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Representation and Weak Convergence of Stochastic Integrals with Fractional Integrator Processes

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Abstract

This paper considers the asymptotic distribution of the covariance of a nonstationary fractionally integrated process with the stationary increments of another such process – possibly, itself. Questions of interest include the relationship between the harmonic representation of these random variables, which we have analysed in a previous paper, and the construction derived from moving average representations in the time domain. The limiting integrals are shown to be expressible in terms of functionals of Itô integrals with respect to two distinct Brownian motions. Their mean is nonetheless shown to match that of the harmonic representation, and they satisfy the required integration by parts rule. The advantages of our approach over the harmonic analysis include the facts that our formulae are valid for the full range of the long memory parameters, and extend to non-Gaussian processes.

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1 Introduction

Let x_t and y_t be linear processes having the MA(∞) forms

$$x_t = \sum_{j=0}^{\infty} b_j u_{t-j}, \qquad y_t = \sum_{j=0}^{\infty} c_j w_{t-j} \tag{1.1}$$

where u_t, w_t are zero mean, independently and identically distributed processes, and the coefficient sequences $\{b_j\}$ and $\{c_j\}$ decay hyperbolically. If X_n and Y_n denote suitably normalized partial sum processes on the unit interval for a sample of size n, it is known under fairly general assumptions that $(X_n, Y_n) \xrightarrow{d} (X, Y)$ where the limit processes are fractional Brownian motions, as defined by Mandelbrot and van Ness (1968). For exemplar case X, the well-known formula is

$$X(\xi) = \frac{1}{\Gamma(d_Y + 1)} \left[\int_0^{\xi} (\xi - \tau)^{d_X} \, dU(\tau) + \int_{-\infty}^{0} \left((\xi - \tau)^{d_X} - (-\tau)^{d_X} \right) dU(\tau) \right] \tag{1.2}$$

where U is regular Brownian motion on $\mathbb{R}$. Fractional noise processes are a well-known simple case, in which

$$b_j = \frac{\Gamma(j + d_X)}{\Gamma(d_X)\Gamma(j+1)} \qquad c_j = \frac{\Gamma(j + d_Y)}{\Gamma(d_Y)\Gamma(j+1)} \tag{1.3}$$

for $-\frac{1}{2} < d_X, d_Y < \frac{1}{2}$. In this case,

$$X_n(\xi) = n^{-1/2 - d_X} \sum_{t=1}^{[n\xi]} x_t, \quad Y_n(\xi) = n^{-1/2 - d_Y} \sum_{t=1}^{[n\xi]} y_t \tag{1.4}$$

for $0 \leq \xi \leq 1$, where $[x]$ denotes the largest integer not exceeding x. Considerably greater generality will be permitted, although parameters d_X and d_Y, subject to these constraints, will in all cases index the rate of lag decay. The best general conditions currently known for these results are given by Davidson and de Jong (2000) (henceforth, DDJ).

In this paper, our concern is the limiting distribution of the random variable

$$G_n = \frac{1}{K(n)} \sum_{t=1}^{n-1} \sum_{s=1}^{t} x_s y_{t+1} \tag{1.5}$$

where $K(n)$ is a function of sample size which, for the case of (1.3) at least, can be set as $n^{1+d_X+d_Y}$. Expressions with the form of G_n arise in the theory of cointegration. For example, in the case $x_t = y_t$ they appear in the formulae for the Dickey-Fuller statistic. In a cointegrating regression they appear in error-of-estimate formulae, with y_t having the interpretation of a stationary error term and x_t the difference of the stochastically trending regressor. In applications we should often wish x_t and y_t to be respectively column and row vectors, and hence G_n to be a vector or matrix. However, this is notationally burdensome and it is more convenient to derive the main results for the scalar case. The required extensions are obtainable by very straightforward generalizations.

A limit distribution for (1.5) has been derived from the harmonic representations of the variables, where defined. In the fractional noise case these are

$$x_t = \int_{-\pi}^{\pi} e^{it\lambda} (i\lambda)^{-d_X} W_u(d\lambda), \qquad y_t = \int_{-\pi}^{\pi} e^{it\lambda} (i\lambda)^{-d_Y} W_w(d\lambda) \tag{1.6}$$

where i is the imaginary unit and (W_u, W_w) is a vector of complex-valued Gaussian random measures with the properties (for $j, k = w, u$) $W_j(-d\lambda) = \overline{W_j(d\lambda)}$, $EW_j(d\lambda) = 0$ and

$$EW_j(d\lambda)\overline{W_k(d\mu)} = \begin{cases} \omega_{jk} d\lambda, & \mu = \lambda \\ 0, & \text{otherwise.} \end{cases}$$

Chan and Terrin (1995) is a well-known study that analyses the weak convergence of fractionally integrated processes under the harmonic representation. The model these authors analyse is different from the usual 'causal' (backward-looking) model considered here, but Davidson and Hashimzade (2008) have extended their analysis and apply it to the causul model in particular. The weak limits of the partial sum processes (1.4) take the form

$$X(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{i\lambda\xi} - 1}{i\lambda} (i\lambda)^{-d_X} W_u(d\lambda)$$
$$Y(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{i\lambda\xi} - 1}{i\lambda} (i\lambda)^{-d_Y} W_w(d\lambda)$$

and G_n has the weak limit

$$\int_0^1 XdY = \frac{1}{2\pi} \int_0^1 \left(\int_{-\infty}^{\infty} \frac{e^{i\lambda r} - 1}{i\lambda} (i\lambda)^{-d_X} W_u\left(d\lambda\right) \int_{-\infty}^{\infty} e^{i\mu r} (i\lambda)^{-d_Y} W_w\left(d\mu\right) \right) dr. \tag{1.8}$$

For the case $d_X + d_Y > 0$, the expected value of this random variable is derived as

$$\begin{aligned} E\int_0^1 XdY &= \frac{\omega_{uw}}{2\pi} \int_0^1 \int_{-\infty}^{\infty} \frac{1 - e^{-i\lambda r}}{i\lambda} \left|\lambda\right|^{-d_X - d_Y} e^{-i\pi(d_X - d_Y)\operatorname{sgn}(\lambda)} d\lambda dr \\ &= \frac{\omega_{uw}\Gamma\left(1 - d_X - d_Y\right)}{\pi(1 + d_X + d_Y)(d_X + d_Y)} \sin \pi d_Y. \end{aligned} \tag{1.9}$$

In this paper, we explore the counterpart of this solution in the time domain. There are several reasons why this alternative approach provides an essential extension. The general weak convergence proofs given by Davidson and Hashimzade (2008) are restricted to the case $d_X + d_Y > 0$, and the 'standard' case $d_X = d_Y = 0$ is especially intractable, because the harmonic representation of the integral breaks down (with undefined expectation) when the processes have summable covariances. While there is no difficulty in constructing more general dependence models than the fractional noise example given, the harmonic representation requires Gaussian, identically distributed shocks – a restrictive requirement for econometric modelling. Working in the time domain allows all these limitations to be relaxed.

The specific assumptions to be adopted are as follows.

Assumption 1 *The collection $\{u_t, w_t; t \in \mathbb{Z}\}$ are identically and independently distributed with zero mean and covariance matrix*

$$E\begin{bmatrix} u_t \\ w_t \end{bmatrix} \begin{bmatrix} u_t & w_t \end{bmatrix} = \mathbf{\Omega} = \begin{bmatrix} \omega_{uu} & \omega_{uw} \\ \omega_{uw} & \omega_{ww} \end{bmatrix} \tag{1.10}$$

and $\mu_{uw}^4 = E(u_t^2 w_t^2) < \infty$. $u_t = w_t$ is an admissible case.

These random variables define the filtered probability space on which our processes live, denoted $(\Omega, \mathcal{F}, P, \boldsymbol{F})$ where

$$\boldsymbol{F} = \{\mathcal{F}_t, t \in \mathbb{Z};\ \mathcal{F}_t \subseteq \mathcal{F} \text{ all } t, \text{ and } \mathcal{F}_t \subseteq \mathcal{F}_s \text{ iff } t \leq s\}. \tag{1.11}$$

The pair (u_t, w_t) are adapted to $\mathcal{F}_t$, and in this setup we may also use the notation $\mathcal{F}_n(r) = \mathcal{F}_{[nr]}$ for $0 \leq r \leq 1$ where n is sample size. Further, letting $\mathcal{F}(r)$ represent the limiting case as $n \to \infty$, $(X(r), Y(r))$ are measurable with respect to $\mathcal{F}(r)$ and accordingly will be called $\boldsymbol{F}$-adapted.

Assumption 2 *The sequences $\{b_j\}_0^\infty$ and $\{c_j\}_0^\infty$ depend on parameters $d_X \in (-\frac{1}{2}, \frac{1}{2})$ and $d_Y \in (-\frac{1}{2}, \frac{1}{2})$, respectively, and sequences $\{L_X(j)\}$ and $\{L_Y(j)\}$ that are at most slowly varying at infinity. These sequences satisfy one of the following conditions, stated for $\{b_j\}$ as exemplar case:*

(a) *If $0 < d_X < \frac{1}{2}$ then $b_j = \Gamma(d_X)^{-1}(j+1)^{d_X-1}L_X(j)$.*

(b) *If $d_X = 0$ then $0 < \left|\sum_{j=0}^\infty b_j\right| < \infty$, and $b_j = O(j^{-1-\delta})$ for $\delta > 0$.*

(c) *If $-\frac{1}{2} < d_X < 0$ then $b_0 = a_0$ and $b_j = a_j - a_{j-1}$ for $j > 0$ where $a_j = \Gamma(1+d_X)^{-1}(j+1)^{d_X}L_X(j)$.*

Under these assumptions, we set $K(n) = n^{1+d_X+d_Y}L_X(n)L_Y(n)$ in (1.5). While the 'pure fractional' cases represented by (1.3) satisfy Assumption 2, the assumption only controls the tail behaviour of the sequences, and allows arbitrary forms for a finite number of the lag coefficients. In particular, the x_t and y_t processes may be stable invertible ARFIMA(p,d,q) processes. Suppose more generally that $x_t = (1-L)^{-d_X}\theta(L)u_t$ where $\theta(L)$ is any lag polynomial with absolutely summable coefficients, specifically, where $\theta_j = O(j^{-1-\delta})$ for $\delta > 0$. Letting for $d_X > 0$ the identity $a(L) = (1-L)^{-d_X}$ define the coefficients a_j, such that[1] $a_j \sim \Gamma(d_X)^{-1}j^{d_X-1}$, note the following result.

Proposition 1.1 *The sequence $\{b_j\}$ defined by $b(L) = a(L)\theta(L)$ satisfies $b_j \sim \theta(1)\Gamma(d_X)^{-1}j^{d_X-1}$ as $j \to \infty$.*

(All proofs are given in the Appendix.) The slowly varying component can be defined to represent the ratio of b_j to the approximating sequence. Also, since $\mathbf{\Omega}$ is unrestricted, we could impose the normalization $\theta(1) = 1$, if desired, with no loss of generality.

The cases $d_X = 0$ and $d_Y = 0$ are deliberately restricted under Assumption 2(b) to rule out the 'knife-edge' non-summable case, to avoid complications of doubtful relevance. Be careful to note that δ is not a fractional differencing coefficient in this case. Also note that the pure fractional model, represented by (1.3) has $b_0 = 1$ and $b_j = 0$ for $j > 0$, in the case $d_X = 0$. The case $d_X < 0$ under Assumption 2(c) has the 'overdifferenced' property, implying in particular that $|\sum_{k=0}^j b_k| = O(j^{d_X})$. In the pure fractional case, note that $b_j < 0$ for all $j > 0$ in this instance.

A multivariate analysis would typically invoke a vector Wold representation of the form (in the bivariate case)

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} (1-L)^{-d_X} & 0 \\ 0 & (1-L)^{-d_Y} \end{bmatrix} \begin{bmatrix} \theta_{XX}(L) & \theta_{XY}(L) \\ \theta_{YX}(L) & \theta_{YY}(L) \end{bmatrix} \begin{bmatrix} u_t \\ w_t \end{bmatrix}.$$

VARFIMA models are a popular example. However, extending our results to general linear models of this type is a simple application of the continuous mapping theorem to the limit distributions we explore in this paper. In the case shown, x_t and y_t are represented as the sums of two terms of the type (1.1), involving $\{u_t\}$ and $\{w_t\}$ respectively. Accordingly, (1.5) becomes a sum of four terms involving respectively the driving pairs $\{u_t, u_t\}$, $\{u_t, w_t\}$, $\{w_t, u_t\}$ and $\{w_t, w_t\}$. Our analysis can be applied to each of these cases in turn, with suitable redefinition of symbols.

[1] The symbol '$\sim$' here denotes that the ratio of the connected sequences converges to 1 as $j \to \infty$.

2 Some Properties of G_n

The key step is the following decomposition of expression (1.5). First, expand by substitution from (1.1), as

$$G_n = \frac{1}{K(n)} \sum_{t=1}^{n-1} \sum_{s=1}^{t} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} b_k c_j u_{s-k} w_{t+1-j}.$$

Decompose this sum as $G_n = G_{1n} + G_{2n} + G_{3n}$ where

$$\begin{aligned} G_{1n} &= \frac{1}{K(n)} \sum_{t=1}^{n-1} \sum_{s=1}^{t} \sum_{k=0}^{\infty} \sum_{j=0}^{k+t-s} b_k c_j u_{s-k} w_{t+1-j} \\ &= \frac{1}{K(n)} \sum_{t=1}^{n-1} \sum_{s=1}^{t} \sum_{j=0}^{\infty} \sum_{k=\max\{0,j+s-t\}}^{\infty} b_k c_j u_{s-k} w_{t+1-j} \end{aligned} \tag{2.1}$$

$$G_{2n} = \frac{1}{K(n)} \sum_{t=1}^{n-1} \sum_{s=1}^{t} \sum_{k=0}^{\infty} b_k c_{k+t-s+1} u_{s-k} w_{s-k} \tag{2.2}$$

and

$$G_{3n} = \frac{1}{K(n)} \sum_{t=1}^{n-1} \sum_{s=1}^{t} \sum_{k=0}^{\infty} \sum_{j=k+t-s+2}^{\infty} b_k c_j u_{s-k} w_{t+1-j}. \tag{2.3}$$

Thus, G_{1n} contains those terms, and only those terms, in which $s - k \leqslant t - j$, so that the time indices of w strictly exceed those of u, and hence $E(G_{1n}) = 0$. In G_{2n}, $s - k = t + 1 - j$ such that the time indices of u and w match. In G_{3n}, $s - k > t + 1 - j$ such that the indices of u lead those of w, and $E(G_{3n}) = 0$.

In this section we consider the behaviour of the sequence G_{2n}. Broadly speaking, its properties depend on the sign of $d_X + d_Y$, and we consider the various cases in turn.

Proposition 2.1 *If $d_X + d_Y > 0$ then $E(G_{2n}) \to \lambda_{XY}$ where*

$$\begin{aligned} \lambda_{XY} = \frac{\omega_{uw}}{\Gamma(d_X+1)\Gamma(d_Y+1)\,(d_X+d_Y)} \Bigg(\frac{d_Y}{(1+d_X+d_Y)} + \\ \int_0^\infty \Big[d_Y(1+\tau)^{d_X+d_Y} + d_X \tau^{d_X+d_Y} - (d_X+d_Y)(1+\tau)^{d_Y}\tau^{d_X} \Big] d\tau \Bigg) \end{aligned} \tag{2.4}$$

Letting λ_{YX} denote the same limit with x_t and y_t interchanged, also note that

$$\begin{aligned} \lambda_{XY} + \lambda_{YX} &= \frac{\omega_{uw}}{\Gamma(d_X+1)\Gamma(d_Y+1)} \times \\ &\quad \left(\frac{1}{(1+d_X+d_Y)} + \int_0^\infty \left((1+\tau)^{d_X} - \tau^{d_X}\right)\left((1+\tau)^{d_Y} - \tau^{d_Y}\right) d\tau \right) \\ &= \psi_{XY} \end{aligned} \tag{2.5}$$

where

$$\psi_{XY} = \lim_{n\to\infty} \frac{1}{K(n)} E\left(\sum_{t=1}^{n} x_t \sum_{t=1}^{n} y_t \right). \tag{2.6}$$

This is the off-diagonal element of $\boldsymbol{\Psi}$, the long-run covariance matrix of the processes, according to equation (3.12) of DDJ. Considering the decomposition

$$E\left(\sum_{t=1}^{n} x_t \sum_{t=1}^{n} y_t \right) = \sum_{t=1}^{n} E(x_t y_t) + \sum_{t=1}^{n-1} \sum_{s=1}^{t} E(x_s y_{t+1}) + \sum_{t=1}^{n-1} \sum_{s=1}^{t} E(y_s x_{t+1}) \tag{2.7}$$

where the second term on the right corresponds to $K(n)E(G_n)$, note that

$$E(x_t y_t) = \sigma_{XY} = \omega_{uw} \sum_{j=0}^{\infty} b_j c_j < \infty. \tag{2.8}$$

The first right-hand side term in (2.7) is $O(n)$, and hence this term is of small order under the normalization $K(n)$. The other two terms converge to λ_{XY} and λ_{YX} respectively under the same normalization, as indicated by (2.5). Observe that λ_{XY} depends only on d_X, d_Y and ω_{uw} since any short-run parameters have been absorbed into the functions L_X and L_Y; compare Lemma 1.1 for example. The sign of λ_{XY} matches that of d_Y, and if $d_Y = 0$, then $\lambda_{XY} = 0$. When $d_X > 0$, the cases where y_t is i.i.d., ($c_j = 0$ for $j > 0$) and is merely weakly dependent ($d_Y = 0$), are equivalent asymptotically.

We give these results in the easily interpretable form of (2.4) but for computational purposes, a closed-form expression is more useful, as follows.

Proposition 2.2 $\lambda_{XY} = \dfrac{\omega_{uw}\Gamma(1 - d_X - d_Y)}{\pi(1 + d_X + d_Y)(d_X + d_Y)} \sin \pi d_Y.$

This formula matches (1.9), indicating that the harmonic and moving average approaches to constructing fractional processes yield equivalent results, at least in mean. The closed form of (2.5)

$$\psi_{XY} = \frac{\omega_{uw}\Gamma(1 - d_X - d_Y)}{(1 + d_X + d_Y)} \left(\frac{\sin \pi d_Y + \sin \pi d_X}{\pi(d_Y + d_X)} \right) \tag{2.9}$$

follows directly.

Next, consider the cases where $d_X + d_Y$ is zero or negative. In the latter case, $E(G_{2n})$ diverges.

Proposition 2.3 *If $d_X + d_Y \le 0$ and $\omega_{wu} \neq 0$, then $E(G_{2n}) = O(n/K(n))$.*

In this instance there is no decomposition of ψ_{YX} into components of the form λ_{XY}, and the three terms in (2.7) are each of $O(n)$. We may write $n^{-1}\sum_{t=1}^n E(x_t y_t) = \sigma_{XY}$ and also

$$\frac{1}{n}\sum_{t=1}^{n-1}\sum_{s=1}^{t} E(x_s y_{t+1}) \to \lambda^*_{XY}$$
$$\frac{1}{n}\sum_{t=1}^{n-1}\sum_{s=1}^{t} E(y_s x_{t+1}) \to \lambda^*_{YX}.$$

These limits are finite constants depending on summable sequences of weights, hence necessarily different from λ_{XY} and λ_{YX}. Note that $E(\sum_{t=1}^n x_t)^2 = O(n^{2d_X+1})$ and $E(\sum_{t=1}^n y_t)^2 = O(n^{2d_Y+1})$ (compare DDJ Lemmas 3.1 and 3.3). For $d_X + d_Y < 0$ the left-hand side of (2.7) is therefore necessarily $o(n)$, by the Cauchy-Schwarz inequality, and so $\sigma_{XY} + \lambda^*_{XY} + \lambda^*_{YX} = 0$. Formula (2.9) is nonetheless well defined for $d_X + d_Y \le 0$. Under the normalization n the covariance vanishes, but under normalization $K(n)$ the limit in (2.6) is well-defined and equal to (2.5) (equivalently, to (2.9)) as shown in DDJ Lemma 3.3. These conclusions assume $\omega_{uw} \neq 0$, but if u_t and w_t are contemporaneously uncorrelated, implying under Assumption 1 that the cross-correlogram is zero at all orders, then each of the terms in (2.7) is zero identically. Then (2.5) holds trivially whatever the sign of $d_X + d_Y$, since $\lambda_{XY} = \lambda_{YX} = 0$.

The following result shows that G_{2n} is a consistent estimator of the mean, albeit not a feasible one.

Theorem 2.1 *If Assumptions 1 and 2 hold, $G_{2n} - E(G_{2n}) \xrightarrow{L_2} 0$.*

The important implication is that the limit distribution of $G_{1n} + G_{3n}$ matches that of the mean deviation of G_n, not forgetting that the mean diverges under the given normalization when $d_X + d_Y < 0$.

One further result concerning the behaviour of the contemporaneous covariance term is generally needed for the analysis of regression models.

Theorem 2.2 *Let Assumptions 1 and 2 hold.*

(i) $n^{-1} \sum_{t=1}^{n} x_t y_t \xrightarrow{L_2} \sigma_{XY}$.

(ii) *If* $\omega_{uw} = 0$, $-\frac{1}{2} < d_Y \leq 0$ *and* $-\frac{1}{2} < d_X \leq 0$, *then* $n^{-1/2} \sum_{t=1}^{n} x_t y_t \xrightarrow{d} N(0, V)$ *where* $V < \infty$.

3 Stochastic Integrals

In this section we use heuristic arguments to construct limiting forms for the terms G_{1n} and G_{3n}, to be denoted respectively by $\Xi_{1,XY}$ and $\Xi_{3,XY}$. Letting $\Xi_{XY} = \Xi_{1,XY} + \Xi_{3,XY}$, we shall subsequently show that $G_n - E(G_n) \xrightarrow{d} \Xi_{XY}$ where $\xrightarrow{d}$ denotes convergence in distribution.

Consider G_{1n} first. Replacing the summation over j in (2.1) by the summation over $m = t + 1 - j$, and the summation over k by the summation over $i = s - k$, rewrite G_{1n} as

$$\begin{aligned} G_{1n} &= \frac{1}{K(n)} \sum_{t=1}^{n-1} \sum_{s=1}^{t} \sum_{m=-\infty}^{t} \sum_{i=-\infty}^{\min\{s,m\}} b_{s-i} c_{t-m} u_i w_{m+1} \\ &= \frac{1}{K(n)} \sum_{m=-\infty}^{n-1} w_{m+1} \sum_{i=-\infty}^{m} u_i \sum_{t=\max\{1,m\}}^{n-1} c_{t-m} \left(\sum_{s=\max\{1,i\}}^{t} b_{s-i} \right) \\ &= \frac{1}{n} \sum_{m=-\infty}^{n-1} q_{nm} w_{m+1} \end{aligned} \tag{3.1}$$

where $q_{nm} = \sum_{i=-\infty}^{m} a_{nim} u_i$ and

$$a_{nim} = \frac{n}{K(n)} \sum_{t=\max\{1-m,0\}}^{n-1-m} c_t \left(\sum_{s=\max\{1-i,0\}}^{t+m-i} b_s \right) \tag{3.2}$$

Lemma 3.1 *For real-valued indices* r, p *with* $-\infty < p \leq r \leq 1$, $a_{n[np][nr]} = A_{XY}(r,p) + o(1)$ *as* $n \to \infty$, *where*

$$\begin{aligned} A_{XY}(r,p) = &\frac{(r-p)^{d_X} (1-r)^{d_Y} F\left(-d_X, d_Y, 1+d_Y; -\dfrac{1-r}{r-p}\right)}{\Gamma(1+d_X)\Gamma(1+d_Y)} \\ &- 1_{\{r<0\}} \frac{(r-p)^{d_X} (-r)^{d_Y} F\left(-d_X, d_Y, 1+d_Y; -\dfrac{-r}{r-p}\right)}{\Gamma(1+d_X)\Gamma(1+d_Y)} \\ &- 1_{\{p<0\}} \frac{(-p)^{d_X} \left((1-r)^{d_Y} - 1_{\{r<0\}}(-r)^{d_Y}\right)}{\Gamma(1+d_X)\Gamma(1+d_Y)}. \end{aligned} \tag{3.3}$$

and

$$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{j=0}^{\infty} \frac{\Gamma(a+j)\Gamma(b+j)}{\Gamma(c+j)j!}(-z)^j$$

represents the hypergeometric function.

Making the substitutions $dU(p)$ for $u_{[np]}/\sqrt{n}$ and $dW(\tau)$ for $w_{[n\tau]}/\sqrt{n}$, the limit of the random variable in (3.1) can be expressed heuristically in the form $\Xi_{1,XY} = \int_{-\infty}^{1} Q(r)dW(r)$ where $Q(r) = \int_{-\infty}^{r} A_{XY}(r,p)dU(p)$. Note that when $d_Y = 0$, $Q(r) = X(r)$ for $r \geq 0$ and 0 for $r < 0$, and $\Xi_{1,XY}$ reduces to the regular Itô integral of a fractional Brownian integrand, as analysed in DDJ. In the general case, we ought to remark on the potential existence issue posed by a functional of Brownian motion with infinitely remote starting point. We shall show in the sequel that these integrals can be constructed as the mean-square limits of integrals on the finite intervals $[-N,r]$ and $[-N,1]$, respectively, as $N \to \infty$. Of course, the fractional Brownian motion (1.2) itself is well-defined on just the same basis.

Next, consider G_{3n}. Proceeding in the same way as before, setting $m = t+1-j$ and $i = s-k$, we obtain from (2.3)

$$\begin{aligned} G_{3n} &= \frac{1}{K(n)} \sum_{t=1}^{n-1} \sum_{s=1}^{t} \sum_{k=0}^{\infty} \sum_{j=k+t-s+2}^{\infty} b_k c_j u_{s-k} w_{t+1-j} \\ &= \frac{1}{K(n)} \sum_{i=-\infty}^{n-1} u_i \sum_{m=-\infty}^{i-1} w_m \sum_{s=\max\{i,1\}}^{n-1} b_{s-i} \sum_{t=s}^{n-1} c_{t+1-m} \\ &= \frac{1}{K(n)} \sum_{i=-\infty}^{n-1} h_{i-1} u_i \end{aligned}$$

where $h_i = \sum_{m=-\infty}^{i} e_{nmi} w_m$ and

$$e_{nmi} = \frac{n}{K(n)} \sum_{s=\max\{0,1-i\}}^{n-1-i} b_s \left(\sum_{t=s+i+1-m}^{n-m} c_t \right). \tag{3.4}$$

Lemma 3.2 *For real-valued indices r,p with $-\infty < r \leq p \leq 1$, $e_{n[nr][np]} = E_{XY}(p,r) + o(1)$ as $n \to \infty$, where*

$$\begin{aligned} E_{XY}(p,r) = {} & \frac{1}{\Gamma(d_Y+1)\Gamma(d_X+1)} \\ & \times \left[(1-p)^{d_X} \left((1-r)^{d_Y} - (p-r)^{d_Y} F\left(-d_Y, d_X; 1+d_X; -\frac{1-p}{p-r}\right) \right) \right. \\ & \left. -1_{\{p<0\}}(-p)^{d_X} \left((1-r)^{d_Y} - (p-r)^{d_Y} F\left(-d_Y, d_X; 1+d_X; \frac{-p}{p-r}\right) \right) \right]. \end{aligned} \tag{3.5}$$

This construction closely parallels the one in Lemma 3.1 except that in this case $p \geq r$. It allows us to express the limit in the form $\Xi_{3,XY} = \int_{-\infty}^{1} H(p)dU(p)$ where $H(p) = \int_{-\infty}^{p} E_{XY}(p,r)dW(r)$. Observe that $E_{XY}(p,r) = 0$ for all p and r when $d_Y = 0$, so that this term arises only the case of fractional integrator functions.

Notice the important fact that both $\Xi_{1,XY}$ and $\Xi_{3,XY}$ are stochastic integrals of $\boldsymbol{F}$-adapted Gaussian integrand processes with respect to $\boldsymbol{F}$-adapted Brownian motions. Therefore, these integrals are of Itô type. Subject to sufficient regularity conditions on the integrands, essentially those of finite variances and almost sure continuity, plus the validity of mean-squared approximations by integrals with finite domain of integration, they may be analysed in the conventional fashion. Section 4 provides the requisite results.

Under assumptions such that both convergence results hold, in particular $d_X > 0$ and $d_Y > 0$, it appears natural to equate the random variable $\Xi_{XY} + \lambda_{XY}$ with the one denoted $\int_0^1 X dY$ in (1.8). We have shown in Lemma 2.2 that the means match. To confirm the representation as an integral, however, we also need to establish that the formulae satisfy the integration by parts rule. In Davidson and Hashimzade (2008), Corollary 4.1, this was shown to hold in expectation for the harmonic representation. Here, we can go further and show the following result, which does not depend on parameter sign restrictions.

Proposition 3.1 $\Xi_{XY} + \Xi_{YX} + \psi_{XY} = X(1)Y(1)$.

4 Weak Convergence

Building on the results in Section 2 on the behaviour of the mean sequence, the general result to be established in this section is the following.

Proposition 4.1 *Let Assumptions 1 and 2 hold.*

(i) If $d_X + d_Y > 0$, then $G_n \xrightarrow{d} \Xi_{XY} + \lambda_{XY}$.

(ii) If $d_X + d_Y = 0$, then $G_n \xrightarrow{d} \Xi_{XY} + \lambda^*_{XY}$.

(iii) If $d_X + d_Y < 0$ and $\lambda^*_{XY} \neq 0$, then $\dfrac{K(n)}{n} G_n \xrightarrow{L_2} \lambda^*_{XY}$.

(iv) If $d_X + d_Y < 0$ and $\lambda^*_{XY} = 0$, then $G_n \xrightarrow{d} \Xi_{XY}$.

Note that case (iii) has already been established in Theorem 2.1, subject to the components G_{1n} and G_{3n} being $O_p(1)$ while $G_{2n} = O_p(n/K(n))$.

Define cadlag processes $X_n = n^{-1/2-d_X} L_X(n)^{-1} \sum_{t=1}^n x_t$ and $Y_n = n^{-1/2-d_Y} L_Y(n)^{-1} \sum_{t=1}^n y_t$. Then, Proposition 4.1 will follow from Propositions 2.1 and 2.3 and Theorem 2.1 in combination with the following result, which is the main concern of this section.

Theorem 4.1 *Under Assumptions 1 and 2,*

$$(X_n, Y_n, G_n - E(G_n)) \xrightarrow{d} (X, Y, \Xi_{XY}) \tag{4.1}$$

where $\Xi_{XY} = \Xi_{1,XY} + \Xi_{3,XY}$*, and* $\xrightarrow{d}$ *denotes joint weak convergence in* $D_{\mathbb{R}^2}[0,1] \times \mathbb{R}$ *where* $D_{\mathbb{R}^2}[0,1]$ *denotes the space of cadlag pairs equipped with the Skorokhod topology.*

The result for the first two members of (4.1) is shown in DDJ. Since the limit processes are almost surely continuous, it is sufficient for joint convergence that arbitrary linear combinations of $(X_n, Y_n, G_n - E(G_n))$ converge to the corresponding combinations of the limit processes (see Davidson 1994, Theorem 29.16). Since the process elements are all defined with respect to the same filtration, these requirements follow directly. In practice, we show $(X_n, Y_n, G_{1n}, G_{3n}) \xrightarrow{d} (X, Y, \Xi_{1,XY}, \Xi_{3,XY})$ where the limit random variables $\Xi_{1,XY}$ and $\Xi_{3,XY}$ can be identified with the Itô integrals on the intervals $(-\infty, 1]$. The continuous mapping theorem then yields Theorem 4.1.

A further rearrangement of (3.1) yields

$$G_{1n} = \frac{1}{n}\sum_{m=-Nn}^{n-1} q_{nm}^N w_{m+1} + \frac{1}{n}\sum_{m=-Nn}^{n-1}(q_{nm} - q_{nm}^N)w_{m+1} + \frac{1}{n}\sum_{m=-\infty}^{-Nn} q_{nm}w_{m+1} \tag{4.2}$$

where $q_{nm}^N = \sum_{i=-Nn}^{m} a_{nim}u_i$, a_{nim} is defined in (3.2) and $N > 0$ is a fixed value to be chosen. In the same way, write

$$G_{3n} = \frac{1}{n}\sum_{i=-Nn}^{n-1} h_{n,i-1}^N u_i + \frac{1}{n}\sum_{i=-Nn}^{n-1}(h_{n,i-1} - h_{n,i-1}^N)u_i + \frac{1}{n}\sum_{i=-\infty}^{-Nn} h_{n,i-1}u_i$$

where $h_{ni}^N = \sum_{m=-Nn}^{i} e_{nmi}w_m$. The strategy of proof of Theorem 4.1 suggested by these decompositions involves three steps, which we describe for G_{1n} as the exemplar case.

1. Define the cadlag arrays

$$Q_n^N(r) = \frac{1}{\sqrt{n}}\sum_{i=-Nn}^{[nr]} a_{ni[nr]}u_i, \qquad W_n^N(r) = \frac{1}{\sqrt{n}}\sum_{m=-Nn}^{[nr]} w_m$$

and show that $Q_n^N \xrightarrow{d} Q^N$, an almost surely continuous Gaussian process on the interval $[-N, 1]$. Also, by standard arguments, $W_n^N \xrightarrow{d} W^N$ where W^N is a Brownian motion on the interval $[-N, 1]$. Since $q_{n,m-1}^N$ is a linear process in i.i.d. shocks, by Assumption 1, Step 1 can be tackled by a minor extension of Theorem 3.1 of de Jong and Davidson (2000) (henceforth, DJD).

2. (Q^N, W^N) are adapted to a common filtration $\boldsymbol{F}$ defined in (1.11), with respect to which W^N is a martingale. We therefore deduce by standard arguments that

$$\left(Q_n^N,\ W_n^N,\ \frac{1}{n}\sum_{m=-Nn}^{n} q_{n,m-1}^N w_m\right) \xrightarrow{d} \left(Q^N,\ W^N,\ \int_{-N}^{1} Q^N dW^N\right). \tag{4.3}$$

3. Show that by taking N large enough, the second and third terms of (4.2) can be made as small as desired in L_2 norm, allowing the limit random process to be formally represented as $\Xi_{1,XY} = \int_{-\infty}^{1} QdW$.

The arguments to establish the validity of these steps are given for the case of G_{1n} in Section 4.1. The case of G_{3n} is on similar lines, replacing a by e, A by E, Q by H, and exchanging w, u and W, U in formulae. These results are given in Section 4.2.

4.1 The Case of G_{1n}

We use Lemma 3.1 to show the following properties, invoking Assumptions 1 and 2 in each case.

Lemma 4.1 *Let $v_{nm}^a = n^{-1}\sum_{i=-\infty}^{m} a_{nim}^2$ for $m \in (-\infty, n)$.*

(i) $\limsup_n v_{n,[nr]}^a < \infty$ *for each fixed $r \in (-\infty, 1]$.*

(ii) $\limsup_n v_{n,[nr]}^a = O((-r)^{2d_Y+2d_X-3})$ *as $r \to -\infty$.*

Lemma 4.2 $\sup_{r\in(-\infty,1]} \limsup_n n^{-1}\sum_{i=[nr]+1}^{[n(r+\delta)]} a_{ni[n(r+\delta)]}^2 = O(\delta^{\min\{1,2d_X+1\}})$.

Lemma 4.3 $\sup_{r\in(-\infty,1]} \limsup_n n^{-1}\sum_{i=-\infty}^{[nr]} (a_{ni[n(r+\delta)]} - a_{ni[nr]})^2 = O(\delta^{2d_X+1})$.

Step 1 is then implemented by means of the following result.

Theorem 4.2 $(Q_n^N, W_n^N) \to (Q^N, W^N)$ *where* $\xrightarrow{d}$ *denotes weak convergence in the space of cadlag functions* $D_{\mathbb{R}^2}[-N,1]$ *endowed with the Skorokhod topology, and* (Q^N, W^N) *are elements of* $C_{\mathbb{R}^2}[-N,1]$ *a.s..*

Be careful to note that the topological space $D_{\mathbb{R}^2}[-N,1]$ is different from $D_{\mathbb{R}}[-N,1]\times D_{\mathbb{R}}[-N,1]$. In the former case, the jump times are assumed to be synchronized in the component spaces so that the Skorokhod distances can be defined in terms of a common change-of-time function, while in the latter case they are not. Since the jumps are always the result of discrete observation dates in our applications, the jump times match by default, and there is no problem about satisfying this requirement in practice.

Given these results, we can proceed directly to Step 2, as follows.

Theorem 4.3 *The convergence in (4.3) holds where* $\xrightarrow{d}$ *denotes weak convergence in the space* $D_{\mathbb{R}^2}[-N,1]\times\mathbb{R}$ *endowed with the Skorokhod topology.*

Theorem 4.3 is a special case of Theorem 2.2 of Kurtz and Protter (1991), see also Theorem 7.4[2] of Kurtz and Protter (1995). These results are given for stochastic processes I on $[0,\infty)$ defined by $I(\xi) = \int_0^\xi H(r)dX(r)$, where H is $\boldsymbol{F}$-adapted and left-continuous, and Y is a $\boldsymbol{F}$-semimartingale satisfying a condition of *uniformly controlled variations* (UCV). This latter condition is directly satisfied by W_n^N since this is a partial sum of independent and identically distributed shocks with finite variance, and our processes are defined on a compact interval. There is no difficulty about considering the interval $[0, N+1]$, and then re-locating the initial date from 0 to $-N$.

We cannot apply the Kurtz-Protter results in full generality, without modification, because in our case the integrands correspond to a family of functionals $Q^N(r,\xi)$, and $\int_{-N}^\xi Q^N(r,\xi)dW(r)$ does not have the form of $I(\xi)$. However, replacing $Q^N(r,\xi)$ by $Q^N(r,1)$ defines an integrand process in the appropriate class, and then extracting the pointwise implication for the case $\xi = 1$ yields the desired distribution. Since Q^N is a.s. continuous according to Lemma 4.2, there is no problem in meeting the left-continuity requirement.

Moving on to Step 3, we show the limiting negligibility of the remainders as follows.

Theorem 4.4 *If Assumptions 1 and 2 hold,*

(i) $\lim_{n\to\infty} E\left(\frac{1}{n}\sum_{m=-Nn}^{n-1}(q_{nm} - q_{nm}^N)w_{m+1}\right)^2 = O(N^{d_X+d_Y-2})$

(ii) $\lim_{n\to\infty} E\left(\frac{1}{n}\sum_{m=-\infty}^{-Nn} q_{nm}w_{m+1}\right)^2 = O(N^{d_X+d_Y-3})$.

[2] This theorem is numbered 34 in an alternate version of these notes posted on the internet.

4.2 The Case of G_{3n}

In this section the arguments are effectively the same as those in Section 4.1, although the results differ in formulae and in the details of proofs. We simply state the counterpart results, in abbreviated form where appropriate. The proofs of these results based on the representation in Lemma 3.2, are treated jointly those of Section 4.1 in the Appendix.

Lemma 4.4 *Let* $v_{ni}^e = n^{-1}\sum_{m=-\infty}^{i} e_{nmi}^2$ *for* $i \in (-\infty, n)$. *Then,*

(i) $\limsup_n v_{n,[np]}^e < \infty$ *for each fixed* $p \in (-\infty, 1]$.

(ii) $\limsup_n v_{n,[np]}^e = O((-p)^{2d_Y+2d_X-3})$ *as* $p \to -\infty$.

Lemma 4.5 $\sup\limits_{p\in(-\infty,1]} \limsup\limits_{n} n^{-1}\sum_{m=[np]+1}^{[n(p+\delta)]} e_{nm[n(p+\delta)]}^2 = O(\delta^{\min\{1,2d_X+1\}})$.

Lemma 4.6 $\sup\limits_{p\in(-\infty,1]} \limsup\limits_{n} n^{-1}\sum_{m=-\infty}^{[np]} (e_{nm[n(p+\delta)]} - e_{nm[np]})^2 = O(\delta^{2d_X+1})$.

Theorem 4.5 $(H_n^N, U_n^N) \xrightarrow{d} (H^N, U^N) \in C_{\mathbb{R}^2}[-N,1]$ *a.s.*

Theorem 4.6 $\left(H_n^N,\ U_n^N,\ \dfrac{1}{n}\sum_{m=-Nn}^{n} u_m\, h_{n,m-1}^N\right) \xrightarrow{d} \left(H^N,\ U^N,\ \displaystyle\int_{-N}^{1} H^N dU^N\right)$.

Theorem 4.7

(i) $\lim_{n\to\infty} E\left(\dfrac{1}{n}\sum_{m=-Nn}^{n-1} (h_{nm} - h_{nm}^N)u_{m+1}\right)^2 = O(N^{d_X+d_Y-2})$

(ii) $\lim_{n\to\infty} E\left(\dfrac{1}{n}\sum_{m=-\infty}^{-Nn} h_{nm}u_{m+1}\right)^2 = O(N^{d_X+d_Y-3})$.

5 Discussion

There exists quite an extensive mathematical literature on the properties of integrals with respect to fractional Brownian motion. See, *inter alia*, Lin (1995), Dai and Heyde (1996), Zähle (1998), Decreusefond and Üstünel (1999), Decreusefond (2001), Pipiras and Taqqu (2000, 2001, 2002), and the references therein, Duncan et. al (2000a, 2000b) and Bender (2003). This literature is chiefly concerned with representation and existence questions for general classes of deterministic and non-adapted integrand. These fractional integrals have been variously represented, applying the Wiener-Ito chaos decomposition of the fractional processes, either as the Skorokhod integrals defined in the Malliavin calculus (see e.g. Øksendal 1997) or as the limits of Riemann sums of the Wick products of the integrand and increments of the integrator process (Duncan et al., 2000a). An important issue in this research, particularly for pricing applications in mathematical finance, has been to find a counterpart of the Itô integral (featuring zero mean, in particular) for fractional Brownian integrators.

However, there has been comparatively little emphasis on deriving these random variables as the weak limits of normalized discrete sums. In this context our results appear to have some novel and interesting features. For the special case of a fractional Brownian motion integrand, we may think of the random variable $\Xi_{XY} + \lambda_{XY}$ defined here, with $d_X + d_Y > 0$, as an integral of Stratonovich type, the counterpart of that derived in the harmonic representation (1.8). The zero-mean component Ξ_{XY} is evidently the counterpart of the Wick integral. We must leave it to future work to establish the relationship between these representations in detail. However, the fact that the latter variable decomposes into a pair of Itô-type terms in which the 'integrator' and 'integrand' processes change places, so that the forcing processes of both 'integrator' and 'integrand' play the role of Brownian integrators is, we suggest, a potentially illuminating way to view the implications of an integrator process that is not a semimartingale.

We have noted that reliance on the results due to Kurtz and Protter (1991, 1995) has limited us to considering the pointwise case of integral convergence. A useful goal would be to extend our results to the stochastic process case on $[0,1]$, for example to show that

$$G_{1n} = \frac{1}{n}\sum_{m=-\infty}^{[n\cdot]} q_{n,m-1}(\cdot)w_m \xrightarrow{d} \int_{-\infty}^{\cdot} Q(\cdot)dW.$$

Our formulae would be unchanged except for the replacement of 1 by $\xi \in [0,1]$ and n by $[n\xi]$, where required, also noting that $E(G_n(\xi)) \to \lambda_{XY}\xi^{1+d_X+d_Y}$. As remarked above, dependence of the integrands Q and H on ξ prevent us from applying the cited results directly. A possible way to achieve the extension from pointwise convergence might be to show tightness using (e.g.) Billingsley (1968) Theorem 12.3. However, we must leave this extension also for future work.

6 Appendix: Proofs

6.1 Proof of Proposition 1.1

The coefficient of L^j in the expansion of $b(L) = \theta(L)a(L)$ is

$$b_j = \sum_{i=0}^{j} \theta_i a_{j-i} \sim \frac{1}{\Gamma(d_X)}\sum_{i=0}^{j-1} \theta_i (j-i)^{d_X-1}. \tag{6.1}$$

Therefore, for any $\eta > 1$ note that

$$b_j \sim \frac{j^{d_X-1}}{\Gamma(d_X)}\Big(\frac{j-j^{1/\eta}}{j}\Big)^{d_X-1}\sum_{i=0}^{j-1}\theta_i\Big(\frac{j-i}{j-j^{1/\eta}}\Big)^{d_X-1}. \tag{6.2}$$

Write

$$\sum_{i=0}^{j-1}\theta_i\Big(\frac{j-i}{j-j^{1/\eta}}\Big)^{d_X-1} = A(j) + B(j)$$

where

$$A(j) = \sum_{i=0}^{[j^{1/\eta}]-1}\theta_i\Big(\frac{j-i}{j-j^{1/\eta}}\Big)^{d_X-1}$$

and

$$B(j) = \sum_{i=[j^{1/\eta}]}^{j-1}\theta_i\Big(\frac{j-i}{j-j^{1/\eta}}\Big)^{d_X-1}.$$

Since the θ_j are summable and

$$\frac{j}{j-j^{1/\eta}} \to 1$$

it is clear that $A(j) \to \theta(1)$ as $j \to \infty$. To show that $B(j) \to 0$, define $k = j - i$. since $\theta_i = O(i^{-1-\delta})$ for $\delta > 0$ by assumption,

$$\begin{aligned} B(j) &\leq \sum_{i=[j^{1/\eta}]}^{j-1} |\theta_i| \Big(\frac{j-i}{j - j^{1/\eta}} \Big)^{d_X - 1} \\ &= O\Bigg((j - j^{1/\eta})^{1-d_X} j^{-(1+\delta)/\eta} \sum_{k=1}^{j-[j^{1/\eta}]} \Big(\frac{j-k}{j^{1/\eta}} \Big)^{-1-\delta} k^{d_X - 1} \Bigg) \\ &= O((j - j^{1/\eta}) j^{-(1+\delta)/\eta}) \end{aligned}$$

in view of the fact that $j - k \geq j^{1/\eta}$ for all the k. Since $\eta > 1$ is arbitrary, pick $\eta < 1 + \delta$ to complete the proof. ∎

6.2 Proof of Proposition 2.1

Under the independence assumption,

$$\begin{aligned} E(G_{2n}) &= \frac{1}{K(n)} \sum_{k=0}^{\infty} b_k \sum_{j=k+1}^{k+n-1} c_j \sum_{i=1-k}^{n-j} E(u_i w_i) \\ &= \frac{\omega_{uw}}{K(n)} \sum_{k=0}^{\infty} b_k \sum_{t=1}^{n-1} (n-t) c_{k+t}. \end{aligned} \tag{6.3}$$

where the second equality makes the substitution $t = j - k$. It can be verified that

$$\begin{aligned} \sum_{k=0}^{\infty} b_k \sum_{t=1}^{n-1} (n-t) c_{k+t} &= \sum_{t=1}^{n-1} \sum_{s=0}^{t-1} \Big(\sum_{k=0}^{s} b_k \Big) c_{s+1} + \sum_{t=1}^{n-1} \sum_{s=t}^{\infty} \Big(\sum_{k=s-t+1}^{s} b_k \Big) c_{s+1} \\ &= \sum_{t=1}^{n-1} \sum_{s=0}^{t-1} a_{n,t-s}(t/n, 0) c_{s+1} + \sum_{t=1}^{n-1} \sum_{s=t}^{\infty} a_{n,t-s}(t/n, 0) c_{s+1} \end{aligned} \tag{6.4}$$

where the expression

$$a_{nt}(s, s') = \sum_{j=\max\{0, [ns'] - t + 1\}}^{[ns]-t} b_j \tag{6.5}$$

is defined in DDJ, equation (3.2). According to a straightforward extension of DDJ Lemma 3.1,

$$a_{n,[ns]-[nx]}(s, 0) \sim \begin{cases} \dfrac{L_X(n)[nx]^{d_X}}{\Gamma(d_X + 1)}, & 0 \leq x \leq s \\ L_X(n) \dfrac{[nx]^{d_X} - ([nx] - [ns])^{d_X}}{\Gamma(d_X + 1)}, & x > s. \end{cases}$$

In the case $d_X + d_Y > 0$ we have, applying Assumption 2 and substituting $d_Y / \Gamma(d_Y + 1)$ for $1/\Gamma(d_Y)$,

$$\frac{1}{K(n)} \sum_{t=1}^{n-1} \sum_{s=0}^{t-1} a_{n,t-s}(t/n, 0) c_{s+1} \sim \frac{d_Y}{n^2 \Gamma(d_X + 1) \Gamma(d_Y + 1)} \sum_{t=1}^{n-1} \sum_{s=1}^{t} \Big(\frac{s}{n} \Big)^{d_X + d_Y - 1}$$

$$\to \frac{d_Y}{\Gamma(d_X+1)\Gamma(d_Y+1)} \int_0^1 \int_0^\tau \zeta^{d_X+d_Y-1} d\zeta d\tau$$

$$= \frac{d_Y}{\Gamma(d_X+1)\Gamma(d_Y+1)(d_Y+d_X)(1+d_Y+d_X)}. \qquad (6.6)$$

Similarly,

$$\frac{1}{K(n)} \sum_{t=1}^{n-1} \sum_{s=t}^{\infty} a_{n,t-s}(t/n,0) c_{s+1}$$

$$\sim \frac{d_Y}{n^2\Gamma(d_X+1)\Gamma(d_Y+1)} \sum_{t=1}^{n-1} \sum_{s=0}^{\infty} \left(\left(\frac{s+t}{n}\right)^{d_X} - \left(\frac{s}{n}\right)^{d_X} \right) \left(\frac{s+t}{n}\right)^{d_Y-1}$$

$$\to \frac{d_Y}{\Gamma(d_X+1)\Gamma(d_Y+1)} \int_0^\infty \int_0^1 ((\zeta+\tau)^{d_X} - \tau^{d_X})(\zeta+\tau)^{d_Y-1} d\zeta d\tau$$

$$= \frac{1}{\Gamma(d_X+1)\Gamma(d_Y+1)\,(d_X+d_Y)}$$

$$\times \int_0^\infty [d_Y\,(1+\tau)^{d_X+d_Y} + d_X \tau^{d_X+d_Y} - (d_X+d_Y)\,(1+\tau)^{d_Y}\,\tau^{d_X}] d\tau. \qquad (6.7)$$

Combining these two limits completes the first part of the proof for the cases with $d_Y \neq 0$. If $d_Y = 0$, Assumption 2(b) does not permit the explicit representation used in (6.6) and (6.7). However, summability of the c_s coefficients implies that

$$\sum_{t=1}^{n-1} \sum_{s=0}^{\infty} a_{n,t-s}(t/n,0) c_{s+1} = o(n^{1+d_X} L_X(n)) \qquad (6.8)$$

and $E(G_{2n})$ vanishes in the limit. These expressions are therefore formally correct in all the cases. ∎

6.3 Proof of Proposition 2.2

Let

$$\mathcal{L}(d_X, d_Y) = \int_0^\infty \left[d_Y\,(1+\tau)^{d_X+d_Y} + d_X \tau^{d_X+d_Y} - (d_X+d_Y)\,(1+\tau)^{d_Y}\,\tau^{d_X} \right] d\tau.$$

Denote the integrand by $f(\tau)$. For $0 < d_X, d_Y < 1/2$, $\lim_{\tau\to\pm\infty} f(\tau) = 0$, $\lim_{\tau\to 0} f(\tau) = 1$, and the function is integrable for both positive and negative τ. For $-1/2 < d_X, d_Y < 0$, we have $\lim_{\tau\to\pm\infty} f(\tau) = 0$, $f(\tau)$ has a singularity at $\tau = 0$ with $\lim_{\tau\to 0} f(\tau)\,\tau^{-(d_X+d_Y)} = 1$, and $f(\tau)$ is integrable for $\tau \geq 0$. It also has a singularity at $\tau = -1$ with $\lim_{\tau\to -1} f(\tau)\,(\tau+1)^{-(d_X+d_Y)} = 1$, and so is also integrable.

Consider an auxiliary integral $\mathcal{L}^*(d_X, d_Y) = \int_{-\infty}^{\infty} f(\tau)\, d\tau$.Changing the variable of integration, $\tau + 1 = -t$, we obtain:

$$\begin{aligned}
\mathcal{L}^*(d_X, d_Y) &= \int_{-\infty}^{\infty} \left[d_Y\,(1+\tau)^{d_X+d_Y} + d_X \tau^{d_X+d_Y} - (d_X+d_Y)\,(1+\tau)^{d_Y}\,\tau^{d_X} \right] d\tau \\
&= \int_{-\infty}^{\infty} \left[d_Y\,(-t)^{d_X+d_Y} + d_X\,(-t-1)^{d_X+d_Y} - (d_X+d_Y)\,(-t)^{d_Y}\,(-t-1)^{d_X} \right] dt \\
&= (-1)^{(d_X+d_Y)} \int_{-\infty}^{\infty} \left[d_Y t^{d_X+d_Y} + d_X\,(t+1)^{d_X+d_Y} - (d_X+d_Y)\,t^{d_Y}\,(t+1)^{d_X} \right] dt \\
&= (-1)^{d_X+d_Y}\, \mathcal{L}^*(d_Y, d_X).
\end{aligned}$$

Note that by interchanging d_X and d_Y we obtain

$$\begin{aligned}\mathcal{L}^*(d_X, d_Y) &= (-1)^{d_X+d_Y}\,\mathcal{L}^*(d_Y, d_X)\\ &= (-1)^{2(d_X+d_Y)}\,\mathcal{L}^*(d_X, d_Y) = 0\end{aligned}$$

and hence also $\mathcal{L}^*(d_X, d_Y) = 0$ unless $d_X + d_Y = 0, \pm 1, \pm 2, \ldots$.

Next, divide the range of integration in $\mathcal{L}^*(d_Y, d_X)$ into $(-\infty, -1)$, $(-1, 0)$, and $(0, \infty)$. For the first interval change of variables $\tau = -t - 1$ gives

$$\begin{aligned}&\int_{-\infty}^{-1}\left[d_Y t^{d_X+d_Y} + d_X\,(t+1)^{d_X+d_Y} - (d_X+d_Y)\,t^{d_Y}\,(t+1)^{d_X}\right] dt\\ &= \int_0^{\infty}\left[d_Y\,(-1-\tau)^{d_X+d_Y} + d_X\,(-\tau)^{d_X+d_Y} - (d_X+d_Y)\,(-1-\tau)^{d_Y}\,(-\tau)^{d_X}\right] d\tau\\ &= (-1)^{d_X+d_Y}\int_0^{\infty}\left[d_Y\,(1+\tau)^{d_X+d_Y} + d_X\tau^{d_X+d_Y} - (d_X+d_Y)\,(1+\tau)^{d_Y}\,\tau^{d_X}\right] d\tau\\ &= (-1)^{d_X+d_Y}\,\mathcal{L}(d_X, d_Y).\end{aligned}$$

For the second interval using $\tau = -t$ we have

$$\begin{aligned}&\int_{-1}^{0}\left[d_Y t^{d_X+d_Y} + d_X\,(t+1)^{d_X+d_Y} - (d_X+d_Y)\,t^{d_Y}\,(t+1)^{d_X}\right] dt\\ &= \int_0^{1}\left[(-1)^{d_X+d_Y}\,d_Y\tau^{d_X+d_Y} + d_X\,(1-\tau)^{d_X+d_Y} - (-1)^{d_Y}\,(d_X+d_Y)\,\tau^{d_Y}\,(1-\tau)^{d_X}\right] d\tau\\ &= \frac{(-1)^{d_X+d_Y}\,d_Y}{d_X+d_Y+1}(1-0) - \frac{d_X}{d_X+d_Y+1}\,(0-1) - (-1)^{d_Y}\,(d_X+d_Y)\,B(d_X+1, d_Y+1)\\ &= \frac{(-1)^{d_X+d_Y}\,d_Y + d_X}{d_X+d_Y+1} - (-1)^{d_Y}\,(d_X+d_Y)\,B\,(d_X+1, d_Y+1)\end{aligned}$$

The integral over the third interval is simply $\mathcal{L}(d_Y, d_X)$. Adding the integrals over these three intervals we obtain

$$\begin{aligned}\mathcal{L}^*(d_Y, d_X) &= (-1)^{d_X+d_Y}\,\mathcal{L}(d_X, d_Y) + \frac{(-1)^{d_X+d_Y}\,d_Y + d_X}{d_X+d_Y+1}\\ &\qquad - (-1)^{d_Y}\,(d_X+d_Y)\,B\,(d_X+1, d_Y+1) + \mathcal{L}(d_Y, d_X)\\ &= 0. \end{aligned} \tag{6.9}$$

By symmetry,

$$\begin{aligned}\mathcal{L}^*(d_X, d_Y) &= (-1)^{d_X+d_Y}\,\mathcal{L}(d_Y, d_X) + \frac{(-1)^{d_X+d_Y}\,d_X + d_Y}{d_X+d_Y+1}\\ &\qquad - (-1)^{d_X}\,(d_X+d_Y)\,B\,(d_X+1, d_Y+1) + \mathcal{L}(d_X, d_Y)\\ &= 0. \end{aligned} \tag{6.10}$$

where we used $B\,(x, y) = B\,(y, x)$. Now we multiply (6.9) by $(-1)^{d_X+d_Y}$ and subtract from (6.10):

$$\begin{aligned}0 \;=\; &\left[1 - (-1)^{2(d_X+d_Y)}\right]\left[\mathcal{L}(d_X, d_Y) + \frac{d_Y}{d_X+d_Y+1}\right]\\ &- (-1)^{d_X}\left[1 - (-1)^{2d_Y}\right](d_X+d_Y)\,B\,(d_X+1, d_Y+1)\,.\end{aligned}$$

Therefore,

$$\mathcal{L}(d_X, d_Y) = -\frac{d_Y}{d_X + d_Y + 1} + (-1)^{d_X} \frac{1 - (-1)^{2d_Y}}{1 - (-1)^{2(d_X + d_Y)}} (d_X + d_Y) B (d_X + 1, d_Y + 1) .$$

Finally, using $(-1)^x = e^{i\pi x}$ we rewrite in the second term

$$\begin{aligned}
(-1)^{d_X} \frac{1 - (-1)^{2d_Y}}{1 - (-1)^{2(d_X + d_Y)}} &= e^{i\pi d_X} \frac{1 - e^{i\pi 2d_Y}}{1 - e^{i\pi 2(d_X + d_Y)}} \\
&= e^{i\pi d_X} \frac{e^{i\pi d_Y} \left(e^{-i\pi d_Y} - e^{i\pi d_Y}\right)}{e^{i\pi (d_X + d_Y)} \left(e^{-i\pi (d_X + d_Y)} - e^{i\pi (d_X + d_Y)}\right)} \\
&= \frac{\sin \pi d_Y}{\sin \pi (d_X + d_Y)}
\end{aligned}$$

and therefore

$$\mathcal{L}(d_X, d_Y) = -\frac{d_Y}{d_X + d_Y + 1} + (d_X + d_Y) B (d_X + 1, d_Y + 1) \frac{\sin \pi d_Y}{\sin \pi (d_X + d_Y)}.$$

To complete the proof, substitute this expression into (2.4) and rearrange using the identities $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$, $\Gamma(1 - x)\Gamma(x) = \pi/\sin \pi x$ and $\Gamma(x + 1) = x\Gamma(x)$:

$$\begin{aligned}
\lambda_{XY} = &\frac{\omega_{uw}}{\Gamma (d_X + 1) \Gamma (d_Y + 1) (d_X + d_Y)} \times \\
&\left(\frac{d_Y}{1 + d_X + d_Y} - \frac{d_Y}{d_X + d_Y + 1} + (d_X + d_Y) B (d_X + 1, d_Y + 1) \frac{\sin \pi d_Y}{\sin \pi (d_X + d_Y)}\right) \\
= &\frac{\omega_{uw}}{\Gamma (d_X + 1) \Gamma (d_Y + 1)} B (d_X + 1, d_Y + 1) \frac{\sin \pi d_Y}{\sin \pi (d_X + d_Y)} \\
= &\frac{\omega_{uw}}{\Gamma (2 + d_X + d_Y)} \frac{\sin \pi d_Y}{\sin \pi (d_X + d_Y)} \\
= &\frac{\omega_{uw} \Gamma(1 - d_X - d_Y)}{\pi (1 + d_X + d_Y) (d_X + d_Y)} \sin \pi d_Y. \qquad \blacksquare
\end{aligned}$$

6.4 Proof of Proposition 2.3

In this case, note that if a_{nt} is defined by (6.5) then

$$a_{n,t-s}(t/n, 0) c_{s+1} = O(s^{d_X + d_Y - 1} L_X(s) L_Y(s))$$

so that these terms are summable by assumption. Considering expression (6.4), the lemma follows since

$$\sum_{t=1}^{n-1} \sum_{s=0}^{t-1} a_{n,t-s}(t/n, 0) c_{s+1} = O(n)$$

and

$$\sum_{t=1}^{n-1} \sum_{s=t}^{\infty} a_{n,t-s}(t/n, 0) c_{s+1} = o(n). \qquad \blacksquare$$

6.5 Proof of Theorem 2.1

Setting $i = s - k$, rewrite (2.2) as

$$\begin{aligned} G_{2n} - E(G_{2n}) &= \frac{1}{K(n)} \sum_{t=1}^{n-1} \sum_{s=1}^{t} \sum_{k=0}^{\infty} b_k c_{k+t-s+1}(u_{s-k}w_{s-k} - \omega_{uw}) \\ &= \frac{1}{K(n)} \sum_{t=1}^{n-1} \sum_{s=1}^{t} P_{ts} \end{aligned}$$

(say) where

$$P_{ts} = \sum_{i=-\infty}^{s} b_{s-i}c_{t+1-i}(u_i w_i - \omega_{uw}).$$

Hence note that

$$E(G_{2n} - E(G_{2n}))^2 \le \frac{2}{K(n)^2} \sum_{t=1}^{n-1} \sum_{s=1}^{t} \sum_{m=0}^{t-s} \sum_{k=0}^{s-1} E(P_{ts}P_{t-m,s-k}).$$

where, setting $j = s - i$ in the third member and letting C denote a generic finite constant,

$$\begin{aligned} E(P_{ts}P_{t-m,s-k}) &= \frac{\mu_{uw}^4 - \omega_{uw}^2}{K(n)^2} \sum_{i=-\infty}^{s-k} b_{s-i}b_{s-k-i}c_{t+1-i}c_{t-m+1-i} \\ &= \frac{\mu_{uw}^4 - \omega_{uw}^2}{K(n)^2} \sum_{j=k}^{\infty} b_j b_{j-k} c_{t+1-s+j} c_{t-m+1-s+j} \\ &\le \frac{C}{n^{2(1+d_X+d_Y)}} \sum_{j=k}^{\infty} j^{d_X-1}(j-k)^{d_X-1}(j+t+1-s)^{d_Y-1}(j+t-m+1-s)^{d_Y-1} \\ &\le \frac{C}{n^{2(1+d_X+d_Y)}} k^{2d_X-1}(k+t+1-s)^{d_Y-1}(k+t-m+1-s)^{d_Y-1}. \end{aligned}$$

Hence,

$$E(G_{2n} - E(G_{2n}))^2 \le \frac{C}{n^{2(1+d_X+d_Y)}} \sum_{t=1}^{n-1} \sum_{s=1}^{t} (s-1)^{2d_X}(t+1-s)^{d_Y-1} \sum_{m=0}^{t-s} (t-m+1-s)^{d_Y-1}.$$

These sums can be bounded by conventional summation arguments (Davidson 1994, Thm 2.27) as follows, also applying Lemma A.1 of DDJ in the case $d_X < 0$.
Case $d_Y > 0$:

$$\begin{aligned} E(G_{2n} - E(G_{2n}))^2 &\le \frac{C}{n^{2(1+d_X+d_Y)}} \sum_{t=1}^{n-1} \sum_{s=1}^{t} (t+1-s)^{2d_Y-1}(s-1)^{2d_X} \\ &= O(n^{-1}). \end{aligned}$$

Case $d_Y \le 0$:

$$\begin{aligned} E(G_{2n} - E(G_{2n}))^2 &\le \frac{C}{n^{2(1+d_X+d_Y)}} \sum_{t=1}^{n-1} \sum_{s=1}^{t} (t+1-s)^{d_Y-1}(s-1)^{2d_X} \\ &= \begin{cases} O(n^{-1}\log n), & d_Y = 0 \\ O(n^{-1-2d_Y}), & d_Y < 0 \end{cases}. \end{aligned}$$

∎

6.6 Proof of Theorem 2.2

First note that

$$E\left(\frac{1}{n}\sum_{t=1}^{n}x_ty_t-\sigma_{XY}\right)^2=\frac{1}{n^2}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}b_kb_lc_jc_i$$
$$\times\sum_{t=1}^{n-1}\sum_{s=1}^{n-1}[E(u_{t-k}w_{t-j}u_{s-l}w_{s-i})-E(u_{t-k}w_{t-j})E(u_{s-l}w_{s-i})]. \quad (6.11)$$

Under the assumptions,

$$E(u_{t-k}w_{t-j}u_{s-l}w_{s-i})-E(u_{t-k}w_{t-j})E(u_{s-l}w_{s-i})$$
$$=\begin{cases}\mu_{uw}^4-\omega_{uw}^2 & t-k=s-l=t-j=s-i\\ \omega_{uu}\omega_{ww}-\omega_{uw}^2 & t-k=s-l \text{ and } t-j=s-i\\ 0 & \text{otherwise.}\end{cases}$$

Collecting these terms, letting sums over an empty index set equal zero, yields

$$\begin{aligned}E\left(\frac{1}{n}\sum_{t=1}^{n}x_ty_t-\sigma_{XY}\right)^2 &= \frac{\mu_{uw}^4-\omega_{uw}^2}{n}\sum_{k=0}^{\infty}b_k^2c_k^2+\frac{\omega_{uu}\omega_{ww}-\omega_{uw}^2}{n}\sum_{k=0}^{\infty}b_k^2\sum_{j=0}^{\infty}c_j^2\\ &= O(n^{-1}).\end{aligned}$$

The conditions of part (ii) overlap with those of part (i), but ensure that the sequences $\{b_k\}$ and $\{c_j\}$ are absolutely summable. In this case,

$$n^{-1/2}\sum_{t=1}^{n}x_ty_t=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}b_kc_j\left(n^{-1/2}\sum_{t=1}^{n}u_{t-k}w_{t-j}\right).$$
$$\xrightarrow{d}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}b_kc_jZ(j,k)=\zeta \quad (6.12)$$

(say) where the $Z(j,k)$ are $N(0,\omega_{uu}\omega_{ww})$ random variables by the application of the standard CLT. Note that $Z(j,k)=Z(j',k')$ if $j-k=j'-k'$ and $E(Z(j,k)Z(j',k'))=0$ if $j-k\neq j'-k'$. Hence ζ is normally distributed with zero mean and finite variance. ∎

6.7 Proof of Lemma 3.1

A preliminary lemma is needed as follows.

Lemma 6.1 *For $0<\alpha<\frac{1}{2}$ and $a>b\geq 0$, and $L_X(n)$ slowly varying,*

$$\left|\frac{1}{n}\sum_{s=1+[nb]}^{[na]}\left(\frac{s}{n}\right)^{\alpha-1}\frac{L_X(s)}{L_X(n)}-\int_b^a u^{\alpha-1}du\right|=o(1).$$

Proof First, note that

$$\frac{1}{n}\sum_{s=1+[nb]}^{[na]}\left(\frac{s}{n}\right)^{\alpha-1}\frac{L_X(s)}{L_X(n)}=\frac{1}{n}\sum_{s=1+[nb]}^{[na]}\left(\frac{s}{n}\right)^{\alpha-1}+o(1).$$

Next,

$$\frac{1}{n}\left|\sum_{s=1+[nb]}^{[na]}\left[\left(\frac{s}{n}\right)^{\alpha-1}-n\int_{s/n}^{(s+1)/n}u^{\alpha-1}du\right]\right| = \frac{1}{n}\left|\sum_{s=1+[nb]}^{[na]}\left[\left(\frac{s}{n}\right)^{\alpha-1}-\frac{n}{\alpha}\left(\left(\frac{s+1}{n}\right)^{\alpha}-\left(\frac{s}{n}\right)^{\alpha}\right)\right]\right|$$

$$\leq \frac{1}{n}\left|\sum_{s=1+[nb]}^{[na]}\left[\left(\frac{s+1}{n}\right)^{\alpha-1}-\left(\frac{s}{n}\right)^{\alpha-1}\right]\right|$$

$$= n^{-\alpha}\left|\sum_{s=1+[nb]}^{[na]}\left[(s+1)^{\alpha-1}-s^{\alpha-1}\right]\right|$$

$$= \frac{n^{-\alpha}}{\alpha-1}\left|\sum_{s=1+[nb]}^{[na]}(s^*)^{\alpha-2}\right|, \quad s\leq s^*\leq s+1,$$

$$= \begin{cases} O(n^{-\alpha}), & b=0, \\ O(n^{-1}), & b>0. \end{cases}$$

where the result for $b=0$ follows directly since the sum converges, and if $b>0$ the tail sum is of order $n^{\alpha-1}$. ∎

Considering the components of equation (3.2), define q and u by $t-m=[nq]$ and $s-i=[nu]$. If $d_X>0$, write

$$\frac{b_{[nu]}}{n^{d_X}L_X(n)} = \frac{(([nu]+1)/n)^{d_X-1}}{n\Gamma\left(d_X\right)}\frac{L_X(nu)}{L_X(n)}$$

Note by Lemma 6.1 that in this case,

$$\frac{1}{n^{d_X}L_X(n)}\sum_{s=\max\{0,1-[np]\}}^{[nq]+[nr]-[np]}b_s = \frac{1}{\Gamma(d_X)}\int_{\max\{0,-p\}}^{q+r-p}u^{d_X-1}du+o(1)$$

$$= \frac{(q+r-p)^{d_X}-1_{\{p<0\}}(-p)^{d_X}}{\Gamma(1+d_X)}+o(1).$$

The actual order of magnitude depends on the average rate of $L_X(nu)/L(n)-1$, but the lemma shows it is bounded by $O(n^{-d_X})$ in the case $p\geq 0$.

If $d_X<0$, use part (c) of Assumption 2 to write

$$\frac{b_{[nu]}}{n^{d_X}L_X(n)} = \frac{(([nu]+1)/n)^{d_X}-([nu]/n)^{d_X}}{\Gamma\left(1+d_X\right)}+o(1)$$

for $[nu]>0$ where $b_0=L_X(0)$, and hence, as before,

$$\frac{1}{n^{d_X}L_X(n)}\sum_{s=\max\{1,[np]\}}^{[nq]}b_{s-[np]} = \frac{(q+r-p)^{d_X}-1_{\{p<0\}}(-p)^{d_X}}{\Gamma\left(1+d_X\right)}+o(1). \tag{6.13}$$

In the case $d_X=0$, under part (b) of Assumption 2, note that

$$\sum_{s=\max\{1,[np]\}}^{[nq]}b_{s-[np]} = \begin{cases} O(1) & p\geq 0 \\ o(1) & p<0 \end{cases}$$

which is formally equivalent to (6.13) when $L_X(n)$ is defined as a constant not depending on n. Moreover, the sum can be assigned the limiting value $1 - 1_{\{p<0\}}$ by choice of normalization, without loss of generality.

Proceeding similarly, for the case $d_Y > 0$ we may now write

$$\frac{c_{[nq]}}{n^{d_Y} L_X(n)} = \frac{(([nq]+1)/n)^{d_Y-1}}{n\Gamma(d_Y)} \frac{L_X(nq)}{L_X(n)}$$

and hence, from (3.2), $a_{n[np][nr]} = A_{XY}(r,p) + o(1)$ where

$$\begin{aligned}
A_{XY}(r,p) &= \frac{1}{\Gamma(1+d_X)\Gamma(d_Y)} \int_{\max\{0,-r\}}^{1-r} q^{d_Y-1} \left[(q+r-p)^{d_X} - 1_{\{p<0\}}(-p)^{d_X} \right] dr \\
&= \frac{1}{\Gamma(1+d_X)\Gamma(d_Y)} \int_{\max\{0,-r\}}^{1-r} q^{d_Y-1} (q+r-p)^{d_X} dr \\
&\qquad - \frac{1_{\{p<0\}}(-p)^{d_X} \left[(1-r)^{d_Y} - 1_{\{r<0\}}(-r)^{d_Y} \right]}{\Gamma(1+d_X)\Gamma(1+d_Y)} \qquad (6.14)
\end{aligned}$$

To verify that this formula matches ((3.3) see Abramovitz and Stegun (1972), 15.3.1.

In the case $d_Y < 0$, on the other hand,

$$\begin{aligned}
\frac{c_{[nq]}}{n^{d_Y} L_X(n)} &= \frac{(([nq]+1)/n)^{d_Y} - ([nq]+1/n)^{d_Y}}{\Gamma(d_Y)} + o(1) \qquad (6.15) \\
&= \frac{(([nq]+\lambda)/n)^{d_Y-1}}{\Gamma(1+d_Y)} + o(1), \quad 0 \le \lambda \le 1.
\end{aligned}$$

The approximation in (3.3) may be applied as before in respect of the first and second terms. However, since the integral of the increments in (6.15) diverges at 0, the terms with factor $1_{\{p<0\}}$ in (3.3) have to be constructed as the limiting case of

$$\frac{1}{n^{d_Y} L_X(n)} \sum_{t=\max\{1-[nr],0\}}^{n-1-[nr]} c_t = \frac{((n-1-[nr])/n)^{d_X} - 1_{\{r<0\}}(-[nr]/n)^{d_Y}}{\Gamma(1+d_Y)} + o(1).$$

The expression in (6.14) nonetheless continues to apply. ∎

6.8 Proof of Lemma 3.2

By arguments closely paralleling those of Lemma 3.1 applied to the formula in (3.4), we arrive at

$$\begin{aligned}
E_{XY}(p,r) &= \frac{1}{\Gamma(d_Y+1)\Gamma(d_X)} \int_{\max\{0,-p\}}^{1-p} u^{d_X-1} \left[(1-r)^{d_Y} - (u+p-r)^{d_Y} \right] du \\
&= \frac{1}{\Gamma(d_Y+1)\Gamma(d_X+1)} \left[(1-r)^{d_Y}(1-p)^{d_X} - d_X \int_0^{1-p} u^{d_X-1}(u+p-r)^{d_Y} du \right. \\
&\qquad \left. -1_{\{p<0\}} \left((1-r)^{d_Y}(-p)^{d_X} - d_X \int_0^{-p} u^{d_X-1}(u+p-r)^{d_Y} du \right) \right]
\end{aligned}$$

This yields the stated result by routine manipulation and application of Abramowitz and Stegun (1972), 15.3.1. ∎

6.9 Proof of Proposition 3.1

For a function F and fractional Brownian motion X as in (1.2), let the notation $\int F\delta X$ be defined by

$$\int F\delta X = \frac{1}{\Gamma(d_X+1)}\int_{-\infty}^{1} F(\tau)\left[(1-\tau)^{d_X} - 1_{\{\tau<0\}}(-\tau)^{d_X}\right] dU(\tau) \tag{6.16}$$

so that in particular, $\int \delta X = X(1)$. Define $\int F\delta Y$ similarly. Observe that

$$A_{XY}(r,p) + E_{YX}(r,p) = \frac{\left[(1-r)^{d_Y} - 1_{\{r<0\}}(-r)^{d_Y}\right]\left[(1-p)^{d_X} - 1_{\{p<0\}}(-p)^{d_X}\right]}{\Gamma(d_Y+1)\Gamma(d_X+1)}$$

with the corresponding identity for $A_{YX}(p,r) + E_{XY}(p,r)$. Therefore, defining processes

$$\begin{aligned}
\tilde{X}(t) &= \frac{1}{\Gamma(d_X+1)}\int_{-\infty}^{t}\left[(1-\tau)^{d_X} - 1_{\{\tau<0\}}(-\tau)^{d_X}\right] dU(\tau) \\
\tilde{Y}(\tau) &= \frac{1}{\Gamma(d_Y+1)}\int_{-\infty}^{\tau}\left[(1-t)^{d_X} - 1_{\{t<0\}}(-t)^{d_X}\right] dW(t).
\end{aligned}$$

note that

$$\Xi_{XY} + \Xi_{YX} = \int \tilde{X}\delta Y + \int \tilde{Y}\delta X. \tag{6.17}$$

Also note that $X(1) = \tilde{X}(t) + \breve{X}(t)$ for any $t<1$ and $Y(1) = \tilde{Y}(\tau) + \breve{Y}(\tau)$ for any $\tau < 1$ where, for example,

$$\breve{X}(t) = \begin{cases} \dfrac{1}{\Gamma(d_X+1)} \int_t^1 (1-\tau)^{d_X} dU(\tau) & t \geq 0 \\[2ex] \dfrac{1}{\Gamma(d_X+1)} \left(\begin{array}{l} \int_0^1 (1-\tau)^{d_X} dU(\tau) \\ \qquad + \int_t^0 \left[(1-\tau)^{d_X} - (-\tau)^{d_X}\right] dU(\tau) \end{array} \right) & t < 0. \end{cases} \tag{6.18}$$

with a complementary expression for $\breve{Y}(\tau)$. Therefore,

$$2X(1)Y(1) = \Xi_{XY} + \Xi_{YX} + \int \breve{X}\delta Y + \int \breve{Y}\delta X. \tag{6.19}$$

Next observe that, since $E(X(1)Y(1)) = \psi_{XY}$ while $E(\Xi_{XY}) = E(\Xi_{YX}) = 0$, (6.19) implies

$$E(\int \breve{X}\delta Y) + E(\int \breve{Y}\delta X) = 2\psi_{XY}. \tag{6.20}$$

Next, using (6.18) and (6.16) write

$$\begin{aligned}
\int \breve{X}\delta Y &= \frac{1}{\Gamma(d_Y+1)}\left(\int_0^1 (1-t)^{d_Y}\breve{X}(t)dW(t) + \int_{-\infty}^{0}\left[(1-t)^{d_Y} - (-t)^{d_Y}\right]\breve{X}(t)dW(t)\right) \\
&= \frac{1}{\Gamma(d_X+1)\Gamma(d_Y+1)}\int_0^1\left(\int_t^1 (1-\tau)^{d_X} dU(\tau)\right)(1-t)^{d_Y} dW(t) \\
&\quad + \frac{1}{\Gamma(d_X+1)\Gamma(d_Y+1)}\int_{-\infty}^{0}\left(\int_0^1 (1-\tau)^{d_X} dU(\tau)\right)\left[(1-t)^{d_Y} - (-t)^{d_Y}\right] dW(t) \\
&\quad + \frac{1}{\Gamma(d_X+1)\Gamma(d_Y+1)}\int_{-\infty}^{0}\left(\int_t^0 \left[(1-\tau)^{d_X} - (-\tau)^{d_X}\right] dU(\tau)\right) \\
&\qquad \times \left[(1-t)^{d_Y} - (-t)^{d_Y}\right] dW(t)
\end{aligned} \tag{6.21}$$

$$
\begin{aligned}
&= E(\textstyle\int \breve{X}\delta Y) + \frac{1}{\Gamma(d_X+1)\Gamma(d_Y+1)} \int_0^1 \left(\int_0^\tau (1-t)^{d_Y} dW(t) \right) (1-\tau)^{d_X} dU(\tau) \\
&\quad + \frac{1}{\Gamma(d_X+1)\Gamma(d_Y+1)} \left(\int_{-\infty}^0 \left[(1-t)^{d_Y} - (-t)^{d_Y} \right] dW(t) \right) \left(\int_0^1 (1-\tau)^{d_X} dU(\tau) \right) \\
&\quad + \frac{1}{\Gamma(d_X+1)\Gamma(d_Y+1)} \int_{-\infty}^0 \left(\int_{-\infty}^\tau \left[(1-t)^{d_Y} - (-t)^{d_Y} \right] dW(t) \right) \\
&\qquad\qquad \times \left[(1-\tau)^{d_X} - (-\tau)^{d_X} \right] dU(\tau). \qquad (6.22)
\end{aligned}
$$

Note how the last equality re-writes the integral in the form that separates the non-stochastic and (zero mean) stochastic components. The second term and fourth term of (6.22) are Itô integrals with respect to $dU(\tau)$ of $\mathcal{F}(\tau)$-measurable processes, while the third term is the product of terms defined on $(-\infty, 0]$ and $(0, \xi]$ respectively. Since each of these terms have mean 0, $E(\int \breve{X}\delta Y)$ must appear explicitly in (6.22).

Finally, consider the expression obtained by taking (6.21) and interchanging the pairs (X, Y), (U, W), and (t, τ). Note that the sum of this expression and (6.22) is $X(1)Y(1) + E(\int \breve{X}\delta Y)$. Clearly, the same equality holds if all the arguments are interchanged. This implies that the two means in (6.20) are equal to each other, and hence to ψ_{XY}, so

$$
\textstyle\int \breve{X}\delta Y + \int \breve{Y}\delta X = X(1)Y(1) + \psi_{XY}. \qquad (6.23)
$$

Equation (6.23) in combination with (6.19) yields the required formula. ∎

6.10 Proof of Lemmas 4.1 and 4.4

The following preliminary lemmas are needed.

Lemma 6.2 *$|A_{XY}(r,p)| \le \bar{A}_{XY}(r,p)$ where $A_{XY}(r,p)$ is defined in (3.3) and*

$$
\bar{A}_{XY}(r,p) = \frac{\left|(1-r)^{d_Y} - 1_{\{r<0\}}(-r)^{d_Y}\right| \left|(g-p)^{d_X} - 1_{\{p<0\}}(-p)^{d_X}\right|}{\Gamma(d_X+1)\Gamma(d_Y+1)}
$$

where

$$
g = \begin{cases} 1, & d_X \ge 0 \text{ or } d_X < 0, p < 0 \\ r, & d_X < 0, \ p \ge 0. \end{cases}
$$

Proof From (3.2),

$$
\begin{aligned}
\left|a_{n[np][nr]}\right| &= \frac{n}{K(n)} \left| \sum_{t=\max\{1-[nr],0\}}^{n-1-[nr]} c_t \left(\sum_{s=\max\{1-[np],0\}}^{t+[nr]-[np]} b_s \right) \right| \\
&\le \frac{n}{K(n)} \left| \sum_{t=\max\{1-[nr],0\}}^{n-1-[nr]} c_t \right| \max_{\max\{1-[nr],0\} \le \tau \le n-1} \left| \sum_{s=\max\{1-[np],0\}}^{\tau+[nr]-[np]} b_s \right| \\
&= \frac{\left|(1-r)^{d_Y} - 1_{\{r<0\}}(-r)^{d_Y}\right| \max_{\max\{0,r\} \le g \le 1} \left|(g-p)^{d_X} - 1_{\{p<0\}}(-p)^{d_X}\right|}{\Gamma(d_X+1)\Gamma(d_Y+1)} + o(1) \qquad (6.24)
\end{aligned}
$$

as $n \to \infty$, using the same arguments as in the proof of Lemma 3.1. When $d_X \ge 0$, $(g-p)^{d_X}$ is monotone nondecreasing in g, and is maximized over $[0,1]$ at $g = 1$. When $d_X < 0$, $(g-p)^{d_X}$ is monotone decreasing. If $p \ge 0$, so that $r \ge 0$, the maximum in (6.24) is achieved at $g = r$. On the other hand, if $p < 0$ then $\left|(g-p)^{d_X} - 1_{\{p<0\}}(-p)^{d_X}\right|$ is maximized at $g = 1$, as indicated. ∎

Lemma 6.3 $|E_{XY}(p,r)| \leq \bar{E}_{XY}(p,r)$ *where* $E_{XY}(p,r)$ *is defined in (3.5) and*

$$\bar{E}_{XY}(p,r) = \frac{\left|(1-p)^{d_X} - 1_{\{p<0\}}(-p)^{d_X}\right| \left|(1-r)^{d_Y} - 1_{\{r<0\}}(-r)^{d_Y}\right|}{\Gamma(d_X+1)\Gamma(d_Y+1)}$$

Proof From (3.4)

$$\begin{aligned}
\left|e_{n[nr][np]}\right| &= \frac{n}{K(n)}\left|\sum_{s=\max\{0,1-[np]\}}^{n-1-[np]} b_s \left(\sum_{t=s+[np]+1-[nr]}^{n-[nr]} c_t\right)\right| \\
&\leq \frac{n}{K(n)}\left|\sum_{s=\max\{0,1-[np]\}}^{n-1-[np]} b_s\right| \max_{\max\{0,1-[np]\}\leq z\leq n-1}\left|\sum_{t=z+[np]+1-[nr]}^{n-[nr]} c_t\right| \\
&= \frac{\left|(1-p)^{d_X} - 1_{\{p<0\}}(-p)^{d_X}\right| \max_{\max\{0,p\}\leq g\leq 1}\left|(1-r)^{d_Y} - (g-r)^{d_Y}\right|}{\Gamma(d_X+1)\Gamma(d_Y+1)} + o(1) \qquad (6.25)
\end{aligned}$$

as $n \to \infty$. First, suppose $p \geq 0$. When $d_Y \geq 0$ then $(1-r)^{d_Y} - (g-r)^{d_Y} > 0$ and is maximized over $[p,1]$ at $g = p$, noting that $r \leq p$ in this case. When $d_Y < 0$, $(1-r)^{d_Y} - (g-r)^{d_Y} < 0$ and is minimized at $g = p$. In the case $p < 0$ the same considerations apply, but the extremum over $[0,1]$ is at $g = 0$ in each case. The proof is completed by noting that for any $p \in [\ r, 1]$, and d_Y of either sign,

$$\left|(1-r)^{d_Y} - (\max\{p,0\}-r)^{d_Y}\right| \leq \left|(1-r)^{d_Y} - 1_{\{r<0\}}(-r)^{d_Y}\right|.$$

∎

To prove Lemma 4.1, first suppose $m \geq 0$. Break the sum nv^a_{nm} into components $\sum_{i=0}^{m-1} a^2_{nim}$ and $\sum_{i=-\infty}^{-1} a^2_{nim}$ where the first term is 0 if $m = 0$. Note that if $m = [nr]$ for $0 \leq r \leq 1$, then since $i = [np] \geq 0$, applying Lemma 6.2,

$$\begin{aligned}
\limsup_n \frac{1}{n}\sum_{i=0}^{m-1} a^2_{nim} &\leq \int_0^r \bar{A}_{XY}(r,p)^2 dp \\
&= \begin{cases} \dfrac{(1-r)^{2d_Y+2d_X+1}}{(2d_X+1)\Gamma(d_X+1)^2\Gamma(d_Y+1)^2}, & d_X \geq 0 \\[2ex] \dfrac{(1-r)^{2d_Y}\, r^{2d_X+1}}{(2d_X+1)\Gamma(d_X+1)^2\Gamma(d_Y+1)^2}, & d_X < 0 \end{cases} \\
&< \infty
\end{aligned}$$

whereas

$$\begin{aligned}
\frac{1}{n}\sum_{i=-\infty}^{0} a^2_{nim} &\leq \int_{-\infty}^0 \bar{A}_{XY}(r,p)^2 dp \\
&= \frac{(1-r)^{2d_Y}}{\Gamma(1+d_X)^2\Gamma(1+d_Y)^2}\int_0^\infty \left[(1+p)^{d_X} - p^{d_X}\right]^2 dp \\
&< \infty
\end{aligned}$$

noting that $\int_0^\infty \left[(1+p)^d - p^d\right]^2 dp < \infty$ for $|d| < \frac{1}{2}$. (See for example Davidson and Hashimzade (2008), Lemma 5.1.) Finally, if $m < 0$ and hence $r < 0$, by Lemma 6.2 there exists n large enough that

$$\frac{1}{n}\sum_{i=-\infty}^{m} a^2_{nim} \leq \int_{-\infty}^r \bar{A}_{XY}(r,p)^2 dp$$

$$
\begin{aligned}
&\leq \frac{\left((1-r)^{d_Y} - (-r)^{d_Y}\right)^2}{\Gamma(1+d_X)^2\Gamma(1+d_Y 1)^2} \int_{-r}^{\infty} \left[(1+p)^{d_X} - p^{d_X}\right]^2 dp \\
&= O((-r)^{2d_Y+2d_X-3}).
\end{aligned}
$$

The argument for Lemma 4.4 is very similar. Letting $i = [np]$, applying Lemma 6.3 leads to

$$
\begin{aligned}
\limsup_n \frac{1}{n} \sum_{m=0}^{i-1} e_{nmi}^2 &\leq \int_0^p \bar{E}_{XY}(p,r)^2 dr \\
&= \frac{\left((1-p)^{d_X} - 1_{\{p<0\}}(-p)^{d_X}\right)^2 (1-p)^{2d_Y+1}}{(2d_Y+1)\Gamma(d_X+1)^2\Gamma(d_Y+1)^2} \\
&< \infty
\end{aligned}
$$

whereas

$$
\frac{1}{n} \sum_{m=-\infty}^{0} e_{nmi}^2 \leq \int_{-\infty}^{0} \bar{E}_{XY}(p,r)^2 dr < \infty
$$

and for $i < 0$,

$$
\frac{1}{n} \sum_{m=-\infty}^{i} e_{nmi}^2 \leq \int_{-\infty}^{p} \bar{E}_{XY}(p,r)^2 dr = O((-p)^{2d_Y+2d_X-3})
$$

follow exactly as for the proof of Lemma 6.2. ∎

6.11 Proof of Lemmas 4.2 and 4.5

In principle there are three cases to consider, depending on the respective signs of r and $r+\delta$. However, if $r < 0$ and $r+\delta > 0$ then the interval may be split into subintervals of widths $-r < \delta$ and $\delta + r$ respectively, and treated separately. Showing the cases $r \geq 0$ and $r < r+\delta \leq 0$ is therefore sufficient. The bounding arguments here are essentially the same as those in the proofs of Lemmas 4.1 and 4.4.

First, for Lemma 4.2,

$$
\begin{aligned}
&\limsup_n \frac{1}{n} \sum_{i=[nr]+1}^{[n(r+\delta)]} a_{ni[n(r+\delta)]}^2 \\
&\leq \int_r^{r+\delta} \bar{A}_{XY}(r+\delta,p)^2 dp \\
&= \frac{\left[(1-r-\delta)^{d_Y} - 1_{\{r+\delta<0\}}(-r-\delta)^{d_Y}\right]^2}{\Gamma(1+d_X)^2\Gamma(1+d_Y)^2} \int_r^{r+\delta} \left[(g-p)^{d_X} - 1_{\{p<0\}}(-p)^{d_X}\right]^2 dp
\end{aligned}
$$

where $\bar{A}_{XY}(r,p)$, and g, are defined in Lemma 6.2.
Case: $r \geq 0$. If $d_X > 0$, then $g = 1$ and

$$
\int_r^{r+\delta} (1-p)^{2d_X} dp = \frac{(1-r)^{2d_X+1} - (1-r-\delta)^{2d_X+1}}{2d_X+1} = O(\delta)
$$

otherwise $g = r+\delta$ and

$$
\int_r^{r+\delta} (r+\delta-p)^{2d_X} dp = \frac{\delta^{2d_X+1}}{2d_X+1}.
$$

Case: $r < r+\delta \le 0$. In this case $g = 1$ and

$$\int_r^{r+\delta} [(1-p)^{d_X} - (-p)^{d_X}]^2 dp \le \int_r^{r+\delta} (-p)^{2d_X} dp + \int_r^{r+\delta} (1-p)^{2d_X} dp$$
$$= O(\delta)$$

since $d_X \ge -\frac{1}{2}$. These bounds are independent of r, and hold uniformly with respect to $r \in (-\infty, 1]$.

The proof of Lemma 4.5 is very similar, noting that

$$\begin{aligned}\limsup_n \frac{1}{n} \sum_{m=[np]+1}^{[n(p+\delta)]} e^2_{nm[n(p+\delta)]} &\le \int_p^{p+\delta} \bar{E}_{XY}(p+\delta, r)^2 dr \\ &= \frac{\left[(1-p-\delta)^{d_X} - 1_{\{p+\delta>0\}}(-p-\delta)^{d_X}\right]^2}{\Gamma(d_X+1)^2\Gamma(d_Y+1)^2} \\ &\qquad \int_p^{p+\delta} \left[(1-r)^{d_Y} - 1_{\{r<0\}}(-r)^{d_Y}\right]^2 dr \\ &= O(\delta). \qquad \blacksquare\end{aligned}$$

6.12 Proof of Lemmas 4.3 and 4.6

First, for case Lemma 4.3, note that

$$\begin{aligned}&a_{n[np][n(r+\delta)]} - a_{n[np][nr]} \\ &\quad= \frac{n}{K(n)} \sum_{t=\max\{1-[n(r+\delta)],0\}}^{n-1-[n(r+\delta)]} c_t \left(\sum_{s=\max\{1-[np],0\}}^{t+[n(r+\delta)]-[np]} b_s\right) \\ &\qquad\qquad - \frac{n}{K(n)} \sum_{t=\max\{1-[nr],0\}}^{n-1-[nr]} c_t \left(\sum_{s=\max\{-[np],0\}}^{t+[nr]-[np]} b_s\right) \\ &\quad= -\frac{n}{K(n)} \sum_{t=n-[n(r+\delta)]}^{n-1-[nr]} c_t \left(\sum_{s=\max\{-[np],0\}}^{t+[nr]-[np]} b_s\right) \\ &\qquad\qquad + \frac{n}{K(n)} \sum_{t=\max\{1-[nr],0\}}^{n-1-[n(r+\delta)]} c_t \left(\sum_{s=t+[nr]-[np]}^{t+[n(r+\delta)]-[np]} b_s\right) \\ &\qquad\qquad\qquad + \frac{n}{K(n)} \sum_{t=\max\{1-[n(r+\delta),0]}^{\max\{1-[nr],0\}-1} c_t \left(\sum_{s=\max\{-[np],0\}}^{t+[n(r+\delta)]-[np]} b_s\right) \\ &\quad= D^a_{1n}(r,\delta,p) + D^a_{2n}(r,\delta,p) + D^a_{3n}(r,\delta,p)\end{aligned}$$

(defining D^a_{1n}, D^a_{2n}, D^a_{3n}) where an empty sum takes the value 0 by convention, so that $D^a_{3n} = 0$ for $r \ge 0$. Now define

$$\bar{D}^a_1(r,\delta,p) = \frac{\left|(1-r)^{d_Y} - (1-r-\delta)^{d_Y}\right| \left|(1-p)^{d_X} - 1_{\{p<0\}}(-p)^{d_X}\right|}{\Gamma(1+d_X)\Gamma(1+d_Y)}$$
$$\bar{D}^a_2(r,\delta,p) = \frac{\left|(1-r-\delta)^{d_Y} - 1_{\{r<0\}}(-r)^{d_Y}\right| \left|(r+\delta-p)^{d_X} - (r-p)^{d_X}\right|}{\Gamma(1+d_X)\Gamma(1+d_Y)}$$

$$\bar{D}_3^a(r,\delta,p) = 1_{\{r<0\}} \frac{\left|(-r)^{d_Y} - (-r-\delta)^{d_Y}\right| \left|(\delta-p)^{d_X} - (-p)^{d_X}\right|}{\Gamma(1+d_X)\Gamma(1+d_Y)}.$$

If $d_X > 0$, $d_Y > 0$ then, using Assumption 2 and Lemma 6.1,

$$\begin{aligned}
&|D_{1n}^a(r,\delta,p)| \\
&= \frac{1}{n^2\Gamma(d_X)\Gamma(d_Y)} \left| \sum_{t=n-[n(r+\delta)]+1}^{n-[nr]} \left(\frac{t}{n}\right)^{d_Y-1} \frac{L_Y(t)}{L_Y(n)} \left(\sum_{s=\max\{-[np],0\}+1}^{t+[nr]-[np]+1} \left(\frac{s}{n}\right)^{d_X-1} \frac{L_X(t)}{L_X(n)} \right) \right| \\
&\leq \frac{1}{\Gamma(d_X)\Gamma(d_Y)} \left| \frac{1}{n} \sum_{t=n-[n(r+\delta)]+1}^{n-[nr]} \left(\frac{t}{n}\right)^{d_Y-1} \right| \left| \frac{1}{n} \sum_{s=\max\{-[np],0\}+1}^{n-[np]+1} \left(\frac{s}{n}\right)^{d_X-1} \right| \\
&= \bar{D}_1^a(r,\delta,p) + O(n^{\max\{-d_X,-d_Y\}});
\end{aligned}$$

$$\begin{aligned}
&|D_{2n}^a(r,\delta,p)| \\
&= \frac{1}{n^2\Gamma(d_X)\Gamma(d_Y)} \left| \sum_{t=\max\{1-[nr],0\}+1}^{n-[n(r+\delta)]} \left(\frac{t}{n}\right)^{d_Y-1} \frac{L_Y(t)}{L_Y(n)} \left(\sum_{s=t+[nr]-[np]+1}^{t+[n(r+\delta)]-[np]+1} \left(\frac{s}{n}\right)^{d_X-1} \frac{L_X(t)}{L_X(n)} \right) \right| \\
&\leq \frac{1}{\Gamma(d_X)\Gamma(d_Y)} \left| \frac{1}{n} \sum_{t=\max\{1-[nr],0\}+1}^{n-[n(r+\delta)]} \left(\frac{t}{n}\right)^{d_Y-1} \right| \left| \frac{1}{n} \sum_{s=[nr]-[np]}^{[n(r+\delta)]-[np]} \left(\frac{s}{n}\right)^{d_X-1} \right| \\
&= \bar{D}_2^a(r,\delta,p) + O(n^{-d_X});
\end{aligned}$$

$$\begin{aligned}
&|D_{3n}^a(r,\delta,p)| \\
&= \frac{1_{\{r<0\}}}{n^2\Gamma(d_X)\Gamma(d_Y)} \left| \sum_{t=-\max\{1-[n(r+\delta),0]+1}^{1-[nr]} \left(\frac{t}{n}\right)^{d_Y-1} \frac{L_Y(t)}{L_Y(n)} \left(\sum_{s=1-[np]}^{t+[n(r+\delta)]-[np]+1} \left(\frac{s}{n}\right)^{d_X-1} \frac{L_X(t)}{L_X(n)} \right) \right| \\
&\leq \frac{1_{\{r<0\}}}{\Gamma(d_X)\Gamma(d_Y)} \left| \frac{1}{n} \sum_{t=1-[n(r+\delta)]}^{1-[nr]} \left(\frac{t}{n}\right)^{d_Y-1} \right| \left| \frac{1}{n} \sum_{s=1-[np]}^{2+[n\delta]-[np]} \left(\frac{s}{n}\right)^{d_X-1} \right| \\
&= \bar{D}_3^a(r,\delta,p) + O(n^{-d_Y}).
\end{aligned}$$

The cases having $d_Y \leq 0$ and/or $d_X \leq 0$ require modification of these formulae on the lines of equation (6.13). Terms of the form $\left(\frac{t}{n}\right)^{d_Y-1}$ and $\left(\frac{s}{n}\right)^{d_X-1}$ are replaced respectively with terms of the form $\left(\frac{t+1}{n}\right)^{d_Y} - \left(\frac{t}{n}\right)^{d_Y}$ and $\left(\frac{s+1}{n}\right)^{d_X} - \left(\frac{s}{n}\right)^{d_X}$. The approximation error rates are modified in the same manner, with $-1-d_Y$ replacing $-d_Y$ when $d_Y < 0$ and $-1-d_X$ replacing $-d_X$ when $d_X < 0$. However, note that although the signs of the sums depend on the signs of d_Y and d_X, the indicated bounds hold in all cases. We therefore have that there exists n large enough that

$$\left|a_{n[np][n(r+\delta)]} - a_{n[np][nr]}\right| \leq \bar{D}_1^a(r,\delta,p) + \bar{D}_2^a(r,\delta,p) + \bar{D}_3^a(r,\delta,p).$$

Therefore,

$$\begin{aligned}
&\limsup_n \frac{1}{n} \sum_{i=-\infty}^{[nr]} (a_{ni[n(r+\delta)]} - a_{ni[nr]})^2 \\
&\qquad \leq \int_{-\infty}^{r} \bar{D}_1^a(r,\delta,p)^2 dp + \int_{-\infty}^{r} \bar{D}_2^a(r,\delta,p))^2 dp + \int_{-\infty}^{r} \bar{D}_3^a(r,\delta,p))^2 dp
\end{aligned}$$

where

$$\int_{-\infty}^{r} \bar{D}_1^a(r,\delta,p)^2 dp \le \frac{(1-r)^{d_Y} - (1-r-\delta)^{d_Y})^2}{\Gamma(1+d_X)^2\Gamma(1+d_Y)^2} \int_{-\infty}^{r} \left[(1-p)^{d_X} - 1_{\{p<0\}}(-p)^{d_X}\right]^2 dp$$
$$= O(\delta^2)$$

$$\int_{-\infty}^{r} \bar{D}_2^a(r,\delta,p))^2 dp \le \frac{\left[(1-r-\delta)^{d_Y} - 1\{r<0\}(-r)^{d_Y}\right]^2}{\Gamma(1+d_X)^2\Gamma(1+d_Y)^2} \int_{-\infty}^{r} \left[(r+\delta-p)^{d_X} - (r-p)^{d_X}\right]^2 dp$$
$$= O(\delta^{2d_X+1})$$

and if $r < 0$,

$$\int_{-\infty}^{r} \bar{D}_3^a(r,\delta,p))^2 dp \le \frac{\left[(-r)^{d_Y} - (-r-\delta)^{d_Y}\right]^2}{\Gamma(1+d_X)^2\Gamma(1+d_Y)^2} \int_{-\infty}^{r} \left[(\delta-p)^{d_X} - (-p)^{d_X}\right]^2 dp$$
$$= O(\delta^{2d_X+3}).$$

These bounds are independent of r, and hold uniformly with respect to $r \in (-\infty, 1]$.

For Lemma 4.6, we have from (3.4)

$$\begin{aligned}
&e_{n[nr][n(p+\delta)]} - e_{n[nr][np]} \\
&\quad = \frac{n}{K(n)} \sum_{s=\max\{1-[n(p+\delta)],0\}}^{n-1-[n(p+\delta)]} b_s \left(\sum_{t=s+[n(p+\delta)]+1-[nr]}^{n-[nr]} c_t \right) \\
&\qquad\qquad - \frac{n}{K(n)} \sum_{s=\max\{0,1-[np]\}}^{n-1-[np]} b_s \left(\sum_{t=s+[np]+1-[nr]}^{n-[nr]} c_t \right) \\
&\quad = -\frac{n}{K(n)} \sum_{s=n-[n(p+\delta)]}^{n-1-[np]} b_s \left(\sum_{t=s+[n(p+\delta)]+1-[nr]}^{n-[nr]} c_t \right) \\
&\qquad\qquad \frac{n}{K(n)} \sum_{s=\max\{0,1-[np]\}}^{n-1-[n(p+\delta)]} b_s \left(\sum_{t=s+[np]+1-[nr]}^{s+[n(p+\delta)]+1-[nr]} c_t \right) \\
&\qquad\qquad + 1_{\{p<0\}} \frac{n}{K(n)} \sum_{s=\max\{0,1-[n(p+\delta)]\}}^{-[np]} b_s \left(\sum_{t=s+[n(p+\delta)]+1-[nr]}^{n-[nr]} c_t \right) \\
&\quad = D_{1n}^e(p,\delta,r) + D_{2n}^e(p,\delta,r) + D_{3n}^e(p,\delta,r).
\end{aligned}$$

Similarly to the previous case, there exists n large enough that

$$\begin{aligned}
|D_{1n}^e(p,\delta,r)| &\le \bar{D}_1^e(p,\delta,r) \\
&= \frac{\left|(1-p)^{d_X} - (1-p-\delta)^{d_X}\right| \left|(1-r)^{d_Y} - (1+\delta-r)^{d_Y}\right|}{\Gamma(1+d_X)\Gamma(1+d_Y)}
\end{aligned}$$

$$\begin{aligned}
|D_{2n}^e(p,\delta,r)| &\le \bar{D}_2^e(p,\delta,r) \\
&= \frac{\left|(1-p-\delta)^{d_X} - 1_{\{p<0\}}(-p)^{d_X}\right| \left|(\max\{p,0\}-\delta-r)^{d_Y} - (\max\{p,0\}-r)^{d_Y}\right|}{\Gamma(1+d_X)\Gamma(1+d_Y)}
\end{aligned}$$

and

$$
\begin{aligned}
|D^e_{3n}(p,\delta,r)| &\le \bar D^e_2(p,\delta,r) \\
&= 1_{\{p<0\}}\frac{\left|(-p)^{d_X}-(-p-\delta)^{d_X}\right|\left|(1-r)^{d_Y}-(-r)^{d_Y}\right|}{\Gamma(1+d_X)\Gamma(1+d_Y)}
\end{aligned}
$$

Therefore, similarly to the previous case,

$$
\begin{aligned}
\limsup_n \frac{1}{n}\sum_{m=-\infty}^{[np]}(e_{nm[n(p+\delta)]}-e_{nm[np]})^2 &\le \int_{-\infty}^p \bar D^e_1(p,\delta,r)^2dp+\int_{-\infty}^p \bar D^e_2(p,\delta,r)^2dp\int_{-\infty}^p \bar D^e_3(p,\delta,r)^2dp \\
&= O(\delta^{2d_Y+1})
\end{aligned}
$$

where the bound is independent of p, and holds uniformly in $p\in(-\infty,1]$. ∎

6.13 Proof of Theorems 4.2 and 4.5

We argue for the exemplar case of Q^N_n. The proof for H^N_n is essentially identical, after swapping H for Q, w for u and e for a, and also substituting the lemmas from Section 4.2 for those of Section 4.1

The proof follows the lines of Theorem 3.1 of Davidson and de Jong (2000) (henceforth, DDJ), which in turn is based on Theorem 3.1 of DJD. Similarly to the DDJ theorem, the sample period is changed from $1,\dots,K_n$ to $-Nn,\dots,n$. For the finite dimensional distributions, we apply the CLT of de Jong (1997). Since the u_i are assumed independent, this is simply a matter of establishing a counterpart of the Lindeberg condition for the process q^N_{nm}.

To translate the conditions of the present setup into those of the DDJ model, note that the process in question has increments

$$
Q^N_n(r+\delta)-Q^N_n(r)=\frac{1}{\sqrt{n}}\sum_{i=[nr]+1}^{[n(r+\delta)]}a_{ni[n(r+\delta)]}u_i+\frac{1}{\sqrt{n}}\sum_{i=-Nn}^{[nr]}(a_{ni[n(r+\delta)]}-a_{ni[nr]})u_i. \tag{6.26}
$$

The notations of this paper and DDJ may be connected by making the equivalences t, ξ and ξ' in DDJ with i, $r+\delta$ and r in this paper, respectively, and so noting that the quantities denoted $a_{nt}(\xi,\xi')$ in DDJ, equation (3.1), correspond in the present notation to $a_{ni[n(r+\delta)]}$ for $i>[nr]$, and to $a_{ni[n(r+\delta)]}-a_{ni[nr]}$ for $-\infty<i\le[nr]$ otherwise. Since the shocks u_i are i.i.d by Assumption 1, bounds on the variances of the increments in (6.26) are found from Lemmas 4.1, 4.2 and 4.3. The finite dimensional distributions of the variates $Q^N_n(r)$ can be determined from Theorem 3.1 of DJD. Note that the variates denoted X_{nt} in that theorem scale correspond in the present case to either $n^{-1/2}a_{ni[n(r+\delta)]}u_i$ or to $n^{-1/2}(a_{ni[n(r+\delta)]}-a_{ni[nr]})u_i$ for $-nN\le i\le[n(r+\delta)]$ and, given the conditions specified in our Assumptions 1 and 2, are sufficient for DJD's Assumption 1. Note further that r is fixed in each application of this theorem, that condition (3.2) in DJD holds in the present case by Lemma 4.1, and that condition 3.3 in DJD holds in the present case by Lemmas 4.2 and 4.3. Finally, to show the tightness of the sequence of measures the argument in DDJ, Theorem 3.1, can be applied with appropriate substitutions. (See also the addendum to this theorem in Davidson (2001)). Noting the equivalences set out above, the condition corresponding to DDJ (B-35) follows directly from Lemmas 4.2 and 4.3. ∎

6.14 Proof of Lemmas 4.4 and 4.7

First note that

$$
\frac{1}{n^2}E\left(\sum_{m=-Nn}^{n-1}(q_{nm}-q^N_{nm})w_{m+1}\right)^2=\frac{1}{n^2}E\left(\sum_{m=-Nn}^{n-1}\sum_{i=-\infty}^{-Nn}u_ia_{nim}w_{m+1}\right)^2
$$

$$= \omega_{uu}\omega_{ww}\frac{1}{n^2}\sum_{m=-Nn}^{n-1}\sum_{i=-\infty}^{-Nn} a_{nim}^2.$$

Copying the argument from the proof of Lemma 4.1, note that

$$\frac{1}{n^2}\sum_{m=-Nn}^{n-1}\sum_{i=-\infty}^{-Nn} a_{nim}^2 = \int_{-N}^{1}\left(\int_{-\infty}^{-N} A_{XY}(r,p)^2 dp\right) dr + o(1)$$

as $n \to \infty$ where

$$\begin{aligned}
\int_{-N}^{1}\left(\int_{-\infty}^{-N} A_{XY}(r,p)^2 dp\right) dr &\leq \int_{-N}^{1}\left(\int_{-\infty}^{-N} \bar{A}_{XY}(r,p)^2 dp\right) dr \\
&= \frac{1}{\Gamma(1+d_X)^2\Gamma(d_Y+1)^2}\int_{-N}^{1}\left((1-r)^{d_Y}-(-r)^{d_Y}\right)^2 dr \\
&\qquad \times \int_{-N}^{\infty}\left[(1+p)^{d_X}-p^{d_X}\right]^2 dp \\
&= O(N^{2d_Y+2d_X-2}).
\end{aligned}$$

Next,

$$\begin{aligned}
\frac{1}{n^2}E\left(\sum_{m=-\infty}^{-Nn} q_{nm}w_{m+1}\right)^2 &= \omega_{uu}\omega_{ww}\frac{1}{n^2}\sum_{m=-\infty}^{-Nn}\sum_{i=-\infty}^{m} a_{nim}^2 \\
&= \int_{-\infty}^{-N}\int_{-\infty}^{r} A_{XY}(r,p)^2 dp dr + o(1)
\end{aligned}$$

as $n \to \infty$, and by previous arguments, letting K denote a finite positive constant, we have

$$\begin{aligned}
&\int_{-\infty}^{N}\left(\int_{-\infty}^{-r} A_{XY}(r,p)^2 dp\right) dr \leq \int_{-\infty}^{N}\left(\int_{-\infty}^{-r} A_{XY}(r,p)^2 dp\right) dr \\
&= \frac{1}{\Gamma(1+d_X)^2\Gamma(d_Y+1)^2}\int_{-\infty}^{-N}\left((1-r)^{d_Y}-(-r)^{d_Y}\right)^2 \\
&\qquad \times \int_{-r}^{\infty}\left[(1+p)^{d_X}-p^{d_X}\right]^2 dp dr \\
&\leq \frac{K}{\Gamma(1+d_X)^2\Gamma(d_Y+1)^2}\int_{-\infty}^{-N}\left((1-r)^{d_Y}-(-r)^{d_Y}\right)^2 (-r)^{2d_X-2} dr \\
&= O(N^{2d_Y+2d_X-3}).
\end{aligned}$$

The case of Lemma 4.7 is essentially similar. ∎

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