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Adding and Subtracting Black-Scholes: A New Approach to Approximating Derivative Prices in Continuous Time Models

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Abstract

This paper develops a new systematic approach to implement approximate solutions to asset pricing models within multi-factor diffusion environments. For any model lacking a closed-form solution, we provide a solution obtained by expanding the analytically intractable model around a known auxiliary pricing function. We derive power series expansions, which provide increasingly improved refinements to the initial mispricing arising from the use of the auxiliary model. In practice, the expansions can be truncated to include only a few terms to generate extremely accurate approximations. We illustrate our methodology in a variety of contexts, including option pricing with stochastic volatility, volatility contracts and the term-structure of interest rates.

Keywords: Asset pricing; stochastic volatility; the term-structure of interest rates, closed-form approximations.

JEL-Classification: G12, G13.

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1 Introduction

The last decade has witnessed an ever increasing demand for new models addressing a number of empirical puzzles in financial economics, which relate to pricing, hedging, and spanning derivatives contracts (e.g., Bakshi and Madan, 2000; Duffie, Pan and Singleton, 2000), the term structure of interest rates (e.g., Ahn, Dittmar and Gallant, 2002; Dai and Singleton, 2002), or the aggregate stock market (e.g., Gabaix, 2008; Menzly, Santos and Veronesi, 2004). The vast majority of these models rely on a continuous time framework, which is by now one of the most celebrated tools in our field. Market practitioners have also increasingly relied on continuous time models (e.g., Brigo and Mercurio, 2006). The reason for this almost unanimous consensus about the benefits of continuous time modeling is that within this framework, we are able to provide elegant representations for the price of a variety of contingent claims. At the same time, continuous time models call for one of the oldest issues in financial economics: how do we go about dealing with models not solved in closed-form?

As is well known, closed-form solutions for asset prices constitute the exception, rather than the norm. This fact has led financial economists and practitioners to single out classes of models for which a solution could indeed be found, the celebrated affine class being the benchmark (Duffie, Pan and Singleton, 2000; Heston, 1993). However, it is an open question as to whether these models clash with the empirical behavior of the state variables in the economy. Quite often, models with closed-form solutions rest on simplifying assumptions that are typically untested, for the sake of analytical tractability. This circumstance might be problematic, once we move towards a quantitative assessment of these models, since we do not know whether, say, the reason for a model's rejection would lie in its very economic rationale or, rather, the mere simplifying assumptions underlying it. The role of simpliflying assumptions has also recently called into question during the 2007 subprime crisis, which has clearly revealed how a small change in the assumptions underlying a model can then have dramatic effects on the ultimate pricing of derivative products (see IMF, 2008).

In dealing with models not solved in closed-form, we typically rely on two standard approaches. The first approach hinges upon the numerical solution to the partial differential equation (through, e.g., finite-difference or Fourier-inversion methods (Schwarz, 1978; Hull and White, 1990; Scott, 1997)). The second approach relies on Monte Carlo simulations, in which a large number of trajectories needs to be generated for the state variables underlying the asset pricing model (Boyle, 1977). Both methods can be cumbersome to implement and, computationally, quite time-consuming.

This paper develops a new conceptual framework to compute asset prices in nonlinear, multi-factor diffusion settings. We develop closed-form approximations to any given contingent claim model, which are easy to implement and require very little computer power. Our main idea is to choose an "auxiliary" pricing model for which a solution is available in closed-form. We derive an expression for the difference between the unknown price of the model of interest and the auxiliary one. This expression takes the form of a conditional moment which, under regularity conditions, can be cast in terms of a power series. We approximate the unknown price by retaining a finite

number of terms from this series. Our method is highly general and therefore applicable in a wide range of settings - for example, the pricing of stock options, computation of the associated Greeks, bond pricing and variance/volatility swaps. We develop several examples demonstrating the use of our general results, and provide numerical results to show that our method is quite precise and easily implemented.

Power series expansions of conditional moments of diffusion processes are widely used in financial econometrics (see, e.g., Aït-Sahalia, 2002; Aït-Sahalia and Yu, 2006; Kimmel, 2008; or Schaumburg, 2004). A key feature in this literature is to expand a conditional moment of a diffusion taken over a small time-span - say, for example, one day or one week at most. These small time expansions are not appropriate to approximate asset pricing models in which payoff functions (i) are not differentiable (as for example in the simple European option pricing case) and/or (ii) occur at long maturity dates (as for example in the term structure of interest rates). For these reasons, small time expansions have not been applied to asset pricing models previously.[1] Our approach still relies on series expansions of conditional moments, but works differently. Rather than being applied directly to payoff functions, our expansions apply to pricing errors that summarize the mispricing between the true pricing function and the auxiliary pricing function we choose to approximate the true model by. These pricing errors are typically differentiable even if the payoffs are not.

Our method can be seen as an expansion of the density implied by the model of interest, around the density of some auxiliary model chosen by the user. As such, our method is similar in spirit to the strand of literature in which option prices are computed through an approximation of the risk-neutral density underlying the true pricing model, as in Abadir and Rockinger (2003) or in the "saddlepoint approximations" considered by, e.g., Aït-Sahalia and Yu (2006), Rogers and Zane (1999) and Xiong et al. (2005). Indeed, we shall explain how to interpret this type of approximation as a special case of our method. Our method carries some advantages over saddlepoint approximations, when applied to asset pricing models. First, saddlepoint approximation rely on expansions of the risk-neutral density, while our method leads to a direct expansion of the asset price. As a practical consequence, our method avoids the numerical computation of Riemann integrals against an approximate density, which is instead a necessary step in the practical implementation of saddlepoint approximations. This feature of our method is particularly attractive when one is concerned with multi-factor models that involve stochastic interest rates, stochastic volatility or macro-finance determinants of the yield curve. Second, our expansion has an appealing and economically meaningful interpretation, since it relies on expanding the model of interest around another asset pricing model. Third, we provide an explicit expression for the difference between the pricing function of the true and the auxiliary model, which leads to a more direct analysis of the pricing error and simpler approximations.

The paper is organized as follows. In the next section, we illustrate our methods through three

[1] One isolated exception appears in Chapman, Long and Pearson (1999), which is indeed a special case of our method, as we shall show. However, this special case does not allow one to deal with derivatives written on non-differentiable payoffs. Another example is Kimmel (2008) who has proposed a method of time-transformation to improve on the long-horizon performance of the power series approximation.

examples. In Section 3, we present the general approximation scheme, provide extensions that allow for the computation of derivatives of pricing functions and, finally, relate our approach to those relying on the expansion of the model implied risk-neutral density. In Section 4, we assess the numerical performance of the method. Section 5 concludes, and an appendix provides technical details.

2 The Approximation Method in Three Examples

We illustrate the basic ideas underlying our method by working out three examples, ranked in order of increasing complexity: (i) the pricing of variance contracts, (ii) the pricing of European options within the generalized Black-Scholes model, and (iii) the pricing of bonds in single-factor models of the short-term rate.

2.1 Log-contracts and Variance Swaps

Our basic example pertains to the recent financial innovation related to variance swaps, which are contracts guaranteeing a payoff linked to the realization of the future variance of some asset price. As is well-known, the forward price of any liquid asset, $F(t)$ say, is a martingale under the risk-neutral probability. Moreover, suppose that $F(t)$ exhibits stochastic volatility, as follows:

$$\frac{dF(t)}{F(t)} = \sigma(t)\, dW(t),$$

where $W(t)$ is a Brownian motion under the risk-neutral probability, and the instantaneous variance follows a continuous time ARCH process (Nelson, 1990),

$$d\sigma^2(t) = \kappa\left(\alpha - \sigma^2(t)\right) dt + \xi\sigma^2(t)\, dW_\sigma(t), \tag{1}$$

for some positive constants κ, α and ξ, and a Brownian motion $W_\sigma(t)$ defined under the risk-neutral probability.

By entering into a variance swap at time t, the holder of the contract will receive, at some time T, a payoff proportional to, $\int_t^T \sigma^2(u)\, du - \sigma^2_{\text{strike}}$, for some constant σ^2_{strike}. Typically, the variance strike, σ^2_{strike}, is set so as to make the contract worthless at the time of origination, t, consistently with the market practice related the more familiar interest rate swaps. Then, if the short-term rate is independent of the forward price variance, it must be that in the absence of arbitrage opportunities, the variance strike equals the expected future integrated variance, viz

$$\sigma^2_{\text{strike}} = \int_t^T \mathbb{E}_{x,t}[\sigma^2(u)]du = -2\mathbb{E}_{x,t}\left(\log\frac{F(T)}{F(t)}\right),$$

where $\mathbb{E}_{x,t}[\sigma^2(u)] = \mathbb{E}[\sigma^2(u)\,|\sigma^2(t) = x]$ denotes the conditional mean, and the last equality follows by a simple application of Itô's lemma. The last term is the payoff of the so-called log-contract introduced by Neuberger (1994) which, as shown in many papers (Demeterfi et al., 1999; Bakshi

and Madan, 2000; Britten-Jones and Neuberger, 2000; Carr and Madan, 2001), is equal to a certain portfolio of options, as being used by the CBOE to compute the new VIX index. Alternatively, we may use the parametric model in Eq. (1) to find σ^2_{strike} and, hence, price the contract. In this case, one can calibrate the parameters κ and α to make σ^2_{strike} consistent with the VIX index, and proceed to use the model in Eq. (1) to price other derivative assets. Needless to say, to perform these tasks, it is crucial to compute the expectation of the future variance, $\mathbb{E}_{x,t}[\sigma^2(u)]$.

In the context of the model in Eq. (1), it is well-known that $\mathbb{E}_{x,t}[\sigma^2(u)]$ has a closed form expression, which leads to a closed-form solution for the variance strike as of any time τ, $w(x,t) = \int_t^T \mathbb{E}_{x,t}[\sigma^2(u)]du$. For the sake of this introductory example, suppose that we ignore this solution, and that we wish to approximate the variance strike for the model in Eq. (1) with another variance strike that we can compute. Consider, for instance, the following variant of the Hull and White (1987) model, in which the instantaneous variance is a martingale,

$$d\sigma^2(t) = \xi\sigma^2(t)\, dW_\sigma(t).$$

For this model, $\mathbb{E}^0_{x,t}\left[\sigma^2(u)\right] = x$, where $\mathbb{E}^0_{x,t}$ denotes the conditional expectation under the Hull and White (1987) model, and the variance strike is just $w_0(x,t) = x(T-t)$.

We now illustrate how to use the "auxiliary" Hull and White model, $w_0(x,t) = x(T-t)$, to approximate the supposedly unknown model, $w(x,t)$. We proceed as follows. First, we note that $w(x,t)$ solves the following partial differential equation,

$$0 = Lw(x,t) + x, \tag{2}$$

with termination value $w(x,T) = 0$, where L denotes the infinitesimal operator of $\sigma^2(t)$:

$$Lw(x,t) = \frac{\partial w(x,t)}{\partial t} + \kappa(\alpha - x)\frac{\partial w(x,t)}{\partial x} + \xi x\frac{\partial^2 w(x,t)}{\partial x^2}.$$

Likewise, $w_0(x,t)$, the variance strike predicted by the auxiliary Hull and White model, satisfies,

$$0 = \frac{\partial w_0(x,t)}{\partial t} + x, \tag{3}$$

with boundary condition $w_0(x,T) = 0$.

Our key idea, now, is to subtract Eq. (3) from Eq. (2). A simple computation, then, shows that the mispricing arising from the use of the wrong model, $\Delta w(x,t) \equiv w(x,t) - w_0(x,t)$, satisfies,

$$0 = L\Delta w(x,t) + (T-t)\,\delta(x),$$

with termination value $\Delta w(x,T) = 0$, and "mispricing function" δ given by: $\delta(x) = \kappa(\alpha - x)$.

The Feynman-Kac solution to the previous equation leads to the following representation of the variance mispricing,

$$\Delta w(x,t) = \int_t^T (T-s)\,\mathbb{E}_{x,t}\left[\delta(x(s))\right] ds, \tag{4}$$

The conditional expectation inside the integral can be written explicitly, in terms of the infinitesimal generator operator associated with the model of interest (1), L, as follows,

$$\mathbb{E}_{x,t}\left[\delta\left(x\left(s\right)\right)\right]=\sum_{n=0}^{\infty}\frac{\left(s-t\right)^{n}}{n!}L^{n}\delta\left(x\right), \tag{5}$$

where, by a direct computation, $L^{n}\delta\left(x\right)=\kappa\left(-\kappa\right)^{n}\left(\alpha-x\right)$.

Our approximation to the variance mispricing, $\Delta w\left(x,t\right)$, is obtained by replacing the expectation $\mathbb{E}_{x,t}\left[\delta\left(x\left(s\right)\right)\right]$ in Eq. (4) with only the first N terms of the series expansion in Eq. (5), as follows:

$$\Delta w_{N}\left(x,t\right)=\int_{t}^{T}\left(T-s\right)\sum_{n=0}^{N}\frac{\left(s-t\right)^{n}}{n!}L^{n}\delta\left(x\right)ds. \tag{6}$$

Accordingly, our approximation to the strike price $w\left(x,t\right)$ is:

$$w_{N}\left(x,t\right)=w_{0}\left(x,t\right)+\Delta w_{N}\left(x,t\right)=x\left(T-t\right)+\kappa\left(\alpha-x\right)\sum_{n=0}^{N}\frac{\left(T-t\right)^{n+2}}{\left(n+2\right)!}\left(-\kappa\right)^{n},$$

where the second equality follows by the evaluation of the integral in Eq. (6). It is easily checked that as N increases, $w_{N}\left(x,t\right)$ approaches the true variance strike, $w\left(x,t\right)$, for all x and t.

2.2 The Generalized Black-Scholes Option Pricing Model

In this example, we illustrate how our method can be exploited to approximate stock option prices. Suppose that the price of the stock, $S\left(t\right)$ say, is the solution to

$$\frac{dS\left(t\right)}{S\left(t\right)}=rdt+\sigma\left(S\left(t\right),t\right)dW\left(t\right), \tag{7}$$

where $W\left(t\right)$ is a standard Brownian motion under the risk-neutral probability, and r is the short-term rate, a constant. A European call option pays $b(S\left(T\right))\equiv\max\left\{S\left(T\right)-K,0\right\}$ at maturity time $T>0$, where $K>0$ is the strike price. We are interested in computing the price of the option at time $t<T$ given the current stock price $S\left(t\right)=x$, denoted as $w\left(x,t\right)$.

Let L be the infinitesimal generator associated with Eq. (7),

$$Lw\left(x,t\right)=\frac{\partial w\left(x,t\right)}{\partial t}+rx\frac{\partial w\left(x,t\right)}{\partial x}+\frac{1}{2}\sigma^{2}\left(x,t\right)x^{2}\frac{\partial^{2}w\left(x,t\right)}{\partial x^{2}}. \tag{8}$$

Under standard regularity conditions on the volatility function $\sigma\left(x,t\right)$, $w\left(x,t\right)$ satisfies the following partial differential equation,

$$0=Lw\left(x,t\right)-rw\left(x,t\right), \tag{9}$$

with boundary condition $w\left(x,T\right)=b\left(x\right)$, for all x. The solution to Eq. (9), provided it exists, can also be represented through the well-known Feynman-Kac representation, $w\left(x,t\right)=e^{-r(T-t)}\mathbb{E}_{x,t}\left[b\left(S\left(T\right)\right)\right]$. In general, no closed-form solutions for this expectation are available. Ac-

cordingly, we need to rely upon numerical methods: typically, we may either rely on solving Eq. (9) through finite-difference methods (e.g., Ames, 1977) or approximating the conditional expectation, $\mathbb{E}_{x,t}[.]$, through simulations.

The starting point of our method is, as in the previous example, the choice of an auxiliary model that can be solved in closed-form. In this context, the Black and Scholes (1973) (BS, henceforth) model is a natural candidate. For this model, the volatility in Eq. (7) is a constant, i.e. $\sigma(x,t) \equiv \sigma_0$, for all x, t. Accordingly, the BS option price, $w^{\text{bs}}(x,t;\sigma_0)$, is solution to,

$$0 = L_0 w^{\text{bs}}(x,t;\sigma_0) - r w^{\text{bs}}(x,t;\sigma_0), \tag{10}$$

where $w^{\text{bs}}(x,T;\sigma_0) = \max\{x-K,0\}$, and the associated infinitesimal operator, L_0, is the same as in Eq. (8), but with σ_0 replacing $\sigma(x,t)$.

Proceeding as we did in Section 2.1, we now subtract Eq. (10) from Eq. (9). The result is that the price difference, $\Delta w(x,t;\sigma_0) \equiv w(x,t) - w^{\text{bs}}(x,t;\sigma_0)$, satisfies,

$$0 = L\Delta w(x,t;\sigma_0) - r\Delta w(x,t;\sigma_0) + \delta(x,t;\sigma_0), \tag{11}$$

with boundary condition $\Delta w(x,t;\sigma_0) = 0$ for all x, where our mispricing function, δ, is:

$$\delta(x,t;\sigma_0) \equiv \frac{1}{2}\left(\sigma(x,t)^2 - \sigma_0^2\right) x^2 \frac{\partial^2}{\partial x^2} w^{\text{bs}}(x,t;\sigma_0). \tag{12}$$

Since $w^{\text{bs}}(x,t;\sigma_0)$ is known, we can compute $\delta(x,t;\sigma_0)$. By relying on the Feynman-Kac theorem, the solution to Eq. (11) can be rearranged to yield the pricing function of interest w, as the sum of the Black-Scholes price plus a conditional moment,

$$w(x,t) = w^{\text{bs}}(x,t;\sigma_0) + \mathbb{E}_{x,t}\left[\int_t^T e^{-r(u-t)}\delta(S(u),u;\sigma_0)\,du\right]. \tag{13}$$

The interpretation of the mispricing function δ in Eq. (12) is related to the hedging cost arising while evaluating and hedging the option through the BS formula. Precisely, suppose a trader sells the option and wishes to hedge against it through a self-financing strategy, in which he trades the underlying stock using the BS delta, $\partial w^{\text{bs}}(x,t;\sigma_0)/\partial x$. Then, as shown by El Karoui, Jeanblanc-Picqué and Shreve (1998), and further elaborated by Corielli (2006), our function δ in Eq. (12) is interpreted as the instantaneous increment in the total hedging cost arising from the use of a wrong model (the BS model) to hedge against the true model in Eq. (7).

The conditional moment in Eq. (13) is taken under the true stock price dynamics given by Eq. (7). Therefore, it is in general impossible to obtain a closed form expression for the second term in Eq. (13). To make this formula operational, we make use of a series expansion of the conditional moment in Eq. (13) in terms of the corresponding infinitesimal generator. As shown

in the Appendix (see Proposition A.3), Eq. (13) is indeed equivalent to:

$$w(x,t) = w^{\text{bs}}(x,t;\sigma_0) + \sum_{n=0}^{\infty} \frac{(T-t)^{n+1}}{(n+1)!} \bar{L}^n \delta(x,t;\sigma_0), \tag{14}$$

where $\bar{L}\phi = L\phi - r\phi$. In practice, this formula needs to be truncated at some point N (say), yielding an N-th order approximation w_N,

$$w_N(x,t;\sigma_0) \equiv w^{\text{bs}}(x,t;\sigma_0) + \sum_{n=0}^{N} \frac{(T-t)^{n+1}}{(n+1)!} \bar{L}^n \delta(x,t;\sigma_0).$$

As an example, a first order approximation ($N = 0$) is given by $w_0(x,t;\sigma_0) \equiv w^{\text{bs}}(x,t;\sigma_0) + (T-t)\,\delta(x,t;\sigma_0)$. Naturally, w in Eq. (14) does not depend on σ_0, although its "truncation" w_N does. In Section 4.1, we discuss choices of the nuisance parameter, σ_0. In our numerical experiments reported in Section 4.1, we find that the numerical accuracy of $w_N(x,t;\sigma_0)$ does not crucially depend on the choice for σ_0.

2.3 Bond Pricing in a Single-Factor Model

For our third example, we consider the pricing of bonds in a single-factor model of the short-term interest rate. Specifically, suppose that the short-term rate r is the solution to

$$dr(t) = \mu(r(t),t)dt + \sigma(r(t),t)dW(t), \tag{15}$$

for some functions μ and σ, and a standard Brownian motion $W(t)$ defined under the risk-neutral probability. Let $w(x,t)$ be the price as of time t of a default-free bond maturing at time $T > t$, when $r(t) = x$. Under standard regularity conditions on μ and σ, $w(x,t)$ is solution to,

$$0 = Lw(x,t) - xw(x,t), \quad Lw = \frac{\partial w}{\partial t} + \mu\frac{\partial w}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 w}{\partial x^2}, \tag{16}$$

with boundary condition $w(x,T) = 1$.

Next, let us introduce, as usual, an auxiliary model,

$$dr(t) = \mu_0(r(t),t)dt + \sigma_0(r(t),t)dW(t).$$

Associated with this model is a bond pricing function, $w_0(x,t)$, which solves the partial differential equation (16), with boundary condition $w_0(x,T) = 1$, but with μ_0 and σ_0 replacing μ and σ.

It is easy to show that the price difference, $\Delta w(x,t) = w(x,t) - w_0(x,t)$, satisfies:

$$0 = L\Delta w(x,t) - x\Delta w(x,t) + \delta(x,t), \tag{17}$$

where $\Delta w(x,T)=0$, and the mispricing function is,

$$\delta(x,t)=(\mu(x,t)-\mu_0(x,t))\frac{\partial w_0(x,t)}{\partial x}+\frac{1}{2}\left(\sigma^2(x,t)-\sigma_0^2(x,t)\right)\frac{\partial^2 w_0(x,t)}{\partial x^2}. \tag{18}$$

Note, the function summarizing the mispricing arising from the use of the auxiliary model, δ, has now a more complex structure than that we find in Section 2.2 in the option pricing case. Its second component, the convexity adjustment, is now familiar, by the results in Section 2.2. Its first term, which is new, arises because the short-term rate is obviously not a traded asset, which makes the two drifts under the risk-neutral probability, μ and μ_0, differ. In the option pricing case dealt with in Section 2.2, instead, the asset underlying the contract was a tradable stock, the price of which has risk-neutral drift equal to rS, independently of the evaluation model. A choice that simplifies the function δ in Eq. (18) is $\mu_0=\mu$, which will be our choice in our numerical work of Section 4.2.

By the Feynman-Kac representation, the solution to Eq. (17) is,

$$\Delta w(x,t)=\int_t^T \mathbb{E}_{x,t}\left[\exp\left(-\int_t^u r(s)ds\right)\delta(r(u),u)\right]du.$$

Using the same type of power series expansions as in Sections 2.1 and 2.2, we obtain the following approximating formula for the bond price function $w(x,t)$:

$$w_N(x,t)=w_0(x,t)+\sum_{n=0}^{N}\frac{(T-t)^{n+1}}{(n+1)!}\bar{L}^n\delta(x,t), \tag{19}$$

where $\bar{L}\delta(x,t)=L\delta(x,t)-x\delta(x,t)$.

3 A General Approximating Pricing Formula

In this section, we derive a general formula for approximating asset prices without a known closed-form solution. In Section 3.1, we introduce notation for the model we wish to approximate and its auxiliary counterpart, and provide our approximating formula. Section 3.2 discusses approximations for derivatives of the pricing functions of interest, which can be useful for hedging purposes. Section 3.3 explains how our approach relates to methods to expand risk-neutral densities, such as saddlepoint approximations.

3.1 The model and its approximation

We consider a multi-factor model in which a d-dimensional vector of state variables $x(t)$ affect all derivative prices in the economy. We assume that under the risk-neutral probability, $x(t)$ satisfies:

$$dx(t)=\mu(x(t),t)\,dt+\sigma(x(t),t)\,dW(t), \tag{20}$$

where $W(t)$ is a d-dimensional standard Brownian motion under the risk-neutral probability. Let $w(x,t)$ be the price of a derivative written on the realization of $x(T)$, for some $T > t$, when the current state is $x(t) = x$. We assume that the payoff function is $w(x,T) = b(x)$. Define the infinitesimal generator operator L associated to Eq. (20),

$$Lw(x,t) = \frac{\partial w(x,t)}{\partial t} + \sum_{i=1}^{d} \mu_i(x,t) \frac{\partial w(x,t)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} \sigma_{ij}^2(x,t) \frac{\partial^2 w(x,t)}{\partial x_i \partial x_j}. \tag{21}$$

Our purpose is to find an approximation to the price $w(x,t)$, solution to the following partial differential equation:

$$a(x,t) w(x,t) = Lw(x,t) + c(x,t), \tag{22}$$

with boundary condition $w(x,T) = b(x)$.

Let us introduce an auxiliary model,

$$dx_0(t) = \mu_0(x_0(t),t)\, dt + \sigma_0(x_0(t),t)\, dW(t), \tag{23}$$

which we wish to expand the d-factor model around. We assume that the auxiliary model has the same dimension as the actual model of interest, that is, $\dim(x(t)) = \dim(x_0(t))$. This assumption does not result in any loss of generality, since we can always add constant components. For example, suppose that a modeler wishes to use as an auxiliary model a lower-dimensional model where the state vector $y(t) \in \mathbb{R}^m$ with $m < d$ solves

$$dy(t) = \mu_Y(y(t),t)\, dt + \sigma_Y(y(t),t)\, dW_1(t),$$

and $W_1(t)$ is a m-dimensional standard Brownian motion. The vector process $x = \left(y^\top, x_{m+1}, ..., x_d\right)^\top$, where the last $d-m$ components remain constant over time, is then a solution to Eq. (23) with

$$\mu_{0,i}(x,t) = \begin{cases} \mu_{Y,i}(y,t), & 1 \le i \le m \\ 0, & \text{otherwise} \end{cases} \qquad \sigma_{0,ij}(x,t) = \begin{cases} \sigma_{Y,ij}(y,t), & 1 \le i,j \le m \\ 0, & \text{otherwise} \end{cases}.$$

In Section 4.3, we use this modeling trick to approximate option prices in stochastic volatility models using the Black-Scholes auxiliary model.

Finally, and crucially, we assume that we have a closed form solution $w_0(x,t)$ for the pricing function in the economy with state vector satisfying Eq. (23). We also assume that the boundary condition $w_0(x,T) = b_0(x)$, thus allowing for the two payoffs functions, $b(x)$ and $b_0(x)$, to differ.

The price difference, $\Delta w(x,t) = w(x,t) - w_0(x,t)$, satisfies,

$$a(x,t) \Delta w(x,t) = L\Delta w(x,t) + \delta(x,t), \tag{24}$$

with boundary condition $\Delta w(x,T) = d(x)$. The two adjustment terms are given by

$$d(x) = b(x) - b_0(x), \tag{25}$$

and

$$\delta(x,t) = \sum_{i=1}^{d} \Delta\mu_i(x,t) \frac{\partial w_0(x,t)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} \Delta\sigma_{ij}^2(x,t) \frac{\partial^2 w_0(x,t)}{\partial x_i \partial x_j}, \tag{26}$$

where

$$\Delta\mu_i(x,t) = \mu_i(x,t) - \mu_{0,i}(x,t), \quad \Delta\sigma_{ij}^2(x,t) = \sigma_{ij}^2(x,t) - \sigma_{0,ij}^2(x,t).$$

The Feynman-Kac representation theorem applied to $\Delta w(x,t)$ now yields:

Theorem 1 (Representation Formula) *Assume that the two solutions, w and Δw to Eq. (22) and (24) exist. Then the following identity holds:*

$$\begin{aligned} w(x,t) - w_0(x,t) = {} & \mathbb{E}_{x,t}\left[\exp\left(-\int_t^T a(x(s),s)\,ds\right) d(x(T))\right] \\ & + \int_t^T \mathbb{E}_{x,t}\left[\delta(x(s),s)\exp\left(-\int_t^s a(x(u),u)\,du\right)\right] ds, \end{aligned} \tag{27}$$

where $x(t)$ satisfies Eq. (20), and d, δ are given in Eq. (25)-(26).

The above representation formula holds under very weak assumptions. The right hand side gives us an exact expression for the error involved when using the auxiliary model to price the claim, instead of the true model. This representation might turn to be useful, as it shows precisely how the pricing error is related to the auxiliary model.

Yet our main goal is to look for an approximation of the error term in order to adjust the price $w_0(x,t)$ for the error involved. Accordingly, our next step is to approximate the two expectations on the right side of Eq. (27) using a series expansion. For the series expansion to hold, we have to impose stronger assumptions. For an N-th order approximation to be valid, we need to assume that $d(x,t)$ and $\delta(x,t)$ are $2N$ times differentiable with respect to x and N times differentiable with respect to t. Under this assumption, we may then make the following definition:

Definition 1 (Approximation Formula) *The N-th order approximation $w_N(x,t)$ is given by:*

$$w_N(x,t) = w_0(x,t) + \sum_{n=0}^{N} \frac{(T-t)^n}{n!} d_n(x,t) + \sum_{n=0}^{N} \frac{(T-t)^{n+1}}{(n+1)!} \delta_n(x,t), \tag{28}$$

where $d_0(x,t) = d(x)$, $\delta_0(x,t) = \delta(x,t)$ and

$$d_n(x,t) = L d_{n-1}(x,t) - a(x,t)\, d_{n-1}(x,t); \quad \delta_n(x,t) = L\delta_{n-1}(x,t) - a(x,t)\,\delta_{n-1}(x,t).$$

Appendix A provides additional regularity conditions under which our approximation formula is valid, in that $w_N(x,t) \to w(x,t)$ as $N \to \infty$. It also provides error bounds for any given $N \geq 1$.

Note that the above expression is only one way to approximate the right hand side of Eq. (27) in Theorem 1. Other methods might be available. For example, one could approximate the two

conditional moments on the right hand side of Eq. (27) through simulations. However, one might instead then just use simulations to directly compute the conditional expectation appearing in the Feynman-Kac representation of $w(x,t)$.

3.2 Approximating Greeks

We outline how our approximation method can be used to obtain closed form approximations for derivatives of the pricing function, $w(x,t)$. If the asset we are concerned with is a European-type option, our results can then be applied to estimate Greeks.

The approximation for derivatives are readily obtained, by differentiating the approximating formula in Eq. (28) of Definition 1 with respect to the variables of interest.

The approximation of the k-th order derivative of $w(x,t)$ is given by,

$$\frac{\partial^k w_N(x,t)}{\partial x^k} = \frac{\partial^k w_0(x,t)}{\partial x^k} + \sum_{n=0}^{N} \frac{(T-t)^n}{n!} d_n^{(k)}(x,t) + \sum_{n=0}^{N} \frac{(T-t)^{n+1}}{(n+1)!} \delta_n^{(k)}(x,t),$$

where

$$d_n^{(k)}(x,t) = \frac{\partial^k d_n(x,t)}{\partial x^k}, \quad \delta_n^{(k)}(x,t) = \frac{\partial_n^k \delta_n(x,t)}{\partial x^k}. \tag{29}$$

The two sequences, $d_n^{(k)}(x,t)$ and $\delta_n^{(k)}(x,t)$, can be evaluated either numerically (using, say, finite-difference methods) or analytically. For example, to compute the approximation to the first-order derivatives, $k=1$, we run the following recursion: $d_0^{(1)}(x,t) = \partial d(x)/\partial x$, $\delta_0^{(1)}(x,t) = \partial \delta(x,t)/\partial x$ and,

$$d_n^{(1)}(x,t) = L d_{n-1}^{(1)}(x,t) - a(x,t) d_{n-1}^{(1)}(x,t) + L^{(1)} d_{n-1}(x,t) - \frac{\partial a(x,t)}{\partial x} d_{n-1}(x,t),$$

$$\delta_n^{(1)}(x,t) = L \delta_{n-1}^{(1)}(x,t) - a(x,t) \delta_{n-1}^{(1)}(x,t) + L^{(1)} \delta_{n-1}(x,t) - \frac{\partial a(x,t)}{\partial x} \delta_{n-1}(x,t),$$

where

$$L^{(1)}\phi(x,t) = \sum_{i=1}^{d} \frac{\partial \mu_i(x,t)}{\partial x} \frac{\partial \phi(x,t)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial \sigma_{ij}^2(x,t)}{\partial x} \frac{\partial^2 \phi(x,t)}{\partial x_i \partial x_j}.$$

The Appendix provides full details on the recursive scheme needed to compute the terms $d_n^{(2)}(x,t)$ and $\delta_n^{(2)}(x,t)$ related to the approximation for the second-order derivative.

3.3 Relation to Saddlepoint Approximations

The goal of this section is two-fold. First, we demonstrate that the expansion of the price $w(x,t)$ in Theorem 1 in terms of the auxiliary function $w_0(x,t)$ is based on an expansion of the underlying density of the true model around the one of the auxiliary model. Second, we show that saddlepoint approximations can be seen as a special case of our expansion method.

To establish the link between the expansion in Theorem 1 and an equivalent expansion in terms of the underlying density, we restrict ourselves to the case where $a(x,t) = c(x,t) \equiv 0$. In this case,

we can represent the two prices, $w(x,t)$ and $w_0(x,t)$, as mere conditional moments, as follows:

$$w(x,t) = \int_{\mathbb{R}^d} b(y)\, p(y,T|x,t)\, dy, \tag{30}$$

$$w_0(x,t) = \int_{\mathbb{R}^d} b(y)\, p_0(y,T|x,t)\, dy, \tag{31}$$

where p and p_0 are the transition densities underlying the two models. Clearly, we have

$$w(x,t) = w_0(x,t) + \int_{\mathbb{R}^d} b(y)\, \Delta p(y,T|x,t)\, dy \tag{32}$$

where $\Delta p \equiv p - p_0$ is the difference between the two transition densities. Then, it is straight forward to see that Theorem 1 implies the following identity:

$$\int_{\mathbb{R}^d} b(y)\, \Delta p(y,T|x,t)\, dy = \int_t^T \mathbb{E}_{x,t}\left[\delta\left(x(s),s\right)\right] ds, \tag{33}$$

where δ is given in Eq. (26). Thus, it should be intuitively clear that our expansion in Definition 1 of $\int_t^T \mathbb{E}_{x,t}\left[\delta\left(x(s),s\right)\right] ds$ is closely related to a corresponding expansion of Δp. In fact, in Appendix B, we derive an explicit representation of $\Delta p(y,T|x,t)$ as a conditional moment, which highlights that the representation of w in Theorem 1 and its approximation in Definition 1 are implicitly based on equivalent representations and approximations of the transition density. However, our methods offer greater flexibility than those based on the approximation of the risk-neutral density, as we argue below.

Next, we give an interpretation of the saddlepoint approximation method as a particular choice of the auxiliary model, and an alternative way to approximate the pricing error. For simplicity, we maintain $a(x,t) = c(x,t) \equiv 0$. As a starting point, we take the first-order saddle point approximation of the transition density $p(y,T|x,t)$. As demonstrated in Aït-Sahalia and Yu (2006, Theorem 1), this takes the form

$$p_0^{(1)}(y,T|x,t;\theta_0) = \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \exp\left[\frac{\left(y-x-\mu_0(T-t)\right)^2}{2\sigma_0^2(T-t)}\right], \tag{34}$$

in the univariate case, $d=1$, where the parameter $\theta_0 = \left(\mu_0, \sigma_0^2\right)$ is chosen as $\theta_0(x) = \left(\mu(x), \sigma^2(x)\right)$. Within our framework, this choice corresponds to selecting an arithmetic Brownian motion as an auxiliary model,

$$dx_0(t) = \mu_0 dt + \sigma_0 dW(t),$$

where the two nuisance parameters $\theta_0 = \left(\mu_0, \sigma_0^2\right)$ are chosen to match the initial values of the drift and diffusion of the true model. In other words, the auxiliary transition density is given by $p_0(y,T|x,t) = p_0^{(1)}(y,T|x,t;\theta_0(x))$. Utilizing Theorem 1, we can give give an explicit expression for the error involved while using a first-order saddlepoint approximation instead of the true density

to evaluate the moment of interest:

$$w(x,t) = w_0(x,t) + \int_t^T \mathbb{E}_{x,t}\left[\delta\left(x(s),s\right)\right] ds, \tag{35}$$

where δ is given in Eq. (26) with $\mu_0(x) = \mu_0$ and $\sigma_0^2(x) = \sigma_0^2$. Thus, higher-order saddlepoint approximations can be interpreted as those taking into account the second term given in Eq. (35). For example, the second-order approximation of p takes the form

$$p_0^{(2)}(y,T|x,t) = p_0^{(1)}(y,T|x,t)\,\frac{1 + c_1(y,T|x,t)(T-t)}{1 + d_{1/2}(y,T|x,t)\sqrt{T-t} + d_1(y,T|x,t)(T-t)},$$

where the expressions for c_1, $d_{1/2}$ and d_1 can be found in Aït-Sahalia and Yu (2006, Theorem 2). Within our framework, this second-order approximation can be interpreted as an adjustment for the presence of the second term in Eq. (35). Namely, the second-order approximation implies the following approximation to the difference between the true and the auxiliary transition density,

$$\Delta\tilde{p}^{(2)}(y,T|x,t) \equiv p_0^{(1)}(y,T|x,t)\left[\frac{1 + c_1(y,T|x,t)(T-t)}{1 + d_{1/2}(y,T|x,t)\sqrt{T-t} + d_1(y,T|x,t)(T-t)} - 1\right].$$

From this expression, we obtain the following second-order approximation for $w(x,t)$,

$$\tilde{w}_2(x,t) = w_0(x,t) + \int_{\mathbb{R}} b(y)\,\Delta\tilde{p}^{(2)}(y,T|x,t)\,dy.$$

The advantage of the approximation given in Definition 1 over the previous one, $\tilde{w}_2$, is that we have closed-form approximations for the adjustment term. In constrast, the approximating formulae above involve the computation of an integral, which might turn out to be a non trivial task, especially when the dimension of the state vector, x, is high. Moreover, the use of saddlepoint approximations might become a bit involved, in some more complicated situations. For example, it is not clear to us how we would use saddlepoint approximations of the risk-neutral density of $x(t)$ to deal with the following conditional expectation, $\mathbb{E}_{x,t}[\exp(-\int_t^T a(x(s),s)\,ds)b(x(T))]$. This case arises, for example, when pricing options with stochastic interest rates or even in the most basic fixed income pricing case. In comparison, our method has no problems with handling this situation.

The above discussion was based on saddlepoint approximations using a Gaussian basis. If the true model is far from the Gaussian one, lower-order approximations will normally work poorly. This leads Aït-Sahalia and Yu (2006, Section 3.3) to propose using a non-Gaussian basis for the saddlepoint approximation. Within our framework, this corresponds to choosing a non-Gaussian auxiliary model. Again, we can give an explicit representation of the error involved in using this model to evaluate moments/prices, and have closed-form approximation for this.

4 Numerical Accuracy of Approximation

We numerically assess the performance of our approximation method in three pricing applications: (i) option pricing when the asset return volatility depends on the level of the asset price; (ii) the term-structure of interest rates; (iii) option pricing when the asset return volatility is stochastic.

4.1 Option Pricing with CEV Volatility

Consider the generalized BS model in Section 2.2. There are several alternatives to deal with the nuisance parameter, σ_0. For example, let $\hat{\sigma}_0$ be some estimate of σ_0. Then, we may approximate $w(x,t)$ with $w_N(x,t;\hat{\sigma}_0)$. Alternatively, we may consider, $\hat{\sigma}_N(x,t) = \arg\min_\sigma \left(w_N(x,t;\sigma) - w_0(x,t;\sigma)\right)^2$. As a simple example, we have that for $N=0$, $\hat{\sigma}_0(x,t) = \sigma(x,t)$, for all x and t. Under regularity conditions, $\lim_{N\to\infty} \hat{\sigma}_N(x,t) = \mathrm{IV}(x,t)$, all x,t, where $\mathrm{IV}(x,t)$ denotes the Black-Scholes implied volatility, defined by $w(x,t) = w^{\mathrm{bs}}(x,t;\mathrm{IV}(x,t))$. The unknown option price, then, can be approximated by $w_N(x,t;\sigma_N(x,t))$.

To examine the numerical performance of the approximation, we choose a CEV-model as the true model, in which case $\sigma(x,t) = \sigma_{\mathrm{cev}} x^{\gamma-1}$, where σ_{cev} is constant and $\gamma > 0$. For this model, the option price is known in closed-form (see Schroder, 1989). Figure 1 depicts the approximation errors made with our method, for different levels of the current stock price. Our approximating price is obtained as $w_N(x,t;\hat{\sigma}_0(x,t))$, which explains why the percentage errors for $N=0$ and $N=1$ coincide. More fundamentally, the approximation errors are several orders of magnitudes lower than one percentage point with only a very small number of terms ($N=3$). Figure 2 shows the approximation error for a longer time to maturity. As is clear from this figure, the approximation is still quite accurate even when we widen the spectrum of the current stock price.

4.2 The Term Structure of Interest Rates

We revisit the approximation for the one-factor model in Section 2.3. Consider the approximating formula given in Eq. (19). This formula comes out of a scheme which is somewhat less general than the approximation designed in the previous section, in which the two payoff functions of the true model and the auxiliary one may differ. In Section 4.2.1, we show that previous work by Chapman, Long and Pearson (1999) can be considered as a specific case of our method, arising when the auxiliary payoff is the zero payoff. In Section 4.2.2, we investigate the performance of our methods when the auxiliary payoff is equal one, which is just the true payoff.

4.2.1 A Simple Power Expansion

As a very simple choice, consider an auxiliary model for which, $\mu = \mu_0$, $\sigma = \sigma_0$, and $b_0 = 0$, in which case $w_0(x,t) = 0$, $d(x) = 1$ and $\delta(x,t) = 0$. In this application, the function $a(x,t) = x$ in

the approximation of Definition 1, such that

$$w_N(x,t) = \sum_{n=0}^{N} \frac{(T-t)^n}{n!} d_n(x,t), \tag{36}$$

where

$$d_n(x,t) = L d_{n-1}(x,t) - x d_{n-1}(x,t), \quad d_0(x,t) = 1.$$

We can easily compute the first few terms:

$$\begin{aligned} d_0(x,t) &= 1, \quad d_1(x,t) = -x, \quad d_2(x,t) = -\left(\mu(x,t) - x^2\right), \\ d_3(x,t) &= -\frac{\partial \mu(x,t)}{\partial t} + \mu(x,t)\left(2x - \frac{\partial \mu(x,t)}{\partial x}\right) + \frac{1}{2}\sigma^2(x,t)\left(2 - \frac{\partial^2 \mu(x,t)}{\partial x^2}\right) + x\left(\mu(x,t) - x^2\right). \end{aligned}$$

The above formula is essentially a slight generalization of the power series expansion appearing in Chapman, Long and Pearson (1999, Proposition 3); see also Wilmott (2003, p. 572).

As a numerical example, consider the Cox, Ingersoll and Ross (1985) (CIR, henceforth) model, which assumes that the short-term rate is solution to,

$$dr(t) = \beta(\alpha - r(t))dt + \sigma\sqrt{r(t)}dW(t).$$

where $\alpha > 0$, $\beta > 0$ and $\sigma > 0$ are constants. As is well-known, the solution to the CIR model can be written on closed form, and so we assess the accuracy of the expansion in Eq. (36) to approximate the CIR bond prices.

Figure 3 plots the percentage approximation error of this expansion for $N = 2, 4, 6, 8$, and 10. Here, the following parameter values were used: $\alpha = \alpha_0 = 0.06$, $\beta = \beta_0 = 0.1$ and $\sigma = 0.12247$ and initial interest rate level $x = 0.10$. From this plot, we see that a truncation of Eq. (36) based on a few terms provides a very accurate approximation to *short maturity* bond prices. Many terms are needed for the resulting approximation to work at longer maturities. As an example, the approximation based on only the first three terms works very poorly for $T - t \geq 3$. So instead it is recommendable to expand around a more suitable candidate pricing function.

4.2.2 A Better Expansion: The Vasicek Model as Auxiliary Pricing Device

The results of the previous example can considerably be improved by using a more informative auxiliary model. Let us consider the Vasicek (1977) model as auxiliary pricing device. In this model, the short-term rate is solution to,

$$dr_0(t) = \beta_0(\alpha_0 - r_0(t))dt + \sigma_0 dW(t),$$

for three constants α_0, β_0 and σ_0^2. The solution for the bond price, denoted with w_0, is well-known as this is the simplest example of an exponential affine model. Next, we choose $b_0(x) = 1$, such

that $d(x) = 0$. The mispricing function, δ, is now,

$$\delta(x,t) = (\mu(x,t) - \beta_0(\alpha_0 - x)) \frac{\partial w_0(x,t)}{\partial x} + \frac{1}{2}\left(\sigma^2(x,t) - \sigma_0^2\right) \frac{\partial^2 w_0(x,t)}{\partial x^2},$$

and our approximation to the CIR model is given by:

$$w_N(x,t;\theta) = w_0(x,t;\theta) + \sum_{n=0}^{N} \frac{(T-t)^{n+1}}{(n+1)!} \delta_n(x,t;\theta),$$

where $\delta_n = \bar{L}^n \delta$. In the previous expansion, we have emphasized the dependence of w_N on the parameter vector $\theta = (\alpha_0, \beta_0, \sigma_0)$. As in the option pricing case considered in the previous section, we have a nuisance parameter to choose. For example, in analogy with the choice we made in Section 4.1, we may use: $\theta_N(x,t) = \arg\min_\theta \left(w_N(x,t;\theta) - w_0(x,t;\theta)\right)^2$. Finally, the CIR model has a linear drift, just as the Vasicek one. Choosing the same drift parameter values for both models, the function δ simplifies to

$$\delta(x,t;\sigma_0) = \frac{1}{2}\left(\sigma^2 x - \sigma_0^2\right) \frac{\partial^2 w_0(x,t)}{\partial x^2}.$$

Figure 4 plots the approximation error as a function of time to maturity for different choices of N for the same parameter values of the CIR model as in Figure 3. We choose the remaining nuisance parameter as $\sigma_0^2 = \sigma^2 x$ for a given short rate level x, i.e. we choose $\theta_N(x,t) = \arg\min_\theta \delta^2(x,t;\theta)$ for all N. Compared to the approximation error of the simple expansion, we see that the approximation based on the Vasicek model works considerably better and only needs a few terms to reach a very high level of precision. The approximation works equally well for other values of x.

4.3 Option Pricing with Stochastic Volatility

Suppose that under the risk-neutral probability, the stock price S is solution to,

$$\left\{ \begin{array}{rcl} \dfrac{dS(t)}{S(t)} & = & r dt + \sqrt{v(t)} dW_1(t) \\ dv(t) & = & \varphi(S(t), v(t))\, dt + \psi(S(t), v(t))\, dW_2(t) \end{array} \right. \tag{37}$$

where $v(t)$ is the stochastic volatility and the two Brownian motions, W_1 and W_2, can be correlated, i.e., $E(dW_1(t) dW_2(t)) = \rho dt$, for some constant ρ.

Let $w(x,t)$, $x = (S,v)$, be the option price at time $t \in [0,T]$ when $S(t) = S$ and $v(t) = v$, with payoff $w(S,v,T) = (S-K)^+$. The pricing function, then, satisfies, $Lw(x,t) - rw(x,t) = 0$, where L is the infinitesimal generator operator associated to (37):

$$Lw = \frac{\partial w}{\partial t} + rS\frac{\partial w}{\partial S} + \frac{1}{2} v S^2 \frac{\partial^2 w}{\partial S^2} + \varphi \frac{\partial w}{\partial v} + \frac{1}{2}\psi^2 \frac{\partial^2 w}{\partial v^2} + \rho\sqrt{v} S \psi \frac{\partial^2 w}{\partial S \partial v}.$$

Let us expand this pricing function around that predicted by the Black and Scholes model,

using our method. Define the auxiliary model as:

$$\begin{cases} \dfrac{dS(t)}{S(t)} &= rdt + \sigma_0 S dW_1(t) \\ dv(t) &= 0 \times dt + 0 \times dW_2(t) \end{cases}$$

Although this way of re-writing the Black-Scholes model appears more complicated than needed, it actually allows us to further illustrate our method. Associated with the Black-Scholes model is the infinitesimal operator $L_0 w = \frac{\partial}{\partial t} w + rS\frac{\partial}{\partial S} w + \frac{1}{2}\sigma_0^2 S^2 \frac{\partial^2}{\partial S^2} w$, and the pricing function solving $L_0 w_0(x,t) - r w_0(x,t) = 0$ is simply $w_0(S,v,t) = w^{\text{bs}}(S,t)$. Now, the price difference, $\Delta w(x,t) = w(x,t) - w_0(x,t)$, satisfies,

$$L\Delta w(S,v,t) - r\Delta w(S,v,t) = (L - L_0) w_0(S,v,t),$$

where $\Delta w(S,v,T) = 0$, for all S and v. Appealing to Theorem 1, we have:

$$\Delta w(x,t) = \int_t^T e^{-r(u-t)} \mathbb{E}_{x,t}\left[\delta(S(u), v(u), t)\, dt\right] du, \tag{38}$$

where

$$\delta(x,t) = \frac{1}{2}\left(v - \sigma_0^2\right) S^2 \frac{\partial w^{\text{bs}}(x,t;\sigma_0)}{\partial S^2}.$$

which in turn leads to the following approximation:

$$w^N(x,t;\sigma_0) = w^{\text{bs}}(x,t;\sigma_0) + \sum_{n=0}^{N} \frac{(T-t)^{n+1}}{(n+1)!} \delta_n(x,t), \tag{39}$$

where $\delta_n(x,t) = \bar{L}^n \delta(x,t)$, $\bar{L} f = Lf - rf$, satisfies

$$\delta_{n+1}(x,t) = L\delta_n(x,t) - r\delta_n(x,t), \qquad \delta_0(x,t) \equiv \delta(x,t).$$

Note that the resulting approximation is seemingly identical to the one of the extended BS-model. In particular, the mispricing function, δ, has the same functional form as the mispricing function in Eq. (12), and still bears the interpretation of an instantaneous hedging cost arising from the use of a wrong model (the BS model). However, the infinitesimal operator L, which $\delta_n(x,t)$ implicitly depends on, has a much more complicated structure since it arrives from a two-factor model, not a simple one-factor model.

To numerically evaluate the performance of our method, we consider the Heston model for which option prices are known in closed-form (Heston, 1993). The Heston's model specifies the drift and diffusion term of $v(t)$ as $\varphi(S,v) = \kappa(\alpha - v)$ and $\psi^2(S,v) = \sigma^2 v$, such that

$$Lw = \frac{\partial w}{\partial t} + rS\frac{\partial w}{\partial S} + \frac{1}{2} v S^2 \frac{\partial^2 w}{\partial S^2} + \kappa(\alpha - v)\frac{\partial w}{\partial v} + \frac{1}{2}\sigma^2 v \frac{\partial^2 w}{\partial v^2} + \sigma\rho v S \frac{\partial^2 w}{\partial S \partial v S}.$$

The parameter values are chosen as: $\sigma = 0.1$, $\kappa = 2$, $\alpha = 0.04$, and $\rho = -0.5$. We choose an option

with $T - t = 1$ year to maturity, strike price $K = 100$, and fix the current level of volatility and interest at $v = 0.05$ and $r = 0.1$. The BS-parameter value is chosen as $\sigma_0 = \sqrt{v}$. In Figure 5, the percentage approximation error is shown as a function of the current price, x. Again, we see that with just 3-4 terms included, the approximation works very well.

5 Conclusion

This paper develops a novel method to obtain closed-form approximations to derivative prices in a quite general setting. The method shows considerable promise, and can be used in many related areas in finance and financial econometrics. The situation where we feel our approach has a strong potential is in the estimation and calibration of asset pricing models. Estimation methods of continuous time models should normally center around a set of conditional moments, which can be readily obtained through the simulation of our approximating formulae. Moreover, one strength of our method is that we left unspecified the auxiliary model to use, in order to come up with our approximations.

In the present paper, we have consistently shown the potential of our methods by relying on affine models as auxiliary devices, although other choices might indeed be possible. For example, in applications arising within the pricing of fixed income securities or credit derivatives, a choice alternative to affine models could be that relying on quadratic models (Ahn, Dittmar and Gallant, 2002). In principle, a topic deserving further investigation relates to the optimal choice of the auxiliary model to use to implement our methods. At the same time, developing this theme is hindered by the availability of a very few candidate models with closed-form solutions falling outside the affine class.

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Figures

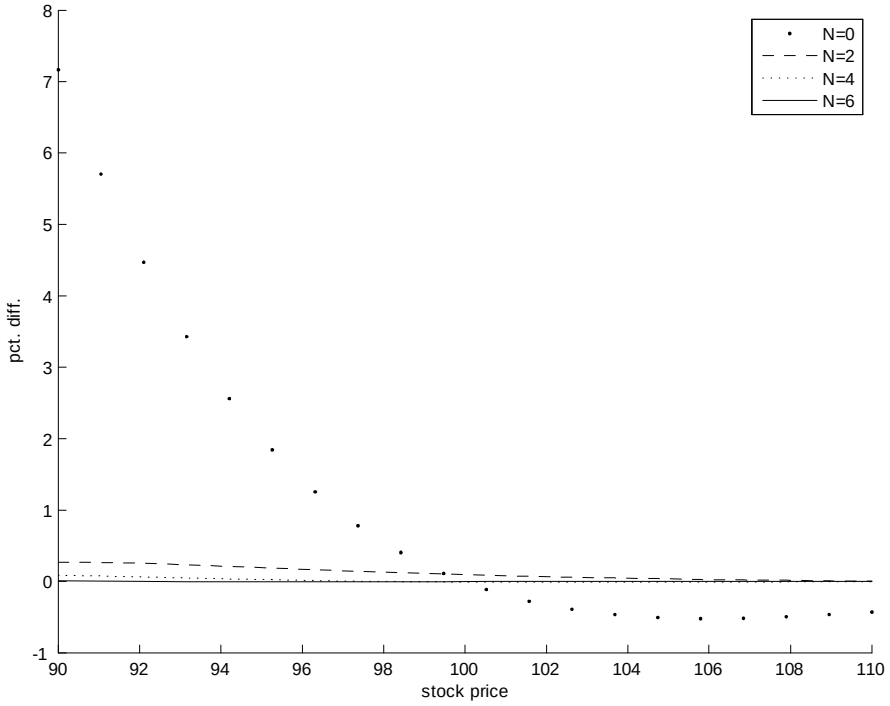

Figure 1: Percentage errors made by approximating the CEV-option pricing model with our method. Parameter values are $\sigma_{\text{cev}} = 10\%$, $\gamma = 1/2$, $r = 5\%$, and $K = 100$. Time-to-maturity: three months. The auxiliary volatility function used in the expansion of the option price when the underlying asset price is s is $\sigma_0(s) = \sigma_{\text{cev}} s^{\gamma-1}$.

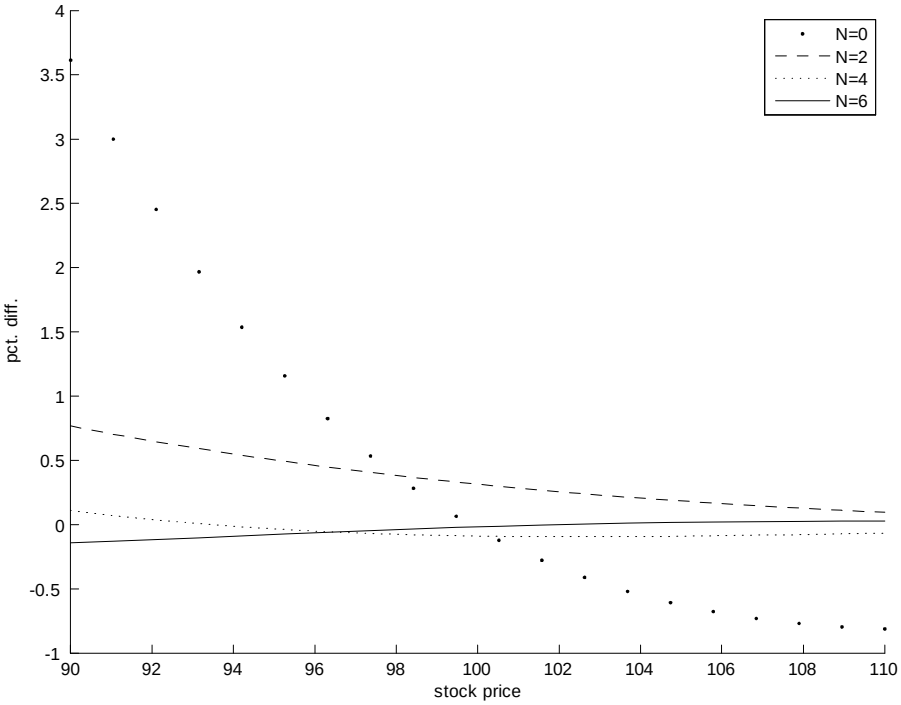

Figure 2: Percentage errors made by approximating the CEV-option pricing model with our methods. Parameters are as in Figure 1. Time-to-maturity: one year.

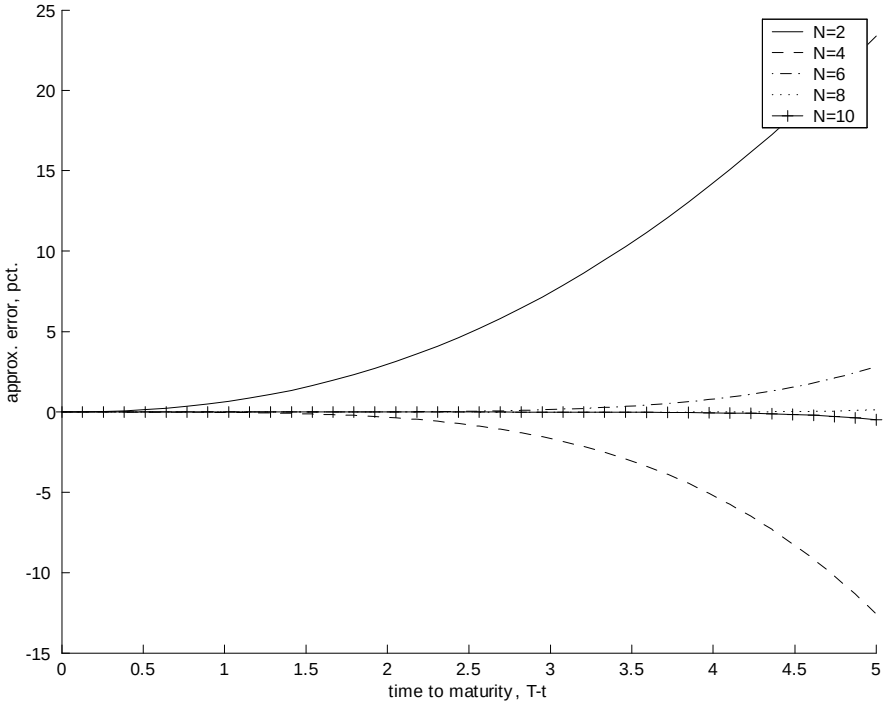

Figure 3: Percentage errors made by approximating the CIR model through a simple power expansion

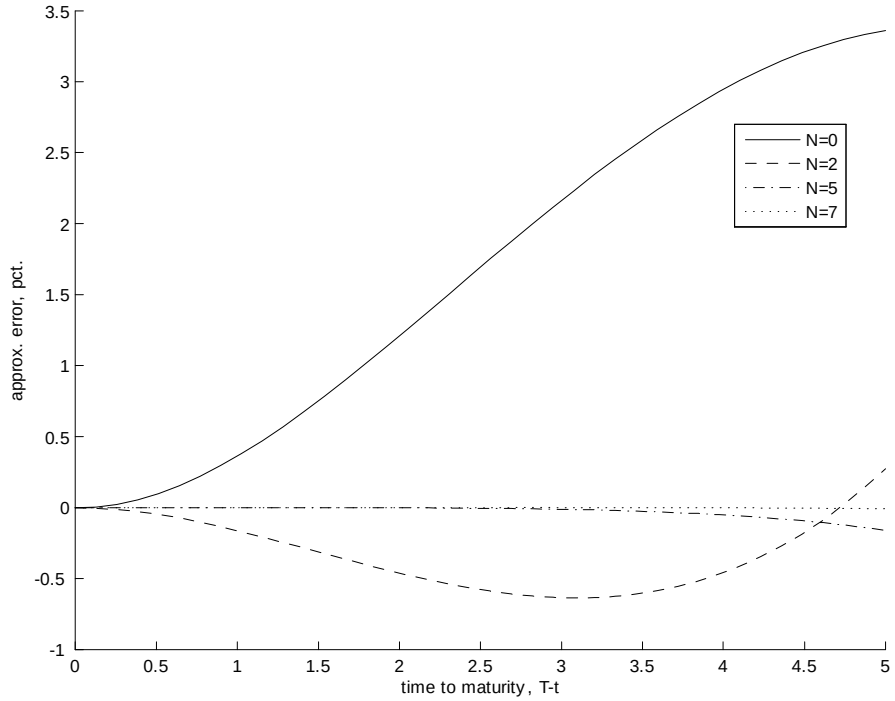

Figure 4: Percentage errors made by approximating the CIR model with the Vasicek model.

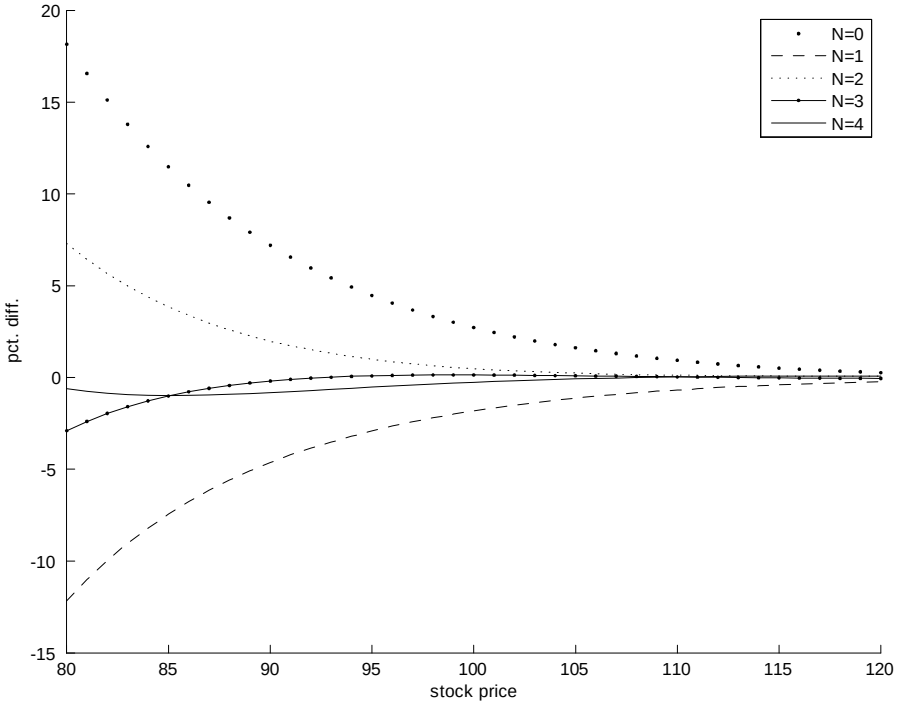

Figure 5: Percentage errors made by approximating the Heston model with the Black-Scholes model. Time-to-maturity: one year.

Technical Appendix

A Theoretical Properties of the Expansion

This Appendix develops theoretical properties of the general approximation formula in Section 3, where the target solution $w(x,t)$ solves the PDE given in Eq. (22) and our proposed approximation $w_N(x,t)$ is defined in Definition 1. In particular, we provide sufficient conditions under which we can state error bounds for fixed N and establish that $w_N(x,t) \to w(x,t)$ as $N \to \infty$. The results and proofs of this section are heavily indebted to previous work by Schaumburg (2004) developed in a different context.

The first result establishes an error bound for the approximation for any fixed $N \geq 1$:

Proposition A.1 *Assume that* $\mu, \sigma^2 \in \mathcal{C}^{2N}\left(\mathbb{R}^d\right)$ *and* $d, \delta \in \mathcal{C}^{2(N+1)}\left(\mathbb{R}^d\right)$. *Then* w_N *given in Definition 1 satisfies:*

$$|w(x,t) - w_N(x,t)| \leq E_N(x) \frac{(T-t)^{N+1}}{(N+1)!}, \text{ for all } (x,t) \in \mathbb{R}^d \times [0,T],$$

where

$$\begin{aligned} E_N(x) \equiv & \sup_{0 \leq s \leq T} \mathbb{E}_{x,t}\left[\left\|A^{N+1} d(x(s))\right\|\right] + \sup_{0 \leq s \leq T} \mathbb{E}_{x,t}\left[\left\|A^{N+1} \delta(x(s), s)\right\|\right] \\ & + \sup_{0 \leq s \leq T} \mathbb{E}_{x,t}\left[\left\|\partial^{N+1} \delta(x(s), s)/\partial s^{N+1}\right\|\right] \end{aligned}$$

In particular, if μ, σ^2, d *and* δ *are polynomially bounded, then for some constants* $c, q > 0$*:*

$$E_N(x) \leq (1 + \|x\|^q) e^{cT},$$

This result tells us that in great generality, the error decreases at a geometric rate uniformly over (x,t) in any compact interval as N increases. Florens-Zmirou (1989, Lemma 1) and Aït-Sahalia (2002) develop results similar to Proposition A.1 in different contexts.

The above result is not informative about what happens asymptotically as $N \to \infty$. To establish results for this case, we first introduce some additional notation: For a given operator A, define its spectrum and its resolvent as

$$\sigma(A) = \{\lambda \in \mathbb{C} : (\lambda - A) \text{ is not a bijection}\},$$

$$R_\lambda(A) = (\lambda - A)^{-1}, \quad \lambda \in \sigma(A).$$

Furthermore, we introduce a function space $\mathcal{H}$, which is equipped with some function norm $\|\cdot\|_{\mathcal{H}}$. We then impose the following conditions on the spectrum and resolvent of the infinitesimal operator L of $\{x(t)\}$ in order to show that our power series expansion converges:

A.1 For some $\delta, \omega > 0$ and $M \in \left(e^{-1}, \infty\right)$, the infinitesimal operator L given in Eq. (21) satisfies

$$\sigma(L) \subset \bar{\sigma} \equiv \{\lambda \in \mathbb{C} : |\arg(\lambda - \omega)| > \pi/2 + \delta\},$$

and its resolvent satisfies $\|R_\lambda(L)\| \leq M/|\lambda|$ for $\lambda \in \mathbb{C} \backslash \bar{\sigma}(L)$.

A.2 There exists $\bar{\tau} > 0$ and $\phi_\delta, \phi_d \in \mathcal{H}$ such that $d : \mathbb{R}^n \times \mathbb{R}_+ \mapsto \mathbb{R}$ and $\delta : \mathbb{R}^n \times \mathbb{R}_+ \mapsto \mathbb{R}$ defined in Eqs. (25)-(26) satisfy:

$$\mathbb{E}\left[\phi_\delta(x(\bar{\tau})) | x(0) = x\right] = \delta(x, \bar{\tau}) \text{ and } \mathbb{E}\left[\phi_d(x(\bar{\tau})) | x(0) = x\right] = d(x).$$

Also, the function $t \mapsto \delta(x,t)$ is analytic uniformly in $\|\cdot\|_{\mathcal{H}}$ and $\|a\|_{\infty} < \infty$.

Condition (A.1) relates to the infinitesimal operator and requires that its spectrum is within $\bar{\sigma}$. The second condition, (A.2), imposes conditions on the two functions, d and δ, whose conditional moments we wish to expand. It basically requires that each of the two functions can be matched through conditional moments.

Both assumptions are rather abstract, and not easily verified for specific models. The following proposition give more primitive conditions below for (A.1) to hold that are met by many standard diffusion models.

Proposition A.2 *Under the following two conditions, the generator L satisfies (A.1):*

(i) It has a transition density $p_t(y|x)$ w.r.t. the Lebesgue measure.

(ii) It has an invariant measure π satisfying

$$\pi(x)\, p_t(y|x) = \pi(y)\, p_t(x|y).$$

Conditions (i)-(ii) in Proposition A.2 are satisfied by many standard processes used in finance. Most known diffusion models have a transition density, while the second condition is a generalization of time-reversibility. In particular, if the process is univariate and stationary, it is necessarily time-reversible and therefore satisfies the second condition. In conclusion, assumption (A.1) holds under fairly weak conditions.

On the other hand, we have unfortunately not been able to find more primitive conditions for (A.2) to hold. For one example where (A.2) is not satisfied, we refer to Schaumburg (2004, Example 1) where a further discussion of this condition also can be found.

The theoretical foundation for the approximation in Definition 1 is stated in the following result:

Proposition A.3 *Let $\{x(t)\}_{t\geq 0}$ be a homogeneous diffusion process with infinitesimal operator L. Assume that L satisfies (A.1), the function $a : \mathbb{R}^d \mapsto \mathbb{R}_+$ is analytic, and d and δ satisfy (A.2).*

Then for any $|t-T| < \bar{\tau}/(Me)$, where M and $\bar{\tau}$ are given in (A.1)-(A.2):

1.

$$\mathbb{E}\left[e^{-\int_t^u a(x(s))ds} \delta(x(u),u) \middle| x(t) = x \right] = \sum_{n=0}^{\infty} \frac{(u-t)^n}{n!} \delta_n(x,t), \tag{A1}$$

and

$$\int_0^T \mathbb{E}\left[e^{-\int_t^u a(x(s))ds} \delta(x(u),u) \middle| x(t) = x \right] du = \sum_{n=0}^{\infty} \frac{(T-t)^{n+1}}{(n+1)!} \delta_n(x,t),$$

where $\delta_0 \equiv \delta$ and

$$\delta_{n+1}(x,t) = L\delta_n(x,t) - a(x,t)\,\delta_n(x,t), \quad n \geq 0,$$

and similarly for the function d.

2. The approximation w_N given in Definition 1 satisfies for all $|t-T| < \bar{\tau}/(Me)$,

$$\|w_N(\cdot,t) \to w(\cdot,t)\|_{\mathcal{H}} \to 0, \quad N \to \infty.$$

To establish Propositions A.1-A.3, we start out with the following abstract Cauchy problem,

$$-\frac{\partial w(x,t)}{\partial t} = Aw(x,t) + b(x,t), \tag{A2}$$

for $(x,t) \in \mathbb{R}^d \times [0,T]$, with termination value

$$w(x,T) = c(x).$$

Here, A is a general linear operator. We define the semigroup associated with A (see, e.g., Pazy, 1983) as

$$U(t) = e^{tA},$$

and let $\mathcal{D}(A)$ denote the domain of A defined as the set of functions for which

$$A\phi(x,0) = \lim_{t \to 0} \frac{U(t)\phi(x,t) - \phi(x,0)}{t}$$

is well-defined. We note that the PDE in Eq. (24) with time-homogenous coefficients can be written on the form (A2) with

$$A\phi(x,t) = L\phi(x,t) - a(x)\phi(x,t), \tag{A3}$$

$$b(x,t) = \delta(x,t), \quad c(x) = d(x). \tag{A4}$$

For this specification of A, we obtain that

$$U(t)\phi(x,t) = \mathbb{E}\left[\exp\left(-\int_0^t a(x(s))\,ds\right)\phi(x(t),t)\middle| x(0) = x\right].$$

It is easily seen that the solution to the inhomogenous problem (A2) can be represented as:

$$w(x,t) = U(T-t)c(x) + \int_0^{T-t} U(s)\,b(x,s)\,ds. \tag{A5}$$

Next, we wish to obtain an approximate solution w_N through a series expansion of $U(t)$. In particular, we wish to give conditions under which $U(t)$ satisfies

$$U(t)\phi(x) = e^{tA}\phi(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n \phi(x), \tag{A6}$$

in which case we define the approximation:

$$U_N(t)\phi(x) = \sum_{n=0}^{N} \frac{t^n}{n!} A^n \phi(x). \tag{A7}$$

Suppose that the function $t \mapsto \phi(x,t)$ is analytic for all x, such that

$$\phi(x,t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} B^k \phi(x,0), \quad B\phi(x,t) \equiv \frac{\partial \phi(x,t)}{\partial t}.$$

Then,

$$U(t)\phi(x,t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n \phi(x,t) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{t^{n+k}}{n!k!} A^n B^k \phi(x,0) = \sum_{n=0}^{\infty} \frac{t^n}{n!} (A+B)^n \phi(x,0)$$

Thus, we shall use the following approximation when ϕ is time-dependent:

$$U_N(t)\,\phi(x,t) = \sum_{n=0}^{N} \frac{t^n}{n!}\,(A+B)^n\,\phi(x,0). \tag{A8}$$

By plugging the two approximations in Eq. (A7) and (A8) into Eq. (A5), we obtain

$$w_N(x,t) = \sum_{n=0}^{N} \frac{(T-t)^n}{n!} A^n c(x) + \sum_{n=0}^{N} \frac{(T-t)^{n+1}}{(n+1)!}\,(A+B)^n\,b(x,0). \tag{A9}$$

The following result gives an upper bound on the approximation error for any given $N \geq 0$:

Proposition A.4 *Assume that the two functions $c(x)$ and $b(x,t)$ both belong to $\mathcal{D}\left(A^{N+1}\right)$ and $t \mapsto b(x,t)$ is $N+1$ times differentiable. Then the approximation error satisfies:*

$$|w(x,t) - w_N(x,t)| \leq E_N(x)\,\frac{(T-t)^{N+1}}{(N+1)!},\ \textit{for all}\ (x,t) \in \mathbb{R}^d \times [0,T],$$

where

$$E_N(x) = \sup_{0\leq s\leq T} \left|A^{N+1}U(s)\,b(x,s)\right| + \sup_{0\leq s\leq T} \left|A^{N+1}U(s)\,c(x)\right| + \sup_{0\leq s\leq T} \left|B^{N+1}U(s)\,b(x,s)\right|.$$

Next, we establish conditions under which the error bound established in Theorem A.4 vanishes as $N \to \infty$. Intuitively, this result will go through if the power expansion in Eq. (A6) is valid. If the operator A is bounded, $\|A\| < \infty$, then the expansion trivially holds. However, the infinitesimal operator is unbounded and we instead have to impose other restrictions to verify the validity of the expansion. We impose restrictions in terms of the operator's spectrum and resolvent as defined above that ensure that A is a so-called analytic operator. This in turn implies that the power expansion is valid.

Proposition A.5 *Assume:*

(i) For some $\delta, \omega > 0$ and $M \in \left(e^{-1}, \infty\right)$:

$$\sigma(A) \subset \bar{\sigma}(A) \equiv \{\lambda \in \mathbb{C} : |\arg(\lambda - \omega)| > \pi/2 + \delta\},$$

and $\|R_\lambda\| \leq M/|\lambda|$ for $\lambda \in \mathbb{C}\backslash\bar{\sigma}(A)$.

(ii) The functions $b(\cdot,\bar{\tau})$ and $c(\cdot)$ both lie in $U(\bar{\tau})\,\mathcal{H}$ for some $\bar{\tau} > 0$, i.e. there exists $\phi_b, \phi_c \in \mathcal{H}$ such that

$$U(\bar{\tau})\,\phi_b(x) = b(x,\bar{\tau}) \quad \textit{and} \quad U(\bar{\tau})\,\phi_c(x) = c(x).$$

Also, $t \mapsto b(x,t)$ is analytic for all x.

Then for all $|t-T| < \bar{\tau}/(Me)$, where M and $\bar{\tau}$ are given in (i) and (ii):

$$\|w_N(\cdot,t) \to w(\cdot,t)\|_{\mathcal{H}} \to 0, \quad N \to \infty.$$

Finally, note that our results only relate to time-homogenous diffusions. It would be of interest to derive results for time-inhomogenous diffusions, where drift and diffusion term vary over time t. This task corresponds to analyzing so-called evolution systems of the form,

$$-\frac{\partial w(x,t)}{\partial t} = A(t)\, w(x,t) + b(x,t),$$

where the linear operator $A(t)$ is now time-inhomogenous. Unfortunately, there are very few results in the literature on the analyticity of this class of operators, and we therefore refrain from trying to extend our results to this more general setting; Pazy (1983, Chapter 5) contains a few preliminary results on this topic.

B Equivalence between Moment and Density Expansion

We give a direct proof of the equality stated in Eq. (33). As a part of the proof, we obtain a direct representation of the difference between the densities of the true and auxiliary model. As a first step, we note that the two transition densities solve the so-called backward Kolmogorov equations,

$$Lp(y,T|x,t) = 0, \qquad L_0 p_0(y,T|x,t) = 0,$$

with boundary conditions $p(y,T|x,T) = p_0(y,T|x,T) = \mathcal{D}(y-x)$, where $\mathcal{D}$ is the Dirac's function. Using the same arguments as those in Section 3, it is easily seen that the "transition discrepancy," Δp, is solution to:

$$L\Delta p(y,T|x,t) + \tilde{\delta}(y,T|x,t) = 0,$$

with boundary condtion $p(y,T|x,T) = 0$, where the adjusment term $\tilde{\delta}(y,T|x,t)$ is given by:

$$\tilde{\delta}(y,T|x,t) = (L_0 - L)\, p_0(y,T|x,t) = \Delta\mu(x,t)\frac{\partial p_0(y,T|x,t)}{\partial x} + \frac{1}{2}\Delta\sigma^2(x,t)\frac{\partial^2 p_0(y,T|x,t)}{\partial x^2}.$$

The Feynman-Kac representation then yields

$$\Delta p(y,T|x,t) = \int_t^T \mathbb{E}_{x,t}\left[\tilde{\delta}(y,T|x(s),s)\right] ds. \tag{B1}$$

Substituting the right hand side of Eq. (B1) back into Eq. (32),

$$\begin{aligned} w(x,t) &= w_0(x,t) + \int_{\mathbb{R}} b(y)\,\Delta p(y,T|x,t)\,dy \\ &= w_0(x,t) + \int_t^T \left[\int_{\mathbb{R}}\int_{\mathbb{R}} b(y)\,\tilde{\delta}(y,T|z,s)\,p(z,s|x,t)\,dydz\right] ds. \end{aligned} \tag{B2}$$

Finally, using that

$$\frac{\partial^k w_0(x,t)}{\partial x^k} = \int_{\mathbb{R}} b(y)\frac{\partial^k p_0(y,T|x,t)}{\partial x^k}dy, \quad k \geq 0,$$

we obtain

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} b(y)\,\tilde{\delta}(y,T|z,s)\,p(z,s|x,t)\,dydz &= \int_{\mathbb{R}}\left[\int_{\mathbb{R}} b(y)\,\frac{\partial p_0(y,T|z,s)}{\partial z}dy\right]\Delta\mu(z,s)\,p(z,s|x,t)\,dz \\ &\quad +\frac{1}{2}\int_{\mathbb{R}}\left[\int_{\mathbb{R}} b(y)\,\frac{\partial^2 p_0(y,T|z,s)}{\partial z^2}dy\right]\Delta\sigma^2(z,t)\,p(z,s|x,t)\,dz \\ &= \int_{\mathbb{R}}\Delta\mu(z,s)\,\frac{\partial w_0(z,s)}{\partial x}p(z,s|x,t)\,dz \\ &\quad +\frac{1}{2}\int_{\mathbb{R}}\Delta\sigma^2(z,t)\,\frac{\partial w_0^2(z,s)}{\partial x^2}p(z,s|x,t)\,dz \\ &= \mathbb{E}_{x,t}\left[\delta(x(s),s)\right],\end{aligned}$$

where δ was defined in Eq. (26).

Note that the above representation of Δp as a conditional moment gives rise to an alternative approximation scheme: the right hand side of Eq. (B1) can be approximated by

$$\Delta p_N(y,T|x,t) = \sum_{n=0}^{N}\frac{(T-t)^{n+1}}{(n+1)!}L\tilde{\delta}(y,T|x,t).$$

Plugging this into the integral in Eq. (B2), we obtain

$$\tilde{w}_N(x,t) = w_0(x,t) + \sum_{n=0}^{N}\frac{(T-t)^{n+1}}{(n+1)!}\int_{\mathbb{R}} b(y)\,L^n\tilde{\delta}(y,T|x,t)\,dy.$$

As noted in the main text, this approximation however involves computation of $\int_{\mathbb{R}} b(y)\,L^n\tilde{\delta}(y,T|x,t)\,dy$, $n=1,...,N$.

C Approximation to Second Order Derivatives of Pricing Functions

For $k=2$, the recursive scheme to compute the second derivatives of $d_n(x,t)$ and $\delta_n(x,t)$ w.r.t. x becomes: $d_0^{(2)}(x,t) = \partial^2 d(x)/\partial x^2$, $\delta_0^{(2)}(x,t) = \partial^2\delta(x,t)/\partial x^2$ and

$$\begin{aligned}d_n^{(2)}(x,t) &= Ld_{n-1}^{(2)}(x,t) - a(x,t)d_{n-1}^{(2)}(x,t) + 2L^{(1)}d_{n-1}^{(1)}(x,t) - 2\frac{\partial a(x,t)}{\partial x}d_{n-1}^{(1)}(x,t) \\ &\quad +L^{(2)}d_{n-1}(x,t) - \frac{\partial^2 a(x,t)}{\partial x^2}d_{n-1}(x,t)\end{aligned}$$

$$\begin{aligned}\delta_n^{(2)}(x,t) &= L\delta_{n-1}^{(2)}(x,t) - a(x,t)\delta_{n-1}^{(2)}(x,t) + 2L^{(1)}\delta_{n-1}^{(1)}(x,t) - 2\frac{\partial a(x,t)}{\partial x}\delta_{n-1}^{(1)}(x,t) \\ &\quad +L^{(2)}\delta_{n-1}(x,t) - \frac{\partial^2 a(x,t)}{\partial x^2}\delta_{n-1}(x,t),\end{aligned}$$

where

$$L^{(2)}\phi(x,t) = \sum_{i=1}^{d}\frac{\partial^2\mu_i(x,t)}{\partial x^2}\frac{\partial\phi(x,t)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2\sigma_{ij}^2(x,t)}{\partial x^2}\frac{\partial^2\phi(x,t)}{\partial x_i\partial x_j}.$$

D Proofs

Proof of Proposition A.1. This is a direct consequence of Proposition A.4 since under the conditions, d and δ clearly lies in the domain of L. Moreover,

$$A^{N+1}U(s)\,\delta(x,s) = \mathbb{E}\left[A^{N+1}\delta(x(s),s)\,|x(0)=x\right]$$

and similar for the other term of $E_N(x)$ defined in Proposition A.4. This establishes the stated error bound. Finally, note that under the polynomial bounds, $\left\|A^{N+1}c(x(s),s)\right\| \leq C\left(\|x(s)\|^q+1\right)$ for some $q \geq 1$, and we then apply Friedman (1975, Theorem 5.2.2-5.2.3) to obtain

$$\mathbb{E}\left[\left|A^{N+1}\delta(x(s),s)\right|\,|x(0)=x\right] \leq C\left(\mathbb{E}\left[\|x(s)\|^q\,|x(0)=z\right]+1\right) \leq \left(1+\|x\|^q\right])\,e^{cs},$$

for some constants $c, q > 0$. Similarly for the other term in $E_N(x)$. ■

Proof of Proposition A.3. The result will follow from Proposition A.5 if we can verify that (A.1)-(A.2) imply (i)-(ii) of that Proposition. It is easily seen that (A.2) implies (ii) given the form of $U(t)$ for the choice of A here. To verify (i), we apply Pazy (1983, Theorem 3.2.1) which will yield the desired result if we can show that the domain of L is contained in the one of the operator F defined as $F\phi(x) = a(x)\,\phi(x)$ - $\mathcal{D}(L) \subset \mathcal{D}(F)$ - and that for some constants c_1 and c_2,

$$\|F\phi\|_{\mathcal{H}} \leq c_1\,\|L\phi\|_{\mathcal{H}} + c_2\,\|\phi\|_{\mathcal{H}}.$$

But clearly $\mathcal{D}(F)$ contains all twice-differentiable functions and the above inequality follows by the fact that $\|a\|_\infty < \infty$. ■

Proof of Proposition A.2. It follows from Schaumburg (2004, Lemma 2.2) that L satisfies (A.1) under the two conditions stated in the proposition. ■

Proof of Proposition A.4. By definition,

$$U(t)\,\phi(x,t) = \phi(x,t) + \int_0^t AU(s)\,\phi(x,s)\,ds.$$

Using this identity iteratively, we obtain

$$\begin{aligned}
U(t)\,\phi(x,t) &= \phi(x,t) + \int_0^t AU(t_1)\,\phi(x,t)\,dt_1 \\
&= \phi(x,t) + \int_0^t A\left[\phi(x,t) + \int_0^{t_1} U(t_2)\,\phi(x,t_2)\right]dt_2 dt_1 \\
&= \phi(x,t) + tA\phi(x,t) + \int_0^t\int_0^{t_1} AU(t_2)\,\phi(x,t_2)\,dt_2 dt_1 \\
&\;\;\vdots \\
&= \sum_{n=0}^{N}\frac{t^n}{n!}A^n\phi(x,t) + R_N(x,t),
\end{aligned}$$

where

$$R_N(x,t) = \frac{1}{N!}\int_0^t\int_0^{t_1}\cdots\int_0^{t_{N+1}} A^{N+1}U(t_{N+1})\,\phi(x,t_{N+1})\,dt_{N+1}\cdots dt_1.$$

The approximation error $R_N(x,t)$ is bounded by:

$$\begin{aligned}
|R_N(x,t)| &= \frac{1}{N!}\left|\int_0^t\int_0^{t_1}\cdots\int_0^{t_{N+1}} A^{N+1}U(t_{N+1})\,\phi(x,t_{N+1})\,dt_{N+1}\cdots dt_1\right| \\
&\leq \frac{1}{N!}\int_0^t (t-s)^N\left|A^{N+1}U(s)\,\phi(x,s)\right| ds \\
&\leq \sup_s\left|A^{N+1}U(s)\,\phi(x,s)\right| \times \frac{1}{(N+1)!}t^{N+1}.
\end{aligned}$$

Next, by an Nth order Taylor expansion of b, there exists $\bar{s}\in[0,s]$ such that

$$b(x,s) - \sum_{k=0}^{N}\frac{t^k}{k!}B^k b(x,0) = \frac{t^{N+1}}{(N+1)}B^{N+1}b(x,\bar{s}).$$

Using the results obtained above,

$$\begin{aligned}
&|w(x,t) - w_N(x,t)| \\
\leq\ & |[U-U_N](T-t)\,c(x)| + \int_0^{T-t}|[U-U_N](s)\,b(x,s)|\,ds + \int_0^{T-t}|U(s)\,[b-b_N](x,s)|\,ds \\
\leq\ & \frac{(T-t)^{N+1}}{(N+1)!}\sup_s\left|A^{N+1}U(s)\,c(x)\right| + \frac{(T-t)^{N+2}}{(N+2)!}\sup_s\left|A^{N+1}U(s)\,b(x,s)\right| \\
&+\frac{(T-t)^{N+2}}{(N+2)!}\sup_s\left|B^{N+1}U(s)\,b(x,s)\right|.
\end{aligned}$$

■

Proof of Proposition A.5. We apply Pazy (1983, Theorem 2.5.2) to obtain that the range of $U(t)$ is dense in $\mathcal{D}(A^\infty)$ and hence in $\mathcal{H}$ under (i). Proposition A.4 supplies an upper bound of the approximation for a given N. By following the same arguments as in Schaumburg (2004, Proof of Theorem 2.1) we obtain that

$$\left\|(T-t)^{N+1}A^{N+1}U(s)\,c\right\| \to 0, \quad \left\|(T-t)^{N+1}A^{N+1}U(s)\,b(\cdot,s)\right\| \to 0,$$

as $N\to\infty$ for all $(T-t) < \bar{\tau}/(Me)$, while it trivially holds for an analytical function that $(T-t)^{N+1}B^{N+1}b(\cdot,s)\to 0$. By the dominated convergence theorem this implies

$$\left\|(T-t)^{N+1}U(s)\,B^{N+1}b(\cdot,s)\right\| \to 0.$$

This shows that the bound goes to zero as $N\to\infty$. ■

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