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Power variation for Gaussian processes with stationary increments *

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Abstract

We develop the asymptotic theory for the realised power variation of the processes $X = \phi \bullet G$, where G is a Gaussian process with stationary increments. More specifically, under some mild assumptions on the variance function of the increments of G and certain regularity condition on the path of the process ϕ we prove the convergence in probability for the properly normalised realised power variation. Moreover, under a further assumption on the Hölder index of the path of ϕ, we show an associated stable central limit theorem. The main tool is a general central limit theorem, due essentially to Hu & Nualart (2005), Nualart & Peccati (2005) and Peccati & Tudor (2005), for sequences of random variables which admit a chaos representation.

Keywords: Central Limit Theorem; Chaos Expansion; Gaussian Processes; High-Frequency Data; Multiple Wiener-Itô Integrals; Power Variation.

JEL Classification: C10, C13, C14.

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1 Introduction

This paper establishes results on convergence in probability and in law stably for (properly normalised) realised power variations of processes of the form $X = \phi \bullet G$. Here G is a Gausssian process with stationary increments whose increments have a variance function that satisfies certain regularity conditions, and G and the process ϕ are defined on one and the same filtered probability space. The special case of $\phi \bullet G$ where ϕ is a constant and G is itself stationary was treated in an early paper [17].

In general, processes of type $\phi \bullet G$ are not semimartingales and the proofs of the limit results use the theory of isonormal processes and techniques developed in [15] for deriving similar limit results for processes $\phi \bullet B^H$ where B^H denotes fractional Brownian motion. For Itô semimartingales, both one- and multi-dimensional, an extensive theory of realised power and multipower variations is available. For discussions of this theory and its applications, see [2], [3], [5],[6], [7], [8], [9], [10], [19], [21], [29] and [30]. General and sharp criteria for when a process of the form $\phi \bullet G$ is a semimartingale are available in [11] and [12].

Section 2 sets up the problem and exemplifies the kind of processes G to which the theory applies, and the convergence in probability and central limit results for processes $\phi \bullet G$ are given in Sections 3 and 5, respectively. Section 4 derives a multivariate central limit theorem (via chaos expansions) which should be of wide general interest. In particular it covers in-fill asymptotics (or triangular array schemes). The main building blocks in the theorem are contained in the recent papers [18], [24] and [26]. The concluding Section 6 indicates lines for further research. Most of the proofs are relegated to an Appendix.

2 The setting

We start with a Gaussian process $(G_t)_{t\geq 0}$ defined on a filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, which has centered and stationary increments. We define R as the variance function of the increments of G, i.e.

$$R(t) = E[|G_{s+t} - G_s|^2] \ , \qquad t \geq 0. \tag{2.1}$$

In this paper we consider a process of the form

$$X_t = X_0 + \int_0^t \phi_s \, dG_s \ , \tag{2.2}$$

defined on the same probability space as G, which is assumed to be observed at time points i/n, $i = 0, 1, \ldots, [nt]$. We are interested in the asymptotic behaviour of the (properly normalized) realised power variation

$$\sum_{i=1}^{[nt]} |\Delta_i^n X|^p \ , \tag{2.3}$$

with $\triangle_i^n X = X_{\frac{i}{n}} - X_{\frac{i-1}{n}}$, for $p > 0$.

Before we proceed with the asymptotic results for the functionals defined in (2.3) we need to ensure that the integral in (2.2) is well-defined in a suitable sense. For this purpose we use the concept of a pathwise Riemann-Stieltjes integral.

Recall that for a real-valued function $f : [0, t] \to I\!R$ the r-variation is defined as

$$\mathrm{var}_r(f; [0, t]) = \sup_{\pi} \Big(\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^r \Big)^{1/r} , \tag{2.4}$$

where the supremum is taken over all partitions $\pi = \{0 = t_0 < t_1 < \ldots < t_n = t\}$. Trivially, when f is α-Hölder continuous it has finite $1/\alpha$-variation on any compact interval. In this case we set

$$||f||_\alpha = \sup_{0 \le s < u \le t} \frac{|f(s) - f(u)|}{|s - u|^\alpha}. \tag{2.5}$$

In [31] it is shown that the Riemann-Stieltjes integral $\int_0^t f(s)dg(s)$ exists if f and g have finite q-variation and r-variation, respectively, in the interval $[0, t]$, where $1/r + 1/q > 1$, and these functions have no common discontinuities.

In order to give a statement about r-variation of the Gaussian process G we require the following assumption on the behaviour of the function R defined in (2.1).

(A1) *$R(t) = t^\beta L_0(t)$ for some $\beta \in (0, 2)$ and some positive slowly varying (at 0) function L_0, which is continuous on $(0, \infty)$.*

Recall that a function $L : (0, \infty) \to I\!R$ is called slowly varying at 0 when the identity

$$\lim_{x \searrow 0} \frac{L(tx)}{L(x)} = 1 \tag{2.6}$$

holds for any fixed $t > 0$. Provided L is continuous on $(0, \infty)$, we have

$$|L(x)| \le C x^{-\alpha} , \qquad x \in (0, t] \tag{2.7}$$

for any $\alpha > 0$ and any $t > 0$ (where the constant $C > 0$ depends on α and t). See [14] (Page 16) for similar properties of slowly varying functions at ∞. Assumption (A1) implies the identity

$$E[|G_t - G_s|^2] = |t - s|^\beta L_0(|t - s|) , \tag{2.8}$$

from which we deduce (by (2.7)) that the trajectories of G are $(\beta/2 - \epsilon)$-Hölder continuous (almost surely) for any $\epsilon \in (0, \beta/2)$. Clearly, G has finite r-variation for any $r > 2/\beta$ and

$$\mathrm{var}_r(G; [a, b]) \le C|b - a|^{1/r} \ \text{ a.s.} \tag{2.9}$$

for any $0 \le a < b \le t$ and for some constant C which depends on t and r. Consequently, the integral in (2.2) is well-defined (as a pathwise Riemann-Stieltjes integral) for any stochastic process ϕ of finite q-variation with $q < \frac{1}{1-\beta/2}$.

In the following we study the asymptotic properties of the process

$$V(X,p)^n_t = \frac{1}{n\tau_n^p}\sum_{i=1}^{[nt]} |\Delta_i^n X|^p \ , \qquad (2.10)$$

where $\tau_n^2 = R(\frac{1}{n}) = E[|\Delta_i^n G|^2]$ and $p > 0$.

3 Convergence in probability

In this section we prove the convergence in probability for the quantity $V(X,p)^n_t$. For this purpose we require the following additional assumptions on the variance function R:

(A2) *$R''(t) = t^{\beta-2}L_2(t)$ for some slowly varying function L_2, which is continuous on $(0,\infty)$.*

(A3) *There exists $b \in (0,1)$ with*

$$K = \limsup_{x\to 0} \sup_{y\in[x,x^b]} \Big|\frac{L_2(y)}{L_0(x)}\Big| < \infty.$$

We start with proving the weak law of large numbers for the sequence $V(G,p)^n_t$. Throughout this paper we write $Y^n \xrightarrow{ucp} Y$ when $\sup_{t\in[0,T]} |Y^n_t - Y_t| \xrightarrow{P} 0$ for any $T > 0$.

Proposition 1 *Assume that conditions (A1)-(A3) are satisfied. Then we have*

$$V(G,p)^n_t \xrightarrow{ucp} \mu_p t \ , \qquad (3.1)$$

where $\mu_p = E[|U|^p]$, $U \sim N(0,1)$.

Proof: see Appendix.

Remark 1 *The rather technical condition (A3) can be replaced by the following (weaker) assumption:*

$$\Big|\frac{R(\frac{j+1}{n}) + R(\frac{j-1}{n}) - 2R(\frac{j}{n})}{2R(\frac{1}{n})}\Big| \le r(j) \ , \qquad \frac{1}{n}\sum_{j=1}^{n} r^2(j) \to 0 \ , \qquad (3.2)$$

for some sequence $r(j)$ (see Lemma 1 and the proof of Theorem 1 in the Appendix).

The main result of this section is the following theorem.

Theorem 2 *Assume that conditions (A1)-(A3) are satisfied and the process $(\phi_t)_{t\ge 0}$ has finite q-variation with $q < \frac{1}{1-\beta/2}$. Then we have*

$$V(X,p)^n_t \xrightarrow{ucp} \mu_p \int_0^t |\phi_s|^p \, ds. \qquad (3.3)$$

Proof: see Appendix.

Example 3 *(Cauchy class) For modelling purposes, an interesting and flexible class of processes is given by* $(G_{\alpha,\gamma})$*, where* $G_{\alpha,\gamma}$*'s are stationary centered Gaussian processes with variance* 1 *and the autocorrelation function*

$$h(t) = (1 + |t|^{\alpha})^{-\gamma/\alpha}.$$

Here the parameters have to satisfy $\alpha \in (0, 2]$ *and* $\gamma > 0$ *(see [16]). With* $\bar{h}(t) = 1 - h(t)$ *we find, for* $t > 0$,

$$\bar{h}(t) = t^{\alpha} L_0(t)$$

with

$$L_0(t) = \frac{(1 + t^{\alpha})^{\gamma/\alpha} - 1}{t^{\alpha}(1 + t^{\alpha})^{\gamma/\alpha}}$$

Further,

$$\bar{h}'(t) = \gamma t^{\alpha-1}(1 + t^{\alpha})^{-\gamma/\alpha-1}$$

and

$$\bar{h}''(t) = t^{\alpha-2} L_2(t) \ , \qquad L_2(t) = -\gamma((\gamma + 1)t^{\alpha} - (\alpha - 1))(1 + t^{\alpha})^{-\gamma/\alpha-2}.$$

Both L_0 *and* L_2 *are slowly varying. Now,*

$$L_2'(t) = -\alpha\gamma t^{\alpha-1}(1 + t^{\alpha})^{-\gamma/\alpha-3} \left[\gamma + 1 + (\alpha - 1)(\gamma/\alpha + 2) - (\gamma + 1)(\gamma/\alpha + 1)t^{\alpha}\right]$$

showing that $L_2(t)$ *is decreasing or increasing in a neighbourhood of* 0 *depending on whether* α *is greater or smaller than* c_γ *where* c_γ *denotes the positive root of the equation* $\gamma + 1 + (\alpha - 1)(\gamma/\alpha + 2) = 0$*. In any case,*

$$\sup_{y \in [x, x^b]} \left| \frac{L_2(y)}{L_0(x)} \right| \to |\alpha - 1|\alpha < \infty \ ,$$

as $x \to 0$*, for any* $b \in (0, 1)$*. Thus conditions (A1)-(A3) are fulfilled for any* $\alpha \in (0, 2)$ *and* $\gamma > 0$*, and Proposition 1 and Theorem 2 apply to the class* $(G_{\alpha,\gamma})_{\alpha\in(0,2),\gamma>0}$.

4 A general multivariate central limit theorem via chaos expansion

In this section we present a multivariate central limit theorem for a sequence of random variables which admit a chaos representation. This result is based on the theory for multiple stochastic integrals developed in [24], [26] and [18] (and it appears implicitly in [15]). The central limit theorem will be used to show the weak convergence of the process $V(G, p)^n_t$. However, the limit results of this section might be of interest for many other applications.

Let us recall the basic notions of the theory of multiple stochastic integrals. Consider a separable Hilbert space $\mathbb{H}$. For any $m \geq 1$, we define $\mathbb{H}^{\otimes m}$ to be the mth tensor product of $\mathbb{H}$ and we write $\mathbb{H}^{\odot m}$ for the mth symmetric tensor product of $\mathbb{H}$, which is endowed with the modified norm $\sqrt{m!}||\cdot||_{I\!H^{\otimes m}}$. A centered Gaussian family $B=\{B(h)|\ h\in I\!H\}$, defined on the probability space $(\Omega,\mathcal{F},P)$, is called an isornormal process on $I\!H$ when

$$E[B(h)B(g)]=<h,g>_{I\!H}\ ,\qquad \forall\ h,g\in I\!H.$$

In this section we assume that $\mathcal{F}$ is generated by B. For $m\geq 1$, we denote by $\mathcal{H}_m$ the mth Wiener chaos associated with B, i.e. the closed subspace of $\mathcal{L}^2(\Omega,\mathcal{F},P)$ generated by the random variables $H_m(B(h))$, where $h\in I\!H$ with $||h||_{I\!H}=1$ and H_m is the mth Hermite polynomial. Recall that the Hermite polynomials $(H_m)_{m\geq 0}$ are defined as follows:

$$\begin{aligned}H_0(x) &= 1\ ,\\ H_m(x) &= (-1)^m e^{\frac{x^2}{2}}\frac{d^m}{dx^m}(e^{-\frac{x^2}{2}})\ ,\qquad m\geq 1.\end{aligned}$$

The first three Hermite polynomials are $H_1(x)=x$, $H_2(x)=x^2-1$ and $H_3(x)=x^3-3x$. By I_m we denote the linear isometry between the symmetric tensor product $I\!H^{\odot m}$, equipped with the norm $\sqrt{m!}\,\|\cdot\|_{I\!H^{\otimes m}}$, and the mth Wiener chaos that is defined by

$$I_m(h^{\otimes m})=H_m(B(h))$$

(see, for instance, Chapter 1 in [23] for more details).

For any $h=h_1\otimes\cdots\otimes h_m$ and $g=g_1\otimes\cdots\otimes g_m\in I\!H^{\otimes m}$, we define the pth contraction of h and g, denoted by $h\otimes_p g$, as the element of $I\!H^{\otimes 2(m-p)}$ given by

$$h\otimes_p g=<h_{m-p+1},g_1>_{I\!H}\cdots<h_m,g_p>_{I\!H}\ h_1\otimes\cdots\otimes h_{m-p}\otimes g_{p+1}\otimes\cdots\otimes g_m.$$

This can be extended by linearity to any element of $I\!H^{\otimes m}$. Note that if h and g belong to $I\!H^{\odot m}$, $h\otimes_p g$ does not necessarily belong to $I\!H^{\odot(2m-p)}$. For any $h=h_1\otimes\cdots\otimes h_m\in I\!H^{\otimes m}$, we denote by $\tilde{h}\in I\!H^{\odot m}$ the symmetrization of h, i.e.

$$\tilde{h}=\frac{1}{m!}\sum_{\zeta\in S_m}h_{\zeta(1)}\otimes\cdots\otimes h_{\zeta(m)}\ ,$$

where S_m is the group of permutations of $\{1,\ldots,m\}$. Moreover, we write $h\widetilde{\otimes}_p g$ for the symmetrization of $h\otimes_p g$.

Now, we present a multivariate central limit theorem which is a straightforward consequence of Theorem 1 and Proposition 2 in [26] (and the proofs therein).

Theorem 4 *Consider a collection of natural numbers $m_1\leq m_2\leq\cdots\leq m_d$ and a collection of elements*

$$\{(f_n^1,\ldots,f_n^d)|\ n\geq 1\}$$

such that $f_n^k\in I\!H^{\odot m_k}$ and the following conditions are satisfied:

(1) For any $k,l=1,\ldots,d$ we have constants C_{kl} such that

$$\lim_{n\to\infty} m_k!||f_n^k||^2_{\mathbb{H}^{\otimes m_k}} = C_{kk} \ ,$$

$$\lim_{n\to\infty} E[I_{m_k}(f_n^k)I_{m_l}(f_n^l)] = C_{kl} \ , \qquad k\neq l \ ,$$

and the matrix $C=(C_{kl})_{1\le k,l\le d}$ is positive definite.

(2) For every $k=1,\ldots,d$ we have

$$\lim_{n\to\infty} ||f_n^k\otimes_p f_n^k||^2_{\mathbb{H}^{\otimes 2(m_k-p)}} = 0$$

for any $p=1,\ldots,m_k-1$.

Then we obtain the central limit theorem

$$\Big(I_{m_1}(f_n^1),\ldots,I_{m_d}(f_n^d)\Big)^T \xrightarrow{\mathcal{D}} N_d(0,C). \tag{4.1}$$

Notice that C_{kl} in (1) of Theorem 4 is equal to 0 when $m_k\neq m_l$, because I_{m_k} and I_{m_l} are orthogonal by construction.

Finally, we consider a d-dimensional process $Y_n=(Y_n^1,\ldots,Y_n^d)^T$, defined on $(\Omega,\mathcal{F},P)$, which has a chaos representation

$$Y_n^k=\sum_{m=1}^{\infty} I_m(f_{m,n}^k) \ , \qquad k=1,\ldots,d \ , \tag{4.2}$$

with $f_{m,n}^k\in\mathbb{H}^{\odot m}$. Notice that $EY_n=0$. The following result provides a central limit theorem for the sequence Y_n.

Theorem 5 *Suppose that the following conditions hold:*

(i) For any $k=1,\ldots,d$ we have

$$\lim_{N\to\infty}\limsup_{n\to\infty}\sum_{m=N+1}^{\infty} m!||f_{m,n}^k||^2_{\mathbb{H}^{\otimes m}}=0.$$

(ii) For any $m\ge 1$, $k,l=1,\ldots,d$ we have constants C_{kl}^m such that

$$\lim_{n\to\infty} m!||f_{m,n}^k||^2_{\mathbb{H}^{\otimes m}} = C_{kk}^m \ ,$$

$$\lim_{n\to\infty} E[I_m(f_{m,n}^k)I_m(f_{m,n}^l)] = C_{kl}^m \ , \qquad k\neq l \ ,$$

and the matrix $C^m=(C_{kl}^m)_{1\le k,l\le d}$ is positive definite for all m.

(iii) $\sum_{m=1}^{\infty} C^m = C\in\mathbb{R}^{d\times d}$.

(iv) For any $m \geq 1$, $k = 1, \ldots, d$ and $p = 1, \ldots, m-1$

$$\lim_{n\to\infty} ||f^k_{m,n} \otimes_p f^k_{m,n}||^2_{\mathbb{H}^{\otimes 2(m-p)}} = 0.$$

Then we have

$$Y_n \xrightarrow{\mathcal{D}} N_d(0, C). \tag{4.3}$$

Proof: Define the "truncated" random variable $Y_{n,N} = (Y^1_{n,N}, \ldots, Y^d_{n,N})^T$ by

$$Y^k_{n,N} = \sum_{m=1}^{N} I_m(f^k_{m,n}) \ , \qquad k = 1, \ldots, d.$$

Since I_{m_1} and I_{m_2} are orthogonal when $m_1 \neq m_2$, Theorem 4 implies (under conditions (ii) and (iv) of Theorem 5) that

$$Y_{n,N} \xrightarrow{\mathcal{D}} \xi_N \sim N_d\Big(0, \sum_{m=1}^{N} C^m\Big)$$

for a fixed N. By assumption (iii) we obtain the convergence in distribution

$$\xi_N \xrightarrow{\mathcal{D}} \xi \sim N_d\Big(0, C\Big)$$

as $N \to \infty$. Finally, condition (i) and the Markov inequality imply

$$\lim_{N\to\infty} \limsup_{n\to\infty} P(||Y_{n,N} - Y_n||_\infty \geq \delta) = 0$$

for any $\delta > 0$ (here $|| \cdot ||_\infty$ denotes the maximum norm). By standard arguments we obtain the desired result. $\square$

5 A stable central limit theorem for power variation

First, we present a functional central limit theorem for the sequence $V(G,p)^n_t$. In the following discussion we use the notation

$$H(x) = |x|^p - \mu_p. \tag{5.1}$$

Notice that the function H has the representation

$$H(x) = \sum_{j=2}^{\infty} a_j H_j(x) \ , \tag{5.2}$$

where $a_2 > 0$ and $(H_j)_{j\geq 0}$ are Hermite polynomials. Under a restriction on the parameter β we obtain the following result.

Theorem 6 *Assume that conditions (A1)-(A3) hold and* $0 < \beta < \frac{3}{2}$. *Then we obtain the weak convergence (in the space* $\mathcal{D}([0,T])^2$ *equipped with the Skorohod topology)*

$$\Big(G_t, \sqrt{n}(V(G,p)_t^n - t\mu_p)\Big) \Longrightarrow \Big(G_t, \tau W_t\Big)\ , \tag{5.3}$$

where W *is a Brownian motion that is defined on an extension of the filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ *and is independent of* $\mathcal{F}$, *and* τ^2 *is given by*

$$\tau^2 = \sum_{j=2}^{\infty} j! a_j^2 \lambda_j^2\ , \qquad \lambda_j^2 = 1 + 2\sum_{l=1}^{\infty} \frac{\Big((l-1)^\beta - 2l^\beta + (l+1)^\beta\Big)^j}{2^j}. \tag{5.4}$$

Proof: see Appendix.

Remark 2 *Theorem 6 applies to the Cauchy class* $(G_{\alpha,\gamma})$ *(with* $\gamma > 0$ *and* $\alpha \in (0,2)$*) of Gaussian processes that has been introduced in Example 3.*

The proof of Theorem 6 relies on the central limit theorem presented in Theorem 5. In [15] the result of Theorem 6 is shown (with the same limit) for the case of fractional Brownian motion with Hurst parameter $H \in (0,1)$ (the parameter β corresponds to $2H$). Their derivation relies on the selfsimilarity of the fractional Brownian motion. The asymptotic theory presented in this section provides a natural extension of their work to general Gaussian processes with stationary increments.

Remark 3 •

(i) A central limit theorem for the quantity $V(G,p)_1^n$ *(i.e. for* $t=1$*) was originally proved in [17] under assumptions (A1)-(A3). For Theorem 6 the technical condition (A3) can be replaced by the weaker assumption:*

$$\Big|\frac{R(\frac{j+1}{n}) + R(\frac{j-1}{n}) - 2R(\frac{j}{n})}{2R(\frac{1}{n})}\Big| \leq \tilde{r}(j)\ , \qquad \sum_{j=1}^{\infty} \tilde{r}^2(j) < \infty\ , \tag{5.5}$$

for some sequence $\tilde{r}(j)$. *Notice that (5.5) implies the condition (3.2) in Remark 1 with* $r(j) = \tilde{r}(j)$ *for all* $j \geq 1$. *See Lemma 1 and the proof of Theorem 6 in the Appendix for more details.*

(ii) Furthermore, in [17] it is shown that the limit of the second component in (5.3) is an element of the second Wiener chaos when G *is a stationary Gaussian process with* $EG_t = 0$, $EG_t^2 = 1$, *and* $\frac{3}{2} < \beta < 2$. *For the covariance function* $\tilde{R}(t) = E[G_s G_{s+t}]$ *they assume the following conditions:* $1 - \tilde{R}(t)$ *satisfies (A1),* $|\tilde{R}''|$ *satisfies (A2) with* $L_2(x) = \beta(1-\beta)L_0(x)(1+o(1))$ *near* 0, $|\tilde{R}''|$ *is decreasing near* 0 *and (A3) holds. Under these assumptions they have proved the convergence*

$$n^{2-\beta} L_0\Big(\frac{1}{n}\Big)(V(G,p)_1^n - \mu_p) \xrightarrow{\mathcal{D}} \frac{p\mu_p}{4}\mathcal{I}_2\ ,$$

where $\mathcal{I}_2$ is the Wiener-Itô integral

$$\mathcal{I}_2 = \int_{\mathbb{R}^2} \frac{e^{i(x_1+x_2)} - 1}{i(x_1 + x_2)} f^{\frac{1}{2}}(x_1) f^{\frac{1}{2}}(x_2) W(dx_1) W(dx_2) ,$$

W is a Brownian motion and f is given by

$$f(x) = -\int_{\mathbb{R}} e^{itx} \tilde{R}''(|t|) dt.$$

If $\beta = \frac{3}{2}$ both limits can appear: when $\tilde{R}''$ is integrable near 0 we obtain an element of the second Wiener chaos in the limit, whereas the limit is normal when $\tilde{R}''$ is not integrable near 0 (although the convergence rate changes). However, functional central limit theorems for $\frac{3}{2} < \beta < 2$ remain an unsolved problem.

Notice that the weak convergence in (5.3) is equivalent to the stable convergence (in $\mathcal{D}([0,T])^2$)

$$\sqrt{n}(V(G,p)^n_t - t\mu_p) \overset{\mathcal{F}^G - st}{\longrightarrow} \tau W_t \ , \tag{5.6}$$

where $\mathcal{F}^G$ denotes the σ-algebra generated by the process G (see [1], [20] or [27] for more details on stable convergence). The latter result is crucial for proving a functional central limit theorem for the sequence $V(X,p)^n_t$ for $\mathcal{F}^G$-measurable processes ϕ.

Theorem 7 *Suppose that ϕ is $\mathcal{F}^G$-measurable and has Hölder continuous trajectories of order $a > 1/2(p \wedge 1)$. When $0 < \beta < \frac{3}{2}$ and assumptions (A1)-(A3) hold we obtain the stable convergence*

$$\sqrt{n}\Big(V(X,p)^n_t - \mu_p \int_0^t |\phi_s|^p \ ds\Big) \overset{\mathcal{F}^G - st}{\longrightarrow} \tau \int_0^t |\phi_s|^p \ dW_s \tag{5.7}$$

in the space $\mathcal{D}([0,T])^2$.

Proof: see Appendix.

Remark 4 *Notice that if $\phi_t = f(G_t)$ for some smooth function f, the conditions of Theorem 7 imply that $p > 1/\beta$ and $\beta \in (1, \frac{3}{2})$. This leads to a serious restriction on the parameters p and β.*

On the other hand, Theorem 7 remains valid when the process ϕ is independent of G (this follows from Theorem 6 if we replace the process G by ϕ). In this case we only require the condition $a > 1/2(p \wedge 1)$.

Applying the properties of stable convergence we can obtain a feasible version of Theorem 7. Since $V(X,2p)^n_t \overset{P}{\longrightarrow} \mu_{2p} \int_0^t \phi_s^{2p} ds$, we deduce the following result.

Corollary 1 *For any fixed $t > 0$, we have*

$$\frac{\sqrt{n}\Big(V(X,p)^n_t - \mu_p \int_0^t |\phi_s|^p \ ds\Big)}{\sqrt{\mu_{2p}^{-1} \tau^2 V(X,2p)^n_t}} \overset{\mathcal{F}^G - st}{\longrightarrow} U \ ,$$

where U is independent of $\mathcal{F}$ and $U \sim N(0,1)$.

6 Conclusion

The results derived in the present paper constitute a natural extension of earlier work on power variation, as indicated in the Introduction. The possibility of further extension to bipower, and more generally multipower, variations is under consideration. From another point of view, the results provide a step in a larger project that aim to develop probabilistic and inferential procedures for the study of volatility modulated Volterra processes, as defined in [4]. Finally, closer links to Malliavin calculus, cf. [22] or [25], offer exciting prospects.

7 Appendix

In the following we denote all constants which do not depend on n by C. Throughout this section we use the notation

$$r_n(j) = \mathrm{Cov}\Big(\frac{\Delta_1^n G}{\tau_n}, \frac{\Delta_{1+j}^n G}{\tau_n}\Big) , \qquad j \geq 0. \tag{7.1}$$

By the triangular identity we know that

$$r_n(j) = \frac{R(\frac{j+1}{n}) + R(\frac{j-1}{n}) - 2R(\frac{j}{n})}{2R(\frac{1}{n})} , \qquad j \geq 1 , \tag{7.2}$$

where the function R is given by (2.1). First, let us prove the following technical lemma which extends Lemma 2 and 3 in [17].

Lemma 1 *Suppose that conditions (A1)-(A3) hold. Let $\epsilon > 0$ with $\epsilon < 2 - \beta$. Define the sequence $r(j)$ by*

$$r(j) = (j-1)^{\beta+\epsilon-2}, \qquad j \geq 2 , \tag{7.3}$$

and $r(0) = r(1) = 1$. Then we obtain the following assertions:

(i) It holds that

$$\frac{1}{n}\sum_{j=1}^{n} r^2(j) \to 0.$$

If, moreover, $\beta + \epsilon - 2 < -\frac{1}{2}$ it holds that

$$\sum_{j=1}^{\infty} r^2(j) < \infty.$$

(ii) For any $0 < \epsilon < 2 - \beta$ from (7.3) there exists a natural number $n_0(\epsilon)$ such that

$$|r_n(j)| \leq Cr(j) , \qquad j \geq 0$$

for all $n \geq n_0(\epsilon)$.

(iii) Set $\rho(0) = 1$ *and* $\rho(j) = \frac{1}{2}\Big((j-1)^\beta - 2j^\beta + (j+1)^\beta\Big)$ *for* $j \geq 1$. *Then it holds that*

$$|r_n(j)| \to \rho(j)$$

for any $j \geq 0$.

(iv) For $0 < \beta < \frac{3}{2}$ *and any* $l \geq 2$ *we have that*

$$\sum_{j=1}^{n-1} r_n^l(j) \to \sum_{j=1}^{\infty} \rho^l(j).$$

Proof of Lemma 1: Part (i) of Lemma 1 is trivial. By assumptions (A1) and (A2) we deduce the identities

$$r_n(1) = -1 + 2^{\beta-1} \frac{L_0(\frac{2}{n})}{L_0(\frac{1}{n})} \ ,$$

$$r_n(j) = -\frac{1}{2}\Big(j + \frac{\theta_j^n}{n}\Big)^{\beta-2} \frac{L_2(\frac{j+\theta_j^n}{n})}{L_0(\frac{1}{n})} \ , \qquad j \geq 2 \ ,$$

where θ_j^n are some real numbers with $|\theta_j^n| < 1$. Recall assumption (A3) and set $a = 1-b \in (0,1)$. When n is large enough we have $r_n(1) < 1$ (because L_0 is a slowly varying function and $\beta \in (0,2)$) and for $2 \leq j \leq [n^a]$ we obtain

$$|r_n(j)| \leq C(j-1)^{\beta-2}$$

by assumption (A3). For $[n^a] \leq j \leq n$ we obtain by (2.7) the following approximation

$$\begin{aligned} |r_n(j+1)| &\leq \frac{1}{2} j^{\beta-2} \frac{L_2(\frac{j+\theta_j^n}{n})}{L_0(\frac{1}{n})} \\ &\leq j^{\beta-2+\epsilon} n^{-a\epsilon} \frac{L_2(\frac{j+\theta_j^n}{n})}{L_0(\frac{1}{n})} \leq C j^{\beta-2+\epsilon}. \end{aligned}$$

Thus, assertion (ii) follows.

Next, by assumption (A1) and (7.2) we obtain the formula

$$r_n(j) = \frac{(j-1)^\beta L_0(\frac{j-1}{n}) - 2j^\beta L_0(\frac{j}{n}) + (j+1)^\beta L_0(\frac{j+1}{n})}{2L_0(\frac{1}{n})} \ , \qquad j \geq 1.$$

We can readily deduce part (iii), because the function L_0 is slowly varying.

Next, assume that $0 < \beta < \frac{3}{2}$. We use $0 < \epsilon < \frac{3}{2} - \beta$ in the definition (7.3). Since $\beta + \epsilon - 2 < -\frac{1}{2}$, we deduce that

$$\sum_{j=1}^{\infty} r^l(j) < \infty$$

for any $l \geq 2$, by part (i). By parts (ii), (iii) and the dominated convergence theorem we obtain (iv), and the proof is complete. $\square$

Proof of Proposition 1: We first show the pointwise convergence $V(G,p)^n_t \stackrel{P}{\longrightarrow} \mu_p t$. Recall the identity

$$E[H_k(U_1)H_l(U_2)] = \delta_{k,l}\rho^l l! \ , \qquad (U_1,U_2) \sim N\left(0, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right) \ , \tag{7.4}$$

where $\delta_{k,l}$ denotes the Kronecker symbol. For any $t>0$ we have

$$E[V(G,p)^n_t] = \mu_p t + O(n^{-1}) \tag{7.5}$$

and by (7.4), (5.1) and (5.2) we obtain the identity

$$\mathrm{Var}(V(G,p)^n_t) = \frac{(\mu_{2p}-\mu_p^2)[nt]}{n^2} + \frac{2}{n^2}\sum_{j=1}^{[nt]-1}([nt]-j)\mathrm{Cov}\Big(\Big|\frac{\Delta^n_1 G}{\tau_n}\Big|^p, \Big|\frac{\Delta^n_{1+j} G}{\tau_n}\Big|^p\Big).$$

$$= \sum_{l=2}^{\infty} l! a_l^2 b_{ln} + O(n^{-1}) \ ,$$

where the coefficients a_l are given by (5.2) and the constants b_{ln} are defined by

$$b_{ln} = \frac{2}{n^2}\sum_{j=1}^{[nt]-1}([nt]-j)r_n^l(j). \tag{7.6}$$

By (i) and (ii) of Lemma 1 we deduce (for $n \geq n_0$) that

$$|b_{ln}| \leq \frac{2t}{n}\sum_{j=1}^{[nt]-1} r^l(j) \leq \frac{2t}{n}\sum_{j=1}^{[nt]-1} r^2(j) \to 0 \ , \tag{7.7}$$

for any $l \geq 2$. This implies the pointwise convergence

$$V(G,p)^n_t \stackrel{P}{\longrightarrow} \mu_p t.$$

The ucp convergence follows immediately, because $V(G,p)^n_t$ is increasing in t and the limit process $\mu_p t$ is continuous. □

Proof of Theorem 2: The basic idea behind the proof of Theorem 2 is the approximation of the process $(\phi_t)_{t\geq 0}$ by a sequence of step functions and the application of Proposition 1. In [15] a proof of (3.3) is given for the case of fractional Brownian motion, and we will basically follow their ideas.

Consider first the case $p \leq 1$. For any $m \geq n$, we obtain the decomposition

$$V(X,p)^n_t - \mu_p \int_0^t |\phi_s|^p \, ds = A_t^{(m)} + B_t^{(n,m)} + C_t^{(n,m)} + D_t^{(n)} \ ,$$

where

$$A_t^{(m)} = \frac{1}{m\tau_m^p}\sum_{i=1}^{[mt]}\Big(|\Delta_i^m X|^p - |\phi_{\frac{i-1}{m}}\Delta_i^m G|^p\Big)\,, \tag{7.8}$$

$$B_t^{(n,m)} = \frac{1}{m\tau_m^p}\Big(\sum_{i=1}^{[mt]}|\phi_{\frac{i-1}{m}}\Delta_i^m G|^p - \sum_{j=1}^{[nt]}|\phi_{\frac{j-1}{n}}|^p\sum_{i\in I_n(j)}|\Delta_i^m G|^p\Big)\,,$$

$$C_t^{(n,m)} = \frac{1}{m\tau_m^p}\sum_{j=1}^{[nt]}|\phi_{\frac{j-1}{n}}|^p\sum_{i\in I_n(j)}|\Delta_i^m G|^p - \mu_p n^{-1}\sum_{j=1}^{[nt]}|\phi_{\frac{j-1}{n}}|^p\,,$$

$$D_t^{(n)} = \mu_p\Big(n^{-1}\sum_{j=1}^{[nt]}|\phi_{\frac{j-1}{n}}|^p - \int_0^t|\phi_s|^p\,ds\Big)\,,$$

and

$$I_n(j) = \Big\{i|\ \frac{i}{m}\in\Big(\frac{j-1}{n},\frac{j}{n}\Big]\Big\}\,, \qquad j\geq 1.$$

For any fixed n, $C_t^{(n,m)}$ converges in probability to 0, uniformly in t, as $m\to\infty$, i.e.

$$\sup_{0\leq t\leq T}|C_t^{(n,m)}| \leq \sum_{j=1}^{[nT]}|\phi_{\frac{j-1}{n}}|^p\Big|\frac{1}{m\tau_m^p}\sum_{i\in I_n(j)}|\Delta_i^m G|^p - \mu_p n^{-1}\Big| \xrightarrow{P} 0$$

thanks to the uniform convergence $V(G,p)_t^m \xrightarrow{ucp} \mu_p t$. Next, observe that the number of jumps of $|\phi_t|^p$ that are bigger than ε is finite (on compact intervals), because $|\phi_t|^p$ is regulated. This implies

$$\sup_{0\leq t\leq T}|D_t^{(n)}| \leq \mu_p n^{-1}\Big(\sup_{0\leq t\leq T}|\phi_t|^p + \sum_{j=1}^{[nT]}\sup_{s\in(\frac{j-1}{n},\frac{j}{n}]}||\phi_{\frac{j-1}{n}}|^p - |\phi_s|^p|\Big) \xrightarrow{P} 0$$

as $n\to 0$. For the term $B_t^{(n,m)}$ we obtain the inequality

$$\begin{aligned}\sup_{0\leq t\leq T}|B_t^{(n,m)}| &\leq \frac{1}{m\tau_m^p}\sum_{j=1}^{[nT]}\sum_{i\in I_n(j)}||\phi_{\frac{j-1}{n}}|^p - |\phi_{\frac{i-1}{m}}|^p||\Delta_i^m G|^p \\ &+ \sup_{0\leq t\leq T}|\phi_t|^p\sup_{0\leq t\leq T}\frac{1}{m\tau_m^p}\sum_{mn^{-1}[nt]\leq i\leq mn^{-1}([nt]+1)}|\Delta_i^m G|^p \\ &\leq \frac{1}{m\tau_m^p}\sum_{j=1}^{[nT]}\sup_{s\in(\frac{j-2}{n},\frac{j}{n}]}||\phi_{\frac{j-1}{n}}|^p - |\phi_s|^p|\sum_{i\in I_n(j)}|\Delta_i^m G|^p \\ &+ \sup_{0\leq t\leq T}|\phi_t|^p\sup_{0\leq t\leq T}\frac{1}{m\tau_m^p}\sum_{mn^{-1}[nt]\leq i\leq mn^{-1}([nt]+1)}|\Delta_i^m G|^p.\end{aligned}$$

By Proposition 1, the latter expression converges in probability to

$$E_n = \mu_p n^{-1}\Big(\sup_{0\leq t\leq T}|\phi_t|^p + \sum_{j=1}^{[nT]}\sup_{s\in(\frac{j-2}{n},\frac{j}{n}]}||\phi_{\frac{j-1}{n}}|^p - |\phi_s|^p|\Big)$$

as $m \to \infty$. As above, we obtain $E_n \xrightarrow{P} 0$ as $n \to \infty$.

For the term $A_t^{(m)}$, we deduce by Young's inequality (for $p \le 1$)

$$\begin{aligned}
\sup_{0\le t\le T} |A_t^{(m)}| &\le \frac{1}{m\tau_m^p} \sup_{0\le t\le T} \Big| \sum_{i=1}^{[mt]} \Big(|\Delta_i^m X|^p - |\phi_{\frac{i-1}{m}} \Delta_i^m G|^p \Big) \Big| \\
&\le \frac{1}{m\tau_m^p} \sum_{i=1}^{[mT]} \Big| \Delta_i^m X - \phi_{\frac{i-1}{m}} \Delta_i^m G \Big|^p \\
&\le \frac{C}{m\tau_m^p} \sum_{i=1}^{[mT]} \Big| \mathrm{var}_q\Big(\phi; \Big(\frac{i-1}{m}, \frac{i}{m}\Big]\Big) \mathrm{var}_{1/(\frac{\beta}{2}-\varepsilon)}\Big(G; \Big(\frac{i-1}{m}, \frac{i}{m}\Big]\Big) \Big|^p = CF_T^{(m)} ,
\end{aligned}$$

where $0 < \varepsilon < \beta/2$. Next, we fix $\delta > 0$ and consider the decomposition

$$\begin{aligned}
F_T^{(m)} &\le \frac{1}{m\tau_m^p} \sum_{i:\mathrm{var}_q(\phi;(\frac{i-1}{m},\frac{i}{m}])>\delta} \Big| \mathrm{var}_q\Big(\phi; \Big(\frac{i-1}{m}, \frac{i}{m}\Big]\Big) \mathrm{var}_{1/(\frac{\beta}{2}-\varepsilon)}\Big(G; \Big(\frac{i-1}{m}, \frac{i}{m}\Big]\Big) \Big|^p \\
&+ \frac{\delta^p}{m\tau_m^p} \sum_{i=1}^{[mT]} \Big| \mathrm{var}_{1/(\frac{\beta}{2}-\varepsilon)}\Big(G; \Big(\frac{i-1}{m}, \frac{i}{m}\Big]\Big) \Big|^p
\end{aligned}$$

Observe that

$$\sum_{i=1}^{[mT]} \Big| \mathrm{var}_q\Big(\phi; \Big(\frac{i-1}{m}, \frac{i}{m}\Big]\Big) \Big|^q \le |\mathrm{var}_q(\phi; [0,T])|^q < \infty ;$$

consequently, the number of indexes i for which $\mathrm{var}_q\Big(\phi; \Big(\frac{i-1}{m}, \frac{i}{m}\Big]\Big) > \delta$ is bounded by $|\mathrm{var}_q(\phi;[0,T])|^q/\delta^q$. Recalling (2.8) and (2.9) we obtain

$$\begin{aligned}
F_T^{(m)} &\le \frac{|\mathrm{var}_q(\phi;[0,T])|^{q+p}}{m\tau_m^p \delta^q} \max_{1\le i\le [mT]} \Big| \mathrm{var}_{1/(\frac{\beta}{2}-\varepsilon)}\Big(G; \Big(\frac{i-1}{m}, \frac{i}{m}\Big]\Big) \Big|^p \\
&+ \frac{\delta^p}{m\tau_m^p} \sum_{i=1}^{[mT]} \Big| \mathrm{var}_{1/(\frac{\beta}{2}-\varepsilon)}\Big(G; \Big(\frac{i-1}{m}, \frac{i}{m}\Big]\Big) \Big|^p \\
&\le C\Big(\frac{|\mathrm{var}_q(\phi;[0,T])|^{q+p}}{m\tau_m^p \delta^q} m^{-p(\frac{\beta}{2}-\varepsilon)} + \frac{\delta^p}{\tau_m^p} m^{-p(\frac{\beta}{2}-\varepsilon)} \Big).
\end{aligned}$$

Choose $0 < \varepsilon < \frac{1}{2p}$, $\varepsilon < \alpha < \frac{1}{p} - \epsilon$ and set $\delta = m^{-\alpha}$. By (2.7) we deduce that

$$F_T^{(m)} \xrightarrow{P} 0 ,$$

which completes the proof of Theorem 2 for $p \le 1$.

For $p > 1$ we use Minkowski's inequality to obtain the approximation

$$\Big|\Big(V(X,p)_t^m\Big)^{1/p} - \Big(\mu_p \int_0^t |\phi_s|^p \, ds\Big)^{1/p}\Big| \le \frac{1}{m^{1/p}\tau_m}\Big(\sum_{i=1}^{[mt]} \Big|\Delta_i^m X - \phi_{\frac{i-1}{m}} \Delta_i^m G\Big|^p\Big)^{1/p}$$

$$+\frac{1}{m^{1/p}\tau_m}\Big(\sum_{j=1}^{[nt]} \sum_{i\in I_n(j)} |(\phi_{\frac{i-1}{m}} - \phi_{\frac{j-1}{n}})\Delta_i^m G|^p\Big)^{1/p}$$

$$+\Big|\frac{1}{m^{1/p}\tau_m}\Big(\sum_{j=1}^{[nt]} |\phi_{\frac{j-1}{n}}|^p \sum_{i\in I_n(j)} |\Delta_i^m G|^p\Big)^{1/p} - \Big(\mu_p n^{-1} \sum_{j=1}^{[nt]} |\phi_{\frac{j-1}{n}}|^p\Big)^{1/p}\Big|$$

$$+\mu_p^{1/p}\Big|\Big(n^{-1}\sum_{j=1}^{[nt]} |\phi_{\frac{j-1}{n}}|^p\Big)^{1/p} - \Big(\int_0^t |\phi_s|^p \, ds\Big)^{1/p}\Big|.$$

By the same methods as presented above we obtain the assertion of Theorem 2 for $p > 1$. □

Proof of Theorem 6: We set

$$Z_t^n = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} H\Big(\frac{\Delta_i^n G}{\tau_n}\Big). \tag{7.9}$$

Step 1: Let us show the tightness of the sequence of processes (G_t, Z_t^n). For any $t > s$ we have

$$E[(Z_t^n - Z_s^n)^4] = \frac{1}{n^2} E\Big[\Big(\sum_{i=1}^{[nt]} H\Big(\frac{\Delta_i^n G}{\tau_n}\Big)\Big)^4\Big].$$

By Proposition 4.2 in [28] and part (iv) of Lemma 1 we know that, for any $N \ge 1$,

$$\frac{1}{N^2} E\Big[\Big(\sum_{i=1}^{N} H\Big(\frac{\Delta_i^n G}{\tau_n}\Big)\Big)^4\Big] \le C\Big(\sum_{i=0}^{\infty} r_n^2(i)\Big)^2 \to C\Big(\sum_{i=0}^{\infty} \rho^2(i)\Big)^2.$$

Since the process G has stationary increments, we obtain

$$E[(Z_t^n - Z_s^n)^4] \le C\Big|\frac{[nt]-[ns]}{n}\Big|^2.$$

For any $t_1 \le t \le t_2$, the Cauchy-Schwarz inequality implies that

$$E[(Z_{t_2}^n - Z_t^n)^2 (Z_t^n - Z_{t_1}^n)^2] \le C\Big(\frac{[nt_2]-[nt]}{n}\Big)\Big(\frac{[nt]-[nt_1]}{n}\Big) \le C(t_2 - t_1).$$

The tightness of (G_t, Z_t^n) follows now by Theorem 15.6 in [13]. □

Step 2: Finally, we need to prove the convergence of finite dimensional distributions of (G_t, Z_t^n). Define the vector $Y_n = (Y_n^1, \ldots, Y_n^d)^T$ by

$$Y_n^k = \frac{1}{\sqrt{n}} \sum_{i=[na_k]+1}^{[nb_k]} H\Big(\frac{\Delta_i^n G}{\tau_n}\Big)\ , \tag{7.10}$$

where $(a_k, b_k]$, $k = 1, \ldots, d$, are disjoint intervals contained in $[0, T]$. Clearly, it suffices to prove that

$$\Big(G_{b_k} - G_{a_k}, Y_n^k\Big)_{1\leq k\leq d} \xrightarrow{\mathcal{D}} \Big(G_{b_k} - G_{a_k}, \tau(W_{b_k} - W_{a_k})\Big)_{1\leq k\leq d},$$

where τ is given by (5.4).

Next, we want to apply Theorem 5. Let $\mathcal{H}_1$ be the first Wiener chaos associated with the triangular array $(\Delta_j^n G/\tau_n)_{n\geq 1, 1\leq j\leq [nt]}$, i.e the closed subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by the random variables $(\Delta_j^n G/\tau_n)_{n\geq 1, 1\leq j\leq [nt]}$. Notice that $\mathcal{H}_1$ can be seen as a separable Hilbert space with a scalar product induced by the covariance function of the process $(\Delta_j^n G/\tau_n)_{n\geq 1, 1\leq j\leq [nt]}$. This means we can apply the theory of Section 4 with the canonical Hilbert space $I\!H = \mathcal{H}_1$. Denote by $\mathcal{H}_m$ the mth Wiener chaos associated with the triangular array $(\Delta_j^n G/\tau_n)_{n\geq 1, 1\leq j\leq [nt]}$ and by I_m the corresponding linear isometry between the symmetric tensor product $\mathcal{H}_1^{\odot m}$ (equipped with the norm $\sqrt{m!}\,\|\cdot\|_{\mathcal{H}_1^{\otimes m}}$) and the mth Wiener chaos. Finally, we will denote by J_m the projection operator on the mth Wiener chaos.

Since

$$E[(G_{b_k} - G_{a_k})Y_n^l] = 0$$

for any $1 \leq k, l \leq d$ (because H is an even function), it is sufficient to check the following conditions.

(i) For any $m \geq 1$ and $k = 1, \ldots, d$, the limit $\lim_{n\to\infty} E[|J_m Y_n^k|^2] = \tau_{m,k}^2$ exists and $\sum_{m=1}^{\infty} \sup_n E[|J_m Y_n^k|^2] < \infty$,

(ii) For any $m \geq 1$ and $k \neq h$, $\lim_{n\to\infty} E[J_m Y_n^k J_m Y_n^h] = 0$,

(iii) For any $m \geq 1$, $k = 1, \ldots, d$ and $1 \leq p \leq m-1$, we have that

$$\lim_{n\to\infty} I_m^{-1} J_m Y_n^k \otimes_p I_m^{-1} J_m Y_n^k = 0.$$

Under conditions (i)-(iii) we then obtain (by Theorem 5) the central limit theorem

$$Y_n \xrightarrow{\mathcal{D}} N_d\Big(0, \tau^2 \mathrm{diag}(b_1 - a_1, \ldots, b_d - a_d)\Big), \tag{7.11}$$

where τ^2 is given by (5.4). Since the increments of the process G are stationary we will prove part (i) and (iii) only for $k = 1$, $a_1 = 0$ and $b_1 = 1$.

(i) We have

$$J_m Y_n^1 = \frac{a_m}{\sqrt{n}} \sum_{i=1}^{n} H_m\Big(\frac{\Delta_i^n G}{\tau_n}\Big).$$

Hence, we obtain (see (7.4))

$$E[|J_m Y_n^1|^2] = m! a_m^2 \Big(1 + 2\sum_{i=1}^{n-1} \frac{n-i}{n} r_n^m(i)\Big)$$

By part (iv) of Lemma 1 we deduce that

$$\lim_{n\to\infty} E[|J_m Y_n^1|^2] = m! a_m^2 \Big(1 + 2\sum_{i=1}^{\infty} \rho^m(i)\Big) ,$$

and

$$\sum_{m=2}^{\infty} \sup_n E[|J_m Y_n^1|^2] < \infty.$$

Furthermore, we obtain that

$$\lim_{n\to\infty} E\Big[\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} H\Big(\frac{\Delta_i^n G}{\tau_n}\Big)\Big)^2\Big] = \lim_{n\to\infty}\sum_{m=2}^{\infty} E[|J_m Y_n^1|^2] = \tau^2.$$

(ii) For any $1 \le k, h \le d$ with $b_k \le a_h$ we have

$$E[J_m Y_n^k J_m Y_n^h] = \frac{m! a_m^2}{n} \sum_{j=[na_k]+1}^{[nb_k]} \sum_{i=[na_h]+1}^{[nb_h]} r_n^m(i-j).$$

Assume w.l.o.g. that $a_k = 0$, $b_k = a_h = 1$ and $b_h = 2$ (the case $b_k < a_h$ is much easier). By part (ii) of Lemma 1 with $0 < \epsilon < \frac{3}{2} - \beta$ in the definition of r (see (7.3)) we obtain the approximation

$$\Big|E[J_m Y_n^k J_m Y_n^h]\Big| \le m! a_m^2 \Big(\frac{1}{n}\sum_{j=1}^{n} j r^m(j) + \sum_{j=1}^{n-1} r^m(n+j)\Big).$$

It follows that $r^m(j) \le (j-1)^{-1-\delta}$ for some $\delta > 0$ and for all $m, j \ge 2$. Hence, we obtain

$$E[J_m Y_n^k J_m Y_n^h] \to 0$$

as $n \to \infty$.

(iii) Fix $1 \le p \le m-1$. We obtain the identity

$$\begin{aligned} I_m^{-1} J_m Y_n^1 \widetilde{\otimes}_p I_m^{-1} J_m Y_n^1 &= \frac{1}{n} \sum_{1\le j,i\le n} \Big(\frac{\Delta_j^n G}{\tau_n}\Big)^{\otimes m} \widetilde{\otimes}_p \Big(\frac{\Delta_i^n G}{\tau_n}\Big)^{\otimes m} \\ &= \frac{1}{n} \sum_{1\le j,i\le n} r_n^p(|j-i|) \Big(\Big(\frac{\Delta_j^n G}{\tau_n}\Big)^{\otimes(m-p)} \widetilde{\otimes} \Big(\frac{\Delta_i^n G}{\tau_n}\Big)^{\otimes(m-p)}\Big), \end{aligned}$$

where "$\sim$" denotes the symmetrization. Consequently, we need to prove that the quantity

$$n^{-2} \sum_{1\le j,l,h,k\le n} r_n^p(|j-l|) r_n^p(|h-k|)$$
$$\times \Big\langle \Big(\frac{\Delta_j^n G}{\tau_n}\Big)^{\otimes(m-p)} \widetilde{\otimes} \Big(\frac{\Delta_l^n G}{\tau_n}\Big)^{\otimes(m-p)}, \Big(\frac{\Delta_h^n G}{\tau_n}\Big)^{\otimes(m-p)} \widetilde{\otimes} \Big(\frac{\Delta_k^n G}{\tau_n}\Big)^{\otimes(m-p)} \Big\rangle_{\mathcal{H}_1^{\otimes 2(m-p)}}$$

converges to zero as $n \to \infty$. It suffices to consider a term of the form

$$n^{-2} \sum_{1 \le j,l,h,k \le n} r_n^p(|j-l|) r_n^p(|h-k|)$$
$$\times r_n^\alpha(|j-h|) r_n^{m-p-\alpha}(|l-h|) r_n^{m-p-\alpha}(|j-k|) r_n^\alpha(|l-k|),$$

where $0 \le \alpha \le m-p$. The latter term is smaller than

$$n^{-1} \sum_{0 \le j,l,k \le n-1} r_n^p(|j-l|) r_n^p(k) r_n^\alpha(j) r_n^{m-p-\alpha}(l) r_n^{m-p-\alpha}(|j-k|) r_n^\alpha(|l-k|).$$

Without any loss of generality we can assume that $p = m-p = 1$ and $\alpha = 0$ or $\alpha = 1$. For $\alpha = 0$ and any $0 < \varepsilon < 1$ we get

$$n^{-1} \sum_{0 \le j \le n-1} \Big(\sum_{0 \le l \le n-1} r_n(|j-l|) r_n(l) \Big)^2 \le n^{-1} \sum_{0 \le j \le [n\varepsilon]} \Big(\sum_{0 \le l \le n-1} r_n(|j-l|) r_n(l) \Big)^2$$
$$+2n^{-1} \sum_{[n\varepsilon] < j \le n-1} \Big(\sum_{0 \le l \le [n\varepsilon/2]} r_n(|j-l|) r_n(l) \Big)^2 + 2n^{-1} \sum_{[n\varepsilon] < j \le n-1} \Big(\sum_{[n\varepsilon/2] < l \le n-1} r_n(|j-l|) r_n(l) \Big)^2$$
$$\le 2\varepsilon \Big(\sum_{0 \le l < n-1} r_n(l)^2 \Big)^2 + 6 \sum_{0 \le l < n-1} r_n(l)^2 \sum_{[n\varepsilon/2] < l < \infty} r_n(l)^2$$

which converges to $2\varepsilon \left(\sum_{0 \le l < \infty} \rho^2(l) \right)^2$ as $n \to \infty$ by Lemma 1. The desired result follows by letting ε tend to zero. This completes the proof of Theorem 6. □

Proof of Theorem 7: Theorem 7 is deduced from Theorem 6 by the same methods as presented in [15] (see Theorem 4 therein).

For any $m \ge n$ we obtain the decomposition

$$\sqrt{m}\Big(V(X,p)_t^m - \mu_p \int_0^t |\phi_s|^p \, ds \Big) = \sqrt{m}(A_t^{(m)} + \tilde{B}_t^{(n,m)} + C_t^{(n,m)} + D_t^{(m)}) \ ,$$

where $A_t^{(m)}$, $C_t^{(n,m)}$ and $D_t^{(m)}$ are defined in (7.8) and $\tilde{B}_t^{(n,m)}$ is given by

$$\begin{aligned} \tilde{B}_t^{(n,m)} &= \frac{1}{m\tau_m^p} \sum_{i=1}^{[mt]} |\phi_{\frac{i-1}{m}} \Delta_i^m G|^p - \mu_p m^{-1} \sum_{i=1}^{[mt]} |\phi_{\frac{i-1}{m}}|^p \\ &- \frac{1}{m\tau_m^p} \sum_{j=1}^{[nt]} |\phi_{\frac{j-1}{n}}|^p \sum_{i \in I_n(j)} |\Delta_i^m G|^p + \mu_p n^{-1} \sum_{j=1}^{[nt]} |\phi_{\frac{j-1}{n}}|^p. \end{aligned}$$

We first prove the stable convergence for the term $\sqrt{m} C_t^{(n,m)}$. Define

$$Y_{n,m}^j = \frac{1}{\sqrt{m}\tau_m^p} \sum_{i \in I_n(j)} |\Delta_i^m G|^p - \frac{\sqrt{m}}{n} \mu_p.$$

For any fixed n, we obtain by Theorem 6 and the properties of stable convergence

$$\Big(|\phi_{\frac{j-1}{n}}|^p, Y^j_{n,m}\Big)_{1\le j\le[nt]} \stackrel{\mathcal{F}^G-st}{\longrightarrow} \Big(|\phi_{\frac{j-1}{n}}|^p, \tau\Delta^n_j W\Big)_{1\le j\le[nt]}$$

as $m\to\infty$. Hence,

$$\sqrt{m}C^{(n,m)}_t \stackrel{\mathcal{F}^G-st}{\longrightarrow} \tau\sum_{j=1}^{[nt]}|\phi_{\frac{j-1}{n}}|^p\Delta^n_j W.$$

For the latter we have

$$\tau\sum_{j=1}^{[nt]}|\phi_{\frac{j-1}{n}}|^p\Delta^n_j W \stackrel{ucp}{\longrightarrow} \tau\int_0^t |\phi_s|^p dW_s.$$

Now, we show that the other terms are negligible. Recalling that ϕ is Hölder continuous of order a we obtain the inequality

$$\begin{aligned}
\sqrt{m}\sup_{0\le t\le T}|D^{(m)}_t| &\le \frac{\mu_p}{\sqrt{m}}\Big(\sup_{0\le t\le T}|\phi_t|^p + \sum_{j=1}^{[mT]}\big||\phi_{\frac{j-1}{m}}|^p - |\phi_{\tilde{t}^m_{j-1}}|^p\big|\Big)\\
&\le \frac{\mu_p}{\sqrt{m}}\Big(\sup_{0\le t\le T}|\phi_t|^p + (p\vee 1)\sup_{0\le t\le T}|\phi_t|^{(p-1)_+}\sum_{j=1}^{[mT]}|\phi_{\frac{j-1}{m}} - \phi_{\tilde{t}^m_{j-1}}|^{p\wedge 1}\Big)\\
&\le \frac{\mu_p}{\sqrt{m}}\sup_{0\le t\le T}|\phi_t|^p + \mu_p T(p\vee 1)||\phi||_a^{p\wedge 1}\sup_{0\le t\le T}|\phi_t|^{(p-1)_+}m^{1/2-a(p\wedge 1)}\ ,
\end{aligned}$$

where $\tilde{t}^m_{j-1}\in(\frac{j-1}{m},\frac{j}{m})$. Hence

$$\sqrt{m}\sup_{0\le t\le T}|D^{(m)}_t| \stackrel{P}{\longrightarrow} 0\ ,$$

because $a(p\wedge 1) > \frac{1}{2}$.

For the term $\sqrt{m}\tilde{B}^{(n,m)}_t$ we obtain the inequality

$$\begin{aligned}
\sqrt{m}|\tilde{B}^{(n,m)}_t| &= \Big|\sum_{j=1}^{[nt]}\sum_{i\in I_n(j)}|\phi_{\frac{i-1}{m}}|^p\Big(\frac{1}{\sqrt{m}\tau^p_m}|\Delta^m_i G|^p - \frac{\mu_p}{\sqrt{m}}\Big)\\
&- \sum_{j=1}^{[nt]}|\phi_{\frac{j-1}{n}}|^p\Big(\frac{1}{\sqrt{m}\tau^p_m}\sum_{i\in I_n(j)}|\Delta^m_i G|^p - \frac{\sqrt{m}}{n}\mu_p\Big) + \sum_{i\ge\frac{m}{n}[nt]}^{[mt]}|\phi_{\frac{i-1}{m}}|^p\Big(\frac{1}{\sqrt{m}\tau^p_m}|\Delta^m_i G|^p - \frac{\mu_p}{\sqrt{m}}\Big)\Big|\\
&\le \Big|\sum_{j=1}^{[nt]}|\phi_{\tilde{s}}|^p\sum_{i\in I_n(j)}\Big(\frac{1}{\sqrt{m}\tau^p_m}|\Delta^m_i G|^p - \frac{\mu_p}{\sqrt{m}}\Big) - \sum_{j=1}^{[nt]}|\phi_{\frac{j-1}{n}}|^p\Big(\frac{1}{\sqrt{m}\tau^p_m}\sum_{i\in I_n(j)}|\Delta^m_i G|^p - \frac{\sqrt{m}}{n}\mu_p\Big)\Big|\\
&+ \sup_{0\le t\le T}\sum_{i\ge\frac{m}{n}[nt]}^{[mt]}\Big||\phi_{\frac{i-1}{m}}|^p\Big(\frac{1}{\sqrt{m}\tau^p_m}|\Delta^m_i G|^p - \frac{\mu_p}{\sqrt{m}}\Big)\Big|
\end{aligned}$$

$$\leq \sum_{j=1}^{[nT]} \sup_{s\in(\frac{j-2}{n},\frac{j}{n}]} ||\phi_s|^p - |\phi_{\frac{j-1}{n}}|^p||Y_{n,m}^j| + \frac{\mu_p}{\sqrt{m}} \sup_{0\leq t\leq T} |\phi_t|^p$$

$$+ \sup_{0\leq t\leq T} \Big| \sum_{i\geq \frac{m}{n}[nt]}^{[mt]} |\phi_{\frac{i-1}{m}}|^p \Big(\frac{1}{\sqrt{m}\tau_m^p} |\Delta_i^m G|^p - \frac{\mu_p}{\sqrt{m}} \Big) \Big| \; ,$$

where $\tilde{s} \in (\frac{j-2}{n}, \frac{j}{n}]$. Then, by Theorem 6, we obtain

$$\limsup_{m\to\infty} P(\sqrt{m} \sup_{0\leq t\leq T} |\tilde{B}_t^{(n,m)}| > \epsilon) \leq P\Big(\tau \sum_{j=1}^{[nT]} \sup_{s\in(\frac{j-2}{n},\frac{j}{n}]} ||\phi_s|^p - |\phi_{\frac{j-1}{n}}|^p||\Delta_j^n W|$$

$$\frac{\tau}{n} \sup_{0\leq t\leq T} |\phi_t|^p \sup_{0\leq t\leq T} |W_t - W_{[nt]/n}| > \epsilon\Big)$$

for any $\epsilon > 0$. Since ϕ is Hölder continuous of order a with $a(p\wedge 1) > 1/2$ it holds, for any $\delta > 0$, that

$$\sum_{j=1}^{[nT]} \sup_{s\in(\frac{j-2}{n},\frac{j}{n}]} ||\phi_s|^p - |\phi_{\frac{j-1}{n}}|^p||\Delta_j^n W| \leq (p\vee 1)C||\phi||_a^{(p\wedge 1)} \sup_{0\leq t\leq T} |\phi_t|^{(p-1)_+} n^{-a(p\wedge 1)+1/2+\delta} \; ,$$

which converges to 0 as $n\to\infty$ if δ is small enough. This implies that

$$\lim_{n\to\infty} \limsup_{m\to\infty} P(\sqrt{m} \sup_{0\leq t\leq T} |\tilde{B}_t^{(n,m)}| > \epsilon).$$

Finally, let us show that $\sqrt{m}\sup_{0\leq t\leq T} |A_t^{(m)}| \xrightarrow{P} 0$. We have

$$\sqrt{m}|A_t^{(m)}| \leq \frac{1}{\sqrt{m}\tau_m^p}(p\vee 1)2^{(p-2)_+} \sum_{j=1}^{[mt]} |\phi_{\frac{j-1}{m}} \Delta_j^m G|^{(p-1)_+} \Big|\Delta_j^m X - \phi_{\frac{j-1}{m}} \Delta_j^m G\Big|^{p\wedge 1}$$

$$+\frac{1}{\sqrt{m}\tau_m^p}(p\vee 1)2^{(p-2)_+} \sum_{j=1}^{[mt]} \Big|\Delta_j^m X - \phi_{\frac{j-1}{m}} \Delta_j^m G\Big|^p.$$

By (2.9) and Young's inequality we deduce (as in Theorem 2)

$$\sqrt{m} \sup_{0\leq t\leq T} |A_t^{(m)}| \leq \frac{C}{\sqrt{m}\tau_m^p} \Big(m^{-(\frac{\beta}{2}-\epsilon)(p-1)_+} \sup_{0\leq t\leq T} |\phi_t|^{(p-1)_+}$$

$$\times \sum_{j=1}^{[mT]} \Big|\mathrm{var}_{1/a}\Big(\phi; \Big(\frac{j-1}{m}, \frac{j}{m}\Big]\Big) \mathrm{var}_{1/(\frac{\beta}{2}-\epsilon)}\Big(G; \Big(\frac{j-1}{m}, \frac{j}{m}\Big]\Big)\Big|^{p\wedge 1}$$

$$+ \sum_{j=1}^{[mT]} \Big|\mathrm{var}_{1/a}\Big(\phi; \Big(\frac{j-1}{m}, \frac{j}{m}\Big]\Big) \mathrm{var}_{1/(\frac{\beta}{2}-\epsilon)}\Big(G; \Big(\frac{j-1}{m}, \frac{j}{m}\Big]\Big)\Big|^{p}\Big)$$

$$\leq \frac{C}{\sqrt{m}\tau_m^p} \Big(m^{-(\frac{\beta}{2}-\epsilon)(p-1)_+ - (p\wedge 1)(\frac{\beta}{2}-\epsilon+a)+1} \sup_{0\leq t\leq T} |\phi_t|^{(p-1)_+} + m^{-p(\frac{\beta}{2}-\epsilon+a)+1}\Big)$$

which converges to 0 as $m\to\infty$, provided $\epsilon < p^{-1}(a(p\wedge 1) - \frac{1}{2})$. This completes the proof. □

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