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Multipower Variation for Brownian Semistationary Processes

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Abstract

In this paper we study the asymptotic behaviour of power and multipower variations of processes Y :

$$Y_t = \int_{-\infty}^t g(t-s)\sigma_s W(ds) + Z_t$$

where $g : (0, \infty) \rightarrow \mathbb{R}$ is deterministic, $\sigma > 0$ is a random process, W is the stochastic Wiener measure, and Z is a stochastic process in the nature of a drift term. Processes of this type serve, in particular, to analyse data of velocity increments of a fluid in a turbulence regime with spot intermittency σ . The purpose of the present paper is to determine the probabilistic limit behaviour of the (multi)power variations of Y , as a basis for studying properties of the intermittency process σ . Notably the processes Y are in general not of the semimartingale kind and the established theory of multipower variation for semimartingales does not suffice for deriving the limit properties. As a key tool for the results a general central limit theorem for triangular Gaussian schemes is formulated and proved. Examples and an application to realised variance ratio are given.

Keywords: Central Limit Theorem; Gaussian Processes; Intermittency; Nonsemimartingales; Turbulence; Volatility; Wiener Chaos.

JEL Classification: C10; C80.

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1 Introduction

The motivation for the development of the results reported in this paper has been the need to construct tools for studying the probabilistic limit behaviour of (realised) quadratic variation and other multipower variations in relation to the class of *Brownian semistationary* ($\mathcal{BSS}$) processes. This class, which was introduced in [BNSch09], consists of the processes $Y = \{Y_t\}_{t \in \mathbb{R}}$ that are defined by

$$Y_t = \mu + \int_{-\infty}^t g(t-s)\sigma_s W(ds) + \int_{-\infty}^t q(t-s)a_s ds \quad (1.1)$$

where μ is a constant, W is the stochastic Wiener measure, g and q are nonnegative deterministic functions on $\mathbb{R}$, with $g(t) = q(t) = 0$ for $t \leq 0$, and σ and a are càdlàg processes. When σ and a are stationary then so is Y . Hence the name Brownian semistationary processes. The $\mathcal{BSS}$ processes form the natural analogue, for stationarity related processes, to the class $\mathcal{BSM}$ of Brownian semimartingales

$$Y_t = \mu + \int_0^t \sigma_s dW_s + \int_0^t a_s ds. \quad (1.2)$$

In the context of stochastic modelling in finance and in turbulence the process σ embodies the *volatility* or *intermittency* of the dynamics, whether the framework is that of $\mathcal{BSM}$ or $\mathcal{BSS}$. For detailed discussion of $\mathcal{BSS}$ and the more general concept of tempo-spatial ambit processes see [BNSch04], [BNSch07], [BNSch08a], [BNSch08b], [BNSch08c], [BNSch09]. Such processes are, in particular, able to reproduce key stylized features of turbulent data.

A main difference between $\mathcal{BSM}$ and $\mathcal{BSS}$ is that in general models of the $\mathcal{BSS}$ form are not semimartingales (for a discussion of this, see Section 3 of [BNSch09]). In consequence, various important techniques developed for semimartingales, such as the calculation of quadratic variation by Ito algebra and those of multipower variation, do not apply or suffice in $\mathcal{BSS}$ settings. The present paper addresses some of the issues that this raises.

The theory of multipower variation was primarily developed as a basis for inference on σ under $\mathcal{BSM}$ models and, more generally Ito semimartingales, with particular focus on inference about the integrated squared volatility σ^{2+} given by

$$\sigma_t^{2+} = \int_0^t \sigma_s^2 ds. \quad (1.3)$$

This quantity is likewise a focal point for the results discussed in the following.

Section 2 introduces common notation for multipower variation and recalls some basic properties of such quantities. A law of large numbers and a central limit theorem for multipower variation of triangular arrays of Gaussian random variables are derived in Section 3, and these limit results are drawn upon in Section 4 to establish probability and central limit theorems for multipower variation for $\mathcal{BSS}$ processes, with most of the proofs postponed to the penultimate Section 7. Section 5 presents several examples, and Section 6 discusses an application

concerning the limit behaviour of the realised variation ratio, i.e. the ratio of realised bipower variation to realised quadratic variation. The final Section 8 concludes and indicates some possible directions for further related work. An appendix summarises a number of properties of the modified Bessel functions of the modified third kind, needed for derivations in Section 5.

2 Multipower Variation

The concept of (realised) multipower variation was originally introduced in [BNS04a] in the context of semimartingales, and the mathematical theory has been studied in a number of papers ([BNGJPS06], [Jac08a], [BNSW06], [KiPo08]) while various applications are the main subjects in ([BNS04b]), [BNS06], [BNS07], [Jac08b], [Woe06]. Multipower variation turns out to be useful for analysing properties of parts of a process that are not directly observable. In this section we present the definition of realised multipower variation and recall its asymptotic properties for some classes of processes.

Let us consider a continuous-time process X , defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$, that is observed at equidistant time points $t_i = i/n$, $i = 0, \dots, [nt]$. A realised multipower variation of the process X is an object of the type

$$\sum_{i=1}^{[nt]-k+1} \prod_{j=1}^k |\Delta_{i+j-1}^n X|^{p_j}, \quad \Delta_i^n X = X_{\frac{i}{n}} - X_{\frac{i-1}{n}}, \quad p_1, \dots, p_k \geq 0, \quad (2.1)$$

for some fixed number $k \geq 1$. We now present an overview of the asymptotic theory for quantities of the form (2.1) for various types of processes X .

We start with the $\mathcal{BSM}$ case

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t \sigma_s dW_s \quad (2.2)$$

where W is a Brownian motion, a is a locally bounded and predictable drift process, and σ is an adapted and càdlàg volatility process. As was established in [BNGJPS06], the convergence in probability

$$n^{p_+/2-1} \sum_{i=1}^{[nt]-k+1} \prod_{j=1}^k |\Delta_{i+j-1}^n X|^{p_j} \xrightarrow{ucp} \mu_{p_1} \cdots \mu_{p_k} \int_0^t |\sigma_s|^{p_+} ds \quad (2.3)$$

holds, where $p_+ = \sum_{j=1}^k p_j$ and $\mu_p = E[|u|^p]$, $u \sim N(0, 1)$ and we write $Z^n \xrightarrow{ucp} Z$ when $\sup_{t \in [0, T]} |Z_t^n - Z_t| \xrightarrow{P} 0$ for any $T > 0$. Under a further condition on the volatility process one obtains the associated stable central limit theorem:

$$\sqrt{n} \left(n^{p_+/2-1} \sum_{i=1}^{[nt]-k+1} \prod_{j=1}^k |\Delta_{i+j-1}^n X|^{p_j} - \mu_{p_1} \cdots \mu_{p_k} \int_0^t |\sigma_s|^{p_+} ds \right) \xrightarrow{st} \sqrt{A} \int_0^t |\sigma_s|^{p_+} dB_s, \quad (2.4)$$

where B is another Brownian motion, defined on an extension of the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ and independent of $\mathcal{F}$, and the constant A is given by

$$A = \prod_{l=1}^k \mu_{2p_l} - (2k-1) \prod_{l=1}^k \mu_{p_l}^2 + 2 \sum_{m=1}^{k-1} \prod_{l=1}^m \mu_{p_l} \prod_{l=k-m+1}^k \mu_{p_l} \prod_{l=1}^{k-m} \mu_{p_l+p_{l+m}}.$$

Recall that the stable convergence of processes is defined as follows. A sequence of processes Z^n converges stably in law towards the process Z (written $Z^n \xrightarrow{st} Z$), that is defined on the extension of the original probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$, if and only if for any bounded and continuous real-valued functional f and any $\mathcal{F}$ -measurable random variable V it holds that

$$\lim_{n \rightarrow \infty} E[f(Z^n)V] = E[f(Z)V].$$

When the latter holds only for $\mathcal{G}$ -measurable random variables V , where $\mathcal{G}$ is a sub- σ -algebra of $\mathcal{F}$, we speak of $\mathcal{G}$ -stable convergence and use the notation $Z^n \xrightarrow{\mathcal{G}-st} Z$.

A crucial property of the realised multipower variation is its robustness to jumps when $\max_i(p_i) < 2$ ([BNSW06], [Jac08c]). Assume for a moment that X is a general Ito semimartingale with continuous part X^c satisfying (2.2). Then, by (2.3) and the robustness property, we obtain the convergence

$$\frac{\mu_1^{-2} \sum_{i=1}^{[nt]-1} |\Delta_i^n X| |\Delta_{i+1}^n X|}{\sum_{i=1}^{[nt]} |\Delta_i^n X|^2} \xrightarrow{P} \frac{[X^c]}{[X]},$$

where $[X]$ denotes the quadratic variation of the semimartingale X and the limit is less than or equal to 1. The latter result, together with the stable convergence in (2.4), can be used to construct a formal test for jumps (see [BNS04a]). On the other hand, we know that if the limit of the left-hand side is greater than 1 (which is the case for some typical turbulence data), the process X can not be an Ito semimartingale.

In another direction, a study [BNCPW09] was made of the asymptotic behaviour of multipower variation for processes of the type

$$X_t = X_0 + \int_0^t \sigma_s dG_s, \quad t \geq 0, \quad (2.5)$$

where G is a continuous Gaussian process with centered and stationary increments (the latter integral is defined as a Riemann-Stieltjes integral). The process defined in (2.5) is, in general, also not a semimartingale, and the theory in [BNGJPS06] does not apply. In particular, a different normalisation is required. Define the (normalised) multipower variation by

$$V(X, p_1, \dots, p_k)_t^n = \frac{1}{n\tau_n^{p_+}} \sum_{i=1}^{[nt]-k+1} \prod_{j=1}^k |\Delta_{i+j-1}^n X|^{p_j}, \quad p_1, \dots, p_k \geq 0,$$

where $\tau_n > 0$ is given by

$$\tau_n^2 = \bar{R}(1/n) \quad (2.6)$$

with

$$\bar{R}(t) = E[(G_{s+t} - G_s)^2]. \quad (2.7)$$

Under some assumptions on $\bar{R}$ and the volatility process σ it was shown that

$$V(X, p_1, \dots, p_k)_t^n \xrightarrow{ucp} \rho_{p_1, \dots, p_k} \int_0^t |\sigma_s|^{p^+} ds$$

for a certain constant $\rho_{p_1, \dots, p_k}$ which depends on the behaviour of $\bar{R}$ near 0. Furthermore, an associated (stable) central limit theorem, of a form similar to (2.4), was derived. Note however that in general there are essential differences between the characters of **BSS** processes and processes of type (2.5).

3 Multipower variation of Gaussian triangular arrays

In this subsection we derive some asymptotic results for functionals of arrays of stationary Gaussian sequences. We consider a triangular array $(X_{i,n})_{n \geq 1, 1 \leq i \leq [nt]}$ ($t > 0$) of row-wise stationary Gaussian variables with mean 0 and variance 1. Let

$$r_n(j) = \text{cor}(X_{1,n}, X_{1+j,n}) , \quad j \geq 0, \quad (3.1)$$

be the correlation function of $(X_{i,n})_{1 \leq i \leq [nt]}$. Assume that the array $(X_{i,n})_{n \geq 1, 1 \leq i \leq [nt]}$ is "non-degenerate", i.e. the covariance matrix of $(X_{i,n}, \dots, X_{i+k,n})$ is invertible for any $k \geq 1$ and $n \geq 1$ (otherwise the results below do not hold).

Now, define the multipower variation associated with the sequence $(X_{i,n})_{n \geq 1, 1 \leq i \leq [nt]}$:

$$V(p_1, \dots, p_k)_t^n = \frac{1}{n} \sum_{i=1}^{[nt]-k+1} \prod_{j=1}^k |X_{i+j-1,n}|^{p_j} , \quad p_1, \dots, p_k \geq 0. \quad (3.2)$$

Our first result is the weak law of large numbers.

Theorem 1 *Assume that there exists a sequence $r(j)$ with*

$$r_n^2(j) \leq r(j) , \quad \frac{1}{n} \sum_{j=1}^{n-1} r(j) \rightarrow 0 \quad (3.3)$$

as $n \rightarrow \infty$. Then it holds that

$$V(p_1, \dots, p_k)_t^n - \rho_{p_1, \dots, p_k}^{(n)} t \xrightarrow{ucp} 0 , \quad (3.4)$$

where

$$\rho_{p_1, \dots, p_k}^{(n)} = E[|X_{1,n}|^{p_1} \dots |X_{k,n}|^{p_k}]. \quad (3.5)$$

Proof of Theorem 1: See Section 7.

Before we present the associated central limit theorem we need to introduce another Gaussian process. Suppose that $r_n(j) \rightarrow \rho(j)$, $j = 1, \dots, k-1$, for some numbers $\rho(j)$. Let $(Q_i)_{i \geq 1}$ be a non-degenerate stationary centered (discrete time) Gaussian process with expectation 0, variance 1 and correlation function

$$\rho(j) = \text{cor}(Q_1, Q_{1+j}) , \quad j \geq 1. \quad (3.6)$$

Define

$$V_Q(p_1, \dots, p_k)_t^n = \frac{1}{n} \sum_{i=1}^{[nt]-k+1} \prod_{j=1}^k |Q_{i+j-1}|^{p_j} \quad (3.7)$$

and let $\rho_{p_1, \dots, p_k} = E(|Q_1|^{p_1} \dots |Q_k|^{p_k})$. Then $\rho_{p_1, \dots, p_k}^{(n)} \rightarrow \rho_{p_1, \dots, p_k}$ and in this case we obtain the ucp convergence

$$V(p_1, \dots, p_k)_t^n \xrightarrow{ucp} \rho_{p_1, \dots, p_k} t.$$

The multivariate central limit theorem for the family $(V(p_1^j, \dots, p_k^j)_t^n)_{1 \leq j \leq d}$ is as follows:

Theorem 2 *Assume that*

$$r_n(j) \rightarrow \rho(j) , \quad j \geq 0, \quad (3.8)$$

where $\rho(j)$ is the correlation function of some stationary centered discrete time Gaussian process $(Q_i)_{i \geq 1}$ with $E[Q_i^2] = 1$. Suppose furthermore that, for any $j, n \geq 1$, there exists a sequence $r(j)$ with

$$r_n^2(j) \leq r(j) , \quad \sum_{j=1}^{\infty} r(j) < \infty. \quad (3.9)$$

Then we have

$$\sqrt{n} \left(V(p_1^j, \dots, p_k^j)_t^n - \rho_{p_1^j, \dots, p_k^j}^{(n)} t \right)_{1 \leq j \leq d} \Rightarrow \beta^{1/2} B_t , \quad (3.10)$$

where B is a d -dimensional Brownian, β is a $d \times d$ -dimensional matrix given by

$$\beta_{ij} = \lim_{n \rightarrow \infty} n \text{ cov} \left(V_Q(p_1^i, \dots, p_k^i)_1^n, V_Q(p_1^j, \dots, p_k^j)_1^n \right) , \quad 1 \leq i, j \leq d , \quad (3.11)$$

and the convergence holds in the space $\mathcal{D}([0, T]^d)$ equipped with the uniform topology.

Proof of Theorem 2: See Section 7.

Remark 1 *It is possible that the condition (3.8) in its present form is not required. However, it leads to an explicit formula for the asymptotic variance in the central limit theorem. In order to allow substitution of $\rho_{p_1^j, \dots, p_k^j}^{(n)}$ by $\rho_{p_1^j, \dots, p_k^j}$ in (3.10) the requirement (3.8) must be strengthened so as to ensure that $\sqrt{n} \left(\rho_{p_1^j, \dots, p_k^j}^{(n)} - \rho_{p_1^j, \dots, p_k^j} \right) \rightarrow 0$. For further discussion of this, see Remark 13 in Section 4 and Section 5.3.*

Remark 2 *Similar asymptotic results can be obtained for general quantities of the form*

$$\frac{1}{n} \sum_{i=1}^{[nt]-k+1} H(X_{i,n}, \dots, X_{i+k-1,n}) \quad (3.12)$$

for some function $H : \mathbb{R}^k \rightarrow \mathbb{R}$. Let m denote the Hermite index of H (notice that the Hermite index of the power function used in (3.2) is 2). Replace the condition (3.3) by

$$|r_n^m(j)| \leq r(j) , \quad \frac{1}{n} \sum_{j=1}^{n-1} r(j) \rightarrow 0$$

and (3.9) by

$$|r_n^m(j)| \leq r(j) , \quad \sum_{j=1}^{\infty} r(j) < \infty.$$

Then Theorem 1 and 2 hold true for the functional (3.12) provided that $EH^2(N_k(0, \Sigma)) < \infty$ for any invertible $\Sigma \in \mathbb{R}^{k \times k}$. We omit the details.

Remark 3 *Ho and Sun [HoSu87] have shown a non-functional version of Theorem 2 for statistics of the type (3.12) when the correlation function r_n does not depend on n . To the best of our knowledge Theorem 2 is the first central limit theorem for (general) multipower variation of a row-wise stationary Gaussian process.*

4 Multipower variation for $\mathcal{BSS}$ processes

Armed with the general theorems proved in Section 3 we are now set to establish laws of large numbers and central limit results for multipower variations in the framework of the Brownian semistationary processes. The regularity conditions invoked are given in a first subsection, while the next states the theorems, the main parts of the proofs being postponed to Section 7; the third subsection discusses the nature of the, rather technical, regularity conditions and describes a set of simpler assumptions that are more amenable to checking.

4.1 Conditions

We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$, assuming the existence thereon of a $\mathcal{BSS}$ process, for the time being without drift term, i.e.

$$Y_t = \int_{-\infty}^t g(t-s) \sigma_s W(ds) \quad (4.1)$$

where W is an $\mathcal{F}_t$ -stochastic Wiener measure, σ is an $\mathcal{F}_t$ -adapted and càdlàg volatility process and $g : (0, \infty) \rightarrow \mathbb{R}$ is a deterministic continuous memory function with $g \in L^2((0, \infty))$. We also require $\int_{-\infty}^t g^2(t-s) \sigma_s^2 ds < \infty$ a.s. to ensure that $Y_t < \infty$ a.s. for all $t \geq 0$. By an

$\mathcal{F}_t$ -stochastic Wiener measure we understand a Gaussian stochastic measure such that, for any Borelian set A with $E(W(A)^2) < \infty$

$$W(A) \sim N(0, m(A)),$$

where m is the Lebesgue measure, and if $A \subseteq [t, +\infty)$ then $W(A)$ is independent of $\mathcal{F}_t$. Note that $\{B_t := \int_a^t W(ds), t \geq a\}$ is a standard Brownian motion starting in a .

The process Y is assumed to be observed at time points $t_i = i/n$, $i = 1, \dots, [nt]$. Now, let G be the stationary Gaussian process defined as

$$G_t = \int_{-\infty}^t g(t-s)W(ds). \quad (4.2)$$

This is an important auxiliary object in the study of $\mathcal{BSS}$ processes. Note that G belongs to the type of processes occurring in (2.5), and that the autocorrelation function of G is

$$r(t) = \int_0^\infty \bar{g}(t+u)\bar{g}(u)du, \quad (4.3)$$

where $\bar{g}(t) = g(t)/\|g\|$ with $\|g\|$ the L^2 norm of g . We are interested in the asymptotic behaviour of the functionals

$$V(Y, p_1, \dots, p_k)_t^n = \frac{1}{n\tau_n^{p_+}} \sum_{i=1}^{[nt]-k+1} \prod_{j=1}^k |\Delta_{i+j-1}^n Y|^{p_j}, \quad p_1, \dots, p_k \geq 0,$$

where $\Delta_i^n Y = Y_{\frac{i}{n}} - Y_{\frac{i-1}{n}}$ and $\tau_n^2 = \bar{R}(1/n)$ with $\bar{R}(t) = E[|G_{s+t} - G_s|^2]$, $t \geq 0$. In the following we assume that the function g is continuously differentiable on $(0, \infty)$, g' is non-increasing on (b, ∞) for some $b > 0$ and $g' \in L^2((\varepsilon, \infty))$ for any $\varepsilon > 0$. Moreover, we assume that for any $t > 0$

$$F_t = \int_1^\infty (g'(s))^2 \sigma_{t-s}^2 ds < \infty \quad (4.4)$$

almost surely. We shall extend the domain of g to $\mathbb{R}$ by taking $g(x) = 0$, for $x \leq 0$.

Remark 4 *The assumption (4.4) ensures that the process Y has the same "smoothness" as the process G (see Lemma 1 in Section 7). It is rather easy to check in practice, because it is implied by the condition $EF_t < \infty$ for $t > 0$. Furthermore, if g has bounded support assumption (4.4) is trivially fulfilled since σ is càdlàg.*

Remark 5 *Let us note again that the process Y is, in general, not a semimartingale. In particular, this is the case when $g' \notin L^2((0, \infty))$. For a closer discussion, see [BNSch09]. On the other hand, the process Y is not of the form (2.5). Thus, we require new methods to prove the asymptotic results for $V(Y, p_1, \dots, p_k)_t^n$. Processes of the form (4.1) are used for modelling velocity of turbulent flows, see [BNSch07], [BNSch08a], [BNSch08b]. The function g , that is used in such models, behaves often as t^δ near the origin. Hence, when $\delta \in (-1/2, 1/2) \setminus \{0\}$, Y is neither a differentiable process nor a semimartingale (because $g' \notin L^2((0, \infty))$). This is the primary case of our interest.*

We define the correlation function of the increments of G :

$$r_n(j) = \text{cov}\left(\frac{\Delta_1^n G}{\tau_n}, \frac{\Delta_{1+j}^n G}{\tau_n}\right) = \frac{\bar{R}(\frac{j+1}{n}) + \bar{R}(\frac{j-1}{n}) - 2\bar{R}(\frac{j}{n})}{2\tau_n^2}, \quad j \geq 0.$$

Next, we introduce a class of measures that is crucial for our purposes. We define (recall that $g(x) := 0$ for $x \leq 0$)

$$\pi^n(A) = \frac{\int_A (g(x - \frac{1}{n}) - g(x))^2 dx}{\int_0^\infty (g(x - \frac{1}{n}) - g(x))^2 dx}, \quad A \in \mathcal{B}(\mathbb{R}). \quad (4.5)$$

We further set $\bar{\pi}^n(x) = \pi^n(\{y : y > x\})$. Note that π^n is a probability measure on $\mathbb{R}_+$.

For the weak law of large numbers we require the following assumptions:

(LLN): There exists a sequence $r(j)$ with

$$r_n^2(j) \leq r(j), \quad \frac{1}{n} \sum_{j=1}^{n-1} r(j) \rightarrow 0.$$

Moreover, it holds that

$$\lim_{n \rightarrow \infty} \bar{\pi}^n(\varepsilon) = 0 \quad (4.6)$$

for any $\varepsilon > 0$.

Remark 6 (i) The first condition of (LLN) is adapted from Theorem 1. It guarantees the ucp convergence of $V(G, p_1, \dots, p_k)_t^n$. The second condition of (LLN) says that the whole mass of the measure π^n concentrates at 0. In particular, it is equivalent to the weak convergence

$$\pi^n \rightarrow \delta_0,$$

where δ_0 is the Dirac measure at 0.

(ii) The condition (4.6) is absolutely crucial for the limit theorems given in the next subsection. When this condition is violated things become more complicated. In particular, it may lead to a different stochastic limit of $V(Y, p_1, \dots, p_k)_t^n$ (see an example in Section 5). Intuitively, this can be explained by the observation that the increments $\Delta_i^n Y$ contain substantial information about the volatility (far) outside of the interval $[\frac{i-1}{n}, \frac{i}{n}]$ when the condition (4.6) does not hold. Thus, in general, we can not expect the limit described in Theorem 3 below.

Now, we introduce the assumptions for the central limit theorem:

(CLT): Assumption **(LLN)** holds, and

$$r_n(j) \rightarrow \rho(j), \quad j \geq 0,$$

where $\rho(j)$ is the correlation function of $(Q_i)_{i \geq 1}$, as introduced in (3.6). Furthermore, there exists a sequence $r(j)$ such that, for any $j, n \geq 1$,

$$r_n^2(j) \leq r(j), \quad \sum_{j=1}^{\infty} r(j) < \infty,$$

and for some $\gamma \in (0, 1]$ we have

$$E[|\sigma_t - \sigma_s|^2] \leq C|t - s|^{2\gamma}. \quad (4.7)$$

Finally, set $p = \min_{1 \leq i \leq k, 1 \leq j \leq d} d(p_i^j)$. Assume that $\gamma(p \wedge 1) > \frac{1}{2}$ and that there exists a constant $\lambda < -\frac{1}{p \wedge 1}$ such that for any $\varepsilon_n = O(n^{-\kappa})$, $\kappa \in (0, 1)$, we have

$$\bar{\pi}^n(\varepsilon_n) = O(n^{\lambda(1-\kappa)}). \quad (4.8)$$

Remark 7 Note that if σ is stationary with increasing complementary autocorrelation function $\bar{\omega}$, say, then condition (4.7) reduces to $\bar{\omega} < C u^{2\gamma}$. In particular, for the realized quadratic variation we require $1/2 < \gamma \leq 1$ meaning that $\varpi(u)$ should go to 0 quite fast or, equivalently, that $\omega(u)$ tends fast to 1. In other words, σ should not vary too fast.

Remark 8 Assumption (4.7) is only one of a variety of possible regularity conditions on σ that can lead to a central limit theorem for multipower variations of the kind we are after, and it is some way away from being necessary. For instance, Theorem 5 below will also hold if σ is a sum of two processes, one of which satisfies (4.7) and the other having the property that (almost) every sample path is of bounded variation on finite intervals.

Remark 9 The first part of assumption (CLT) ensures the weak convergence of the standardized version of $V(G, p_1, \dots, p_k)_t^n$. The condition (4.8) is certainly stronger than (4.6) in (LLN). In Section 4.3 we will explain how (4.8) can be checked in practice.

4.2 Limit theorems

In this section we present the limit laws of multipower variations of $\mathcal{BSS}$ processes, in part widening the scope slightly to allow more general drift terms. Recall that the (realised) multipower variation of a process Y of the form (4.1) is defined as

$$V(Y, p_1, \dots, p_k)_t^n = \frac{1}{n\tau_n^{p_+}} \sum_{i=1}^{[nt]-k+1} \prod_{j=1}^k |\Delta_{i+j-1}^n Y|^{p_j}, \quad p_1, \dots, p_k \geq 0, \quad (4.9)$$

where $\tau_n^2 = \bar{R}(\frac{1}{n})$ and $p_+ = \sum_{j=1}^k p_j$. Our first result is the following probability limit theorem.

Theorem 3 Assume that the condition (**LLN**) holds with Y given by (4.1). Define

$$\rho_{p_1, \dots, p_k}^{(n)} = E \left[\left| \frac{\Delta_1^n G}{\tau_n} \right|^{p_1} \dots \left| \frac{\Delta_k^n G}{\tau_n} \right|^{p_k} \right].$$

Then we have

$$V(Y, p_1, \dots, p_k)_t^n - \rho_{p_1, \dots, p_k}^{(n)} \int_0^t |\sigma_s|^{p+} ds \xrightarrow{ucp} 0. \quad (4.10)$$

Proof of Theorem 3: See section 7.

Next, we demonstrate that the functional $V(Y, p_1, \dots, p_k)_t^n$ is robust to some types of drift processes, which includes the kind of drift term occurring in $\mathcal{BSS}$ processes.

Proposition 4 *Consider a process $Z = Z_1 + Z_2$, where $Z_2 = Y$ is given by (4.1). Assume that the condition **(LLN)** holds and define*

$$V(Z_1, Z_2; p_1, \dots, p_k; \iota_1, \dots, \iota_k)_t^n = \frac{1}{n\tau_n^{p+}} \sum_{i=1}^{[nt]-k+1} \prod_{j=1}^k |\Delta_{i+j-1}^n Z_{\iota_j}|^{p_j}, \quad p_1, \dots, p_k \geq 0,$$

where $\iota_1, \dots, \iota_k \in \{1, 2\}$. If for any $t > 0$ and any $(\iota_1, \dots, \iota_k) \neq (2, \dots, 2)$,

$$V(Z_1, Z_2; p_1, \dots, p_k; \iota_1, \dots, \iota_k)_t^n \xrightarrow{P} 0, \quad (4.11)$$

then

$$V(Z, p_1, \dots, p_k)_t^n - \rho_{p_1, \dots, p_k}^{(n)} \int_0^t |\sigma_s|^{p+} ds \xrightarrow{ucp} 0.$$

Proof of Proposition 4: For simplicity we consider the case $k = 2$. Since $V(Z, p_1, p_2)_t^n$ is increasing in t and the process $\rho_{p_1, p_2}^{(n)} t$ is continuous in t , it is sufficient to prove $V(Z, p_1, p_2)_t^n - \rho_{p_1, p_2}^{(n)} t \xrightarrow{P} 0$ for a fixed $t > 0$.

Assume first that $0 \leq p_1, p_2 \leq 1$. We have $||x_1 + y_1|^{p_1} |x_2 + y_2|^{p_2} - |y_1|^{p_1} |y_2|^{p_2}| \leq C(|x_1|^{p_1} |x_2|^{p_2} + |x_1|^{p_1} |y_2|^{p_2} + |y_1|^{p_1} |x_2|^{p_2})$ (here and elsewhere C denotes a constant the value of which may change from line to line). Hence we deduce

$$\begin{aligned} & |V(Z, p_1, p_2)_t^n - V(Y, p_1, p_2)_t^n| \\ & \leq C(V(Z_1, Z_2; p_1, p_2; 1, 1)_t^n + V(Z_1, Z_2; p_1, p_2; 1, 2)_t^n + V(Z_1, Z_2; p_1, p_2; 2, 1)_t^n), \end{aligned}$$

and we obtain Proposition 4 by (4.11).

Next, assume that $p_1 \leq p_2$, $p_2 > 1$. We deduce that

$$\begin{aligned} & |(V(Z, p_1, p_2)_t^n)^{1/p_2} - (V(Y, p_1, p_2)_t^n)^{1/p_2}| \\ & \leq C \left((V(Z_1, Z_2; p_1, p_2; 1, 1)_t^n)^{1/p_2} + (V(Z_1, Z_2; p_1, p_2; 1, 2)_t^n)^{1/p_2} + (V(Z_1, Z_2; p_1, p_2; 2, 1)_t^n)^{1/p_2} \right), \end{aligned}$$

which completes the proof of Proposition 4. $\square$

Remark 10 *The multipower variation is robust to drift processes Z_1 that are smoother than the process Y . Assume, for instance, that the process Z_1 satisfies*

$$E[|Z_1(t) - Z_1(s)|^p] = o(\bar{R}^{p/2}(|t - s|))$$

for every $p > 0$. In this case condition (4.11) is obviously satisfied.

Next, we demonstrate a joint central limit theorem for a family $(V(Y, p_1^j, \dots, p_k^j)_t^n)_{1 \leq j \leq d}$ of multipower variations. Let $\mathcal{G}$ be the σ -algebra generated by the auxiliary process G .

Theorem 5 *Assume that the process σ is $\mathcal{G}$ -measurable and the condition (CLT) holds. Then we obtain the stable convergence*

$$\sqrt{n} \left(V(Y, p_1^j, \dots, p_k^j)_t^n - \rho_{p_1^j, \dots, p_k^j}^{(n)} \int_0^t |\sigma_s|^{p_+^j} ds \right)_{1 \leq j \leq d} \xrightarrow{\mathcal{G}-st} \int_0^t A_s^{1/2} dB_s, \quad (4.12)$$

where B is a d -dimensional Brownian motion that is defined on an extension of the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ and is independent of $\mathcal{F}$, and A is a $d \times d$ -dimensional process given by

$$A_s^{ij} = \beta_{ij} |\sigma_s|^{p_+^i + p_+^j}, \quad 1 \leq i, j \leq d, \quad (4.13)$$

and the $d \times d$ matrix β is defined in (3.11).

Proof of Theorem 5: See Section 7.

Remark 11 *We require the $\mathcal{G}$ -measurability of the process σ for the following reason. In fact, in Section 7, we will first prove the joint weak convergence*

$$\left(G_t, \sqrt{n} \left(V(G, p_1^j, \dots, p_k^j)_t^n - \rho_{p_1^j, \dots, p_k^j}^{(n)} t \right)_{1 \leq j \leq d} \right) \Rightarrow \left(G_t, \beta^{1/2} B_t \right).$$

This implies the joint convergence

$$\left(\sigma_{t_l}, \sqrt{n} \left(V(G, p_1^j, \dots, p_k^j)_{t_l}^n - \rho_{p_1^j, \dots, p_k^j}^{(n)} t_l \right)_{1 \leq j \leq d} \right)_{1 \leq l \leq m} \Rightarrow \left(\sigma_{t_l}, \beta^{1/2} B_{t_l} \right)_{1 \leq l \leq m},$$

for any fixed m , because σ is $\mathcal{G}$ -measurable. From the latter we deduce the $\mathcal{G}$ -stable convergence of Theorem 5 by a certain approximation technique.

From the above argument it is easy to see that Theorem 5 remains true when σ is independent of W . Hence, Theorem 5 holds for any process $\sigma = \sigma^{(1)} + \sigma^{(2)}$ such that $\sigma^{(1)}$ is $\mathcal{G}$ -measurable, $\sigma^{(2)}$ is independent of W and both processes $\sigma^{(1)}$ and $\sigma^{(2)}$ satisfy the conditions of assumption (CLT).

For completeness we provide a condition under which the above central limit theorem is robust to a potential drift process.

Proposition 6 *Consider a process $Z = Z_1 + Z_2$, where $Z_2 = Y$ is given by (4.1). Assume that the conditions of Theorem 5 hold and $d = 1$. If*

$$\sqrt{n} V(Z_1, Z_2; p_1, \dots, p_k; \iota_1, \dots, \iota_k)_t^n \xrightarrow{P} 0, \quad (4.14)$$

for any $t > 0$ and any $(\iota_1, \dots, \iota_k) \neq (2, \dots, 2)$, then we obtain

$$\sqrt{n} \left(V(Z, p_1, \dots, p_k)_t^n - \rho_{p_1, \dots, p_k}^{(n)} \int_0^t |\sigma_s|^{p_+} ds \right)_{1 \leq j \leq d} \xrightarrow{\mathcal{G}-st} \sqrt{\beta} \int_0^t |\sigma_s|^{p_+} dB_s.$$

Proof of Proposition 6: Proposition 6 can be proved by the same methods as Proposition 4 (details omitted). $\square$

Clearly, Proposition 6 extends in a direct manner to the multivariate setting ($d \geq 1$). Concerning the possibility of substituting $\rho_{p_1^j, \dots, p_k^j}^{(n)}$ by $\rho_{p_1^j, \dots, p_k^j}$ in the above conclusions, see Remark 1.

4.3 Discussion of assumptions

We start our discussion again by considering the auxiliary, centered stationary Gaussian, process

$$G_t = \int_{-\infty}^t g(t-s)W(ds).$$

First of all, we want to demonstrate how Theorems 1 and 2 apply for the multipower variation of the process G . In other words, we will give a hint how to check the conditions of these theorems.

Recall the definition (2.7) of the variance function $\bar{R}$ of the increments of G and note that

$$\bar{R}(t) = E[|G_{s+t} - G_s|^2] = \int_0^t g^2(x)dx + \int_0^\infty (g(t+x) - g(x))^2 dx, \quad t \geq 0.$$

Clearly, the asymptotic behaviour of the multipower variation of the process G is fully determined by the behaviour of the function $\bar{R}$ near 0. As we deal with a continuous process G , it is natural to assume that $\bar{R}(t)$ behaves essentially as t^α (for some $\alpha > 0$) near 0 (later on we will formalize this assumption). Since the case where the paths of G are differentiable (a.s.) is not very interesting for us (because the consistency can be deduced by the mean value theorem), we concentrate on the region $0 < \alpha < 2$ (the corresponding $g(t)$ behaving as $t^{(\alpha-1)/2}$).

Let us introduce a new set of assumptions that correspond to the previous discussion. These assumptions were proposed by Guyon and Leon in [GuyLe89] (those authors considered the case of centered stationary Gaussian processes X ; this relates to the $\mathcal{BSS}$ setting with σ constant) and the same assumptions were used in [BNCP09] and [BNCPW09].

(A1) $\bar{R}(t) = t^\alpha L_0(t)$ for some $\alpha \in (0, 2)$ and some positive slowly varying (at 0) function L_0 , which is continuous on $(0, \infty)$.

(A2) $\bar{R}''(t) = t^{\alpha-2} L_2(t)$ for some slowly varying function L_2 , which is continuous on $(0, \infty)$.

(A3) There exists $b \in (0, 1)$ with

$$K = \limsup_{x \rightarrow 0} \sup_{y \in [x, x^b]} \left| \frac{L_2(y)}{L_0(x)} \right| < \infty.$$

Recall that a function $L : (0, \infty) \rightarrow \mathbb{R}$ is called slowly varying at 0 when the identity

$$\lim_{x \searrow 0} \frac{L(tx)}{L(x)} = 1$$

holds for any fixed $t > 0$.

Now, note that under assumption **(A1)** we have, for any $j \geq 1$,

$$\begin{aligned} r_n(j) &= \text{cov}\left(\frac{\Delta_1^n G}{\tau_n}, \frac{\Delta_{1+j}^n G}{\tau_n}\right) \\ &= \frac{\bar{R}(\frac{j+1}{n}) + \bar{R}(\frac{j-1}{n}) - 2\bar{R}(\frac{j}{n})}{2\bar{R}(\frac{1}{n})} \rightarrow \rho(j) = \frac{1}{2}((j+1)^\alpha - 2j^\alpha + (j-1)^\alpha), \end{aligned} \quad (4.15)$$

because L_0 is slowly varying at 0. It is obvious that $\rho(j)$ is the correlation function of the discrete time stationary Gaussian process $Q_i = B_i^{\alpha/2} - B_{i-1}^{\alpha/2}$, where $B^{\alpha/2}$ is a fractional Brownian motion with parameter $\alpha/2$.

As shown in [GuyLe89] and [BNCP09] assumptions **(A1)**-**(A3)** imply that for any $\alpha \in (0, 2)$ there exists a number $h = h(\alpha) > 0$ such that

$$r_n^2(j) \leq r(j) = Cj^{-h}$$

for all $j \geq 1$ and all, but finitely many, n . Hence, under **(A1)**-**(A3)**, the condition of Theorem 1 is satisfied for all $\alpha \in (0, 2)$, because $\frac{1}{n} \sum_{j=1}^n r(j) \rightarrow 0$.

Moreover, the conditions **(A1)**-**(A3)** imply that for any $\alpha \in (0, 3/2)$ there exists a number $\bar{h} = \bar{h}(\alpha) > 1$ such that

$$r_n^2(j) \leq r(j) = Cj^{-\bar{h}-1}$$

for all $j \geq 1$ and all, but finitely many, n . Consequently, when $\alpha \in (0, 3/2)$ and **(A1)**-**(A3)** hold, the conditions of Theorem 2 are satisfied with $Q_i = B_i^{\alpha/2} - B_{i-1}^{\alpha/2}$, $i \geq 1$, since $\sum_{j=1}^\infty r(j) < \infty$.

Remark 12 *It is easy to see that $\frac{\bar{R}(\frac{j+1}{n}) + \bar{R}(\frac{j-1}{n}) - 2\bar{R}(\frac{j}{n})}{2\bar{R}(\frac{1}{n})} \rightarrow \rho(j)$ implies that for all $j \geq 1$ there exists an $s(j)$ such that*

$$\frac{\bar{R}(\frac{j}{n})}{\bar{R}(\frac{1}{n})} \rightarrow s(j).$$

Since the result in Theorem 5 is independent of the scale of time we use, we must have

$$\frac{\bar{R}(j\Delta)}{\bar{R}(\Delta)} \xrightarrow{\Delta \downarrow 0} s(j),$$

for any Δ and then $s(jk) = s(j)s(k)$; consequently $s(j) = j^\alpha$, for a certain $\alpha \in \mathbb{R}$. Moreover, since $(j+1)^\alpha - 2j^\alpha + (j-1)^\alpha$ is a covariance function we have $0 < \alpha < 2$. So in the present setting $(Q_i)_{i \geq 1}$, as defined in Section 3, is always a standard fractional Gaussian noise.

Remark 13 In Theorem 5 the quantity $\rho_{p_1, \dots, p_k}^{(n)}$ can be replaced by its limit $\rho_{p_1, \dots, p_k}$ (which necessarily exists because $r_n(j) \rightarrow \rho(j)$ for all $j \geq 1$) whenever

$$\sqrt{n}(\rho_{p_1, \dots, p_k}^{(n)} - \rho_{p_1, \dots, p_k}) \rightarrow 0.$$

As the quantity $\rho_{p_1, \dots, p_k}^{(n)}$ is a continuously differentiable function of $r_n(1), \dots, r_n(k-1)$ (recall that we are in the non-degenerate case) the latter follows from

$$\sqrt{n} \max_{j=1, \dots, k-1} |r_n(j) - \rho(j)| \rightarrow 0.$$

To study the above convergence let us introduce the notion of the second order regular variation (however, we restrict ourselves to the second order regular variation of slowly varying functions). A slowly varying function $L : (0, \infty) \rightarrow \mathbb{R}$ is called second order regular varying (at 0) with parameter $\gamma > 0$ if there exists a function $A : (0, \infty) \rightarrow \mathbb{R}$, positive or negative, with $\lim_{x \rightarrow 0} A(x) = 0$ such that

$$\lim_{x \rightarrow 0} \frac{\frac{L(tx)}{L(x)} - 1}{A(x)} = \frac{t^\gamma - 1}{\gamma}$$

holds for all $t > 0$. It is known that if the limit on the left-hand side exists for all $t > 0$ it must be essentially of the form $\frac{t^\gamma - 1}{\gamma}$. Furthermore, if L is second order regular varying with parameter $\gamma > 0$ then the function A must be regular varying with parameter γ , i.e.

$$\lim_{x \rightarrow 0} \frac{A(tx)}{A(x)} = t^\gamma$$

for all $t > 0$. Notice that the parameter γ essentially gives the rate of convergence for $\frac{L(tx)}{L(x)} \rightarrow 1$.

Observing the convergence in (4.15) we deduce that

$$\sqrt{n} \max_{j=1, \dots, k-1} |r_n(j) - \rho(j)| \rightarrow 0$$

holds when the slowly varying function L_0 , which appears in the assumption **(A1)**, is second order regular varying with parameter $\gamma > \frac{1}{2}$.

Now, let us see what the conditions **(A1)**-**(A3)** mean for the memory function g . For simplicity let us consider functions of the form

$$g(x) = x^\delta 1_{(0,1]}(x), \quad x > 0. \quad (4.16)$$

For such functions we readily obtain assumptions **(A1)**-**(A2)** with

$$\alpha = 2\delta + 1, \quad \delta \in \left(-\frac{1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right).$$

(the technical assumption **(A3)** has to be checked separately; for an example see Section 5). Note that for $\delta = 0$, for which assumption **(A2)** does not hold, the process G is a semimartingale and the multipower variations can be treated as in [BNGJPS06].

Next, we discuss the assumptions of subsection 4.1 for the function g defined in (4.16). Recall that condition (4.4) is automatically satisfied for functions g with compact support (as in (4.16)). A straightforward calculation shows that

$$\bar{\pi}^n(\varepsilon) = O((n\varepsilon)^{2\delta-1}).$$

for any $\varepsilon > \frac{1}{n}$. Thus, the condition (4.6) of **(LLN)** is satisfied (because $2\delta - 1 < 0$) and Theorem 3 is valid for all $\delta \in (-1/2, 0) \cup (0, 1/2)$.

Finally, we explain how to verify the condition (4.8) of **(CLT)**. Recall that $p = \min_{1 \leq i \leq k, 1 \leq j \leq d} (p_i^j)$. Let $\varepsilon_n = n^{-\kappa}$, $\kappa \in (0, 1)$. We readily deduce that

$$\bar{\pi}^n(\varepsilon) = O(n^{\lambda(1-\kappa)}), \quad \lambda = 2\delta - 1.$$

Thus, condition (4.8) is satisfied if

$$\lambda < -\frac{1}{1 \wedge p}.$$

We immediately deduce that Theorem 5 holds if

$$\begin{aligned} p \geq 1 : \quad & \gamma > \frac{1}{2}, \quad \delta \in (-\frac{1}{2}, 0) \\ \frac{1}{2} < p < 1 : \quad & \gamma > \frac{1}{2p}, \quad \delta \in (-\frac{1}{2}, \frac{p-1}{2p}). \end{aligned}$$

Remark 14 Clearly, we can deal with a larger class of functions g than $g(x) = x^\delta 1_{(0,1]}(x)$. Assume that condition (4.4) holds. In the following we consider functions L_g , $L_{g'}$, which are continuous on $(0, \infty)$ and slowly varying at 0. We assume the following conditions:

Assumption: $g \in L^2((0, \infty))$ and for some $\delta \in (-1/2, 0) \cup (0, 1/2)$ it holds that

$$(i) \quad g(x) = x^\delta L_g(x).$$

$$(ii) \quad g'(x) = x^{\delta-1} L_{g'}(x) \text{ and, for any } \varepsilon > 0, g' \in L^2((\varepsilon, \infty)). \text{ Moreover, } g' \text{ is non-increasing on } (b, \infty) \text{ for some } b > 0.$$

We further assume that the function

$$\bar{R}(t) = \int_0^t g^2(x) dx + \int_0^\infty (g(t+x) - g(x))^2 dx$$

satisfies the conditions (A1)-(A3) with $\alpha = 2\delta + 1$.

Under these assumptions we conclude (as for the simple example $g(x) = x^\delta 1_{(0,1]}(x)$) that Theorem 3 holds for any $\delta \in (-1/2, 0) \cup (0, 1/2)$, and Theorem 5 holds when further

$$\begin{aligned} p \geq 1 : \quad & \gamma > \frac{1}{2}, \quad \delta \in (-\frac{1}{2}, 0) \\ \frac{1}{2} < p < 1 : \quad & \gamma > \frac{1}{2p}, \quad \delta \in (-\frac{1}{2}, \frac{p-1}{2p}). \end{aligned}$$

In both cases we have $Q_i = B_i^{\delta+\frac{1}{2}} - B_{i-1}^{\delta+\frac{1}{2}}$, $i \geq 1$.

5 Examples

The present Section discusses two examples of choice of the damping function g and the associated probabilistic limit behaviour.

As above let r denote the autocorrelation function of $G = \int_{-\infty}^{\cdot} g(\cdot - s)W(ds)$, and we write $\bar{r}$ for $1 - r$. Note that assumptions **(A1)**-**(A3)** could equivalently have been formulated in terms of $\bar{r}$ rather than $\bar{R}$ (since $\bar{R}(t) = 2 \|g\|^2 \bar{r}(t)$).

Suppose first that

$$g(t) = e^{-\lambda t} 1_{(0,1)}(t)$$

with $\lambda > 0$. This example (for a detailed discussion see [BNSch09]) is a non-semimartingale case, and it can be shown that $\pi^n \rightarrow \pi$, with π given by

$$\pi = \frac{1}{1 + e^{-2\lambda}} \delta_0 + \frac{1}{1 + e^{2\lambda}} \delta_1,$$

where δ_x is the Dirac measure at x . Moreover,

$$V(Y, 2)_t^n \xrightarrow{P} \sigma_t^{2+} - \left(1 + e^{2\lambda}\right)^{-1} \sigma_{-t}^{2+}.$$

Thus, in particular, we do not have $V(Y, 2)_t^n \xrightarrow{P} \sigma_t^{2+}$. Note that in this example assumption **(A2)** is not satisfied.

Our main example is

$$g(t) = t^{\nu-1} e^{-\lambda t} 1_{(0,\infty)}(t) \tag{5.1}$$

for $\lambda > 0$ and with $\nu > \frac{1}{2}$. (So, for t near 0, $g(t)$ behaves as t^δ with $\delta = \nu - 1$.) The following two subsections discuss the properties of the autocorrelation function r for this g , presenting exact formulae (in terms of the Bessel functions K_ν) for r and its derivatives in subsection 5.1 and deriving asymptotic properties of $\bar{r}(t) = 1 - r(t)$ for $t \rightarrow 0$ in subsection 5.2. Armed with these results we show in subsection 5.3 that assumptions **(A1)**-**(A3)** are met provided $\alpha = 2\nu - 1 \in (0, 2)$, i.e. $\nu \in (\frac{1}{2}, \frac{3}{2})$ and that $\rho_{p_1^j, \dots, p_k^j}^{(n)}$ may be substituted by $\rho_{p_1^j, \dots, p_k^j}$ in the central limit theory provided $\nu \in (\frac{1}{2}, \frac{5}{4})$.

Remark 15 *The derivative g' of g is not square integrable if $\frac{1}{2} < \nu < 1$ or $1 < \nu \leq \frac{3}{2}$; hence, in these cases Y is not a semimartingale. For $\frac{1}{2} < \nu < 1$ we have $g(0+) = \infty$ while $g(0+) = 0$ when $1 < \nu \leq \frac{3}{2}$. Of course, for $\nu = 1$ the process $Y = \int_{-\infty}^{\cdot} g(\cdot - s)\sigma_s W(ds)$ is simply a modulated version of the Gaussian Ornstein-Uhlenbeck process, and in particular, a semimartingale. Note also that when $\nu > \frac{3}{2}$ then Y is of finite variation and hence, trivially, a semimartingale.*

Remark 16 *With $Y = \int_{-\infty}^{\cdot} g(\cdot - s)\sigma_s W(ds)$, suppose that the volatility process is constant, $\sigma_t = \sigma$. This is a special case of the class of stationary Gaussian processes discussed by*

[GuyLe89]. These authors showed that, under conditions **(A1)**-**(A3)**,

$$\frac{(2\lambda)^{2\nu-1}}{\Gamma(2\nu-1)} \frac{1}{2n\tau_n} V(Y, 2)_t^n \xrightarrow{p} \sigma^2$$

More generally, they derived associated (nonfeasible) limit law results which in the present example (of (5.1) (not considered by [GuyLe89]), implies that the limit distribution is normal if $\frac{1}{2} < \nu < \frac{5}{4}$, with rate $n^{3/2}\bar{r}(\frac{1}{n})$, while for $\frac{5}{4} < \nu < \frac{3}{2}$ it belongs to the second order chaos, the rate of convergence being then $n^{-2\nu+3}$. Extension to the power variations $V(Y, q)_t^n$, $q > 0$, are also given in [GuyLe89]. In [BNCP09] extensions of the normal limit results to processes of the form $\int_0^\cdot \sigma_s dG_s$ where G is a Gaussian process with stationary increments are established. The earlier paper [Guy87] contains a set of sufficient conditions for convergence in probability of normalised versions of $V(Y, q)_t^n$ in cases where the process Y is nonstationary Gaussian. The conditions are rather restrictive; in particular, they can only apply to certain types of processes for which $\bar{r}(\frac{1}{n})$ behaves as a constant times n^{-1} as $n \rightarrow \infty$.

For the following analysis we need a number of, mostly well known, properties of modified Bessel functions of the third type K_ν . These are given in the Appendix.

5.1 Formulae for r and its derivatives

With g given by (5.1) the autocorrelation function r of $G = \int_{-\infty}^\cdot g(\cdot - s)W(ds)$ has the form (cf. formula (4.3))

$$r(t) = \frac{(2\lambda)^{2\nu-1}}{\Gamma(2\nu-1)} e^{-\lambda t} \int_0^\infty (t+u)^{\nu-1} u^{\nu-1} e^{-2\lambda u} du. \quad (5.2)$$

Hence by formulae (A.1.4), (A.1.5) and (A.1.10) in the Appendix we find

$$\begin{aligned} \|g\|^2 r(t) &= \sqrt{\frac{2}{\pi}} \Gamma(\nu) 2^{-\nu} \lambda^{-\nu+\frac{1}{2}} t^{\nu-\frac{1}{2}} K_{\nu-\frac{1}{2}}(\lambda t) \\ &= \sqrt{\frac{2}{\pi}} \Gamma(\nu) 2^{-\nu} \lambda^{-2\nu+1} \bar{K}_{\nu-\frac{1}{2}}(\lambda t) \\ &= \frac{1}{2} \frac{\Gamma(\nu) \Gamma(\nu-\frac{1}{2})}{\Gamma(\frac{1}{2})} \lambda^{-2\nu+1} \check{K}_{\nu-\frac{1}{2}}(\lambda t). \end{aligned} \quad (5.3)$$

where we have used the doubling formula

$$\Gamma(2\nu-1) = 2^{2\nu-2} \frac{\Gamma(\nu) \Gamma(\nu-\frac{1}{2})}{\Gamma(\frac{1}{2})}. \quad (5.4)$$

It follows, by (A.1.7), that

$$r(t) = \check{K}_{\nu-\frac{1}{2}}(\lambda t) .. \quad (5.5)$$

Now, let

$$a(\nu) = 2^{-\nu+1} \Gamma(\nu)^{-1}.$$

so that

$$\frac{a(\nu)}{a(\nu+1)} = 2\nu \quad (5.6)$$

and

$$\check{K}(x) = a(\nu) \bar{K}_\nu(x).$$

Supposing for notational simplicity that $\lambda = 1$ and using (A.1.9) and (A.1.8) we find, for $\nu \in (\frac{1}{2}, \frac{3}{2})$,

$$\begin{aligned} \bar{r}'(t) &= -a\left(\nu - \frac{1}{2}\right) t \bar{K}_{\nu-\frac{3}{2}}(t) \\ &= -a\left(\nu - \frac{1}{2}\right) t^{\nu-\frac{1}{2}} K_{\nu-\frac{3}{2}}(t) \\ &= -a\left(\nu - \frac{1}{2}\right) t^{\nu-\frac{1}{2}} K_{\frac{3}{2}-\nu}(t) \\ &= -a\left(\nu - \frac{1}{2}\right) t^{2\nu-2} \bar{K}_{\frac{3}{2}-\nu}(t) \\ &= -\frac{a(\nu-\frac{1}{2})}{a(\frac{3}{2}-\nu)} t^{2\nu-2} \check{K}_{\frac{3}{2}-\nu}(t) \end{aligned} \quad (5.7)$$

$$\begin{aligned} \bar{r}''(t) &= a\left(\nu - \frac{1}{2}\right) \left\{ t^2 \bar{K}_{\nu-\frac{5}{2}}(t) - \bar{K}_{\nu-\frac{3}{2}}(t) \right\} \\ &= a\left(\nu - \frac{1}{2}\right) \left\{ t^{\nu-\frac{1}{2}} K_{\frac{5}{2}-\nu}(t) - t^{\nu-\frac{3}{2}} K_{\frac{3}{2}-\nu}(t) \right\} \\ &= a\left(\nu - \frac{1}{2}\right) t^{2\nu-3} \left\{ \bar{K}_{\frac{5}{2}-\nu}(t) - \bar{K}_{\frac{3}{2}-\nu}(t) \right\} \end{aligned} \quad (5.8)$$

$$\begin{aligned} \bar{r}'''(t) &= -a\left(\nu - \frac{1}{2}\right) \left\{ t^3 \bar{K}_{\nu-\frac{7}{2}}(t) - 3t \bar{K}_{\nu-\frac{5}{2}}(t) \right\} \\ &= -a\left(\nu - \frac{1}{2}\right) \left\{ t^{\nu-\frac{1}{2}} K_{\frac{7}{2}-\nu}(t) - 3t^{\nu-\frac{3}{2}} K_{\frac{5}{2}-\nu}(t) \right\} \\ &= -a\left(\nu - \frac{1}{2}\right) t^{2\nu-4} \left\{ \bar{K}_{\frac{7}{2}-\nu}(t) - 3\bar{K}_{\frac{5}{2}-\nu}(t) \right\}. \end{aligned} \quad (5.9)$$

5.2 Behaviour of $\bar{r} = 1 - r$ near 0

From the asymptotic expansions (A.2.4), (A.2.5) and (A.2.6) we find that for $t \rightarrow 0$ the complementary autocorrelation function $\bar{r}(t) = 1 - r(t)$ behaves as

$2^{-2\nu+1} \frac{\Gamma(\frac{3}{2}-\nu)}{\Gamma(\nu+\frac{1}{2})} (\lambda t)^{2\nu-1} + O(t^2) \quad \text{for} \quad \frac{1}{2} < \nu < \frac{3}{2}$
$\bar{r}(t) \sim \frac{1}{2} (\lambda t)^2 \log t \quad \text{for} \quad \nu = \frac{3}{2}$
$\frac{1}{4(\nu-\frac{3}{2})} (\lambda t)^2 + O(t^{2\nu-1}) \quad \text{for} \quad \frac{3}{2} < \nu$

Table 1

Remark 17 So for $\frac{3}{2} < \nu \leq 2$ the autocorrelation function is twice differentiable at 0 and consequently Y has continuously differentiable sample paths, while for $\frac{1}{2} < \nu \leq \frac{3}{2}$ the sample paths are Lipschitz of order λ for every $0 < \lambda < \nu - \frac{1}{2}$ (cf. [CL67] Section 9.2).

5.3 Verification of assumptions (A1)-(A3)

This subsection establishes that conditions **(A1)**-(**A3**) are satisfied (with $\alpha = 2\nu - 1$ and $\nu \in (\frac{1}{2}, \frac{3}{2})$, i.e. $\alpha \in (0, 2)$). Recall that in those conditions we may substitute $\bar{r}$ for $\bar{R}$.

On account of (5.5) and Table 1 we find that $\bar{r}$ is of the form

$$\bar{r}(t) = t^{2\nu-1} L_0(t)$$

with

$$L_0(t) = t^{-2\nu+1} \left(1 - \check{K}_{\nu-\frac{1}{2}}(\lambda t)\right)$$

and

$$L_0(t) \rightarrow 2^{-2\nu+1} \frac{\Gamma(\frac{3}{2}-\nu)}{\Gamma(\nu+\frac{1}{2})} \quad \text{for } t \rightarrow 0.$$

It follows that L_0 is slowly varying at 0, and hence assumption **(A1)** is met.

In fact, more is true: L_0 is second order slowly varying. To see this, note that

$$\frac{L_0(tx)}{L_0(x)} = t^{-2\nu+1} \frac{1 - K_{\nu-\frac{1}{2}}(\lambda tx)}{1 - K_{\nu-\frac{1}{2}}(\lambda x)}$$

from which we find, using formula (A.2.4),

$$\begin{aligned} \frac{L_0(tx)}{L_0(x)} &= \frac{2^{-2\nu+1} \frac{\Gamma(\frac{3}{2}-\nu)}{\Gamma(\frac{1}{2}+\nu)} (\lambda x)^{2\nu-1} - \frac{1}{4} \frac{1}{\frac{3}{2}-\nu} (\lambda x)^2 t^{3-2\nu} + O(x^{2\nu+1})}{2^{-2\nu+1} \frac{\Gamma(\frac{3}{2}-\nu)}{\Gamma(\frac{1}{2}+\nu)} (\lambda x)^{2\nu-1} - \frac{1}{4} \frac{1}{\frac{3}{2}-\nu} (\lambda x)^2 + O(x^{2\nu+1})} \\ &= \frac{1 - 2^{2(\nu-1)} \frac{\Gamma(\frac{1}{2}+\nu)}{\Gamma(\frac{3}{2}-\nu)} (\lambda x)^{3-2\nu} \frac{t^{3-2\nu}}{3-2\nu} + O(x^2)}{1 - 2^{2(\nu-1)} \frac{\Gamma(\frac{1}{2}+\nu)}{\Gamma(\frac{3}{2}-\nu)} (\lambda x)^{3-2\nu} + O(x^2)}. \end{aligned}$$

Thus

$$\frac{L_0(tx)}{L_0(x)} - 1 = A(x) \frac{t^{3-2\nu} - 1}{3-2\nu} + o(A(x))$$

where

$$A(x) = -2^{-2(\nu-1)} \frac{\Gamma(\frac{1}{2}+\nu)}{\Gamma(\frac{3}{2}-\nu)} (\lambda x)^{3-2\nu}$$

which tends to 0 as $x \rightarrow 0$. Hence L_0 is second order regular varying with parameter $\gamma = 3-2\nu$.

It follows that, for the present example, Remark 13 applies provided $3 - 2\nu > \frac{1}{2}$, i.e. $\frac{1}{2} < \nu < \frac{5}{4}$.

Next, we note that

$$\bar{r}''(t) = t^{2\nu-3} L_2(t)$$

with

$$L_2(t) = a\left(\nu - \frac{1}{2}\right) \left\{ \bar{K}_{\frac{5}{2}-\nu}(t) - \bar{K}_{\frac{3}{2}-\nu}(t) \right\},$$

where the function a is defined by $a(\nu) = 2^{-\nu+1} \Gamma(\nu)^{-1}$ and L_2 is slowly varying at 0 with

$$L_2(t) \rightarrow -2^3 (\nu - 1) \frac{\Gamma\left(\frac{3}{2} - \nu\right)}{\Gamma\left(\nu - \frac{1}{2}\right)} \quad \text{for } t \rightarrow 0.$$

The latter follows from the rewrite

$$\begin{aligned} \bar{r}''(t) &= t^{2\nu-3} a\left(\nu - \frac{1}{2}\right) a\left(\frac{3}{2} - \nu\right)^{-1} \left\{ (3 - 2\nu) \check{K}_{\frac{5}{2}-\nu}(t) - \check{K}_{\frac{3}{2}-\nu}(t) \right\} \\ &= t^{2\nu-3} 2^2 \frac{\Gamma\left(\frac{3}{2} - \nu\right)}{\Gamma\left(\nu - \frac{1}{2}\right)} \left\{ (3 - 2\nu) \check{K}_{\frac{5}{2}-\nu}(t) - \check{K}_{\frac{3}{2}-\nu}(t) \right\}. \end{aligned}$$

Thus **(A2)** holds.

Finally, we find

$$\begin{aligned} L_2'(t) &= a\left(\nu - \frac{1}{2}\right) \left\{ \bar{K}'_{\frac{5}{2}-\nu}(t) - \bar{K}'_{\frac{3}{2}-\nu}(t) \right\} \\ &= a\left(\nu - \frac{1}{2}\right) t \left\{ \bar{K}_{\frac{1}{2}-\nu}(t) - \bar{K}_{\frac{3}{2}-\nu}(t) \right\} \\ &= a\left(\nu - \frac{1}{2}\right) t \left\{ t^{\frac{1}{2}-\nu} K_{\nu-\frac{1}{2}}(t) - \bar{K}_{\frac{3}{2}-\nu}(t) \right\} \\ &= a\left(\nu - \frac{1}{2}\right) t \left\{ t^{-2\nu+1} \bar{K}_{\nu-\frac{1}{2}}(t) - \bar{K}_{\frac{3}{2}-\nu}(t) \right\} \\ &= a\left(\nu - \frac{1}{2}\right) t^{-2\nu+2} \left\{ a\left(\nu - \frac{1}{2}\right)^{-1} t^{-2\nu+1} \check{K}_{\nu-\frac{1}{2}}(t) - a\left(\nu - \frac{3}{2}\right)^{-1} t^{2\nu-1} \check{K}_{\frac{3}{2}-\nu}(t) \right\} \end{aligned}$$

Hence (for $\nu \in (\frac{1}{2}, \frac{3}{2})$) $L_2(t)$ is increasing near 0. Consequently

$$\limsup_{x \rightarrow 0} \sup_{y \in [x, x^b]} \left| \frac{L_2(y)}{L_0(x)} \right| \leq \limsup_{x \rightarrow 0} \left| \frac{L_2(x^b)}{L_0(x)} \right|.$$

Here, as $x \rightarrow 0$,

$$L_0(x) \rightarrow 2^{-2\nu+1} \frac{\Gamma\left(\frac{3}{2} - \nu\right)}{\Gamma\left(\nu + \frac{1}{2}\right)}.$$

while

$$L_2(x^b) \rightarrow a\left(\nu - \frac{1}{2}\right) \left\{ a\left(\frac{5}{2} - \nu\right)^{-1} - a\left(\frac{3}{2} - \nu\right)^{-1} \right\}.$$

Therefore also condition **(A3)** is satisfied.

6 An application

Let us consider the realised variation ratio (RVR) defined for a stochastic process X as

$$RV R_t^n := \frac{\frac{\pi}{2} V(X, 1, 1)_t^n}{V(X, 2, 0)_t^n}. \quad (6.1)$$

The RVR is of interest as a diagnostic tool concerning the nature of empirical processes.

In particular, it can be used to test the hypothesis that such a process is a Brownian semimartingale (with nontrivial local martingale component) against the possibility that it is of this type plus a jump process, see [BNSW06] and [Jac08c] (some related work is discussed in [Woe08]). If a jump component is present then the limit of $RV R_t^n$ is smaller than 1.

However, in the course of the turbulence project, mentioned earlier, when calculating the RVR for an extensive high quality data set from atmospheric turbulence it turned out that the values of RVR were consistently higher than 1. The wish to understand this phenomenon has been a strong motivation for the theoretical developments described in the present paper. As a consequence of Theorem 3, we obtain the following probability limit result for the realised variation ratio of $\mathcal{BSS}$ processes:

$$RV R_t^n - \psi(r_n(1)) \xrightarrow{ucp} 0 \quad (6.2)$$

where

$$\psi(\rho) = \sqrt{1 - \rho^2} + \rho \arcsin \rho, \quad (6.3)$$

which equals $\frac{\pi}{2}$ times the mean $E\{|UV|\}$ of two standard normal variables U and V with correlation ρ .

Remark 18 Notice that under the assumptions of Theorem 2 we have $r_n(1) \rightarrow \rho(1)$ and thus

$$RV R_t^n \xrightarrow{ucp} \psi(\rho(1)).$$

The latter can be used for parameter estimation. Under assumptions (A1)-(A3) we have that $\rho(1) = 2^{\alpha-1} - 1$. Consequently, the parameter α can be consistently estimated, because the function ψ is invertible on $(0,1)$.

Moreover, we have that

$$\begin{aligned} \sqrt{n} \left(RV R_t^n - \psi(r_n(1)) \right) &= \sqrt{n} \left(\frac{\frac{\pi}{2} V(Y, 1, 1)_t^n - \psi(r_n(1)) \int_0^t \sigma_s^2 ds}{\int_0^t \sigma_s^2 ds} \right) \\ &\quad - \sqrt{n} RV R_t^n \left(\frac{V(Y, 2, 0)_t^n - \int_0^t \sigma_s^2 ds}{\int_0^t \sigma_s^2 ds} \right), \end{aligned}$$

so, if the parameter $\alpha \in (0, 1)$, by applying Theorems 3 and 5, we obtain

$$\sqrt{n} \left(RV R_t^n - \psi(r_n(1)) \right) \xrightarrow{st} \left(\frac{\pi}{2}, -\psi(\rho(1)) \right) \beta^{1/2} \frac{\int_0^t \sigma_s^2 dB_s}{\int_0^t \sigma_s^2 ds}, \quad (6.4)$$

where ψ is as above and the matrix β is given in Theorem 2. Specifically we find $\beta = (\beta_{ij})_{1 \leq i, j \leq 2}$ where

$$\begin{aligned} \beta_{11} &= \lim_{n \rightarrow \infty} n \operatorname{var} \left(V_Q(1, 1)_1^n \right), \\ \beta_{22} &= \lim_{n \rightarrow \infty} n \operatorname{var} \left(V_Q(2, 0)_1^n \right), \\ \beta_{12} &= \lim_{n \rightarrow \infty} n \operatorname{cov} \left(V_Q(2, 0)_1^n, V_Q(1, 1)_1^n \right) \end{aligned}$$

with Q as defined in Theorem 2. Thus, we obtain

$$\beta_{22} = \operatorname{var}(Q_1^2) + 2 \sum_{k=1}^{\infty} \operatorname{cov}(Q_1^2, Q_{1+k}^2) = 2 + 4 \sum_{k=1}^{\infty} \rho^2(k).$$

Similarly, we have that

$$\beta_{12} = \operatorname{cov}(Q_1^2, |Q_1||Q_2|) + 2 \sum_{k=1}^{\infty} \operatorname{cov}(Q_1^2, |Q_{1+k}||Q_{2+k}|).$$

To compute the latter, we use the following formula:

$$E[|X_1^2 X_2 X_3|] = \frac{2}{\pi} \left(\sqrt{1 - \rho_{23}^2} (1 + \rho_{12}^2 + \rho_{13}^2) + (\rho_{23} + 2\rho_{12}\rho_{13}) \arcsin(\rho_{23}) \right) := h(\rho_{12}, \rho_{13}, \rho_{23}),$$

where X_1, X_2, X_3 are standard normal with $\operatorname{cov}(X_i, X_j) = \rho_{ij}$, see [Nab52]. Consequently, we obtain the identity

$$\beta_{12} = \left(h(1, \rho(1), \rho(1)) - f(\rho(1)) \right) + 2 \sum_{k=1}^{\infty} \left(h(\rho(k), \rho(k+1), \rho(1)) - f(\rho(1)) \right).$$

For the remaining term we deduce

$$\beta_{11} = \operatorname{var}(|Q_1||Q_2|) + 2 \sum_{k=1}^{\infty} \operatorname{cov}(|Q_1||Q_2|, |Q_{1+k}||Q_{2+k}|).$$

However, there is no explicit formula available for the latter expression but it can be easily computed numerically.

7 Proofs

All positive constants (which do not depend on n) in the proof are denoted by C , although they might change from line to line.

Before we proceed with the proofs of the main results we review the basic concepts of the Wiener chaos expansion. Consider a complete probability space $(\Omega, \mathcal{F}, P)$ and a subspace $\mathcal{H}_1$ of $L^2(\Omega, \mathcal{F}, P)$ whose elements are zero-mean Gaussian random variables. Let $\mathcal{H}$ be a separable

Hilbert space with scalar product denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and norm $\| \cdot \|_{\mathcal{H}}$. We will assume that there is an isometry

$$\begin{aligned} W &: \mathcal{H} \rightarrow \mathcal{H}_1 \\ h &\mapsto W(h) \end{aligned}$$

in the sense that

$$E[W(h_1)W(h_2)] = \langle h_1, h_2 \rangle_{\mathcal{H}}.$$

It is easy to see that this map has to be linear.

For any $m \geq 2$, we denote by $\mathcal{H}_m$ the m -th Wiener chaos, that is, the closed subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by the random variables $H_m(X)$, where $X \in \mathcal{H}_1$, $E[X^2] = 1$, and H_m is the m -th Hermite polynomial, i.e. $H_0(x) = 1$ and $H_m(x) = (-1)^m e^{\frac{x^2}{2}} \frac{d^m}{dx^m} (e^{-\frac{x^2}{2}})$.

Suppose that $\mathcal{H}$ is infinite-dimensional and let $\{e_i, i \geq 1\}$ be an orthonormal basis of $\mathcal{H}$. Denote by Λ the set of all sequences $a = (a_1, a_2, \dots)$, $a_i \in \mathbb{N}$, such that all the terms, except a finite number of them, vanish. For $a \in \Lambda$ we set $a! = \prod_{i=1}^{\infty} a_i!$ and $|a| = \sum_{i=1}^{\infty} a_i$. For any multindex $a \in \Lambda$ we define

$$\Phi_a = \frac{1}{\sqrt{a!}} \prod_{i=1}^{\infty} H_{a_i}(W(e_i)).$$

The family of random variables $\{\Phi_a, a \in \Lambda\}$ is an orthonormal system. In fact

$$E[\prod_{i=1}^{\infty} H_{a_i}(W(e_i)) \prod_{i=1}^{\infty} H_{b_i}(W(e_i))] = \delta_{ab} a!,$$

where δ_{ab} denotes the Kronecker symbol. Moreover, $\{\Phi_a \mid a \in \Lambda, |a| = m\}$ is a complete orthonormal system in $\mathcal{H}_m$.

Let $a \in \Lambda$ with $|a| = m$. The mapping

$$\begin{aligned} I_m : \mathcal{H}^{\odot m} &\rightarrow \mathcal{H}_m \\ \tilde{\otimes}_{i=1}^{\infty} e_i^{\otimes a_i} &\mapsto \prod_{i=1}^{\infty} H_{a_i}(W(e_i)), \end{aligned}$$

between the symmetric tensor product $\mathcal{H}^{\odot m}$, equipped with the norm $\sqrt{m!} \|\cdot\|_{\mathcal{H}^{\otimes m}}$, and the m -th chaos $\mathcal{H}_m$ is a linear isometry. Here $\tilde{\otimes}$ denotes the symmetrization of the tensor product $\otimes$ and I_0 is the identity in $\mathbb{R}$. For $h \in \mathcal{H}^{\otimes m}$ we set $I_m(h) := I_m(\tilde{h})$. For any $g \in \mathcal{H}^{\otimes m}$, $h \in \mathcal{H}^{\otimes n}$, $n, m \geq 0$, it holds that

$$E[I_m(g)I_n(h)] = \delta_{mn} m! \langle \tilde{g}, \tilde{h} \rangle_{\mathcal{H}^{\otimes m}}.$$

For any $h = h_1 \otimes \dots \otimes h_m$ and $g = g_1 \otimes \dots \otimes g_m \in \mathcal{H}^{\otimes m}$, we define the p -th contraction of h and g , denoted by $h \otimes_p g$, as the element of $\mathcal{H}^{\otimes 2(m-p)}$ given by

$$h \otimes_p g = \langle h_1, g_1 \rangle_{\mathcal{H}} \dots \langle h_p, g_p \rangle_{\mathcal{H}} h_{p+1} \otimes \dots \otimes h_m \otimes g_{p+1} \otimes \dots \otimes g_m.$$

This definition can be extended by linearity to any element of $\mathcal{H}^{\otimes m}$.

Now, let $\mathcal{G}$ be the σ -field generated by the random variables $\{W(h) \mid h \in \mathcal{H}\}$. Any square integrable random variable $F \in L^2(\Omega, \mathcal{G}, P)$ has a unique chaos decomposition

$$F = \sum_{m=0}^{\infty} I_m(h_m) ,$$

where $h_m \in \mathcal{H}^{\odot m}$ (see [Nu06] for more details).

Finally, we adapt the theory of Wiener chaos expansion to the set up of Section 3. Let $\mathcal{G}$ be the σ -field generated by the random variables $(X_{i,n})_{n \geq 1, 1 \leq i \leq [nt]}$ and $\mathcal{H}_1$ be the first Wiener chaos associated with $(X_{i,n})_{n \geq 1, 1 \leq i \leq [nt]}$, i.e. the closed subspace of $L^2(\Omega, \mathcal{G}, P)$ generated by the random variables $(X_{i,n})_{n \geq 1, 1 \leq i \leq [nt]}$. Notice that $\mathcal{H}_1$ can be seen as a separable Hilbert space with a scalar product induced by the covariance function of the process $(X_{i,n})_{n \geq 1, 1 \leq i \leq [nt]}$. This means we can apply the above theory of Wiener chaos expansion with the canonical Hilbert space $\mathcal{H} = \mathcal{H}_1$. Denote by $\mathcal{H}_m$ the m th Wiener chaos associated with the triangular array $(X_{i,n})_{n \geq 1, 1 \leq i \leq [nt]}$ and by I_m the corresponding linear isometry between the symmetric tensor product $\mathcal{H}_1^{\odot m}$ (equipped with the norm $\sqrt{m!} \|\cdot\|_{\mathcal{H}_1^{\otimes m}}$) and the m th Wiener chaos.

7.1 Preliminary results

First of all, let us note that w.l.o.g. the volatility process σ can be assumed to be bounded on compact intervals because σ is càdlàg. This follows by a standard localization procedure presented in [BNGJPS06]. Furthermore, the process F_t , defined by (4.4), is continuous, because σ is càdlàg. Hence, F_t is locally bounded and can be assumed to be bounded on compact intervals w.l.o.g. by the same localization procedure.

Next we establish three lemmas.

Lemma 1 *Under assumption (4.4) it holds that*

$$E[|\Delta_i^n Y|^p] \leq C_p \tau_n^p , \quad i = 0, \dots, [nt] \quad (7.1)$$

for all $p > 0$.

Proof of Lemma 1: Recall that g' is non-increasing on (b, ∞) for some $b > 0$. Assume w.l.o.g. that $b > 1$. Observe the decomposition

$$\Delta_i^n Y = \int_{\frac{i-1}{n}}^{\frac{i}{n}} g\left(\frac{i}{n} - s\right) \sigma_s W(ds) + \int_{-\infty}^{\frac{i-1}{n}} \left(g\left(\frac{i}{n} - s\right) - g\left(\frac{i-1}{n} - s\right)\right) \sigma_s W(ds).$$

Since σ is bounded on compact intervals we deduce by Burkholder's inequality

$$E[|\Delta_i^n Y|^p] \leq C_p \left(\tau_n^p + E\left(\int_0^\infty \left(g\left(\frac{1}{n} + s\right) - g(s)\right)^2 \sigma_{\frac{i-1}{n}-s}^2 ds \right)^{p/2} \right)$$

We immediately obtain the estimates

$$\int_0^1 \left(g\left(\frac{1}{n} + s\right) - g(s)\right)^2 \sigma_{\frac{i-1}{n}-s}^2 ds \leq C \tau_n^2 ,$$

$$\int_1^b \left(g\left(\frac{1}{n} + s\right) - g(s) \right)^2 \sigma_{\frac{i-1}{n}-s}^2 ds \leq \frac{C}{n^2},$$

because g' is continuous on $(0, \infty)$ and σ is bounded on compact intervals. On the other hand, since g' is non-increasing on (b, ∞) , we get

$$\int_b^\infty \left(g\left(\frac{1}{n} + s\right) - g(s) \right)^2 \sigma_{\frac{i-1}{n}-s}^2 ds \leq \frac{F_{\frac{i-1}{n}}}{n^2}.$$

The boundedness of the process F implies (7.1). $\square$

Next, for any stochastic process f and any $s > 0$, we define the (possibly infinite) measure (recall that $g(x) := 0$ for $x \leq 0$)

$$\pi_{f,s}^n(A) = \frac{E \int_A (g(x - \frac{1}{n}) - g(x))^2 f_{s-x}^2 dx}{\int_0^\infty (g(x - \frac{1}{n}) - g(x))^2 dx}, \quad A \in \mathcal{B}(\mathbb{R}). \quad (7.2)$$

We further define $\bar{\pi}_{f,s}^n(x) = \pi_{f,s}^n(\{y : y > x\})$.

Lemma 2 *Under assumption (4.4) it holds that*

$$\sup_{s \in [0, t]} \bar{\pi}_{\sigma, s}^n(\varepsilon) \leq C \bar{\pi}^n(\varepsilon) \quad (7.3)$$

for any $\varepsilon > 0$, where $\bar{\pi}^n$ is given by (4.5).

Proof of Lemma 2: Recall again that g' is non-increasing on (b, ∞) for some $b > 0$, and assume w.l.o.g. that $b > \varepsilon$. Since the processes σ and F are bounded we deduce exactly as in the previous proof that

$$\begin{aligned} \int_\varepsilon^\infty (g(x - \frac{1}{n}) - g(x))^2 \sigma_{s-x}^2 dx &= \int_\varepsilon^b (g(x - \frac{1}{n}) - g(x))^2 \sigma_{s-x}^2 dx + \int_b^\infty (g(x - \frac{1}{n}) - g(x))^2 \sigma_{s-x}^2 dx \\ &\leq C \left(\int_\varepsilon^\infty (g(x - \frac{1}{n}) - g(x))^2 dx + n^{-2} \right). \end{aligned}$$

This completes the proof of Lemma 2. $\square$

Finally, we present the following technical Lemma.

Lemma 3 *Under the assumption (CLT) there exists a number $l \geq 1$ and positive sequences $\varepsilon_n^{(j)} \rightarrow 0$, $j = 1, \dots, l$, such that $0 < \varepsilon_n^{(1)} < \dots < \varepsilon_n^{(l)}$ and*

$$\varepsilon_n^{(1)} = o(n^{-\frac{1}{2\gamma(p \wedge 1)}}), \quad \bar{\pi}^n(\varepsilon_n^{(l)}) = o(n^{-\frac{1}{p \wedge 1}}) \quad (7.4)$$

$$(\varepsilon_n^{(j+1)})^{2\gamma} \bar{\pi}^n(\varepsilon_n^{(j)}) = o(n^{-\frac{1}{p \wedge 1}}), \quad j = 1, \dots, l-1, \quad (7.5)$$

where $p = \min_{1 \leq i \leq k, 1 \leq j \leq d} (p_i^j)$.

Proof of Lemma 3: Assume first that $p \geq 1$. Recall that $\gamma > 1/2$. Set $\varepsilon_n^{(j)} = n^{-\kappa_j}$, $j = 1, \dots, l$, with $1 > \kappa_1 > \dots > \kappa_l > 0$. The condition $\bar{\pi}^n(\varepsilon_n^{(j)}) = O(n^{\lambda(1-\kappa_j)})$ for some $\lambda < -1$, presented in (4.8), implies that conditions (7.4) and (7.5) are satisfied if we find $1 > \kappa_1 > \dots > \kappa_l > 0$ such that

$$\begin{aligned} \kappa_1 &> \frac{1}{2\gamma}, \\ \kappa_l &< 1 + \frac{1}{\lambda}, \\ (1 + \lambda) - \kappa_j \lambda - 2\kappa_{j+1}\gamma &< 0, \quad 1 \leq j \leq l-1. \end{aligned}$$

From the first and the last inequality we deduce by induction that

$$\frac{1}{2\gamma} < \kappa_1 < \frac{1+\lambda}{\lambda} \sum_{i=0}^{l-1} \left(-\frac{2\gamma}{\lambda}\right)^i + \left(-\frac{2\gamma}{\lambda}\right)^l \kappa_l$$

must hold.

When $2\gamma \geq -\lambda$ the term on the right-hand side converges to ∞ as $l \rightarrow \infty$. In that case it is easy to find constants $1 > \kappa_1 > \dots > \kappa_l > 0$ such that (7.4) and (7.5) are satisfied.

When $2\gamma < -\lambda$ the limit of $\frac{1+\lambda}{\lambda} \sum_{i=0}^{l-1} \left(-\frac{2\gamma}{\lambda}\right)^i$ is $\frac{1+\lambda}{\lambda+2\gamma}$ (as $l \rightarrow \infty$) and the restriction on κ_1 becomes

$$\frac{1}{2\gamma} < \kappa_1 < \frac{1+\lambda}{\lambda+2\gamma}.$$

Notice that $\frac{1}{2\gamma} < \frac{1+\lambda}{\lambda+2\gamma}$ because $\gamma > 1/2$. The existence of the positive powers κ_j , $j = 2, \dots, l$ that satisfy the original inequality follows by an induction argument.

Assume now that $p < 1$. Recall that γ must satisfy

$$\gamma > \frac{1}{2p}$$

and $\lambda < -\frac{1}{p}$. Again the conditions (7.4) and (7.5) are satisfied if we find $1 > \kappa_1 > \dots > \kappa_l > 0$ such that

$$\begin{aligned} \kappa_1 &> \frac{1}{2\gamma p}, \\ \kappa_l &< \frac{1+\lambda p}{\lambda p}, \\ \left(\frac{1}{p} + \lambda\right) - \kappa_j \lambda - 2\kappa_{j+1}\gamma &< 0, \quad 1 \leq j \leq l-1. \end{aligned}$$

Notice that the second inequality has solutions because $\lambda < -\frac{1}{p}$. Moreover, we deduce as above that the inequality

$$\frac{1}{2\gamma p} < \kappa_1 < \frac{\frac{1}{p} + \lambda}{\lambda} \sum_{i=0}^{l-1} \left(-\frac{2\gamma}{\lambda}\right)^i + \left(-\frac{2\gamma}{\lambda}\right)^l \kappa_l$$

must hold. Again the more complicated case is $2\gamma < -\lambda$. By letting $l \rightarrow \infty$ the restriction on κ_1 becomes

$$\frac{1}{2\gamma p} < \kappa_1 < \frac{\frac{1}{p} + \lambda}{\lambda + 2\gamma}$$

Note that $\frac{1}{2\gamma p} < \frac{\frac{1}{p} + \lambda}{\lambda + 2\gamma}$ because $\gamma > \frac{1}{2p}$. As before the existence of the positive powers κ_j , $j = 2, \dots, l$ that satisfy the original inequality follows by an induction argument. $\square$

7.2 Some notation

Recall that the covariance matrix of $(X_{i,n}, \dots, X_{i+l,n})$ is invertible for any $l \geq 1$ and $n \geq 1$. Let $X_i^n(1), \dots, X_i^n(k)$ be an i.i.d. $N(0, 1)$ sequence that spans the same linear space as $X_{i,n}, \dots, X_{i+k-1,n}$ (such a sequence can be constructed by the Gram-Schmidt method). Thus, it has the representation

$$X_i^n(j) = \sum_{l=1}^k a_{lj}^n X_{i+l-1,n}, \quad j = 1, \dots, k, \quad (7.6)$$

for some real numbers a_{lj}^n . Note that

$$|a_{lj}^n| \leq C,$$

for all l, j, n , because $E[X_{i,n}^2] = 1$ for all i, n .

For any $1 \leq j \leq d$, we obtain the Wiener chaos representation

$$V(p_1^j, \dots, p_k^j)_t - \rho_{p_1^j, \dots, p_k^j}^{(n)} t = \sum_{m=2}^{\infty} I_m \left(\frac{1}{n} \sum_{i=1}^{[nt]} f_{m,j}^n(i) \right) + O_p(n^{-1}), \quad (7.7)$$

where the $f_{m,j}^n(i) \in \mathcal{H}^{\odot m}$ are given by

$$f_{m,j}^n(i) = \sum_{k_l \in \{1, \dots, k\}} c_{k_1, \dots, k_m}^n(j) X_i^n(k_1) \otimes \dots \otimes X_i^n(k_m) \quad (7.8)$$

for some coefficients $c_{k_1, \dots, k_m}^n(j)$. We set

$$c_m^n(j) = \|f_{m,j}^n(i)\|_{\mathcal{H}^{\otimes m}}^2 = \sum_{k_l \in \{1, \dots, k\}} |c_{k_1, \dots, k_m}^n(j)|^2. \quad (7.9)$$

Note that

$$\text{var} \left(|X_{i,n}|^{p_1^j} \dots |X_{i+k-1,n}|^{p_k^j} \right) = \sum_{m=2}^{\infty} m! c_m^n(j) < C \quad (7.10)$$

for all n, j , because $E[X_{i,n}^2] = 1$ for all i, n . Finally, when $f_{m,j}^n(i)$, $c_{k_1, \dots, k_m}^n(j)$ and $c_m^n(j)$ correspond to some particular choice of powers $p_1, \dots, p_k$ we use the notation $f_m^n(i)$, $c_{k_1, \dots, k_m}^n$ and c_m^n .

Now assume that the assumptions (3.8) and (3.9) of Theorem 2 hold. Since a_{lj}^n in (7.6) is a continuous function of $r(1), \dots, r_n(k-1)$ and the Gaussian process Q is non-degenerate, we have that

$$a_{lj}^n \rightarrow a_{lj},$$

and the sequence $Q_i(1), \dots, Q_i(k)$ given by

$$Q_i(j) = \sum_{l=1}^k a_{lj} Q_{i+l-1}, \quad j = 1, \dots, k, \quad (7.11)$$

is an i.i.d. $N(0, 1)$ sequence. Now, let us associate $f_{m,j}(i)$, $c_{k_1, \dots, k_m}(j)$ and $c_m(j)$ with the functional $V_Q(p_1^j, \dots, p_k^j)_t^n - \rho_{p_1^j, \dots, p_k^j} t$, where

$$\rho_{p_1^j, \dots, p_k^j} = E \left[|Q_1|^{p_1^j} \dots |Q_k|^{p_k^j} \right],$$

by (7.7), (7.8) and (7.9). By a repeated application of the multiplication formula (see [Nu06]), we know that $c_{k_1, \dots, k_m}^n(j)$ is a continuous function of $r_n(1), \dots, r_n(k-1)$. Since $r_n(j) \rightarrow \rho(j)$ we obtain

$$c_{k_1, \dots, k_m}^n(j) \rightarrow c_{k_1, \dots, k_m}(j), \quad c_m^n(j) \rightarrow c_m(j), \quad (7.12)$$

$$\langle f_{m,j_1}^n(i), f_{m,j_2}^n(i+l) \rangle_{\mathbb{H}^{\otimes m}} \rightarrow \langle f_{m,j_1}(i), f_{m,j_2}(i+l) \rangle_{\mathbb{H}^{\otimes m}}, \quad (7.13)$$

$$\begin{aligned} & \text{cov} \left(|X_{i,n}|^{p_1^{j_1}} \dots |X_{i+k-1,n}|^{p_k^{j_1}}, |X_{i,n}|^{p_1^{j_2}} \dots |X_{i+k-1,n}|^{p_k^{j_2}} \right) = \sum_{m=2}^{\infty} m! \langle f_{m,j_1}^n(1), f_{m,j_2}^n(1) \rangle_{\mathbb{H}^{\otimes m}} \\ & \rightarrow \text{cov} \left(|Q_i|^{p_1^{j_1}} \dots |Q_{i+k-1}|^{p_k^{j_1}}, |X_{i,n}|^{p_1^{j_2}} \dots |Q_{i+k-1}|^{p_k^{j_2}} \right) \\ & = \sum_{m=2}^{\infty} m! \langle f_{m,j_1}(1), f_{m,j_2}(1) \rangle_{\mathbb{H}^{\otimes m}}. \end{aligned} \quad (7.14)$$

7.3 Proof of Theorem 1 and 3

Proof of Theorem 1: Since $V(p_1, \dots, p_k)_t^n$ is increasing in t and the process $\rho_{p_1, \dots, p_k}^{(n)} t$ is continuous in t , it is sufficient to prove $V(p_1, \dots, p_k)_t^n - \rho_{p_1, \dots, p_k}^{(n)} t \xrightarrow{P} 0$ for a fixed $t > 0$.

Note that

$$|\langle f_m^n(1), f_m^n(1+l) \rangle_{\mathbb{H}^{\otimes m}}| \leq c_m^n, \quad |\langle f_m^n(1), f_m^n(1+l) \rangle_{\mathbb{H}^{\otimes m}}| \leq c_m^n C^m (|r_n(l)|^m + \dots + |r_n(l-k+1)|^m) \quad (7.15)$$

Now, due to assumption (3.3), $r(j) \rightarrow 0$ as $j \rightarrow \infty$. Thus, there exists a H such that $|Cr^{1/2}(j-k+1)| < 1$ for $j \geq H$ (for any fixed C). By (7.15) we have (for any $m \geq 2$)

$$\begin{aligned} & \sum_{l=1}^{n-1} |\langle f_m^n(1), f_m^n(1+l) \rangle_{\mathbb{H}^{\otimes m}}| \leq C \left(H c_m^n + \sum_{l=H}^{n-1} |\langle f_m^n(1), f_m^n(1+l) \rangle_{\mathbb{H}^{\otimes m}}| \right) \\ & \leq C c_m^n \left(H + \sum_{l=H}^{n-1} (C |r_n(l)|)^2 \right) \leq C c_m^n \sum_{l=1}^{n-1} r(l). \end{aligned} \quad (7.16)$$

Hence,

$$\text{var}(V(p_1, \dots, p_k)_t^n) \leq \frac{C}{n} \sum_{m=2}^{\infty} m! c_m^n \left(1 + \sum_{l=1}^{n-1} r(l) \right).$$

The latter converges to 0 due to (7.10) and assumption (3.3). $\square$

Proof of Theorem 3: First of all, recall that

$$E[|\Delta_i^n Y|^q] \leq C\tau_n^q, \quad E[|\Delta_i^n G|^q] \leq C\tau_n^q, \quad (7.17)$$

for any $q \geq 0$, due to Lemma 1.

In the following we will prove Theorem 3 only for $k = 1$, $p_1 = p$. The general case can be obtained in a similar manner by (7.17) and an application of the Hölder inequality.

Since $V(Y, p)_t^n$ is increasing in t and the limit process is continuous in t , it suffices to prove the pointwise convergence $V(Y, p)_t^n \xrightarrow{P} \mu_p \int_0^t |\sigma_s|^p ds$. For any $l \leq n$, we have

$$V(Y, p)_t^n - \mu_p \int_0^t |\sigma_s|^p ds = \frac{1}{n\tau_n^p} \sum_{i=1}^{[nt]} \left(|\Delta_i^n Y|^p - |\sigma_{\frac{i-1}{n}} \Delta_i^n G|^p \right) + R_t^{n,l},$$

where

$$\begin{aligned} R_t^{n,l} &= \frac{1}{n\tau_n^p} \left(\sum_{i=1}^{[nt]} |\sigma_{\frac{i-1}{n}} \Delta_i^n G|^p - \sum_{j=1}^{[lt]} |\sigma_{\frac{j-1}{l}}|^p \sum_{i \in I_l(j)} |\Delta_i^n G|^p \right) \\ &+ \frac{1}{n\tau_n^p} \sum_{j=1}^{[lt]} |\sigma_{\frac{j-1}{l}}|^p \sum_{i \in I_l(j)} |\Delta_i^n G|^p - \mu_p l^{-1} \sum_{j=1}^{[lt]} |\sigma_{\frac{j-1}{l}}|^p \\ &+ \mu_p \left(l^{-1} \sum_{j=1}^{[lt]} |\sigma_{\frac{j-1}{l}}|^p - \int_0^t |\sigma_s|^p ds \right), \end{aligned}$$

and

$$I_l(j) = \left\{ i \mid \frac{i}{n} \in \left(\frac{j-1}{l}, \frac{j}{l} \right] \right\}, \quad j \geq 1.$$

The assumption **(LLN)** implies that $V(G, p)_t^n \xrightarrow{ucp} \mu_p t$. Since σ is càdlàg and bounded on compact intervals, we deduce that

$$\lim_{l \rightarrow \infty} \lim_{n \rightarrow \infty} P(|R_t^{n,l}| > \epsilon) = 0,$$

for any $\epsilon > 0$. Hence, we are left to prove that

$$\frac{1}{n\tau_n^p} \sum_{i=1}^{[nt]} \left(|\Delta_i^n Y|^p - |\sigma_{\frac{i-1}{n}} \Delta_i^n G|^p \right) \xrightarrow{P} 0.$$

By applying the inequality $||x|^p - |y|^p| \leq p|x - y|(|x|^{p-1} + |y|^{p-1})$ for $p > 1$ and $||x|^p - |y|^p| \leq |x - y|^p$ for $p \leq 1$, (7.17) and the Cauchy-Schwarz inequality we can conclude that the above convergence follows from

$$\frac{1}{n\tau_n^2} \sum_{i=1}^{[nt]} E[|\Delta_i^n Y - \sigma_{\frac{i-1}{n}} \Delta_i^n G|^2] \rightarrow 0. \quad (7.18)$$

Observe the decomposition

$$\Delta_i^n Y - \sigma_{\frac{i-1}{n}} \Delta_i^n G = A_i^n + B_i^{n,\varepsilon} + C_i^{n,\varepsilon},$$

where

$$\begin{aligned} A_i^n &= \int_{\frac{i-1}{n}}^{\frac{i}{n}} g\left(\frac{i}{n} - s\right) (\sigma_s - \sigma_{\frac{i-1}{n}}) W(ds) \\ B_i^{n,\varepsilon} &= \int_{\frac{i-1}{n}-\varepsilon}^{\frac{i-1}{n}} \left(g\left(\frac{i}{n} - s\right) - g\left(\frac{i-1}{n} - s\right)\right) \sigma_s W(ds) - \sigma_{\frac{i-1}{n}} \int_{\frac{i-1}{n}-\varepsilon}^{\frac{i-1}{n}} \left(g\left(\frac{i}{n} - s\right) - g\left(\frac{i-1}{n} - s\right)\right) W(ds) \\ C_i^{n,\varepsilon} &= \int_{-\infty}^{\frac{i-1}{n}-\varepsilon} \left(g\left(\frac{i}{n} - s\right) - g\left(\frac{i-1}{n} - s\right)\right) \sigma_s W(ds) - \sigma_{\frac{i-1}{n}} \int_{-\infty}^{\frac{i-1}{n}-\varepsilon} \left(g\left(\frac{i}{n} - s\right) - g\left(\frac{i-1}{n} - s\right)\right) W(ds) \end{aligned}$$

By Lemma 2 and the boundedness of σ on compact intervals we deduce

$$\frac{1}{n\tau_n^2} \sum_{i=1}^{[nt]} E[|C_i^{n,\varepsilon}|^2] \leq C\bar{\pi}^n(\varepsilon). \quad (7.19)$$

and by (4.6) we obtain that

$$\lim_{n \rightarrow \infty} \frac{1}{n\tau_n^2} \sum_{i=1}^{[nt]} E[|C_i^{n,\varepsilon}|^2] = 0.$$

Next, we get

$$\frac{1}{n\tau_n^2} \sum_{i=1}^{[nt]} E[|A_i^n|^2] \leq \frac{C}{n\tau_n^2} E\left[\sum_{i=1}^{[nt]} \int_{\frac{i-1}{n}}^{\frac{i}{n}} g^2\left(\frac{i}{n} - s\right) (\sigma_s - \sigma_{\frac{i-1}{n}})^2 ds\right]. \quad (7.20)$$

Set $v(s, \eta) = \sup\{|\sigma_s - \sigma_r|^2 \mid s, r \in [-t, t], |r - s| \leq \eta\}$. Then we obtain

$$\frac{1}{n\tau_n^2} \sum_{i=1}^{[nt]} E[|A_i^n|^2] \leq \frac{1}{n} \sum_{i=1}^{[nt]} E\left[v\left(\frac{i-1}{n}, n^{-1}\right)\right]. \quad (7.21)$$

Moreover, for any $\kappa > 0$, since σ is cadlag, there exists n big enough such that

$$v\left(\frac{i-1}{n}, n^{-1}\right) \leq \kappa + \left(\Delta\sigma_{\frac{i-1}{n}}\right)^2 \mathbf{1}_{\left\{\left(\Delta\sigma_{\frac{i-1}{n}}\right)^2 \geq \kappa\right\}},$$

so

$$\begin{aligned} \frac{1}{n\tau_n^2} \sum_{i=1}^{[nt]} E[|A_i^n|^2] &\leq \kappa + \frac{1}{n} \sum_{i=1}^{[nt]} E\left[\left(\Delta\sigma_{\frac{i-1}{n}}\right)^2 \mathbf{1}_{\left\{\left(\Delta\sigma_{\frac{i-1}{n}}\right)^2 \geq \kappa\right\}}\right] \\ &\leq \kappa + E\left[\frac{1}{n} \sum_{-t \leq s \leq t} (\Delta\sigma_s)^2 \mathbf{1}_{\{(\Delta\sigma_s)^2 \geq \kappa\}}\right], \end{aligned}$$

then

$$\lim_{n \rightarrow \infty} \frac{1}{n\tau_n^2} \sum_{i=1}^{[nt]} E[|A_i^n|^2] \leq \kappa$$

and the convergence to zero follows letting κ tend to zero.

Finally, observe the decomposition $B_i^{n,\varepsilon} = B_i^{n,\varepsilon}(1) + B_i^{n,\varepsilon}(2)$ with

$$\begin{aligned} B_i^{n,\varepsilon}(1) &= \int_{\frac{i-1}{n}-\varepsilon}^{\frac{i-1}{n}} \left(g\left(\frac{i}{n} - s\right) - g\left(\frac{i-1}{n} - s\right) \right) (\sigma_s - \sigma_{\frac{i-1}{n}-\varepsilon}) W(ds) \\ B_i^{n,\varepsilon}(2) &= (\sigma_{\frac{i-1}{n}-\varepsilon} - \sigma_{\frac{i-1}{n}}) \int_{\frac{i-1}{n}-\varepsilon}^{\frac{i-1}{n}} \left(g\left(\frac{i}{n} - s\right) - g\left(\frac{i-1}{n} - s\right) \right) W(ds).. \end{aligned}$$

We obtain the inequalities

$$\begin{aligned} \frac{1}{n\tau_n^2} \sum_{i=1}^{[nt]} E[|B_i^{n,\varepsilon}(1)|^2] &\leq \frac{1}{n} \sum_{i=1}^{[nt]} E\left[v\left(\frac{i-1}{n}, \varepsilon\right)\right] \\ \frac{1}{n\tau_n^2} \sum_{i=1}^{[nt]} E[|B_i^{n,\varepsilon}(2)|^2] &\leq \frac{1}{n} \sum_{i=1}^{[nt]} E\left[v\left(\frac{i-1}{n}, \varepsilon\right)^2\right]^{1/2}. \end{aligned} \quad (7.22)$$

By using the same arguments as above we have that both terms converge to zero and we obtain (7.18), which completes the proof of Theorem 3. $\square$

7.4 Proof of Theorem 2 and 4

Proof of Theorem 2: We first show the weak convergence of finite dimensional distributions and then prove the tightness of the sequence $\sqrt{n} \left(V(p_1^j, \dots, p_k^j)_t^n - \rho_{p_1^j, \dots, p_k^j}^{(n)} t \right)_{1 \leq j \leq d}$.

Step 1: Define the vector $Z_n(j) = (Z_n^1(j), \dots, Z_n^e(j))^T$, $1 \leq j \leq d$, by

$$Z_n^l(j) = \frac{1}{\sqrt{n}} \sum_{i=[nc_l]+1}^{[nb_l]} \left(|X_{i,n}|^{p_1^j} \cdots |X_{i+k-1,n}|^{p_k^j} - \rho_{p_1^j, \dots, p_k^j}^{(n)} \right), \quad (7.23)$$

where $(c_l, b_l]$, $l = 1, \dots, e$, are disjoint intervals contained in $[0, T]$. Set $Z_n^l = (Z_n^l(1), \dots, Z_n^l(d))$, $l = 1, \dots, e$. Clearly, it suffices to prove that

$$\left(Z_n^l \right)_{1 \leq l \leq e} \xrightarrow{\mathcal{D}} \left(\beta^{1/2} (B_{b_l} - B_{c_l}) \right)_{1 \leq l \leq e},$$

where the matrix β is given in Theorem 2. By (7.7) we have the representation

$$Z_n^l(j) = \sum_{m=2}^{\infty} I_k \left(\frac{1}{\sqrt{n}} \sum_{i=[nc_l]+1}^{[nb_l]} f_{m,j}^n(i) \right).$$

Set $F_{m,l}^n(j) = \frac{1}{\sqrt{n}} \sum_{i=[nc_l]+1}^{[nb_l]} f_{m,j}^n(i)$. By Theorem 2 in [BNCPW09] we obtain the weak convergence of finite dimensional distributions when we show that

(i) For any $1 \leq l \leq e$, $1 \leq j \leq d$ we have

$$\lim_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{m=N+1}^{\infty} m! \|F_{m,l}^n(j)\|_{\mathcal{H}^{\otimes k}}^2 = 0.$$

(ii) For any $m \geq 2$, $1 \leq l \leq e$ and $1 \leq j_1, j_2 \leq d$, we have constants $C_{k,l}$ such that

$$\lim_{n \rightarrow \infty} m! \langle F_{m,l}^n(j_1), F_{m,l}^n(j_2) \rangle_{\mathbb{H}^{\otimes m}} = C_{m,l}(j_1, j_2),$$

$$\text{and } \sum_{m=2}^{\infty} C_{m,l}(j_1, j_2) = \beta_{j_1, j_2}(b_l - c_l).$$

(iii) For any $1 \leq l_1 \neq l_2 \leq e$ and $1 \leq j_1, j_2 \leq d$, we have

$$\lim_{n \rightarrow \infty} \langle F_{m,l_1}^n(j_1), F_{m,l_2}^n(j_2) \rangle_{\mathbb{H}^{\otimes m}} = 0.$$

(iv) For any $m \geq 2$, $1 \leq l \leq e$, $1 \leq j \leq d$ and $p = 1, \dots, m-1$

$$\lim_{n \rightarrow \infty} \|F_{m,l}^n(j) \otimes_p F_{m,l}^n(j)\|_{\mathbb{H}^{\otimes 2(m-p)}}^2 = 0.$$

Note that it is sufficient to prove (i), (ii) and (iv) for $l = 1$, $b_l = 1$ and $a_l = 0$. In this case we use the notation $F_m^n(j) = F_{m,1}^n(j)$.

(i) and (ii): As in (7.16) we have

$$\begin{aligned} m! \langle F_m^n(j_1), F_m^n(j_2) \rangle_{\mathbb{H}^{\otimes m}} &= m! \left(\langle f_{m,j_1}^n(1), f_{m,j_2}^n(1) \rangle_{\mathbb{H}^{\otimes m}} + \frac{2}{n} \sum_{l=1}^{n-1} (n-l) \langle f_{m,j_1}^n(1), f_{m,j_2}^n(1+l) \rangle_{\mathbb{H}^{\otimes m}} \right) \\ &\leq C m! \langle f_{m,j_1}^n(1), f_{m,j_2}^n(1) \rangle_{\mathbb{H}^{\otimes m}} \left(1 + \sum_{l=1}^{n-1} r(l) \right) \end{aligned} \quad (7.24)$$

Since $\sum_{l=1}^{\infty} r(l) < \infty$, we obtain by (7.12)-(7.14) and the dominated convergence theorem

$$\begin{aligned} \lim_{n \rightarrow \infty} m! \langle F_{m,l}^n(j_1), F_{m,l}^n(j_2) \rangle_{\mathbb{H}^{\otimes m}} &= C_m(j_1, j_2) \\ &= m! \left(\langle f_{m,j_1}(1), f_{m,j_2}(1) \rangle_{\mathbb{H}^{\otimes m}} + 2 \sum_{l=1}^{\infty} \langle f_{m,j_1}^n(1), f_{m,j_2}^n(1+l) \rangle_{\mathbb{H}^{\otimes m}} \right), \end{aligned}$$

and $\sum_{m=2}^{\infty} C_m(j_1, j_2) = \beta_{j_1, j_2}$ (notice that β_{j_1, j_2} is finite due to the dominated convergence theorem). Hence, we deduce (ii). On the other hand, we have

$$\limsup_{n \rightarrow \infty} \sum_{m=N+1}^{\infty} m! \langle f_{m,j_1}^n(1), f_{m,j_2}^n(1) \rangle_{\mathbb{H}^{\otimes m}} = \sum_{m=N+1}^{\infty} m! \langle f_{m,j_1}(1), f_{m,j_2}(1) \rangle_{\mathbb{H}^{\otimes m}} < \infty.$$

Thus, we obtain (i) by (7.24).

(iii): W.l.o.g. consider the case $j = j_1 = j_2$. For any $l_1 < l_2$, as in (7.16), we have

$$|\langle F_{m,l_1}^n(j), F_{m,l_2}^n(j) \rangle_{\mathbb{H}^{\otimes m}}| \leq \frac{C}{n} \sum_{h=[nc_{l_1}]+1}^{[nb_{l_1}]} \sum_{i=[nc_{l_2}]+1}^{[nb_{l_2}]} |r_n^m(i-h)|.$$

Assume w.l.o.g. that $c_{l_1} = 0$, $b_{l_1} = c_{l_2} = 1$ and $b_{l_2} = 2$ (the case $b_{l_1} < c_{l_2}$ is much easier). Then, by condition (3.9), we obtain the approximation (as in (7.24))

$$|\langle F_{m,l_1}^n(j), F_{m,l_2}^n(j) \rangle_{\mathcal{H}^{\otimes m}}| \leq C \left(\frac{1}{n} \sum_{j=1}^n j r(j) + \sum_{j=1}^{n-1} r(n+j) \right) \rightarrow 0,$$

since $\sum_{j=1}^{\infty} r(j) < \infty$.

(iv): A straightforward computation shows that

$$\|F_m^n(j) \otimes_p F_m^n(j)\|_{\mathcal{H}^{\otimes 2(m-p)}}^2 = \frac{C_{m,j}}{n^2} \sum_{i_1, i_2, i_3, i_4=1}^{n-1} r_n^p(|i_1 - i_2|) r_n^p(|i_4 - i_3|) r_n^{m-p}(|i_1 - i_4|) r_n^{m-p}(|i_2 - i_3|).$$

for some constant $C_{m,j}$. The latter is smaller than

$$\frac{C}{n} \sum_{i,h,l=1}^{n-1} |r_n^p(i)| |r_n^p(l)| |r_n^{m-p}(|i-h|)| |r_n^{m-p}(|l-h|)| = \frac{C}{n} \sum_{h=1}^{n-1} \left(\sum_{i=1}^{n-1} |r_n^p(i)| |r_n^{m-p}(|i-h|)| \right)^2.$$

Now, for any $0 < \varepsilon < 1$, we obtain by the Hölder inequality

$$\begin{aligned} & n^{-1} \sum_{0 \leq h \leq n-1} \left(\sum_{0 \leq i \leq n-1} |r_n^p(i)| |r_n^{m-p}(|i-h|)| \right)^2 \leq n^{-1} \sum_{0 \leq h \leq [n\varepsilon]} \left(\sum_{0 \leq i \leq n-1} |r_n^p(i)| |r_n^{m-p}(|i-h|)| \right)^2 \\ & + 2n^{-1} \sum_{h=[n\varepsilon]}^{n-1} \left(\sum_{i=0}^{[n\varepsilon/2]} |r_n^p(i)| |r_n^{m-p}(|i-h|)| \right)^2 + 2n^{-1} \sum_{h=[n\varepsilon]}^{n-1} \left(\sum_{h=[n\varepsilon/2]}^{n-1} |r_n^p(i)| |r_n^{m-p}(|i-h|)| \right)^2 \\ & \leq C \left(\varepsilon \left(\sum_{0 \leq i \leq n-1} |r_n^m(i)| \right)^2 + \left(\sum_{0 \leq i \leq n-1} |r_n^m(i)| \right)^{2p/m} \left(\sum_{[n\varepsilon/2] < h \leq n-1} |r_n^m(h)| \right)^{2(m-p)/m} \right) \end{aligned}$$

The latter is smaller (again by (3.9)) than

$$C \left(\varepsilon \left(\sum_{0 \leq i \leq n-1} r(i) \right)^2 + \left(\sum_{0 \leq i \leq n-1} r(i) \right)^{2p/m} \left(\sum_{[n\varepsilon/2] < h \leq n-1} r(h) \right)^{2(m-p)/m} \right)$$

that converges to $C\varepsilon (\sum_{i=0}^{\infty} r(i))^2$ as $n \rightarrow \infty$. Thus, we obtain (iv) by letting $\varepsilon \rightarrow 0$. $\square$

Step 2: Clearly, it suffices to consider the case $d = 1$, $p_l^1 = p_l$. Set

$$\sqrt{n} \left(V(p_1, \dots, p_k)_t^n - \rho_{p_1, \dots, p_k}^{(n)} t \right) = \sum_{m=2}^{\infty} I_m \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} f_m^n(i) \right) + O_p(n^{-1/2}) =: Z_t^n + O(n^{-1/2})$$

(where the approximation holds locally uniformly in t) and

$$Z_t^{n,N} = \sum_{m=2}^N I_m \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} f_m^n(i) \right).$$

In Step 1 we have proved that conditions (i)-(iii) of Theorem 2 in [BNCPW09] are satisfied. Then by (2.3) of Theorem 2 in [BNCPW09] and the Cauchy-Schwarz inequality we obtain the approximation

$$\begin{aligned} P\left(|Z_t^{n,N} - Z_{t_1}^{n,N}| \geq \lambda, |Z_{t_2}^{n,N} - Z_t^{n,N}| \geq \lambda\right) &\leq \frac{E^{1/2}[|Z_t^{n,N} - Z_{t_1}^{n,N}|^4] E^{1/2}[|Z_{t_2}^{n,N} - Z_t^{n,N}|^4]}{\lambda^4} \\ &\leq C \frac{\beta^2([nt] - [nt_1])([nt_2] - [nt])}{\lambda^4} \leq C \frac{\beta^2(t_2 - t_1)^2}{\lambda^4} \end{aligned}$$

for any $t_1 \leq t \leq t_2$ and $\lambda > 0$. On the other hand (7.14) and (7.24) imply that

$$\lim_{N \rightarrow \infty} E[|Z_t^n - Z_t^{n,N}|^2] = 0$$

for any n and any t . Using this we conclude that

$$P\left(|Z_t^n - Z_{t_1}^n| \geq \lambda, |Z_{t_2}^n - Z_t^n| \geq \lambda\right) \leq C \frac{\beta^2(t_2 - t_1)^2}{\lambda^4}$$

for any $t_1 \leq t \leq t_2$ and $\lambda > 0$, from which we deduce the tightness of the sequence Z_t^n by Theorem 15.6 in [Bil68]. This completes the proof of Theorem 2. $\square$

Proof of Theorem 5: We only consider the case $d = 1$, $k = 1$, $p_1 = p$ (the general result is obtained by analogous arguments). We use the decomposition from the proof of Theorem 3:

$$\begin{aligned} \sqrt{n}\left(V(Y, p)_t^n - \mu_p \int_0^t |\sigma_s|^p ds\right) &= \sqrt{n}\left(\frac{1}{n\tau_n^p} \sum_{j=1}^{[lt]} |\sigma_{\frac{j-1}{l}}|^p \sum_{i \in I_l(j)} |\Delta_i^n G|^p - \mu_p l^{-1} \sum_{j=1}^{[lt]} |\sigma_{\frac{j-1}{l}}|^p\right) \\ &\quad + \frac{1}{\sqrt{n}\tau_n^p} \sum_{i=1}^{[nt]} \left(|\Delta_i^n Y|^p - |\sigma_{\frac{i-1}{n}}|^p |\Delta_i^n G|^p\right) + \bar{R}_t^{n,l}, \end{aligned} \quad (7.25)$$

for any $l \leq n$, with

$$\begin{aligned} \bar{R}_t^{n,l} &= \frac{1}{\sqrt{n}\tau_n^p} \left(\sum_{i=1}^{[nt]} |\sigma_{\frac{i-1}{n}}|^p |\Delta_i^n G|^p - \sum_{j=1}^{[lt]} |\sigma_{\frac{j-1}{l}}|^p \sum_{i \in I_l(j)} |\Delta_i^n G|^p \right) \\ &\quad + \sqrt{n}\mu_p \left(l^{-1} \sum_{j=1}^{[lt]} |\sigma_{\frac{j-1}{l}}|^p - \int_0^t |\sigma_s|^p ds \right). \end{aligned}$$

Observe that under the assumption **(CLT)** we obtain the weak convergence

$$\sqrt{n}\left(V(G, p)_t^n - \mu_p t\right) \Rightarrow \sqrt{\beta} B_t$$

(see Theorem 2). Since $E[G_t(V(G, p)_t^n - \mu_p t)] = 0$ for any $t > 0$, because G has a symmetric distribution, we deduce (by Theorem 5 in [BNCP09]) that

$$\left(G_t, \sqrt{n}\left(V(G, p)_t^n - \mu_p t\right)\right) \Rightarrow \left(G_t, \sqrt{\beta} B_t\right).$$

It follows that

$$\sqrt{n} \left(\frac{1}{n\tau_n^p} \sum_{j=1}^{[lt]} |\sigma_{\frac{j-1}{l}}|^p \sum_{i \in I_l(j)} |\Delta_i^n G|^p - \mu_p l^{-1} \sum_{j=1}^{[lt]} |\sigma_{\frac{j-1}{l}}|^p \right) \xrightarrow{\mathcal{G}-st} \sqrt{\beta} \sum_{j=1}^{[lt]} |\sigma_{\frac{j-1}{l}}|^p \Delta_j^l B$$

for any fixed l , because σ is assumed to be $\mathcal{G}$ -measurable. On the other hand, we have

$$\sqrt{\beta} \sum_{j=1}^{[lt]} |\sigma_{\frac{j-1}{l}}|^p \Delta_j^l B \xrightarrow{P} \sqrt{\beta} \int_0^t |\sigma_s|^p dB_s$$

as $l \rightarrow \infty$.

Now we need to prove that the other summands in the decomposition (7.25) are negligible. The negligibility of the term $\overline{R}_t^{n,l}$ is shown as in the proof of Theorem 7 in [BNCP09] but by using condition (4.7) instead of Hölder continuity of index γ . So we are left to prove that

$$\frac{1}{\sqrt{n}\tau_n^p} \sum_{i=1}^{[nt]} \left(|\Delta_i^n Y|^p - |\sigma_{\frac{i-1}{n}} \Delta_i^n G|^p \right) \xrightarrow{P} 0.$$

By applying, for $p \geq 1$, the inequality $||x|^p - |y|^p| \leq p|x - y|(|x|^{p-1} + |y|^{p-1})$, (7.17) and the Cauchy-Schwarz inequality, and, for $p \leq 1$, $||x|^p - |y|^p| \leq |x - y|^p$ and the Jensen inequality, we have

$$\frac{1}{\sqrt{n}\tau_n^p} \sum_{i=1}^{[nt]} E \left| |\Delta_i^n Y|^p - |\sigma_{\frac{i-1}{n}} \Delta_i^n G|^p \right| \leq \frac{1}{\sqrt{n}\tau_n^{p \wedge 1}} \sum_{i=1}^{[nt]} \left(E |\Delta_i^n Y - \sigma_{\frac{i-1}{n}} \Delta_i^n G|^2 \right)^{\frac{p \wedge 1}{2}}.$$

Now we use a similar decomposition as presented in the proof of Theorem 3:

$$\Delta_i^n Y - \sigma_{\frac{i-1}{n}} \Delta_i^n G = A_i^n + B_i^{n, \varepsilon_n^{(1)}} + \sum_{j=1}^l C_i^{n, \varepsilon_n^{(j)}, \varepsilon_n^{(j+1)}}$$

where A_i^n , $B_i^{n, \varepsilon_n^{(1)}}$ are defined as above, $0 < \varepsilon_n^{(1)} < \dots < \varepsilon_n^{(l)} < \varepsilon_n^{(l+1)} = \infty$ and

$$\begin{aligned} C_i^{n, \varepsilon_n^{(j)}, \varepsilon_n^{(j+1)}} &= \int_{\frac{i-1}{n} - \varepsilon_n^{(j+1)}}^{\frac{i-1}{n} - \varepsilon_n^{(j)}} \left(g\left(\frac{i}{n} - s\right) - g\left(\frac{i-1}{n} - s\right) \right) \sigma_s W(ds) \\ &- \sigma_{\frac{i-1}{n}} \int_{\frac{i-1}{n} - \varepsilon_n^{(j+1)}}^{\frac{i-1}{n} - \varepsilon_n^{(j)}} \left(g\left(\frac{i}{n} - s\right) - g\left(\frac{i-1}{n} - s\right) \right) W(ds). \end{aligned}$$

By assumption **(CLT)** and Lemma 2 we obtain the following inequalities (since σ is bounded on compact intervals)

$$\begin{aligned} \frac{1}{\sqrt{n}\tau_n^{p\wedge 1}} \sum_{i=1}^{[nt]} (E|A_i^n|^2)^{\frac{p\wedge 1}{2}} &\leq Cn^{-\gamma(p\wedge 1)+\frac{1}{2}} \\ \frac{1}{\sqrt{n}\tau_n^{p\wedge 1}} \sum_{i=1}^{[nt]} \left(E|B_i^{n,\varepsilon_n^{(1)}}|^2\right)^{\frac{p\wedge 1}{2}} &\leq Cn^{1/2}|\varepsilon_n^{(1)}|^{\gamma(p\wedge 1)} \\ \frac{1}{\sqrt{n}\tau_n^{p\wedge 1}} \sum_{i=1}^{[nt]} \left(E|C_i^{n,\varepsilon_n^{(j)},\varepsilon_n^{(j+1)}}|^2\right)^{\frac{p\wedge 1}{2}} &\leq Cn^{1/2}|\varepsilon_n^{(j+1)}|^{\gamma(p\wedge 1)}|\bar{\pi}^n(\varepsilon_n^{(j+1)}) - \bar{\pi}^n(\varepsilon_n^{(j)})|^{\frac{p\wedge 1}{2}}, \quad j = 1, \dots, l-1, \\ \frac{1}{\sqrt{n}\tau_n^{p\wedge 1}} \sum_{i=1}^{[nt]} \left(E|C_i^{n,\varepsilon_n^{(l)},\varepsilon_n^{(l+1)}}|^2\right)^{\frac{p\wedge 1}{2}} &\leq Cn^{1/2}\bar{\pi}^n(\varepsilon_n^{(l)})^{\frac{p\wedge 1}{2}}. \end{aligned}$$

Then we deduce by **(CLT)** and Lemma 3

$$\frac{1}{\sqrt{n}\tau_n^p} \sum_{i=1}^{[nt]} |\Delta_i^n Y - \sigma_{\frac{i-1}{n}} \Delta_i^n G|^p \xrightarrow{P} 0.$$

This completes the proof of Theorem 5. $\square$

8 Conclusion and outlook

In this paper we have derived convergence in probability and normal asymptotic limit results for multipower variations of processes Y that, up to a drift-like term, has the form

$$Y_t = \int_{-\infty}^t g(t-s) \sigma_s W(ds)$$

where the kernel g is deterministic, $\sigma > 0$ is an adapted càdlàg process and W is the stochastic Wiener measure. A key type of example has $g(t)$ behaving as t^δ for $t \downarrow 0$ and $\delta \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$. In those instances Y is not a semimartingale and the limit theory of multipower variation developed for semimartingales does not suffice to derive the desired kind of limit results. The basic tool we establish and apply for this is a normal central limit theorem for triangular arrays of dependent Gaussian variables. As a case of some special interest for applications, particularly in turbulence, the central limit behaviour of the realised variation ratio, i.e. the ratio of bipower variation to quadratic variation, is briefly discussed. Some specific examples of choice of g are also considered.

The turbulence context referred to concerns time-wise observations of velocities at a single location x in space. More generally it would be of interest to develop the theory of multipower variation corresponding to a setting where velocities are observed along a curve τ in space-time. More specifically, suppose that velocity $Y_t(x)$ at position x and time t is defined by

$$Y_t(x) = \int_{A+(x,t)} g(t-s, x-\xi) \sigma_s(\xi) W(d\xi ds)$$

where W denotes white noise, $\sigma_t(x)$ is a positive stationary random field on $\mathbb{R}^2$, g is a deterministic damping function and A is a subset of space-time involving only points with negative time coordinate. (For realism in turbulence modelling a drift term should be added to this expression for Y , but we ignore that here.) Then, with the curve τ parametrised as $\tau(w) = (x(w), t(w))$, say, the problem is to study multipower variations of the process X defined as

$$X_w = \int_{A+\tau(w)} g(t(w) - s, x(w) - \xi) \sigma_s(\xi) W(d\xi ds).$$

Among the questions that this raises is that of proper definition of filtrations. As to the latter, the concept of alignment, introduced in [BNSch09], is relevant. The definition is as follows.

Definition The curve τ and the ambit set A , with rectifiable and parametrised boundary $C = \{c(\gamma) : \gamma \in \Gamma\}$, are said to be aligned if the following conditions are satisfied. Let $c^\perp$ denote the transversal of $\dot{c}$, i.e. $c^\perp = (\dot{c}_2, -\dot{c}_1)$.

- (i) For all w there exists a partition of C into two sets C_w^+ and C_w^- such that $\dot{\tau}(w) \cdot c^\perp(\gamma) \geq 0$ for all γ with $c(\gamma) \in C_w^+$ while $\dot{\tau}(w) \cdot c^\perp(\gamma) \leq 0$ for all γ with $c(\gamma) \in C_w^-$.
- (ii) The subsets Γ_w^+ and Γ_w^- of Γ corresponding to C_w^+ and C_w^- are connected.
- (iii) For all w the curve lengths of C_w^+ and C_w^- are positive.

Note that the sets C_w^+ and C_w^- constitute the ‘front’ and the ‘rear’ of $A_{t(w)}(x(w))$ as $(x(w), t(w))$ moves along the curve τ .

In another direction it would be of interest to extend results of the present paper to power and multipower variations of higher order differences of Y . In particular, this might yield normal central limit theorems for the whole range of values of δ and it could also lead to more robustness against drift processes. For some recent work on quadratic variation of higher order differences, see [Beg07a] and [Beg07b] and references given there.

Appendix: Properties of the Bessel functions K

The Bessel functions K_ν are defined by

$$K_\nu(x) = \frac{1}{2} \int_0^\infty y^{\nu-1} e^{-\frac{1}{2}x(y^{-1}+y)} dy$$

where the index ν may take any real value. The known formulas for the K_ν recalled below are all given in [GrRy95].

A.1 Exact properties

Elementary exact properties are

$$K_\nu(x) = K_{-\nu}(x) \quad (\text{A.1.1})$$

$$K_{\nu+1}(x) = 2\nu x^{-1} K_\nu(x) + K_{\nu-1}(x) \quad (\text{A.1.2})$$

$$K'_\nu(x) = -K_{\nu-1}(x) - \nu x^{-1} K_\nu(x) \quad (\text{A.1.3})$$

Let

$$\bar{K}_\nu(x) = x^\nu K_\nu(x) \quad (\text{A.1.4})$$

and, for $\nu > 0$,

$$\check{K}_\nu(x) = 2^{-\nu+1} \Gamma(\nu)^{-1} \bar{K}_\nu(x). \quad (\text{A.1.5})$$

Here $\bar{K}_\nu$ may be reexpressed as

$$\bar{K}_\nu(x) = \frac{1}{2} \int_0^\infty y^{\nu-1} e^{-\frac{1}{2}y} e^{-\frac{1}{2}x^2 y^{-1}} dy, \quad (\text{A.1.6})$$

and, by a well known limit property of K_ν , we have (for $\nu > 0$)

$$\check{K}_\nu(x) \rightarrow 1 \quad \text{as } x \downarrow 0. \quad (\text{A.1.7})$$

Note further that expressed in terms of $\bar{K}$, the relations (A.1.1), (A.1.2) and (A.1.3) take the form

$$\begin{aligned} \bar{K}_\nu(x) &= x^{2\nu} \bar{K}_{-\nu}(x) \\ \bar{K}_{\nu+1}(x) &= 2\nu \bar{K}_\nu(x) + x^2 \bar{K}_{\nu-1}(x) \end{aligned} \quad (\text{A.1.8})$$

and

$$\bar{K}'_\nu(x) = -x \bar{K}_{\nu-1}(x). \quad (\text{A.1.9})$$

The latter implies

$$\begin{aligned} \bar{K}''_\nu(x) &= x^2 \bar{K}_{\nu-2}(x) - \bar{K}_{\nu-1}(x) \\ \bar{K}'''_\nu(x) &= x \bar{K}_{\nu-2}(x) + 2x \bar{K}_{\nu-2}(x) - x^3 \bar{K}_{\nu-3}(x) \\ &= -(x^3 \bar{K}_{\nu-3}(x) - 3x \bar{K}_{\nu-2}(x)). \end{aligned}$$

We shall also need the following formula (formula 3.383.8 in [GrRy95]). For $t, \psi, \nu > 0$,

$$\int_0^\infty (t+s)^{\nu-1} s^{\nu-1} e^{-2\psi s} ds = \sqrt{\frac{2}{\pi}} \Gamma(\nu) 2^{-\nu} \psi^{-\nu+\frac{1}{2}} t^{\nu-\frac{1}{2}} K_{\nu-\frac{1}{2}}(\psi t) e^{\psi t}. \quad (\text{A.1.10})$$

Finally, for $n = 0, 1, 2, \dots$ we have

$$K_{n+\frac{1}{2}}(x) = K_{\frac{1}{2}}(x) \left(1 + \sum_{i=1}^n \frac{(n+i)!}{i!(n-i)!} 2^{-i} x^{-i} \right) \quad (\text{A.1.11})$$

with

$$K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2}} x^{-\frac{1}{2}} e^{-x} \quad (\text{A.1.12})$$

while, when the index is a natural number,

$$\begin{aligned} K_n(x) &= \frac{1}{2} \sum_{k=0}^{n-1} (-1)^k \frac{(n-k-1)!}{k!} \left(\frac{x}{2}\right)^{2k-n} \\ &\quad + (-1)^{n+1} \sum_{k=0}^{\infty} \frac{1}{k!(n+k)!} \left[\log \frac{x}{2} - \frac{1}{2} \psi(k+1) - \frac{1}{2} \psi(n+k+1) \right] \left(\frac{x}{2}\right)^{2k+n} \end{aligned} \quad (8.1)$$

or, equivalently,

$$\begin{aligned} \bar{K}_n(x) &= \frac{1}{2} \sum_{k=0}^{n-1} (-1)^k \frac{(n-k-1)!}{k!} \left(\frac{x}{2}\right)^{2k} \\ &\quad + (-1)^{n+1} \sum_{k=0}^{\infty} \frac{1}{k!(n+k)!} \left[\log \frac{x}{2} - \frac{1}{2} \psi(k+1) - \frac{1}{2} \psi(n+k+1) \right] \left(\frac{x}{2}\right)^{2(k+n)} \end{aligned} \quad (8.2)$$

where ψ is the digamma function.

A.2 Asymptotic properties

The Bessel functions K_ν and I_ν are connected by

$$K_\nu(x) = \frac{1}{2} \frac{\pi}{\sin(\pi\nu)} (I_{-\nu}(x) - I_\nu(x)) \quad (\text{A.2.1})$$

and we have

$$I_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n}}{n! \Gamma(n+\nu+1)}.$$

Using these relations and the fact that

$$\Gamma(1-\nu) \Gamma(\nu) = \frac{\pi}{\sin(\pi\nu)} \quad (\text{A.2.2})$$

we find

$$\begin{aligned} \bar{K}_\nu(x) &= 2^{\nu-1} \frac{\pi}{\sin(\pi\nu)} \left\{ \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n}}{n! \Gamma(n-\nu+1)} - \left(\frac{x}{2}\right)^{2\nu} \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n}}{n! \Gamma(n+\nu+1)} \right\} \\ &= 2^{\nu-1} \frac{\pi}{\sin(\pi\nu)} \left\{ \frac{1}{\Gamma(1-\nu)} + \frac{1}{\Gamma(2-\nu)} \left(\frac{x}{2}\right)^2 + \dots \right. \\ &\quad \left. - \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^{2\nu} - \frac{1}{\Gamma(\nu+2)} \left(\frac{x}{2}\right)^{2(\nu+n)} - \dots \right\}. \end{aligned}$$

Alternatively we may write

$$\begin{aligned}\check{K}_\nu(x) &= 1 + \sum_{n=1}^{\infty} \frac{\Gamma(1-\nu) \left(\frac{x}{2}\right)^{2n}}{n! \Gamma(n-\nu+1)} - \left(\frac{x}{2}\right)^{2\nu} \sum_{n=0}^{\infty} \frac{\Gamma(1-\nu) \left(\frac{x}{2}\right)^{2n}}{n! \Gamma(n+\nu+1)} \\ &= 1 + \sum_{n=1}^{\infty} \frac{\Gamma(1-\nu) \left(\frac{x}{2}\right)^{2n}}{n! \Gamma(n-\nu+1)} - \left(\frac{x}{2}\right)^{2\nu} \sum_{n=0}^{\infty} \frac{\Gamma(1-\nu) \left(\frac{x}{2}\right)^{2n}}{n! \Gamma(n+\nu+1)}\end{aligned}\quad (8.3)$$

From these expansions, distinguishing between the cases $0 < \nu < 1$ and $\nu > 1$, we obtain:

If $0 < \nu < 1$ then

$$\check{K}_\nu(x) = 1 - 2^{-2\nu} \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)} x^{2\nu} + \frac{1}{4} \frac{1}{1-\nu} x^2 - O\left(x^{2(1+\nu)}\right). \quad (A.2.4)$$

When $\nu = 1$ we have, from (8.1),

$$\check{K}_1(x) = 1 - \frac{1}{2} |\log x| x^2 + O(x^2). \quad (A.2.5)$$

For $\nu > 1$

$$\check{K}_\nu(x) = 1 - \frac{1}{4} \frac{1}{\nu-1} x^2 - 2^{-2\nu} x^{2\nu} + O(x^4). \quad (A.2.6)$$

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