. \tag{34}$$

In Walsh [46], Theorem 3.2, it is proved that there exists a weak solution v to (33) which is bounded in variance on $[0,K]\times[0,T]$ for each $T>0$. The proof goes by application of the Green's function and Picard iterations.

To see the connection to (33) note that formal differentiation of (34) with respect to t gives

$$\int_0^K v_t(t,x)\phi(x)dx \quad = \quad \int_0^K v(t,x)\left(\phi''(x)-\phi(x)\right)dx \; + \; \int_0^K f(t,v(t,x))\phi(x)W(dx,dt)\,.$$

An integration–by–parts applied formally to the first integral on the right hand side and application of the initial conditions essentially leads to (33).

The homogeneous form of (33) is known as the *cable equation*, and Walsh [46] presents the Green's function of this as

$$G_t(x,y) = \frac{\mathrm{e}^{-t}}{\sqrt{4\pi t}}\sum_{n=-\infty}^{\infty} \exp\left(-\frac{(y-x-2nK)^2}{4t}\right) + \exp\left(-\frac{(y+x-2nK)^2}{4t}\right)\,.$$

A solution to the case $f=1$ can be represented as

$$v(t,x) = \int_0^K G_t(x,y)v_0(y)\,dy + \int_0^t\int_0^K G_{t-s}(x,y)\,W(dy,ds)\,. \tag{35}$$

Note that if the last integral was computed over $(-\infty,t]$ rather than over $[0,t]$, the Wold–Karhunen representation with respect to a Brownian motion could be used in principle.

The solution in (35) represents the solution to an SPDE which can be related to physical processes. Walsh [46] interprets the problem (33) to description of the nervous system, and another interpretation is diffusion of heat. These physical systems may be described directly through an ambit process rather than via an SPDE. As such, we could model the phenomena using a general Lévy basis Λ instead of the particular white noise W. Thus, a generalisation of v in (35) is to consider

$$v(t,x)=\int_0^K G_t(x,y)v_0(y)\,dy+\int_0^t\int_0^K G_{t-s}(x,y)\,L(dy,ds)\,. \tag{36}$$

One may also take this further, and consider "stochastic intermittency" described by a random field $\sigma(t,x)$. Thus,

$$v(t,x)=\int_0^K G_t(x,y)v_0(y)\,dy+\int_0^t\int_0^K G_{t-s}(x,y)\sigma(s,y)\,L(dy,ds)\,. \tag{37}$$

The intermittency field σ may be defined as an ambit field, and as such, we have that $v(t,x)$ is an ambit field over the ambit set $\mathcal{A}_t(x)=[0,t]\times[0,K]$ under appropriate regularity conditions ensuring the existence of the integrals in (37). In fact, we have that $v(t,x)$ in (37) is by definition a *mild* solution of the parabolic problem

$$\begin{cases} \frac{\partial v}{\partial t}=\frac{\partial^2 v}{\partial x^2}-v+\sigma(t,x)\dot{\Lambda}\,, & t>0, 0<x<K, \\ \frac{\partial v}{\partial x}(t,0)=\frac{\partial v}{\partial x}(t,K)=0, & t>0, \\ v(x,0)=v_0(x)\,, & 0<x<K\,. \end{cases} \tag{38}$$

Here, $\dot{\Lambda}$ is a suggestive notation for the noise of the Lévy basis L (see Section 4 for a mathematical formulation of this). The definition of a mild solution of a parabolic stochastic partial differential equation is introduced in Da Prato and Zabczyk [26, p. 152] and is in general weaker than a weak solution. By Theorem 6.5 in Da Prato and Zabczyk [26], we have that the mild solution $v(t,x)$ in (37) is a weak solution under natural integrability conditions on σ and v_0.

It is important to notice that we can generalise the solution $v(t,x)$ in (37) to hold for very general specifications of σ, in fact, by going to the general integration concept of Rajput and Rosinski [38], we can make sense of $v(t,x)$ as an ambit field. By weakening the integration, we can still interpret v as a mild solution to the parabolic problem. A further generalisation is of course to allow for more general ambit sets $\mathcal{A}_t(x)$, leaving the specification $\mathcal{A}_t(x)=[0,t]\times[0,K]$. This will allow for a great deal of flexibility in modelling the physical phenomena in question, in particular how the dependency structure in time and space evolves.

4 Lévy noise analysis

The white noise analysis introduced by Hida in the 80ties has become a popular tool for analysing SPDEs which are singular in the sense of not admitting regular solutions. Hida proposed an analysis based on white noise, that is, the time–derivative of Brownian motion, with applications from quantum mechanics and Feynman path integrals in mind. In Hida, Kuo, Potthoff and Streit [29] one can find a detailed account of the so–called *white noise analysis* and its applications to physics. In this paper we are concerned with SPDEs, and will base our further discussion on the Lévy noise analysis for Lévy processes introduced in Holden, Øksendal, Ubøe and Zhang [31]. In particular, we link Lévy bases and ambit processes with the Lévy noise analysis framework, and finally discuss SPDEs in this context.

4.1 Lévy bases and Lévy noise

Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$, and define $\Omega = \mathcal{S}'(\mathbb{R}^d)$ where $\mathcal{S}'(\mathbb{R}^d)$ is its dual. Denote by $\mathcal{F}$ the Borel σ–algebra on Ω, and let ℓ be a Lévy measure on $\mathbb{R}\setminus\{0\}$ satisfying the condition of square–integrability

$$C := \int_{\mathbb{R}\setminus\{0\}} z^2\,\ell(dz) < \infty\,. \tag{39}$$

By the Bochner–Minlos Theorem (see Definition 5.4.1 in [31]) there exists a probability measure P on $(\Omega, \mathcal{F})$ such that

$$\int_\Omega \mathrm{e}^{\mathrm{i}\langle\omega,\phi\rangle}\,dP(\omega) = \exp\left(-\frac{1}{2}\sigma^2|\phi|_2^2 + \int_{\mathbb{R}^d}\int_{\mathbb{R}\setminus\{0\}}\left\{\mathrm{e}^{\mathrm{i}\phi(y)z} - 1 - \mathrm{i}\phi(y)z\right\}\,\ell(dz)\,dy\right)\,, \tag{40}$$

where $\langle\omega,\phi\rangle := \omega(\phi)$, that is, the action of $\omega \in \mathcal{S}'(\mathbb{R}^d)$ on $\phi \in \mathcal{S}(\mathbb{R}^d)$, and $|\cdot|_2$ is the norm in $L^2(\mathbb{R}^d)$. The probability space $(\Omega, \mathcal{F}, P)$ is called the d–parameter *Lévy noise probability space* by Holden *et al.* [31][4]. This probability space will support a d–parameter Lévy process and is the basis for defining its derivative, the *Lévy noise*[5].

Introduce the cylindrical random variables N_ϕ by

$$N_\phi(\omega) = \langle\omega,\phi\rangle\,, \tag{41}$$

for $\phi \in \mathcal{S}(\mathbb{R}^d)$. Observe, that since (40) gives an explicit form of the characteristic function of N_ϕ in terms of the Lévy measure, we easily find that

$$\mathbb{E}[N_\phi] = 0\,,$$

and

$$\mathrm{Var}[N_\phi] = (\sigma^2 + C)\int_{\mathbb{R}^d}\phi^2(y)\,dy\,.$$

We can extend these random variables to $\phi \in L^2(\mathbb{R}^d)$ by a standard limit argument choosing a sequence $\{\phi_n\} \subset \mathcal{S}(\mathbb{R}^d)$ converging in $L^2(\mathbb{R}^d)$ to ϕ. The limit of N_{ϕ_n} exists in $L^2(P)$ and will be denoted N_ϕ. The limit is independent of the choice of approximating sequence. In particular, we can define $N_A := N_{\mathbf{1}_A}$ for bounded Borel sets $A \subset \mathbb{R}^d$. We make the following definition.

Definition 4. *For every bounded Borel subset A of $\mathbb{R}^d$, define the random measure*

$$\Lambda(A) = N_A\,.$$

We show that Λ defines a Lévy basis (see Proposition 7) and that it is homogeneous (see Proposition 8).

[4]We note that in Holden *et al.* [31] one constructs this probability space for Brownian motion and a pure–jump Lévy process separately. We merge this into a more general Lévy process with both jumps and continuous martingale part. Further note that the representation result (40) was originally introduced in [28]. See also [1] for related work.

[5]Note that Holden *et al.* [31] call such noise *Lévy coloured noise*.

Proposition 7. *The random measure Λ is a Lévy basis, with mean zero and variance $(\sigma^2+C)\cdot leb(A)$, where $leb(A)$ is the Lebesgue measure of A, and the associated control measure of Λ is*

$$\lambda(A) = \sigma^2 leb(A) + \int_{\mathbb{R}} \min(1, z^2)\, \ell(dz) leb(A)\,.$$

Proof. The random measure $\Lambda(A)$ has mean zero and variance equal to $M\ leb(A)$, where $leb(A)$ is the Lebesgue measure of the set A. We show that Λ has the additivity and independence properties.

Let A and B be two disjoint bounded Borel sets, and let $\phi_n \to 1_A$ and $\psi_n \to 1_B$ in $L^2(\mathbb{R}^d)$. Since obviously $1_{A\cup B} = 1_A + 1_B$ and $\phi_n + \xi_n$ converges to $1_A + 1_B$ in $L^2(\mathbb{R}^d)$, $\phi_n + \xi_n$ converges to $1_{A\cup B}$ in $L^2(\mathbb{R}^d)$. Hence, by independence of the approximating sequence, we find that that $N_{\phi+\xi_n}$ converges in $L^2(P)$ to $\Lambda(A\cup B)$, and since

$$N_{\phi+\xi_n}(\omega) = \langle \omega, \phi_n + \psi_n\rangle = \langle \omega, \phi_n\rangle + \langle \omega, \psi_n\rangle = N_{\phi_n}(\omega) + N_{\psi_n}(\omega)\,,$$

it holds that

$$\Lambda(A\cup B) = \Lambda(A) + \Lambda(B)\,.$$

This proves the additivity. To prove independence, we have to show that for two disjoint bounded sets A and B, $\Lambda(A)$ is independent of $\Lambda(B)$, or equivalently, N_A is independent of N_B. To this end, choose two approximating sequences ϕ_n and ξ_n in $\mathcal{S}(\mathbb{R}^d)$ converging to 1_A and 1_B, respectively, in $L^2(\mathbb{R}^d)$. Use the characteristic function of N_{ϕ_n} and N_{ξ_x} to find

$$\begin{aligned}
&\ln \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\theta N_{\phi_n}} \mathrm{e}^{\mathrm{i}\eta N_{\xi_n}}\right] = \ln \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\langle\cdot, \theta\phi_n + \eta\xi_n\rangle}\right] \\
&= -\frac{1}{2}\sigma^2 |\theta\phi_n + \eta\xi_n|_2^2 \\
&\quad + \int_{\mathbb{R}^d}\int_{\mathbb{R}\setminus\{0\}} \{\mathrm{e}^{\mathrm{i}(\theta\phi_n(y)+\eta\xi_n(y))z} - 1 - \mathrm{i}(\theta\phi_n(y) + \eta\xi_n(y))z\}\, \ell(dz)\, dy\,.
\end{aligned}$$

We can write the $L^2(\mathbb{R}^d)$–norm as follows,

$$\begin{aligned}
|\theta\phi_n + \eta\xi_n|_2^2 = \theta^2 &\int_{\mathrm{supp}\phi_n\setminus\mathrm{supp}\xi_n} \phi_n^2(y)\, dy \\
&+ \int_{\mathrm{supp}\phi_n\cap\mathrm{supp}\xi_n} (\theta\phi_n(y) + \eta\xi_n(y))^2\, dy + \eta^2 \int_{\mathrm{supp}\xi_n\setminus\mathrm{supp}\phi_n} \xi_n^2(y)\, dy\,.
\end{aligned}$$

The set $\mathrm{supp}\phi_n \cap \mathrm{supp}\xi_n$ must go to a set of Lebesgue measure zero since $A\cap B = \emptyset$; otherwise the two sequences will not converge to their respective indicator functions in $L^2(\mathbb{R}^d)$. Hence, passing to the limit, we find that

$$\lim_{n\to\infty} |\theta\phi_n + \eta\xi_n|_2^2 = \theta^2 leb(A) + \eta^2 leb(B)\,.$$

A similar argument shows that

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\mathbb{R}^d}\int_{\mathbb{R}\setminus\{0\}} \{\mathrm{e}^{\mathrm{i}(\theta\phi_n(y)+\eta\xi_n(y))z} - 1 - \mathrm{i}(\theta\phi_n(y) + \eta\xi_n(y))z\}\, \ell(dz)\, dy \\
&\quad = leb(A)\int_{\mathbb{R}\setminus\{0\}} \{\mathrm{e}^{\mathrm{i}\theta z} - 1 - \mathrm{i}\theta z\}\, \ell(dz) + leb(B)\int_{\mathbb{R}\setminus\{0\}} \{\mathrm{e}^{\mathrm{i}\theta z} - 1 - \mathrm{i}\theta z\}\, \ell(dz)\,.
\end{aligned}$$

Thus, after taking limits, we find

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\theta\Lambda(A)}\mathrm{e}^{\mathrm{i}\eta\Lambda(B)}\right] = \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\theta\Lambda(A)}\right] \times \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\eta\Lambda(B)}\right] .$$

This proves independence.

In fact, the above limit argument shows that the (log-)characteristic function of $\Lambda(A)$ is

$$\ln \mathbb{E}[\mathrm{e}^{\mathrm{i}\theta\Lambda(A)}] = \left(-\frac{1}{2}\theta^2\sigma^2 + \int_{\mathbb{R}\setminus\{0\}} \{\mathrm{e}^{\mathrm{i}\theta z} - 1 - \mathrm{i}\theta z\}\, \ell(dz)\right) leb(A)\,,$$

This is the Lévy–Kintchine formula where we can read off the control measure for the Lévy basis as being

$$\lambda(A) = \sigma^2 leb(A) + \int_{\mathbb{R}} \min(1, z^2)\ell(dz) leb(A)\,,$$

(see Appendix A for the definition of the control measure for a Lévy basis). □

By letting $\ell(dz) = 0$ and $\sigma = 1$, we recover the case of white noise and the setting for the white noise analysis. Note that here we consider only Lévy bases with no drift and being square integrable.

The Lévy basis has a stationarity property, as shown in the next Proposition.

Proposition 8. *For each $x \in \mathbb{R}^d$, $\Lambda(\cdot)$ and $\Lambda(\cdot + x)$ has the same distribution, i.e. Λ is homogeneous.*

Proof. Give $\phi \in \mathcal{S}(\mathbb{R}^d)$, we prove that N_ϕ and N_{ϕ_x} have the same distribution, where $\phi_x(y) = \phi(y - x)$. It follows from the translation invariance of the Lebesgue measure that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}\setminus\{0\}} \{\mathrm{e}^{\mathrm{i}\phi(y-x)z} - 1 - \mathrm{i}\phi(y-x)z\}\, \ell(dz)\, dy \quad = \int_{\mathbb{R}^d}\int_{\mathbb{R}\setminus\{0\}} \{\mathrm{e}^{\mathrm{i}\phi(y)z} - 1 - \mathrm{i}\phi(y)z\}\, \ell(dz)\, dy\,.$$

Similarly we have that $|\phi|_2 = |\phi_x|_2$. Hence, the characteristic function of N_ϕ and of N_{ϕ_x} is the same. By a limit argument, it follows that N_A and N_{A+x} has the same characteristic function as well, implying that their distributions are coinciding. The Proposition is proved. □

In Lévy noise analysis, one is interested in the noise process of the smoothed random variables N_ϕ. Introduce the object $\dot{N}_x$ for $x \in \mathbb{R}^d$ by

$$\dot{N}_x(\omega) = \langle \omega, \delta_x \rangle\,, \tag{42}$$

where δ_x is the Dirac δ–function. Obviously, δ_x is not an element of $L^2(\mathbb{R}^d)$ (and definitely not a Schwartz function), however, it is a tempered distribution. The notation $\langle \omega, \delta_x \rangle$ is just suggestive, since it only makes sense in an operator context as we now discuss. By conveniently introducing spaces of *smooth* random variables as certain subspaces of $L^2(P)$ one can look at their duals and in fact manage to embed $\dot{N}_x$ into one of these. Thus, if X is a smooth random variable, then $\dot{N}_x$ makes sense as a linear functional on this (we refer to [31] for details). As a simple example, we have that N_ϕ is a smooth random variable, and in this case

$$\langle\langle \dot{N}_x, N_\phi \rangle\rangle = \langle \delta_x, \phi \rangle = \phi(x)\,.$$

From this we can do the following: Interpreting the integral in the sense of Pettis or Bochner, we can define for $\phi \in \mathcal{S}(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \phi(x)\dot{N}_x\, dx\,, \tag{43}$$

as an integral with values in a suitable space of linear functionals on smooth random variables. However, as it turns out, this integral will coincide with a smoothed white noise,

$$\int_{\mathbb{R}^d} \phi(x)\dot{N}_x \, dx = N_\phi \, .$$

But then we can interpret $\dot{N}_x$ as the noise of Λ, since we can write by limit arguments

$$\Lambda(A) = \int_A \dot{N}_x \, dx \, .$$

So, $\dot{N}$ is an extension of the previously introduced object $\dot{\Lambda}$. Note that there is no nuclear condition given here in order to introduce $\dot{N}_x$. Indeed, we have that

$$\int_{\mathbb{R}^d} \phi(x)\Lambda(dx) = \int_{\mathbb{R}^d} \phi(x)\dot{N}_x \, dx = N_\phi,$$

for a function $\phi \in L^2(\mathbb{R})$, and thus,

$$\sum_{k=1}^{\infty} \mathbb{E}\left[\left(\int_{\mathbb{R}^d} e_k(x)\,\Lambda(dx)\right)^2\right] = \sum_{k=1}^{\infty} \mathbb{E}[N_{e_k}^2] = \sum_{k=1}^{\infty} |e_k|_2^2 = \sum_{k=1}^{\infty} 1 = \infty \, .$$

Here, $\{e_k\}$ is a complete orthonormal system in $L^2(\mathbb{R}^d)$. Hence, we have that the nuclear covariance condition does not hold. This means that we have a Lévy basis which has finite variance, but is not sufficiently smooth to admit a Hilbert space valued Radon–Nikodym derivate $\dot{N}_x$. This links Lévy bases to the Lévy noise analysis.

4.2 Stochastic partial differential equations and Lévy noise analysis

Consider the stochastic Poisson equation

$$\begin{aligned} \Delta u(x) &= -\dot{N}_x \, , \qquad && x \in D \\ u(x) &= 0 \, , && x \in \partial D \, , \end{aligned}$$

where $D \subset \mathbb{R}^d$ is a bounded domain with regular boundary and Δ is the Laplace operator in $\mathbb{R}^d$. In order to make sense out of this equation, Holden *et al.* [31] introduce the space of Hida distributions $(\mathcal{S})^*$, which plays much the same role for stochastic processes as the space of tempered Schwartz distributions plays for functions. The space of Hida distributions, is the dual of the space of Hida test functions $(\mathcal{S})$, which is the space of smooth random variables. This space consists of square integrable random variables for which the terms in the chaos expansion decays rapidly in variance. A precise definition of $(\mathcal{S})$ and $(\mathcal{S})^*$ is found in Holden *et al.* [31], but important to notice is that $(\mathcal{S})^*$ consists of linear operators on the space $(\mathcal{S})$, and as such can not be understood as random variables (i.e., if $X \in (\mathcal{S})^*$, $X(\omega)$ does not make sense in general for $\omega \in \Omega$). A prominent example is $\dot{N}_x \in (\mathcal{S})^*$. As is well–known, the noise of a Lévy process can not be regarded as a classical random variable.

The Poisson equation is interpreted as an SPDE in $(\mathcal{S})^*$. More precisely, we say that u is a generalised solution of the stochastic Poisson equation if $u : \overline{D} \mapsto (\mathcal{S})^*$ is twice differentiable, satisfies the boundary conditions and the SPDE. By differentiability of a $(\mathcal{S})^*$–valued mapping from D we mean that the limit $(u(x+h) - u(x))/h$ exists in $(\mathcal{S})^*$.

Letting $G(x, y)$ be the Green's function of Δ on D with zero boundary conditions, Løkka, Øksendal and Proske [35] shows that the unique solution is

$$u(x) = \int_D G(x, y)\dot{N}_y \, dy \,. \tag{44}$$

Note that the integral is interpreted as a Pettis integral, that is, defining an operator on the space of smooth random variables $(\mathcal{S})$. If $d \leq 3$, it is shown in Løkka *et al.* [35] that $u \in L^2(P)$, but in general dimensions we have to interpret the solution in a weak sense.

Since for $d \leq 3$ the solution u is square–integrable, we may write the solution as

$$u(x) = \int_D G(x, y) \, \Lambda(dy) \,. \tag{45}$$

Therefore, u is in fact an ambit process, with the ambit set being the domain D. The reason for u losing its square–integrability when going beyond dimension 3 lies in the fact that $G(x, y)$ has a singularity at $x = y$ of order $|x - y|^{2-d}$ for $d \geq 3$. By using ambit processes, we may define more general expressions

$$\widetilde{u}(x) = \int_{D_x} G(x, y)\sigma(y)\Lambda(dy) \,, \tag{46}$$

for general random fields $\sigma(x)$ sufficiently regular to make the stochastic integral well–defined. The set D_x denotes some ambit set which can be defined to incorporate complex spatial dependency structures. In fact, such a specification $\widetilde{u}(x)$ may go beyond what can be linked to a stochastic partial differential equation, and still make sense as a random field (in particular, a real–valued random field).

Note that the theory of white noise permits the study of SPDEs driven by noise in both time and space, and provide a theory for defining the noise of Lévy processes (or, in our context, Lévy bases). Hence, one can interpret the SPDEs in a strong sense, with the price that the solutions must be understood as operators rather than random fields. This is in contrast to the theory of Walsh presented above, where the solution is formulated in terms of an integral equation moving all derivatives to test functions. Ambit processes appear as a natural object in the theory of Lévy noise, as well.

5 Conclusions

We have considered ambit processes and their building blocks, Lévy bases, in view of two classical theories for studying stochastic partial differential equations: the Walsh theory of martingale measures and the Lévy noise analysis. Lévy bases can be naturally connected to both theories by introducing concepts of noise of Lévy bases and processes. We show that the solutions of some stochastic partial differential equations can be represented by integrals of random fields with respect to Lévy bases, naturally relating to ambit processes. In this respect, ambit processes provide a class of random fields which generalise the solutions of these physical dynamical systems and provide new and interesting models that include the additional elements of volatility fields and time dependent ambit sets. A further key point is that the extended integration theory allows the handling of objects such as the main term in (3) by means of integration w.r.t. martingale measures.

A Lévy bases and integration

This section reviews the integration theory of [38] (for a survey, see also [40]), since this concept of integration is used for defining stochastic integrals in the context of ambit fields.

A.1 Introduction

Throughout the text, let S denote a non–empty set and let $\mathcal{A}$ denote a σ–finite δ–ring on S, i.e. $\mathcal{A}$ is a family of subsets of S such that for every pair of sets in $\mathcal{A}$, the union, the intersection and the set difference is in $\mathcal{A}$ (hence $\mathcal{A}$ is a ring) and if $(A_n)_{n\geq 1} \subseteq \mathcal{A}$ then $\cap A_n \in \mathcal{A}$; also, there exists a sequence $(A_n^*)_{n\geq 1} \subseteq \mathcal{A}$ such that $\cup A_n^* = S$.

Note that we call a real stochastic process $\Lambda = \{\Lambda(A) : A \in \mathcal{A}\}$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ an *independently scattered random measure*, if for every sequence of disjoint sets, $(A_n)_{n\geq 1}$ say, the random variables $\Lambda(A_n)$ are independent, for $n = 1, 2, \ldots$, and if $\cup_n A_n \in \mathcal{S}$, then $\Lambda(\cup_n A_n) = \sum_n \Lambda(A_n)$ almost surely.

A.2 Representation of the characteristic function of a Lévy basis

If $\Lambda(A)$ is infinitely divisible for every $A \in \mathcal{A}$, we call it a *Lévy basis*. Its characteristic function for $A \in \mathcal{A}$ is then given by

$$\mathbb{E}\left(\exp(it\Lambda(A))\right) = \exp\left(it\nu_0(A) - \frac{1}{2}t^2\nu_1(A) + \int_{\mathbb{R}} \left(e^{itx} - 1 - it\tau(x)\right) F_A(dx)\right), \quad (47)$$

where $\nu_0 : \mathcal{S} \to \mathbb{R}$ is a signed measure, $\nu_1 : \mathcal{A} \to [0, \infty)$ is a measure and F_A is a Lévy measure on $\mathbb{R}$ for every $A \in \mathcal{A}$ while $A \mapsto F_A(B) \in [0, \infty)$ is a measure for every $B \in \mathcal{B}(\mathbb{R})$, whenever $0 \notin \overline{B}$. Also, the centering function τ is defined by $\tau(x) = x$ if $||x|| \leq 1$ and by $\tau(x) = x/||x||$, if $||x|| > 1$.

Further, let

$$\lambda(A) = |\nu_0|(A) + \nu_1(A) + \int_{\mathbb{R}} \min(1, x^2) F_A(dx), \quad A \in \mathcal{A}.$$

It can be shown that $\lambda : \mathcal{A} \to [0, \infty)$ is a measure on $\mathcal{A}$ such that if, for every $(A_n)_{n\geq 1} \subset \mathcal{A}$, $\lambda(A_n) \to 0$, then $\Lambda(A_n) \to 0$ in probability. Also, if, for every sequence $(A'_n)_{n\geq 1} \subset \mathcal{A}$ with $A'_n \subset A_n \in \mathcal{A}$, we have $\Lambda(A'_n) \to 0$ in probability, then $\lambda(A_n) \to 0$.

Note that the measure λ satisfies $\lambda(A_n^*) < \infty$ for $n = 1, 2, \ldots$. Hence, it can be extended to a σ–finite measure on $(S, \sigma(\mathcal{A}))$. This measure is then called the *control measure* of Λ.

It turns out, that the characteristic function of an infinitely divisible random measure has also an alternative representation than the one given above.

In order to state it, we need a preliminary result first (see [38, Lemma 2.3]). Let $F_\cdot$ be as above. Then there exists a unique σ–finite measure F on $\sigma(\mathcal{A}) \times \mathcal{B}(\mathbb{R})$ such that $F(A \times B) = F_A(B)$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}(\mathbb{R})$. Furthermore, there exists a function $\rho : S \times \mathcal{B}(\mathbb{R}) \to [0, \infty]$ such that

1. $\rho(s, \cdot)$ is a Lévy measure on $\mathcal{B}(\mathbb{R})$, for every $s \in S$,
2. $\rho(\cdot, B)$ is a Borel measurable function, for every $B \in \mathcal{B}(\mathbb{R})$,
3. $\int_{S\times\mathbb{R}} h(s, x) F(ds, dx) = \int_S \left(\int_{\mathbb{R}} h(s, x)\rho(s, dx)\right) \lambda(ds)$, for every $\sigma(\mathcal{A}) \times \mathcal{B}(\mathbb{R})$–measurable function $h : S \times \mathbb{R} \to [0, \infty]$. Under some restrictions regarding the behaviour at $\pm\infty$, this equality can be extended to real and complex–valued functions h.

Using the above notation, we can now rewrite the characteristic function of $\Lambda(A)$ (see [38, Proposition 2.4]):

$$\mathbb{E}\left(\exp\left(it\Lambda(A)\right)\right) = \exp\left(\int_A K(t, s)\lambda(ds)\right), \quad t \in \mathbb{R}, A \in \mathcal{A}, \quad (48)$$

where

$$K(t,s) = ita(s) - \frac{1}{2}t^2\sigma^2(s) + \int_{\mathbb{R}} \left(e^{itx} - 1 - it\tau(x)\right)\rho(s,dx),$$

where $a(s) = \frac{d\nu_0}{d\lambda}(s)$, $\sigma^2(s) = \frac{d\nu_1}{d\lambda}(s)$ and ρ is defined as above. Furthermore,

$$|a(s)| + \sigma^2(s) + \int_{\mathbb{R}} \min(1,x^2)\rho(s,dx) = 1, \quad \lambda - a.e..$$

A.3 Integration with respect to a Lévy basis

Next, we review the definition of a stochastic integral with respect to an infinitely divisible random measure Λ as defined in [38].

First, we define integration of a real simple function on S, which is given by $f = \sum_{j=1}^n x_j 1_{A_j}$ for disjoint $A_j \in \mathcal{A}$. Then, for every $A \in \sigma(\mathcal{A})$, the stochastic integral with respect to Λ is defined by

$$\int_A f d\Lambda = \sum_{j=1}^n x_j \Lambda(A \cap A_j).$$

The generalisation to general functions works as follows. We call a measurable function $f : (S, \sigma(\mathcal{A})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ Λ–integrable, if there exists a sequence of simple functions, $(f_n)_{n\geq 1}$ such that $f_n \to f$ λ–a.e. and for every $A \in \sigma(\mathcal{A})$, the sequence $\left(\int_A f_n d\Lambda\right)_{n\geq 1}$ converges in probability as $n \to \infty$. In that case, we define

$$\int_A f d\Lambda = \mathbb{P} - \lim_{n\to\infty} \int_A f_n d\Lambda.$$

The above integral is well defined in the sense that it does not depend on the approximating sequence $(f_n)_{n\geq 1}$.

A.4 Criteria for integrability

Now we provide a characterisation of Λ–integrable functions. The necessary and sufficient conditions will depend on the characteristics given in the Lévy form of the characteristic function of Λ.

According to [38, Theorem 2.7], the integrability conditions are as follows.

Let $f : S \to \mathbb{R}$ be a $\sigma(\mathcal{A})$–measurable function. Then f is integrable w.r.t. Λ if and only if the following three conditions are satisfied:

1. $\int_S |U(f(s), s)|\lambda(ds) < \infty$,
2. $\int_S |f(s)|^2\sigma^2(s)\lambda(ds) < \infty$, and
3. $\int_S V_0(f(s), s)\lambda(ds) < \infty$, where

$$U(u,s) = ua(s) + \int_{\mathbb{R}} (\tau(xu) - u\tau(x))\,\rho(s,dx),$$
$$V_0(u,s) = \int_{\mathbb{R}} \min(1, |xu|^2)\rho(s,dx).$$

Further, if f is integrable w.r.t. Λ, then the characteristic function of $\int_S f d\Lambda$ can be expressed as

$$\mathbb{E}\left(\exp\left(it\int_S f d\Lambda\right)\right) = \exp\left(ita_f - \frac{1}{2}t^2\sigma_f^2 + \int_{\mathbb{R}}\left(e^{itx} - 1 - it\tau(x)\right)F_f(dx)\right),$$

where

$$a_f = \int_S U(f(s), s)\lambda(ds), \qquad \sigma_f = \int_S |f(s)|^2\sigma^2(s)\lambda(ds),$$

and

$$F_f(B) = F(\{(s,x) \in S \times \mathbb{R} : f(s)x \in B \setminus \{0\}\}), \quad B \in \mathcal{B}(\mathbb{R}).$$

Acknowledgement

We would like to thank Andreas Basse–O'Connor and Jan Pedersen for helpful discussions and constructive comments. Financial support by the Center for Research in Econometric Analysis of Time Series, CREATES, funded by the Danish National Research Foundation is gratefully acknowledged by the third author.

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