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Local Whittle estimation of multivariate fractionally integrated processes

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Abstract

This paper derives a semiparametric estimator of multivariate fractionally integrated processes covering both stationary and non-stationary values of d. We utilize the notion of the extended discrete Fourier transform and periodogram to extend the multivariate local Whittle estimator of Shimotsu (2007) to cover non-stationary values of d. We show consistency and asymptotic normality for $d \in (-1/2, \infty)$. A simulation study illustrates the performance of the proposed estimator for relevant sample sizes. Empirical justification of the proposed estimator is shown through an empirical analysis of log spot exchange rates. We find that the log spot exchange rates of Germany, United Kingdom, Japan, Canada, France, Italy, and Switzerland against the US Dollar for the period January 1974 until December 2001 are well decribed as $I(1)$ processes.

Keywords: fractional integration, local Whittle, long memory, multivariate semiparametric estimation, exchange rates.

JEL Classification: C14, C32

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1 Introduction

This paper considers semiparametric estimation of the multivariate extension of the scalar fractionally integrated process as analyzed in Shimotsu (2007). Shimotsu (2007) recently introduced a Gaussian semiparametric estimator of multivariate stationary fractionally integrated processes, i.e. $I(d)$ processes, by extending the work by Robinson (1995*a*) on the univariate local Whittle (LW) estimator initially proposed by Künsch (1987), to cover multivariate fractional processes.

The contribution of the paper is to establish consistency and asymptotic normality when considering potentially non-stationary multivariate fractionally integrated processes. It is an important topic as evident from the number of papers in economics that have derived (mostly in the univariate setting) estimators that are robust to non-stationary values of d. In the scalar case a common semiparametric estimator is the LW estimator. Robinson (1995*a*) shows its consistency and asymptotic normality for $d \in (-1/2, 1/2)$. Velasco (1999*a*) extended Robinson's (1995*a*) results to show that the estimator is consistent for $d \in (-1/2, 1)$ and asymptotically normally distributed for $d \in (-1/2, 3/4)$, given that the fractional process is of Type I, see Marinucci & Robinson (1999) for a definition of Type I and Type II fractional processes. Phillips & Shimotsu (2004) show that the LW estimator is consistent for $d \in (1/2, 1]$ and has a nonnormal limit distribution for $d \in (3/4, 1)$, and a mixed normal limit distribution for $d = 1$. When $d > 1$ the LW estimator converges to unity in probability and therefore is inconsistent, given that the fractional process is of Type II, Phillips & Shimotsu (2004). This convergence in probability to unity when $d > 1$ also holds for log periodogram estimators as shown in simulations studies by Hurvich & Ray (1995) and Velasco (1999*b*), and theoretically by Kim & Phillips (2006). That is, in general the LW (or log periodogram) estimator is not a good general purpose estimator when d takes on values in the non-stationary region beyond 3/4. The asymptotic theory is discontinuous at $d \in \{3/4, 1\}$ and the estimator is not consistent for $d > 1$. Several methods are available to avoid the problems when entering the non-stationary region. A simple one is to first difference the series before using the semiparametric estimator and then add one to the estimate (or fractional differencing). Tapering the data is another method often implemented and suggested, see Velasco (1999*a*) and Hurvich & Chen (2000). Shimotsu & Phillips (2005) introduce what they call an exact local Whittle estimator which is consistent and has the same $N(0, 1/4)$ limit distribution for all values of d if the $I(d)$ series is generated by a linear sequence and the range of the estimator is not wider than 9/2.[1] Instead of using fractional differencing of the data, Abadir, Distaso & Giraitis (2007) use a different approach first noted by Phillips (1999). They extend the discrete Fourier transform to the non-stationary case and use this in whitening of the periodogram. Abadir et al. (2007) show that when the $I(d)$ series is generated by a linear sequence the extended discrete Fourier transform and periodogram have the same asymptotic behavior for $d \in (-3/2, \infty)$. In the context of multivariate estimation of long memory processes Lobato (1999) derived a semiparametric two-step estimator in a multivariate long memory model. Shimotsu (2007) instead used a more general form of the spectral density and from this derive a semiparametric estimator of multivariate fractionally integrated processes. The class of spectral densitites included in Shimotsu's (2007) specification in-

[1] The assumption concerning the width of the admissible parameter space is needed to ensure that the difference in the criteria function is uniformly bounded away from zero, see Shimotsu & Phillips (2005).

cludes those of multivariate fractionally integrated processes, whereas the specification used in Lobato (1999) is an alternate local form of the spectral density that neglects the information in phase shifts and which will lead to less efficient estimates of the integration orders. Shimotsu (2007) notes that there is no apparent realizable time domain model which has the spectral density representation that Lobato (1999) uses (except the cases where G, see later for definition, is diagonal implying no long-run covariance between the variables of interest). Shimotsu (2007) shows that the estimator of Lobato (1999) is consistent given the more precise sprectral density representation, but the limiting distribution is more evolved. Therefore, it follows that the estimator of Shimotsu (2007) has a smaller limiting distribution than the two-step estimator of Lobato (1999). Lobato & Velasco (2000) extended the results of Lobato (1999) by using tapering, and thereby allowing for non-stationary values of d and potential trends in the data generating process. In this paper, we focus on the general local form of the spectral density employed by Shimotsu (2007) and extend his results to cover non-stationary values of d by using the notion of the extended discrete Fourier transform and periodogram as in Abadir et al. (2007). We call the new estimator the extended multivariate local Whittle (ExtMLW) estimator. Given that the generating process is linear, the same central limit theorem argument as in the stationary case $|d| < \frac{1}{2}$ derived by Robinson (1995a) (for the univariate case) and Shimotsu (2007) (for the multivariate case) holds; although, not for $d = \left\{\frac{1}{2}, \frac{3}{2}, ...\right\}$. We establish consistency and asymptotic normality for $d \in (-1/2, \infty)$. In addition, we could also have shown consistency and asymptotic normality for non-stationary values of d by extending the results of Shimotsu (2007) either by fractional differencing (Shimotsu & Phillips (2005)) or tapering (Velasco (1999a)). The reason for not using fractional differencing in setting up the likelihood was that we want to stay in the setup of Shimotsu (2007). Furthermore, it is shown by Abadir et al. (2007) in their simulations that when there is no trends in the data generating process there is an efficiency gain over tapering in using the extended discrete Fourier transform.

In an empirical application of the proposed multivariate semiparametric estimator we analyze the long range dependence of log spot exchange rates. In the case of the log spot exchange rates of Germany, United Kingdom, Japan, Canada, France, Italy, and Switzerland against the US Dollar for the period January 1974 until December 2001, measured on a monthly basis, we find that they are well decribed as $I(1)$ processes and that there is a high degree of coherence.

The remainder of the paper is structured as follows: Section 2 gives a short introduction to the multivariate semiparametric estimation of multivariate fractionally integrated processes. Section 3 expands the usual stationary framework to the non-stationary framework and thereby defining our proposed estimator. Section 4 and 5 derives consistency and the Gaussian limiting distributional results. Section 6 presents the results from a small simulation study. Section 7 contains an empirical investigations of potential long memory properties in exchange rates. Section 8 concludes. Proofs to Theorem 1, 2, and Lemma 1 are situated in Appendix A.

2 Multivariate local Whittle estimation

Consider the spectral density representation of the following covariance stationary $q-$vector process

$$\begin{pmatrix} (1-L)^{d_1} & & 0 \\ & \ddots & \\ 0 & & (1-L)^{d_q} \end{pmatrix} \begin{pmatrix} X_{1t} - E[X_{1t}] \\ \vdots \\ X_{qt} - E[X_{qt}] \end{pmatrix} = \begin{pmatrix} u_{1t} \\ \vdots \\ u_{qt} \end{pmatrix}, \quad t = 1, ..., n, \tag{1}$$

where $(1-L)^{d_a}$ is the fractional difference operator defined by its binomial expansion in the lag operator (see e.g. Hosking (1981)), $d_a \in (-1/2, 1/2)\,, a = 1, ..., q$, and u_t is covariance stationary with spectral density matrix that is finite and bounded away from zero at the origin, i.e. $I(0)$. This induces the following spectral density representation, see Shimotsu (2007)

$$f(\lambda) \sim \Lambda(\lambda)\, G \Lambda^*(\lambda)\,, \text{ as } \lambda \to 0^+, \tag{2}$$

where the * denotes the conjugate transpose, $\Lambda(\lambda) = diag(\Lambda_{aj}(d))$, $\Lambda_{aj}(d) = e^{i(\pi-\lambda)d_a/2}\lambda^{-d_a}$ and G is a real, symmetric, finite, and positive definite matrix.[2] We note that (2) differs from the representation used in e.g. Lobato (1999) where

$$\tilde{f}_{ab}(\lambda) \sim G_{ab}\lambda^{-da-d_b}, \text{ as } \lambda \to 0^+, \tag{3}$$

where G_{ab} is the (a,b)th element of G (the long-run covariance matrix). $f(\lambda)$ has a non-zero complex part even at the origin unless the integration orders are equal, i.e. $d_a = d_b$. More specifically, we have from (2) that

$$f_{ab}(\lambda) \sim G_{ab}\lambda^{-d_a-d_b} e^{i(\pi-\lambda)(d_a-d_b)/2}, \text{ as } \lambda \to 0^+. \tag{4}$$

That is, the integration orders appear in both the power decay and the phase shift. Therefore, the phase spectrum of X_{at} and X_{bt} is nonzero and depends on d_a and d_b even at the zero frequency. Neglecting the information in phase shifts will lead to less efficient estimation of the integration orders. The two representations are identical when G is itself diagonal in which case there is no long-run covariance between the elements in X_t or when $d_a = d_b$, $a, b = 1, ..., q$. See Shimotsu (2007) and Robinson (2008) for detailed comparison between $f_{ab}(\lambda)$ and $\tilde{f}_{ab}(\lambda)$.

We can write the Gaussian log-likelihood localized to the origin where we have concentrated G out as

$$L_n(d) = \log\det \hat{G}(d) - 2\sum_{a=1}^{q} d_a \frac{1}{m}\sum_{j=1}^{m}\log\lambda_j, \quad \hat{G}(d) = \frac{1}{m}\sum_{j=1}^{m}\mathrm{Re}\left\{\Lambda_j^{-1} I(\lambda_j)\left(\Lambda_j^*\right)^{-1}\right\}, \tag{5}$$

where we denote the true parameter values by d^0 and G^0. Furthermore, the space of admissible estimates of d^0 is $D = [\Delta_1, \Delta_2]^q$, with $-1/2 < \Delta_1 < \Delta_2 < 1/2$, $m = o(n)$ is a bandwidth number which tends to infinity as $n \to \infty$, but at a slower rate than n, $\lambda_j = 2\pi j/n$ are the Fourier frequencies, and $I(\lambda) = w(\lambda)\, w^*(\lambda)\,, w(\lambda) = \frac{1}{\sqrt{2\pi n}}\sum_{t=1}^{n} X_t e^{it\lambda}$ is the periodogram matrix. Note that the estimator is invariant to a possible non-zero mean since $j = 0$ is left out of the minimization and it enjoys robustness to short-memory dynamics since it uses only information from the periodogram ordinates

[2]Note that the condition of positive definiteness of G rules out the possibility of cointegration.

in the vicinity of the origin. Additionally, we note that the difference between the Lobato (1999) and Shimotsu (2007) multivariate estimator is in the definition of $\Lambda(\lambda)$.

Lobato (1999) estimates the long memory estimates in two steps (based on (2) where $\Lambda(\lambda) = diag\left(\lambda^{-d_a}\right)$), i.e.

$$\tilde{d}^{(2)} = \tilde{d}^{(1)} - \left(\left.\frac{\partial^2 L(d)}{\partial d \partial d'}\right|_{\tilde{d}^{(1)}} \right)^{-1} \left(\left.\frac{\partial L(d)}{\partial d}\right|_{\tilde{d}^{(1)}} \right),$$

where $\tilde{d}^{(1)}$ is the vector of univariate local Whittle estimates of Robinson (1995a) and the estimate of the long-run covariance matrix is $\tilde{G}\left(\tilde{d}^{(2)}\right)$. Given the spectral density representation in Lobato (1999), the distribution of $\tilde{d}^{(2)}$ is extremely simple

$$\sqrt{m}\left(\tilde{d}^{(2)} - d^0\right) \xrightarrow{d} N\left(0, E^{-1}\right), \tag{6}$$

where $E = 2\left(G^0 \circ \left(G^0\right)^{-1} + I_2\right)$, $\circ$ denotes the Hadamard product and e.g. in the bivariate case the asymptotic variance is

$$E^{-1} = \frac{1}{8}\begin{pmatrix} 2-c^2 & c^2 \\ c^2 & 2-c^2 \end{pmatrix},$$

where $c^2 = \frac{G_{12}^2}{G_{11}G_{22}}$ is the squared coherence as $\lambda \to 0^+$. Since, we in the univariate case have $E^{-1} = 1/4$, the efficiency increase is $c^2/8$. Shimotsu (2007) establishes consistency and asymptotic normality of the two-step estimator of Lobato (1999). Shimotsu (2007) finds that $\tilde{d}$ is consistent even though the estimator is based on a misspecified model, i.e. where $\Lambda = diag\left(\lambda^{-d_a}\right)$ in (2). The asymptotic variance of $\tilde{d}$ is more complicated because

$$\tilde{G}(d) \xrightarrow{p} \tilde{G}^0, \quad \text{as } n \to \infty,$$

where

$$\begin{aligned}\tilde{G}(d) &= \frac{1}{m}\sum_{j=1}^{m} \mathrm{Re}\left[diag\left(\lambda_j^{d_a}\right) I(\lambda_j)\, diag\left(\lambda_j^{d_a}\right)\right], \\ \tilde{G}^0 &= \mathrm{Re}\left[diag\left(e^{i\pi d_a^0/2}\right) G^0 diag\left(e^{i\pi d_a^0/2}\right)^*\right],\end{aligned}$$

and hence is an inconsistent estimate of G^0. This is because we omit the phase shift in the spectral density representation and therefore $\tilde{G}^0$ underestimates the true off-diagonal elements of G^0. If $d_1^0 = ... = d_q^0$, (3) and (4) are identical, and the asymptotic variance for the two different multivariate estimators coincide.

Shimotsu (2007) establishes consistency and asympotic normality under the spectral density representation (2), i.e

$$\sqrt{m}\left(\hat{d} - d_0\right) \xrightarrow{d} N\left(0, \Omega^{-1}\right), \quad \Omega = 2\left[G^0 \circ \left(G^0\right)^{-1} + I_2 + \frac{\pi^2}{4}\left(G^0 \circ \left(G^0\right)^{-1} - I_2\right)\right].$$

Since $G^0 \circ \left(G^0\right)^{-1} - I_2$ is a positive semi-definite matrix, $\hat{d}$ has a smaller limiting variance matrix than $\tilde{d}$, except of course when G^0 is diagonal.

In the next section, we will extend the results of Shimotsu (2007) to cover potential non-stationarity.

3 The extended multivariate local Whittle estimator

In this paper we use the framework of Abadir et al. (2007) to expand the setup of Shimotsu (2007) to cover non-stationary values of d, and therefore we define a fractional process as a Type I process, see Marinucci & Robinson (1999) for a thorough description of this type of $I\,(d)$ process compared to other types. Because we are not only interested in the stationary region, it is not enough just to expand the filter $(1-L)^d$ and express it as an infinite order moving average of the innovations which results in a stationary process when $d < 1/2$. When we move into the non-stationary region, i.e. $d \geq 1/2$, this procedure breaks down because the infinite order moving average of the innovations does not converge. This is circumvented by modeling the process as the partial sum of the component $I(d-p)$ process for some $p \in \mathbb{Z}$ and expanding $(1-L)^{p-d}$ in terms of the innovations. This results in a stationary integer differenced series. The disadvantage is that it introduces discontinuities at $d = 1/2, 3/2, ..p - 1/2$, where $p \in \mathbb{Z}$. Therefore, we expand Definiton 1 in Abadir et al. (2007) and Nielsen (2008) to the multivariate setup.

Definition 1 *For $d = p + d^u$, where $p \in \mathbb{Z}^q$ and $d^u \in (-1/2, 1/2)^q$, we say that $\{X_t\}$ is a matrix of $I(d)$ processes, i.e. $X_t \sim I(d)$, if*

$$diag\left((1-L)^{p_a}\right) X_t = u_t, \quad t = 1-p, 2-p, ..., \tag{7}$$

for $a = 1, ..., q$ and u_t is second order stationary with spectral density for the (a,b)th element $a, b = 1, ..., q$

$$f^u_{ab}(\lambda) = e^{i(\pi-\lambda)\left(d^u_a - d^u_b\right)/2} G^0_{ab} \lambda^{-d^u_a - d^u_b} + o\left(\lambda^{-d^u_a - d^u_b}\right), \quad as\ \lambda \to 0^+, \tag{8}$$

where G^0 is a real, symmetric, finite, and positive definite matrix.

Define the extended DFT and the extended periodogram matrix of $\{X_t\}$ evaluated at the Fourier frequencies $\lambda_j = \frac{2\pi j}{n}$, where $j = 1, ..., n$, by

$$\begin{aligned} w_j\,(d) &= w(\lambda_j, d) = w^x(\lambda_j) + c(\lambda_j, d), & (9)\\ I_j\,(d) &= I(\lambda_j, d) = w(\lambda_j, d) w^*(\lambda_j, d), & (10)\end{aligned}$$

where $w^x(\lambda_j)$ is the usual DFT defined as

$$w_x(\lambda_j) = \frac{1}{\sqrt{(2\pi n)}} \sum_{t=1}^{n} X_t e^{it\lambda_j}, \tag{11}$$

and the correction term for the ath element $c_a(\lambda_j, d)$ takes on constant values on the intervals $d_a \in D_{p_a} := [p_a - 1/2, p_a + 1/2)$, $p_a \in \mathbb{N}_0$, $a = 1, ..., q$ and is defined by

$$c_a(\lambda_j, d) = \begin{cases} 0 \text{ if } d_a \in D^0 = [-1/2, 1/2), \\ e^{i\lambda_j} \sum_{\ell=1}^{p_a} (1 - e^{i\lambda_j})^{-\ell} Z_{a\ell} \text{ if } d_a \in D_{p_a} \quad \text{for } p_a = 1, 2, ..., \end{cases} \tag{12}$$

and $a = 1, ..., q$ where

$$\begin{aligned} Z_{a0} &= w_{ax}(0) = \frac{1}{\sqrt{(2\pi n)}} \sum_{t=1}^{n} X_{at}, & (13)\\ Z_{a\ell} &= \frac{1}{\sqrt{(2\pi n)}} \left\{(1-L)^{\ell-1} X_{an} - (1-L)^{\ell-1} X_{a0}\right\}, \quad \ell = 1, 2, ..., p_a. & (14)\end{aligned}$$

In the computation of the step function $c_a(\lambda_j, d)$, we have to enumerate the data depending on what subspace of $D = [d_1, d_2]^q$ we are interested in. This is apparent from looking at (14), for example when $p_a = 2$. That is, $X_{a,-i+1}, X_{a,-i+2}, ..., X_{an}$ where $i = (0 \vee \lfloor d_2 - 1/2 \rfloor)$ and $a = 1, ..., q$. The usual DFT, (11) is always computed using the enumeration $\{X_t\}_{t=1}^n$.

This notion of the extended DFT allows us to estimate the usual MLW estimator in the context of non-stationary values for d by minimizing the criteria function defined as (5) over the admissible parameter space. The extension of the DFT to the non-stationary case is based on the work of Phillips (1999), Lahiri (2003), Dalla, Giraitis & Hidalgo (2006) and Abadir et al. (2007). Define the pseudo spectral density of the (a, b)th element of the sequence $\{X_t\} \sim I(d^0)$, where $d^0 = p^0 + d^u$ and $d^u \in (-1/2, 1/2)^q$ as

$$f_{ab}(\lambda) = |1 - \exp(i\lambda)|^{-p_a^0 - p_b^0} f_{ab}^u(\lambda), \quad |\lambda| \leq \pi. \tag{15}$$

From this definition it is clear that

$$f_{ab}(\lambda) \sim G_{ab} \lambda^{-d_a^0 - d_b^0} e^{i(\pi - \lambda)\left(d_a^0 - d_b^0\right)/2}, \text{ as } \lambda \to 0^+. \tag{16}$$

Then following Abadir et al. (2007, Lemma 4.4), Definition 1, and (9), the extended DFT has the property that for the ath element of $w(\lambda_j, d^0)$

$$w_a(\lambda_j, d_a^0) = (1 - \exp(i\lambda_j))^{-p_a^0} \omega_a^u(\lambda_j), \quad a = 1, ..., q, \; j = 1, ..., n, \tag{17}$$

where $\omega_a^u(\lambda_j)$ is the DFT of the ath stationary sequence of u_t. From Abadir et al. (2007, Lemma 4.4(i)), it follows that

$$\begin{aligned} w_a^x(\lambda_j) &= (1 - \exp(i\lambda_j))^{-p_a^0} w_{\Delta^{p_a^0} x_a}(\lambda_j) - \exp(i\lambda_j) \sum_{r_a=1}^{p_a^0} (1 - \exp(i\lambda_j))^{-r_a} w_{\Delta^{r_a} x_a} && (18) \\ &= (1 - \exp(i\lambda_j))^{-p_a^0} w_a^u(\lambda_j) - \exp(i\lambda_j) \sum_{r_a=1}^{p_a^0} (1 - \exp(i\lambda_j))^{-r_a} w_{\Delta^{r_a} x_a}, \quad a = 1, ..., q, && (19) \end{aligned}$$

where the second equality follows from Definition 1. Then the definition in (9) follows trivially. Denote the rescaled extended DFT for the ath element by

$$v_{aj} = v_a(\lambda_j, d_0) = \frac{w_a\left(\lambda_j, d_a^0\right)}{\left(G_{aa}^0\right)^{1/2} e^{i(\pi - \lambda_j) d_a^0/2} \lambda_j^{-d_a^0}}, \quad a = 1, ..., q, \; 1 \leq j \leq m, \tag{20}$$

where G_{aa}^0 denotes the (a, a)th element of G^0. Given that the generating process is linear, equation (17) and Lemma 4.6 in Abadir et al. (2007) show that the asymptotic behavior of the rescaled extended DFT and periodogram is the same for all $d_a^0 \in (-1/2, \infty)$ for $a = 1, ..., q$. Furthermore, given consistency, $\hat{d} \xrightarrow{p} d^0$ and the definition of the extended DFT, we get

$$w\left(\lambda_j, \hat{d}\right) \xrightarrow{p} w\left(\lambda_j, d^0\right). \tag{21}$$

This follows because $c(\lambda_j, d)$ is a step function and therefore constant on the intervals $d \in (p - 1/2, p + 1/2)^q$ for $p \in \mathbb{N}_0^q$. This considerably eases the estimation as we are left with the same estimation procedure as in the stationary case.

If the process is stationary the ExtMLW estimator is identical to the MLW estimator of Shimotsu (2007). Similarly to other semiparametric (univariate) estimators of Robinson (1995*a*), Andrews & Sun (2004), and Abadir et al. (2007) this estimator is based on the whitening principle of the periodogram. That is, similarly to the stationary case, Shimotsu (2007), the ExtMLW estimator is based on the behavior of

$$\eta_j = \eta(\lambda_j) = \frac{I^u(\lambda_j)}{f^u(\lambda_j)},\ 1 \le j \le m. \tag{22}$$

Then given the spectral density of the second order stationary sequence $\{u_t\}$,(8), it follows that (see Robinson (1995*b*, Theorem 2) and Shimotsu (2007)) for the (a,b)th element of η_j

$$E\left[\eta_{abj}\right] = 1 + O(j^{-1}\log(j)),\quad \forall\ 1 \le j \le m,\ \text{as } n \to \infty. \tag{23}$$

Additionally, under regularity assumptions, see Lahiri (2003) and Abadir et al. (2007), the random variable for the (a,b)th element also satisfy

$$var\left[\eta_{abj}\right] \le C,\quad \forall\ 1 \le j \le m,\ a,b = 1,...,q, \tag{24}$$

where C is a positive finite constant and

$$cov\left[\eta_{abj}, \eta_{abs}\right] \to 0,\ \text{for } a,b = 1,...,q,\ j,s \to \infty \text{ and } j \ne s. \tag{25}$$

In the proof to Lemma 4.6 in Abadir et al. (2007), the above equations are proven (for the univariate case. The multivariate setting follows straightforwardly).

Then given the equations (23), (24) and (25), η_j satisfy a weak law of large numbers (WLLN), i.e. for the (a,b)th element

$$\frac{1}{m}\sum_{j=1}^{m}\eta_{abj} \xrightarrow{p} 1,\quad \text{for } a,b = 1,...,q,\ \text{as } n \to \infty. \tag{26}$$

Given additional assumptions, this result is sufficient to ensure consistency of the estimator $\hat{d}$. The WLLN for the random variables η_{abj} is equivalent to a WLLN for the random variables $v_{aj}v^*_{bj}$, i.e.

$$\frac{1}{m}\sum_{j=1}^{m}v_{aj}v^*_{bj} \xrightarrow{p} 1,\quad \text{as } n \to \infty. \tag{27}$$

Then given the nature of the spectral density (8) and (17)

$$v_{aj}v^*_{bj} = \eta_{abj}\left(1 + o(1)\right),\ \forall\ 1 \le j \le m,\ a,b = 1,...,q,\ \text{as } n \to \infty. \tag{28}$$

Furthermore, given equation (23)

$$E\left[v_{aj}v^*_{bj}\right] \le C,\ \forall\ 1 \le j \le m,\ a,b = 1,...,q. \tag{29}$$

For a detailed decription of the extended DFT, see Phillips (1999), Lahiri (2003), Dalla et al. (2006), and Abadir et al. (2007).

4 Consistency

In this section, we introduce the assumptions needed to establish consistency. Let $f_{ab}^{u}(\lambda)$ and G_{ab}^{0} denote the (a,b)th element of $f^{u}(\lambda)$ and G^{0}, respectively. In general, the assumptions are multivariate extensions of Assumptions A1-A4 of Robinson (1995*a*) and Assumptions 1, A, and B of Abadir et al. (2007). They are similar to the assumptions imposed by Robinson (1995*b*), Lobato (1999), and Shimotsu (2007). Assumptions 1 and 6 are analogous to Assumptions 1 and B in Abadir et al. (2007). Assumptions 2-5 are identical to Assumptions 1-4 of Shimotsu (2007).

Assumption 1 D is a compact and convex subset of $\mathbb{R}^q$ and $d^0 \in \mathcal{D} = [d_1, d_2]^q \subset [-1/2, \infty]^q$ where $d^0 \neq p^0 - 1/2$, $p^0 \in \mathbb{N}^q$.

Assumption 2 The spectral density of u_t for the (a,b)th element is

$$f_{ab}^{u}(\lambda) = e^{i(\pi-\lambda)\left(d_a^u - d_b^u\right)/2} G_{ab}^{0} \lambda^{-d_a^u - d_b^u} + o\left(\lambda^{-d_a^u - d_b^u}\right), \quad \text{as } \lambda \to 0^+, \tag{30}$$

where $d_a^u \in (-1/2, 1/2)$ and $d_b^u \in (-1/2, 1/2)$, $a, b = 1, ..., q$.

This is a smoothness condition that imposes a rate of convergence for $f_{ab}^{u}(\lambda)$, and this is more restrictive than imposed by Robinson (1995*b*, Assumption 1).

Assumption 3 $\{X_t\}$ is generated by the linear process $\{u_t\}$

$$u_t = A(L)\varepsilon_t = \sum_{j=0}^{\infty} A_j \varepsilon_{t-j}, \quad \sum_{j=0}^{\infty} \|A_j\|^2 < \infty, \tag{31}$$

where $\|\cdot\|$ denotes the supremum norm and $E\left[\varepsilon_t | \Im_{t-1}\right] = 0$, $E\left[\varepsilon_t \varepsilon_t' | \Im_{t-1}\right] = I_q$ a.s. $\forall\ t = 0, \pm 1, \pm 2, ...$, and $\Im_{t-1}$ is the $\sigma-$ field generated by $\{\varepsilon_s : s < t\}$. Furthermore, there exists a scalar random variable ε with $E\varepsilon^2 < \infty$ such that for all $\upsilon > 0$ and some generic constant $K > 0$, $\Pr(\|\varepsilon_t\|^2 > \upsilon) \leq K \Pr(\varepsilon^2 > \upsilon)$.

Assumption 3 tells us that that X_t is generated by a linear fourth order stationary process u_t. Additionally, Assumption 3 allows for non-Gaussian processes.

Assumption 4 In some neighborhood $(0, \delta)$ of the origin $A(\lambda) = \sum_{j=0}^{\infty} A_j e^{ij\lambda}$ is differentiable and

$$\frac{d}{d\lambda} A_a(\lambda) = O\left(\|A(\lambda)\| / \lambda\right), \quad \text{as } \lambda \to 0^+, \tag{32}$$

where $A_a(\lambda)$ is the ath row of $A(\lambda)$.

Assumption 5 As $n \to \infty$,

$$\frac{1}{m} + \frac{m}{n} = o(1). \tag{33}$$

Assumption 5 imposes an upper bound on the the rate at which m can increase with n.

Assumption 6 For $m = o(n)$ the renormalized periodogram for the (a,b)th element, η'_{abj}, $\forall\ 1 \leq j \leq m$ and $a, b = 1, ..., q$, satisfies a WLLN

$$\frac{1}{m}\sum_{j=1}^{m}\eta'_{abj} \xrightarrow{p} 1, \text{ as } m, n \rightarrow \infty, \tag{34}$$

where $\eta'_{abj} = \frac{I^u_{ab}(\lambda_j)}{\Lambda_{aj}(d^u_a)G^0_{ab}\Lambda^*_{bj}(d^u_b)}$, $\forall 1 \leq j \leq m$.

Assumption 6 is a consequence of equations (23), (24), and (25), and is a base for the consistency of the proposed estimator, see the previous section. Furthermore, Assumption 6 states that if Assumption 2 and equation (17) hold then for the (a,b)th element

$$\eta'_{abj} = \eta_{abj}\left(1 + o(1)\right), \ \forall\ 1 \leq j \leq m, \text{ as } n \rightarrow \infty. \tag{35}$$

Furthermore, (23) implies that

$$E\left[\eta'_{abj}\right] \leq C, \ \forall\ 1 \leq j \leq m, \text{ as } n \rightarrow \infty, \tag{36}$$

for $a, b = 1, ..., q$ where C is a positive constant.

Under these assumptions we can setup the following theorem which delivers consistency of the extended multivariate local Whittle estimator, $\hat{d}$.

Theorem 1 *Given Definition 1 and assume that Assumptions 1 through 6 hold. Then,* $\hat{d} \xrightarrow{p} d^0$, *as* $n \rightarrow \infty$.

5 Asymptotic normality

In this section, we list some further assumptions which are needed in deriving asymptotic normality of the proposed multivariate estimator. Assumptions 2'-4' are analogous to the ones found in Lobato (1999) and Shimotsu (2007) in their derivation of asymptotic normality.

Assumption 1' Assume that Assumption 1 holds and d^0 is an interior point of D.

Assumption 2' The spectral density of the stationary sequence $\{u_t\}$ is for $\beta \in (0, 2]$

$$f^u_{ab}(\lambda) - e^{i(\pi-\lambda)\left(d^u_a - d^u_b\right)/2}G^0_{ab}\lambda^{-d^u_a - d^u_b} = O\left(\lambda^{-d^u_a - d^u_b + \beta}\right), \quad \text{as } \lambda \rightarrow 0^+, \tag{37}$$

where $d^u_a \in (-1/2, 1/2)$ and $d^u_b \in (-1/2, 1/2)$, $a, b = 1, ..., q$.

This Assumption is similar to the one used in Robinson (1995*b*, pp. 1056).

Assumption 3' Assume that Assumption 3 holds and further, we need

$$\begin{aligned} E\left(\varepsilon_{at}\varepsilon_{bt}\varepsilon_{ct}\left|\Im_{t-1}\right.\right) &= \mu_{abc} \text{ a.s.}, \\ E\left(\varepsilon_{at}\varepsilon_{bt}\varepsilon_{ct}\varepsilon_{dt}\left|\Im_{t-1}\right.\right) &= \mu_{abcd}, \quad \forall t = 0, \pm 1, \pm 2, ..., \end{aligned}$$

for $a, b, c, d = 1, 2$ where $|\mu_{abc}| < \infty$ and $|\mu_{abcd}| < \infty$.

Assumption 4' As $n \to \infty$,

$$\frac{1}{m} + \frac{m^{1+2\beta} (\log m)^2}{n^{2\beta}} + \frac{\log n}{m^\gamma} = o(1), \tag{38}$$

for any $\gamma > 0$.

The bandwidth restriction in Assumption 4' is stronger than in e.g. Robinson (1995*a*) and Lobato (1999) as we have an additional third term in eqn. (38) which is necessary in showing convergence of the Hessian, see Shimotsu (2007).

Assumption 5' There exists a finite real matrix H such that

$$\Lambda\left(\lambda_j; (d^u)^0\right)^{-1} A(\lambda_j) = H + o(1), \quad \text{as } \lambda_j \to 0^+,$$

for $1 \leq j \leq m$.

Assumption 5' implies that $HH' = 2\pi G^0$.

Theorem 2 *Given Definition 1 and assume that Assumption 4, 6, and 1'-5' hold. Then, as $n \to \infty$,*

$$m^{1/2}\left(\hat{d} - d^0\right) \xrightarrow{d} N\left(0, \Omega^{-1}\right), \quad \Omega = 2\left[G^0 \odot \left(G^0\right)^{-1} + I_q + \frac{\pi^2}{4}\left(G^0 \odot \left(G^0\right)^{-1} - I_q\right)\right],$$
$$\hat{G}\left(\hat{d}\right) \xrightarrow{p} G^0.$$

From the limiting distributional results, we can consider the null hypothesis of a linear set of q restrictions, i.e.

$$W = \left(\hat{d} - d\right)' \hat{\Omega}\left(\hat{d} - d\right) \xrightarrow{d} \chi_q^2, \tag{39}$$

where $\left(\hat{d} - d\right)$ is a $q \times 1$ vector, $\hat{\Omega}$ is the $q \times q$ covariance matrix obtained by replacing G^0 in the definition of the covariance matrix by the estimate $\hat{G}\left(\hat{d}\right)$. If we want to test for equality of the q fractional integration parameters we can consider the following feasible test statistic

$$W_f = \left(T\hat{d}\right)' \left(T\hat{\Omega}^{-1}T'\right)^{-1} \left(T\hat{d}\right) \xrightarrow{d} \chi_{q-1}^2, \tag{40}$$

where we use the mean value of the estimates as an approximation of d in (39), and the $(q-1) \times q$ matrix T is defined with elements $[T]_{aa} = 1 - 1/q$ for $a = 1, \ldots, q-1$ and $-1/q$ elsewhere.[3]

[3] In the simulation study and the empirical application we will use the finite-sample approximation of the variance covariance matrix in Theorem 2. This is done by multiplying equations (39) and (40) by a number c_m defined as

$$c_m = \sum_{j=1}^{m} v_j^2, \quad v_j = \log \lambda_j - \frac{1}{m}\sum_{j=1}^{m} \log \lambda_j.$$

This was shown by Hurvich & Chen (2000) (and also Shimotsu (2007)) to improve the finite sample properties of the test.

6 Simulation study

We compare our derived estimators to the stationary multivariate local Whittle (MLW) estimator of Shimotsu (2007). Of course it is an unfair comparison when we consider non-stationary values of d, i.e. $d \geq 1/2$. But it is still of interest to see what happens with the precision of the MLW estimator for these cases. We conjecture that as $d > 1$, the estimates of the fractional integration parameters become severely negative biased and inflation of the long-run variance as in the univariate setting, see Phillips & Shimotsu (2004), Shimotsu & Phillips (2005), Abadir et al. (2007), and Nielsen (2008) among others. Furthermore, we also compare the multivariate methods to univariate counterparts, i.e. the local Whittle (LW) estimator of Robinson (1995*a*) and extended local Whittle (ExtLW) estimator of Abadir et al. (2007) through a variance comparison.

6.1 Setup

This sections concerns the finite sample performance of the extended multivariate local Whittle estimator (ExtMLW). We generate $I(d)$ processes by truncating the moving average representation in eqn. (1). Specifically,

$$\begin{aligned} X_t &= \begin{pmatrix} (1-L)^{-d_1} & 0 \\ 0 & (1-L)^{-d_2} \end{pmatrix} \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix} I\left(t \geq 1\right), \\ & \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix} \overset{iid}{\sim} N\left(0, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right), \end{aligned}$$

where we generated $n+2000$ observations of X_t, and discard the first 2000 observations.[4] The correlation between u_{1t} and u_{2t} were set equal to $0, 0.4$, and 0.8. We set the fractional parameters of interest equal to $(d_1, d_2) = \{(0.2, 0.2), (0.2, 0.4), (0.2, 0.8), (0.8, 0.8), (1, 1), (1, 1.2), (1, 1.4), (1.4, 1.4)\}$. Sample size is set equal to $n \in \{512, 1024\}$ and bandwidth $m = \lfloor n^a \rfloor$ where $a \in \{0.5, 0.65\}$. The bias and root mean squared error (RMSE) were computed using 1000 replications. Simulations were done in Matlab v7.2. The optimization procedure was implemented using the Nelder-Mead simplex method (fminsearch) for the multivariate estimators, whereas the univariate estimators used golden section search and parabolic interpolation (fminbnd) to find a minimum. As initial values for the multivariate estimators we used the univariate extended local Whittle (ExtLW) estimates of d_1 and d_2.

To conserve space we present only a subset of the results. The left-out results (univariate semiparametric estimates of bias and RMSE are similar to the ones obtained in Abadir et al. (2007) and Nielsen (2008), and thefore omitted) are available upon request.

6.2 Simulation results

When there is no correlation between X_{1t} and X_{2t}, there is no efficiency gain in using the semiparametric multivariate estimators. Furthermore, as $\{u_t\}$ is generated with no short-run contamination, the precision of the semiparametric estimators should increase as a function of the bandwidth.

Insert Table 1 about here

[4] By discarding 2000 observations we should be able to approximate the Type I process to a desired degree of accuracy, see Davidson & Hashimzade (2009).

In Table 1, results with $\rho = 0$ are presented. For the stationary region the two multivariate semiparametric estimators are seemingly unbiased, and we clearly see that the extended estimator is (in a statistical sense) equal to their non-extended counterpart. The RMSE shows that the fractional parameters are estimated quite accurately. Moving on to the non-stationary region, i.e. $d \geq 1/2$, we see that the MLW estimates are unbiased and precise for $d \leq 1$, and that the bias of $\hat{d}_2^{MLW}$ increases quite considerably as the true fractional integration parameter, d_2^0, increases. This is to be expected as we know that the LW estimator of Künsch (1987) and Robinson (1995*a*) is not consistent for $d > 1$ and the LW estimator is biased towards unity, thereby confirming the results of Phillips & Shimotsu (2004). It should be noted that the bias from estimation d_2 does not have an impact on the estimation of d_1 for the MLW estimator (which also holds for increasing ρ). Not reported results show, in addition, that the MLW parameter estimates of d_1 and d_2 are not as downward biased as the univariate LW estimator. For example, in the case where $d_1 = d_2 = 1.4$ the bias for the LW estimator is double that of the MLW estimator. For the ExtMLW estimator, regardless of which region we are in, the estimator is unbiased and the RMSE indicates that the fractional parameter of interest is estimated accurately. Additionally, the RMSE does not vary much in the given range of d. We conclude that (as expected) there is not much gained efficiency wise by doing multivariate estimation (when $\rho = 0$).

Insert Tables 2 and 3 about here

Looking at the case where $\rho = 0.4$ and $\rho = 0.8$, Tables 2 and 3, we observe the same magnitude of bias and a bit lower RMSE than in Table 1. The more precision in the estimates is also seen when comparing the variances of the univariate and multivariate estimators, as we get an efficiency gain from doing multivariate estimation. Concluding Tables 1-3, we see that the ExtMLW estimator is unbiased and precise. There is an efficiency gain in comparison to the univariate ExtLW estimator for $\rho > 0$, and in addition the results are stable across d.

Insert Tables 4 and 6 about here

In Tables 4-6, we analyze the behavior of the long-run covariance matrix G. The ExtMLW estimates are stable across d and the estimate of G becomes more precise for $\rho = 0.4$ and $\rho = 0.8$ in comparison to when $\rho = 0$. This also holds for the absolute coherence (long-run correlation), i.e. $|c|$. Looking at the MLW estimate of G, we know that the limiting distribution does not hold for non-stationary values of d and this is especially evident when $d_2 > 1$.

Insert Tables 7 and 8 about here

Table 7 displays the rejection frequency from testing the null hypothesis $H_0 : d_1 = d_1^0$ and $d_2 = d_2^0$ obtained at a 5% asymptotic critical value for $n = 512$ and $n = 1024$. Overall, the modified Wald statistic overrejects, and the overrejection gets less profound as the sample increases. In addition, Table 8 contain results from a small power study where we test the null hypothesis $d_1 = d_2 = 1$ when $d_1^0 = 1$ and $d_2^0 \in \{0.4, 0.7, 0.75, 0.80, 0.90, 0.95\}$. It is seen that the power increases as we move away from the null and is increasing the sample size. Furthermore, when $3/4 < d_2^0 < 1$ the ExtMLW outperforms the MLW estimator, which is expected.

The simulation results for the MLW estimator is similar to that of Shimotsu (2007) for $(d_1, d_2) \in \{(0.2, 0.2), (0.2, 0.4)\}$. Shimotsu (2007) compares his estimator with that of Lobato (1999) and shows that because Lobato (1999) bases his estimator on (3), we get a downward bias in the off-diagonal elements of G. Therefore, the proposed extension of the estimator of Shimotsu (2007) will also outperform the multivariate estimator of Lobato (1999) if we were to extend this to cover non-stationary values of d. In addition, we could have included the univariate exact local Whittle estimator of Shimotsu & Phillips (2005) in the above simulation experiments, but as the simulation experiments of the ExtLW estimator in Abadir et al. (2007) are qualitatively similar to the results obtained for the exact local Whittle estimator in Shimotsu & Phillips (2005), this is omitted.[5] Furthermore, Shimotsu & Phillips (2005) compare their univariate estimator to the tapered local Whittle estimator of Velasco (1999*a*) and Hurvich & Chen (2000). They show that their estimator achieves a MSE reduction compared to the tapered versions, and therefore the ExtLW also has a lower MSE than the tapered estimator.

7 Empirical application

There has been a lot of attention drawn to the analysis of exchange rate dynamics. A sound knowledge of these dynamics are of interest as this is the basis for appropriate inference, modeling, and forecasting. Several papers make it clear that the exchange rates can be well described as $I(1)$ processes, e.g. Baillie & Bollerslev (1989) using conventional unit root tests, whereas Baillie (1996), Nielsen (2004), and Nielsen & Shimotsu (2007) find evidence of an unit root using methods that are robust to the fractional alternative.

The contribution of this section is to analyze potential long memory in log spot exchange rates and test if they share a common order of integration as this is a necessary condition for there to exist potential (long-run) relations (see Baillie & Bollerslev (1989), Diebold, Gardeazabal & Yilmaz (1994), Baillie & Bollerslev (1994), Nielsen (2004), and Nielsen & Shimotsu (2007) among others). We consider monthly averages of noon buying rates for the log spot exchange rates of Germany, United Kingdom, Japan, Canada, France, Italy, and Switzerland against the US Dollar. The data set is obtained from the Federal Reserve Board of Governers G.5 release. The sample runs from January 1974 to December 2001 for a total of $n = 336$ observations. This is the same set of currencies considered in; Baillie & Bollerslev (1989, 1994) and Diebold et al. (1994) who analyze daily observations running from March 1, 1980 until January 28, 1985, Kim & Phillips (2000) who considers quarterly observations from 1957 through 1997, and Nielsen (2004) and Nielsen & Shimotsu (2007) who analyze monthly observations from January 1974 through December 2001.

Insert Figure 1 about here

Figure 1 shows a time series plot of the seven log spot exchange rates.

Insert Table 9 about here

Table 9 presents the fractional integration orders from the MLW and ExtMLW estimators. The standard errors are reported in parentheses. We compute the estimates for two different bandwidth

[5] Because of the little impact that initial values and fractional differencing have when the sample size is large.

choices, i.e. $m = \lfloor n^{0.5} \rfloor$ and $m = \lfloor n^{0.65} \rfloor$. The multivariate estimates clearly indicate that the exchange rate dynamics are well described as $I(1)$ processes.

Insert Table 10 about here

In Table 10 we display the estimated covariance matrix for the two multivariate semiparametric estimators. From the asymptotics we know that the MLW estimator is not asymptotically normal when the fractional orders of interest are in the non-stationary region[6] which is also seen from the simulation study (especially when $d > 1$) in the previous section. Therefore, most weight should be put on the covariance estimates of the ExtMLW estimator.

Insert Table 11 about here

Table 11 reports the normalized $\hat{G}$ (the coherence at the zero frequency) which measures the long-run correlation between the log spot exchange rates, and this can be seen as informal evidence of the existence of potential long-run relations. First of all, the results are in some cases quit different across the two estimators. E.g. for the MLW estimator the JPY/USD log spot exchange is negatively correlated at the zero frequency with the CAD/USD, FRF/USD, ITL/USD, and GBP/USD log spot exchange rates, whereas for the ExtMLW estimator this is only the case for the CAD/US when $m = \lfloor n^{0.5} \rfloor$. As we are in the non-stationary region we should put the emphazis on the results obtained from the ExtMLW estimator. Looking at the zero frequency correlations it is clear that the European currencies (as expected) have a high degree of coherence, which is a sign of potential long-run relations between the log spot exchange rates.

Insert Table 12 about here

Table 12 tests the null hypothesis; $H_1 : d_1 = d_2 = ... = d_7 = 1$ and $H_2 : d_1 = d_2 = ... = d_7$. The critical values of the test statistics are $\chi^2_7(0.95) = 14.067$ and $\chi^2_6(0.95) = 12.592$. First, considering the null H_1, we can only reject that all seven fractional integration orders are identical to an unit root in one case, i.e. for the MLW estimator when the bandwidth is equal to $m = \lfloor n^{0.5} \rfloor$. Furthermore, when considering the null H_2 we cannot reject that the fractional integration estimates are identical.

Insert Figure 2 about here

In Figure 2, we present the multivariate estimates for $m \in \{20, 21, ..., 100\}$ of the GBP/USD log spot exchange rate. It is obvious that increasing m inference becomes more accurate. But including more and more frequencies we will include medium and short term behavior of the process, which could bias our estimates, see e.g. Robinson (1994).

8 Concluding remarks

In this paper, we propose an extension of the multivariate local Whittle estimator of Shimotsu (2007) to cover potentially non-stationary multivariate fractional integrated processes using the notion of the

[6] Presumably, it holds that the MLW is consistent for $d \in (-1/2, 1)$ and asymptotic normal for $d \in (-1/2, 3/4)$ as for the univariate counterpart, which there is evidence of in the simulation study.

extended DFT and periodogram. The multivariate framework is based on a spectral density that has both a real and complex part even at the origin, and the long memory parameters affect both the slope and the phase of the spectral density at the origin. Consistency and asymptotic normality of the estimator is shown. Furthermore, there is potentially considerable efficiency gain over univariate semipametric estimators depending on whether or not there is dependence between the fractionally integrated processes.

A simulation study confirms the asymptotic results. In addition, we have applied the proposed multivariate semiparametric estimators the analysis of log exchange rates, confirming that for the given sample of log spot exchange rates they are well decribed as $I(1)$ processes, and that there is a high degree of coherence between the European currencies.

Appendix

The Appendix section is structured as follows: In the first section the proof to Theorem 1 and 2 are given. Section 2 presents a technical lemma adapted from Abadir et al. (2007).

Proof of Theorems

Proof of Theorem 1. Define $\theta = (\theta_1, ..., \theta_q) = d - d^0$, where $d = (d_1, ..., d_q)'$ and $d = \left(d_1^0, ..., d_q^0\right)'$, and $S_n(d) = L_n(d) - L_n\left(d^0\right)$. Then it suffices to prove consistency that for any $\zeta > 0$

$$\Pr\left(\inf_{d:\|d-d^0\|\geq\zeta:d\in D} S(d) \geq \delta\right) \to 1, \tag{41}$$

as $n \to \infty$ for some $\delta > 0$. Since $m^{-1}\sum_{j=1}^{m}\log\left(\frac{j}{m}\right) = -1 + O\left(m^{-1}\log m\right)$ it follows that

$$S_n(d) = \log\det K_n(d) - \log\det K_n\left(d^0\right) + 2\sum_{a=1}^{q}\theta_a + o(1), \tag{42}$$

where

$$\begin{aligned} K_n(d) &= \frac{1}{m}\sum_{j=1}^{m}\mathrm{Re}\left[M_j(d)\left(G^0\right)^{-1}I_j(d)\,M_j^*(d)\right], \\ M_j(d) &= diag\left(M_{ja}(d)\right), M_{ja}(d) = (j/m)^{d_a}\,e^{i(\lambda_j-\pi)d_a/2}, \end{aligned} \tag{43}$$

and * denotes conjugate transpose. What we know is that by definiton the extended periodogram matrix is given by

$$\begin{aligned} I_j(d) &= I(\lambda_j; d) = w(\lambda_j; d)\,w^*(\lambda_j; d) \\ &= \left(w^x(\lambda_j) + c(\lambda_j; d)\right)\left(w^x(\lambda_j) + c(\lambda_j; d)\right)^*. \end{aligned}$$

Then, from Abadir et al. (2007, Lemma 4.4), it follows that for $d^0 = p^0 + d^u$

$$w\left(\lambda_j; d^0\right) = \left(1 - e^{i\lambda_j}\right)^{-p^0} w^u(\lambda_j). \tag{44}$$

Therefore, by setting $\tau(\lambda_j, d) = c(\lambda_j, d) - c(\lambda_j, d^0)$ we can write

$$I_j(d) = \left(w\left(\lambda_j; d^0\right) + \tau(d)\right)\left(w\left(\lambda_j; d^0\right) + \tau(d)\right)^*. \tag{45}$$

We note that if $d^0 = p^0 + d^u$ then by definition $\tau(\lambda_j, d) = 0$ for $d \in D_{p^0}$ as $c(\lambda_j, d)$ is a step function. Rewrite (43) as

$$K_n(d) = \frac{1}{m}\sum_{j=1}^{m} \mathrm{Re}\left[M_j(d)\left(G^0\right)^{-1}\left\{\left(w\left(\lambda_j; d^0\right) + \tau(d)\right)\left(w\left(\lambda_j; d^0\right) + \tau(d)\right)^*\right\} M_j^*(d)\right], \tag{46}$$

for $d \in D$. Set

$$F_n(d) = \frac{1}{m}\sum_{j=1}^{m} \mathrm{Re}\left[M_j(d)\left(G^0\right)^{-1}\left\{w\left(\lambda_j; d^0\right) w^*\left(\lambda_j; d^0\right)\right\} M_j^*(d)\right], \tag{47}$$

where we for $d \in D_{p^0}$ have that $K_n(d) = F_n(d)$ as $\tau(\lambda_j, d) = 0$. Now define, because of the non-uniform behavior of $L_n(d)$, $\varepsilon > 0$ and $d^\varepsilon = d^0 - \frac{1-\varepsilon}{2}$, where the ath element of ε, is denoted $\varepsilon_a > 0$ which is chosen sufficiently small. Then from Lemma 1

$$\begin{aligned} F_n(d \vee d^\varepsilon) &= \Lambda_m\left(d^0\right)\left(\Sigma(\theta^\varepsilon) \odot M_\infty(\theta^\varepsilon) + o_p(1)\right)\Lambda_m^*\left(d^0\right), & (48)\\ K_n(d) &\geq (1 + o_p(1)) \odot F_n(d \vee d^\varepsilon), & (49) \end{aligned}$$

where $o_p(1) \xrightarrow{p} 0$, as $n \to \infty$, uniformly on $d \in D$, and $\theta^\varepsilon = d \vee d^\varepsilon - d^0$, $\Sigma(\theta^\varepsilon)$ and $M_\infty(\theta^\varepsilon)$ are defined to be matrices where the (a,b)th elements are $e^{-i\pi\left(\theta_a^\varepsilon - \theta_b^\varepsilon\right)/2}$ and $\left(1 + \theta_a^\varepsilon + \theta_b^\varepsilon\right)^{-1} = \int_0^1 x^{\theta_a^\varepsilon + \theta_b^\varepsilon} dx$, respectively, and $\Lambda_m\left(d^0\right) = diag\left(\Lambda_{ma}\left(d_a^0\right)\right), \Lambda_{ma}\left(d_a^0\right) = \lambda_m^{-d_a^0} e^{i(\pi - \lambda_j)d_a^0/2}$ for $a = 1, ..., q$. Hence, for any $\varepsilon > 0$

$$\begin{aligned} S_n(d) &\geq \log\det F_n(d \vee d^\varepsilon) - \log\det F_n\left(d^0\right) + 2\sum_{a=1}^{q}\theta_a + o_p(1) & (50)\\ &= 2\sum_{a=1}^{q}\theta_a - \log\det\left(\Sigma(\theta^\varepsilon) \odot M_\infty(\theta^\varepsilon)\right) + o_p(1), & (51) \end{aligned}$$

uniformly on $d \in D$ as

$$\begin{aligned} \log\det F_n(d \vee d^\epsilon) - \log\det F_n\left(d^0\right) &= \log\left(\frac{\det F_n(d \vee d^\varepsilon)}{\det F_n\left(d^0\right)}\right)\\ &= \log\left(\frac{\det\left\{\Lambda_m\left(d^0\right)\left(\Sigma(\theta^\varepsilon) \odot M_\infty(\theta^\varepsilon) + o_p(1)\right)\Lambda_m^*\left(d^0\right)\right\}}{\det\left\{\Lambda_m(d^0)\left(\Sigma(d^0 - d^0) \odot M_\infty(d^0 - d^0) + o_p(1)\right)\Lambda_m^*(d^0)\right\}}\right)\\ &= \log\left(\frac{\det\left\{\Sigma(\theta^\varepsilon) \odot M_\infty(\theta^\varepsilon)\right\}}{\det\left\{\Sigma(d^0 - d^0) \odot M_\infty(d^0 - d^0)\right\}}\right) + o_p(1)\\ &= -\log\det\left\{\Sigma(\theta^\varepsilon) \odot M_\infty(\theta^\varepsilon)\right\} + o_p(1). \end{aligned}$$

Given Assumption 1, if $d_1 \leq d \leq d^\varepsilon$, then $2\left(d \vee d^\varepsilon - d^0\right) = -1 + \varepsilon$, and therefore we have for the (a,b)th element (the results of Abadir et al. (2007) hold for the ath element)

$$\begin{aligned} & 2(\theta_a + \theta_b) - \log\left(\det\left\{\mathrm{Re}\left[e^{-i\pi\left(\theta_a^\varepsilon - \theta_b^\varepsilon\right)/2}\left(1 + \theta_a^\varepsilon + \theta_b^\varepsilon\right)^{-1}\right]\right\}\right) & (52)\\ \geq\ & 2\left(\left(d_1 - d_a^0\right) + \left(d_1 - d_b^0\right)\right) - \log\left(\det\left\{\mathrm{Re}\left[e^{-i\pi(\varepsilon_a - \varepsilon_b)/2}\left(1 + \varepsilon_a + \varepsilon_b\right)^{-1}\right]\right\}\right) \geq 1. & (53) \end{aligned}$$

If instead $d^\varepsilon \leq d \leq d_2$ then $2\left(d \vee d^\varepsilon - d^0\right) \geq 2\left(d^\varepsilon - d^0\right) = -1+\varepsilon$, and therefore we have for the (a,b)th element

$$\inf_{\{d^\varepsilon \leq d \leq d_2, |d-d^0|\geq \xi\}} 2\left(\left(d_a - d_a^0\right) + \left(d_b - d_b^0\right)\right) - \log\left(\det\left\{\mathrm{Re}\left[e^{-i\pi(\theta_a-\theta_b)/2}\left(1+\theta_a+\theta_b\right)^{-1}\right]\right\}\right) \geq \delta(\xi) > 0, \tag{54}$$

as $y - \log(y+1) > 0$ for $y \neq 0$, $y > -1$. Then from (51), (52), and (54), we have shown that (41) holds, and therefore the proof Theorem 1 is complete. ■

Proof of Theorem 2. To show asymptotic normality, observe that given the current assumptions, we have that $\hat{d} \xrightarrow{p} d^0$, together with $d^0 \neq p - 1/2$, $p = -1, 0, 1, 2...$, and $w(\lambda_j; d) = w_X(\lambda_j) + c(\lambda_j; d)$, implies that as $n \to \infty$, $w\left(\lambda_j; \hat{d}\right) - w\left(\lambda_j; d^0\right) = o_p(1)$. This enables us to use Shimotsu (2007) and his proof of asymptotic normality. The proof is analogous to Lobato (1999). Therefore, as $n \to \infty$, $\hat{d}$ satisfies

$$0 = \left.\frac{\partial L_n(d)}{\partial d}\right|_{\hat{d}} = \left.\frac{\partial L_n(d)}{\partial d}\right|_{d^0} + \left(\left.\frac{\partial^2 L_n(d)}{\partial d \partial d'}\right|_{\bar{d}}\right)\left(\hat{d} - d^0\right),$$

where $\left\|\bar{d} - d^0\right\| \leq \left\|\hat{d} - d^0\right\|$. Therefore, the argument is that, $\hat{d}$ is asymptotic normal with zero mean and asymptotic variance, Ω^{-1} where

$$\Omega = 2\left[G^0 \odot \left(G^0\right)^{-1} + I_q + \frac{\pi^2}{4}\left(G^0 \odot \left(G^0\right)^{-1} - I_q\right)\right],$$

if, for any $q \times 1$ vector η, as $n \to \infty$,

$$\eta' m^{1/2} \left.\frac{\partial L_n(d)}{\partial d}\right|_{d^0} = \sum_{a=1}^{q} \eta_a m^{1/2} \left.\frac{\partial L_n(d)}{\partial d_a}\right|_{d_a^0} \xrightarrow{d} N\left(0, \eta'\Omega\eta\right), \tag{55}$$

$$\left(\left.\frac{\partial^2 L_n(d)}{\partial d \partial d'}\right|_{\bar{d}}\right) \xrightarrow{p} \Omega. \tag{56}$$

(55) is proved by using the score approximation arguments in Shimotsu (2007, A.2.1 (pp. 295-299)). (56) is proved by using the Hessian approximation arguments in Shimotsu (2007, A.2.2 (pp. 299-302)). This completes the proof. ■

Lemma

Lemma 1 *Suppose assumptions of Theorem 1 hold. Then for any $\varepsilon > 0$, as $n \to \infty$, uniformly in $d \in D$ it holds*

$$\det\left\{\Lambda_m^{-1}\left(d^0\right) F_n(d) \Lambda_m^*\left(d^0\right)^{-1}\right\} \geq \det\left\{\Lambda_m\left(d^0\right)^{-1} F_n\left(d \vee d^\varepsilon\right) \Lambda_m^*\left(d^0\right)^{-1}\right\} \tag{57}$$

$$\geq \det\left\{\Sigma\left(\theta^\varepsilon\right) \odot M_\infty\left(\theta^\varepsilon\right)\right\} + o_p(1), \tag{58}$$

where

$$F_n(d) = \frac{1}{m}\sum_{j=1}^{m} \mathrm{Re}\left[M_j(d)\left(G^0\right)^{-1} w_j\left(d^0\right) w_j^*\left(d^0\right) M_j^*(d)\right],$$

where $M_j(d) = diag\left\{M_{aj}(d)\right\}, M_{aj}(d) = (j/m)^{d_a} e^{i(\lambda_j - \pi)d_a/2}$ for $a = 1, ..., q$, $w_j\left(d^0\right) = w\left(\lambda_j; d^0\right)$, $\theta^\varepsilon = d \vee d^\varepsilon - d^0$ and $\Sigma\left(\theta^\varepsilon\right)$ and $M_\infty\left(\theta^\varepsilon\right)$ are defined to be matrices where the (a,b)th elements are $e^{-i\pi\left(\theta_a^\varepsilon - \theta_b^\varepsilon\right)/2}$ and $\left(1+\theta_a^\varepsilon+\theta_b^\varepsilon\right)^{-1} = \int_0^1 x^{\theta_a^\varepsilon+\theta_b^\varepsilon}dx$, respectively, and

$$\det\left\{K_n(d)\right\} \geq \left(1 + o_p(1)\right)\det\left\{F_n\left(d \vee d^\varepsilon\right)\right\}, \tag{59}$$

where $d^{\varepsilon} = d^0 - \frac{1-\varepsilon}{2}$ *and* $o_p(1)$, *uniformly in* $d \in D$, *as* $n \to \infty$.

Proof. Proof of (58) follows from noting that for the ath element the proof follows from Abadir et al. (2007) and noting that we get the extra complex term as we have moved from the univariate fractional integrated model to the multivariate fractional integrated model. What is left is to prove (58) for the (a,b)th element. Note that the (a,b)th element inside of $F_n(d)$ is

$$\begin{aligned}
& \frac{1}{m}\sum_{j=1}^{m} \mathrm{Re}\left[(j/m)^{d_a+d_b} e^{i(\lambda_j-\pi)(d_a-d_b)/2} \left(G^0\right)^{-1} w_{aj}\left(d^0\right) w_{bj}^*\left(d^0\right)\right] \\
= & \frac{1}{m}\sum_{j=1}^{m} \mathrm{Re}\left[\lambda_m^{-d_a^0-d_b^0} e^{i(\pi-\lambda_j)\left(d_a^0-d_b^0\right)/2} (j/m)^{(\theta_a+\theta_b)} e^{i(\lambda_j-\pi)(\theta_a-\theta_b)/2} v_{aj}v_{bj}^*\right] \\
\geq & \frac{1}{m}\sum_{j=1}^{m} \mathrm{Re}\left[(j/m)^{\left(d_a\vee d_a^{\varepsilon}+d_b\vee d_b^{\varepsilon}\right)} e^{i(\lambda_j-\pi)\left(d_a\vee d_a^{\varepsilon}-d_b\vee d_b^{\varepsilon}\right)/2} v_{aj}v_{bj}^*\right].
\end{aligned}$$

Since $v_{aj}v_{bj}^*$ satisfies a WLLN argument, $E\left[v_{aj}v_{bj}^*\right] \leq C, 1 \leq j \leq m$ and, we have that $2\left(d \vee d^{\varepsilon} - d^0\right) \geq -1+\varepsilon$, then by Abadir et al. (2007, Lemma 4.5)

$$\begin{aligned}
& \frac{1}{m}\sum_{j=1}^{m} \mathrm{Re}\left[\lambda_m^{d_a^0+d_b^0} e^{i(\pi-\lambda_j)\left(d_a^0-d_b^0\right)/2} (j/m)^{\left(d_a\vee d_a^{\varepsilon}+d_b\vee d_b^{\varepsilon}\right)} e^{i(\lambda_j-\pi)\left(d_a\vee d_a^{\varepsilon}-d_b\vee d_b^{\varepsilon}\right)/2} v_{aj}v_{bj}^*\right] \\
= & \frac{1}{m}\sum_{j=1}^{m} \mathrm{Re}\left[(j/m)^{\left(\theta_a^{\varepsilon}+\theta_b^{\varepsilon}\right)} e^{i(\lambda_j-\pi)\left(\theta_a^{\varepsilon}-\theta_b^{\varepsilon}\right)/2} v_{aj}v_{bj}^*\right] \\
& \xrightarrow{p} \frac{1}{m}\sum_{j=1}^{m} \mathrm{Re}\left[e^{i(\lambda_j-\pi)\left(\theta_a^{\varepsilon}-\theta_b^{\varepsilon}\right)/2}\right] \int_0^1 x^{\theta_a^{\varepsilon}+\theta_b^{\varepsilon}} \\
= & \frac{1}{m}\sum_{j=1}^{m} \mathrm{Re}\left[e^{i(\lambda_j-\pi)\left(\theta_a^{\varepsilon}-\theta_b^{\varepsilon}\right)/2}\right] \left(1+\theta_a^{\varepsilon}+\theta_b^{\varepsilon}\right)^{-1},
\end{aligned}$$

uniformly in $d \in D$ which proves (58). Proof of (59) follows from the proof in Abadir et al. (2007, Lemma 4.2) for the ath diagonal element. For the (a,b)th element, we notice that to show (59), we estimate for the (a,b)th element of $K_n(d)$ is

$$\begin{aligned}
& \frac{1}{m}\sum_{j=1}^{m} \mathrm{Re}\left[(j/m)^{d_a+d_b} e^{i(\lambda_j-\pi)(d_a-d_b)} \left(G^0\right)^{-1} \left(w_j\left(d^0\right)+\tau(d)\right)\left(w_j\left(d^0\right)+\tau(d)\right)^*\right] \\
\geq & \frac{1}{m}\sum_{j=1}^{m} \mathrm{Re}\left[(j/m)^{d_a\vee d_a^{\varepsilon}+d_b\vee d_b^{\varepsilon}} e^{i(\lambda_j-\pi)\left(d_a\vee d_a^{\varepsilon}-d_b\vee d_b^{\varepsilon}\right)} \left(G^0\right)^{-1} \left(w_{aj}\left(d^0\right)+\tau_{aj}(d)\right)\left(w_{bj}\left(d^0\right)+\tau_{bj}(d)\right)^*\right] \\
\geq & \frac{1}{m}\sum_{j=1}^{m} \mathrm{Re}\left[(j/m)^{d_a\vee d_a^{\varepsilon}+d_b\vee d_b^{\varepsilon}} e^{i(\lambda_j-\pi)\left(d_a\vee d_a^{\varepsilon}-d_b\vee d_b^{\varepsilon}\right)} \left(G^0\right)^{-1} w_{aj}\left(d^0\right) w_{bj}^*\left(d^0\right)\right] \\
& -\frac{2}{m}\sum_{j=1}^{m} \mathrm{Re}\left[(j/m)^{d_a\vee d_a^{\varepsilon}+d_b\vee d_b^{\varepsilon}} e^{i(\lambda_j-\pi)\left(d_a\vee d_a^{\varepsilon}-d_b\vee d_b^{\varepsilon}\right)} \left(G^0\right)^{-1} w_{aj}\left(d^0\right) \tau_{bj}^*(d)\right] \\
& +\frac{1}{m}\sum_{j=1}^{m} \mathrm{Re}\left[(j/m)^{d_a\vee d_a^{\varepsilon}+d_b\vee d_b^{\varepsilon}} e^{i(\lambda_j-\pi)\left(d_a\vee d_a^{\varepsilon}-d_b\vee d_b^{\varepsilon}\right)} \left(G^0\right)^{-1} \tau_{aj}(d)\tau_{bj}^*(d)\right] \\
= & F_{ab,n}\left(d\vee d^{\varepsilon}\right) - 2C_{ab,n}(d) + B_{ab,n}(d).
\end{aligned}$$

That is,

$$C_{ab,n}(d) = \frac{1}{m}\sum_{j=1}^{m}\mathrm{Re}\left[(j/m)^{d_a\vee d_a^\varepsilon+d_b\vee d_b^\varepsilon}e^{i(\lambda_j-\pi)\left(d_a\vee d_a^\varepsilon-d_b\vee d_b^\varepsilon\right)}\left(G^0\right)^{-1}w_{aj}\left(d^0\right)\tau_{bj}^*(d)\right],$$

$$B_{ab,n}(d) = \frac{1}{m}\sum_{j=1}^{m}\mathrm{Re}\left[(j/m)^{d_a\vee d_a^\varepsilon+d_b\vee d_b^\varepsilon}e^{i(\lambda_j-\pi)\left(d_a\vee d_a^\varepsilon-d_b\vee d_b^\varepsilon\right)}\left(G^0\right)^{-1}\tau_{aj}(d)\tau_{bj}^*(d)\right].$$

Next, we will show that

$$C_{ab,n}(d) = o_p(1)\left(F_{ab,n}(d\vee d^\varepsilon)+B_{ab,n}(d)\right), \tag{60}$$

uniformly in $d\in D$, as $n\to\infty$, which implies that

$$K_{ab,n}(d)\geq(1+o_p(1))\,F_{ab,n}(d\vee d^\varepsilon),$$

uniformly in $d\in D$ as $B_{ab,n}(d\vee d^\varepsilon)\geq 0$. To show (60) let $d^0\in D_{p^0}$, and since D is a finite set, it suffices to show validity for $d\in D_p$ for any fixed integer $p\geq -1$. Set for the (a,b)th element

$$B_{ab}(d) = \sum_{r_a=p_a^0\wedge p_a+1}^{p_a^0\vee p_a}\sum_{r_b=p_b^0\wedge p_b+1}^{p_b^0\vee p_b}\mathrm{Re}\left[\lambda_m^{-r_a-r_b}e^{i(\pi-\lambda_j)(r_a-r_b)/2}w_{\nabla^{r_a}X}(0)\,w_{\nabla^{r_b}X}^*(0)\right],$$

if $p\neq p^0$, and set $B_{ab}(d)=0$ if $p_a=p_a^0$ and $p_b=p_b^0$. We will next show that, as $n\to\infty$ $\exists\eta>0:$

$$\begin{aligned}
&\frac{1}{m}\sum_{j=1}^{m}\mathrm{Re}\left[(j/m)^{d_a\vee d_a^\varepsilon+d_b\vee d_b^\varepsilon}e^{i(\lambda_j-\pi)\left(d_a\vee d_a^\varepsilon-d_b\vee d_b^\varepsilon\right)}\left(G^0\right)^{-1}\tau_{aj}(d)\tau_{bj}^*(d)\right]\\
&\geq \eta\sum_{r_a=p_a^0\wedge p_a+1}^{p_a^0\vee p_a}\sum_{r_b=p_b^0\wedge p_b+1}^{p_b^0\vee p_b}\mathrm{Re}\left[\lambda_m^{-r_a-r_b}e^{i(\pi-\lambda_j)(r_a-r_b)/2}w_{\nabla^{r_a}X}(0)\,w_{\nabla^{r_b}X}^*(0)\right],
\end{aligned}$$

i.e.,

$$B_{ab,n}(d)\geq\eta B_{ab}(d), \tag{61}$$

uniformly in $d\in D$, and

$$\begin{aligned}
&\frac{1}{m}\sum_{j=1}^{m}\mathrm{Re}\left[\left(\lambda_m^{d_a^0+d_b^0}e^{i(\lambda_j-\pi)\left(d_a^0-d_b^0\right)/2}\right)^{1/2}(j/m)^{d_a\vee d_a^\varepsilon+d_b\vee d_b^\varepsilon}e^{i(\lambda_j-\pi)\left(d_a\vee d_a^\varepsilon-d_b\vee d_b^\varepsilon\right)}\left(G^0\right)^{-1}w_{aj}\left(d^0\right)\tau_{bj}^*(d)\right]\\
&= o_p(1)\left(\sum_{r_a=p_a^0\wedge p_a+1}^{p_a^0\vee p_a}\sum_{r_b=p_b^0\wedge p_b+1}^{p_b^0\vee p_b}\mathrm{Re}\left[\lambda_m^{-r_a-r_b}e^{i(\pi-\lambda_j)(r_a-r_b)/2}w_{\nabla^{r_a}X}(0)\,w_{\nabla^{r_b}X}^*(0)\right]\right)^{1/2},
\end{aligned}$$

i.e.,

$$C_{ab,n}(d) = o_p(1)\left(B_{ab}(d)\right)^{1/2}, \tag{62}$$

where $o_p(1)\to 0$ in probability uniformly in $d\in D$, as $n\to\infty$. Then, by (58) (focusing on the (a,b)th element) $\exists c>0:$

$$\begin{aligned}
&\frac{1}{m}\sum_{j=1}^{m}\mathrm{Re}\left[(j/m)^{d_a\vee d_a^\varepsilon+d_b\vee d_b^\varepsilon}e^{i(\lambda_j-\pi)\left(d_a\vee d_a^\varepsilon-d_b\vee d_b^\varepsilon\right)/2}\left(G^0\right)^{-1}w_{aj}\left(d^0\right)w_{bj}^*\left(d^0\right)\right]\\
&\geq c\,(1+o_p(1))\frac{1}{m}\sum_{j=1}^{m}\mathrm{Re}\left[\lambda_m^{-d_a^0-d_b^0}e^{i(\pi-\lambda_j)\left(d_a^0-d_b^0\right)/2}\right],
\end{aligned}$$

i.e.,

$$F_{ab,n}\left(d \vee d^{\varepsilon}\right) \geq \Lambda_{ab,m}\left(d^{0}\right) \Lambda_{ab,m}^{*}\left(d^{0}\right) c\left(1+o_{p}(1)\right),$$

uniformly in $d \in D$. Therefore, (61) and (62) implies that

$$\begin{aligned}
C_{ab,n}(d) &= \frac{1}{m}\sum_{j=1}^{m} \operatorname{Re}\left[(j/m)^{d_a \vee d_a^{\varepsilon}+d_b \vee d_b^{\varepsilon}}\, e^{i(\lambda_j-\pi)\left(d_a \vee d_a^{\varepsilon}-d_b \vee d_b^{\varepsilon}\right)}\left(G^{0}\right)^{-1} w_{aj}\left(d^{0}\right) \tau_{bj}^{*}(d)\right] \\
&= o_p(1)\begin{pmatrix} \frac{1}{m}\sum_{j=1}^{m} \operatorname{Re}\left[\lambda_m^{-d_a^0-d_b^0} e^{i(\pi-\lambda_j)\left(d_a^0-d_b^0\right)/2}\right] \\ \times \sum_{r_a=p_a^0 \wedge p_a+1}^{p_a^0 \vee p_a} \sum_{r_b=p_b^0 \wedge p_b+1}^{p_b^0 \vee p_b} \operatorname{Re}\left[\lambda_m^{-r_a-r_b} e^{i(\pi-\lambda_j)(r_a-r_b)/2} w_{\nabla^{r_a}X}(0)\, w_{\nabla^{r_b}X}^{*}(0)\right] \end{pmatrix}^{1/2} \\
&= o_p(1)\begin{pmatrix} \frac{1}{m}\sum_{j=1}^{m} \operatorname{Re}\left[\lambda_m^{-d_a^0-d_b^0} e^{i(\pi-\lambda_j)\left(d_a^0-d_b^0\right)/2}\right] \\ +\sum_{r_a=p_a^0 \wedge p_a+1}^{p_a^0 \vee p_a} \sum_{r_b=p_b^0 \wedge p_b+1}^{p_b^0 \vee p_b} \operatorname{Re}\left[\lambda_m^{-r_a-r_b} e^{i(\pi-\lambda_j)(r_a-r_b)/2} w_{\nabla^{r_a}X}(0)\, w_{\nabla^{r_b}X}^{*}(0)\right] \end{pmatrix} \\
&= o_p(1)\begin{pmatrix} \frac{1}{m}\sum_{j=1}^{m} \operatorname{Re}\left[(j/m)^{d_a+d_b}\, e^{i(\lambda_j-\pi)(d_a-d_b)/2}\left(G^{0}\right)^{-1} w_{aj}\left(d^{0}\right) w_{bj}^{*}\left(d^{0}\right)\right] \\ +\frac{1}{m}\sum_{j=1}^{m} \operatorname{Re}\left[(j/m)^{d_a \vee d_a^{\varepsilon}+d_b \vee d_b^{\varepsilon}}\, e^{i(\lambda_j-\pi)\left(d_a \vee d_a^{\varepsilon}-d_b \vee d_b^{\varepsilon}\right)}\left(G^{0}\right)^{-1} \tau_{aj}(d)\, \tau_{bj}^{*}(d)\right] \end{pmatrix} \\
&= o_p(1)\left(F_{ab,n}(d)+B_{ab,n}(d)\right),
\end{aligned}$$

uniformly in $d \in D$, which proves (60), and hence completes the proof of the lemma. Now what is left is to prove (61) and (62). (61) follows from extending eqn. (4.34) and using Lemma 4.3 in Abadir et al. (2007) to take account of the phase shift from moving from the univariate setup to the multivariate setup. (62) follows from eqn. (4.35)-(4.39) in Abadir et al. (2007). ■

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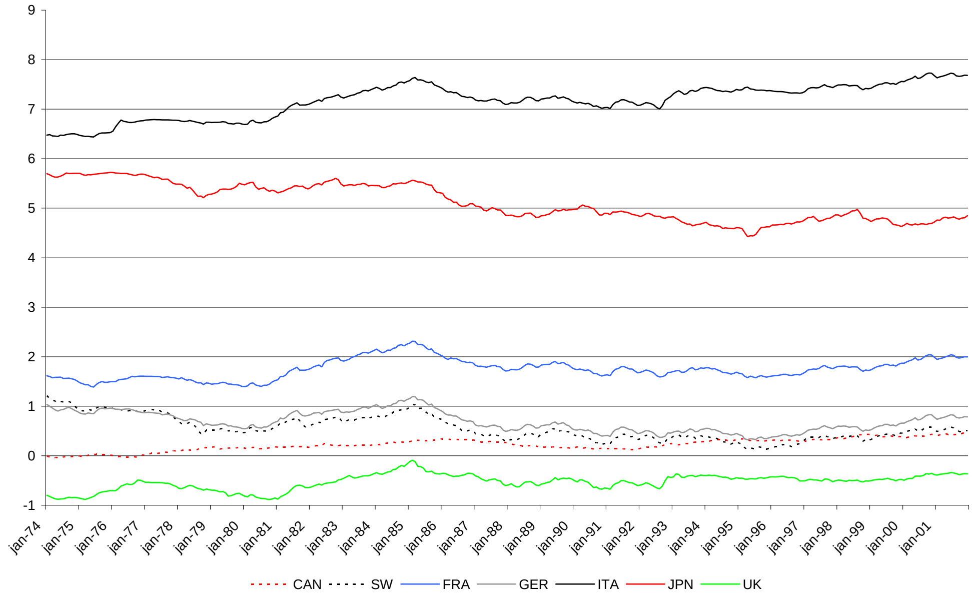

Figure 1: Time series plot of the CAN/USD, CHF/USD, FRF/USD, DEM/USD, ITL/USD, JPY/USD, and GBP/USD log spot exchange rates.

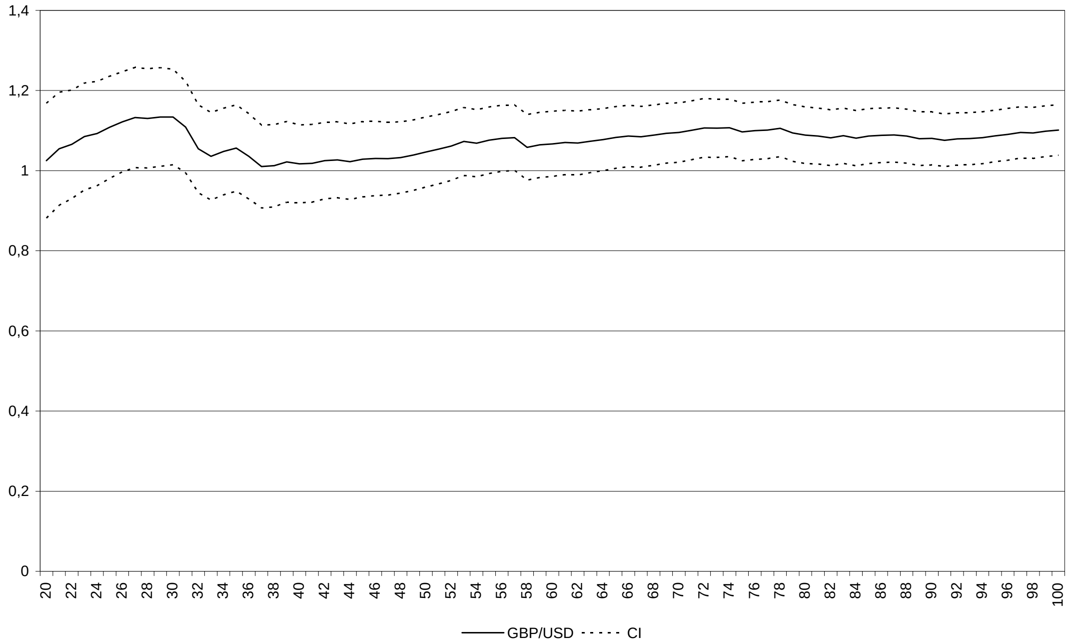

Figure 2: Plot of the extended multivariate local Whittle estimate of the log spot exchange rate of British Pund against the US Dollar.

Table 1: Simulation results for bias, RMSE, and variance comparison where $m = n^{0.65}$ and $\rho = 0$

		MLW				ExtMLW				Variance comparison		
		d_1		d_2		d_1		d_2				
d_1	d_2	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	$\frac{\text{var(LW)}}{\text{var(MLW)}}$	$\frac{\text{var(ExtLW)}}{\text{var(ExtMLW)}}$	$\frac{\text{var(ExtMLW)}}{\text{var(MLW)}}$
Panel A: $n = 512$												
0.2	0.2	−0.006	0.073	−0.007	0.074	−0.006	0.073	−0.007	0.074	(1.014, 1.001)	(1.014, 1.001)	(1.000, 1.000)
	0.4	−0.009	0.075	−0.005	0.077	−0.009	0.075	−0.001	0.079	(1.011, 1.007)	(1.011, 1.010)	(0.999, 1.059)
	0.8	−0.007	0.076	0.013	0.083	−0.007	0.076	−0.006	0.077	(1.016, 1.027)	(1.011, 1.017)	(0.995, 0.892)
0.8	0.8	0.011	0.081	0.009	0.078	−0.012	0.082	−0.012	0.078	(0.954, 0.991)	(1.013, 1.025)	(1.023, 0.979)
1	1	−0.011	0.071	−0.011	0.070	−0.012	0.079	−0.010	0.079	(0.977, 0.926)	(1.015, 1.016)	(1.243, 1.268)
	1.2	0.007	0.065	−0.126	0.148	−0.011	0.074	−0.011	0.074	(0.923, 1.013)	(1.005, 1.010)	(1.287, 0.867)
	1.4	0.009	0.073	−0.330	0.348	−0.015	0.081	−0.011	0.074	(1.063, 1.050)	(1.003, 1.009)	(1.217, 0.438)
1.4	1.4	−0.164	0.183	−0.158	0.177	−0.010	0.074	−0.007	0.073	(0.627, 0.522)	(0.996, 1.010)	(0.807, 0.820)
Panel B: $n = 1024$												
0.2	0.2	−0.007	0.060	−0.007	0.060	−0.007	0.060	−0.007	0.060	(1.015, 1.006)	(1.015, 1.006)	(1.000, 1.000)
	0.4	−0.005	0.058	0.000	0.061	−0.005	0.058	0.004	0.065	(1.019, 1.017)	(1.019, 1.017)	(0.999, 1.135)
	0.8	−0.004	0.059	0.019	0.068	−0.004	0.059	−0.003	0.058	(1.004, 1.007)	(1.004, 1.015)	(0.999, 0.805)
0.8	0.8	0.017	0.064	0.015	0.064	−0.003	0.058	−0.006	0.060	(0.973, 0.953)	(1.023, 1.006)	(0.887, 0.907)
1	1	−0.007	0.053	−0.006	0.052	−0.009	0.059	−0.010	0.061	(0.926, 0.944)	(1.011, 1.006)	(1.260, 1.320)
	1.2	0.009	0.054	−0.127	0.147	−0.009	0.059	−0.006	0.059	(1.016, 0.976)	(1.011, 1.016)	(1.176, 0.648)
	1.4	0.015	0.060	−0.330	0.344	−0.008	0.059	−0.003	0.060	(1.212, 0.996)	(1.003, 1.014)	(1.018, 0.370)
1.4	1.4	−0.156	0.171	−0.154	0.168	−0.005	0.056	−0.002	0.058	(0.439, 0.423)	(1.001, 1.005)	(0.656, 0.714)

Notes: MLW and ExtMLW denote the multivariate local Whittle estimator of Shimotsu (2007) and our proposed extended multivariate local Whittle estimator, respectively.

Table 2: Simulation results for bias, RMSE, and variance comparison where $m = n^{0.65}$ and $\rho = 0.4$

		MLW				ExtMLW				Variance comparison		
		d_1		d_2		d_1		d_2				
d_1	d_2	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	$\frac{\text{var(LW)}}{\text{var(MLW)}}$	$\frac{\text{var(ExtLW)}}{\text{var(ExtMLW)}}$	$\frac{\text{var(ExtMLW)}}{\text{var(MLW)}}$
Panel A: $n = 512$												
0.2	0.2	−0.005	0.072	−0.001	0.068	−0.005	0.072	−0.001	0.068	(0.832, 0.867)	(0.832, 0.867)	(1.000, 1.000)
	0.4	−0.001	0.069	−0.002	0.072	0.000	0.070	0.001	0.075	(0.805, 0.813)	(0.820, 0.837)	(1.018, 1.088)
	0.8	0.000	0.070	0.014	0.075	−0.004	0.069	−0.004	0.070	(0.836, 0.844)	(0.807, 0.805)	(0.965, 0.901)
0.8	0.8	0.011	0.074	0.013	0.073	−0.007	0.072	−0.006	0.069	(0.774, 0.799)	(0.804, 0.799)	(0.974, 0.924)
1	1	−0.008	0.063	−0.014	0.066	−0.008	0.068	−0.013	0.073	(0.845, 0.830)	(0.832, 0.809)	(1.158, 1.225)
	1.2	0.000	0.067	−0.125	0.151	−0.009	0.069	−0.013	0.069	(0.916, 0.988)	(0.770, 0.804)	(1.014, 0.670)
	1.4	0.001	0.071	−0.328	0.346	−0.011	0.069	−0.009	0.066	(1.063, 1.039)	(0.832, 0.834)	(0.938, 0.348)
1.4	1.4	−0.162	0.181	−0.162	0.181	−0.008	0.067	−0.009	0.067	(0.565, 0.581)	(0.831, 0.817)	(0.721, 0.679)
Panel B: $n = 1024$												
0.2	0.2	−0.003	0.055	−0.005	0.053	−0.003	0.055	−0.005	0.053	(0.834, 0.785)	(0.834, 0.785)	(1.000, 1.000)
	0.4	−0.004	0.054	0.000	0.054	−0.002	0.055	0.004	0.059	(0.839, 0.801)	(0.863, 0.861)	(1.029, 1.215)
	0.8	0.001	0.055	0.014	0.060	−0.003	0.054	−0.003	0.055	(0.835, 0.826)	(0.808, 0.806)	(0.967, 0.885)
0.8	0.8	0.014	0.058	0.016	0.058	−0.005	0.053	−0.002	0.054	(0.774, 0.771)	(0.810, 0.847)	(0.881, 0.920)
1	1	−0.008	0.049	−0.007	0.048	−0.009	0.054	−0.007	0.053	(0.839, 0.800)	(0.825, 0.799)	(1.226, 1.259)
	1.2	0.004	0.053	−0.130	0.149	−0.006	0.052	−0.007	0.053	(1.015, 0.980)	(0.810, 0.831)	(0.972, 0.525)
	1.4	0.010	0.058	−0.329	0.345	−0.005	0.052	−0.004	0.051	(1.227, 1.037)	(0.803, 0.803)	(0.814, 0.244)
1.4	1.4	−0.151	0.166	−0.154	0.169	−0.002	0.050	−0.001	0.051	(0.434, 0.433)	(0.775, 0.858)	(0.520, 0.539)

Notes: MLW and ExtMLW denote the multivariate local Whittle estimator of Shimotsu (2007) and our proposed extended multivariate local Whittle estimator, respectively.

Table 3: Simulation results for bias, RMSE, and variance comparison where $m = n^{0.65}$ and $\rho = 0.8$

		MLW				ExtMLW				Variance comparison		
		d_1		d_2		d_1		d_2				
d_1	d_2	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	$\frac{\text{var(LW)}}{\text{var(MLW)}}$	$\frac{\text{var(ExtLW)}}{\text{var(ExtMLW)}}$	$\frac{\text{var(ExtMLW)}}{\text{var(MLW)}}$
Panel A: $n = 512$												
0.2	0.2	0.001	0.056	0.001	0.056	0.001	0.056	0.001	0.056	$(0.529, 0.528)$	$(0.529, 0.528)$	$(1.000, 1.000)$
	0.4	-0.001	0.057	0.001	0.057	0.003	0.060	0.005	0.062	$(0.576, 0.602)$	$(0.654, 0.677)$	$(1.135, 1.187)$
	0.8	0.014	0.060	0.017	0.066	0.002	0.058	0.003	0.061	$(0.599, 0.649)$	$(0.583, 0.620)$	$(0.973, 0.900)$
0.8	0.8	0.014	0.062	0.014	0.060	-0.005	0.059	-0.005	0.057	$(0.562, 0.515)$	$(0.609, 0.562)$	$(0.944, 0.945)$
1	1	-0.011	0.054	-0.012	0.056	-0.012	0.057	-0.014	0.059	$(0.599, 0.622)$	$(0.539, 0.571)$	$(1.095, 1.123)$
	1.2	-0.024	0.067	-0.129	0.152	-0.010	0.059	-0.011	0.060	$(0.830, 1.079)$	$(0.559, 0.597)$	$(0.855, 0.521)$
	1.4	-0.019	0.076	-0.333	0.352	-0.001	0.057	-0.002	0.058	$(1.135, 1.097)$	$(0.582, 0.631)$	$(0.598, 0.267)$
1.4	1.4	-0.160	0.176	-0.161	0.177	-0.007	0.057	-0.007	0.057	$(0.452, 0.474)$	$(0.575, 0.628)$	$(0.579, 0.592)$
Panel B: $n = 1024$												
0.2	0.2	-0.001	0.045	-0.002	0.044	-0.001	0.045	-0.002	0.044	$(0.570, 0.588)$	$(0.570, 0.588)$	$(1.000, 1.000)$
	0.4	0.001	0.044	0.002	0.045	0.003	0.047	0.005	0.049	$(0.559, 0.589)$	$(0.634, 0.616)$	$(1.133, 1.176)$
	0.8	0.017	0.050	0.020	0.055	0.003	0.044	0.001	0.045	$(0.670, 0.675)$	$(0.583, 0.604)$	$(0.869, 0.790)$
0.8	0.8	0.017	0.050	0.017	0.049	-0.003	0.045	-0.003	0.044	$(0.564, 0.550)$	$(0.587, 0.574)$	$(0.909, 0.933)$
1	1	-0.008	0.043	-0.009	0.042	-0.010	0.045	-0.009	0.046	$(0.588, 0.581)$	$(0.560, 0.551)$	$(1.101, 1.199)$
	1.2	-0.018	0.053	-0.130	0.150	-0.002	0.045	-0.002	0.044	$(0.903, 1.099)$	$(0.615, 0.590)$	$(0.828, 0.343)$
	1.4	-0.016	0.060	-0.337	0.353	0.000	0.043	0.001	0.045	$(1.255, 1.123)$	$(0.586, 0.644)$	$(0.561, 0.182)$
1.4	1.4	-0.155	0.168	-0.151	0.164	-0.002	0.043	-0.002	0.042	$(0.435, 0.366)$	$(0.587, 0.577)$	$(0.440, 0.460)$

Notes: MLW and ExtMLW denote the multivariate local Whittle estimator of Shimotsu (2007) and our proposed extended multivariate local Whittle estimator, respectively.

Table 4: Simulation results for the mean of $2\pi G$ and the mean of the coherence, $c = \sqrt{G_{12}^2/G_{11}G_{22}}$ where $m = n^{0.65}$ and $\rho = 0$

		MLW				ExtMLW			
d_1	d_2	$2\pi\hat{G}_{11}$	$2\pi\hat{G}_{12}$	$2\pi\hat{G}_{22}$	$\hat{c}$	$2\pi\hat{G}_{11}$	$2\pi\hat{G}_{12}$	$2\pi\hat{G}_{22}$	$\hat{c}$
Panel A: $n = 512$									
0.2	0.2	1.026	0.001	1.034	0.081	1.026	0.001	1.034	0.081
	0.4	1.042	-0.003	1.050	0.079	1.042	-0.002	1.037	0.079
	0.8	1.038	-0.002	1.154	0.078	1.039	-0.003	1.037	0.076
0.8	0.8	1.162	-0.003	1.165	0.127	1.064	0.000	1.051	0.078
1	1	2.068	0.013	2.085	0.280	1.060	-0.001	1.059	0.077
	1.2	1.968	-0.141	11.95	0.401	1.050	0.001	1.060	0.079
	1.4	1.906	0.624	189.79	0.474	1.058	0.002	1.070	0.076
1.4	1.4	110.44	-0.584	107.04	0.826	1.067	-0.003	1.059	0.078
Panel B: $n = 1024$									
0.2	0.2	1.036	0.007	1.032	0.060	1.036	0.007	1.032	0.060
	0.4	1.026	-0.002	1.025	0.059	1.026	-0.002	1.011	0.059
	0.8	1.026	0.003	1.098	0.059	1.027	0.002	1.029	0.060
0.8	0.8	1.103	-0.003	1.126	0.106	1.027	0.003	1.038	0.062
1	1	2.115	-0.016	2.057	0.280	1.052	-0.001	1.050	0.059
	1.2	1.911	0.026	16.09	0.411	1.044	-0.004	1.040	0.060
	1.4	1.882	-0.108	267.84	0.487	1.047	-0.003	1.049	0.060
1.4	1.4	155.79	4.427	168.54	0.846	1.049	-0.003	1.042	0.060

Notes: MLW and ExtMLW denote the multivariate local Whittle estimator of Shimotsu (2007) and our proposed extended multivariate local Whittle estimator, respectively. Furthermore, note that $2\pi G_{11} = 2\pi G_{22} = 1, 2\pi G_{12} = 2\pi G_{21} = \rho$.

Table 5: Simulation results for the mean of $2\pi G$ and the mean of the coherence, $c = \sqrt{G_{12}^2/G_{11}G_{22}}$ where $m = n^{0.65}$ and $\rho = 0.4$

		MLW				ExtMLW			
d_1	d_2	$2\pi\hat{G}_{11}$	$2\pi\hat{G}_{12}$	$2\pi\hat{G}_{22}$	$\hat{c}$	$2\pi\hat{G}_{11}$	$2\pi\hat{G}_{12}$	$2\pi\hat{G}_{22}$	$\hat{c}$
Panel A: $n = 512$									
0.2	0.2	1.027	0.406	1.012	0.399	1.027	0.406	1.012	0.399
	0.4	1.008	0.408	1.037	0.400	1.004	0.404	1.022	0.399
	0.8	1.016	0.406	1.143	0.380	1.029	0.413	1.031	0.402
0.8	0.8	1.162	0.461	1.155	0.395	1.045	0.419	1.037	0.402
1	1	2.161	0.876	2.106	0.439	1.049	0.421	1.056	0.402
	1.2	2.046	1.911	13.43	0.489	1.052	0.429	1.064	0.406
	1.4	1.888	4.013	168.83	0.491	1.056	0.431	1.073	0.405
1.4	1.4	106.76	45.60	107.73	0.837	1.070	0.431	1.060	0.405
Panel B: $n = 1024$									
0.2	0.2	1.021	0.413	1.029	0.403	1.021	0.413	1.029	0.403
	0.4	1.026	0.410	1.024	0.401	1.022	0.405	1.008	0.399
	0.8	1.014	0.405	1.116	0.383	1.027	0.414	1.032	0.402
0.8	0.8	1.127	0.452	1.111	0.402	1.038	0.415	1.026	0.402
1	1	2.063	0.868	2.086	0.437	1.049	0.418	1.046	0.401
	1.2	1.989	1.985	17.61	0.488	1.031	0.414	1.040	0.400
	1.4	1.972	5.768	265.13	0.508	1.027	0.409	1.039	0.397
1.4	1.4	147.16	58.04	147.96	0.849	1.029	0.408	1.032	0.397

Notes: MLW and ExtMLW denote the multivariate local Whittle estimator of Shimotsu (2007) and our proposed extended multivariate local Whittle estimator, respectively. Furthermore, note that $2\pi G_{11} = 2\pi G_{22} = 1, 2\pi G_{12} = 2\pi G_{21} = \rho$.

Table 6: Simulation results for the mean of $2\pi G$ and the mean of the coherence, $c = \sqrt{G_{12}^2/G_{11}G_{22}}$ where $m = n^{0.65}$ and $\rho = 0.8$

		MLW				ExtMLW			
d_1	d_2	$2\pi\hat{G}_{11}$	$2\pi\hat{G}_{12}$	$2\pi\hat{G}_{22}$	$\hat{c}$	$2\pi\hat{G}_{11}$	$2\pi\hat{G}_{12}$	$2\pi\hat{G}_{22}$	$\hat{c}$
Panel A: $n = 512$									
0.2	0.2	1.008	0.807	1.011	0.798	1.008	0.807	1.011	0.798
	0.4	1.015	0.814	1.025	0.797	1.008	0.806	1.013	0.797
	0.8	0.972	0.782	1.133	0.751	1.001	0.798	1.003	0.797
0.8	0.8	1.149	0.922	1.154	0.797	1.031	0.827	1.034	0.801
1	1	2.082	1.686	2.097	0.790	1.049	0.843	1.054	0.801
	1.2	2.123	3.429	12.42	0.686	1.041	0.837	1.054	0.799
	1.4	2.077	9.593	185.46	0.558	1.021	0.822	1.046	0.795
1.4	1.4	107.54	82.97	104.84	0.863	1.062	0.847	1.054	0.800
Panel B: $n = 1024$									
0.2	0.2	1.019	0.817	1.020	0.801	1.019	0.817	1.020	0.801
	0.4	1.003	0.802	1.012	0.796	0.997	0.797	1.003	0.796
	0.8	0.965	0.776	1.096	0.758	1.006	0.808	1.016	0.799
0.8	0.8	1.106	0.883	1.100	0.798	1.021	0.816	1.018	0.800
1	1	2.050	1.643	2.061	0.778	1.042	0.830	1.037	0.798
	1.2	2.118	3.962	16.40	0.684	1.018	0.814	1.022	0.797
	1.4	2.189	12.84	292.01	0.581	1.014	0.812	1.024	0.797
1.4	1.4	149.98	117.58	149.46	0.877	1.042	0.833	1.039	0.800

Notes: MLW and ExtMLW denote the multivariate local Whittle estimator of Shimotsu (2007) and our proposed extended multivariate local Whittle estimator, respectively. Furthermore, note that
$2\pi G_{11} = 2\pi G_{22} = 1, 2\pi G_{12} = 2\pi G_{21} = \rho$.

Table 7: Rejection frequency with 0.05 asymptotic critical value for $m = n^{0.65}$

		$\rho = 0$		$\rho = 0.4$		$\rho = 0.8$	
d_1	d_2	W_{MLW}	W_{ExtMLW}	W_{MLW}	W_{ExtMLW}	W_{MLW}	W_{ExtMLW}
Panel A: $n = 512$							
0.2	0.2	0.045	0.045	0.074	0.074	0.049	0.049
	0.4	0.060	0.063	0.078	0.079	0.056	0.061
	0.8	0.081	0.068	0.075	0.070	0.078	0.075
0.8	0.8	0.086	0.079	0.076	0.063	0.060	0.061
1	1	0.067	0.070	0.063	0.073	0.061	0.061
	1.2	0.552	0.052	0.591	0.058	0.628	0.072
	1.6	0.925	0.067	0.912	0.065	0.924	0.057
1.4	1.4	0.821	0.056	0.827	0.047	0.822	0.048
Panel B: $n = 1024$							
0.2	0.2	0.068	0.068	0.080	0.080	0.053	0.053
	0.4	0.064	0.064	0.066	0.068	0.070	0.071
	0.8	0.094	0.063	0.090	0.080	0.106	0.065
0.8	0.8	0.092	0.058	0.102	0.065	0.086	0.056
1	1	0.045	0.062	0.041	0.069	0.057	0.071
	1.2	0.682	0.070	0.722	0.057	0.739	0.064
	1.4	0.957	0.058	0.948	0.057	0.953	0.058
1.4	1.4	0.948	0.041	0.931	0.044	0.931	0.051

Notes: MLW and ExtMLW denote the multivariate local Whittle estimator of Shimotsu (2007) and our proposed extended multivariate local Whittle estimator, respectively. W denotes the modified Wald statistic of Hurvich & Chen (2000) for the different semiparametric multivariate estimators.

Table 8: Rejection frequency with 0.05 asymptotic critical value for $m = n^{0.65}$ where we test that $d_1 = d_2 = 1$

		$\rho = 0$		$\rho = 0.4$		$\rho = 0.8$	
d_1	d_2	W_{MLW}	W_{ExtMLW}	W_{MLW}	W_{ExtMLW}	W_{MLW}	W_{ExtMLW}
Panel A: $n = 512$							
1	0.40	1.000	1.000	1.000	1.000	1.000	1.000
	0.70	0.938	0.959	0.967	0.990	1.000	1.000
	0.75	0.814	0.886	0.864	0.938	0.991	1.000
	0.80	0.589	0.696	0.697	0.815	0.952	0.996
	0.90	0.201	0.294	0.215	0.321	0.469	0.683
	0.95	0.089	0.137	0.099	0.138	0.150	0.248
Panel B: $n = 1024$							
1	0.40	1.000	1.000	1.000	1.000	1.000	1.000
	0.70	0.995	0.999	0.998	1.000	1.000	1.000
	0.75	0.950	0.980	0.982	1.000	0.999	1.000
	0.80	0.828	0.914	0.877	0.953	0.996	1.000
	0.90	0.268	0.389	0.311	0.465	0.673	0.885
	0.95	0.100	0.144	0.100	0.149	0.209	0.345

Notes: MLW and ExtMLW denote the multivariate local Whittle estimator of Shimotsu (2007) and our proposed extended multivariate local Whittle estimator, respectively. W denotes the modified Wald statistic of Hurvich & Chen (2000) for the different semiparametric multivariate estimators.

Table 9: Long memory estimates

	CAN	SW	FRA	GER	ITA	JPN	UK
Panel A: $m = \lfloor n^{0.5} \rfloor$							
MLW	0.975	0.916	1.005	0.965	0.928	1.043	0.895
	(0.059)	(0.052)	(0.049)	(0.052)	(0.048)	(0.061)	(0.063)
ExtMLW	1.178	1.011	1.129	1.052	1.082	1.197	0.996
	(0.099)	(0.060)	(0.053)	(0.055)	(0.061)	(0.090)	(0.074)
Panel B: $m = \lfloor n^{0.65} \rfloor$							
MLW	1.000	0.940	1.001	0.967	0.966	0.998	0.985
	(0.037)	(0.034)	(0.033)	(0.034)	(0.032)	(0.040)	(0.041)
ExtMLW	1.017	0.972	1.061	1.014	1.020	1.045	1.026
	(0.069)	(0.038)	(0.037)	(0.037)	(0.042)	(0.056)	(0.048)

Notes: The table shows long memory estimates from the MLW and ExtMLW estimators with standard errors in parentheses.

Table 10: Estimated covariance matrix

	$\hat{\Omega}_{MLW}$							$\hat{\Omega}_{ExtMLW}$						
Panel A: $m = \lfloor n^{0.5} \rfloor$														
	CAN	SW	FRA	GER	ITA	JPN	UK	CAN	SW	FRA	GER	ITA	JPN	UK
CAN	0.063	0.029	0.031	0.028	0.037	0.028	0.031	0.097	0.012	0.015	0.012	0.013	0.016	0.011
SW		0.049	0.033	0.042	0.032	0.035	0.026		0.066	0.042	0.048	0.032	0.026	0.020
FRA			0.044	0.035	0.038	0.030	0.035			0.051	0.045	0.044	0.016	0.033
GER				0.049	0.032	0.033	0.027				0.055	0.039	0.021	0.025
ITA					0.042	0.031	0.034					0.069	0.010	0.039
JPN						0.067	0.022						0.148	0.019
UK							0.071							0.099
Panel B: $m = \lfloor n^{0.65} \rfloor$														
CAN	0.060	0.030	0.032	0.030	0.035	0.026	0.034	0.101	0.015	0.014	0.024	0.016	0.023	0.025
SW		0.050	0.033	0.040	0.031	0.037	0.025		0.064	0.041	0.047	0.032	0.033	0.025
FRA			0.049	0.036	0.039	0.029	0.030			0.060	0.048	0.045	0.021	0.027
GER				0.050	0.032	0.033	0.026				0.059	0.037	0.024	0.027
ITA					0.045	0.029	0.035					0.077	0.013	0.037
JPN						0.069	0.022						0.139	0.015
UK							0.074							0.090

Notes: The table shows the estimated covariance matrix for the MLW and ExtMLW estimates.

Table 11: Normalized G

	Normalized $\hat{G}_{MLW}$							Normalized $\hat{G}_{ExtMLW}$						
Panel A: $m = \lfloor n^{0.5} \rfloor$														
	CAN	SW	FRA	GER	ITA	JPN	UK	CAN	SW	FRA	GER	ITA	JPN	UK
CAN	1.000	-0.607	0.424	-0.278	0.794	-0.728	0.576	1.000	-0.012	-0.172	-0.050	-0.103	-0.056	0.022
SW		1.000	0.225	0.856	-0.294	0.764	-0.050		1.000	0.856	0.914	0.709	0.484	0.564
FRA			1.000	0.617	0.826	-0.184	0.807			1.000	0.929	0.884	0.376	0.705
GER				1.000	0.139	0.509	0.312				1.000	0.819	0.444	0.643
ITA					1.000	-0.602	0.822					1.000	0.234	0.709
JPN						1.000	-0.311						1.000	0.353
UK							1.000							1.000
Panel B: $m = \lfloor n^{0.65} \rfloor$														
CAN	1.000	-0.605	0.483	-0.280	0.822	-0.685	0.632	1.000	0.083	0.013	0.046	0.077	0.068	0.245
SW		1.000	0.172	0.844	-0.357	0.813	-0.053		1.000	0.856	0.905	0.734	0.607	0.639
FRA			1.000	0.577	0.799	-0.131	0.762			1.000	0.924	0.840	0.508	0.647
GER				1.000	0.068	0.579	0.297				1.000	0.796	0.541	0.661
ITA					1.000	-0.565	0.797					1.000	0.406	0.688
JPN						1.000	-0.266						1.000	0.409
UK							1.000							1.000

Notes: The table shows the normalized long-run covariance matrix for the MLW and ExtMLW estimates.

Table 12: Wald statistics for testing the null hypothesis H1 and H2.

Bandwidth	MLW		ExtMLW	
	W_1	W_2	W_1	W_2
$m = \lfloor n^{0.5} \rfloor$	14.55*	14.09	13.65	11.03
$m = \lfloor n^{0.65} \rfloor$	7.85	7.49	8.20	7.73

Notes: $W_1 \xrightarrow{d} \chi^2_7(0.95) = 14.067$ and $W_2 \xrightarrow{d} \chi^2_6(0.95) = 12.592$.

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