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New tests for jumps: a threshold-based approach

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New tests for jumps: a threshold-based approach *

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Abstract

In this paper we propose a test to determine whether jumps are present in a discretely sampled process or not. We use the concept of truncated power variation to construct our test statistics for (i) semimartingale models and (ii) semimartingale models with noise. The test statistics converge to infinity if jumps are present and have a normal distribution otherwise. Our method is valid (under very weak assumptions) for all semimartingales with absolute continuous characteristics and rather general model for the noise process. We finally implement the test and present the simulation results. Our simulations suggest that for semimartingale models the new test is much more powerful then tests proposed by Barndorff-Nielsen and Shephard (2006) and Aït-Sahalia and Jacod (2008).

Keywords: Central Limit Theorem; High-Frequency Data; Microstructure Noise; Semimartingale Theory; Tests for Jumps; Truncated Power Variation.

JEL Classification: C10, C13, C14.

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1 Introduction

The last years have seen a rapidly growing literature on statistical methods for high frequency data (see e.g. Barndorff-Nielsen and Shephard (2004a,b), Barndorff-Nielsen, Graversen, Jacod, Podolskij and Shephard (2006) or Jacod (2008)). In econometrics price processes are typically modeled by semimartingales, which constitute a natural class of models under the assumption of no-leverage (see Delbaen and Schachermayer (1994)). In general, semimartingales are càdlàg processes, which can be written as a sum of a continuous and a discontinuous component. For various applications it is important to be able to separate these two parts based on discrete (high frequency) observations. In particular, practitioners want to decide whether the discretely observed path of a semimartingale is continuous or not.

Quite recently, several methods have been proposed to test for jumps in semimartingale models. Barndorff-Nielsen and Shephard (2006) use the concept of bipower variation to construct a consistent estimator of the quadratic variation of the discontinuous part. This estimator is then applied to test whether a semimartingale has jumps or not. On the other hand, Aït-Sahalia and Jacod (2008) compare the power variation at different sampling frequencies to construct a test for jumps. Both tests apply for general Itô semimartingales when additionally the volatility process is also a semimartingale. Some further approaches can be found in Jiang and Oomen (2005) or in Lee and Mykland (2007).

In this paper we propose a threshold-based procedure to test for jumps. Our method is based upon the truncated power variation which has been originally introduced by Mancini (2001,2004) to obtain jump-robust estimates of some functionals of the volatility process. We combine this approach with the wild bootstrap idea (see Wu (1986)) to define a new class of test statistics. Our test statistics converge to a standard normal distribution when the semimartingale is continuous, whereas they tend to infinity for semimartingales with non-vanishing jump part. Furthermore, we construct tests for jumps in semimartingale models with noise, which are now intensively studied in the econometric literature (see e.g. Zhang, Mykland and Aït-Sahalia (2005) or Hansen and Lunde (2006)).

The advantage of our method is twofold. On the one hand, our test procedure applies for all Itô semimartingales and we require no further assumptions on the volatility process. On the other hand, the threshold-based class of statistics has very good finite sample properties. The power of our tests is much higher compared with the tests of Barndorff-Nielsen and Shephard (2006) and Aït-Sahalia and Jacod (2008), while we also obtain a reasonable approximation of the level.

This paper is organised as follows. In Chapter 2 we present the asymptotic results for the threshold-based class of test statistics in the pure semimartingale setting. The theoretical comparison (via local alternatives) with the tests developed by Barndorff-Nielsen and Shephard (2006) and Aït-Sahalia and Jacod (2008) is given in Section 3. The construction of test statistics for semimartingale models with noise is discussed in Section 4. Finally, we illustrate the finite sample performance of our procedure in Sections 5 and 6. All proofs are given in the Appendix.

2 The main setting, the statistical problem and the new class of test statistics

We consider a semimartingale $(X_t)_{t\geq 0}$ of the form

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t \sigma_s dW_s + (x1_{\{|x|\leq 1\}}) * (\mu - \nu) + (x1_{\{|x|>1\}}) * \mu, \tag{2.1}$$

defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. Here W denotes a one-dimensional Brownian motion, a is a locally bounded and predictable drift term, σ is an adapted and càdlàg volatility process, μ is a jump measure and ν is its predictable compensator. Moreover, we make the following assumption on the compensator ν:

(H) ν is of the form

$$\nu(dt, dx) = dt\ F_t(dx)$$

with $\int (1 \wedge x^2) dF_t(x)$ being a locally bounded and predictable process.

We observe the time continuous process X over a given interval $[0, t]$ at equidistant time points $t_i = \frac{i}{n}$, $i = 1, ..., [nt]$. Based on discrete observations $(X_{\frac{i}{n}}(\omega))_{0\leq i\leq [nt]}$ we want to decide whether the unobserved path $(X_s(\omega))_{s\in[0,t]}$ is continuous or not. As it has been already mentioned in Aït-Sahalia and Jacod (2008) we are only able to make statistical decisions about the particular (unobserved) path $(X_s(\omega))_{s\in[0,t]}$. It is impossible to say whether the semimartingale model allows for jumps, because there is a positive probability that the path $(X_s(\omega))_{s\in[0,t]}$ has no jumps although the model (2.1) allows the process X to jump (this is the case for compound Poisson processes). Consequently, we want to decide to which of the following two complementary sets the path $(X_s(\omega))_{s\in[0,t]}$ belongs:

$$\begin{cases} \Omega_t^j = \{\omega : s \mapsto X_s(\omega) \text{ is discontinuous on [0,t]}\} \\ \\ \Omega_t^c = \{\omega : s \mapsto X_s(\omega) \text{ is continuous on [0,t]}\}. \end{cases} \tag{2.2}$$

2.1 Realised truncated power variation

To construct a new class of test statistics we use the concept of *realised power variation* and *realised truncated power variation*. Recall that the realised power variation of the process X is given by

$$V(X,p)_t^n = n^{\frac{p}{2}-1} \sum_{i=1}^{[nt]} |\Delta_i^n X|^p \ , \tag{2.3}$$

with $\Delta_i^n X = X_{\frac{i}{n}} - X_{\frac{i-1}{n}}$. It is well-known (see, for instance, Barndorff-Nielsen, Graversen, Jacod, Podolskij and Shephard (2006)) that

$$V(X,2)_t^n \xrightarrow{P} [X]_t = \int_0^t \sigma_s^2 ds + \sum_{0\le s\le t} |\Delta X_s|^2 \ , \tag{2.4}$$

where $\Delta X_s = X_s - X_{s-}$, whereas

$$V(X,p)_t^n \xrightarrow{P} \begin{cases} \mu_p \int_0^t |\sigma_s|^p ds \text{ on } \Omega_t^c \\ \\ \infty \text{ on } \Omega_t^j \end{cases} \tag{2.5}$$

when $p > 2$ ($\mu_p = E[|u|^p]$ with $u \sim N(0,1)$).

The realised truncated power variation, originally proposed by Mancini (2001,2004), is given by

$$\overline{V}(X,p)_t^n = n^{\frac{p}{2}-1} \sum_{i=1}^{[nt]} |\Delta_i^n X|^p 1_{\{|\Delta_i^n X| \le c n^{-\varpi}\}} \ , \tag{2.6}$$

where $c > 0$ and $\varpi \in (0, 1/2)$. The threshold given in (2.6) eliminates the increments $\Delta_i^n X$ which are affected by jumps, while the increments $\Delta_i^n X$ are (asymptotically) not influenced by the threshold when there are no jumps on the interval $[\frac{i-1}{n}, \frac{i}{n}]$. Consequently, $\overline{V}(X,p)_t^n$ is robust to jumps, i.e.

$$\overline{V}(X,p)_t^n \xrightarrow{P} \mu_p \int_0^t |\sigma_s|^p ds \tag{2.7}$$

for any $p \ge 2$ (this is a straightforward extension of the results presented in Jacod (2008) and Cont and Mancini (2007)). Moreover, under a further assumption on the activity of the jump part of X (and on the parameter ϖ), the efficiency of $\overline{V}(X,p)_t^n$ is the same as the efficiency of $V(X,p)_t^n$ (see again Jacod (2008)).

The results of (2.4), (2.5) and (2.7) suggest to use the statistic $V(X,p)_t^n - \overline{V}(X,p)_t^n$ (or $\frac{V(X,p)_t^n}{\overline{V}(X,p)_t^n} - 1$) for $p \ge 2$ to decide whether the process X jumps or not. However, the derivation of the distribution theory (on Ω_t^c) for the above statistics turns out to be a difficult task.

2.2 New class of test statistics

Inspired by the wild bootstrap procedure (see Wu (1986)) we introduce external (i.e. independent of $\mathcal{F}$) positive i.i.d. random variables $(\eta_i)_{1\leq i\leq[nt]}$ with $E[\eta_i]=1$ and $E[|\eta_i|^2]<\infty$, and define a new class of test statistics by

$$T(X,p)^n_t = n^{\frac{p-1}{2}}\sum_{i=1}^{[nt]}|\triangle^n_i X|^p\Big(1-\eta_i 1_{\{|\triangle^n_i X|\leq cn^{-\varpi}\}}\Big)\ , \qquad p\geq 2. \tag{2.8}$$

The choice of the distribution of η crucially influences the level and power performance of the test statistic $T(X,p)^n_t$. In the next section we will explain how to choose the distribution of η.

All processes are now defined on a canonical extension $(\Omega^*,\mathcal{F}^*,(\mathcal{F}^*_t)_{t\geq 0},P^*)$ of the original filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},P)$, which also supports the random variables $(\eta_i)_{1\leq i\leq[nt]}$.

In what follows we will intensively use the concept of stable convergence. Recall that a sequence (Y_n) is said to converge towards Y $\mathcal{F}$-stably in law ($Y_n \stackrel{\mathcal{F}-st}{\longrightarrow} Y$) when the weak convergence

$$(Y_n,Z)\stackrel{\mathcal{D}}{\longrightarrow}(Y,Z)$$

holds for any $\mathcal{F}$-measurable variable Z. This is obviously a slightly stronger mode of convergence than convergence in law (see Renyi (1963), Aldous and Eagleson (1978) or Jacod and Shiryaev (2003) for more details on stable convergence).

The next theorem demonstrates the stable limit of $T(X,p)^n_t$ on Ω^c_t and Ω^j_t.

Theorem 1 *Assume that condition (H) holds and $E[|\eta_i|^{2+\delta}]<\infty$ for some $\delta>0$. For any $p\geq 2$ and any $t>0$, we obtain the following results:*

(i) On Ω^c_t we have

$$T(X,p)^n_t \stackrel{\mathcal{F}-st}{\longrightarrow}\sqrt{Var[\eta_i]\mu_{2p}}\int_0^t|\sigma_s|^p dW^{'}_s\ , \tag{2.9}$$

where $W^{'}$ is a new Brownian motion, defined on the extension $(\Omega',\mathcal{F}',(\mathcal{F}'_t)_{t\geq 0},P')$ of the probability space $(\Omega^*,\mathcal{F}^*,(\mathcal{F}^*_t)_{t\geq 0},P^*)$, which is independent of $\mathcal{F}^*$.

(ii) On Ω^j_t we have

$$T(X,p)^n_t\stackrel{P^*}{\longrightarrow}\infty. \tag{2.10}$$

Proof: see Appendix.

Note that the limiting random variable in (2.9) is mixed normal with $\mathcal{F}$-conditional variance given by the expression

$$\rho^2(p)_t=\mathrm{Var}[\eta_i]\mu_{2p}\int_0^t|\sigma_s|^{2p}ds. \tag{2.11}$$

By (2.7) we obtain a jump-robust estimate of $\rho^2(p)_t$, i.e.

$$\rho^2(p)_t^n = \mathrm{Var}[\eta_i]\overline{V}(X,2p)_t^n \stackrel{P}{\longrightarrow} \rho^2(p)_t.$$

Finally let us define the standardized statistics

$$S(p)_t^n = \frac{T(X,p)_t^n}{\rho(p)_t} \ , \qquad \hat{S}(p)_t^n = \frac{T(X,p)_t^n}{\rho(p)_t^n}. \tag{2.12}$$

By the properties of stable convergence we obtain the following corollary.

Corollary 1 *Assume that condition (H) holds and $E[|\eta_i|^{2+\delta}] < \infty$ for some $\delta > 0$. For any $p \geq 2$ and any $t > 0$, we obtain the following results:*

(i) On Ω_t^c we have

$$S(p)_t^n \stackrel{\mathcal{F}-st}{\longrightarrow} U \ , \qquad \hat{S}(p)_t^n \stackrel{\mathcal{F}-st}{\longrightarrow} U \ , \tag{2.13}$$

where U is a standard normal random variable, defined on the extension $(\Omega', \mathcal{F}', (\mathcal{F}_t')_{t\geq 0}, P')$ of the probability space $(\Omega^*, \mathcal{F}^*, (\mathcal{F}_t^*)_{t\geq 0}, P^*)$, which is independent of $\mathcal{F}^*$.

(ii) On Ω_t^j we have

$$S(p)_t^n \stackrel{P^*}{\longrightarrow} \infty \ , \qquad \hat{S}(p)_t^n \stackrel{P^*}{\longrightarrow} \infty. \tag{2.14}$$

Using again the properties of stable convergence and applying Corollary 1 we deduce that

$$\begin{aligned} P^*(\hat{S}(p)_t^n > c_{1-\alpha} |\ \Omega_t^c) &\rightarrow \alpha \ , \\ P^*(\hat{S}(p)_t^n > c_{1-\alpha} |\ \Omega_t^j) &\rightarrow 1 \ , \end{aligned}$$

where $c_{1-\alpha}$ is the $(1-\alpha)$-quantile of a standard normal distribution.

2.3 The choice of the distribution of η_i

Here we use the motivation from Section 2.1. As we have already mentioned before it is natural to use the statistic $V(X,p)_t^n - \overline{V}(X,p)_t^n$ for $p \geq 2$ to decide whether the process X jumps or not. Since the distribution theory for the afore-mentioned statistic is not available we require a "small perturbation" of the increments. Therefore we suggest to sample $(\eta_i)_{1\leq i\leq [nt]}$ from the following distribution

$$P^\eta = \frac{1}{2}(\delta_{1-\tau} + \delta_{1+\tau}) \ , \tag{2.15}$$

where δ stands for the Dirac measure. We propose to choose the constant τ relatively small, e.g. $\tau = 0.1$ or 0.05. Note that for small values of τ our class of statistics $T(X,p)_t^n$

is quite close to $\sqrt{n}(V(X,p)_t^n - \overline{V}(X,p)_t^n)$. This feature ensures a very good power performance of our test statistics.

On the other hand the symmetry of the distribution of η_i around 1 is responsible for a reasonable level approximation of our test. This is partially justified by the following proposition.

Proposition 2 *Assume that $X_t = \sigma W_t$. Then, for any $p \geq 2$, it holds that*

$$P^*(\hat{S}(p)_t^n \leq x) = \Phi(x) + O(n^{-1}) \ , \tag{2.16}$$

where Φ denotes the standard normal distribution.

Notice the absence of the term of order $n^{-1/2}$ on the right-hand side of (2.16). This means that we have a second-order refinement.

3 Comparison with other test procedures via local alternatives

In this section we discuss the behaviour of the statistic $T(X,p)_t^n$ under local alternatives and compare it with the behaviour of the tests proposed by Barndorff-Nielsen and Shephard (2006) and Aït-Sahalia and Jacod (2008). Let us briefly recall the ideas of these tests.

Barndorff-Nielsen and Shephard (2006) propose to use the $(1,1)$-bipower variation, i.e.

$$V(X,1,1)_t^n = \sum_{i=1}^{[nt]-1} |\Delta_i^n X||\Delta_{i+1}^n X| \ , \tag{3.1}$$

to construct a test for jumps. Indeed, under assumption (H), $V(X,1,1)_t^n$ is robust to jumps of the process X (see e.g. Aït-Sahalia and Jacod (2008)) and it holds that

$$V(X,1,1)_t^n \xrightarrow{P} \mu_1^2 \int_0^t \sigma_s^2 ds. \tag{3.2}$$

The authors propose to use the simple statistic

$$T^{BS}(X)_t^n = \sqrt{n}(V(X,2)_t^n - \mu_1^{-2} V(X,1,1)_t^n) \tag{3.3}$$

to decide whether the process X has jumps or not. To show a stable central limit theorem (on Ω_t^c) the following assumption is required:

(V) The volatility process σ is itself a semimartingale with absolute continuous characteristics and it does not vanish on $[0,t]$.

On Ω_t^c, under assumption (V), it holds for any $t > 0$

$$\frac{T^{BS}(X)_t^n}{\sqrt{\kappa QQ_t^n}} \stackrel{\mathcal{F}-st}{\longrightarrow} U \ , \tag{3.4}$$

where U is defined as in Corollary 1, $\kappa = \frac{\pi^2}{4} + \pi - 5$ and QQ_t^n is given by

$$QQ_t^n = n\mu_1^{-4} \sum_{i=1}^{[nt]-3} |\Delta_i^n X||\Delta_{i+1}^n X||\Delta_{i+2}^n X||\Delta_{i+3}^n X|.$$

On the other hand $\frac{T^{BS}(X)_t^n}{\sqrt{\kappa QQ_t^n}}$ converges to infinity on Ω_t^j.

Remark 1 In fact, Barndorff-Nielsen and Shephard (2006) propose to use the ratio statistic

$$T^{BS,r}(X)_t^n = \sqrt{n}\Big(1 - \frac{\mu_1^{-2}V(X,1,1)_t^n}{V(X,2)_t^n}\Big) \Big/ \sqrt{\kappa \max(QQ_t^n/(\mu_1^{-2}V(X,1,1)_t^n)^2, 1/t)} \tag{3.5}$$

to test for jumps. The above statistic turns out to have better finite sample properties. However, it has the same behaviour as $T^{BS}(X)_t^n$ under local alternatives.

Aït-Sahalia and Jacod (2008) compare $V(X,p)_t^n$ (with $p > 3$) at different sampling frequences to construct a test for jumps. In particular, they analyze the behaviour of the statistic

$$T^{AJ}(X)_t^n = \sqrt{n}\Big(\frac{2V(X,4)_t^{n/2}}{V(X,4)_t^n} - 2\Big). \tag{3.6}$$

On Ω_t^c, under assumption (V), it holds for any $t > 0$

$$\frac{T^{AJ}(X)_t^n}{\sqrt{\hat{V}_t^n}} \stackrel{\mathcal{F}-st}{\longrightarrow} U \ , \tag{3.7}$$

where U is defined as in Corollary 1 and $\hat{V}_t^n$ is given by

$$\hat{V}_t^n = \kappa' \frac{\overline{V}(X,8)_t^n}{(\overline{V}(X,4)_t^n)^2}$$

with $\kappa' = \frac{32}{7}$. On Ω_t^j Aït-Sahalia and Jacod (2008) also showed the stable convergence of the statistic (3.6) when 2 is replaced by 1 in the definition of $T^{AJ}(X)_t^n$.

Now we consider local alternatives of the form

$$X_t^{(n)} = X_0 + \int_0^t a_s ds + \int_0^t \sigma_s dW_s + \gamma_n J_t \ ,$$

where J_t is a compound Poisson process and γ_n is some sequence with $\gamma_n \to 0$. We obtain the following theorem.

Theorem 3 *For any $t > 0$ we have the following results.*

(i) *Consider the assumptions of Theorem 1 and set $\gamma_n = n^{-\frac{p-1}{2p}}$. If $\varpi < \frac{p-1}{2p}$ it holds*

$$T(X^{(n)}, p)_t^n \xrightarrow{\mathcal{F}-st} \sqrt{Var[\eta_i]\mu_{2p}} \int_0^t |\sigma_s|^p dW_s' + \sum_{0 \le s \le t} |\Delta J_s|^p$$

for any $p \ge 2$ and $t > 0$.

(ii) *Consider the assumption (V) and set $\gamma_n = n^{-\frac{1}{4}}$. It holds*

$$T^{BS}(X^{(n)})_t^n \xrightarrow{\mathcal{F}-st} U_t^c + \sum_{0 \le s \le t} |\Delta J_s|^2 ,$$

where U_t^c is the stable limit of $T^{BS}(X^{(n)})_t^n$ when $J = 0$ given by

$$U_t^c = \sqrt{\kappa \int_0^t \sigma_s^4 ds}\; U ,$$

where U is defined as in Corollary 1 (see Barndorff-Nielsen and Shephard (2006) for the proof in the continuous case [i.e. when $J = 0$]).

(iii) *Consider the assumption (V) and set $\gamma_n = n^{-\frac{3}{8}}$. It holds*

$$T^{AJ}(X^{(n)})_t^n \xrightarrow{\mathcal{F}-st} U_t'^c + \frac{2\sum_{0 \le s \le t} |\Delta J_s|^4}{3\int_0^t \sigma_s^4 ds} ,$$

where $U_t'^c$ is the stable limit of $T^{AJ}(X^{(n)})_t^n$ when $J = 0$ given by

$$U_t'^c = \frac{\sqrt{\kappa' \mu_8 \int_0^t \sigma_s^8 ds}}{\mu_4 \int_0^t \sigma_s^4 ds} U ,$$

where U is defined as in Corollary 1 (see Aït-Sahalia and Jacod (2008) for the proof in the continuous case).

Proof: see Appendix.

Notice that the rate at which our class of test statistics uncovers local alternatives is varying between $\gamma_n = n^{-1/4}$ (for $p = 2$) and $\gamma_n = n^{-1/2}$ (for $p \to \infty$). In this respect $T(X^{(n)}, p)_t^n$ outperforms $T^{BS}(X^{(n)})_t^n$ for $p > 2$, while $T(X^{(n)}, p)_t^n$ outperforms $T^{AJ}(X^{(n)})_t^n$ for $p > 4$.

However, the main reason for a better power performance of our test statistic $T(X, p)_t^n$ (see the simulation results) is different. At moderate sampling frequencies the power of the test crucially depends on the robustness properties of statistic $\overline{V}(X, p)_t^n$, which is implicitly used to construct $T(X, p)_t^n$. Once the threshold in the definition of $\overline{V}(X, p)_t^n$ uncovers a jump it is immediately eliminated by the indicator function. The test statistics proposed by Barndorff-Nielsen and Shephard (2006) and Aït-Sahalia and Jacod (2008) do not have this property.

4 The noise case

In this section we present an extension of our theory to noisy semimartingales. Assume first that the semimartingale X given in (2.1) is defined on some filtered probability space $(\Omega^0, \mathcal{F}^0, (\mathcal{F}_t^0)_{t\geq 0}, P^0)$. However, we do not directly observe the process X, but a process Z which is contaminated by the noise. The modelling of the noise process is adapted from Jacod, Li, Mykland, Podolskij and Vetter (2007). More precisely, we consider the process Z, observed at time points i/n, $i = 0, 1, \dots, [nt]$, which is given as

$$Z_t = X_t + \epsilon_t \ , \tag{4.1}$$

where ϵ_t's are the errors which are, conditionally on X, centered and independent. This can be formally constructed as follows. For any $t \geq 0$, consider a transition probability $Q_t(\omega^0, dz)$ from $(\Omega^0, \mathcal{F}_t^0)$ into $\mathbb{R}$. We endow the space $\Omega^1 = \mathbb{R}^{[0,\infty)}$ with the product (Borel) σ-field $\mathcal{F}^1$ ($(\epsilon_t)_{t\geq 0}$ is regarded as the canonical process on this space). The probability measure $Q(\omega^0, d\omega^1)$ is given as a product $\otimes_{t\geq 0} Q_t(\omega^0, \cdot)$. The filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, on which the process Z lives, is defined as

$$\left.\begin{array}{lll} \Omega = \Omega^0 \times \Omega^1, & \mathcal{F} = \mathcal{F}^0 \times \mathcal{F}^1, & \mathcal{F}_t = \cap_{s>t} \mathcal{F}_s^0 \times \mathcal{F}_s^1, \\ P(d\omega^0, d\omega^1) = P^0(d\omega^0) Q(\omega^0, d\omega^1). & & \end{array}\right\} \tag{4.2}$$

Furthermore, we assume that

$$\int z Q_t(\omega^0, dz) = X_t(\omega^0), \quad \text{and} \ \ \alpha_t^2(\omega^0) = E[Z_t^2 | \mathcal{F}^0](\omega^0) - X_t^2(\omega^0) \ \ \text{is càdlàg.} \tag{4.3}$$

Finally, we define the process

$$N_t(q) = \int |z|^q Q_t(\omega^0, dz). \tag{4.4}$$

Remark 2 Typical examples of a process Z which satisfies the above construction and condition (4.3) are the following.

(i) (*Additive i.i.d. process*) When

$$Z_{\frac{i}{n}} = X_{\frac{i}{n}} + \epsilon_{\frac{i}{n}} \ ,$$

where $(\epsilon_{\frac{i}{n}})_i$ is an i.i.d. process with expectation 0 and variance α^2, condition (4.3) is obviously satisfied.

(ii) (*Additive i.i.d. process + rounding*) Consider the process

$$Z_{\frac{i}{n}} = \gamma \Big[\frac{X_{\frac{i}{n}} + \epsilon_{\frac{i}{n}}}{\gamma} \Big] \ ,$$

where $\gamma > 0$, $(\epsilon_{\frac{i}{n}})_i$ is as in (i) and has an $U([0,\gamma])$ distribution. Then

$$\alpha_t^2 = \gamma^2\Big(\Big\{\frac{X_t}{\gamma}\Big\} - \Big\{\frac{X_t}{\gamma}\Big\}^2\Big) ,$$

and condition (4.3) is fulfilled (here $\{x\}$ denotes the fractional part of x).

Since the "true" process X is contaminated by noise we need to "pre-filter" the data. For this purpose we use the method which has been proposed in Jacod, Li, Mykland, Podolskij and Vetter (2007) and Podolskij and Vetter (2008) (see also Podolskij and Vetter (2006)).

First, we choose a sequence k_n of integers, which satisfies

$$\frac{k_n}{\sqrt{n}} = \theta + o(n^{-\frac{1}{4}}) \tag{4.5}$$

for some $\theta > 0$, and a nonzero real-valued function $g : I\!R \to I\!R$, which fulfills the following conditions

(i) g vanishes outside of $(0,1)$

(ii) g is continuous and piecewise C^1

(iii) Its derivative g' is piecewise Lipschitz.

We associate with g the following real valued numbers

$$\psi_1 = \int_0^1 (g'(s))^2 ds , \qquad \psi_2 = \int_0^1 (g(s))^2 \, ds. \tag{4.6}$$

Furthermore, we define the quantity

$$\overline{Z}_i^n = \sum_{j=1}^{k_n-1} g\Big(\frac{j}{k_n}\Big)\Delta_{i+j}^n Z. \tag{4.7}$$

Next, we choose the constants $c > 0$ and $\varpi \in (0, 1/4)$. Finally, we introduce external (i.e. independent of $\mathcal{F}$) positive i.i.d. random variables $(\eta_i)_{0\le i\le[nt]}$ with $E[\eta_i] = 1$ and $E[|\eta_i|^2] < \infty$, and define a class of test statistics by

$$T^{noise}(Z,p)_t^n = n^{(p-2)/4} \sum_{i=0}^{[nt]-k_n+1} |\overline{Z}_i^n|^p\Big(1 - \eta_i 1_{\{|\overline{Z}_i^n|\le cn^{-\varpi}\}}\Big) , \qquad p \ge 2. \tag{4.8}$$

Note that $T^{noise}(X,p)_t^n$ has the same structure as $T(X,p)_t^n$.

All processes are defined on a canonical extension $(\Omega^*, \mathcal{F}^*, (\mathcal{F}_t^*)_{t\ge0}, P^*)$ of the original filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, P)$, which also supports the random variables $(\eta_i)_{0\le i\le[nt]}$.

Remark 3 If V is a continuous semimartingale or a noise process constructed as above, then

$$\overline{V}_i^n = O_p(n^{-1/4})$$

(see e.g. Jacod, Li, Mykland, Podolskij and Vetter (2007) for more details). This explains the condition on ϖ, i.e. $\varpi \in (0, 1/4)$.

To formulate the theoretical results we need to introduce the following sets

$$\Omega_t^c = \Omega_t^{0,c} \times \Omega^1 \qquad \text{and} \qquad \Omega_t^j = \Omega_t^{0,j} \times \Omega^1$$

with

$$\begin{aligned} \Omega_t^{0,c} &= \{\omega^0 |\ s \mapsto X_s(\omega^0) \text{ is continuous on } [0,t]\}, \\ \Omega_t^{0,j} &= \{\omega^0 |\ s \mapsto X_s(\omega^0) \text{ is discontinuous on } [0,t]\}. \end{aligned}$$

Finally, we define the statistic

$$\Gamma(p)_t^n = \text{Var}[\eta_i] n^{(p-2)/2} \sum_{i=0}^{[nt]-k_n+1} |\overline{Z}_i^n|^{2p} 1_{\{|\overline{Z}_i^n| \le c n^{-\varpi}\}} \tag{4.9}$$

and set

$$S^{noise}(p)_t^n = \frac{T^{noise}(Z,p)_t^n}{\sqrt{\Gamma(p)_t^n}}. \tag{4.10}$$

The main result of this section is the following theorem.

Theorem 4 *Assume that condition (H) holds, $E[|\eta_i|^{2+\delta}] < \infty$ for some $\delta > 0$ and the process $N_t(q)$ defined in (4.4) is locally bounded for some $q > \frac{2p}{1-4\varpi}$. For any $p \ge 2$ and any $t > 0$, we obtain the following results:*

(i) On Ω_t^c we have

$$S^{noise}(p)_t^n \xrightarrow{\mathcal{F}-st} U\ , \tag{4.11}$$

where U is a standard normal random variable, defined on the extension $(\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t\ge0}, P')$ of the probability space $(\Omega^*, \mathcal{F}^*, (\mathcal{F}^*_t)_{t\ge0}, P^*)$, which is independent of $\mathcal{F}^*$.

(ii) On Ω_t^j we have

$$S^{noise}(p)_t^n \xrightarrow{P^*} \infty. \tag{4.12}$$

Proof: see Appendix.

Now we deduce from Theorem 4 that

$$\begin{aligned} P^*(S^{noise}(p)_t^n > c_{1-\alpha} |\ \Omega_t^c) &\to \alpha\ , \\ P^*(S^{noise}(p)_t^n > c_{1-\alpha} |\ \Omega_t^j) &\to 1\ , \end{aligned}$$

where $c_{1-\alpha}$ is the $(1-\alpha)$-quantile of a standard normal distribution.

Remark 4 As in Section 2 we suggest to generate the external variables $(\eta_i)_{1\leq i\leq [nt]}$ from the distribution

$$P^\eta = \frac{1}{2}(\delta_{1-\tau} + \delta_{1+\tau}) \tag{4.13}$$

for relatively small values of τ (e.g. $\tau = 0.1$ or 0.05).

5 The choice of the threshold

In this section we investigate how to choose the threshold in our test statistic. A sensible choice of c and ϖ is crucial for the finite sample performance of our test.

5.1 Semimartingale model

Although the asymptotic results are valid for all $c > 0$ and $\varpi \in (0, 1/2)$, it is very important for the finite sample performance to choose both constants in a reasonable way. We have to ensure that the threshold is sharp enough to detect and eliminate jumps, while increments of the continuous part should not be affected by the threshold. Here we present an easy but effective way to determine c and ϖ.

First, we compute a robust estimator for the integrated volatility $\int_0^t \sigma_s^2 ds$. A suitable estimator for this quantity is given by $\mu_1^{-2} V(X, 1, 1)_t^n$ (see (3.2)). Next, we choose

$$c = 2.3\sqrt{V(X, 1, 1)_t^n}\ ,$$

where the quantity 2.3 is approximately the 99%-quantile of the standard normal distribution. Notice that $\sqrt{V(X, 1, 1)_t^n}$ represents the "average" level of volatility. Therefore, we expect that the most increments of the continuous part do not exceed the threshold $cn^{-1/2}$.

The constant ϖ obviously controls the rate of convergence of the threshold. This means, the bigger ϖ is the faster converges the threshold to zero. Consequently, jumps become faster eliminated for large values of ϖ. On the other hand too large values of ϖ increase the probability to declare an increments of the continuous part as a jump. As a balance between this effects we suggest to use $\varpi = 0.4$.

5.2 Semimartingale model with noise

The choice of the constants become more involved in the semimartingale model with noise. Notice the strong dependence between neighbored $\bar{Z}_i^n$'s. Consequently, there is a high probability of eliminating more than one summand if the threshold classifies a big increment of the continuous part as a jump. Therefore, we have to choose the threshold very carefully.

We suggest the following procedure. First, we choose

$$c = 2.3\sqrt{\mu_1^{-2}BT(1,1)_t^n + \frac{\psi_1}{\theta}NV_t^n}\ , \tag{5.1}$$

where

$$BT(1,1)_t^n = n^{-\frac{1}{2}} \sum_{i=0}^{[nt]-2k_n+1} |\bar{Z}_i^n||\bar{Z}_{i+k_n}^n|$$

and

$$NV_t^n = \frac{V(Z,2,0)_t^n}{2n}.$$

The structure of c is similar to the case without noise. By the results of Podolskij and Vetter (2008) we have that

$$\mu_1^{-2}BT(1,1)_t^n + \frac{\psi_1}{\theta}NV_t^n \xrightarrow{P^*} \int_0^t (\theta\psi_2\sigma_s^2 + \frac{\psi_1}{\theta}\alpha_s^2)ds.$$

On the other hand the quantity $\bar{Z}_i^n$ is asymptotically distributed as $n^{-1/4}N(0, \theta\psi_2\sigma_{\frac{i}{n}}^2 + \frac{\psi_1}{\theta}\alpha_{\frac{i}{n}}^2)$ when Z does not jump on the interval $[\frac{i}{n}, \frac{i+k_n}{n}]$. Consequently, when Z has no jumps the most quantities $\bar{Z}_i^n$ should not exceed the threshold $cn^{-1/4}$ (if the processes σ^2 and α^2 are not very volatile). By applying the same intuition as for pure semimartingale models we recommend to use $\varpi = 0.17$.

6 Simulation results

In this section we investigate the performance of the different test statistics in finite samples. First, we compare our test statistic $\hat{S}(p)_t^n$ with $p = 2$ and $p = 4$ with the test statistics $T^{BS}(X)_t^n$, $T^{BS,r}(X)_t^n$ of Barndorff-Nielsen and Shephard (2006) and the test statistic $T^{AJ}(X)_t^n$ proposed by Aït-Sahalia and Jacod (2008). After that we investigate the behavior of the test statistic $S^{noise}(p)_t^n$ (with $p = 2$) in semimartingale models with noise.

We assume that all processes live on the interval $[0, 1]$. We consider two different continuous semimartingale models. The first model is a Brownian semimartingale with constant volatility $\sigma = 2$, i.e.

$$X_t = 2W_t. \tag{6.1}$$

Second, we consider a two factor model. It is specified by the stochastic differential equation

$$dX_t = \mu dt + \sigma_t dW_t \tag{6.2}$$

with

$$\sigma_t = \exp(\beta_0 + \beta_1\tau_t), \qquad d\tau_t = \alpha\tau_t dt + dB_t, \qquad \mathrm{corr}(dW_t, dB_t) = \rho.$$

The parametes are chosen as $\mu = 0$, $\beta_0 = 0.3125$, $\beta_1 = 0.125$, $\alpha = -0.025$ and $\rho = -0.3$. The two factor model with the afore-mentioned parameters has been adapted from Barndorff-Nielsen, Hansen, Lunde and Shephard (2006) and Podolskij and Vetter (2006). In this model the level of the volatility process σ^2 varies from 1.4 to 2.1. Furthermore, we generate the external i.i.d. sequence $(\eta_i)_{1\leq i\leq n}$ from the distribution (2.15) with $\tau = 0.05$.

We consider three different types of jump models:

(i) One jump with a fixed jump size

(ii) Two jumps with fixed jump sizes

(iii) Three jumps with random $N(0, a^2)$-distributed jump sizes

All jump times are independent and $U([0, 1])$-distributed. Moreover, we adapt the jump size(s) to the particular model to make it comparable with the magnitude of the volatility process σ.

To study the performance of the test statistic $S^{noise}(p)^n_t$ defined in (4.10) for semimartingales with noise, we consider an i.i.d. model for the noise process ϵ, which is assumed to be independent of the semimartingale. These random variables $(\epsilon_{\frac{i}{n}})$ are generated according to a $N(0, 0.0005^2)$ distribution. Moreover, we use $g(x) = (\min(x, 1-x))^+$ and $\theta = 1/3$ as proposed in Jacod, Li, Mykland, Podolskij and Vetter (2007).

We did 10000 simulation runs for each model. The simulation results are reported in Tables 1-10.

6.1 Level performance

We start with the pure semimartingale models. We compare the level performance of test statistics for different levels ($\alpha = 1\%, 2.5\%, 5\%, 10\%, 25\%$) and different sample sizes ($n = 100, 200, 500, 1000, 3000, 10000$). The simulated level results are listed in Table 1 (for the constant volatility model (6.1)) and Table 5 (for the two factor model (6.2)).

We observe that the test statistics (3.4), (3.5) proposed by Barndorff-Nielsen and Shephard (2006) and our test statistics $\hat{S}(2)^n_t$ and $\hat{S}(4)^n_t$ tend to overestimate the true level, while the testing procedure (3.7) proposed by Aït-Sahalia and Jacod (2008) underestimates it. The particular performance of the tests depends on the sample size. While the ratio statistic $T^{BS,r}(X)^n_t$ of Barndorff-Nielsen and Shephard (2006) and the test statistic $T^{AJ}(X)^n_t$ of Aït-Sahalia and Jacod (2008) yield better results for small sample sizes, our test statistics $\hat{S}(2)^n_t$ and $\hat{S}(4)^n_t$ have the best performance for $n = 1000$ and larger ($\hat{S}(4)^n_t$ is slightly better than $\hat{S}(2)^n_t$). However, all test statistics perform rather well.

Now we consider the noisy semimartingale. The corresponding results for $S^{noise}(2)_t^n$ are reported in Tables 9 and 10. We observe that the asymptotic theory starts to work for relatively large sample sizes, i.e. for $n = 900, 1600, 2500$. It is not surprising, because the semimartingale process is corrupted by noise, so we expect a slower speed of convergence. Quite interestingly, the performance of $S^{noise}(2)_t^n$ looks rather good for very small sample sizes. However, this issue is due to the fact that different finite sample effects seem to eliminate each other as it has been reported in Podolskij and Vetter (2006).

6.2 Power performance

We start with the no-noise case. The continuous part of the semimartingale is generated according to the models (6.1) and (6.2). We add to the continuous part the following jump processes: (i) one jump with the jump size 0.4 for (6.1) and 0.26 for (6.2), (ii) two jumps with jump sizes $\sqrt{0.08}$ and $-\sqrt{0.08}$ for (6.1) and $\sqrt{0.26^2/2}$ and $-\sqrt{0.26^2/2}$ for (6.2), (iii) three jumps with $N(0, \frac{0.16}{3})$-distributed jump sizes for (6.1) and with $N(0, \frac{0.26^2}{3})$-distributed jump sizes for (6.2). All jump times are independent and $U([0,1])$-distributed. Notice that the quadratic variation of the jump is kept (approximately) constant (0.16 for model (6.1) and 0.26^2 for model (6.2)). The corresponding power performance is reported in Tables 2 - 4 and 6 - 8.

The results are striking. Our test statistics $\hat{S}(2)_t^n$ and $\hat{S}(4)_t^n$ yield by far the best power performance for all models. More precisely, our method detects the jumps at relatively small sample frequencies (i.e. $n = 500, 1000$), whereas the testing procedures of Barndorff-Nielsen and Shephard (2006) and Aït-Sahalia and Jacod (2008) start to work at quite high sample frequencies (i.e. $n = 3000, 10000$). Besides, the results show that it is more difficult to find small jumps than one big jump (which is not surprising).

Finally, let us consider the semimartingale model with noise. We generate the same semimartingale processes as described above. The power performance of the test statistic $S^{noise}(2)_t^n$ is presented in Tables 9 and 10.

We observe that the jumps are much harder to detect in models with noise. This is due to the slower convergence rate of the threshold. Consequently, much more data points are required to detect jumps. Our test yields good results for the case of one big jump when the sample size is rather high (i.e. $n = 4900 - 22500$). If the jumps are small it takes extremely large samples to uncover them. Nevertheless, the power performance seems to be quite reasonable since we consider noisy observations of semimartingales.

7 Appendix

7.1 Proofs

By standard truncation technique (see e.g. Barndorff-Nielsen, Graversen, Jacod, Podolskij and Shephard (2006)) we can assume w.l.o.g. that the processes a and σ are bounded. We denote all constants which appear in the proofs by C or by C_p when they depend on an additional parameter p.

In the following we will often use the inequality

$$E[|\Delta_i^n X^c|^l] \le C_l n^{-\frac{l}{2}} , \qquad l > 0 , \tag{7.1}$$

where X^c denotes the continuous part of X, which is deduced by the Burkholder inequality.

Proof of Theorem 1: (i) Assume first that the process X has no jumps on the interval $[0,t]$, i.e. we are on the set Ω_t^c. It suffices to show that

$$S(p)_t^n \stackrel{\mathcal{F}-st}{\longrightarrow} U$$

(see (2.13) of Corollary 1). Then, by the properties of stable convergence, we also obtain

$$T(X,p)_t^n \stackrel{\mathcal{F}-st}{\longrightarrow} \sqrt{\mathrm{Var}[\eta_i]\mu_{2p}} \int_0^t |\sigma_s|^p dW_s^{'}.$$

Note that by (7.1) we have

$$n^{\frac{p-1}{2}} \sum_{i=1}^{[nt]} |\triangle_i^n X|^p \eta_i 1_{\{|\triangle_i^n X| > cn^{-\varpi}\}} \le C_l n^{\frac{p-1}{2}+l\varpi} \sum_{i=1}^{[nt]} |\triangle_i^n X|^{p+l} \eta_i = O_{P^*}(n^{l(\varpi-1/2)+1/2})$$

for any $l > 0$. Choosing $l > \frac{1}{2(\varpi-1/2)}$ we obtain the approximation

$$T(X,p)_t^n = n^{\frac{p-1}{2}} \sum_{i=1}^{[nt]} |\triangle_i^n X|^p \Big(1-\eta_i\Big) + o_{P^*}(1) =: \tilde{T}(X,p)_t^n + o_{P^*}(1).$$

From Theorem 2 in Podolskij and Ziggel (2007) we deduce that

$$P^*\Big(\frac{\tilde{T}(X,p)_t^n}{\rho(p)_t} \le x|\ \mathcal{F}\Big) \stackrel{P}{\longrightarrow} \Phi(x) ,$$

where Φ is the distribution function of a standard normal variable and $\rho(p)_t$ is defined in (2.11). Set

$$Y_n = \frac{\tilde{T}(X,p)_t^n}{\rho(p)_t}$$

and consider a $\mathcal{F}$-measurable variable Z. Then we obtain as $n \to \infty$

$$E^*[1_{\{Y_n \le x, Z \le z\}}] = E^*[1_{\{Z \le z\}} P^*(Y_n \le x |\ \mathcal{F})] \longrightarrow \Phi(x) P(Z \le z) = P'(U \le x, Z \le z) \ ,$$

where U is defined in (2.13). It follows by definition that

$$S(p)_t^n \stackrel{\mathcal{F}-st}{\longrightarrow} U$$

and we are done.

(ii) By the results of Aït-Sahalia and Jacod (2008) we obtain under assumption (H), for any $p \ge 2$,

$$\sum_{i=1}^{[nt]} |\triangle_i^n X|^p \Big(1 - \eta_i 1_{\{|\triangle_i^n X| \le c n^{-\varpi}\}}\Big) \stackrel{P^*}{\longrightarrow} \sum_{0 \le s \le t} |\Delta X_s|^p.$$

Hence, on Ω_t^j we have

$$T(X,p)_t^n \stackrel{P^*}{\longrightarrow} \infty \ ,$$

which completes the proof of Theorem 1. □

Proof of Proposition 2: Recall that $X = \sigma W$ and the distribution of η is given by (2.15). As in the proof of Theorem 1 we obtain the approximation

$$\hat{S}(p)_t^n = \frac{n^{\frac{p-1}{2}} \sum_{i=1}^{[nt]} |\triangle_i^n X|^p \Big(1 - \eta_i\Big)}{\sqrt{\mathrm{Var}[\eta_i] V(X, 2p)_t^n}} + O_{P^*}(n^{-1}) =: \tilde{S}(p)_t^n + O_{P^*}(n^{-1})$$

on Ω_t^c. For the conditional cumulants of $\tilde{S}(p)_t^n$ we deduce the following identities

$$\begin{aligned} k_1 &:= \mathrm{plim}_{n\to\infty} \sqrt{n} E[\tilde{S}(p)_t^n | \mathcal{F}] = 0 \ , \\ k_3 &:= \mathrm{plim}_{n\to\infty} \sqrt{n} \Big(E[(\tilde{S}(p)_t^n)^3 | \mathcal{F}] - 3E[(\tilde{S}(p)_t^n)^2 | \mathcal{F}] E[(\tilde{S}(p)_t^n) | \mathcal{F}] + 2(E[(\tilde{S}(p)_t^n) | \mathcal{F}])^3 \Big) \\ &= 0 \ , \end{aligned}$$

where the second identity follows from the fact that η has a symmetric distribution (around 1). Using a standard Edgeworth expansion (see Hall (1992), p. 48) we conclude that

$$P^*\Big(\tilde{S}(p)_t^n \le x |\ \mathcal{F}\Big) = \Phi(x) + R_n(x) \ ,$$

where $R_n(x)$ satisfies $E[|R_n(x)|] = O(n^{-1})$. By taking the expectation we obtain

$$P^*\Big(\tilde{S}(p)_t^n \le x\Big) = \Phi(x) + O(n^{-1}) \ ,$$

which completes the proof of Proposition 2. □

Proof of Theorem 3: First, we introduce the decomposition

$$X_t^{(n)} = X_t^c + X_t^{j,(n)} ,$$

where X_t^c denotes the continuous part of $X_t^{(n)}$ and $X_t^{j,(n)} = \gamma_n J_t$.

(i) Set $\gamma_n = n^{-\frac{p-1}{2p}}$. Observe that

$$\begin{aligned} T(X^{(n)}, p)_t^n &= n^{\frac{p-1}{2}} \sum_{i \in I_t^n} |\triangle_i^n X^{(n)}|^p \Big(1 - \eta_i 1_{\{|\triangle_i^n X^{(n)}| \le cn^{-\varpi}\}}\Big) \\ &+ n^{\frac{p-1}{2}} \sum_{i \in (I_t^n)^c} |\triangle_i^n X^{(n)}|^p \Big(1 - \eta_i 1_{\{|\triangle_i^n X^{(n)}| \le cn^{-\varpi}\}}\Big) \end{aligned}$$

with $I_t^n = \{i|$ the process J jumps on $[\frac{i-1}{n}, \frac{i}{n}]\}$. Note that the first sum is finite (a.s.), because J is a compound Poisson process. By Theorem 1 we have

$$n^{\frac{p-1}{2}} \sum_{i \in (I_t^n)^c} |\triangle_i^n X^{(n)}|^p \Big(1 - \eta_i 1_{\{|\triangle_i^n X^{(n)}| \le cn^{-\varpi}\}}\Big) \stackrel{\mathcal{F}-st}{\longrightarrow} \sqrt{\mathrm{Var}[\eta_i]\mu_{2p}} \int_0^t |\sigma_s|^p dW_s^{'} ,$$

and, since $\varpi < \frac{p-1}{2p}$,

$$n^{\frac{p-1}{2}} \sum_{i \in I_t^n} |\triangle_i^n X^{(n)}|^p \Big(1 - \eta_i 1_{\{|\triangle_i^n X^{(n)}| \le cn^{-\varpi}\}}\Big) \stackrel{P^*}{\longrightarrow} \sum_{0 \le s \le t} |\Delta J_s|^p.$$

Consequently, it holds that

$$T(X^{(n)}, p)_t^n \stackrel{\mathcal{F}-st}{\longrightarrow} \sqrt{\mathrm{Var}[\eta_i]\mu_{2p}} \int_0^t |\sigma_s|^p dW_s^{'} + \sum_{0 \le s \le t} |\Delta J_s|^p$$

for any $p \ge 2$ and $t > 0$.

(ii) Set $\gamma_n = n^{-\frac{1}{2}}$. Observe that (recall (7.1))

$$T^{BS}(X)_t^n = \sqrt{n}(V(X^c, 2)_t^n - \mu_1^{-2} V(X^c, 1, 1)_t^n + V(X^{j,(n)}, 2)_t^n) + o_{P^*}(1).$$

As above we obtain

$$T^{BS}(X^{(n)})_t^n \stackrel{\mathcal{F}-st}{\longrightarrow} U_t^c + \sum_{0 \le s \le t} |\Delta J_s|^2 ,$$

where U_t^c is given in Theorem 3.

(iii) Set $\gamma_n = n^{-\frac{3}{8}}$. Since J has only finitely many jumps, we deduce by (7.1)

$$T^{AJ}(X^{(n)})_t^n = \sqrt{n}\Big(\frac{2V(X^c, 4)_t^{n/2}}{V(X^c, 4)_t^n} - 2\Big) + 2\sqrt{n} \frac{V(X^{j,(n)}, 2)_t^{n/2}}{V(X^c, 4)_t^n} + o_{P^*}(1).$$

Consequently, we have

$$T^{AJ}(X^{(n)})^n_t \xrightarrow{\mathcal{F}-st} U'^c_t + \frac{2\sum_{0\le s\le t}|\Delta J_s|^4}{3\int_0^t \sigma_s^4 ds} \ ,$$

where U'^c_t is given in Theorem 3. □

Proof of Theorem 4: Since the process (α_t^2) defined in (4.3) is supposed to be càdlàg, we can assume without loss of generality that (α_t^2) is bounded.

Notice the identity

$$\overline{Z}^n_i = \sum_{j=1}^{k_n-1} g\Big(\frac{j}{k_n}\Big)\Delta^n_{i+j}Z = \sum_{j=0}^{k_n-1}\Big(g\Big(\frac{j}{k_n}\Big) - g\Big(\frac{j+1}{k_n}\Big)\Big)Z_{\frac{i+j}{n}}. \tag{7.2}$$

By Burkholder inequality we obtain

$$E[|\overline{X^c}^n_i|^q] \le Cn^{-q/4} \tag{7.3}$$

for all $q \ge 0$ and uniformly in i (X^c denotes the continuous part of X). On the other hand, using the right-hand side of (7.2), we deduce the following inequality for the noise process (when the process $N_t(2p)$ is locally bounded):

$$E[|\overline{\epsilon}^n_i|^q] \le Cn^{-q/4} \ , \tag{7.4}$$

which holds for any $q < 2p$ and uniformly in i.

(i) Assume that the process X has no jumps, i.e. we are on Ω^c_t. Due to inequalities (7.3) and (7.4) we obtain the approximations (see the proof of Theorem 1 (i))

$$T^{noise}(Z,p)^n_t = n^{(p-2)/4}\sum_{i=0}^{[nt]-k_n+1}|\overline{Z}^n_i|^p\Big(1-\eta_i\Big) + o_{P*}(1) =: \overline{T}^{noise}(Z,p)^n_t + o_{P*}(1)$$

and

$$\Gamma(p)^n_t = \mathrm{Var}[\eta_i]n^{(p-2)/2}\sum_{i=0}^{[nt]-k_n+1}|\overline{Z}^n_i|^{2p} + o_{P*}(1) =: \overline{\Gamma}(p)^n_t + o_{P*}(1).$$

Since $\mathrm{Var}\Big(\overline{T}^{noise}(Z,p)^n_t|\ \mathcal{F}\Big) = \overline{\Gamma}(p)^n_t$ we deduce that

$$P^*\Big(S^{noise}(p)^n_t \le x|\ \mathcal{F}\Big) \xrightarrow{P} \Phi(x) \ ,$$

where Φ is the distribution function of a standard normal variable (this follows by the same methods that are used in the proof of Theorem 2 in Podolskij and Ziggel (2007)). The arguments of the proof of Theorem 1 yield the convergence

$$S^{noise}(p)^n_t \xrightarrow{\mathcal{F}-st} U \ ,$$

where U is defined in (4.11), and we are done.

(ii) By the results of Podolskij and Vetter (2008) (see the proof of Lemma 1 therein) we obtain under assumption (H), for any $p \geq 2$,

$$\frac{1}{k_n} \sum_{i=0}^{[nt]-k_n+1} |\overline{Z}_i^n|^p \Big(1 - \eta_i 1_{\{|\overline{Z}_i^n| \leq c n^{-\varpi}\}}\Big) \stackrel{P^*}{\longrightarrow} \int_0^1 |g(u)|^p du \sum_{0 \leq s \leq t} |\Delta X_s|^p,$$

and

$$\Gamma(p)_t^n \stackrel{P^*}{\longrightarrow} \mathrm{Var}[\eta_i] \mu_{2p} \int_0^t \Big(\psi_2 \theta \sigma_s^2 + \frac{\psi_1}{\theta} \alpha_s^2\Big)^p ds.$$

Hence, on Ω_t^j we have

$$S^{noise}(p)_t^n \stackrel{P^*}{\longrightarrow} \infty \ ,$$

which completes the proof of Theorem 4. □

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7.1 Simulation Results

n	100	200	500	1000	3000	10000
$T^{BS}(X)_t^n$-1%	5.7	3.93	2.77	2.23	2.01	1.5
$T^{BS}(X)_t^n$-2.5%	7.78	5.81	4.39	3.75	3.48	2.99
$T^{BS}(X)_t^n$-5%	11.37	9.05	7.32	6.65	6.42	5.71
$T^{BS}(X)_t^n$-10%	17.1	14.41	12.88	11.58	11.51	10.58
$T^{BS}(X)_t^n$-25%	30.83	29.12	27.54	25.76	26.21	25.77
$T^{BS,r}(X)_t^n$-1%	2.36	1.9	1.79	1.53	1.57	1.37
$T^{BS,r}(X)_t^n$-2.5%	4.01	3.42	3.1	2.9	3.02	2.64
$T^{BS,r}(X)_t^n$-5%	7.12	6.4	5.71	5.62	5.77	5.41
$T^{BS,r}(X)_t^n$-10%	12.86	11.76	11.18	10.54	10.91	10.19
$T^{BS,r}(X)_t^n$-25%	28.55	27.52	26.54	25.21	25.72	25.53
$\hat{S}(2)_t^n$-1%	4.68	3.12	2.45	1.78	1.41	1.38
$\hat{S}(2)_t^n$-2.5%	5.94	4.37	3.58	3.13	2.53	2.61
$\hat{S}(2)_t^n$-5%	8.65	6.78	5.99	5.41	4.93	5.01
$\hat{S}(2)_t^n$-10%	13.67	11.72	10.65	9.94	10.14	9.74
$\hat{S}(2)_t^n$-25%	28.27	26.75	25.89	24.75	25.28	24.51
$\hat{S}(4)_t^n$-1%	4.21	2.75	2.16	1.71	1.2	1.26
$\hat{S}(4)_t^n$-2.5%	5.38	3.93	3.45	2.87	2.46	2.55
$\hat{S}(4)_t^n$-5%	8.26	6.65	6.06	5.35	4.99	5.21
$\hat{S}(4)_t^n$-10%	14.53	12.18	11.4	10.37	10.14	10.16
$\hat{S}(4)_t^n$-25%	30.47	27.85	26.8	25.79	25.39	24.72
$T^{AJ}(X)_t^n$-1%	0.35	0.38	0.56	0.71	0.96	0.86
$T^{AJ}(X)_t^n$-2.5%	1.05	1.1	1.35	1.88	2.04	2.06
$T^{AJ}(X)_t^n$-5%	3.18	3.3	3.69	4.27	4.35	4.37
$T^{AJ}(X)_t^n$-10%	8.77	8.62	9.36	9.45	9.62	9.19
$T^{AJ}(X)_t^n$-25%	27.41	25.99	26.38	25.19	24.43	25.09

Table 1: *This table shows the level performance for the model (6.1).*

n	100	200	500	1000	3000	10000
$T^{BS}(X)_t^n$-1%	6.66	6.07	9.71	16.1	49.17	97.09
$T^{BS}(X)_t^n$-2.5%	9.5	8.6	13.81	22.17	59.18	98.4
$T^{BS}(X)_t^n$-5%	13.3	13.01	19.98	31	69.06	99.23
$T^{BS}(X)_t^n$-10%	19.61	20.04	29.28	42.71	78.81	99.68
$T^{BS}(X)_t^n$-25%	35.09	36.85	48.41	63.83	91.07	99.94
$T^{BS,r}(X)_t^n$-1%	2.92	3.2	6.53	13.15	46.03	96.78
$T^{BS,r}(X)_t^n$-2.5%	4.95	5.55	10.69	19.08	56.68	98.28
$T^{BS,r}(X)_t^n$-5%	8.79	9.48	16.89	28.05	67.55	99.19
$T^{BS,r}(X)_t^n$-10%	15.15	16.75	26.61	40.78	77.97	99.68
$T^{BS,r}(X)_t^n$-25%	32.79	35.03	47.53	63.09	90.91	99.94
$\hat{S}(2)_t^n$-1%	8.6	15.24	56.6	95.11	100	100
$\hat{S}(2)_t^n$-2.5%	9.88	16.37	57.24	95.18	100	100
$\hat{S}(2)_t^n$-5%	12.73	18.76	58.29	95.28	100	100
$\hat{S}(2)_t^n$-10%	17.34	23.05	60.63	95.55	100	100
$\hat{S}(2)_t^n$-25%	31.18	35.94	66.7	96.28	100	100
$\hat{S}(4)_t^n$-1%	8.05	14.8	56.27	95.1	100	100
$\hat{S}(4)_t^n$-2.5%	9.31	15.8	56.82	95.14	100	100
$\hat{S}(4)_t^n$-5%	12.06	18.37	58.02	95.27	100	100
$\hat{S}(4)_t^n$-10%	18.05	23.74	60.69	95.61	100	100
$\hat{S}(4)_t^n$-25%	32.7	37.66	68.03	96.4	100	100
$T^{AJ}(X)_t^n$-1%	0.42	1.05	5.65	26.49	87.34	99.89
$T^{AJ}(X)_t^n$-2.5%	1.26	2.46	9.57	34.24	89.91	99.93
$T^{AJ}(X)_t^n$-5%	3.55	5.34	15.14	43.21	92	99.96
$T^{AJ}(X)_t^n$-10%	9.44	12.21	24.91	53.62	93.79	99.97
$T^{AJ}(X)_t^n$-25%	28.83	32.26	45.21	70.38	96.21	99.99

Table 2: *This table shows the power performance for a jump-diffusion process. The process is generated according to the model (6.1) plus one jump with the jump size* 0.4.

n	100	200	500	1000	3000	10000
$T^{BS}(X)^n_t$-1%	5.88	5.4	7.1	12.44	41.24	95.65
$T^{BS}(X)^n_t$-2.5%	8.39	7.76	10.4	17.39	50.8	97.41
$T^{BS}(X)^n_t$-5%	11.92	11.82	15.77	25.06	61.8	98.77
$T^{BS}(X)^n_t$-10%	17.84	18.13	24.24	36.18	73.72	99.62
$T^{BS}(X)^n_t$-25%	32.47	34.42	43.16	56.77	88.5	99.92
$T^{BS,r}(X)^n_t$-1%	2.46	2.81	4.68	9.59	38.36	95.27
$T^{BS,r}(X)^n_t$-2.5%	4.27	4.88	7.81	14.88	48.28	97.23
$T^{BS,r}(X)^n_t$-5%	7.66	8.49	12.96	22.47	60.03	98.73
$T^{BS,r}(X)^n_t$-10%	13.82	15.13	21.72	34.07	72.73	99.58
$T^{BS,r}(X)^n_t$-25%	30.3	32.59	42.2	56.19	88.3	99.92
$\hat{S}(2)^n_t$-1%	6.09	7.44	24.18	68.5	99.95	99.99
$\hat{S}(2)^n_t$-2.5%	7.27	8.26	25.1	68.86	99.95	99.99
$\hat{S}(2)^n_t$-5%	9.82	11.02	26.86	69.63	99.95	99.99
$\hat{S}(2)^n_t$-10%	14.96	15.75	30.81	71.1	99.95	99.99
$\hat{S}(2)^n_t$-25%	29.55	30.02	42.1	75.83	99.97	99.99
$\hat{S}(4)^n_t$-1%	5.61	7.01	23.81	68.31	99.95	99.99
$\hat{S}(4)^n_t$-2.5%	6.73	8.24	24.73	68.72	99.95	99.99
$\hat{S}(4)^n_t$-5%	9.26	10.78	26.81	69.57	99.95	99.99
$\hat{S}(4)^n_t$-10%	15.1	16.42	31.19	71.49	99.95	99.99
$\hat{S}(4)^n_t$-25%	31.25	31.75	43.46	76.46	99.95	99.99
$T^{AJ}(X)^n_t$-1%	0.29	0.58	1.49	7.11	73.07	99.86
$T^{AJ}(X)^n_t$-2.5%	0.95	1.52	2.81	11.2	78.11	99.89
$T^{AJ}(X)^n_t$-5%	3.31	3.93	6.39	17.69	83.33	99.92
$T^{AJ}(X)^n_t$-10%	9.21	9.96	13.4	28.47	88.04	99.93
$T^{AJ}(X)^n_t$-25%	28.32	28.58	34.87	52.16	93.26	99.96

Table 3: *This table shows the power performance for a jump-diffusion process. The process is generated according to the model (6.1) plus two jumps with jump sizes* $\sqrt{0.08}$ *and* $-\sqrt{0.08}$.

n	100	200	500	1000	3000	10000
$T^{BS}(X)^n_t$-1%	6.3	6.41	10.09	17.37	37.88	64.97
$T^{BS}(X)^n_t$-2.5%	8.66	8.97	13.7	21.89	42.76	68.98
$T^{BS}(X)^n_t$-5%	12.5	13.37	19.18	28.09	49.22	73.78
$T^{BS}(X)^n_t$-10%	18.96	19.66	26.95	36.85	57.11	78.82
$T^{BS}(X)^n_t$-25%	33.52	36.3	44.11	54.04	71.47	86.84
$T^{BS,r}(X)^n_t$-1%	2.38	3.85	7.68	15.03	36.2	64.41
$T^{BS,r}(X)^n_t$-2.5%	4.5	5.83	11.07	19.53	41.48	68.36
$T^{BS,r}(X)^n_t$-5%	8.05	9.64	16.52	25.96	56.4	73.25
$T^{BS,r}(X)^n_t$-10%	14.35	16.51	24.77	35.07	71.1	78.51
$T^{BS,r}(X)^n_t$-25%	30.95	34.58	43.18	53.41	80.82	86.77
$\hat{S}(2)^n_t$-1%	8.14	14.07	33.02	53.5	80.98	94.6
$\hat{S}(2)^n_t$-2.5%	9.44	15.24	33.94	54.09	81.44	94.68
$\hat{S}(2)^n_t$-5%	12.36	17.38	35.56	55.11	82.51	94.78
$\hat{S}(2)^n_t$-10%	17.05	21.85	38.95	57.5	82.57	94.99
$\hat{S}(2)^n_t$-25%	31.23	34.47	49	65.02	85.57	95.73
$\hat{S}(4)^n_t$-1%	7.67	13.74	32.84	53.27	80.81	94.64
$\hat{S}(4)^n_t$-2.5%	8.79	14.82	33.68	53.8	81.02	94.69
$\hat{S}(4)^n_t$-5%	11.55	17.09	35.49	55.19	81.5	94.75
$\hat{S}(4)^n_t$-10%	17.6	22.31	39.18	57.79	82.46	95
$\hat{S}(4)^n_t$-25%	33.53	36.51	49.82	65.22	85.56	95.74
$T^{AJ}(X)^n_t$-1%	0.4	0.98	5.12	15.85	50.49	83.57
$T^{AJ}(X)^n_t$-2.5%	1.38	2.22	7.97	19.56	54.19	85.05
$T^{AJ}(X)^n_t$-5%	3.82	5.23	12.27	25.2	58.94	86.79
$T^{AJ}(X)^n_t$-10%	9.88	11.63	19.96	33.42	64.62	88.78
$T^{AJ}(X)^n_t$-25%	29.8	31.23	39.14	51.47	75.1	92.24

Table 4: *This table shows the power performance for a jump-diffusion process. The process is generated according to the model (6.1) plus three* $N(0, \frac{0.16}{3})$*-distributed jumps.*

n	100	200	500	1000	3000	10000
$T^{BS}(X)^n_t$-1%	5.56	4	2.69	2.07	1.85	1.35
$T^{BS}(X)^n_t$-2.5%	7.6	5.9	4.49	3.67	3.25	2.67
$T^{BS}(X)^n_t$-5%	10.99	9.17	7.23	6.98	5.95	5.19
$T^{BS}(X)^n_t$-10%	16.3	14.83	12.43	11.74	11.46	10.37
$T^{BS}(X)^n_t$-25%	30.9	29.29	27.22	26.7	25.95	25.18
$T^{BS,r}(X)^n_t$-1%	2.29	1.9	1.65	1.37	1.5	1.1
$T^{BS,r}(X)^n_t$-2.5%	3.93	3.49	3.12	2.85	2.9	2.44
$T^{BS,r}(X)^n_t$-5%	6.92	6.28	5.69	5.83	5.45	4.9
$T^{BS,r}(X)^n_t$-10%	12.55	12.22	10.9	10.92	10.88	10.06
$T^{BS,r}(X)^n_t$-25%	28.57	27.52	26.43	25.99	25.6	25.03
$\hat{S}(2)^n_t$-1%	5.08	3.79	2.71	1.97	1.68	1.25
$\hat{S}(2)^n_t$-2.5%	6.63	4.91	3.91	3.14	2.78	2.4
$\hat{S}(2)^n_t$-5%	9.24	7.4	6.53	5.21	4.95	4.9
$\hat{S}(2)^n_t$-10%	14.48	12.14	11.39	10.12	9.84	9.92
$\hat{S}(2)^n_t$-25%	29.58	26.66	25.62	25.12	24.63	24.91
$\hat{S}(4)^n_t$-1%	4.4	3.51	2.32	1.68	1.37	1.37
$\hat{S}(4)^n_t$-2.5%	5.69	4.57	3.73	3.06	2.5	2.54
$\hat{S}(4)^n_t$-5%	8.74	7.09	6.27	5.48	4.91	4.86
$\hat{S}(4)^n_t$-10%	15.1	12.56	11.34	10.57	9.74	9.93
$\hat{S}(4)^n_t$-25%	31.23	27.9	26.63	25.86	24.82	24.78
$T^{AJ}(X)^n_t$-1%	0.37	0.41	0.47	0.6	0.7	1.13
$T^{AJ}(X)^n_t$-2.5%	1.1	1.17	1.27	1.53	1.6	2.29
$T^{AJ}(X)^n_t$-5%	3.33	3.62	3.6	3.78	3.74	4.6
$T^{AJ}(X)^n_t$-10%	9.3	9.83	9.1	9.1	8.88	9.55
$T^{AJ}(X)^n_t$-25%	28.36	28.03	25.62	25.8	25.68	25.1

Table 5: *This table shows the level performance for the model (6.2).*

n	100	200	500	1000	3000	10000
$T^{BS}(X)^n_t$-1%	6.61	6.11	8.39	14.69	41.41	92.03
$T^{BS}(X)^n_t$-2.5%	8.96	9.07	12.1	20.12	50.89	94.67
$T^{BS}(X)^n_t$-5%	12.74	13.09	18	27.98	61.15	96.87
$T^{BS}(X)^n_t$-10%	18.81	19.59	26.66	39.22	72.28	98.31
$T^{BS}(X)^n_t$-25%	33.81	36.09	45.57	59.48	86.85	99.52
$T^{BS,r}(X)^n_t$-1%	2.62	3.33	6.02	11.85	38.52	91.45
$T^{BS,r}(X)^n_t$-2.5%	4.94	5.51	9.52	17.23	48.58	94.39
$T^{BS,r}(X)^n_t$-5%	8.25	9.74	15.2	25.66	59.64	96.7
$T^{BS,r}(X)^n_t$-10%	14.64	16.72	24.53	37.22	71.38	98.3
$T^{BS,r}(X)^n_t$-25%	31.83	34.45	44.75	58.78	86.64	99.51
$\hat{S}(2)^n_t$-1%	7.83	13.41	48.41	90.1	100	100
$\hat{S}(2)^n_t$-2.5%	9.03	14.63	49.02	90.3	100	100
$\hat{S}(2)^n_t$-5%	11.57	16.78	50.4	90.54	100	100
$\hat{S}(2)^n_t$-10%	16.68	21.21	52.96	91.04	100	100
$\hat{S}(2)^n_t$-25%	30.73	34.39	60.66	92.52	100	100
$\hat{S}(4)^n_t$-1%	7.57	13.01	48.12	90.06	100	100
$\hat{S}(4)^n_t$-2.5%	8.52	14.06	48.88	90.12	100	100
$\hat{S}(4)^n_t$-5%	11.13	16.43	50.32	90.4	100	100
$\hat{S}(4)^n_t$-10%	17.15	21.51	53.25	91.04	100	100
$\hat{S}(4)^n_t$-25%	32.74	35.54	61.54	92.97	100	100
$T^{AJ}(X)^n_t$-1%	0.37	0.82	5.13	21.3	81.54	99.81
$T^{AJ}(X)^n_t$-2.5%	1.08	1.87	8.13	28.06	84.87	99.85
$T^{AJ}(X)^n_t$-5%	3.6	4.73	13.37	36.82	88.19	99.94
$T^{AJ}(X)^n_t$-10%	10.26	10.98	21.59	47.45	91.18	99.96
$T^{AJ}(X)^n_t$-25%	30.98	30.83	42.2	65.11	94.66	99.97

Table 6: *This table shows the power performance for a jump-diffusion process. The process is generated according to the model (6.2) plus one jump with the jump size* 0.26.

n	100	200	500	1000	3000	10000
$T^{BS}(X)^n_t$-1%	6.15	5.47	6.87	11.14	34.5	89.2
$T^{BS}(X)^n_t$-2.5%	8.24	7.92	9.71	16.3	43.05	92.79
$T^{BS}(X)^n_t$-5%	11.92	12.03	14.94	22.95	53.41	95.58
$T^{BS}(X)^n_t$-10%	17.69	18.47	22.87	32.86	65.88	97.94
$T^{BS}(X)^n_t$-25%	32.7	34.36	41.99	53.68	82.36	99.47
$T^{BS,r}(X)^n_t$-1%	2.43	2.92	4.76	8.88	31.69	88.55
$T^{BS,r}(X)^n_t$-2.5%	4.55	5.02	7.65	13.6	40.82	92.41
$T^{BS,r}(X)^n_t$-5%	7.8	8.57	12.3	20.78	51.75	95.4
$T^{BS,r}(X)^n_t$-10%	13.8	15.33	20.72	30.97	64.92	97.85
$T^{BS,r}(X)^n_t$-25%	30.22	32.82	40.82	53.07	82.17	99.46
$\hat{S}(2)^n_t$-1%	6.26	7.9	20.84	57.82	99.84	100
$\hat{S}(2)^n_t$-2.5%	7.36	9.22	21.76	58.37	99.85	100
$\hat{S}(2)^n_t$-5%	9.93	11.56	23.7	59.37	99.86	100
$\hat{S}(2)^n_t$-10%	14.82	16.36	27.77	61.83	99.86	100
$\hat{S}(2)^n_t$-25%	29.08	30.26	39.49	68.29	99.87	100
$\hat{S}(4)^n_t$-1%	5.71	7.56	20.61	57.64	99.84	100
$\hat{S}(4)^n_t$-2.5%	6.78	8.78	21.46	58.16	99.84	100
$\hat{S}(4)^n_t$-5%	9.68	11.35	23.61	59.48	99.84	100
$\hat{S}(4)^n_t$-10%	15.58	16.35	28.29	62.09	99.85	100
$\hat{S}(4)^n_t$-25%	31.13	31.55	40.54	69.19	99.89	100
$T^{AJ}(X)^n_t$-1%	0.33	0.61	1.54	5.86	61.82	99.61
$T^{AJ}(X)^n_t$-2.5%	1.15	1.73	2.91	9.35	68.18	99.71
$T^{AJ}(X)^n_t$-5%	3.47	4.3	6.29	15.39	74.6	99.77
$T^{AJ}(X)^n_t$-10%	9.36	9.94	12.99	25.23	81.16	99.82
$T^{AJ}(X)^n_t$-25%	28.65	29.44	34.3	47.87	89.57	99.93

Table 7: *This table shows the power performance for a jump-diffusion process. The process is generated according to the model (6.2) plus two jumps with jump sizes* $\sqrt{0.26^2/2}$ *and* $-\sqrt{0.26^2/2}$.

n	100	200	500	1000	3000	10000
$T^{BS}(X)_t^n$-1%	6.55	6.48	9.64	15.27	34.94	60.08
$T^{BS}(X)_t^n$-2.5%	8.74	9.02	12.69	19.65	40.27	64.42
$T^{BS}(X)_t^n$-5%	12.23	13.17	17.59	25.8	46.87	69.41
$T^{BS}(X)_t^n$-10%	18.17	19.37	25	34.04	55.03	75.42
$T^{BS}(X)_t^n$-25%	32.96	36.06	42.74	52.01	69.74	84.39
$T^{BS,r}(X)_t^n$-1%	2.77	3.59	7.39	13.22	33.51	59.24
$T^{BS,r}(X)_t^n$-2.5%	4.75	5.86	10.37	17.39	38.88	63.89
$T^{BS,r}(X)_t^n$-5%	8.31	9.86	15.18	23.91	45.94	69.03
$T^{BS,r}(X)_t^n$-10%	14.03	16.4	22.98	32.49	54.11	75.24
$T^{BS,r}(X)_t^n$-25%	30.66	34.4	41.85	51.41	69.53	84.33
$\hat{S}(2)_t^n$-1%	8.14	12.98	29.72	50.57	79.03	93.4
$\hat{S}(2)_t^n$-2.5%	9.27	14.03	30.54	51.35	79.24	93.52
$\hat{S}(2)_t^n$-5%	11.37	16.13	32.05	52.72	79.7	93.74
$\hat{S}(2)_t^n$-10%	16.52	20.72	35.85	55.32	80.84	94.12
$\hat{S}(2)_t^n$-25%	31.03	34.69	45.96	63.07	84.02	95.08
$\hat{S}(4)_t^n$-1%	7.52	12.64	29.37	50.33	78.98	93.44
$\hat{S}(4)_t^n$-2.5%	8.69	13.59	30.32	50.89	79.22	93.54
$\hat{S}(4)_t^n$-5%	11.22	16.35	32.06	52.59	79.86	93.68
$\hat{S}(4)_t^n$-10%	16.53	21.41	36.11	55.57	81.05	94.04
$\hat{S}(4)_t^n$-25%	32.79	35.7	47.36	63.47	84.6	94.98
$T^{AJ}(X)_t^n$-1%	0.34	0.89	4.42	13.89	46.56	79.57
$T^{AJ}(X)_t^n$-2.5%	0.97	1.95	6.74	17.59	50.79	81.21
$T^{AJ}(X)_t^n$-5%	3.38	4.59	10.9	23.02	55.64	83.28
$T^{AJ}(X)_t^n$-10%	9.25	11.02	18.47	31.13	61.73	85.6
$T^{AJ}(X)_t^n$-25%	28.92	30.36	38.21	49.43	73.14	90.25

Table 8: *This table shows the power performance for a jump-diffusion process. The process is generated according to the model (6.2) plus three jumps with* $N(0, \frac{0.26^2}{3})$*-distributed jump sizes.*

n	100	400	900	1600	2500	4900	10000	22500
$\hat{S}(2)_t^n$-1%	1.71	3.74	4.03	3.26	2.8	1.91	1.41	1.18
$\hat{S}(2)_t^n$-2.5%	3.06	4.96	5.13	4.5	3.97	3.06	2.54	2.53
$\hat{S}(2)_t^n$-5%	5.79	7.44	7.7	7.06	6.39	5.59	4.71	4.97
$\hat{S}(2)_t^n$-10%	10.82	12.24	12.69	11.73	11.2	10.87	10.03	10.23
$\hat{S}(2)_t^n$-25%	26.05	27.23	27.48	26.46	25.66	26.13	24.12	25.4
n	100	400	900	1600	2500	4900	10000	22500
$\hat{S}(2)_t^n$-1%	2.82	10.43	23.42	31.3	39.05	63.27	84.64	97.85
$\hat{S}(2)_t^n$-2.5%	4.23	11.66	24.34	32.18	39.83	64.19	85.33	97.97
$\hat{S}(2)_t^n$-5%	6.94	13.94	26.35	34.13	41.36	65.26	86.16	98.1
$\hat{S}(2)_t^n$-10%	12.01	18.66	30.42	37.75	44.47	67.35	87.04	98.27
$\hat{S}(2)_t^n$-25%	26.85	32.42	42.14	48.05	53.73	72.94	89.58	98.7
n	100	400	900	1600	2500	4900	10000	22500
$\hat{S}(2)_t^n$-1%	2.14	6.63	11.02	13.09	15.39	26.12	42.73	72.36
$\hat{S}(2)_t^n$-2.5%	3.37	7.68	12.08	14.07	16.68	27.34	44.22	73.28
$\hat{S}(2)_t^n$-5%	5.64	10.19	14.26	16.16	18.87	29.05	46.52	74.58
$\hat{S}(2)_t^n$-10%	10.72	14.59	18.83	20.89	23.14	32.92	49.84	76.58
$\hat{S}(2)_t^n$-25%	26.21	28.88	32.57	34.25	35.93	43.79	58.95	81.13
n	100	400	900	1600	2500	4900	10000	22500
$\hat{S}(2)_t^n$-1%	2.53	9.89	18.34	22.37	25.88	35.08	42.44	54.36
$\hat{S}(2)_t^n$-2.5%	3.91	11.28	19.34	23.38	26.73	36.07	43.61	55.26
$\hat{S}(2)_t^n$-5%	6.51	13.7	21.49	25.25	28.53	37.68	45.42	56.83
$\hat{S}(2)_t^n$-10%	11.72	18.24	25.96	29	32.64	41.23	48.39	59.28
$\hat{S}(2)_t^n$-25%	26.78	32.64	38.29	41.17	43.58	51	57.15	66.16

Table 9: *This table shows the level (upper panel) and power (lower panels) performance for the model (6.1) which is corrupted by noise. First, we added one jump with jump size* 0.4 *(second panel). Then we added two jumps with jump sizes* $\sqrt{0.08}$ *and* $-\sqrt{0.08}$ *(third panel) and three* $N(0, \frac{0.16}{3})$*-distributed jumps (fourth panel).*

n	100	400	900	1600	2500	4900	10000	22500
$\hat{S}(2)_t^n$-1%	1.98	4.42	4.95	4.35	2.96	2.69	1.69	1.46
$\hat{S}(2)_t^n$-2.5%	3.31	5.66	6.08	5.52	4.1	4.11	2.93	2.68
$\hat{S}(2)_t^n$-5%	6.16	8.1	8.41	7.93	6.59	6.66	5.43	5.16
$\hat{S}(2)_t^n$-10%	10.99	13.11	13.22	12.84	11.3	11.56	10.35	10
$\hat{S}(2)_t^n$-25%	25.62	27.89	27.08	27.3	25.99	26.07	24.68	24.84
n	100	400	900	1600	2500	4900	10000	22500
$\hat{S}(2)_t^n$-1%	2.6	9.86	20.88	27.14	33.1	53.65	74.48	93.75
$\hat{S}(2)_t^n$-2.5%	3.99	11.07	21.88	28.08	33.96	54.88	75.45	94.02
$\hat{S}(2)_t^n$-5%	6.74	13.33	23.74	29.78	35.63	56.35	76.85	94.48
$\hat{S}(2)_t^n$-10%	11.74	17.66	27.92	33.26	38.91	58.74	78.45	95.12
$\hat{S}(2)_t^n$-25%	26.24	31.76	39.83	44.76	48.91	65.7	82.12	96.07
n	100	400	900	1600	2500	4900	10000	22500
$\hat{S}(2)_t^n$-1%	2.23	6.41	10.33	11.52	13.61	21.17	35.15	60.13
$\hat{S}(2)_t^n$-2.5%	3.3	7.62	11.53	12.66	14.73	22.46	36.66	61.25
$\hat{S}(2)_t^n$-5%	5.98	10.01	14.12	15.01	16.74	24.6	38.81	62.99
$\hat{S}(2)_t^n$-10%	11.08	14.69	18.62	19.54	20.86	28.51	42.6	65.81
$\hat{S}(2)_t^n$-25%	26.62	29.21	32.25	32.89	34.54	40.58	52.02	71.86
n	100	400	900	1600	2500	4900	10000	22500
$\hat{S}(2)_t^n$-1%	2.82	8.99	16.54	19.9	22.61	31.73	38.89	50.49
$\hat{S}(2)_t^n$-2.5%	4.16	10.2	17.42	20.92	23.69	32.86	40.02	51.3
$\hat{S}(2)_t^n$-5%	6.51	12.4	19.06	22.8	25.81	34.96	41.95	52.87
$\hat{S}(2)_t^n$-10%	11.91	17.26	23.35	26.89	29.81	38.43	45.05	55.66
$\hat{S}(2)_t^n$-25%	26.52	30.46	36.71	38.44	41.41	48.9	54.49	63.25

Table 10: *This table shows the level (upper panel) and the power (lower panels) performance for the model (6.2) which is corrupted by noise. First, we added one jump with the jump size* 0.26 *(second panel). Then we added two jumps with jump sizes* $\sqrt{0.26^2/2}$ *and* $-\sqrt{0.26^2/2}$ *(third panel) and three* $N(0, \frac{0.26^2}{3})$*-distributed jumps. (fourth panel).*

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