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Impact of time–inhomogeneous jumps and leverage type effects on returns and realised variances

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Abstract

This paper studies the effect of time–inhomogeneous jumps and leverage type effects on realised variance calculations when the logarithmic asset price is given by a Lévy–driven stochastic volatility model. In such a model, the realised variance is an inconsistent estimator of the integrated variance. Nevertheless it can be used within a quasi–maximumlikelihood setup to draw inference on the model parameters. In order to do that, this paper introduces a new methodology for deriving all cumulants of the returns and realised variance in explicit form by solving a recursive system of inhomogeneous ordinary differential equations.

Keywords: Lévy processes; stochastic volatility, leverage effect; superposition; realised variance.
JEL classification: C10, C13, C14, G10, G12.

1 Introduction

Realised variance and its use for estimating and forecasting stochastic volatility has been studied extensively in the finance literature in the last decade (see e.g. [2, 10, 11, 37, 52]). So far, such studies have mainly focused on asset price models given by Brownian semimartingales or, more generally, Itô semimartingales. This paper is now devoted to studying the impact of *time-inhomogeneous jumps* and *leverage type effects* on realised variance calculations when the logarithmic asset price is given by a stochastic integral with respect to a Lévy process.

Lévy–driven stochastic volatility models are able to cope with many stylised facts of asset returns particularly well, e.g. they reflect the skewness and fat tails of asset return distributions more appropriately and can handle jumps and volatility smiles much better than models based on Brownian motions alone. Recently, several types of such Lévy–based stochastic volatility models have been studied in the financial literature. Basically, they can be divided into two groups: time–changed Lévy processes (see e.g. [12, 21, 22]) and stochastic integrals with respect to a Lévy process (see e.g. [28, 40, 51, 53]). Here we restrict our attention to the latter class of Lévy–based stochastic volatility models. We note that this class of models does not generally fall into the class of affine models (as time–changed Lévy

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processes, see e.g. [26, 39]) and, hence, explicit computations in such a modelling framework turn out to be analytically more involved. Apart from inhomogeneous jumps, our model also allows for very general leverage type effects or asymmetric volatility. During the last decades, many empirical studies have revealed the fact that past stock returns tend to be negatively correlated with innovations of future volatilities. This property is often called the *leverage effect* — an expression which has been derived from the hypothesis that a negative stock return might increase financial leverage and, hence, leads to a riskier stock which results in higher volatility. [17] was probably the first to investigate this effect, and his finding was further supported by studies by [23, 43] among others and, more recently, by [18, 20, 35, 49, 50, 54]. While the existence of asymmetric volatility is rarely questioned, its main determinant is still subject to vivid discussions (see e.g. [15] and the references therein). Besides the previously mentioned leverage–hypothesis, there is also the time–varying risk premium theory or volatility feedback theory, which essentially relies on the converse causality when stating that increasing volatility leads to decreasing stock price returns. However, regardless of where asymmetric volatility originates from, it is definitely an important fact, which has to be accounted for in asset pricing, especially in the context of option pricing since the asymmetric relationship is directly associated with implied volatility smiles. So the leverage effect is often regarded as a natural tool for explaining smirks in option price data (see [31, 36]). Unfortunately, previous work on the econometric properties of Lévy –driven stochastic volatility has so far only been carried out under the no–leverage assumption ([12, 53]) which is, particularly in equity markets, not realistic. So the main contribution of this paper is, that it overcomes this restrictive assumption and studies the impact of both time–inhomogeneous jumps and general leverage type effects on returns and realised variance simultaneously.

The remaining part of the paper is structured as follows. Section 2 sets up the notation and defines the Lévy–driven stochastic volatility model, which we study in this paper. Following recent research on stochastic volatility models, we use the so–called realised variance as a proxy for the accumulated variance over a day. This quantity will be defined in Section 3. Section 4 and 5 contain the main theoretical results of the paper. In Section 4, we present explicit formulae for the moments and second order properties of the returns, the actual variance and the quadratic variation of the price process. Section 5 addresses the first and second order properties of the realised variance where we study in detail the influence of the jumps and the leverage effect on volatility estimation. All these results are derived explicitly by using a novel methodology which involves solving a recursive system of inhomogeneous ordinary differential equation. Finally, we give a brief outlook on parameter estimation and inference in Section 6. Throughout the text, all of the mathematical proofs are relegated to the Appendix (Section A).

2 Model definition and technical assumptions

Let $(\Omega, \mathcal{A}, \mathbb{P})$ denote a probability space with filtration $\mathcal{F} = \{\mathcal{F}_t\}_{0\leq t<\infty}$, satisfying the usual conditions (see e.g. [47]). Let $S = (S_t)_{t\geq 0}$ denote the logarithmic asset price and $\sigma = (\sigma_t)_{t\geq 0}$ the stochastic volatility (SV). We will study models of the form

$$S_t = \mu t + \int_0^t \sigma_{s-} d(vW_s + X_s),$$

where $X = (X_t)_{t\geq 0}$ denotes a pure jump Lévy processes (see e.g. [16, 45, 48]) and $W = (W_t)_{t\geq 0}$ is a standard Brownian motion and $v \in \mathbb{R}$ is a constant, which could be 0. In that case, we would be in a pure jump setting. Solely for ease of exposition, we will assume that $\mathbb{E}X_1 = 0$. Furthermore, we assume that $\text{Var}(X_1) < \infty$. More precisely, in order to make sure that the model is uniquely identified, we set $VarX_1 + Var(vW_1) = 1$. Otherwise, one could always multiply $vW + X$ by a constant and scale σ appropriately and one would still obtain the same value for the price process S.

Here we will focus on Ornstein–Uhlenbeck (OU) type stochastic volatility models. I.e., we shall model the volatility process by a stationary Lévy–driven OU process which satisfies the following stochastic differential equation

$$d\sigma_t = -\lambda\sigma_{t-}dt + dY_{\lambda t},$$

where $Y = (Y_t)_{t\geq 0}$ denotes a pure jump subordinator (i.e. a non–decreasing Lévy process) and $\lambda > 0$ is the memory parameter. Throughout we assume that σ_0 is drawn from its stationary distribution.

Similar models have been studied in detail in the Brownian motion framework (i.e. when X is a Brownian motion) by [9] who chose σ^2 to be a OU process. However, studies by [41] have shown that choosing σ or σ^2 to be an OU process leads to similar results. So for reasons of mathematical tractability we have chosen the volatility process rather than the variance to be of OU–type. In such a modelling framework, the dynamics of the squared volatility process are given by

$$d\sigma_t^2 = -2\lambda\sigma_{t-}^2dt + 2\sigma_{t-}dY_{\lambda t} + d[Y]_{\lambda t},$$

which might remind us on a jump–driven version of a square root process. Clearly, such a model for the volatility process satisfies the essential requirement that volatility has to be non–negative. By time–changing the subordinator Y by the memory parameter λ, we obtain a stationary distribution of σ which does not depend on λ. Note in particular that we do not assume that σ and X are independent. We rather choose a bivariate Lévy process $(vW + X, Y_\lambda)'$ as driving process of $(S, \sigma)'$ and, hence, can capture the leverage effect with this model. In order to be able to choose such a bivariate Lévy process, we have to make sure that both driving processes run on the same time scale. The choice of Y_λ rather than Y in the second component is hence essential. Otherwise it would be possible that there was already information about the price process available before there was any information about the volatility process and vice versa, and this would possibly lead to arbitrage opportunities. Throughout the text, we will set $\mu = 0$, although the results can be easily generalised for a price process which includes a drift term. Finally, we introduce the notation for the cumulants of the the driving Lévy processes. We define $Z = (Z_t)_{t\geq 0}$ by $Z_t = (X_t, Y_{\lambda t})'$ for $\lambda > 0$, which is a bivariate pure jump Lévy process of which the second component is a subordinator. Let ν denote the Lévy measure of Z and ν_X and ν_{Y_λ} denote the Lévy measure of X and Y_λ, respectively. Note that if X and Y are pure jump Lévy processes of finite variation, then they can be represented as sum of their jumps: $X_t = \sum_{0\leq s\leq t} \Delta X_s$,and $Y_{\lambda t} = \sum_{0\leq s\leq t} \Delta Y_{\lambda s}$.

Remark If $v = 0$ and X is of finite variation, S can be written as $S_t = \sum_{0\leq s\leq t} \Delta S_s$. From [38, Chapter IX, Proposition 3] we can deduce that its characteristic triplet is given by $(0, 0, \nu_S)$, where ν_S is defined by $\mathbb{I}_F \star \nu_S = \mathbb{I}_F(\sigma_{s-}x) \star \nu_X$ for all $F \subseteq \mathbb{R}\backslash\{0\}$, where we denote by $f \star \nu$ for a function f on a subset of $\mathbb{R}^2$ and random measure ν the following integral process $(f \star \nu)_t = \int_{\mathbb{R}\times[0,t]} f(x, s)\nu(dx, ds)$.

Recall that the n–th cumulant of a stochastic process $Z = (Z_t)_{t\geq 0}$ is defined by (provided it exists)

$$\kappa_n(Z_t) = \frac{1}{i^n}\frac{\partial^n}{\partial u^n}\log\left(\mathbb{E}\left(\exp(iuZ_t)\right)\right).$$

Furthermore, it is well–known (see e.g. [24, p.32, 92]) that the moments of a Lévy process can then be expressed in terms of the corresponding cumulants by $\mathbb{E}(Z_t) = \kappa_1(Z_1)t$, $\mathbb{E}(Z_t^2) = \kappa_2(Z_1)t +$

$(\kappa_1(Z_1)t)^2$, $\mathbb{E}(Z_t^3) = \kappa_3(Z_1)t + 3\kappa_1(Z_1)\kappa_2(Z_1)t^2 + (\kappa_1(Z_1)t)^3$, $\mathbb{E}(Z_t^4) = \kappa_4(Z_1)t + 3(\kappa_2(Z_1)t)^2 + 4\kappa_1(Z_1)\kappa_3(Z_1)t^2 + 6(\kappa_1(Z_1)^2)\kappa_2(Z_1)t^3 + (\kappa_1(Z_1)t)^4$. For the cumulants of these processes, we will use the following notation. The cumulants (denoted by $\kappa_i(\cdot)$, $i = 1, 2, \ldots$) of the random variable X_1 are denoted by ξ and the cumulants of the process Y_1 are denoted by η. Hence,

$$\xi_i = \kappa_i(X_1) = \int_{\mathbb{R}} u^i \nu_X(du), \qquad \eta_i = \kappa_i(Y_1) = \int_{[0,\infty)} v^i \nu_{Y_1}(dv), \quad \text{for } i = 1, 2, \ldots$$

Note that $\kappa_i(Y_\lambda) = \lambda\eta_i$, for $i = 1, 2, \ldots$. Furthermore,

$$\kappa_{n,m} = \int_{\mathbb{R}\times[0,\infty)} u^n\, v^m\, \nu(du, dv), \quad \text{for } n,\ m\ \in\ \mathbb{N}.$$

Throughout the text, we will assume that at least the first four cumulants of the Lévy process Z are finite. From the Cauchy–Schwarz inequality, we obtain the following constraints for the cumulants of Z. For $n, m \in \mathbb{N}$, the cumulants (if they exist) satisfy $\kappa_{n,m} \leq \sqrt{\xi_{2n}\,\lambda\eta_{2m}}$.

The cumulants of the bivariate Lévy process can be regarded as a measure of the dependence between the two driving processes, which obviously includes the leverage effect: the measure of dependence of first order. In the following, we will deal with the following five cumulants.

$$\begin{aligned}
\kappa_{1,1} &= \mathbb{E}\left\{(X_1 - \mathbb{E}X_1)(Y_\lambda - \mathbb{E}Y_\lambda)\right\} = Cov(X_1,\ Y_\lambda),\\
\kappa_{1,2} &= \mathbb{E}\left\{(X_1 - \mathbb{E}X_1)(Y_\lambda - \mathbb{E}Y_\lambda)^2\right\},\\
\kappa_{1,3} &= \mathbb{E}\left\{(X_1 - \mathbb{E}X_1)(Y_\lambda - \mathbb{E}Y_\lambda)^3\right\} - 3\mathbb{E}\left\{(X_1 - \mathbb{E}X_1)(Y_\lambda - \mathbb{E}Y_\lambda)\right)\mathbb{E}\left((Y_\lambda - \mathbb{E}Y_\lambda)^2\right\},\\
\kappa_{2,1} &= \mathbb{E}\left\{(X_1 - \mathbb{E}X_1)^2(Y_\lambda - \mathbb{E}Y_\lambda)\right\},\\
\kappa_{2,2} &= \mathbb{E}(X_1Y_\lambda) - 2\mathbb{E}(X_1)\mathbb{E}\left(X_1Y_\lambda^2\right) - 2\mathbb{E}(Y_\lambda)\mathbb{E}\left(X_1^2Y_\lambda\right) - \mathbb{E}\left(X_1^2\right)\mathbb{E}\left(Y_\lambda^2\right)\\
&\quad - 2\left\{\mathbb{E}(X_1Y_\lambda)\right\}^2 + 2\left\{\mathbb{E}(X_1)\right\}^2\mathbb{E}\left(Y_\lambda^2\right) + 2\left\{\mathbb{E}(Y_\lambda)\right\}^2\mathbb{E}\left(X_1^2\right)\\
&\quad + 8\,\mathbb{E}(Y_\lambda)\mathbb{E}(X_1)\mathbb{E}(X_1Y_\lambda) - 6\left\{\mathbb{E}(X_1)\right\}^2\left\{\mathbb{E}(Y_\lambda)\right\}^2.
\end{aligned}$$

Note that if X and Y_λ are independent, they have no common jumps and, hence, $\nu\{(u,v) \in \mathbb{R} \times [0,\infty) : uv \neq 0\} = 0$. That means that, if X and Y_λ are independent and, hence, there is no leverage effect, then all joint cumulants $\kappa_{n,m} = 0$.

3 Returns and realised variance

Our aim is to study the econometric properties of the Lévy–type SV model defined above. So far, we have defined a model for an asset price in continuous time. We can now use the increments of such a process for modelling (high frequency) returns. Recall that $(S_t)_{t\geq 0}$ denotes the continuous–time log–price process of an asset. Further, let $h > 0$ denote the length of a fixed time interval, typically one day. The returns of the asset price are then given by

$$s_i = S_{ih} - S_{(i-1)h}, \quad i = 1, 2, \ldots,$$

where i indexes the day. Due to the availability of high frequency data, one is often interested in modelling returns at a higher frequency than just daily data. Suppose that we are given M intra–h observations during each time interval of length h. The time gap between these high frequency observations is denoted by $\delta = h/M$. Then

$$s_{j,i} = S_{(i-1)h+j\delta} - S_{(i-1)h+(j-1)\delta}, \quad j = 1, \ldots, M$$

denotes the j–th intra–h high frequency return on the i–th period of length h. Often we work with $h = 1$, representing one day. Based on these high frequency returns, one can then define the *realised variance* for the i–th day by

$$[S_\delta]_{[(i-1)h,ih]} = \sum_{j=1}^{M} s_{j,i}^2.$$

This quantity is often used to proxy the variability in financial markets in SV models (see e.g. [4, 10]). In this paper, we will compute the realised variance based on five–minute returns and will therefore ignore possible market microstructure effects which come into play when analysing really high frequency returns (i.e. one minute returns, tick by tick data). Such effects can be caused by e.g. bid/ask spreads, irregular trading and the fact that prices are recorded in discreet time. Since studying realised variance in the presence of market frictions is beyond the scope of this paper, we just refer to articles by e.g. [1, 3, 5, 6, 8, 13, 33, 55–57], for recent and very detailed studies of this research topic. Recall that the quadratic variation of a semimartingale $S = (S_t)_{t\geq 0}$ is defined by $[S]_t = S_t^2 - 2\int_0^t S_{u-}dS_u$. It is well–known that

$$[S_\delta]_{[(i-1)h,ih]} \overset{ucp}{\longrightarrow} [S]_{ih} - [S]_{(i-1)h}, \qquad \text{as } M \to \infty \text{ (i.e. } \delta \to 0),$$

where the convergence is uniform on compacts in probability (ucp) (see [45]). So the realised variance can be used to estimate the (increments of the) quadratic variation of the price process consistently. Hence, in our modelling framework, the realised variance can be used as a consistent estimator of

$$[S]_{ih} - [S]_{(i-1)h} = \int_{(i-1)h}^{ih} \sigma_{s-}^2 d[X]_s.$$

However, one is rather interested in estimating and forecasting the integrated variance $IV_t = \int_0^t \sigma_s^2 ds$. From Lévy's theorem, we can deduce that $[X]_t = t$ if and only if X is a Brownian motion. I.e. as soon as the Brownian motion is replaced by a more general Lévy process or semimartingale, the quadratic variation of the SV model is not given by the integrated variance. Hence, it is important to study the bias and the degree of inconsistency of the realised variance as proxy for the integrated variance. This is the task we tackle in the next section.

4 Cumulants of returns, actual variance and incremental quadratic variation

Here we study the statistical properties of the following three quantities: the price process S, the integrated variance IV and the quadratic variation of the price process $[S]$. From these properties, we can then directly derive properties of the increments of the corresponding stochastic processes: the returns of the log–price (the increments of S), the actual variance (the increments of IV) and the incremental quadratic variation (the increments of $[S]$).

4.1 Cumulants of returns

We start our theoretical study by computing the moments of the log–price process S. Since these moments depend not only on the moments of X but also on those of σ, we calculate the moments of σ first. In order to do that, we derive a general representation formula for the nth power of σ, for $n \in \mathbb{N}$.

Proposition 4.1 *Let $n \in \mathbb{N}$. As long as $\int_0^t \sigma_s^n ds < \infty$, the n–th power of σ_t satisfies*

$$\sigma_t^n - \sigma_0^n = -\lambda\, n \int_0^t \sigma_{s-}^n ds + \sum_{k=1}^n \binom{n}{k} \sum_{0 \le s \le t} \sigma_{s-}^{n-k} \left(\Delta Y_{\lambda s}\right)^k .$$

Proof. Given in the Appendix. From the formula above, one can deduce the moments of σ.

Corollary 4.2 *Recall that $\eta_i = \kappa_i(Y_1)$. The first four moments of the stationary distribution of σ are, hence, given by: $\mathbb{E}\left(\sigma_t\right) = \eta_1$, $\mathbb{E}\left(\sigma_t^2\right) = \eta_1^2 + \frac{1}{2}\eta_2$, $\mathbb{E}\left(\sigma_t^3\right) = \eta_1^3 + \frac{1}{3}\eta_3 + \frac{3}{2}\eta_2\eta_1$, $\mathbb{E}\left(\sigma_t^4\right) = \frac{1}{4}\eta_4 + \eta_1^4 + 3\eta_2\eta_1^2 + \frac{4}{3}\eta_1\eta_3 + \frac{3}{4}\eta_2^2$.*

Now we focus on the moments of the price process S and, also, on its joint moments with the volatility process σ. It turns out that by repeated applications of Itô's formula and the use of the compensation formulas for jump processes, we obtain a recursive system of inhomogeneous ordinary differential equations, which can be solved explicitely. This methodology is described in detail in the Appendix and will be used extensively in the remaining paper. So although our model is generally not affine and, hence, might look complicated to tackle at first sight – at least compared to the affine models of time changed Lévy processes (see e.g. [39]) – the new methodology proposed in this paper enables us to derive all cumulants of interest explicitely.

Proposition 4.3 (Recursive formulae for the moments of S)
Let $k, n \in \mathbb{N}$, with $k \le n$. If $\eta_k < \infty$ for $k \le n$, we get the following results.

1. *The joint moments of S and σ are given by*

$$\mathbb{E}\left(S_t^{n-k}\sigma_t^k\right) = e^{-k\lambda t} \int_0^t g(u;n,k)\, e^{k\lambda u} du,$$

where

$$g(u;n,k) = \sum_{j=1}^k \binom{k}{j} \lambda\eta_j \int_0^t \mathbb{E}\left(S_{s-}^{n-k}\sigma_{s-}^{k-j}\right) ds + \frac{(n-k)(n-k-1)v^2}{2} \int_0^t \mathbb{E}\left(S_s^{n-k-2}\sigma_s^2\right) ds$$
$$+ \sum_{j=2}^{n-k} \binom{n-k}{j} \xi_j \int_0^t \mathbb{E}\left(\sigma_s^{k+j} S_s^{n-k-j}\right) + \sum_{j=0}^{n-1} \binom{n}{j} \xi_{n-j} \int_0^t \mathbb{E}\left(S_{s-}^j \sigma_{s-}^{m+n-j}\right) ds$$
$$+ \sum_{j=1}^{n-k} \sum_{l=1}^{k} \binom{n-k}{j} \binom{k}{l} \kappa_{j,l} \int_0^t \mathbb{E}\left(S_s^{n-k-j}\sigma_s^{j+k-l}\right) ds.$$

2. *The moments of S are given by*

$$\mathbb{E}\left(S_t^n\right) = \frac{n(n-1)}{2} v^2 \int_0^t \mathbb{E}\left(S_{s-}^{n-2}\sigma_{s-}^2\right) ds + \sum_{k=2}^n \binom{n}{k} \xi_k \int_0^t \mathbb{E}\left(\sigma_{s-}^k S_{s-}^{n-k}\right).$$

Proof. Given in the Appendix.

Now we can recursively solve the equation above and we obtain the corresponding moments of S.

Corollary 4.4 *If $\xi_1 = 0$, the first two moments are given by:*

$$\mathbb{E}\left(S_t\right) = 0, \qquad \mathbb{E}\left(S_t^2\right) = \left(v^2 + \xi_2\right)\left(\eta_1^2 + \frac{\eta_2}{2}\right) t.$$

Using a Taylor expansion around 0 for moments of higher order, one obtains for $t \downarrow 0$:

$$\begin{aligned}
\mathbb{E}\left(S_t^3\right) &= \xi_3\left(\eta_1^3 + \frac{1}{3}\eta_3 + \frac{3}{2}\eta_1\eta_2\right)\ t + \frac{3}{2}\left(v^2+\xi_2\right)\left(\kappa_{1,1}(2\eta_1^2+\eta_2) + \kappa_{1,2}\eta_1\right) t^2 + O(t^3),\\
\mathbb{E}\left(S_t^4\right) &= \xi_4\left(\frac{1}{4}\eta_4 + \eta_1^4 + \frac{3}{4}\eta_2^2 + 3\eta_1^2\eta_2 + \frac{4}{3}\eta_1\eta_3\right) t\\
&\quad + \left\{\left(3\eta_1^4 + 4\eta_1\eta_3 + \frac{3}{4}\eta_4 + 9\eta_1^2\eta_2 + \frac{9}{4}\eta_2^2\right)\left(v^2+\xi_2\right)^2\right.\\
&\quad + \left(\left(6\eta_1^3 + 2\eta_3 + 9\eta_1\eta_2\right)\kappa_{2,1} + \left(3\eta_1^2 + \frac{3}{2}\eta_2\right)\kappa_{2,2}\right)\left(v^2+\xi_2\right)\\
&\quad \left. + \xi_3\ \left(\left(6\eta_1^3 + 9\eta_2\eta_1 + 2\eta_3\right)\kappa_{1,1} + \left(3\eta_2 + 6\eta_1^2\right)\kappa_{1,2} + 2\eta_1\kappa_{1,3}\right)\right\} t^2 + O(t^3).
\end{aligned}$$

Further, one obtains

$$\begin{aligned}
Var(S_t^2) &= \xi_4\left(\frac{1}{4}\eta_4 + \frac{3}{4}\eta_2^2 + 3\eta_1^2\eta_2 + \frac{4}{3}\eta_1\eta_3 + \eta_1^4\right) t\\
&\quad + \left\{\left(8\eta_1^2\eta_2 + 2\eta_2^2 + 2\eta_1^4 + 4\eta_1\eta_3 + \frac{3}{4}\eta_4\right)\left(v^2+\xi_2\right)^2\right.\\
&\quad + \left(\left(6\eta_1^3 + 2\eta_3 + 9\eta_1\eta_2\right)\kappa_{2,1} + \left(3\eta_1^2 + \frac{3}{2}\eta_2\right)\kappa_{2,2}\right)\left(v^2+\xi_2\right)\\
&\quad \left. +\xi_3\left(\left(6\eta_1^3 + 9\eta_2\eta_1 + 2\eta_3\right)\kappa_{1,1} + \left(3\eta_2 + 6\eta_1^2\right)\kappa_{1,2} + 2\kappa_{1,3}\eta_1\right)\right\} t^2 + O(t^3).
\end{aligned}$$

Since S has stationary increments, we can deduce the moments of the corresponding returns s_i over a time interval of length h by setting $t = h$ and, hence,

$$\mathbb{E}(s_i) = \mathbb{E}(S_h), \quad \mathbb{E}(s_i^2) = \mathbb{E}(S_h^2), \quad \mathbb{E}(s_i^3) = \mathbb{E}(S_h^3), \quad \mathbb{E}(s_i^4) = \mathbb{E}(S_h^4), \quad Var(s_i^2) = Var(S_h^2).$$

4.2 First and second order properties of the actual variance

Recent research has focused on integrated variance as a measure for the variability of financial markets. The integrated variance is defined by $IV_t = \int_0^t \sigma_s^2 ds$. Often, one is interested in studying the increments of this process over a time interval of length h, say. So, we will denote these increments by

$$\sigma^2_{[(i-1)h,ih]} = IV_{ih} - IV_{(i-1)h} = \int_{(i-1)h}^{ih} \sigma_s^2 ds,$$

which is generally called the *actual variance* (AV) on the ith interval of length h. Basically, it measures the accumulated variance over a time interval (often chosen to be one day). Now we can compute the mean, variance and covariance of the AV as given in the following proposition.

Proposition 4.5 *The mean, variance and covariance of the actual variance are given by the following formulae:*

$$\mathbb{E}(\sigma^2_{[(i-1)h,ih]}) = \left(\eta_1^2 + \frac{\eta_2}{2}\right) h,$$

and

$$Var(\sigma^2_{[(i-1)h,ih]}) = \frac{1}{\lambda^2}\left\{\left(\frac{1}{2}\eta_2^2 + 4\eta_1^2\eta_2 + 2\eta_1\eta_3 + \frac{1}{4}\eta_4\right)\lambda\, h\right.$$
$$\left. + \left(\frac{4}{3}\eta_1\eta_3 + 4\eta_1^2\eta_2\right)e^{-\lambda h} + \left(\frac{1}{4}\eta_2^2 + \frac{1}{8}\eta_4 + \frac{1}{3}\eta_1\eta_3\right)e^{-2\lambda h} - \frac{1}{8}\eta_4 - 4\eta_1^2\eta_2 - \frac{5}{3}\eta_1\eta_3 - \frac{1}{4}\eta_2^2\right\},$$

and

$$\begin{aligned} Cov(\sigma^2_{[(i-1)h,ih]}, \sigma^2_{[(i+s-1)h,(i+s)h]}) \\ = \frac{1}{\lambda^2}\left\{\left(2\eta_1^2\eta_2 + \frac{2}{3}\eta_1\eta_3\right)\left(e^{-\lambda(s+1)h} + e^{-\lambda(s-1)h} - 2e^{-\lambda hs}\right)\right. \\ \left. + \left(\frac{1}{8}\eta_2^2 + \frac{1}{16}\eta_4 + \frac{1}{6}\eta_1\eta_3\right)\left(e^{-2\lambda(s+1)h} + e^{-2\lambda(s-1)h} - 2e^{-2\lambda hs}\right)\right\}. \end{aligned}$$

Proof. Given in the Appendix.

As already mentioned above, in a SV model based on a Brownian motion, the actual variance can be consistently estimated by the realised variance. However, in a more general Lévy–based model, the quadratic variation of the price process does not equal the integrated variance. Hence, we will turn our attention to the quadratic variation of the log–price process and study its first and second order properties. We will then be able to compare those with the results we have just obtained for the integrated variance.

4.3 First and second order properties of the quadratic variation

The first and second order properties of the quadratic variation are described in the following proposition.

Proposition 4.6 *Let $t, s > 0$ and $\xi_1 = 0$. Then:*

$$\begin{aligned} \mathbb{E}\left([S]_t\right) &= \left(v^2 + \xi_2\right)\left(\eta_1^2 + \frac{1}{2}\eta_2\right)t, \\ Var([S]_t) &= c_1 + c_2 t + c_3 e^{-\lambda\, t} + c_4 e^{-2\,\lambda\, t}, \\ Cov([S]_t, [S]_{t+s}) &= \frac{1}{2}\, c_1 + c_2 t + \frac{1}{2}\, c_3(e^{-\lambda t} - e^{-\lambda s} + e^{-\lambda(t+s)}) \\ &\quad + \frac{1}{2}\, c_4(e^{-2\lambda t} - e^{-2\lambda s} + e^{-2\lambda(t+s)}), \end{aligned}$$

where $c_i = c_i(\lambda, v, \xi_2, \xi_4, \eta_1, \eta_2, \eta_3, \eta_4, \kappa_{2,1}, \kappa_{2,2})$ for $i = 1, \ldots, 4$, with

$$\begin{aligned} c_1 &= \frac{1}{24\lambda^2}\left[\left(-40\eta_1\eta_3 - 96\eta_1^2\eta_2 - 3\eta_4 - 6\eta_2^2\right)\left(v^2 + \xi_2\right)^2\right. \\ &\quad \left. + \left\{\left(-8\eta_3 - 72\eta_1\eta_2 - 96\eta_1^3\right)\kappa_{2,1} + \left(-6\eta_2 - 12\eta_1^2\right)\kappa_{2,2}\right\}\left(v^2 + \xi_2\right)\right], \\ c_2 &= \frac{1}{24\lambda^2}\left[\left(48\eta_1\eta_3 + 12\eta_2^2 + 96\eta_1^2\eta_2 + 6\eta_4\right)\lambda\left(v^2 + \xi_2\right)^2\right. \\ &\quad + \left\{\left(96\eta_1\eta_2 + 96\eta_1^3 + 16\eta_3\right)\kappa_{2,1} + \left(24\eta_1^2 + 12\eta_2\right)\kappa_{2,2}\right\}\lambda\left(v^2 + \xi_2\right) \\ &\quad \left. + \left(24\eta_1^4 + 18\eta_2^2 + 6\eta_4 + 72\eta_1^2\eta_2 + 32\eta_1\eta_3\right)\lambda^2\xi_4\right], \end{aligned}$$

$$c_3 = \frac{1}{24\lambda^2}\left\{\left(96\eta_1^2\eta_2 + 32\eta_1\eta_3\right)\left(v^2+\xi_2\right)^2 + \left(48\eta_1\eta_2 + 96\eta_1^3\right)\kappa_{2,1}\left(v^2+\xi_2\right)\right\}$$
$$c_4 = \frac{1}{24\lambda^2}\Big[\left(6\eta_2^2 + 3\eta_4 + 8\eta_1\eta_3\right)\left(v^2+\xi_2\right)^2 + \left\{\left(8\eta_3 + 24\eta_1\eta_2\right)\kappa_{2,1} + \left(12\eta_1^2 + 6\eta_2\right)\kappa_{2,2}\right\}\left(v^2+\xi_2\right)\Big].$$

From this proposition, we can easily deduce the first and second order properties for the *incremental quadratic variation* (IQV):

$$[s]_{[(i-1)h,ih]} = [S]_{ih} - [S]_{(i-1)h} = \int_{(i-1)h}^{ih} \sigma_{u-}^2 d[vW+X]_u.$$

They are given by the following.

Theorem 4.7 *Let $t,s>0$ and $\xi_1=0$ and $v^2+\xi_2=1$. Then:*

$$\mathbb{E}([s]_{[(i-1)h,ih]}) = \mathbb{E}(\sigma^2_{[(i-1)h,ih]}),$$
$$Var([s]_{[(i-1)h,ih]}) = Var(\sigma^2_{[(i-1)h,ih]}) + \xi_4\left(3\eta_1^2\eta_2 + \frac{3}{4}\eta_2^2 + \eta_1^4 + \frac{4}{3}\eta_1\eta_3 + \frac{1}{4}\eta_4\right)h$$
$$+\frac{1}{\lambda^2}\Big[\kappa_{2,1}\left(4\eta_1^3 + 2\eta_1\eta_2\right)\left(e^{-\lambda h} - 1 + \lambda h\right) + \left\{\kappa_{2,1}\left(\eta_1\eta_2 + \frac{1}{3}\eta_3\right) + \frac{1}{2}\kappa_{2,2}\left(\eta_1^2 + \frac{1}{2}\eta_2\right)\right\}\left(e^{-2\lambda h} - 1 + 2\lambda h\right)\Big],$$

and

$$Cov([s]_{[(i-1)h,ih]}, [s]_{[(i+s-1)h,(i+s)h]}) = Cov(\sigma^2_{[(i-1)h,ih]}, \sigma^2_{[(i+s-1)h,(i+s)h]})$$
$$+\frac{1}{\lambda^2}[\kappa_{2,1}\left(2\eta_1^3 + \eta_1\eta_2\right)(e^{-\lambda((s+1)h)} - 2e^{-\lambda hs} + e^{-\lambda((s-1)h)})$$
$$+\left\{\kappa_{2,1}\left(\frac{1}{6}\eta_3 + \frac{1}{2}\eta_1\eta_2\right) + \kappa_{2,2}\frac{1}{4}\left(\eta_1^2 + \frac{1}{2}\eta_2\right)\right\}\cdot (e^{-2\lambda(s+1)h} - 2e^{-2\lambda hs} + e^{-2\lambda(s-1)h})].$$

When we compare the first and second order properties of the AV with those of the IQV, we observe the following. Firstly, in the variance of IQV there is an extra summand given by $\xi_4\mathbb{E}\sigma^4 h$. Note here that we have assumed that X is a pure jump Lévy process without drift. Hence, $\nu_{X_1}\neq 0$ and, hence, $\xi_4 = \int_{\mathbb{R}} x^4\nu_{X_1}(dx) > 0$, so this factor will never disappear. Secondly, both the variance and the covariance of the IQV have an extra term which is due to a possible leverage–type effect in the model. Clearly, in the absence of this effect (i.e. when X and Y are independent), then $\kappa_{2,1}=\kappa_{2,2}=0$, and hence this extra term would not exist. So, altogether, we can say that, by choosing a pure jump Lévy process as a driving process for the asset price, we observe an extra term in the variance of the IQV compared to the variance of the AV. If one additionally allows for a leverage type effects, both the variance and the covariance of the IQV have to be generalised by an additional leverage term.

4.4 Covariation of returns

We conclude this section by studying the covariance between returns, squared returns and IQV. We will consider returns over a time interval of length h, which are denoted by $s_i = S_{ih} - S_{(i-1)h}$.

Proposition 4.8 *Let $i, s \in \mathbb{N}$. For $h \to 0$, one obtains with $\xi_1 = 0$:*

$$
\begin{aligned}
Cov(s_i, s_{i+s}) &= 0,\\
Cov(s_i, s_{i-s}^2) &= 0,\\
Cov(s_i, s_{i+s}^2) &= \frac{1}{4\lambda^2}\Big\{8\eta_1^2\kappa_{1,1}\left(e^{-\lambda(s+1)h} - 2e^{-\lambda hs} + e^{-\lambda(s-1)h}\right)\\
&\quad + (\kappa_{1,1}\eta_2 + \kappa_{1,2}\eta_1)\left(e^{-2\lambda(s+1)h} - 2e^{-2\lambda hs} + e^{-2\lambda(s-1)h}\right)\Big\}\left(v^2 + \xi_2\right)\\
Cov(s_i^2, s_{i+s}^2) &= Cov([s]_{[(i-1)h, ih]}, [s]_{[(i+s-1)h,(i+s)h]})\\
&\quad + \kappa_{1,1}\left(2(\eta_2 + 2\eta_1^2)\kappa_{1,1} + 3\kappa_{1,2}\eta_1\right)\left(v^2 + \xi_2\right)h^3 + O(h^4).
\end{aligned}
$$

Besides,

$$
Cov(s^2_{[(i-1)h,ih]} - [s]_{[(i-1)h,ih]}, s^2_{[(i+s-1)h,(i+s)h]} - [s]_{[(i+s-1)h,(i+s)h]}) = 0.
$$

Proof. Given in the Appendix.

So we see that the asset returns are uncorrelated (under the assumption that $\xi_1 = 0$). Further, we observe that the covariance between returns and squared returns basically depends on the two leverage parameters $\kappa_{1,1},\ \kappa_{2,1}$, which denote the covariation between X_1 and Y_λ and the joint centred moment of X_1^2 and Y_λ, respectively. Recall that the squared returns can also be used for estimating the variance (although such an estimate is noisier than one based on realised variance). This covariation will damp down exponentially with the lag length s and so will the influence of a possible leverage effect. Finally, we observe that the covariance between squared returns can be approximated by the covariance between the IQV and by terms of lower order which depend on parameters κ of possible leverage .

5 First and second order properties of the realised variance

Finally, we can apply the results we have deduced so far for computing the first and second order properties of the realised variance (RV) and for studying the degree of inconsistence of the RV as estimator for the integrated variance.

5.1 First and second order properties of Realised Variance Error

Let

$$
H_t = S_t^2 - IV_t = S_t^2 - \int_0^t \sigma_s^2 ds.
$$

Proposition 5.1 *Let $t, s > 0$ and $\xi_1 = 0$ and $v^2 + \xi_2 = 1$. Then (for $t \to 0$):*

$$
\begin{aligned}
\mathbb{E}(H_t) &= 0,\\
\mathbb{E}(H_t^2) &= (\xi_4 - 2)\left(\frac{1}{4}\eta_4 + \frac{3}{4}\eta_2^2 + 3\eta_1^2\eta_2 + \frac{4}{3}\eta_1\eta_3 + \eta_1^4\right)t\\
&\quad + \Big\{\left(\left(6\eta_1^3 + 2\eta_3 + 9\eta_1\eta_2\right)\kappa_{1,1} + \left(6\eta_1^2 + 3\eta_2\right)\kappa_{1,2} + 2\kappa_{1,3}\eta_1\right)\xi_3\\
&\quad + \left(\frac{4}{3}\eta_3 + 4\eta_1^3 + 6\eta_1\eta_2\right)\kappa_{2,1} + \left(2\eta_1^2 + \eta_2\right)\kappa_{2,2} + 3\eta_1^4 + 4\eta_1\eta_3 + 9\eta_1^2\eta_2 + \frac{9}{4}\eta_2^2 + \frac{3}{4}\eta_4\Big\}t^2
\end{aligned}
$$

Proof. Given in the Appendix.

Since both S^2 and IV are stationary, we can easily deduce the results for the corresponding increments of the returns and the IV by setting $t = h$ and, hence,

$$\mathbb{E}\left(s_i^2 - \sigma^2_{[(i-1)h,ih]}\right) = \mathbb{E}\left(H_h\right), \qquad \mathbb{E}\left(\left(s_i^2 - \sigma^2_{[(i-1)h,ih]}\right)^2\right) = \mathbb{E}\left(H_h^2\right).$$

So we observe that even in the presence of leverage the expectation of H is zero and, hence, there is no bias. However, leverage–type effects (of higher order) do affect the mean square error. We start by studying the properties of the difference between the squared log–price process and the quadratic variation.

Proposition 5.2 *Let $t, s > 0$ and $\xi_1 = 0$. Define $G_t = S_t^2 - [S]_t$. Then (for $t \to 0$):*

$$\begin{aligned}
\mathbb{E}(G_t) &= 0,\\
Var(G_t) &= \left[\left(\frac{8}{3}\eta_1\eta_3 + 2\eta_1^4 + \frac{1}{2}\eta_4 + \frac{3}{2}\eta_2^2 + 6\eta_1^2\eta_2\right)\left(v^2+\xi_2\right)^2\right.\\
&\quad \left. + \left\{\left(\frac{4}{3}\eta_3 + 4\eta_1^3 + 6\eta_1\eta_2\right)\kappa_{2,1} + \left(2\eta_1^2 + \eta_2\right)\kappa_{2,2}\right\}\left(v^2+\xi_2\right)\right] t^2 + O(t^3),\\
Cov(G_t, [S]_t) &= \xi_3\left\{\kappa_{1,1}\left(3\eta_1^3 + \eta_3 + \frac{9}{2}\eta_1\eta_2\right) + \kappa_{1,2}\left(3\eta_1^2 + \frac{3}{2}\eta_2\right) + \kappa_{1,3}\eta_1\right\} t^2 + O(t^3).
\end{aligned}$$

Proof. Given in the Appendix.

Recall that the realised variance error (when estimating the IQV by the RV) is given by

$$[S_\delta]_{[(i-1)h,ih]} - \sigma^2_{[(i-1)h,ih]} = \sum_{j=1}^{M}(s_{j,i}^2 - \sigma_{j,i}^2).$$

Using

$$s_{j,i} \stackrel{L}{=} S_\delta \qquad \text{and} \qquad [s]_{j,i} \stackrel{L}{=} [S]_\delta,$$

we obtain the following result for the squared returns and the IQV.

Corollary 5.3 *Let $\xi_1 = 0$ and $v^2 + \xi_2 = 1$. Then:*

$$\begin{aligned}
&\mathbb{E}([S_\delta]_{[(i-1)h,ih]} - [s]_{[(i-1)h,ih]}) = 0,\\
&Var([S_\delta]_{[(i-1)h,ih]} - [s]_{[(i-1)h,ih]}) = \left[\left(\frac{8}{3}\eta_1\eta_3 + 2\eta_1^4 + \frac{1}{2}\eta_4 + \frac{3}{2}\eta_2^2 + 6\eta_1^2\eta_2\right)\right.\\
&\qquad \left. + \left\{\kappa_{2,1}\left(\frac{4}{3}\eta_3 + 4\eta_1^3 + 6\eta_1\eta_2\right) + \kappa_{2,2}\left(2\eta_1^2 + \eta_2\right)\right\}\right] h^2 M^{-1} + O(M^{-2}),
\end{aligned}$$

and

$$Cov([S_\delta]_{[(i-1)h,ih]} - [s]_{[(i-1)h,ih]}, [S_\delta]_{[(i+s-1)h,i+sh]} - [s]_{[(i+s-1)h,(i+s)h]}) = 0.$$

So we observe the following. RV is an unbiased estimate for IQV; the variance of the RV error is of O(M^{-1}) and the RV errors are uncorrelated. These findings correspond to similar results in the Brownian motion case (see [10]).

5.2 Cumulants of realised variance

Using the results above, we can now derive the mean, variance and covariance of the realised variance.

Proposition 5.4 *Let $\xi_1 = 0$ and $v^2 + \xi_2 = 1$. The first and second order properties of the realised variance are then given by*

$$\mathbb{E}[S_\delta]_{[(i-1)h,ih]} = \left(\eta_1^2 + \frac{1}{2}\eta_2\right) h,$$

and

$$\begin{aligned} Var([S_\delta]_{[(i-1)h,ih]}) &= Var([s]_{[(i-1)h,ih]}) + \left[\left(\frac{8}{3}\eta_1\eta_3 + 2\eta_1^4 + \frac{1}{2}\eta_4 + \frac{3}{2}\eta_2^2 + 6\eta_1^2\eta_2\right)\right. \\ &+ \left\{2\frac{\kappa_{1,1}^2}{\lambda}\left(\eta_2 + 3\eta_1^2\right) + 3\frac{\kappa_{1,2}\kappa_{1,1}}{\lambda}\eta_1 + \kappa_{2,1}\left(\frac{4}{3}\eta_3 + 4\eta_1^3 + 6\eta_1\eta_2\right) + \kappa_{2,2}\left(2\eta_1^2 + \eta_2\right)\right\} \\ &\left. + \xi_3\left\{\kappa_{1,1}\left(6\eta_1^3 + 2\eta_3 + 9\eta_1\eta_2\right) + \kappa_{1,2}\left(6\eta_1^2 + 3\eta_2\right) + 2\kappa_{1,3}\eta_1\right\}\right] h^2 M^{-1} \\ &+ \left[\frac{\kappa_{1,1}^2}{\lambda^2}\left\{\eta_1^2\left(e^{-2\lambda h} + 4e^{-\lambda h} - 5\right) + \eta_2\left(e^{-2\lambda h} - 1\right)\right\}\right. \\ &\left. + \frac{3}{2\lambda^2}\kappa_{1,2}\kappa_{1,1}\eta_1\left(-1 + e^{-2\lambda h}\right)\right] hM^{-1} + O\left(M^{-2}\right), \end{aligned}$$

and

$$\begin{aligned} Cov([S_\delta]_{[(i-1)h,ih]}, [S_\delta]_{[(i+s-1)h,(i+s)h]}) &= Cov([s]_{[(i-1)h,ih]}, [s]_{[(i+s-1)h,(i+s)h]}) \\ &+ \frac{\kappa_{1,1}}{4\lambda^2}\left[8\kappa_{1,1}\eta_1^2\left(-2e^{-\lambda hs} + e^{-\lambda(s-1)h} + e^{-\lambda(s+1)h}\right)\right. \\ &\left. + \left\{2\kappa_{1,1}(\eta_1^2 + \eta_2) + 3\kappa_{1,2}\eta_1\right\}\left(-2e^{-2\lambda hs} + e^{-2\lambda(s+1)h} + e^{-2\lambda(s-1)h}\right)\right] hM^{-1} + O\left(M^{-2}\right). \end{aligned}$$

Proof. Given in the Appendix.

5.3 Comparing the autocorrelation functions of realised variance, quadratic variation and integrated variance

Now we briefly study some implications of our results for autocorrelations of RV, QV, and IV. Hereby we follow [12], who have studied the same question in the framework of a time–changed Lévy process. From our results above, we can deduce that:

$$\begin{aligned} \lim_{M\to\infty} Cor\left([S_\delta]_{[(i-1)h,ih]}, [S_\delta]_{[(i+s-1)h,(i+s)h]}\right) &= Cor\left([s]_{[(i-1)h,ih]}, [s]_{[(i+s-1)h,(i+s)h]}\right) \\ &= \frac{Cov(\sigma^2_{[(i-1)h,ih]}, \sigma^2_{[(i+s-1)h,(i+s)h]}) + L_C}{Var\left(\sigma^2_{[(i-1)h,ih]}\right) + \xi_4\,\mathbb{E}\sigma^4 + L_V}, \end{aligned}$$

where the leverage part in the covariance is denoted by

$$\begin{aligned} L_C = \frac{1}{\lambda^2}&\left[\kappa_{2,1}\left(2\eta_1^3 + \eta_1\eta_2\right)\left(e^{-\lambda((s+1)h)} - 2e^{-\lambda hs} + e^{-\lambda((s-1)h)}\right)\right. \\ &+ \left\{\kappa_{2,1}\left(\frac{1}{6}\eta_3 + \frac{1}{2}\eta_1\eta_2\right) + \kappa_{2,2}\frac{1}{4}\left(\eta_1^2 + \frac{1}{2}\eta_2\right)\right\} \end{aligned}$$

$$\left(e^{-2\lambda(s+1)h} - 2e^{-2\lambda hs} + e^{-2\lambda(s-1)h}\right)\Big],$$

and the leverage part in the variance is denoted by

$$L_V = \frac{1}{\lambda^2}\Big[\kappa_{2,1}\left(4\eta_1^3 + 2\eta_1\eta_2\right)\left(e^{-\lambda h} - 1 + \lambda h\right) + \left\{\kappa_{2,1}\left(\eta_1\eta_2 + \frac{1}{3}\eta_3\right) + \frac{1}{2}\kappa_{2,2}\left(\eta_1^2 + \frac{1}{2}\eta_2\right)\right\}\left(e^{-2\lambda h} - 1 + 2\lambda h\right)\Big].$$

In the absence of leverage type effects (hence $L_C = L_V = 0$), we obtain exactly the same results as derived by [12] for time–changed Lévy processes:

- The acf of the RV is monotonically decreasing in ξ_4.
- For $M \to \infty$, the acf of the RV is given by

$$\lim_{M\to\infty} Cor\left([S_\delta]_{[(i-1)h,ih]}, [S_\delta]_{[(i+s-1)h,(i+s)h]}\right) = \frac{Cov(\sigma^2_{[(i-1)h,ih]}, \sigma^2_{[(i+s-1)h,(i+s)h]})}{Var\left(\sigma^2_{[(i-1)h,ih]}\right) + \xi_4\,\mathbb{E}\sigma^4}$$
$$< Cor\left(\sigma^2_{[(i-1)h,ih]}, \sigma^2_{[(i+s-1)h,(i+s)h]}\right), \quad \text{since } \xi_4 > 0.$$

  which implies that the ACF of RV systematically underestimates the ACF of the actual variance.
- And for $\xi_4 \to \infty$, we obtain

$$\lim_{\xi_4\to\infty} Cor\left([S_\delta]_{[(i-1)h,ih]}, [S_\delta]_{[(i+s-1)h,(i+s)h]}\right) = 0.$$

However, if we allow for leverage–type effects, we observe the following for the acf of the RV:

- Dependencies between X and (higher) moments of Y (i.e. $\kappa_{1,1},\ \kappa_{1,2},\ \kappa_{1,3}$) are asymptotically negligible. In particular, the quantity $\kappa_{1,1}$, which describes the *classical* leverage effect, has asymptotically no influence on the acf of the RV.
- Dependencies between X^2 and (higher) moments of Y (i.e. $\kappa_{2,1},\ \kappa_{2,2}$) do influence the acf of the RV.

5.4 Superposition model

Let us briefly mention a method for generalising our model slightly. Many empirical studies have indicated that one–factor stochastic volatility models cannot fit empirical data very satisfactorily. Hence, a standard approach for tackling this problem is to study at least a two–factor (or a multi–factor) stochastic volatility model (see e.g. [14], [18]). Often, one uses the class of so–called superposition models where the volatility is not just given by a single OU process (as in our modelling framework), but rather by a convex combination of independent OU processes (see e.g. [7], [10] and the references therein). We assume that the volatility process is given by a weighted sum of independent OU processes. For $J \in \mathbb{N}$ and $i = 1..J$, let $w_i \geq 0$ and $\sum_{i=1}^J w_i = 1$. Then we define

$$\sigma_t = \sum_{i=1}^{J} w_i \tau_t^{(i)}, \qquad d\tau^{(i)} = -\lambda^{(i)}\tau_t^{(i)}dt + dY^{(i)}_{\lambda^{(i)}t},$$

where we assume that the Y_i are independent (but not necessarily identically distributed). However, as in the one–factor model, we allow for dependence between X_t and $Y^{(i)}_{\lambda^{(i)}\ t}$. In particular, since X is a Lévy process, there is a sequence of independent identically distributed random variables $X_{J,k}$ for $k = 1, \ldots, J$ such that $X \stackrel{\text{law}}{=} X_{J,1} + \cdots + X_{J,J}$. So, we write

$$\kappa^{(i)}_{n,m} = \int_{\mathbb{R}\times\mathbb{R}_+} u^n\ v^m \nu_{X_{J,i},Y^{(i)}}(du, dv),$$

for the corresponding cumulants of the bivariate Lévy process. The mean, variance and covariance of the realised variance when the volatility process is given by a superposition model can be derived in a similar way as in the one–factor model. However, due to the fact that we allow for dependencies between the driving process of the asset price and the driving processes of the different components of σ, these formulae become rather lengthy. So, we will just present them in the Appendix.

6 Model estimation and inference

6.1 Quasi–likelihood estimation based on realised variance

Finally, we turn our attention to estimating the parameters of our Lévy –driven stochastic volatility model. It is well–known that parameter estimation in such a model framework is difficult since one cannot easily compute the exact likelihood function. Here we follow [12] and use a quasi–maximum likelihood approach (see e.g. [30, Chapter 5]) based on the Gaussian density function. This methodology leads to consistent and asymptotically normally distributed set of estimators. Alternative estimation techniques include method of moment (e.g. [19]) and simulation based methods. For instance, independent work by [46], [29] and [32] have focused on Markov chain Monte Carlo methodology for Bayesian inference in OU stochastic volatility models. Recall that we have shown that we can write the mean, the variance and the covariance of the vector of realised variances $[\underline{S_\delta}] = ([S_\delta]_1, \ldots, [S_\delta]_n)'$ as function of the model parameters, which we write in terms of a vector θ, say. We choose the following quasi–maximum likelihood (QML) approach for estimating the parameters. Let

$$\begin{aligned} l(\theta) = \log L(\theta) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\log\det(Cov([\underline{S_\delta}])) \\ - \frac{1}{2}([\underline{S_\delta}] - \mathbb{E}([\underline{S_\delta}]))'(Cov([\underline{S_\delta}]))^{-1}([\underline{S_\delta}] - \mathbb{E}([\underline{S_\delta}])) \end{aligned} \quad (1)$$

denote the Gaussian realised quasi–likelihood function and let $\hat{\theta} = arg \max_\theta \log l(\theta)$ denote the QML estimate. In order to find this estimate, one has to compute the inverse and the determinant of the RV vector, which would be in general an operation of order O(n^3). However, since σ is stationary, $[S_\delta]$ is itself stationary. Hence, $Cov([\underline{S_\delta}])$ is a Toeplitz matrix, which can be inverted by using the Levinson–Durbin algorithm (see [42] and [27]) in O(n^2). Basically, one uses a Choleski decomposition of the covariance matrix (see e.g. [25]) with $Cov([\underline{S_\delta}]) = LDL' = PP'$, where L is lower diagonal, with ones on the diagonal and D is a diagonal matrix with the variances of the residuals (which are denoted by E) as entries. So the likelihood function (1) can be written as

$$l(\theta) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\ \log(det(D)) - \frac{1}{2}\ E'E,$$

where the residuals E are given by

$$E = D^{-1/2}L^{-1}\left([\underline{S_\delta}] - \mathbb{E}\left([\underline{S_\delta}]\right)\right) = P^{-1}\left([\underline{S_\delta}] - \mathbb{E}\left([\underline{S_\delta}]\right)\right).$$

Remark We can express the likelihood function in terms of the mean, variance and covariance of the linear predictions of the RV. Assume that f denotes the joint density of the time series of the RV. Straightforwardly, we get

$$f([S_\delta]_1, \ldots, [S_\delta]_n) = f([S_\delta]_1) \prod_{i=2}^{n} f([S_\delta]_i \mid [S_\delta]_{i-1}, \ldots, [S_\delta]_1).$$

Now let $\mathbb{E}_L(y_i|\mathcal{F}_{i-1})$ and $Var_L(y_i|\mathcal{F}_{i-1})$ denote the mean and the variance of the linear prediction. In order to construct a quasi–likelihood, we assume that f is given by a Gaussian density and hence

$$l(\theta) = \log L(\theta, y) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{n} \log Var_L([S_\delta]_i|\mathcal{F}_{i-1}) - \frac{1}{2}\sum_{i=1}^{n} \frac{([S_\delta]_i - \mathbb{E}_L([S_\delta]_i|\mathcal{F}_{i-1}))^2}{Var_L([S_\delta]_i|\mathcal{F}_{i-1})}.$$

So we observe that the entries in the diagonal matrix D in the Choleski decomposition are exactly the variances of the best linear, unbiased one–step ahead forecast of the RV.

So far, we have only discussed how the model parameters can be estimated. However, in the remaining part of this section we will briefly describe how one can make inference on the model parameters. Let

$$\mathcal{J} = \lim_{n\to\infty} \frac{1}{n} Cov\left(\frac{\partial l(\theta)}{\partial \theta}\right), \qquad \text{and} \qquad \mathcal{I} = \lim_{n\to\infty} -\frac{1}{n}\mathbb{E}\left(\frac{\partial^2 l(\theta)}{\partial\theta\partial\theta'}\right).$$

It is well–known (see e.g. [30]) that not only the maximum likelihood estimator but also the QML estimator is asymptotically normally distributed with an adjusted covariance matrix (compared to the MLE setting) and hence

$$\sqrt{n}(\hat{\theta} - \theta) \stackrel{d}{\to} N(0, \mathcal{I}^{-1}\mathcal{J}\,\mathcal{I}^{-1}).$$

Based on this asymptotic result, we can construct 95 % confidence intervals for θ, which are of the form

$$\hat{\theta} - \frac{1.96}{\sqrt{n}}\left(\mathcal{I}^{-1}\mathcal{J}\mathcal{I}^{-1}\right)^{1/2} \leq \theta \leq \hat{\theta} + \frac{1.96}{\sqrt{n}}\left(\mathcal{I}^{-1}\mathcal{J}\mathcal{I}^{-1}\right)^{1/2},$$

where the square root of a positive (semi-)definite matrix Σ, say, is defined by the matrix $\Sigma^{1/2}$ such that $\Sigma^{1/2}\Sigma^{1/2\,\prime} = \Sigma$. Estimating the so–called sandwich matrices I, which only appear in a QML setting and accounting for the fact, that the estimation was not based on the true density function, does not cause any problems, whereas estimating the covariance matrix $\mathcal{J}$ is more complicated. Here we have used spectral methods based on the approach popularised by [44]. I.e., let $\hat{h}_t = \frac{\partial}{\partial\theta} l(x_t, \hat{\theta})$ and let m denote the number of nonzero autocorrelations of $h_t(h)$. Then,

$$\hat{\mathcal{J}} = \hat{\Omega}_0 + \sum_{j=1}^{m}\left(1 - \frac{j}{m+1}\right)\left(\hat{\Omega}_j + \hat{\Omega}_j'\right), \quad \hat{\Omega}_j = \frac{1}{n}\sum_{t=j+1}^{n} \hat{h}_t'\hat{h}_{t-j}.$$

The sandwich matrices can be estimated straightforwardly by

$$\hat{\mathcal{I}} = -\frac{1}{n}\frac{\partial^2}{\partial\theta\partial\theta'} l(x, \hat{\theta}) = -\frac{1}{n}\sum_{i=1}^{n}\frac{\partial^2}{\partial\theta\partial\theta'} l(x_i, \hat{\theta}).$$

6.2 Empirical study

In order to illustrate our results further, we have also carried out an empirical study. We have used General Motors (GM) intra–day TAQ database, available at WRDS, from 2 January 2001 to 28 April 2006. Before analysing the data, we have cleaned the data. Following methods used by [34], we concentrate on quote data from one stock exchange only. Here we have chosen the NYSE. We only consider quotes, where both the bid–size and the ask–size are greater than 0, and which are quoted in a normal trading environment (quote condition = 12 in the TAQ database). Since data at the beginning and at the end of a trading day differ quite a lot from the quotes during the day, we concentrate on data from 9.35 am until 15.55pm only. Further, if there were not any quote data for more than five minute, we have interpolated the missing data via linear interpolation. Besides, we have deleted obvious outliers, which we have chosen to be data points which differ by more than 0.15 from the prevailing (log-) price level. Finally, we have deleted data from those days where there were too many data missing at the beginning of a day (no data before 9.45 am) or which were just half trading days. Furthermore, we have focused on the bid–prices only. In order to construct a time series of five minute returns of the log–bid–prices, we use the previous tick sampling method. After we have cleaned the data, we have a data set consisting of 1,308 business days with 76 five minute returns per day, hence 99,408 returns. We provide a plot of the cleaned GM log–price data (based on five

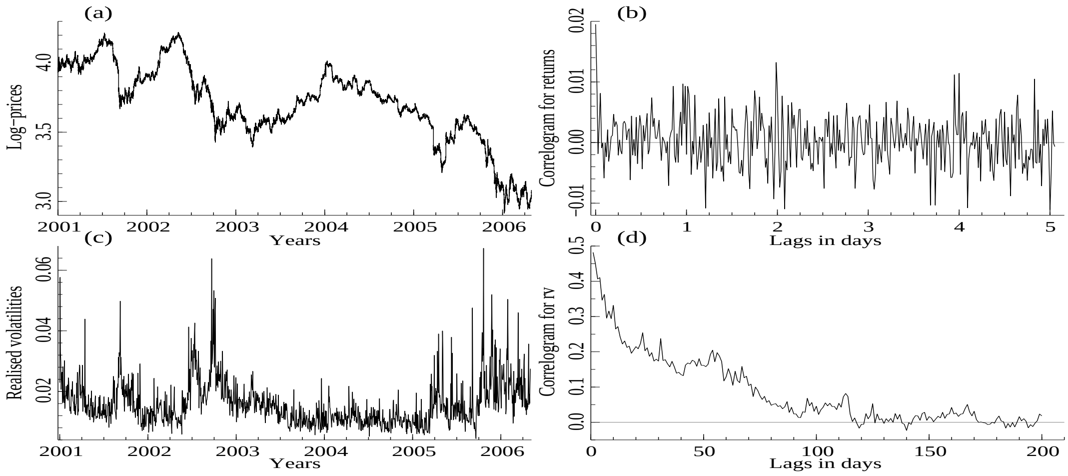

Figure 1: *GM data: (a) cleaned log–bid–prices (plotted every 5 minutes for 1308 business days from 2 January 2001 – 28 April 2006; (b) autocorrelation between five minute returns; (c) time series of realised volatilities (i.e. square root of realised variances) based on five minute returns; (d) autocorrelation between realised variances.*

minute data), the corresponding autocorrelation function and the time series of the realised variances with their autocorrelation in Figure 1. We have estimated the model parameters of the one–factor model and the superposition model (with two factors). We have carried out the estimation three times: once, when $\xi_2 = \xi_3 = \xi_4 = 0$ and all the leverage parameters are set to 0 (this corresponds to a Brownian semimartingale framework), once when allowing for jumps but setting all leverage parameters to zero (and, hence, assuming that there is no leverage) and once with jumps and arbitrary leverage parameters (which just satisfy the moment conditions described previously). The model fit of the Brownian motion based model was very poor, hence we do not report the exact details. By allowing for jumps, the model fit has improved significantly and the corresponding estimation results

Table 1.
Estimation results for the one–factor model, without κ

Parameter	**Estimate**	**Robust standard error**
λ	0.037	0.026
ξ_4	0.353	0.141
η_1	1.770	0.249
η_2	0.153	0.198
η_3	2.197	0.891
η_4	3.195	4.441
Quasi-L	-3264.13	
BP	28.789	
m	35	

Table 2.
Estimation results for the one–factor model, with κ

Parameter	**Estimate**	**Robust standard error**
λ	0.037	0.026
ξ_3	-0.009	1.153
ξ_4	0.356	0.132
η_1	1.771	0.223
η_2	0.145	0.269
η_3	2.153	0.261
η_4	3.188	0.950
κ_{11}	-0.009	0.181
κ_{12}	-0.009	2.459
κ_{13}	-0.009	1.078
κ_{21}	0.008	0.074
κ_{22}	0.011	0.668
Quasi-L	-3264.13	
BP	28.789	
m	35	

are given in the following tables. Note here that in the superposition model the QML estimates for the third and fourth cumulant of Y, i.e. η_3 and η_4 are on the boundaries rather than in the interior of the parameter space. So in these cases, we do not report the robust standard errors. In order to assess the model fit, we provide the following plots and statistics: Figure 2 and Figure 3 show the empirical versus the fitted autocorrelation function of the realised variance (a), the estimated (based on the one step ahead forecast) and the empirical realised variance (b), the time series of the scaled residuals (c) and their autocorrelation function (d). Furthermore we have computed the Box–Pierce statistic based on 20 lags which measured the degree of dependence in the scaled residuals. It is really striking that the model fit seems to be already quite good for the one–factor model, when we allow for jumps in the asset price. Incorporating a second factor leads to another (small) improvement. One could also study multifactor models with more than two factors. But from our results for one and two factors, we would not expect that this would lead to a big improvement anymore. Note here that, in order to estimate the parameters of the superposition model, we have used $O(M^{-1})$–approximations (as given in the Appendix) for the variance and covariance of the realised variance (rather than $O(M^{-2})$–approximations), since this reduces the number of parameters which have to be estimated from 20 to 12. Studying the higher order approximation and carrying out a numerical optimisation over 20 unknown parameters, which are partly very weakly identified, will be subject to future research. Let us now turn our attention to the estimated leverage parameters. We observe that accounting for the leverage parameters in this particular example does not lead to an improvement (but also not to a

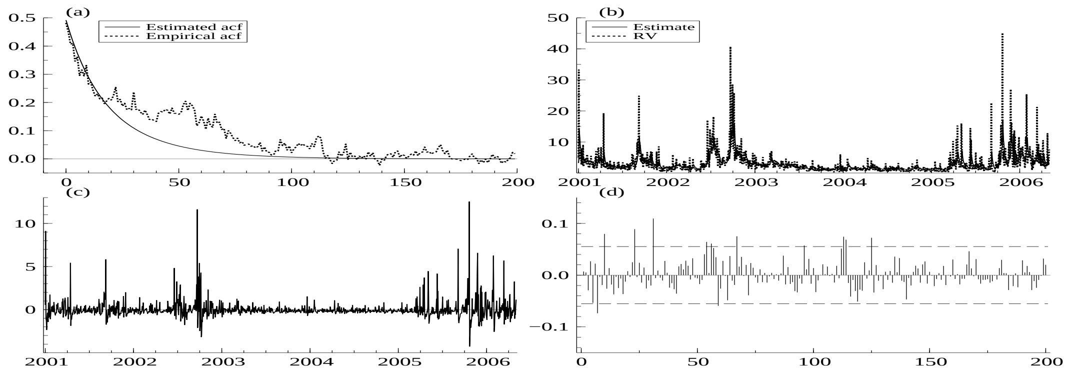

Figure 2: *(a)Empirical autocorrelation of realised variance and estimated (for J=1) acf for 01/2001–04/2006; (b) estimated variance and realised variance; (c) scaled residuals; (d) acf of scaled residuals.*

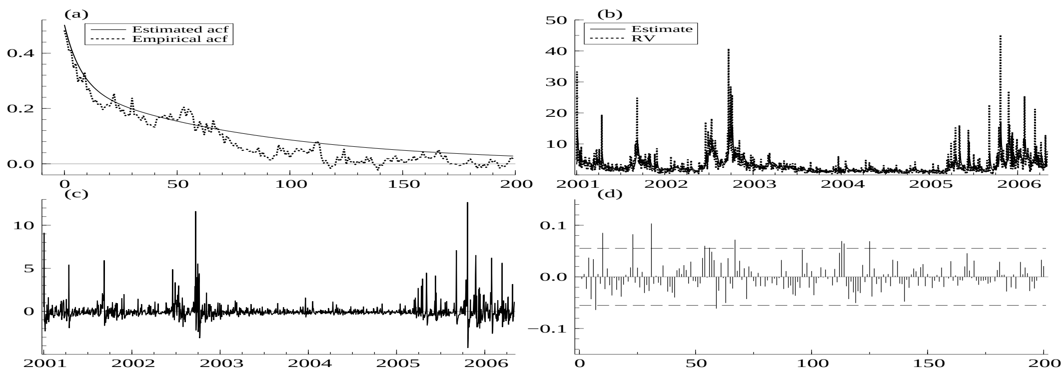

Figure 3: *(a)Empirical autocorrelation of realised variance and estimated (for J=2) acf for 01/2001–04/2006; (b) estimated variance and realised variance; (c) scaled residuals; (d) acf of scaled residuals.*

deterioration) of the model fit. We hypothesise here that estimating the leverage parameters by our QML method is difficult since they only appear in terms of low order. So they are not easy to identify and since their corresponding confidence intervals all cover the 0, it is not possible to deduce clearly whether leverage (in the sense of non–zero joint cumulants) is present in the data. These findings are in line with other empirical studies, which focused on estimating the leverage effect in single stocks. However, when studying index data, one often observes a leverage effect, which is present for several days (see e.g. [20]). Finally, we provide a plot (Figure 4) of the empirical cross–correlation between returns and realised variances which we denote by

$$L(s) = \frac{\sum_i((s_i - \overline{s})([S_\delta]_{i+s} - \overline{[S_\delta]}))}{\sqrt{\sum_i((s_i - \overline{s})^2 \sum_i([S_\delta]_{i+s} - \overline{[S_\delta]}))^2}}$$

for $s \in \{-20, \ldots, 20\}$, where $\overline{s}$ and $\overline{[S_\delta]}$ denote the sample mean of the returns and the realised variance, respectively. Similarly to [20], this function can be interpreted as a kind of leverage correlation function. In addition to this function, we plot the Bartlett confidence bounds of the hypothesis that there is no leverage, which are given by $1.96/\sqrt{T} = 0.054$. Although we can clearly spot that

Table 3.
Estimation results for the two–factor model, without κ

Parameter	Estimate	Robust standard error
$\lambda^{(1)}$	0.130	0.175
$\lambda^{(2)}$	0.011	0.017
w_1	0.478	0.156
ξ_4	0.344	0.210
η_1	1.667	0.482
η_2	2.360	0.863
η_3	1e-007	
η_4	1e-007	
Quasi-L	-3261.956	
BP	27.949	
m	35	

Table 4.
Estimation results for the two–factor model, with κ

Parameter	Estimate	Robust standard error
$\lambda^{(1)}$	0.130	0.198
$\lambda^{(2)}$	0.011	0.039
w_1	0.478	0.893
ξ_4	0.344	0.356
η_1	1.667	0.511
η_2	2.360	3.536
η_3	1.0002e-007	
η_4	1.0374e-007	
$\kappa_{21}^{(1)}$	2.3994e-008	1.263
$\kappa_{22}^{(1)}$	2.4014e-008	5.570
$\kappa_{21}^{(2)}$	2.3972e-008	3.630
$\kappa_{22}^{(2)}$	4.1144e-007	22.086
Quasi-L	-3261.956	
BP	27.948	
m	35	

returns and future realised variance seem to be slightly negatively correlated for a couple of days, the correlation is really small and not statistically significant.

7 Conclusion

In this paper, we have studied the impact of time–inhomogeneous jumps and leverage type effects on returns and realised variance in Lévy–driven stochastic volatility models. In particular, we have derived explicit expressions for the cumulants of the returns and the realised variance by solving a recursive system of inhomogeneous ordinary differential equations. This seems to be a very powerful technique and might be applicable to a much wider class of asset price models. This aspect will be investigated further in future research. Although the realised variance is an inconsistent estimator of the integrated variance in the Lévy–driven stochastic volatility model studied in this paper, we have shown how it can be used in a quasi–maximumlikelihood framework for estimating the model parameters of our model.

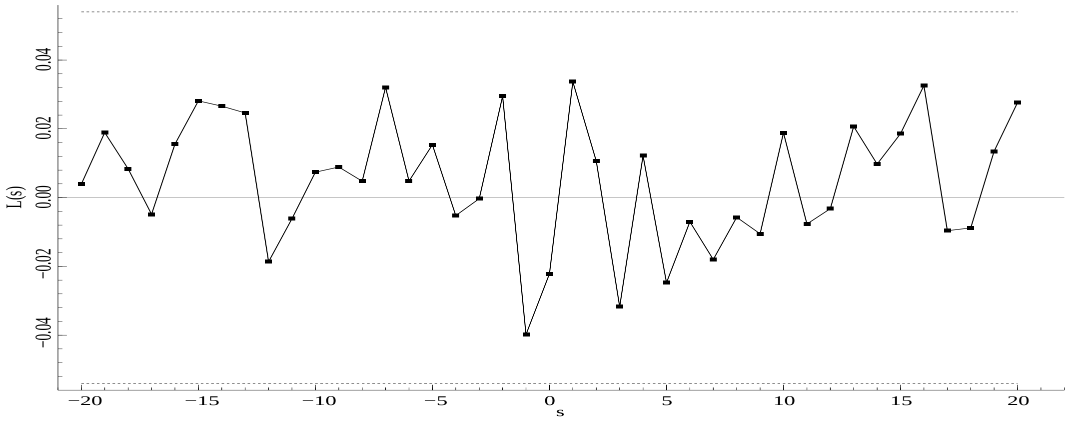

Figure 4: *This plot shows the leverage correlation function* $L(s) = Cor(s_i, [S_\delta]_{i+s})$ *for* $s \in \{-20, \ldots, 20\}$.

A Technical appendix

A.1 Proofs

Proof of Proposition 4.1 From the binomial formula one obtains:

$$\begin{aligned}\Delta\sigma_t^n &= \sigma_t^n - \sigma_{t-}^n = (\sigma_{t-} + \Delta\sigma_t)^n - \sigma_{t-}^n = \sum_{k=0}^{n} \binom{n}{k} \sigma_{t-}^{n-k} (\Delta\sigma_t)^k - \sigma_{t-}^n \\ &= \sum_{k=1}^{n} \binom{n}{k} \sigma_{t-}^{n-k} (\Delta Y_{\lambda t})^k.\end{aligned}$$

Applying Itô's formula to $f(x) = x^n$ with $f'(x) = nx^{n-1}, \;\; f''(x) = n(n-1)x^{n-2}$, one gets

$$\begin{aligned}\sigma_t^n - \sigma_0^n &= f(\sigma_t) - f(\sigma_0) \\ &= \int_0^t f'(\sigma_{s-}) d\sigma_s + \frac{1}{2} \int_0^t f''(\sigma_{s-}) d[\sigma, \sigma]_s^c + \sum_{0<s\le t} (f(\sigma_s) - f(\sigma_{s-}) - f'(\sigma_{s-})\Delta\sigma_s) \\ &= -\lambda\, n \int_0^t \sigma_{s-}^n ds + n \int_0^t \sigma_{s-}^{n-1} dY_{\lambda s} + \sum_{0<s\le t} (\Delta\sigma_s^n - n\sigma_{s-}^{n-1} \Delta Y_{\lambda s}) \\ &= -\lambda\, n \int_0^t \sigma_{s-}^n ds + \sum_{k=1}^{n} \binom{n}{k} \sum_{0\le s\le t} \sigma_{s-}^{n-k} (\Delta Y_{\lambda s})^k.\end{aligned}$$

□

Proof of Proposition 4.3 Note that, for $n \in \mathbb{N}$,

$$\Delta S_t^n = \sum_{k=1}^{n} \binom{n}{k} (\sigma_{t-} \Delta X_t)^k \, S_{t-}^{n-k},$$

and, by Itô's formula:

$$S_t^n = n\int_0^t S_{s-}^{n-1}\sigma_{s-}d(vW_s+X_s) + \frac{n(n-1)}{2}v^2\int_0^t S_{s-}^{n-2}\sigma_{s-}^2 ds + \sum_{0\le s\le t}\sum_{k=2}^{n}\binom{n}{k}(\sigma_{s-}\Delta X_s)^k S_{s-}^{n-k}.$$

Hence,

$$\mathbb{E}\left(S_t^n\right) = \frac{n(n-1)}{2}v^2\int_0^t \mathbb{E}\left(S_{s-}^{n-2}\sigma_{s-}^2\right)ds + \sum_{k=2}^{n}\binom{n}{k}\xi_k\int_0^t \mathbb{E}\left(\sigma_{s-}^k S_{s-}^{n-k}\right).$$

From the integration by parts formula, it follows that, for $k,n\in\mathbb{N}$ and $k\le n$,

$$S_t^{n-k}\sigma_t^k = \int_0^t S_{s-}^{n-k}d\sigma_s^k + \int_0^t \sigma_{s-}^k dS_{s-}^{n-k} + [S^{n-k},\sigma^k]_t = I + II + III.$$

From Proposition 4.1, one can deduce that

$$I = -k\lambda\int_0^t S_{s-}^{n-k}\sigma_{s-}^k ds + \sum_{j=1}^{k}\binom{k}{j}\sum_{0\le s\le t}S_{s-}^{n-k}\sigma_{s-}^{k-j}(\Delta Y_{\lambda s})^j,$$

and

$$II = (n-k)\int_0^t \sigma_{s-}^{k+1}S_{s-}^{n-k-1}d(vW_s+X_s) + \frac{(n-k)(n-k-1)v^2}{2}\int_0^t S_{s-}^{n-k-2}\sigma_{s-}^2 ds + \sum_{j=2}^{n-k}\binom{n-k}{j}\sum_{0\le s\le t}\sigma_{s-}^{k+j}S_{s-}^{n-k-j}(\Delta X_s)^j,$$

and

$$III = \sum_{0\le s\le t}\Delta S_s^{n-k}\Delta\sigma_s^k = \sum_{j=1}^{n-k}\sum_{l=1}^{k}\binom{n-k}{j}\binom{k}{l}\sum_{0\le s\le t}S_{s-}^{n-k-j}\sigma_{s-}^{j+k-l}(\Delta X_s)^j(\Delta Y_{\lambda s})^l.$$

When taking the expectation and applying the Master formula and Fubini's theorem, one obtains:

$$\begin{aligned}\mathbb{E}\left(S_t^{n-k}\sigma_t^k\right) &= -k\lambda\int_0^t \mathbb{E}\left(S_{s-}^{n-k}\sigma_{s-}^k\right)ds + \sum_{j=1}^{k}\binom{k}{j}\lambda\eta_j\int_0^t \mathbb{E}\left(S_{s-}^{n-k}\sigma_{s-}^{k-j}\right)ds \\ &+ \frac{(n-k)(n-k-1)v^2}{2}\int_0^t \mathbb{E}\left(S_s^{n-k-2}\sigma_s^2\right)ds + \sum_{j=2}^{n-k}\binom{n-k}{j}\xi_j\int_0^t \mathbb{E}\left(\sigma_s^{k+j}S_s^{n-k-j}\right) \\ &+ \sum_{j=0}^{n-1}\binom{n}{j}\xi_{n-j}\int_0^t \mathbb{E}\left(S_{s-}^j\sigma_{s-}^{m+n-j}\right)ds + \sum_{j=1}^{n-k}\sum_{l=1}^{k}\binom{n-k}{j}\binom{k}{l}\kappa_{j,l}\int_0^t \mathbb{E}\left(S_s^{n-k-j}\sigma_s^{j+k-l}\right)ds.\end{aligned}$$

The equation above can be written as

$$\frac{d}{dt}\mathbb{E}\left(S_t^{n-k}\sigma_t^k\right) + k\lambda\mathbb{E}\left(S_{t-}^{n-k}\sigma_{t-}^k\right) = g(t;n,k), \tag{2}$$

where $g(\cdot; n, k)$ is defined as in Proposition 4.3. Since the processes above are continuous in probability, we can write

$$\mathbb{E}\left(S_{t-}^{n-k}\sigma_{t-}^{k}\right) = \mathbb{E}\left(S_{t}^{n-k}\sigma_{t}^{k}\right).$$

Clearly, equation (2) is a inhomogeneous ordinary differential equation of first order. From solving (2) with initial value 0 at 0, one obtains

$$\mathbb{E}\left(S_t^n\sigma_t^k\right) = e^{-k\lambda t} \int_0^t g(u; n, k)e^{k\lambda u} du.$$

□

Proof of Proposition 4.5 First, we have to compute the mean, variance and covariance of the integrated variance IV. We use Fubini's Theorem and the results we have already derived for computing the moments of the volatility process σ. Hence,

$$\mathbb{E}\left(IV_t\right) = \mathbb{E}\left(\int_0^t \sigma_{s-}^2 ds\right) = \int_0^t \mathbb{E}\left(\sigma_s^2\right) ds = \left(\eta_1^2 + \frac{\eta_2}{2}\right) t.$$

The second moment is given by

$$\mathbb{E}\left(IV_t^2\right) = \int_0^t \int_0^t \mathbb{E}\left(\sigma_s^2\sigma_u^2\right) ds\, du,$$

where for $u \geq s$,

$$\mathbb{E}\left(\sigma_s^2\sigma_u^2\right) = \mathbb{E}\left(\sigma_s^4\right) + \mathbb{E}\left(\sigma_s^2\left(\sigma_u^2 - \sigma_s^2\right)\right).$$

Using the SDE representation of σ^2 and Itô's formula, one obtains the SDE

$$\frac{d}{du}\,\mathbb{E}\left(\sigma_s^2\sigma_u^2\right) = -2\lambda\,\mathbb{E}\left(\sigma_s^2\sigma_u^2\right) + 2\eta_1\mathbb{E}\left(\sigma_s^2\sigma_u\right) + \eta_2\mathbb{E}\left(\sigma_s^2\right),$$

which can be solved using the initial value $\left(s, \mathbb{E}\left(\sigma_s^4\right)\right)$. Similarly,

$$\frac{d}{du}\,\mathbb{E}\left(\sigma_s^2\sigma_u\right) = -\lambda\,\mathbb{E}\left(\sigma_s^2\sigma_u\right) + \eta_1\mathbb{E}\left(\sigma_s^2\right),$$

with initial value $\left(s, \mathbb{E}\left(\sigma_s^3\right)\right)$. By writing

$$\mathbb{E}\left(IV_t^2\right) = \int_0^t \int_0^u \mathbb{E}\left(\sigma_s^2\,\sigma_u^2\right) ds\, du + \int_0^t \int_u^t \mathbb{E}\left(\sigma_s^2\,\sigma_u^2\right) ds\, du,$$

and

$$\mathbb{E}\left(IV_t\, IV_{t+s}\right) = \mathbb{E}\left(IV_t^2\right) + \int_0^t \int_t^{t+s} \mathbb{E}\left(\sigma_x^2\,\sigma_u^2\right) dx\, du,$$

we can immediately deduce the variance and the covariance of IV. The mean, variance and covariance of $\sigma_{[(i-1)h,ih]}$ can be directly derived from the corresponding results for IV. □

Proof of Proposition 4.6 First, we compute the mean of $[S]$. Using the formula for the quadratic variation of a stochastic integral (see e.g. [45] Theorem II.29), the Master Formula and Fubini's Theorem, one can deduce that

$$\mathbb{E}\left([S]_t\right) = \mathbb{E}\left(\int_0^t \sigma_{s-}^2 d[vW+X]_s\right) = \left(v^2+\xi_2\right)\int_0^t \mathbb{E}\left(\sigma_s^2\right) ds = \left(v^2+\xi_2\right)\left(\frac{\eta_1^2}{\lambda^2}+\frac{1}{2}\frac{\eta_2}{\lambda}\right) t.$$

Second, we compute second moment of $[S]$. Using Itô's formula, we deduce that

$$\begin{aligned}
[S]_t^2 &= 2\int_0^t [S]_{u-} d[S]_u + [[S]]_u = 2\int_0^t [S]_{u-}\sigma_{u-}^2 d[vW+X]_u + [[S]]_u \\
&= 2v^2\int_0^t [S]_{u-}\sigma_{u-}^2 ds + 2\sum_{0\le u\le t}[S]_{u-}\sigma_{u-}^2(\Delta X_u)^2 + \sum_{0\le u\le t}\sigma_{u-}^4(\Delta X_u)^4.
\end{aligned}$$

Hence (using again the Master Formula and Fubini's Theorem),

$$\mathbb{E}\left([S]_t^2\right) = 2\left(v^2+\xi_2\right)\int_0^t \mathbb{E}\left([S]_u\sigma_u^2\right) du + \xi_4\int_0^t \mathbb{E}\left(\sigma_u^4\right) du.$$

So we have to compute $\mathbb{E}\left([S]_u\sigma_u^2\right)$. We apply the integration by parts formula and obtain

$$\begin{aligned}
[S]_u\sigma_u^2 &= \int_0^u [S]_{x-} d\sigma_x^2 + \int_0^u \sigma_{x-}^2 d[S]_x + [[S],\sigma^2]_u \\
&= -2\lambda\int_0^u [S]_{x-}\sigma_x^2 dx + 2\int_0^u [S]_{x-}\sigma_x dY_{\lambda x} + \int_0^u [S]_{x-} d[Y]_{\lambda x} \\
&\quad + v^2\int_0^u \sigma_{x-}^4 dx + \int_0^u \sigma_{x-}^4 d[X]_x + \int_0^u \sigma_{x-}^2 d[[X],\sigma^2]_x.
\end{aligned}$$

Applying the Master formula and Fubini's theorem, we get

$$\begin{aligned}
\mathbb{E}\left([S]_u\sigma_u^2\right) &= -2\lambda\int_0^u \mathbb{E}\left([S]_x\sigma_x^2\right) dx + 2\eta_1\int_0^u \mathbb{E}\left([S]_x\sigma_x\right) dx + \eta_2\int_0^u \mathbb{E}\left([S]_x\right) dx \\
&\quad + \left(v^2+\xi_2\right)\int_0^u \mathbb{E}\left(\sigma_x^4\right) dx + 2\kappa_{2,1}\int_0^u \mathbb{E}\left(\sigma_x^3\right) dx + \kappa_{2,2}\int_0^u \mathbb{E}\left(\sigma_x^2\right) dx.
\end{aligned}$$

So we have to compute $\mathbb{E}\left([S]_u\sigma_u\right)$ first and, then, we can solve the differential equation

$$\begin{aligned}
(\mathbb{E}[S]_u\sigma_u^2)' &= -2\lambda\mathbb{E}\left([S]_u\sigma_u^2\right) + 2\eta_1\mathbb{E}\left([S]_u\sigma_u\right) + \eta_2\mathbb{E}\left([S]_u\right) \\
&\quad + \left(v^2+\xi_2\right)\mathbb{E}\left(\sigma_u^4\right) + 2\kappa_{2,1}\mathbb{E}\left(\sigma_u^3\right) + \kappa_{2,2}\mathbb{E}\left(\sigma_u^2\right),
\end{aligned}$$

with initial value $(0,0)$. So similarly as above, we can compute $\mathbb{E}\left([S]_u\sigma_u\right)$. From the integration by parts formula, we get

$$\begin{aligned}
[S]_u\sigma_u &= \int_0^u [S]_{x-} d\sigma_x + \int_0^u \sigma_{x-} d[S]_x + [[S],\sigma]_x \\
&= -\lambda\int_0^u [S]_{x-}\sigma_{x-} dx + \int_0^u [S]_{x-} dY_{\lambda x} + \int_0^u \sigma_{x-}^3 d[vW+X]_x + \int_0^u \sigma_{x-}^2 d[[X],\sigma]_x.
\end{aligned}$$

Hence, we deduce from the Master formula and Fubini's theorem that

$$\mathbb{E}\left([S]_u\sigma_u\right) = -\lambda\int_0^u \mathbb{E}\left([S]_x\sigma_x\right)dx + \eta_1\int_0^u \mathbb{E}\left([S]_x\right)dx + \left(v^2+\xi_2\right)\int_0^u \mathbb{E}\left(\sigma_x^3\right)dx + \kappa_{2,1}\int_0^u \mathbb{E}\left(\sigma_x^2\right)dx.$$

So one obtains $\mathbb{E}\left([S]_u\sigma_u\right)$ by solving the differential equation

$$\frac{d}{du}\mathbb{E}\left([S]_u\sigma_u\right) = -\lambda\mathbb{E}\left([S]_u\sigma_u\right) + \eta_1\mathbb{E}\left([S]_u\right) + \left(v^2+\xi_2\right)\mathbb{E}\left(\sigma_u^3\right) + \kappa_{2,1}\mathbb{E}\left(\sigma_u^2\right),$$

with initial value $(0,0)$. Finally, we derive the covariance of $[S]$.

$$\mathbb{E}\left([S]_t([S]_{t+s}-[S]_t)\right) = \mathbb{E}\left([S]_t\int_t^{t+s}\sigma_u^2 d[vW+X]_u\right) = \left(v^2+\xi_2\right)\int_t^{t+s}\mathbb{E}\left([S]_t\sigma_u^2\right)du.$$

In order to compute $\mathbb{E}\left([S]_t\sigma_u^2\right)$, we use once again integration by parts, the Master Formula and Fubini's Theorem so that in the end we just have to solve the following two differential equations for $u \geq t$:

$$\frac{d}{du}\mathbb{E}\left([S]_t\sigma_u\right) = -\lambda\mathbb{E}\left([S]_t\sigma_u\right) + \lambda\eta_1\mathbb{E}\left([S]_t\right),$$

with initial value $(t, \mathbb{E}[S]_t\sigma_t)$ and

$$\frac{d}{du}\mathbb{E}[S]_t\sigma_u^2 = -2\lambda\mathbb{E}[S]_t\sigma_u^2 + 2\lambda\eta_1\mathbb{E}[S]_t\sigma_u + \lambda\eta_2\,\mathbb{E}[S]_t,$$

with initial value $\left(t, \mathbb{E}[S]_t\sigma_t^2\right)$. Combining these results, we obtain the covariance from

$$\begin{aligned}\mathrm{Cov}\left([S]_t,[S]_{t+s}\right) &= \mathbb{E}\{[S]_t[S]_{t+s}\} - \mathbb{E}([S]_t)\,\mathbb{E}([S]_{t+s}) \\ &= \mathbb{E}([S]_t)^2 + \mathbb{E}\{[S]_t([S]_{t+s}-[S]_t) - \mathbb{E}([S]_t)\,\mathbb{E}([S]_{t+s}).\end{aligned}$$

□

Proof of Proposition 4.8 In order to proof Proposition 4.8, i.e. the covariations between, returns and squared returns, we have to compute the corresponding covariations of the price process S.

In the following, we will only sketch the proof. Basically, we will always apply Itô's formula, the Master Formula and Fubini's Theorem in order to derive a first order ordinary differential equations (ODE) for the expectation we want to compute. This ODE can then be easily solved.

Covariation of returns

$$\begin{aligned}Cov(S_t,S_{t+s}) &= \mathbb{E}S_t^2 + \mathbb{E}(S_t(S_{t+s}-S_t)) \\ &= \mathbb{E}\left(S_t^2\right) + \mathbb{E}\left(\int_t^{t+s} S_t\sigma_{s-}d(vW_s+X_s)\right) = \mathbb{E}\left(S_t^2\right) = Var\left(S_t\right).\end{aligned} \tag{3}$$

From equation (3), we can immediately deduce that

$$\begin{aligned}Cov(s_i,s_{i+s}) &= \mathbb{E}\left(s_is_{i+s}\right) = \mathbb{E}(S_{ih}-S_{(i-1)h})(S_{(i+s)h}-S_{(i+s-1)h}) \\ &= Cov(S_{ih},S_{(i+s)h}) - Cov(S_{ih},S_{(i+s-1)h}) - Cov(S_{(i-1)h},S_{(i+s)h}) \\ &\quad + Cov(S_{(i-1)h},S_{(i+s-1)h}) = 0.\end{aligned}$$

□

Covariation of returns and squared returns

$$\begin{aligned} Cov(S_t, S_{t+s}^2) &= \mathbb{E}\left(S_t^3\right) + \mathbb{E}\left(S_t(S_{t+s}^2 - S_t^2)\right) \\ &= \mathbb{E}\left(S_t^3\right) + \mathbb{E}\left(S_t\left(2\int_t^{t+s} S_{u-}dS_u + \int_t^{t+s} \sigma_{u-}^2 d[vW+X]_u\right)\right) \\ &= \mathbb{E}\left(S_t^3\right) + \int_t^{t+s} \mathbb{E}\left(S_t\sigma_u^2\right) du. \end{aligned} \tag{4}$$

In the following, we will always assume that $u \geq t$. □

Computing $\mathbb{E}\left(S_t\sigma_u\right)$

$$S_t\sigma_u = S_t\sigma_t + S_t(\sigma_t - \sigma_u) = S_t\sigma_t + \int_t^u S_t d\sigma_u = S_t\sigma_t + \int_t^u S_t(-\lambda\sigma_t dt + dY_{\lambda t}).$$

Hence,

$$\mathbb{E}\left(S_t\sigma_u\right) = \mathbb{E}\left(S_t\sigma_t\right) - \lambda\int_t^u \mathbb{E}\left(S_t\sigma_u\right) du.$$

So, $\mathbb{E}\left(S_t\sigma_u\right)$ is the solution to

$$\frac{d}{du}\mathbb{E}\left(S_t\sigma_u\right) = -\lambda\mathbb{E}\left(S_t\sigma_u\right),$$

with initial value $(t, \mathbb{E}\left(S_t\sigma_t\right))$. □

Computing $\mathbb{E}\left(S_t\sigma_u^2\right)$ Similarly,

$$S_t\sigma_u^2 = S_t\sigma_t^2 + \int_t^u S_t d\sigma_x^2 = S_t\sigma_t^2 - 2\lambda\int_t^u S_t\sigma_{x-}^2 dx + 2\int_t^u S_t\sigma_{x-}dY_{\lambda x} + \int_t^u S_t d[\sigma]_x.$$

Hence,

$$\mathbb{E}\left(S_t\sigma_u^2\right) = \mathbb{E}\left(S_t\sigma_t^2\right) - 2\lambda\int_t^u \mathbb{E}\left(S_t\sigma_x^2\right) dx + 2\eta_1\int_t^u \mathbb{E}\left(S_t\sigma_x\right) dx.$$

So $\mathbb{E}\left(S_t\sigma_u^2\right)$ is the solution to

$$\frac{d}{du}\mathbb{E}\left(S_t\sigma_u^2\right) = -2\lambda\mathbb{E}\left(S_t\sigma_u^2\right) + 2\eta_1\mathbb{E}\left(S_t\sigma_u\right),$$

with initial value $(t, \mathbb{E}\left(S_t\sigma_t^2\right))$. □

Computing $\mathbb{E}\left(S_aS_bS_c\right)$ **for** $a \leq b \leq c$

$$S_aS_bS_c = S_aS_b(S_b + (S_c - S_b)) = S_aS_b^2 + \int_b^c S_aS_b\sigma_{x-}d(vW_x + X_x).$$

Hence, $\mathbb{E}S_aS_bS_c = \mathbb{E}S_aS_b^2$. □

So we get

$$\begin{aligned}Cov(s_i, s_{i+s}^2) =&\mathbb{E}\left(S_{ih}S_{(i+s)h}^2\right) - 2\mathbb{E}\left(S_{ih}S_{(i+s-1)h}^2\right) + \mathbb{E}\left(S_{ih}S_{(i+s-1)h}^2\right) - \mathbb{E}\left(S_{(i-1)h}S_{(i+s)h}^2\right)\\ &+ 2\mathbb{E}\left(S_{(i-1)h}S_{(i+s-1)h}^2\right) - \mathbb{E}\left(S_{(i-1)h}S_{(i+s-1)h}^2\right)\\ =&\frac{1}{4\lambda^4}\left(\eta_2\left(-2\,\lambda\,e^{-2\,\lambda\,sh} + \lambda\,e^{-2\,\lambda\,(sh-h)} + \lambda\,e^{-2\,\lambda\,(h+sh)}\right)\right.\\ &\left.+8\eta_1^2\kappa_{1,1}\left(-2\,e^{-\lambda\,sh} + \,e^{-\lambda\,(h+sh)} + \,e^{-\lambda\,(sh-h)}\right)\right)\\ &+ \kappa_{1,2}\left(\lambda\,e^{-2\,\lambda\,(h+sh)}\eta_1 + \lambda\,e^{-2\,\lambda\,(sh-h)}\eta_1 - 2\,\lambda\,e^{-2\,\lambda\,sh}\eta_1\right) = O(h^2).\end{aligned}$$

Covariation between squared returns

$$\begin{aligned}Cov(S_t^2, S_{t+s}^2) &= \mathbb{E}\left(S_t^4\right) + \mathbb{E}\left(S_t^2(S_{t+s}^2 - S_t^2)\right) - \mathbb{E}\left(S_t^2\mathbb{E}S_{t+s}^2\right)\\ &= \mathbb{E}\left(S_t^4\right) + \left(v^2+\xi_2\right)\int_t^{t+s}\mathbb{E}\left(S_t^2\sigma_u^2\right)du - \mathbb{E}\left(S_t^2\right)\mathbb{E}\left(S_{t+s}^2\right),\end{aligned}$$

and

$$Cov(s_i^2, s_{i+s}^2) = \mathbb{E}(s_i^2 s_{i+s}^2) - \mathbb{E}\left(s_i^2\right)\mathbb{E}\left(s_{i+s}^2\right),$$

and

$$\begin{aligned}\mathbb{E}(s_i^2 s_{i+s}^2) = \mathbb{E}(&S_{ih}^2 S_{(i+s)h}^2 - 2\,S_{ih}^2 S_{(i+s)h}S_{(i+s-1)h} + S_{ih}^2 S_{(i+s-1)h}^2 - 2\,S_{ih}S_{(i-1)h}S_{(i+s)h}^2\\ &+ 4\,S_{ih}S_{(i-1)h}S_{(i+s)h}S_{(i+s-1)h} - 2\,S_{ih}S_{(i-1)h}S_{(i+s-1)h}^2 + S_{(i-1)h}^2 S_{(i+s)h}^2\\ &- 2\,S_{(i-1)h}^2 S_{(i+s)h}S_{(i+s-1)h} + S_{(i-1)h}^2 S_{(i+s-1)h}^2).\end{aligned}$$

□

Computing $\mathbb{E}\left(S_a^2 S_b S_c\right)$ **for** $a \le b \le c$

$$S_a^2 S_b S_c = S_a^2 S_b(S_b + (S_c - S_b)) = S_a^2 S_b^2 + \int_b^c S_a^2 S_b \sigma_{x-} d(vW_x + X_x).$$

Hence,

$$\mathbb{E}\left(S_a^2 S_b S_c\right) = \mathbb{E}\left(S_a^2 S_b^2\right).$$

□

Computing $\mathbb{E}\left(S_a S_b S_c^2\right)$ **for** $a \le b \le c$ Similarly,

$$S_a S_b S_c^2 = S_a S_b(S_b^2 + (S_c^2 - S_b^2)) = S_a S_b^3 + 2\int_b^c S_a S_b S_{x-}\sigma_{x-} d(vW_x + X_x) + \int_b^c S_a S_b \sigma_{x-}^2 d[X]_x.$$

Hence,

$$\mathbb{E}\left(S_a S_b S_c^2\right) = \mathbb{E}\left(S_a S_b^3\right) + \left(v^2 + \xi_2\right)\int_b^c \mathbb{E}\left(S_a S_b \sigma_x^2\right)dx$$

□

Computing $\mathbb{E}S_aS_b^3$ **for** $a \leq b$ $S_aS_b^3 = S_a^4 + S_a(S_b^3 - S_a^3)$. From Itô's formula, we deduce that

$$S_b^3 - S_a^3 = 3\int_a^b S_{t-}^2 dS_t + 3v^2 \int_0^t S_{t-}\sigma_{t-}^2 ds + \sum_{a\leq t\leq b} (\Delta S_t^3 - 3S_{t-}^2\Delta S_t),$$

where

$$\Delta S_t^3 = \sum_{j=0}^{2} \binom{n}{j} S_{t-}^j \sigma_{t-}^{2-j} (\Delta X_t)^{2-j} = \sigma_{t-}^3(\Delta X_t)^3 + 3S_{t-}\sigma_{t-}^2(\Delta X_t)^2 + 3S_{t-}^2\sigma_{t-}\Delta X_t.$$

Hence,

$$\mathbb{E}S_aS_b^3 = \mathbb{E}S_a^4 + \xi_3 \int_a^b S_a\sigma_t^3 dt + 3\left(v^2 + \xi_2\right)\int_a^b \mathbb{E}S_aS_t\sigma_t^2 dt.$$

□

Computing $\mathbb{E}\left(S_t\sigma_u^3\right)$ **for** $u \geq t$

$$S_t\sigma_u^3 = S_t\sigma_t^3 + S_t(\sigma_u^3 - \sigma_t^3) = S_t\sigma_t^3 + S_t\Bigg(-3\lambda\int_t^u \sigma_{x-}^3 dx$$
$$+3\int_t^u \sigma_{x-}^2 dY_{\lambda x} + \sum_{t\leq x\leq u} (\Delta Y_{\lambda x})^3 + 3\sigma_{x-}(\Delta Y_{\lambda x})^2 + 3\sigma_{x-}^2\Delta Y_{\lambda x}\Bigg).$$

Hence,

$$\mathbb{E}\left(S_t\sigma_u^3\right) = \mathbb{E}\left(S_t\sigma_t^3\right) - 3\lambda\int_t^u \mathbb{E}\left(S_t\sigma_x^3\right)dx + 3\eta_1\int_t^u \mathbb{E}\left(S_t\sigma_x^2\right)dx$$
$$+ 3\eta_2\int_t^u \mathbb{E}\left(S_t\sigma_x\right)dx + 3\eta_1\int_t^u \mathbb{E}\left(S_t\sigma_x^2\right)dx.$$

So $\mathbb{E}\left(S_t\sigma_u^3\right)$ is the solution to the the ODE

$$\frac{d}{du}\mathbb{E}\left(S_t\sigma_u^3\right) = -3\lambda\mathbb{E}\left(S_t\sigma_u^3\right) + 3\eta_1\mathbb{E}\left(S_t\sigma_u^2\right) + 3\eta_2\mathbb{E}\left(S_t\sigma_u\right) + 3\eta_1\mathbb{E}\left(S_t\sigma_u^2\right),$$

with initial value $\left(t, \mathbb{E}\left(S_t\sigma_t^3\right)\right)$. □

Computing $\mathbb{E}\left(S_tS_u\sigma_u\right)$ **for** $u \geq t$

$$S_tS_u\sigma_u = S_t^2\sigma_t + S_t(S_t\sigma_t - S_u\sigma_u)$$
$$= S_t^2\sigma_t + S_t(-\lambda\int_t^u S_{x-}\sigma_{x-}dx + \int_t^u S_{x-}dY_{\lambda x} + \int_t^u \sigma_{x-}dS_x + \int_t^u \sigma_{x-}d[X,\sigma]_x).$$

Hence,

$$\mathbb{E}\left(S_tS_u\sigma_u\right) = \mathbb{E}\left(S_t^2\sigma_t\right) - \lambda\int_t^u \mathbb{E}\left(S_tS_x\sigma_x\right)dx + \eta_1\int_t^u \underbrace{\mathbb{E}\left(S_tS_x\right)}_{=\mathbb{E}\left(S_t^2\right)}dx + \kappa_{1,1}\int_t^u \mathbb{E}\left(S_t\sigma_x\right)dx.$$

□

Computing $\mathbb{E}\left(S_t S_u \sigma_u^2\right)$ **for** $u \geq t$

$$S_t S_u \sigma_u^2 = S_t^2 \sigma_t^2 + S_t (S_u \sigma_u^2 - S_t \sigma_t^2),$$

where

$$\begin{aligned} S_u \sigma_u^2 - S_t \sigma_t^2 = & -2\lambda \int_t^u S_{x-} \sigma_{x-}^2 \, dx + 2 \int_t^u S_{x-} \sigma_{x-} dY_{\lambda x} + \int_t^u S_{x-} d[Y]_{\lambda x} \\ & + \int_t^u \sigma_{x-}^3 d(vW_x + X_x) + 2 \sum_{t \leq x \leq u} \sigma_{x-}^2 \Delta X_x \Delta Y_{\lambda x} + \sum_{t \leq x \leq u} \sigma_{x-} \Delta X_x (\Delta Y_{\lambda x})^2. \end{aligned}$$

Hence,

$$\begin{aligned} \mathbb{E}\left(S_t S_u \sigma_u^2\right) = & \mathbb{E}\left(S_t^2 \sigma_t^2\right) - 2\lambda \int_t^u \mathbb{E}\left(S_t S_x \sigma_x^2\right) dx + 2\eta_1 \int_t^u \mathbb{E}\left(S_t S_x \sigma_x\right) dx + \eta_2 \int_t^u \mathbb{E}\left(S_t S_x\right) dx \\ & + 2\kappa_{1,1} \int_t^u \mathbb{E}\left(S_t \sigma_x^2\right) dx + \kappa_{1,2} \int_t^u \mathbb{E}\left(S_t \sigma_x\right) dx. \end{aligned}$$

□

Computing $S_a S_b \sigma_u$ **for** $u \geq b \geq a$

$$\begin{aligned} S_a S_b \sigma_u &= S_a S_b \sigma_b + S_a S_b (\sigma_u - \sigma_b) \\ &= S_a S_b \sigma_b + S_a S_b (\int_b^u -\lambda \sigma_{x-} dx + dY_{\lambda x}). \end{aligned}$$

Hence,

$$\mathbb{E}\left(S_a S_b \sigma_u\right) = \mathbb{E}\left(S_a S_b \sigma_b\right) - \lambda \int_b^u \mathbb{E}\left(S_a S_b \sigma_x\right) dx + \eta_1 \int_b^u \mathbb{E}\left(S_a^2\right) dx.$$

□

Computing $S_a S_b \sigma_u^2$ **for** $u \geq b \geq a$

$$\begin{aligned} S_a S_b \sigma_u^2 &= S_a S_b \sigma_b^2 + S_a S_b (\sigma_u^2 - \sigma_b^2) \\ &= S_a S_b \sigma_b^2 + S_a S_b (-2\lambda \int_b^u \sigma_{x-}^2 dx + 2 \int_b^u \sigma_{x-} dY_{\lambda x} + [\sigma]_u - [\sigma]_b). \end{aligned}$$

Hence,

$$\mathbb{E}\left(S_a S_b \sigma_u^2\right) = \mathbb{E}\left(S_a S_b \sigma_b^2\right) - 2\lambda \int_b^u \mathbb{E}\left(S_a S_b \sigma_x^2\right) dx + 2\eta_1 \int_b^u \mathbb{E}\left(S_a S_b \sigma_x\right) dx + \eta_2 \int_b^u \mathbb{E}\left(S_a S_b\right) dx.$$

□

Computing $\mathbb{E}\left(S_a S_b S_c S_d\right)$ **for** $a \leq b \leq c \leq d$

$$S_a S_b S_c S_d = S_a S_b S_c^2 + S_a S_b S_c (S_d - S_c),$$

so,

$$\mathbb{E}\left(S_a S_b S_c S_d\right) = \mathbb{E}\left(S_a S_b S_c^2\right).$$

□

From the results above, $Cov(s_i^2, s_{i+s}^2)$ follows straightforwardly .

□

A.1.1 Proofs for Section 5

Proof of Proposition 5.1 Recall that $H_t = S_t^2 - IV_t$. Since we have already computed the moments of S and IV, we only have to compute the joint moments of these two processes. The computations are analogous to the ones carried out in the proof of Proposition 5.2, so we will refer to that proof for more details. □

Proof of Proposition 5.2 Recall that $G_t = S_t^2 - [S]_t$. Since we have already computed the moments of S^2 and $[S]$, we basically just have to compute the joint moments of these two processes in order to derive the mean, variance and covariance of G.

Mean The mean is given by

$$\mathbb{E}\left(G_t\right) = \mathbb{E}\left(S_t^2\right) - \mathbb{E}\left([S]_t\right) = 0.$$

□

Variance Next, we compute the second moment of G.

$$G_t^2 = S_t^4 - 2S_t^2[S]_t + [S]_t^2,$$

where

$$S_t^2[S]_t = \int_0^t S_{s-}^2 \sigma_{s-}^2 d[vW+X]_s + \int_0^t [S]_{s-}(2S_{s-}\sigma_{s-}d(vW_s+X_s) + \sigma_{s-}^2 d[vW+X]_s) + [S^2,[S]]_t,$$

and

$$\begin{aligned}[S^2,[S]]_t &= \int_0^t \sigma_{s-}^2 d[S^2,[vW+X]]_s = \sum_{0\leq s\leq t} \sigma_{s-}^2(2S_{s-}\sigma_{s-}\Delta X_s + \sigma_{s-}^2(\Delta X_s)^2)(\Delta X_s)^2 \\ &= 2\sum_{0\leq s\leq t} S_{s-}\sigma_{s-}^3(\Delta X_s)^3 + \sum_{0\leq s\leq t}\sigma_{s-}^4(\Delta X_s)^4.\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{E}\left(S_t^2[S]_t\right) = \int_0^t \mathbb{E}\left(S_s^2\sigma_s^2\right) ds + \left(v^2+\xi_2\right)\int_0^t \mathbb{E}\left([S]_s\sigma_s^2\right) ds \\ + 2\xi_3\int_0^t \mathbb{E}\left(S_s\sigma_s^3\right) ds + \xi_4\int_0^t \mathbb{E}\left(\sigma_s^4\right) ds.\end{aligned}$$

All quantities above are already known, so the result for the variance follows directly from above. □

Covariance

$$\begin{aligned}Cov(G_t, G_{t+s}) &= Cov(S_t^2 - [S]_t, S_{t+s}^2 - [S]_{t+s})\\ &= \mathbb{E}((S_t^2 - [S]_t)(S_{t+s}^2 - [S]_{t+s})) - \mathbb{E}(S_t^2 - [S]_t)\mathbb{E}(S_{t+s}^2 - [S]_{t+s}),\end{aligned}$$

where

$$\mathbb{E}((S_t^2 - [S]_t)(S_{t+s}^2 - [S]_{t+s})) = \mathbb{E}S_t^2 S_{t+s}^2 - \mathbb{E}[S]_t S_{t+s}^2 - \mathbb{E}[S]_{t+s} S_t^2 + \mathbb{E}[S]_t[S]_{t+s}.$$

□

Computing $\mathbb{E}S_t^2 S_{t+s}^2$

$$\begin{aligned}\mathbb{E}\left(S_t^2 S_{t+s}^2\right) &= \mathbb{E}\left(S_t^2 (S_t + (S_{t+s} - S_t))^2\right)\\ &= \mathbb{E}\left(S_t^4\right) + 2\underbrace{\mathbb{E}\left(S_t^3 (S_{t+s} - S_t)\right)}_{=0} + \mathbb{E}(S_t^2 (S_{t+s} - S_t)^2).\end{aligned}$$

$$(S_{t+s} - S_t)^2 = 2\int_t^{t+s} S_{u-} dS_u - 2\int_t^{t+s} S_t dS_u + \int_t^{t+s} \sigma_{u-}^2 d[vW + X]_u$$

Hence,

$$\mathbb{E}\left(S_t^2 (S_{t+s} - S_t)^2\right) = \left(v^2 + \xi_2\right)\int_t^{t+s} \mathbb{E}\left(S_t^2 \sigma_u^2\right) du.$$

□

Computing $\mathbb{E}\left(S_t^2 \sigma_u^2\right)$ **for** $u \geq t$

$$\begin{aligned}S_t^2 \sigma_u^2 &= S_t^2 \sigma_t^2 + S_t^2(\sigma_u^2 - \sigma_t^2) = S_t^2 \sigma_t^2 + \int_t^u S_t^2 d\sigma_s^2\\ &= S_t^2 \sigma_t^2 + 2\int_t^u S_t^2 \sigma_{s-} d\sigma_s + \int_t^u S_t^2 d[\sigma]_s\\ &= S_t^2 \sigma_t^2 - 2\lambda\int_t^u S_t^2 \sigma_{s-}^2 ds + 2\int_t^u S_t^2 \sigma_{s-} dY_{\lambda s} + \int_t^u S_t^2 d[\sigma]_s.\end{aligned}$$

Hence,

$$\frac{d}{du}\mathbb{E}\left(S_t^2 \sigma_u^2\right) = -2\lambda\mathbb{E}\left(S_t^2 \sigma_u^2\right) + 2\eta_1\mathbb{E}\left(S_t^2 \sigma_u\right) + \eta_2\mathbb{E}\left(S_t^2\right).$$

□

Computing $\mathbb{E}\left(S_t^2 \sigma_u\right)$ **for** $u \geq t$

$$\begin{aligned}S_t^2 \sigma_u &= S_t^2 \sigma_t + S_t^2(\sigma_u - \sigma_t) = S_t^2 \sigma_t + \int_t^u S_t^2 d\sigma_s\\ &= S_t^2 \sigma_t - \lambda\int_t^u S_t^2 \sigma_{s-} ds + \int_t^u S_t^2 dY_{\lambda s}.\end{aligned}$$

Hence,

$$\frac{d}{du}\mathbb{E}\left(S_t^2 \sigma_u\right) = -\lambda\mathbb{E}\left(S_t^2 \sigma_u\right) + \eta_1\mathbb{E}\left(S_t^2\right).$$

□

Computing $\mathbb{E}\left([S]_t S_{t+s}^2\right)$

$$\begin{aligned}[S]_t S_{t+s}^2 &= [S]_t S_t^2 + [S]_t (S_{t+s}^2 - S_t^2) \\ &= [S]_t S_t^2 + [S]_t (2\int_t^{t+s} S_{u-} dS_u + [S]_{t+s} - [S]_t).\end{aligned}$$

Hence,

$$\mathbb{E}[S]_t S_{t+s}^2 = \mathbb{E}[S]_t S_t^2 + \left(v^2 + \xi_2\right) \int_t^{t+s} \mathbb{E}[S]_t \sigma_{u-}^2 du.$$

□

Computing $\mathbb{E}\left([S]_{t+s} S_t^2\right)$

$$\begin{aligned}[S]_{t+s} S_t^2 &= [S]_t S_t^2 + S_t^2 ([S]_{t+s} - [S]_t) \\ &= [S]_t S_t^2 + \int_0^t S_t^2 \sigma_{u-}^2 d[vW + X]_u.\end{aligned}$$

Hence,

$$\mathbb{E}\left([S]_{t+s} S_t^2\right) = \mathbb{E}\left([S]_t S_t^2\right) + \left(v^2 + \xi_2\right) \int_t^{t+s} \mathbb{E}\left(S_t^2 \sigma_u^2\right) du.$$

□

Computing $\mathbb{E}\left([S]_t [S]_{t+s}\right)$ We have already shown that

$$\mathbb{E}\left([S]_t [S]_{t+s}\right) = \mathbb{E}\left([S]_t^2\right) + \left(v^2 + \xi_2\right) \int_t^{t+s} \mathbb{E}\left([S]_t \sigma_u^2\right) du.$$

So altogether, we obtain:

$$\begin{aligned}\mathbb{E}((S_t^2 - [S]_t)(S_{t+s}^2 - [S]_{t+s})) &= \mathbb{E}\left(S_t^2 S_{t+s}^2\right) - \mathbb{E}\left([S]_t S_{t+s}^2\right) - \mathbb{E}\left([S]_{t+s} S_t^2\right) + \mathbb{E}\left([S]_t [S]_{t+s}\right) \\ &= \mathbb{E}\left((S_t^2 - [S]_t)^2\right) = \mathbb{E}\left(G_t^2\right).\end{aligned}$$

□

So

$$Cov(G_t, G_{t+s}) = \mathbb{E}\left(G_t^2\right) - \mathbb{E}\left(G_t\right) \mathbb{E}\left(G_{t+s}\right) = Var(G_t) = O(t^2),$$

for $t \to 0$, which does not depend on s.

Computing $Cov(G_t, [S]_t)$

$$\begin{aligned}Cov(G_t, [S]_t) &= \mathbb{E}\left(G_t [S]_t\right) - \mathbb{E}\left(G_t\right) \mathbb{E}\left([S]_t\right) = \mathbb{E}\left(G_t [S]_t\right) \\ &= \mathbb{E}\left(S_t^2 [S]_t\right) - \mathbb{E}\left([S]_t^2\right).\end{aligned}$$

□

Note here that the corresponding properties for the increments of G follow directly from the results above. □

Proof of Proposition 5.4 The first and second order properties of the realised variance can be derived from the corresponding results of the squared returns:

$$\mathbb{E}[S_\delta]_{[(i-1)h,ih]} = \sum_{j=1}^{M} \mathbb{E}\left(s_{j,i}^2\right),$$

$$Var\left([S_\delta]_{[(i-1)h,ih]}\right) = \sum_{i=1}^{M} Var(s_{j,i}^2) + 2 \sum_{1\leq j<k\leq M} Cov(s_{j,i}^2, s_{k,i}^2).$$

And for $s > 0$:

$$Cov\left([S_\delta]_{[(i-1)h,ih]}, S_\delta]_{[(i+s-1)h,(i+s)h]}\right) = \sum_{1\leq j,\, k\leq M} Cov(s_{j,i}^2, s_{k,i+s}^2).$$

Finally, we express the (quite lengthy) formulae of the variance and covariance of the realised variance in form of a Taylor series expansion and focus on the first terms only. These leads to the results given in Proposition 5.4. □

A.1.2 Superposition Model

Now we assume that σ is given by a superposition model as defined in Section 5.4. As already mentioned, we can derive the mean, variance and covariance even in that more general model, but due to the fact that the driving processes of the asset price and the volatility components are dependent, the formulae become quite lengthy. However, here we focus the case $J = 2$ and we compute the quadratic variation of the price process and use that as an $O(M^{-1})$–approximation of the corresponding formulae of the realised variance. Since we only consider the case $J = 2$ here, we can write the weights as $w_1 = w$, $w_2 = 1 - w$.

Proposition A.1 *Let* $v^2 + \xi_2 = 1$. *Then:*

$$\mathbb{E}\left([S]_t\right) = \left\{\left(w^2 - w + \frac{1}{2}\right)\eta_2 + \eta_1^2\right\} h,$$

and

$$\begin{aligned} Var([S]) = \Big(& a_1 + a_2 h + a_3 e^{-\lambda^{(1)}h} + a_4 e^{-\lambda^{(2)}h} + a_5 e^{-2\lambda^{(1)}h} + a_6 e^{-2\lambda^{(2)}h} \\ &+ a_7 e^{-(\lambda^{(1)}+\lambda^{(2)}h)}\Big) + \Big(a_8 + a_9 h + a_{10} e^{-\lambda^{(1)}h} + a_{11} e^{-\lambda^{(2)}h} + a_{12} e^{-2\lambda^{(1)}h} \\ &+ a_{13} e^{-2\lambda^{(2)}h} + a_{14} e^{-(\lambda^{(1)}+\lambda^{(2)}h)}\Big) + \xi_4 h\, a_{15}, \end{aligned}$$

where

$$a_1 = \left(-\frac{4}{\lambda^{(2)\,2}} + 8\,\frac{w}{\lambda^{(2)\,2}} - 4\,\frac{w^2}{\lambda^{(2)\,2}} - 4\,\frac{w^2}{\lambda^{(1)\,2}}\right)\eta_2\eta_1^2 + \left(-\frac{5}{3\lambda^{(2)\,2}} - 5\,\frac{w^2}{\lambda^{(2)\,2}} + 5\,\frac{w}{\lambda^{(2)\,2}}\right.$$

$$
\begin{aligned}
&+\frac{5}{3}\,\frac{w^3}{\lambda^{(2)\,2}}-\frac{5}{3}\,\frac{w^3}{\lambda^{(1)\,2}}\Big)\,\eta_3\eta_1-\frac{1}{4}\,\Big(-4\,\lambda^{(1)\,4}w^3+14\,\lambda^{(1)\,2}\lambda^{(2)\,2}w^2+\lambda^{(1)\,4}w^4\\
&-4\,\lambda^{(1)\,4}w+\lambda^{(1)\,4}+10\,\lambda^{(1)\,2}\lambda^{(2)\,2}w^4-4\,\lambda^{(1)\,2}\lambda^{(2)\,2}w+6\,\lambda^{(1)\,4}w^2+\lambda^{(2)\,4}w^4\\
&+2\,\lambda^{(1)}\,\lambda^{(2)\,3}w^4+\lambda^{(1)\,2}\lambda^{(2)\,2}+2\,\lambda^{(1)\,3}\lambda^{(2)}-20\,\lambda^{(1)\,2}\lambda^{(2)\,2}w^3+2\,\lambda^{(1)\,3}\lambda^{(2)}\,w^4\\
&-8\,\lambda^{(1)\,3}\lambda^{(2)}\,w^3+12\,\lambda^{(1)\,3}\lambda^{(2)}\,w^2-8\,\lambda^{(1)\,3}\lambda^{(2)}\,w\Big)\,\eta_2^2\left\{\lambda^{(1)\,2}\lambda^{(2)\,2}\left(\lambda^{(1)}+\lambda^{(2)}\right)^2\right\}^{-1}\\
&+\left\{\left(-\frac{1}{8\,\lambda^{(2)\,2}}-\frac{1}{8\,\lambda^{(1)\,2}}\right)w^4+\frac{1}{2}\,\frac{w^3}{\lambda^{(2)\,2}}-\frac{3}{4}\,\frac{w^2}{\lambda^{(2)\,2}}+\frac{1}{2}\,\frac{w}{\lambda^{(2)\,2}}-\frac{1}{8\,\lambda^{(2)\,2}}\right\}\eta_4,\\
a_2=&\ \frac{1}{\lambda^{(1)}\lambda^{(2)}\left(\lambda^{(1)}+\lambda^{(2)}\right)}\Big\{\Big(16\,\lambda^{(2)\,2}w^2+16\,\lambda^{(1)\,2}-32\,\lambda^{(1)\,2}w-32\,\lambda^{(1)}\,\lambda^{(2)}\,w\\
&+16\,\lambda^{(1)}\,\lambda^{(2)}+32\,\lambda^{(1)}\,\lambda^{(2)}\,w^2+16\,\lambda^{(1)\,2}w^2\Big)\,\eta_2\eta_1^2+\Big(8\,\lambda^{(1)\,2}+8\,\lambda^{(1)}\,\lambda^{(2)}\\
&+24\,\lambda^{(1)\,2}w^2+8\,w^3\lambda^{(2)\,2}+24\,\lambda^{(1)}\,\lambda^{(2)}\,w^2-24\,\lambda^{(1)}\,\lambda^{(2)}\,w-8\,w^3\lambda^{(1)\,2}\\
&-24\,\lambda^{(1)\,2}w\Big)\,\eta_3\eta_1+\Big(12\,\lambda^{(1)}\,w^4\lambda^{(2)}-8\,w^3\lambda^{(1)\,2}+2\,\lambda^{(1)\,2}+12\,\lambda^{(1)\,2}w^2\\
&-8\,\lambda^{(1)}\,\lambda^{(2)}\,w+2\,w^4\lambda^{(1)\,2}+2\,\lambda^{(1)}\,\lambda^{(2)}-8\,\lambda^{(1)\,2}w+20\,\lambda^{(1)}\,\lambda^{(2)}\,w^2\\
&-24\,\lambda^{(1)}\,\lambda^{(2)}\,w^3+2\,w^4{\lambda^{(2)}}^2\Big)\,{\eta_2}^2+\Big(\lambda^{(1)}\,\lambda^{(2)}-4\,{\lambda^{(1)}}^2w-4\,w^3{\lambda^{(1)}}^2\\
&+6\,\lambda^{(1)\,2}w^2+w^4\lambda^{(1)\,2}+6\,\lambda^{(1)}\,\lambda^{(2)}\,w^2+2\,\lambda^{(1)}\,w^4\lambda^{(2)}-4\,\lambda^{(1)}\,\lambda^{(2)}\,w^3\\
&-4\,\lambda^{(1)}\,\lambda^{(2)}\,w+\lambda^{(1)\,2}+w^4\lambda^{(2)\,2}\Big)\,\eta_4\Big\},\\
a_3=&\ \frac{4\,(3\,\eta_1\eta_2+\eta_3 w)\,w^2\eta_1}{3\,\lambda^{(1)\,2}},\\
a_4=&-\frac{4\left(6\,w-3\,w^2-3\right)\eta_2\eta_1^2}{3\lambda^{(2)\,2}}-\frac{4\left(-3\,w^2+3\,w+w^3-1\right)\eta_3\eta_1}{3\lambda^{(2)\,2}},\\
a_5=&\ \frac{w^3\eta_3\eta_1}{3\lambda^{(1)\,2}}+\frac{w^4\eta_2^2}{4\lambda^{(1)\,2}}+\frac{w^4\eta_4}{8\lambda^{(1)\,2}},\\
a_6=&\ \frac{\left(8-24\,w+24\,w^2-8\,w^3\right)\eta_3\eta_1}{24\lambda^{(2)\,2}}+\frac{\left(-24\,w^3+36\,w^2+6\,w^4-24\,w+6\right)\eta_2^2}{24\lambda^{(2)\,2}}\\
&+\frac{-12\,\eta_4 w+3\,w^4\eta_4-12\,\eta_4 w^3+18\,w^2\eta_4+3\,\eta_4}{24\lambda^{(2)\,2}}\\
a_7=&\ 2\,\frac{\eta_2^2w^2\left(1-2\,w+w^2\right)}{\left(\lambda^{(1)}+\lambda^{(2)}\right)^2},\\
a_8=&\ \Bigg[-4\,\frac{w\,\left(w\,\lambda^{(1)\,2}+2\,\lambda^{(1)}\,\lambda^{(2)}+\lambda^{(2)\,2}\right)\eta_1^3}{\lambda^{(1)\,2}\left(\lambda^{(1)}+\lambda^{(2)}\right)^2}-w\,\Big(4\,\lambda^{(1)\,2}w^3-3\,w^2\lambda^{(1)\,2}+5\,\lambda^{(2)\,2}w^2\\
&+10\,w^2\lambda^{(1)}\,\lambda^{(2)}+2\,w\,\lambda^{(1)\,2}-8\,w\,\lambda^{(1)}\,\lambda^{(2)}-4\,\lambda^{(2)\,2}w+4\,\lambda^{(1)}\,\lambda^{(2)}+2\,\lambda^{(2)\,2}\Big)\,\eta_2\eta_1\\
&\left\{\lambda^{(1)\,2}\left(\lambda^{(1)}+\lambda^{(2)}\right)^2\right\}^{-1}-\frac{1}{3}\,\frac{w^4\eta_3}{\lambda^{(1)\,2}}\Bigg]\,\kappa_{2,1}^{(1)}\\
&+\left\{-\frac{1}{2}\,\frac{w^2{\eta_1}^2}{{\lambda^{(1)}}^2}-\frac{1}{4}\,\frac{\left(2\,w^2-2\,w+1\right)w^2\eta_2}{{\lambda^{(1)}}^2}\right\}\kappa_{2,2}^{(1)}
\end{aligned}
$$

$$+ \Big\{4 \left(-\lambda^{(2)\,2} w^2 + w\,\lambda^{(1)\,2} + 2\,w\,\lambda^{(1)}\,\lambda^{(2)} + 2\,\lambda^{(2)\,2} w - \lambda^{(1)\,2} - 2\,\lambda^{(1)}\,\lambda^{(2)} - \lambda^{(2)\,2}\right)$$

$$\eta_1^3 \left(\lambda^{(2)\,2} \left(\lambda^{(1)} + \lambda^{(2)}\right)^2\right)^{-1} + \left(-4\,\lambda^{(2)\,2} w^4 + 10\,w^3 \lambda^{(1)}\,\lambda^{(2)} + 5\,\lambda^{(1)\,2} w^3 + 13\,\lambda^{(2)\,2} w^3\right.$$

$$-17\,\lambda^{(2)\,2} w^2 - 22\,w^2 \lambda^{(1)}\,\lambda^{(2)} - 11\,w^2 \lambda^{(1)\,2} + 9\,w\,\lambda^{(1)\,2} + 18\,w\,\lambda^{(1)}\,\lambda^{(2)} + 11\,\lambda^{(2)\,2} w$$

$$\left.-3\,\lambda^{(1)\,2} - 6\,\lambda^{(1)}\,\lambda^{(2)} - 3\,\lambda^{(2)\,2}\right)$$

$$\eta_2 \eta_1 \left\{\lambda^{(2)\,2} \left(\lambda^{(1)} + \lambda^{(2)}\right)^2\right\}^{-1} - \frac{1}{3}\,\frac{\left(w^4 - 4\,w^3 + 6\,w^2 - 4\,w + 1\right)\eta_3}{\lambda^{(2)\,2}}\Bigg\}\,\kappa_{2,1}^{(2)}$$

$$+ \left\{-\frac{1}{2}\,\frac{\left(w^2 - 2\,w + 1\right)\eta_1^2}{\lambda^{(2)\,2}} - \frac{1}{4}\,\frac{\left(2\,w^4 - 6\,w^3 + 7\,w^2 - 4\,w + 1\right)\eta_2}{\lambda^{(2)\,2}}\right\}\kappa_{2,2}^{(2)},$$

$$a_9 = \frac{2}{3}\,\kappa_{2,1}^{(1)} \left(6\,\lambda^{(2)} + 6\,\lambda^{(1)}\,w\right) w\,\eta_1^3 \left\{\lambda^{(1)}\,\left(\lambda^{(1)} + \lambda^{(2)}\right)\right\}^{-1} + \frac{2}{3}\,\kappa_{2,1}^{(1)} \Big(6\,\lambda^{(1)}\,w^3$$

$$+3\,\lambda^{(1)}\,w - 3\,w^2 \lambda^{(1)} - 6\,\lambda^{(2)}\,w + 9\,\lambda^{(2)}\,w^2 + 3\,\lambda^{(2)}\Big)\,w\,\eta_2 \eta_1 \left\{\lambda^{(1)}\,\left(\lambda^{(1)} + \lambda^{(2)}\right)\right\}^{-1}$$

$$+\frac{2}{3}\,\frac{\kappa_{2,1}^{(1)}\left(\lambda^{(1)}\,w^3 + \lambda^{(2)}\,w^3\right) w\,\eta_3}{\lambda^{(1)}\,\left(\lambda^{(1)} + \lambda^{(2)}\right)} + \frac{\kappa_{2,2}^{(1)} w^2 \eta_1^2}{\lambda^{(1)}} + \frac{1}{2}\,\frac{\kappa_{2,2}^{(1)}\left(2\,w^2 - 2\,w + 1\right) w^2 \eta_2}{\lambda^{(1)}}$$

$$+\frac{2}{3}\,\frac{\kappa_{2,1}^{(2)}\left(6\,\lambda^{(2)}\,w^2 + 6\,\lambda^{(2)} - 12\,\lambda^{(2)}\,w + 6\,\lambda^{(1)} - 6\,\lambda^{(1)}\,w\right)\eta_1^3}{\lambda^{(2)}\,\left(\lambda^{(1)} + \lambda^{(2)}\right)}$$

$$+\frac{2}{3}\,\kappa_{2,1}^{(2)} \Big(6\,w^4 \lambda^{(2)} + 21\,w^2 \lambda^{(1)} - 18\,\lambda^{(1)}\,w + 6\,\lambda^{(2)} - 9\,\lambda^{(1)}\,w^3 - 21\,\lambda^{(2)}\,w^3$$

$$-21\,\lambda^{(2)}\,w + 6\,\lambda^{(1)} + 30\,\lambda^{(2)}\,w^2\Big)\,\eta_2 \eta_1 \left\{\lambda^{(2)}\,\left(\lambda^{(1)} + \lambda^{(2)}\right)\right\}^{-1}$$

$$+\frac{2}{3}\,\kappa_{2,1}^{(2)} \Big(-4\,\lambda^{(1)}\,w + 6\,w^2 \lambda^{(1)} + \lambda^{(1)}\,w^4 + \lambda^{(1)} - 4\,\lambda^{(1)}\,w^3 - 4\,\lambda^{(2)}\,w^3 + \lambda^{(2)}$$

$$+6\,\lambda^{(2)}\,w^2 - 4\,\lambda^{(2)}\,w + w^4 \lambda^{(2)}\Big)\,\eta_3 \left\{\lambda^{(2)}\,\left(\lambda^{(1)} + \lambda^{(2)}\right)\right\}^{-1}$$

$$+\frac{1}{2}\,\frac{\kappa_{2,2}^{(2)}\left(2\,w^2 - 4\,w + 2\right)\eta_1^2}{\lambda^{(2)}} + \frac{1}{2}\,\frac{\kappa_{2,2}^{(2)}\left(2\,w^4 - 6\,w^3 + 7\,w^2 - 4\,w + 1\right)\eta_2}{\lambda^{(2)}},$$

$$a_{10} = \left\{4\,\frac{w\,\eta_1^3}{\lambda^{(1)\,2}} + 2\,\frac{\left(2\,w^2 - 2\,w + 1\right) w\,\eta_2 \eta_1}{\lambda^{(1)\,2}}\right\}\kappa_{2,1}^{(1)},$$

$$a_{11} = \left\{-2\,\frac{\left(-2 + 2\,w\right)\eta_1^3}{\lambda^{(2)\,2}} - 2\,\frac{\left(-1 + 2\,w^3 - 4\,w^2 + 3\,w\right)\eta_2 \eta_1}{\lambda^{(2)\,2}}\right\}\kappa_{2,1}^{(2)},$$

$$a_{12} = \left(\frac{w^3 \eta_1 \eta_2}{\lambda^{(1)\,2}} + \frac{1}{3}\,\frac{w^4 \eta_3}{\lambda^{(1)}}^2\right)\kappa_{2,1}^{(1)} + \left\{\frac{1}{2}\,\frac{w^2 \eta_1^2}{\lambda^{(1)\,2}} + \frac{1}{12}\,\frac{\left(3 - 6\,w + 6\,w^2\right) w^2 \eta_2}{\lambda^{(1)\,2}}\right\}\kappa_{2,2}^{(1)},$$

$$a_{13} = \left\{\frac{1}{12}\,\frac{\left(12 - 12\,w^3 - 36\,w + 36\,w^2\right)\eta_2 \eta_1}{\lambda^{(2)\,2}} + \frac{1}{12}\,\frac{\left(4 + 24\,w^2 - 16\,w^3 - 16\,w + 4\,w^4\right)\eta_3}{\lambda^{(2)\,2}}\right\}$$

$$\kappa_{2,1}^{(2)} + \left\{\frac{1}{12}\,\frac{\left(6 + 6\,w^2 - 12\,w\right)\eta_1^2}{\lambda^{(2)\,2}} + \frac{1}{12}\,\frac{\left(3 - 12\,w - 18\,w^3 + 21\,w^2 + 6\,w^4\right)\eta_2}{\lambda^{(2)\,2}}\right\}\kappa_{2,2}^{(2)},$$

$$a_{14} = \left\{ 2\,\frac{w\,\left(-2+2\,w\right)\eta_1{}^3}{\left(\lambda^{(1)}+\lambda^{(2)}\right)^2} + 2\,\frac{w\,\left(-1+2\,w^3-4\,w^2+3\,w\right)\eta_2\eta_1}{\left(\lambda^{(1)}+\lambda^{(2)}\right)^2} \right\}\kappa_{2,1}^{(1)}$$
$$+ \left\{ 2\,\frac{w\,\left(-2+2\,w\right)\eta_1{}^3}{\left(\lambda^{(1)}+\lambda^{(2)}\right)^2} + 2\,\frac{w\,\left(-1+2\,w^3-4\,w^2+3\,w\right)\eta_2\eta_1}{\left(\lambda^{(1)}+\lambda^{(2)}\right)^2} \right\}\kappa_{2,1}^{(2)},$$
$$a_{15} = \left\{ \eta_1^4 + \left(3-6\,w+6\,w^2\right)\eta_2\eta_1^2 + \left(4\,w^2+\frac{4}{3}-4\,w\right)\eta_3\eta_1 \right.$$
$$\left. + \left(6\,w^2-3\,w+\frac{3}{4}-6\,w^3+3\,w^4\right)\eta_2^2 + \left(-w^3+\frac{1}{4}-w+\frac{3}{2}\,w^2+\frac{1}{2}\,w^4\right)\eta_4 \right\},$$

$$Cov([S]_t,[S]_{t+s}) = b_1\left(e^{-\lambda^{(1)}\,(s+1)h} - 2\,e^{-\lambda^{(1)}\,sh} + e^{-\lambda^{(1)}\,(s-1)h}\right)$$
$$+ b_2\left(e^{-\lambda^{(2)}\,(s+1)h} - 2\,e^{-\lambda^{(2)}\,sh} + e^{-\lambda^{(2)}\,(s-1)h}\right)$$
$$+ b_3\left(e^{-2\lambda^{(1)}\,(s+1)h} - 2\,e^{-2\lambda^{(1)}\,sh} + e^{-2\lambda^{(1)}\,(s-1)h}\right)$$
$$+ b_4\left(e^{-2\lambda^{(2)}\,(s+1)h} - 2\,e^{-2\lambda^{(2)}\,sh} + e^{-2\lambda^{(2)}\,(s-1)h}\right)$$
$$+ b_5\left(e^{-(s+1)h\left(\lambda^{(1)}+\lambda^{(2)}\right)} - 2\,e^{-sh\left(\lambda^{(1)}+\lambda^{(2)}\right)} + e^{-(s-1)h\left(\lambda^{(1)}+\lambda^{(2)}\right)}\right),$$

where

$$b_1 = \frac{1}{\lambda^{(1)\,2}}\left[\left(2\,w^2\eta_2\eta_1^2+\frac{2}{3}\,w^3\eta_3\eta_1\right) + \left(2\,w\,\eta_1^3 + \left\{2\,w^3-2\,w^2+w\right)\eta_2\eta_1\right\}\kappa_{2,1}^{(1)}\right],$$
$$b_2 = \frac{1}{\lambda^{(2)\,2}}\left[\left\{\left(2\,w^2-4\,w+2\right)\eta_2\eta_1^2 + \left(-2\,w+\frac{2}{3}-\frac{2}{3}\,w^3+2\,w^2\right)\eta_3\eta_1\right\}\right.$$
$$\left. + \left\{(-2\,w+2)\,\eta_1^3 + \left(4\,w^2-2\,w^3-3\,w+1\right)\eta_2\eta_1\right\}\kappa_{2,1}^{(2)}\right],$$
$$b_3 = \frac{1}{\lambda^{(1)\,2}}\left[\left(\frac{1}{6}\,w^3\eta_3\eta_1+\frac{1}{8}\,w^4\eta_2^2+\frac{1}{16}\,w^4\eta_4\right) + \left(\frac{1}{2}\,w^3\eta_2\eta_1+\frac{1}{6}\,w^4\eta_3\right)\kappa_{2,1}^{(1)}\right.$$
$$\left. + \left\{\frac{1}{4}\,w^2\eta_1^2 + \left(\frac{1}{4}\,w^4-\frac{1}{4}\,w^3+\frac{1}{8}\,w^2\right)\eta_2\right\}\kappa_{2,2}^{(1)}\right],$$
$$b_4 = \frac{1}{\lambda^{(2)\,2}}\left[-36\,\lambda^{(1)\,4}\kappa_{2,2}^{(2)}\eta_2 w^3\left\{\left(-\frac{1}{6}\,w^3+\frac{1}{6}+\frac{1}{2}\,w^2-\frac{1}{2}\,w\right)\eta_3\eta_1\right.\right.$$
$$+ \left(-\frac{1}{2}\,w^3-\frac{1}{2}\,w+\frac{1}{8}+\frac{3}{4}\,w^2+\frac{1}{8}\,w^4\right)\eta_2^2 - \frac{1}{4}\,\eta_4 w + \frac{1}{16}\,\eta_4$$
$$\left. + \frac{1}{16}\,w^4\eta_4 - \frac{1}{4}\,\eta_4 w^3 + \frac{3}{8}\,w^2\eta_4\right\} + \left\{\left(\frac{3}{2}\,w^2-\frac{3}{2}\,w+\frac{1}{2}-\frac{1}{2}\,w^3\right)\eta_2\eta_1\right.$$
$$\left. + \left(-\frac{2}{3}\,w^3+w^2-\frac{2}{3}\,w+\frac{1}{6}\,w^4+\frac{1}{6}\right)\eta_3\right\}\kappa_{2,1}^{(2)}$$
$$\left. + \left\{\left(-\frac{1}{2}\,w+\frac{1}{4}+\frac{1}{4}\,w^2\right)\eta_1^2 + \left(\frac{7}{8}\,w^2-\frac{3}{4}\,w^3-\frac{1}{2}\,w+\frac{1}{8}+\frac{1}{4}\,w^4\right)\eta_2\right\}\kappa_{2,2}^{(2)}\right],$$
$$b_5 = \frac{1}{\left(\lambda^{(1)}+\lambda^{(2)}\right)^2}\left[\left(w^4-2\,w^3+w^2\right)\eta_2^2 + \left\{\left(2\,w^2-2\,w\right)\eta_1^3 + \left(2\,w^4-4\,w^3+3\,w^2\right.\right.\right.$$
$$\left.\left.\left. -w\right)\eta_2\eta_1\right\}\kappa_{2,1}^{(1)} + \left\{\left(2\,w^2-2\,w\right)\eta_1^3 + \left(2\,w^4-4\,w^3+3\,w^2-w\right)\eta_2\eta_1\right\}\kappa_{2,1}^{(2)}\right].$$

Proof of Proposition A.1 In the following we will assume that $1 \leq i,\ j \leq J$ and $v^2 + \xi_2 = 1$.

Moments of σ The moments of σ can be straightforwardly derived from the corresponding moments of $\tau^{(i)}$ by

$$\sigma_t = \sum_{j=1}^{J} w_j \ \tau_t^{(j)}.$$

In the following we use the notation $\eta_k^{(i)} = \lambda^{(i)}\eta_k$, for $i = 1, \ldots, J$ and $k \in \mathbb{N}$. $\square$

Mean

$$\mathbb{E}\left([S]_t\right) = \mathbb{E}\left(\int_0^t \sigma_{u-}^2 d[X]_u\right) = \mathbb{E}\left(\sigma_0^2\right) t.$$

$\square$

Variance

$$\mathbb{E}\left([S]_t^2\right) = 2\int_0^t \mathbb{E}\left([S]_u \sigma_u^2\right) du + \xi_4 \int_0^t \mathbb{E}\left(\sigma_u^4\right) du.$$

$\square$

Computing $\mathbb{E}\left([S]_u \sigma_u\right)$

$$[S]_u \sigma_u = \sum_{i=1}^{J} [S]_u w_i \ \tau_u^{(i)},$$

where

$$\mathbb{E}\left([S]_u \tau_u^{(i)}\right) = -\lambda^{(i)} \int_0^u \mathbb{E}\left([S]_x \tau_x^{(i)} dx\right) + \eta_1^{(i)} \int_0^u \mathbb{E}\left([S]_x\right) dx + \int_0^u \mathbb{E}\left((\tau_x^{(i)})^3\right) dx + \kappa_{2,1}^{(i)} \int_0^u \mathbb{E}\left((\tau_x^{(i)})^2\right) dx.$$

$\square$

Computing $\mathbb{E}\left([S]_u \sigma_u^2\right)$

$$[S]_u \sigma_u^2 = [S]_u \left(\sum_{i=1}^{J} w_i \ \tau_u^{(i)}\right)^2 = \sum_{i=1}^{J} [S]_u w_i^2 \ \left(\tau_u^{(i)}\right)^2 + 2 \sum_{1 \leq j < k \leq J} [S]_u w_j \ \tau_u^{(j)} w_k \ \tau_u^{(k)},$$

where

$$\frac{d}{du}\,\mathbb{E}\left([S]_u \tau_u^{(i)\,2}\right) = -2\lambda^{(i)} \mathbb{E}\left([S]_u \tau_u^{(i)\,2}\right) + 2\eta_1^{(i)} \mathbb{E}\left([S]_u \tau_u^{(i)}\right) + \mathbb{E}\left(\sigma_u^2 \tau_u^{(i)\,2}\right) + 2\kappa_{2,1}^{(i)} \mathbb{E}\left(\sigma_u^2 \tau_u^{(i)}\right) + \kappa_{2,2}^{(1)} \mathbb{E}\left(\sigma_u^2\right).$$

$\square$

Computing $\mathbb{E}\left([S]_u\tau_u^{(i)}\tau_u^{(k)}\right)$ **for** $i\neq k$

$$\begin{aligned}\tau_u^{(i)}\tau_u^{(k)} &= \int_0^u \tau_s^{(i)}d\tau_s^{(k)} + \int_0^t \tau_s^{(k)}d\tau_s^{(i)} + [\tau^{(i)},\tau^{(k)}]_u \\ &= \int_0^u \tau_s^{(i)}\left(-\lambda^{(k)}\tau_{s-}^{(k)}ds + dY^{(k)}_{\lambda^{(k)}s}\right) + \int_0^u \tau_s^{(k)}\left(-\lambda^{(i)}\tau_{s-}^{(i)}ds + dY^{(i)}_{\lambda^{(i)}s}\right) \\ &\quad + \sum_{0\le s\le u}\Delta Y^{(i)}_{\lambda^{(i)}}\Delta Y^{(k)}_{\lambda^{(k)}}. \\ &= -\left(\lambda^{(i)}+\lambda^{(k)}\right)\int_0^u \tau_{s-}^{(i)}\tau_{s-}^{(k)}ds + \int_0^u \tau_{s-}^{(i)}dY^{(k)}_{\lambda^{(k)}s} + \int_0^u \tau_{s-}^{(k)}dY^{(k)}_{\lambda^{(i)}s} + [\tau^{(i)},\tau^{(k)}]_u.\end{aligned}$$

Note: From the independence of the $Y^{(i)}$ we get:

$$\{(u,v)\in\mathbb{R}_+^2 : \nu_{\left((Y^{(i)}_{\lambda^{(i)}},Y^{(k)}_{\lambda^{(k)}}\right)}(u,v)>0\}\subset\{(u,v)\in\mathbb{R}_+^2 : uv=0\}.$$

So we obtain

$$\begin{aligned}[S]_u\tau_u^{(i)}\tau_u^{(k)} &= \int_0^u [S]_{s-}d\left(\tau_s^{(i)}\tau_s^{(k)}\right) + \int_0^u \tau_{s-}^{(i)}\tau_{s-}^{(k)}d[S]_s + [[S],\tau^{(i)}\tau^{(k)}]_u \\ &= \int_0^u [S]_{s-}\left(-\left(\lambda^{(i)}+\lambda^{(k)}\right)\tau_{s-}^{(i)}\tau_{s-}^{(k)}ds + \tau_{s-}^{(i)}dY^{(k)}_{\lambda^{(k)}s} + \tau_{s-}^{(k)}dY^{(k)}_{\lambda^{(i)}s} + d[\tau^{(i)},\tau^{(k)}]_s\right) \\ &\quad + \int_0^u \tau_{s-}^{(i)}\tau_{s-}^{(k)}\sigma_{s-}^2 d[X]_s + \int_0^u \sigma_{s-}^2 d[[X],\tau_{s-}^{(i)}\tau_{s-}^{(k)}]_s.\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{E}\left([S]_u\tau_u^{(i)}\tau_u^{(k)}\right) &= \int_0^u -\left(\lambda^{(i)}+\lambda^{(k)}\right)\mathbb{E}\left([S]_s\tau_s^{(i)}\tau_s^{(k)}\right)ds + \eta_1^{(k)}\int_0^u \mathbb{E}\left([S]_s\tau_s^{(i)}\right)ds \\ &\quad + \eta_1^{(i)}\int_0^u \mathbb{E}\left([S]_s\tau_s^{(k)}\right)ds + \int_0^u \mathbb{E}\left(\tau_s^{(i)}\tau_s^{(k)}\sigma_s^2\right)ds \\ &\quad + \kappa_{2,1}^{(k)}\int_0^u \mathbb{E}\left(\sigma_s^2\tau_s^{(i)}\right)ds + \kappa_{2,1}^{(i)}\int_0^u \mathbb{E}\sigma_s^2\tau_s^{(k)}ds,\end{aligned}$$

since

$$\int_0^u \sigma_{s-}^2 d[[X]],\tau^{(i)}\tau^{(k)}] = \sum_{0\le s\le u}\sigma_{s-}^2(\Delta X_s)^2(\tau_{s-}^{(i)}\Delta\tau_s^{(k)} + \tau_{s-}^{(k)}\Delta\tau_s^{(i)} + \Delta\left(\tau_s^{(i)}\tau_s^{(k)}\right)).$$

□

Computing $\mathbb{E}\left(\sigma_s^2\tau_s^{(i)}\right)$

$$\begin{aligned}[S]_u\tau_u^{(i)} &= \int_0^u [S]_{s-}d\tau_s^{(i)} + \int_0^u \tau_{s-}^{(i)}d[S]_s + [[S],\tau^{(i)}]_u \\ &= -\lambda^{(i)}\int_0^u [S]_{s-}\tau_{s-}^{(i)}ds + \int_0^u [S]_{s-}dY^{(i)}_{\lambda^{(i)}s} + \int_0^u \sigma_{s-}^2\tau_{s-}^{(i)}d[X]_s + \int_0^u \sigma_{s-}^2 d[[X],\tau^{(i)}]_s.\end{aligned}$$

Hence,

$$\mathbb{E}\left([S]_u\tau_u^{(i)}\right) = -\lambda^{(i)}\int_0^u \mathbb{E}\left([S]_s\tau_s^{(i)}\right)ds + \eta_1^{(i)}\int_0^u \mathbb{E}\left([S]_s\right)ds + \int_0^u \mathbb{E}\left(\sigma_s^2\tau_s^{(i)}\right)ds + \kappa_{2,1}^{(i)}\int_0^u \mathbb{E}\sigma_s^2 ds.$$

□

The computations for the covariance are completely analogous to the ones above and the corresponding calculations in the one–factor setting.

□

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