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Structural estimation of jump-diffusion processes in macroeconomics

Olaf Posch

School of Economics and Management
University of Aarhus
Building 1322, DK-8000 Aarhus C
Denmark

Aarhus School of Business
University of Aarhus

Handelshøjskolen
Aarhus Universitet

UNIVERSITY OF COPENHAGEN

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Olaf Posch*
University of Aarhus and CREATES

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Abstract

Understanding the process of economic growth involves comparing competing theoretical models and evaluating their empirical relevance. Our approach is to take the neoclassical stochastic growth model directly to the data and make inferences about the model parameters of interest. In this paper, output follows a jump-diffusion process. By imposing parameter restrictions we derive two solutions in explicit form. Based on them, we obtain transition densities in closed form and employ maximum likelihood techniques to estimate the model parameters. In extensive Monte Carlo simulations we demonstrate that population parameters of the underlying data generating process can be recovered. We find empirical evidence for jumps in monthly and quarterly data on industrial production for the UK, the US, Germany, and the euro area (Euro12).

JEL classification: C13; E32; O40

Keywords: Jump-diffusion estimation; Stochastic growth; Closed form solutions

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*University of Aarhus, School of Economics and Management, Aarhus, Denmark (oposch@econ.au.dk).
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1 Introduction

Research on technological change and economic growth has been always on top of the research agenda. A better understanding involves comparing competing theoretical approaches and evaluating their empirical relevance.

From a theoretical perspective, discrete changes in factor productivity - or jumps - are the key ingredients in macro models where growth and fluctuations can emerge endogenously (cyclical growth models in Bental and Peled 1996, Matsuyama 1999, Francois and Lloyd-Ellis 2003, Wälde 2005). In these models, output volatility is something natural, inherently linked to a country's growth process. This is in sharp contrast to the literature with steady technological improvements at the aggregate level (acyclical models à la Romer 1990).[1] While output growth emerges either through accumulation of knowledge or product variety, all models of the latter type share the view of a balanced growth path. An econometric model therefore should account for both, a steady accumulation and for jumps in technology.

From an econometric perspective, often jumps are introduced in finance and financial econometrics to account for empirical features of the data which would not be captured by pure diffusions (among others Das 2002, Johannes 2004). Statistically, the presence of jumps would imply that diffusion models are misspecified. Macroeconomic series of growth rates also demonstrate considerable non-normalities such as skewness and excess kurtosis which makes a case for jump models.

Although theoretical contributions as well as empirical features support the presence of jumps in growth rates, an empirical investigation or econometric validation of growing through cycles models in macro is pending. In this line of research, the empirical question should be to what extent the growth process actually is driven by discrete innovations. Fortunately, the literature on estimating jump-diffusions is large, and the methods currently widely used in financial economics/econometrics (recent contributions include Bandi and Nguyen 2003, Johannes 2004, Aït-Sahalia 2004).

The main contribution of this paper is the structural estimation of parameters of the underlying stochastic processes. Our approach is to take the continuous-time stochastic macro model directly to the data and make inferences about the model parameters of interest. Using continuous time allows to solve a much broader class of interesting models and enables us to handle stochastic dynamic models easily without linearization and certainty analysis.[2] For this purpose, the stochastic neoclassical growth model seems to be a natural

[1] In Francois and Lloyd-Ellis (2003), the outcome is either cyclically or acyclically.

[2] The is a tradition of stochastic models in continuous-time macroeconomics (starting with Brock and Mirman 1972, Merton 1975, Eaton 1981). An introduction can be found in Turnovsky (2000). Moreover, examples of stochastic processes in economics provide Dixit and Pindyck (1994) and Wälde (2006).

first choice. In this paper, output follows a jump-diffusion process. In that view, we extend diffusion models (Merton 1975), allowing for jumps in productivity. By imposing parameter restrictions, we obtain two solutions in explicit form.

As a starting point, we abstract from the endogeneity of jumps as in the cyclical growth models. Nonetheless, our econometric model incorporates the idea of discrete changes by introducing an exogenous jump process. It therefore captures the dynamics of intrinsically different models, as the interpretations of the impulses are multi-facetted. For example, while in Wälde (2005) a new technology arrives in form of a prototype machine opening the possibility of accumulating better capital goods, in Francois and Lloyd-Ellis (2003) the joint implementation of productivity improvements across many sectors cause jumps in aggregate technology. Nevertheless, both models have in common that discrete changes are rare events. Similarly, the idea of steady improvements in technology (or variety) of Romer type models is captured by an exogenous deterministic drift component.

For estimation of the model parameters of interest, we make use of output growth rates. In this paper, our focus is on (i) approximate methods as well as (ii) exact methods. Both approaches make use of the Markov property of original or 'modified' output growth rates, respectively. As a drawback, the latter approach demands the growth rate of the capital stock to be observable. Based on explicit solutions, we obtain transition densities in closed form and employ maximum likelihood (ML) estimation. Standard likelihood ratio tests are used to test for the presence of jumps. In extensive simulation studies we demonstrate that population parameters of the underlying data generating process indeed can be recovered. Using data on monthly industrial production we find strong empirical evidence for jumps and obtain reasonable parameter estimates.

Beside there is a tradition in macro estimating continuous-time models (Phillips 1972, 1974, Hansen and Sargent 1980) at least two apparent questions emerge. First, how can we estimate continuous-time macro models given the stochastic process is observed only at discrete dates? Second, with discretely sampled data all changes are jumps: how to distinguish small jumps resulting from the Brownian noise from large jumps of rare technological improvements? In recent years, new methods emerged in the financial literature to estimate continuous-time models using discrete-time observations (among others Aït-Sahalia 2002b). Moreover, Aït-Sahalia (2004) demonstrates that it is possible to disentangle Brownian noise from jumps even if the jump process exhibits an infinite number of small jumps in any finite time interval. Our Monte Carlo experiments demonstrate that these estimation methods can be applied to macroeconomics and macro data as well.

A related popular research field is the (discrete-time) real business cycle model (Long and Plosser 1983) and its various extensions. Using the traditional approach, one considers a stationary system of equations and analyzes linearized transitional dynamics around some non-stochastic steady state (see King et al. 1988, Uhlig 1995, Ireland 2004). In this paper we show that particularly with asymmetric shocks, one should avoid linearizing accordingly since non-stochastic steady-state values do not coincide with their mean values.

The remainder of the paper is structured as follows: In Section 2, we set out the model used for both simulations and estimations. Section 3 focuses on the estimation strategy. In Section 4, we report the results of Monte Carlo experiments, while we present the empirical results in Section 5. Some concluding remarks are in Section 6.

2 The model

We consider a standard stochastic continuous-time neoclassical model.

2.1 Technology and households

Production possibilities. The single production good is produced according to a standard Cobb-Douglas function,

$$Y_t = A_t K_t^{\alpha}(X_t L)^{1-\alpha}, \tag{1}$$

where L denotes total constant labor supply. In the tradition of standard macro models (King et al. 1988), A_t denotes total factor productivity and X_t labor augmenting technology. Output Y_t is used for producing consumption goods C_t and investment goods I_t. Aggregate capital stock increases if gross investment I_t exceeds depreciation δK_t,

$$dK_t = (I_t - \delta K_t)dt. \tag{2}$$

To keep the model simple, technological progress is exogenous. Motivated by competing growth theories, uncertainty enters via two independent stochastic processes: a (geometric) diffusion with drift, A_t, driven by a standard Brownian motion z_t, and a (geometric) jump process, X_t, driven by a standard Poisson process q_t,

$$\begin{aligned} dA_t &= \mu A_t dt + \eta A_t dz_t, & (3)\\ dX_t &= \left((\exp(J_t))^{\frac{1}{1-\alpha}} - 1\right) X_{t-} dq_t, & (4) \end{aligned}$$

respectively.[3] Both stochastic processes are homogenous linear SDEs (multiplicative noise) with explicit solutions (Kloeden and Platen 1999), and belong to the family of Lévy processes

[3]Note that the standard Poisson process q_t can either be zero or one with mean and variance λt. Since z_t is a standard Brownian motion, $z_0 = 0$, $z_{t+\Delta} - z_t \sim \mathcal{N}(0, \Delta)$, $t \in [0, \infty[, \Delta > 0$.

with stationary and independent increments.[4] We model the jump size proportional to its value an instant before the jump, X_{t-}, while the independent random variable J_t with constant mean ν and variance γ specifies the jump size distribution.

Preferences. The economy is populated by a large number of infinitely-lived identical individuals, each sufficiently small to neglect effects on aggregate variables. Each consumer maximizes expected utility, U_0, given by the integral over instantaneous utilities, u, resulting from consumption flows, c_t, discounted at the rate of time preference, ρ,

$$U_0 = E_0 \int_0^\infty e^{-\rho t} u(c_t) dt, \tag{5}$$

where instantaneous utility is characterized by constant relative risk aversion,

$$u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}, \quad \sigma > 0. \tag{6}$$

The standard budget constraint of the representative household reads

$$da_t = ((r_t - \delta)a_t + w_t - c_t)dt, \tag{7}$$

where r_t denotes the rental rate of capital and w_t the real wage rate, respectively.

Equilibrium properties. In equilibrium, factors are rewarded by $w_t = Y_L$, and $r_t = Y_K$ (value marginal product), respectively. The market clearing condition demands

$$Y_t = C_t + I_t. \tag{8}$$

Solving the model requires the first order condition for consumption, the aggregate capital accumulation constraint (2), the goods market equilibrium (8), and optimality conditions of perfectly competitive firms. Thus we obtain a system of differential equations determining, given initial conditions, the time paths of C_t, K_t, Y_t, as well as of w_t and r_t.

2.2 Explicit solutions

The traditional approach is to consider a stationary system of equations and linearize the system to analyze transitional dynamics often around the non-stochastic steady state (e.g. Uhlig 1995). In this paper we restrict ourself to particular parameter restrictions under which the model has explicit solutions. Thus, simulations can be done without relying on numerical methods, linearization or certainty analysis.[5] Given a realization of stochastic

[4]An extension to multiple jump and diffusion terms poses no conceptional difficulty but is notationally more cumbersome. In that, the aggregate jump process q_t can also be thought of the sum of several poisson processes $q_t = \sum_i q_{t,i}$ with arrival rates λ_i (for a multisector version see also Wälde 2005).

[5]Since most of the literature neglect uncertainty, care must be taken in using certainty analysis even as a first-order approximation theory as the first-moment of the steady-state distribution does not equal the certainty estimate (Merton 1975, p.382).

processes, the complete equilibrium path can easily be computed. Most notably, we can directly take the model to the data in order to estimate the parameters of interest.

Applying Itô's formula (or change of variables formula, cf. Protter 2004, Sennewald 2007), the assumed technology in (1) implies that output evolves according to

$$\begin{aligned} dY_t &= Y_A dA_t + (Y_t - Y_{t-})\, dq_t + Y_K dK_t, \\ &= (\mu + \alpha(dK_t/dt)/K_t)\, Y_t dt + \eta Y_t dz_t + (\exp(J_t) - 1) Y_{t-} dq_t. \end{aligned} \tag{9}$$

It describes a stochastic differential equation (SDE), more precisely a jump-diffusion process which, for solving, demands more information about the behavior of households. In that the instantaneously growth rate of the capital stock is determined by the consumers. The effects of shocks or impulses will be propagated contemporaneously as well as inter-temporally via capital accumulation. In the absence of shocks, standard textbooks show that this economy would converge to a balanced growth path (Barro and Sala-i-Martín 2003). Along such non-stochastic path $(dK_t/dt)/K_t = \frac{1}{1-\alpha}\mu$ is constant (see Appendix A.6). Below we use this result to generalize the econometric approach.

As it is well known from deterministic continuous-time models, at least for two cases we obtain unique explicit analytical solutions. In both cases log-output is a linear SDE in the *narrow sense* (additive noise). Note that restrictions on the parameter range are widely used in economics to study explicit dynamics (among others Xie 1991). We focus on the $\alpha = \sigma$ solution in the text and refer another solution to the appendix.[6]

Theorem 2.1 *If the output elasticity of the capital stock equals the parameter of the utility function, $\alpha = \sigma$, consumption is a linear function of the capital stock, $C_t = \phi K_t$, where*

$$\phi = \frac{\rho + (1-\sigma)\delta}{\sigma}. \tag{10}$$

Proof. Appendix A.7. ■

Corollary 2.2 *The rental rate of capital follows a reducible SDE,*

$$dr_t = c_1 r_t (c_2 - r_t)\, dt + \eta r_t dz_t + (\exp(J_t) - 1) r_{t-} dq_t, \tag{11}$$

where we defined $c_1 \equiv \frac{1-\alpha}{\alpha}$, and $c_2 \equiv \frac{\alpha}{1-\alpha}\mu + \rho + \delta$.

[6] The parameter restriction $\alpha = \sigma$ implies a relatively high inter-temporal elasticity of substitution σ^{-1}. While there is supporting empirical evidence (Vissing-Jørgensen 2002, Gruber 2006), our results are not intrinsically tied to this assumption. As shown in Appendix A.9, another solution with $\sigma > 1$ gives the same structure of the SDE for the rental rate of capital as well as for the growth rate of output.

The SDE in (11) is a geometric reverting jump-diffusion process known as stochastic Verhulst equation (Sørensen 1991, p.97). Most notably, since (11) denotes a reducible SDE (with polynomial drift $n = 2$), its solution reads (cf. Theorem A.1),

$$r_t = \Theta_t \left(r_0^{-1} + c_1 \int_0^t \Theta_s ds \right)^{-1} \tag{12}$$

with

$$\Theta_s = \exp\left(\left(c_1 c_2 - \frac{1}{2}\eta^2 \right) s + \eta z_s + \int_0^s J_s dq_s \right).$$

Similar reverting processes are well studied and widely applied in financial economics to model the short-rate and the behavior of prices (as in Sundaresan 1984, Metcalf and Hassett 1995). Accordingly, c_2 defines the non-stochastic steady state or tendency parameter to which r_t reverts, and c_1 is the speed of reversion. Note that capital rewards, r_t, cannot become negative: a jump induces a discontinuity in the sample path whose size depends on its value an instant before the jump, $r_t = \exp(J_t) r_{t_-}$, while between jumps r_t grows logistically augmented by lognormal distributed noise.

Corollary 2.3 *The growth rate of output per unit of time, g_Δ, reads*

$$g_\Delta = \Delta \left(\mu - \rho - \delta - \frac{1}{2}\eta^2 \right) + \int_{t-\Delta}^t r_s ds + \eta(z_t - z_{t-\Delta}) + \int_{t-\Delta}^t J_s dq_s. \tag{13}$$

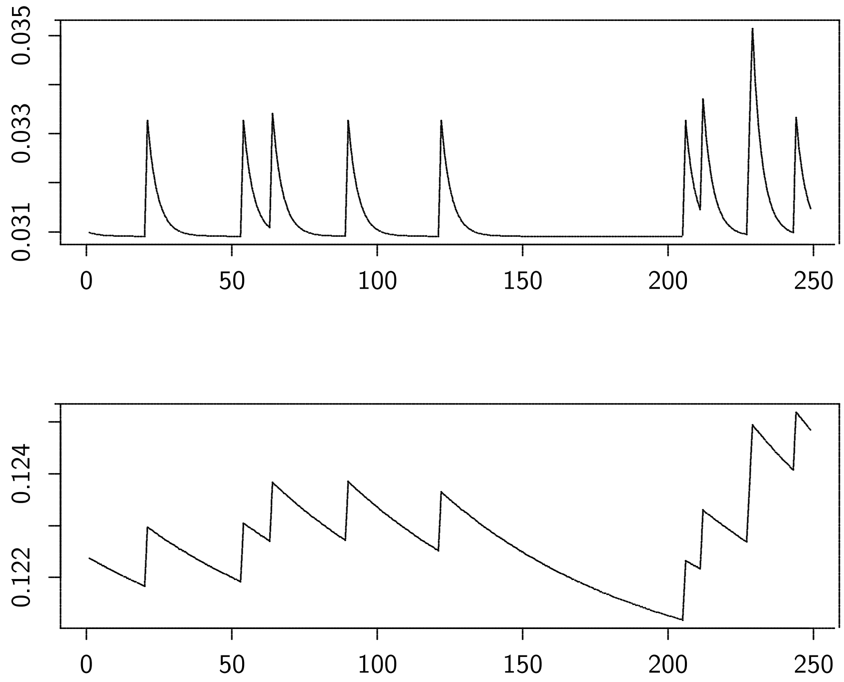
Intuitively, the growth rate consists of an obvious deterministic part, the integral over the rental rate of capital which propagates impulses inter-temporally, the Brownian motion component, and the integral over the jump size distribution per unit of time.

2.3 Theory meets empirical facts

A stylized fact in the literature is that output growth is positively autocorrelated over short horizons and has weak and possibly insignificant negative autocorrelation over longer horizons (Cochrane 1988). An often addressed critique to RBC models is that if they rely on capital accumulation and inter-temporal substitution only, as in this paper, fail to reproduce the positive autocorrelation pattern (Cogley and Nason 1995). Below we show that the degree of autocorrelation in neoclassical models crucially depends on the speed of reversion of capital rewards. Because estimates of the underlying parameter of risk aversion range substantially (cf. Gruber 2006), the proposition that neoclassical models suffer from weak internal propagation mechanism does not hold in general.[7]

[7] Others suggest to rely on external sources of dynamics, e.g. various kinds of adjustment lags or costs, temporal aggregation, correlated increments to total factor productivity, and higher-order autoregressive representations for transitory shocks (Cogley and Nason 1995).

Figure 1: Speed of reversion and the evolution of capital rewards.



Notes: This figure illustrates the evolution of capital rewards, r_t, in the case of high speed of reversion, $\alpha = \sigma = .1$ (upper panel), and low speed of reversion, $\alpha = \sigma = .9$ (lower panel). The parameters are $(\nu, \lambda, \eta, \gamma, \mu) = (.05, .05, 0, 0, .01)$ and $(\rho, \alpha, \sigma, \delta) = (.03, \cdot, \cdot, 0)$. Note that the non-stochastic equilibrium, c_2, does not necessarily reflect the the mean.

For illustration, we set the variance for the Brownian motion and jump-size distribution to zero, $(\eta, \gamma) = 0$, and simulate the evolution of the rental rate of capital for different values of $\alpha = \sigma$ which determines the speed of reversion in (11). Since the Poisson jumps are rare, the effect is easily visible (cf. Figure 1). This intuition helps to understand why positive autocorrelation and 'mean reversion' of growth rates in (13) could be observed. Summarizing, if the speed of reversion was high, the persistence in capital rewards is low. In contrast, a low speed of reversion implies that the rental rate of capital has a longer memory with output growth rates being positively autocorrelated over short horizons (with weak and insignificant negative coefficients over longer horizons). This exactly resembles the empirical observation for output (as in Cogley and Nason 1995). To proof our conjecture, we use a ceteris paribus analysis with parameters implying high speed of reversion. Now the autocorrelation pattern is similar to that of an unit root process (cf. Appendix A.10).

3 Estimation strategy

In this paper, we tie in with a tradition estimating continuous-time macro models starting with Phillips (1972). To estimate the model parameters of interest, we make use of output growth rates and the Markov property. This approach leads to a considerable simplification in the calculation of the likelihood function. We obtain the Markov property via approximations for the general case as well as exact for specific cases using the Solow residual. In subsequent sections we demonstrate that both methods are able to recover the parameters of the underlying stochastic processes. Using Monte Carlo experiments we illustrate the power of these estimation strategies to distinguish small jumps resulting from the Brownian noise from large jumps of rare technological improvements (Aït-Sahalia 2004).

Since we will be working with maximum likelihood techniques, the jump-size distribution, J_t, has to be fully specified. In this study we assume a binomial distribution, where in a lottery a positive jump in factor productivity (success), $\nu_s \geq 0$, occurs with probability p, and a negative jump (failure), $-\nu_f \leq 0$ occurs with probability $1-p$,

$$J_t = \begin{cases} \nu_s & \text{with} \quad p \\ -\nu_f & \text{with} \quad 1-p \end{cases} . \tag{14}$$

Although this assumption goes beyond growing through cycles models with (degenerated) positive jumps only, there is clear evidence of negative jumps in the data. A non-degenerated jump distribution therefore seems more plausible for estimation purposes. Nevertheless, the degenerated case is captured as a special case with $p = 1$. Below we report estimates based on both, the more empirically motivated binomial jump-size distribution, as well as on the degenerated case. It is straightforward to show that the assumption of constant mean ν and variance γ is fulfilled for both assumptions.[8]

3.1 Approximate methods

Basically, we assume that $(dK_t/dt)/K_t$ from (9) in general, or $\int_{t-\Delta}^{t} r_s ds$ from (13) in specific, are constants and thus we simply neglect autocorrelation in the data. Two scenarios are plausible. First, as in traditional approaches, we assume that (a) capital rewards are near to their non-stochastic equilibrium value (to which they tend to revert). Second, we consider (b) the variables near to their mean values. While the first assumption may imply inconsistent estimates, the second assumption assures consistency which therefore is the preferred strategy. Unfortunately, the mean value of capital rewards cannot be obtained in the general case but relies on our explicit solutions.

[8]Note that J_t is stationary with mean $\nu = \nu_s p - \nu_f(1-p)$ and variance $\gamma = (\nu_s + \nu_f)^2 p(1-p)$.

In either approach, growth rates can be approximated by (cf. Appendix B.1),

$$g_\Delta \;\approx\; \Delta\omega + \eta(z_t - z_{t-\Delta}) + \int_{t-\Delta}^{t} J_s dq_s, \tag{15}$$

where $\omega \in (\omega_1, \omega_2)$ is $\omega_1 \equiv \frac{\mu}{1-\alpha} - \frac{1}{2}\eta^2$ and $\omega_2 \equiv \omega_1 + \frac{\alpha}{1-\alpha}\left(\nu\lambda - \frac{1}{2}\eta^2\right)$, respectively. While ω_1 emerges from the first assumption (in the general case), ω_2 is implied by the second assumption (for both explicit solutions). Obviously, ω_2 denotes a refined approximation of $\int_{t-\Delta}^{t} r_s ds$ which accounts for the shift in the mean caused by (asymmetric) jumps as well as caused by the Jensen's inequality term weighted by the inverse of the speed of reversion.

It is remarkably that the equation (15) with ω_1 also emerges if we used another explicit solution (cf. Appendix A.9), using the assumption that capital rewards are at their non-stochastic steady state. Indeed, similarly we could assume variables for any given policy function either near their tendency values for stationary variables, or near their balanced growth paths of non-stationary variables, in the absence of shocks. As mentioned above, along such non-stochastic paths, $(dK_t/dt)/K_t = \frac{1}{1-\alpha}\mu$. Inserting in equation (9), solving and computing the output growth rate, we obtain approximate growth rates (15) with ω_1 also for the general case. Thus, although motivated by explicit solutions, the first approach can well be used to estimate the underlying parameters for the general model.

3.2 Exact methods

For solutions where the growth rate of the capital stock is known explicitly, the Markov property of output growth rates can be obtained exact.[9] Using Monte Carlo experiments as well as empirically, we therefore can study the power of approximate versus exact methods. Recalling that consumption is linear in the capital stock (Theorem 2.1), growth rates in (13) can be simplified further. In this case, $\int_{t-\Delta}^{t} r_s ds$ corresponds to the growth rate of consumption (cf. Appendix B.1.2), which is observable,

$$\alpha(\ln C_t - \ln C_{t-\Delta}) \;=\; -\Delta\left(\rho + \delta\right) + \int_{t-\Delta}^{t} r_s ds.$$

Simply speaking, it describes a 'discrete version' of the optimal consumption rule. We use this result to obtain the Markov property without imposing additional assumptions. For this we use an intuitive modification of original growth rates,

$$g_\Delta^{mod} \;\equiv\; g_\Delta - \alpha(\ln C_t - \ln C_{t-\Delta}) \;=\; \Delta\omega_3 + \eta(z_t - z_{t-\Delta}) + \int_{t-\Delta}^{t} J_s dq_s, \tag{16}$$

[9] A similar empirical approach is to construct a series of capital rewards and use this to compute $\int_{t-\Delta}^{t} r_s ds$.

where $\omega_3 \equiv \mu - \frac{1}{2}\eta^2$. Note that modified growth rates simply denote the growth rates of the Solow residuals: from (1), use $\ln SR_t = \ln A_t + (1-\alpha)\ln X_t$ and insert the solutions to the SDEs in (4) and (3), respectively. Hence, the log-differences, $\ln SR_t - \ln SR_{t-\Delta}$, are Markovian consisting of a deterministic drift component, $\Delta(\omega_3 + \lambda\nu)$, and white noise, $\eta(z_t - z_{t-\Delta}) + \int_{t-\Delta}^{t}(J_s - \nu)dq_s$. Intuitively, by subtracting consumption growth rates from original growth rates as in (16), we remove autocorrelation patterns or simply get rid of all movements in the data captured by the underlying model. What remains is a constant and white noise, or the proportional change of the Solow residual. In that sense, the approach can be understood as a reduced form of a more complex model using, after correcting for various effects, its Solow residuals for estimation. In contrast to approximative methods, all cyclical effects resulting from capital accumulation are now considered.

3.3 The distribution of output growth rates

For reading convenience we define

$$X_\Delta \equiv \Delta\omega + \eta(z_t - z_{t-\Delta}) + \int_0^\Delta J_s dq_s, \tag{17}$$

where $X_\Delta \in (g_\Delta, g_\Delta^{mod})$ either denotes growth rates in (15) or modified growth rates in (16), and $\omega \in (\omega_1, \omega_2, \omega_3)$. Given the assumed distribution in (14), we can define the parameter vector $\theta = (\nu_s, \nu_f, \lambda, \eta, \mu, p)'$, where ν_s and ν_f are the jump-sizes for positive and negative jumps (ν is the average size of jumps), λ the arrival rate of the Poisson process, η the standard deviation of the Brownian process, μ the drift of the Brownian process, and p the probability that a jump is of size ν_s. For the general estimation problem to be well defined, we restrict $(\nu_s, \nu_f, \lambda, \eta, p) \geq 0$ to be positive.

To obtain the transition density, we follow a similar approach as in Aït-Sahalia (2004). Conditioned on the event $Q_\Delta \equiv q_t - q_{t-\Delta} = n$, there must have been exactly n times, say τ_i, $i = 0, ..., n$, between 0 and Δ such that $dq_{\tau_i} = 1$. Thus, $\int_0^\Delta J_s dq_s = \sum_{i=0}^n J_{\tau_i}$ is the sum of n independent jumps. Further using (14), we condition on the event, $S_\Delta \equiv k$ (number of successful jumps), to obtain $\sum_{i=0}^n J_{\tau_i} = \nu_s k - \nu_f(n-k)$.

Using this result, we condition on the number of jumps as well as on the number of successful jumps per unit of time, and apply Bayes' rule to obtain

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(\mathbf{P}(X_\Delta \leq x | Q_\Delta = n, S_\Delta = k; \theta) \times \mathbf{P}(Q_\Delta = n, S_\Delta = k; \theta)\right)$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(\mathbf{P}(X_\Delta \leq x, Q_\Delta = n, S_\Delta = k; \theta)\right) \;=\; \mathbf{P}(X_\Delta \leq x, 0 \leq Q_\Delta < \infty, 0 \leq S_\Delta < \infty; \theta),$$

which simply is the marginal distribution, $\mathbf{P}(X_\Delta \le x; \theta)$. Now we can determine

$$\begin{aligned}
\mathbf{P}(X_\Delta \le x | Q_\Delta = n, S_\Delta = k; \theta) &= \mathbf{P}(z_t - z_{t-\Delta} \le (x - \omega\Delta - (\nu_s k - \nu_f(n-k)))/\eta; \theta) \\
&= \Phi\left\{(x - \omega\Delta - \nu_s k + \nu_f(n-k))/\eta\right\} \\
&= \int_{-\infty}^{(x-\omega\Delta-\nu_s k+\nu_f(n-k))/\eta} \frac{1}{\sqrt{2\pi\Delta}} \exp\left(-\frac{u^2}{2\Delta}\right) du,
\end{aligned}$$

where $\Phi\left\{(x - \omega\Delta - \nu_s k + \nu_f(n-k))/\eta\right\}$ denotes the Normal distribution with mean zero and variance Δ. Inserting this expression back into the marginal distribution, we finally obtain the unconditional distribution of X_Δ as

$$\mathbf{P}(X_\Delta \le x; \theta) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left(\int_{-\infty}^{(x-\omega\Delta-\nu_s k+\nu_f(n-k))/\eta} \frac{e^{-\frac{u^2}{2\Delta}}}{\sqrt{2\pi\Delta}} du \frac{e^{-\lambda\Delta}(\lambda\Delta)^n}{k!(n-k)!} p^k (1-p)^{n-k} \right),$$

where we used

$$\begin{aligned}
\mathbf{P}(Q_\Delta = n, S_\Delta = k; \theta) &= \mathbf{P}\left(S_\Delta = k | Q_\Delta = n; \theta\right) \times \mathbf{P}\left(Q_\Delta = n; \theta\right) \\
&= \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \times \frac{\exp(-\lambda\Delta)(\lambda\Delta)^n}{n!} \\
&= \frac{\exp(-\lambda\Delta)(\lambda\Delta)^n}{k!(n-k)!} p^k (1-p)^{n-k}.
\end{aligned}$$

Applying the Leibnitz rule, we now can derive the density (transition density of X_Δ) as

$$f_{X_\Delta}(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left(e^{-\frac{(x-\omega\Delta-\nu_s k+\nu_f(n-k))^2}{2\Delta\eta^2}} \frac{1}{\eta\sqrt{2\pi\Delta}} \frac{e^{-\lambda\Delta}(\lambda\Delta)^n}{k!(n-k)!} p^k (1-p)^{n-k} \right). \tag{18}$$

This expression shows that the distribution of growth rates is a *mixture density.* Intuitively, three components are involved, (1) the density of the normal distribution, augmented by (2) terms of the Poisson distribution as well as (3) elements of the binomial distribution. Similar formulas for the transition density are contained in Lo (1988) or in Aït-Sahalia (2004).

3.4 Inferring jumps from large realized growth rates

In this section, we assign 'probabilities' of occurred jumps to realized output growth rates. In principle, we could use these empirical probabilities to identify jumps in the data: Given an observation of magnitude x, what is the likelihood that such a change involved a jump?

For this, we use Bayes' rule to compute the probability of one positive jump as

$$\begin{aligned}
\mathbf{P}(Q_\Delta = 1, S_\Delta = 1 | X_\Delta \geq x; \theta) &= \frac{\mathbf{P}(Q_\Delta = 1, S_\Delta = 1, X_\Delta \geq x; \theta)}{\mathbf{P}(X_\Delta \geq x; \theta)} \\
&= \frac{\mathbf{P}(X_\Delta \geq x | Q_\Delta = 1, S_\Delta = 1; \theta) \times \mathbf{P}(Q_\Delta = 1, S_\Delta = 1; \theta)}{\mathbf{P}(X_\Delta \geq x; \theta)} \\
&= \frac{(1 - \mathbf{P}(X_\Delta \leq x | Q_\Delta = 1, S_\Delta = 1; \theta)) \times \mathbf{P}(Q_\Delta = 1, S_\Delta = 1; \theta)}{1 - \mathbf{P}(X_\Delta \leq x; \theta)} \\
&= \frac{\left(1 - \Phi\left(\frac{x - \omega\Delta - \nu_s}{\eta}\right)\right) p e^{-\lambda\Delta} \lambda\Delta}{1 - \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left(\Phi\left\{\frac{x - \omega\Delta - \nu_s k + \nu_f (n-k)}{\eta}\right\} \frac{e^{-\lambda\Delta}(\lambda\Delta)^n}{k!(n-k)!} p^k (1-p)^{n-k}\right)},
\end{aligned}$$

given that

$$\begin{aligned}
\mathbf{P}(Q_\Delta = 1, S_\Delta = 1; \theta) &= p e^{-\lambda\Delta} \lambda\Delta, \\
\mathbf{P}(X_\Delta \leq x | Q_\Delta = 1, S_\Delta = 1; \theta) &= \int_{-\infty}^{(x - \omega\Delta - \nu_s k)/\eta} \frac{e^{-\frac{u^2}{2\Delta}}}{\sqrt{2\pi\Delta}} du.
\end{aligned}$$

Similarly, the probability of one negative jump is

$$\mathbf{P}(Q_\Delta = \frac{\Phi\left\{\frac{x - \omega\Delta + \nu_f}{\eta}\right\} (1-p) e^{-\lambda\Delta} \lambda\Delta}{\sum_{n=0}^{\infty} \sum_{k=0}^{n} \left(\Phi\left\{\frac{x - \omega\Delta - \nu_s k + \nu_f (n-k)}{\eta}\right\} \frac{e^{-\lambda\Delta}(\lambda\Delta)^n}{k!(n-k)!} p^k (1-p)^{n-k}\right)}.$$

Hence, empirical probabilities could be used to identify jumps in the data as follows. After obtaining estimates for the parameter vector θ, we can assign the likelihood of one positive and one negative jump to the observed growth rates. Note that using quarterly data, in principle, one should also consider computing the probability of observing two or more jumps. However, our empirical estimates suggests that multiple jumps would be identified by visual examination already.

3.5 Maximum likelihood estimation

This section presents the methods used for estimation parameters of the underlying model making use of growth rates of macroeconomic variables only.

Because the stochastic processes, X_Δ, in (15) and (16) are Markovian, and because that property carries over to any discrete subsample from the continuous-time path, the log-likelihood function has the form

$$\ell_N(\theta) \equiv N^{-1} \sum_{i=1}^{N} \ln\{f_{X_\Delta}(x)\}, \tag{19}$$

where $\theta = (\nu_s, \nu_f, \lambda, \eta, \mu, p)'$ denotes the parameter vector that maximizes $\ell_N(\theta)$. Owing to difficulties that arise as a result of the complexity of the infinite series representation, the ML estimation approach does not yield explicit estimators in this problem (Press 1967).

We obtain the parameter estimates via numerical evaluation of the log-likelihood in (19). Asymptotic standard errors are obtained via the *outer-product estimate* using estimates of the information matrix (Hamilton 1994, Aït-Sahalia 2004),

$$\hat{\mathcal{J}}_{OP} = N^{-1} \sum_{i=1}^{N} \{h(\hat{\theta}, X_\Delta)\}\{h(\hat{\theta}, X_\Delta)\}',$$

where

$$h(\hat{\theta}, X_\Delta) = \left. \frac{\partial \ln\{f_{X_\Delta}(x)\}}{\partial \theta} \right|_{\theta=\hat{\theta}}.$$

If the sample size is sufficiently large, the central limit theorem implies that the distribution of the ML estimate, $\hat{\theta}$, is approximately $\mathcal{N}(\theta_0, \Delta N^{-1}\mathcal{J}^{-1})$. Our results have been checked by computing standard errors via the *second-derivative estimate* of the information matrix. Both approaches give for most cases values reasonably close to each other indicating that the empirical data were really generated from the assumed density (Hamilton 1994).

Using the closed form transition density in (18), we can apply standard techniques to test for jumps. Hence, denoting $\hat{\theta}$ as the unrestricted ML estimate, and $\tilde{\theta}$ as the estimate satisfying the restriction $H_0: \ \lambda = 0$, asymptotically (Sørensen 1991, Hamilton 1994, p.144)

$$2(\ell_N(\hat{\theta}) - \ell_N(\tilde{\theta})) \approx \chi^2(1). \tag{20}$$

In case of a single restriction, if the test statistic in (20) exceeds 3.8415 (2.7055), we can reject the null hypothesis of no jumps ($H_0 : \lambda = 0$) in favor of the hypothesis that jumps are present at the 5% (10%) significance level.

4 Monte Carlo evidence

In this section we report the results of Monte Carlo experiments in order to recover sample parameters. These experiments answer two separate questions. Firstly, how accurate are the asymptotic distributions in practice? The asymptotic distribution of ML estimators does not imply that the resulting parameter estimates would in practice necessarily be close to the sample parameter if the sample size is small or moderate. While this question is fairly standard emerging in any study that makes use of asymptotic properties, the second question is more peculiar for macroeconomics. Do general equilibrium effects complicate or circumvent the estimation of parameters? We tackle the latter question by comparing the performance of approximate versus exact estimation methods.

In finance, there is a large literature on disentangling the jump component from the diffusion (among others Aït-Sahalia 2004). We also make use of the fact that the jump size does not depend on the frequency of observations, so increasing the frequency of observations reduces the Brownian noise holding the jump size constant. Aït-Sahalia (2004) demonstrates that it is possible to perfectly disentangle Brownian noise from jumps when using high frequency data. Macroeconomists, however, can make use of monthly or quarterly data only, thus a simulation study where the parameters of the continuous-time process are estimated using discrete observations seems inevitable.

Using the explicit solution in Theorem 2.1, we perform $M = 5000$ simulations of the sample paths generated by the model, each containing 1050 observations where we cut off the first 500 as a burn in period ($N = 550$). Hence, the length of each series coincides with observable monthly data on which most of the empirical results will be based on. The population parameters of the model, $(\nu_s, \nu_f, \lambda, \eta, \mu, p)' = (.025, .025, .8, .025, .02, .5)$, and other parameters, $(\rho, \alpha, \sigma, \delta)' = (.03, .5, .5, .02)$, were chosen such that they denote 'annual' rates, while the frequency was set to $\Delta = 1/10$. Given the realizations of stochastic processes $\{A_t\}$, $\{X_t\}$, and $\{J_t\}$, sample parameters, $(\nu_s, \nu_f, \lambda^s, \eta^s, \mu^s, p^s)$, are computed (sample and population parameters for the jump-size are identical by construction).

From output growth rates in (13) with the assumption of a binomial jump-distribution in (14), the data generating process is uniquely determined. Nevertheless, we employ two models with different assumptions on the jump size to study the effects on other parameter estimates. For this purpose, on the one hand, we simply neglect negative jumps and falsely assume a degenerated distribution (model I). On the other hand, we correctly assume that the jump size has a binomial distribution (model II). We then employ different estimation strategies, i.e. the *Approximate* method with approximation around the non-stochastic steady state (a) as well as around mean values (b), and the *Exact* method based on modified growth rates, respectively (cf. Section 3). In Table 1 we summarize results of the Monte Carlo averages for the sample parameters and their estimates.

As a result, model I on average clearly underestimates the sample arrival rate while on average it overestimates the sample noise: $1/M \sum_i \hat{\lambda}_i = .2457 < 1/M \sum_i \lambda_i^s = .7898$, while $1/M \sum_i \hat{\eta}_i = .0305 > 1/M \sum_i \eta_i^s = .0250$ (model Ib). It is even substantially lower than the average arrival rate of successful jumps, $1/M \sum_i (p_i^s \lambda_i^s) = .3953$. Obviously, neglected negative shocks simply are captured by a too high estimate for the diffusion term, $\hat{\eta}$. Because this term is overestimated, fewer realizations are implicitly referred to as jumps, thus $\hat{\lambda}$ on average is too low. Further, neglecting negative jumps also has implications for the estimate of the deterministic trend: $1/M \sum_i \hat{\mu}_i = .0153 < 1/M \sum_i \mu_i^s = .0199$ (model Ib), that is on average it is estimated too small. This is because a substantial part of the growth simply

Table 1: Comparison of Monte Carlo estimates and sample parameters

Method			*Parameter estimate averages*					
			$\hat{\nu}_s$	$\hat{\nu}_f$	$\hat{\lambda}$	$\hat{\eta}$	$\hat{\mu}$	$\hat{p}$
Sample averages			0.0250	0.0250	0.7898	0.0250	0.0199	0.5005
			(0.0000)	(0.000)	(0.1218)	(0.0008)	(0.0033)	(0.0774)
MLE	*Approximate*	*Ia*	0.0248		0.2400	0.0305	0.0173	
			(0.0168)		(0.1074)	(0.0016)	(0.0040)	
		Ib	0.0230		0.2457	0.0305	0.0153	
			(0.0192)		(0.1104)	(0.0016)	(0.0056)	
		IIa	0.0247	0.0247	1.0206	0.0246	0.0198	0.5003
			(0.0039)	(0.0039)	(1.2244)	(0.0016)	(0.0049)	(0.1357)
		IIb	0.0224	0.0272	0.9352	0.0247	0.0212	0.4998
			(0.0040)	(0.0034)	(0.4284)	(0.0013)	(0.0055)	(0.1313)
MLE	*Exact*	*I*	0.0221		0.2478	0.0303	0.0155	
			(0.0201)		(0.1053)	(0.0015)	(0.0059)	
		II	0.0246	0.0247	1.0224	0.0245	0.0199	0.4997
			(0.0039)	(0.0039)	(1.2470)	(0.0016)	(0.0079)	(0.1355)

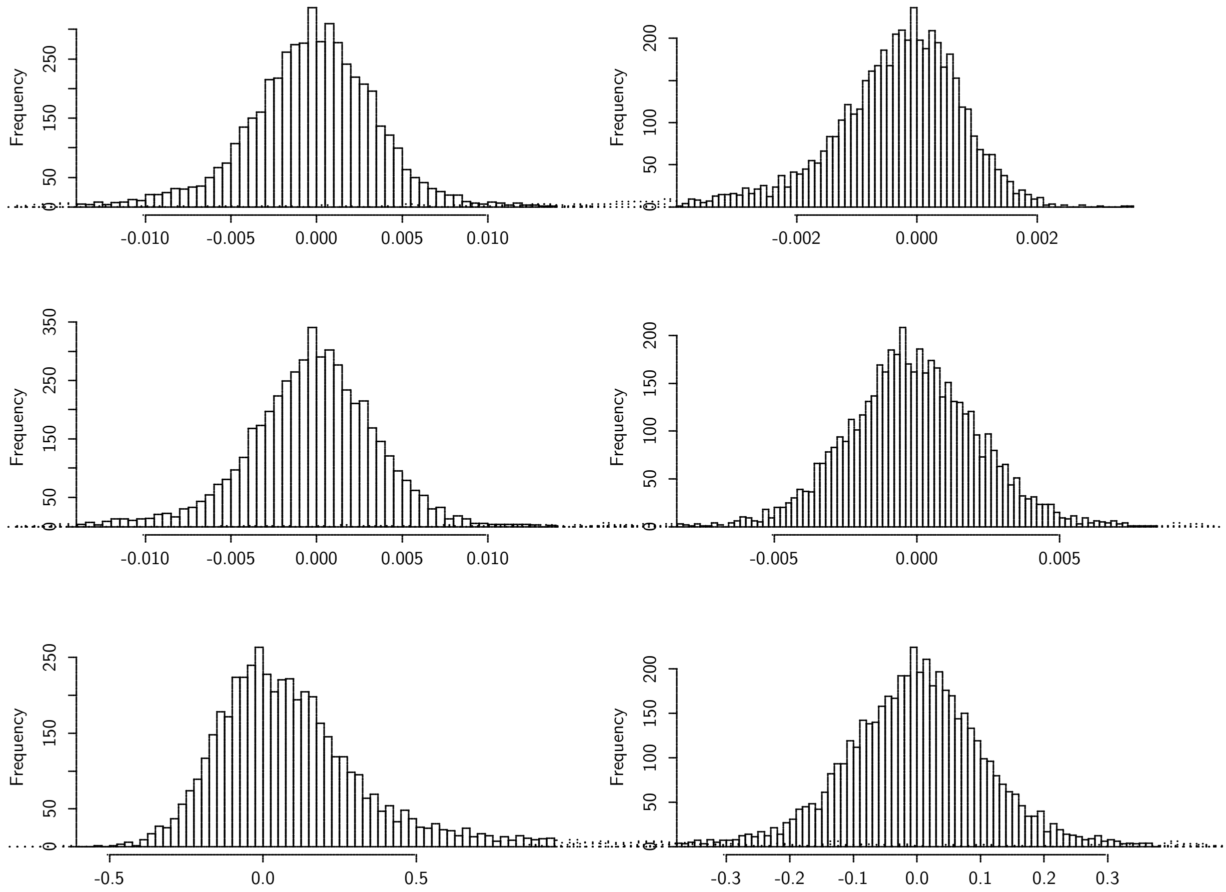
Notes: This table reports the averages over $M = 5000$ Monte Carlo simulations. It compares averages of MLE for degenerated jumps (model I), and binomial jump-size distribution (model II) using *Approximate* methods: (a) around the non-stochastic steady-state and (b) around the mean values, and the *Exact* method based on Solow residuals, respectively, with sample averages. Standard deviations of estimates are in parentheses. Calibrated parameters are $(\alpha, \sigma)' = (.5, .5)$; $\Delta = 1/10$; $N = 550$.

is attributed to the stochastic trend (though actually there is no stochastic growth involved in the sample path). Given the discussion and problems due to the false specification, the jump size, ν_s, on average is surprisingly accurately estimated.[10]

In the correct model specification (model II), we obtain an arrival rate which on average is estimated too high: $1/M \sum_i \hat{\lambda}_i = 0.9352 > 1/M \sum_i \lambda_i^s = .7898$ (model IIb). As shown below, this phenomenon can be attributed to a well known identification problem that may arise if the sample size is small or moderate (cf. Aït-Sahalia 2004). From Figure 2, the finite sample distribution (histogram) of estimates for the arrival rate, $\hat{\lambda}$, is skewed right while for the Brownian noise, $\hat{\eta}$, the histogram is skewed left. This is because in some cases, the jumps cannot be correctly disentangled from the Brownian noise. In such cases we obtain an (extremely) high estimate for the arrival rate along with tiny estimates for the jump-size, while the estimate of the Brownian noise is too small. Fortunately, such problems of identification arise not very often in practice (cf. Figure 2). For illustration, ordering the

[10]Note, however, that in a few cases, the estimated size of 'successful jumps' is negative. In that simply speaking the local maximum around the jump-size of negative jumps becomes the global maximum.

Figure 2: Finite sample distribution (histogram) of estimation errors



Notes: These figures reports the histogram of differences for estimates and sample parameters over $M = 5000$ Monte Carlo simulations for ν_s, ν_f, λ, η, μ, and p (column by column, from top left to bottom right) resulting from the MLE for the binomial jump-size distribution (model *IIa*) with approximation around the non-stochastic steady-state (compare with Table 1).

estimates with respect to $\hat{\lambda}$, and considering the .95 quantile of estimates only, this 'small sample bias' is negligible (cf. Appendix C, Table 8). Roughly speaking, in 95 out of 100 cases, we are able to correctly disentangle jumps from diffusions.[11]

Interestingly, the alleged advantage of approximations around mean values (model IIb) are compensated by a sensitivity with respect to the arrival rate and jump sizes (which enters distortionary if the arrival rate was not estimated correctly). Thus, the deterministic component of the growth rate as well as the jump-sizes are estimated on average more precisely using approximations around the non-stochastic steady state (model IIa). But these findings, of course, depend on the parametrization determining the speed of reversion,

[11]Often, problems of identification can be detected in practice from an insignificant parameter estimate along with an implausible high arrival rates and tiny jump sizes.

Table 2: Comparison of Monte Carlo estimates and sample parameters (fixed jump-sizes)

Method			*Parameter estimate averages*					
			$\hat{\nu}_s$	$\hat{\nu}_f$	$\hat{\lambda}$	$\hat{\eta}$	$\hat{\mu}$	$\hat{p}$
Sample averages			0.025 (0.0000)	0.025 (0.000)	0.7898 (0.1218)	0.0250 (0.0008)	0.0199 (0.0033)	0.5005 (0.0774)
MLE	*Approximate*	*Ia*	0.0250 (0.0000)		0.2680 (0.1104)	0.0307 (0.0017)	0.0165 (0.0037)	
		Ib	0.0250 (0.0000)		0.2680 (0.1104)	0.0307 (0.0017)	0.0134 (0.0039)	
		IIa	0.0250 (0.0000)	0.0250 (0.0000)	0.8022 (0.1648)	0.0251 (0.0010)	0.0198 (0.0035)	0.4992 (0.1009)
		IIb	0.0250 (0.0000)	0.0250 (0.0000)	0.7781 (0.1560)	0.0252 (0.0011)	0.0213 (0.0042)	0.4515 (0.1118)
MLE	*Exact*	*I*	0.0250 (0.0000)		0.2675 (0.1100)	0.0306 (0.0016)	0.0134 (0.0042)	
		II	0.0250 (0.0000)	0.0250 (0.0000)	.7989 (0.1641)	0.0250 (0.0010)	0.0199 (0.0041)	0.4993 (0.1009)

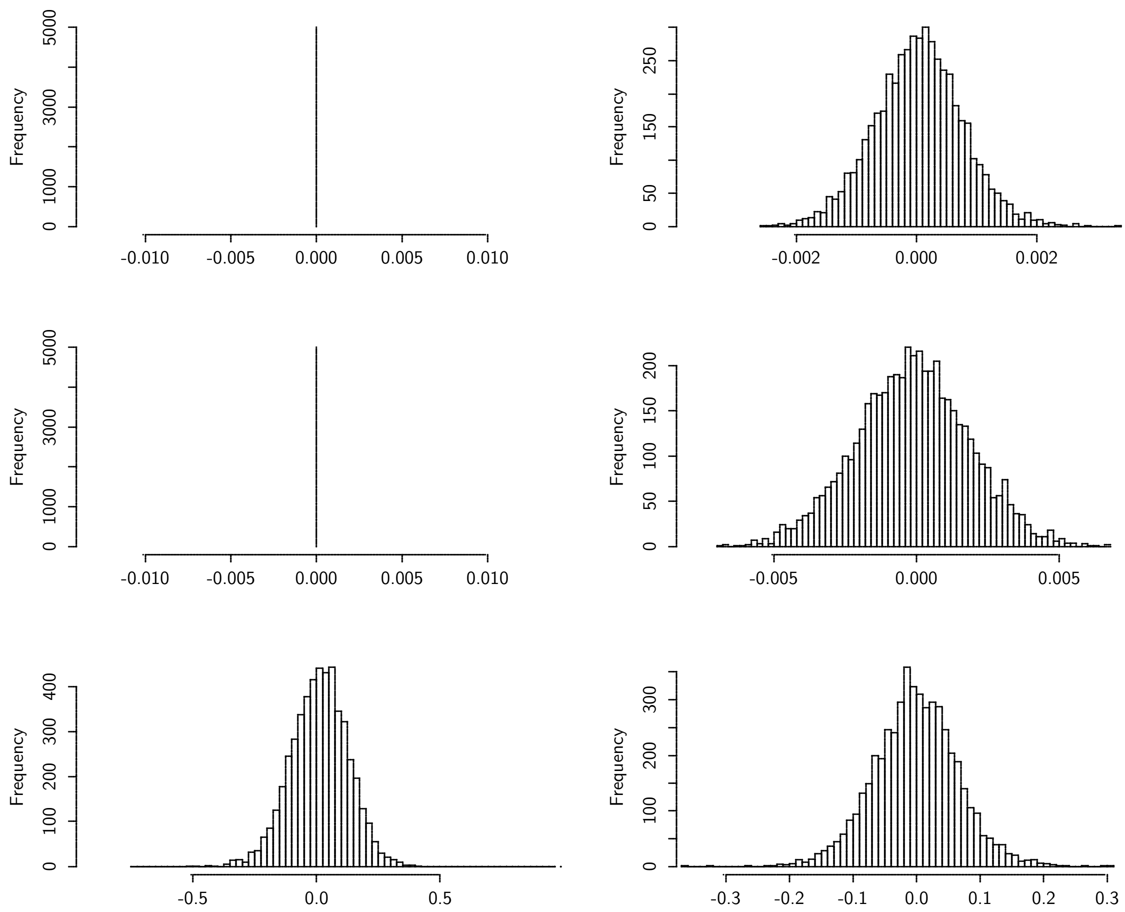
Notes: This table reports the averages over $M = 5000$ Monte Carlo simulations with fixed jump sizes. It compares averages of MLE for degenerated jumps (model *I*) and binomial jump-size distribution (model *II*) using *Approximate* methods: (a) around the non-stochastic steady-state and (b) around the mean values, and the *Exact* method based on Solow residuals, respectively, with sample averages. Standard deviations of estimates are in parentheses. Calibrated parameters are $(\nu_s, \nu_f, \alpha, \sigma)' = (.025, .025, .5, .5)$; $\Delta = 1/10$; $N = 550$.

as well as on the assumption of symmetric jumps. For example, asymmetric jumps moves the non-stochastic steady state farther away from the expected value, which would favor model IIb to the more general model IIa. While these considerations should be explored in more detailed for richer models, for the present model and with the available sample sizes, both estimation strategies should be employed and are useful.

Comparing the estimates of *Approximate* method versus the *Exact* estimation methods from Table 1, we conclude that correctly accounting for the reversion in capital rewards does not necessarily give better estimates. The identification problem seems by far more important in the Monte Carlo experiments of the given model specification. In that, approximative methods are powerful and general equilibrium effects in macroeconomics do not complicate the estimation of parameters too much.

Another way of accommodating the identification problem in practice is fixing jump-sizes to plausible values, while estimating other parameters by maximum likelihood. The results are summarized in Table 2 and Figure 3. Intuitively, the bias due to the wrong jump-size

Figure 3: Finite sample distribution (histogram) of estimation errors (fixed jump-sizes)



Notes: These figures reports the histogram of estimation errors over $M = 5000$ Monte Carlo simulations for ν_s, ν_f, λ, η, μ, and p (column by column, from top left to bottom right) with fixed jump-sizes resulting from the MLE for the binomial jump-size distribution (model IIa) with approximation around the non-stochastic steady-state (compare with Table 2).

distribution assumption (model I) still remains. Though the sample size is still moderate, however, there are no longer problems of identification in the correct jump-size specification (model II), and parameter estimates on average are unbiased (cf. Table 2). Hence, fixing the size of the jumps or equivalently giving more a priori information, does not only yield in better finite sample distributions of estimates in the sense of lower dispersion of estimates, but also removes identification problems that arise due to the sample size.

Summarizing the Monte Carlo experiments, in principle, jumps in macro series can be detected. General equilibrium effects do not prevent us from estimating the model parameters of interest. For illustration, we plot an arbitrary realization of the Monte Carlo experiment with 'identified' and realized jumps (cf. Appendix C, Figure 6). Identification of jumps is done in assigning probabilities to jump arrivals. Most of the jumps are identified using an appropriate probability threshold. We obtain reliable estimates for the sample parameters also for moderate sample sizes if 'monthly frequency' was considered (results on 'quarterly frequency' are available on request). In cases with problems of identification, further restrictions on the jump-size are useful and should be employed for correctly disentangling the jumps from the Brownian noise. It seems that for our model specification approximate methods are powerful, while exact methods do not necessarily improve on the accuracy of estimates if the sample size is small or moderate.

5 Empirical results

In this section, we obtain empirical estimates and search for jumps in output growth rates using industrial production (IP).[12] For convenience, descriptive statistics of IP growth rates for the United Kingdom, United States, Germany, and the euro area (Euro12) are contained in the data appendix (cf. Appendix D, Table 9). Before estimating, the only parameter we have to calibrate is the output elasticity of capital, α. An extensive sensitivity analysis on the effects of α on parameter estimates, $\hat{\theta}$, is provided below (cf. Section 5.3). For the empirical part, without loss in generality we set it to $\alpha = .5$.

5.1 Taking the model to the data

Before we apply our estimation techniques and interpret the results, we have to keep in mind the limitations of our simple model.

For simplicity we use random walks specifications for both stochastic processes in (4) and (3). Thus, impulses with temporary effects on productivity are ruled out by construction. Phenomena such as strikes with inventory building in anticipation of the strike, inventory depletion during the strike, and replacement production afterwards cannot be captured (Gunderson and Melino 1987). Interpreting those events as temporary (not permanent) shocks to productivity, we would expect a negative jump during the strike, possibly positive jumps before and after the strike. In that, impulses display patterns of autocorrelation themselves. Furthermore, structural changes such as the adoption of a new exchange rate regime (Bretton-Woods) affect the volatility of shocks and their propagation (Posch 2007). Such phenomena could be addressed with time varying volatility models.

The aim of this paper, however, is to adopt methods developed for high-frequency data to macroeconomics to estimate the parameters of the stochastic continuous-time growth model. Several extensions such as the introduction of autocorrelated impulses with temporary effects

[12] Despite the fact that industrial production contributes only minor parts to total output, nevertheless production indices are often used and play an important role in assessing the state of the economy.

on variables and/or time-varying volatility could be implemented in the future but go beyond the scope of this paper. The challenge for empirical research therefore is to filter the data to get rid of effects that are not captured by the model, while the challenge for theoretical research is to model all important effects without loosing the big picture.

5.2 Empirical estimates using industrial production

In this section we report empirical estimates using data on industrial production for the United Kingdom (UK), the United States (US), Germany (DE), and the euro area (Euro12). In most cases, we use monthly observations instead of quarterly data, though consumption data (as needed for the *Exact* method) often is not available at this frequency, for the following reasons. As from the Monte Carlo study, the problem of disentangling jumps from diffusions using quarterly data becomes more severe (Aït-Sahalia 2004). As an example, we report estimates for quarterly UK data (starting in 1948Q1). Moreover, the experiments suggest that methods based on Solow residuals do not necessarily improve on the accuracy of estimates at least for our model specification and sample size. For the US where monthly consumption data is available, we also provide estimates using modified growth rates.

All plots of (modified) growth rates as well as exemplarily computed jump probabilities can be found in Appendix D. Because our estimation methods are sensitive to the correct model specification and outliers (observations not explained by the model), we omit observations during the oil crises in the case of the specification with degenerated jumps (model I). Omitted observations coincide with the related US business cycle contractions identified by the NBER's Business Cycle Dating Committee.[13] Proceeding suchlike, we get rid of singular effects and biases in the estimates, as negative jumps are not captured by model I. For the binomial jump-size specification (model II) all observations are included.

UK industrial production

Based on quarter-to-quarter UK industrial production (manufacturing sector) of the period from 1948Q1 to 2005Q4, we find strong empirical evidence of jumps (cf. Table 3).

As a result, using model I, *Approximate* methods suggest that positive jumps in output growth rates occur about every $1/.2066 = 4.8$ years, that means at business cycle frequency.[14] Accordingly, the annual growth rate jumps by .032 or 3.2 percentage points. Statistically, the likelihood ratio test (20) rejects the null of no jumps in favor of the jump-diffusion model,

[13]In the case of quarterly data we omit periods 1973Q4 to 1975Q2 and 1980Q1 to 1980Q4 (11 observations), for monthly data we discard 1973:11 to 1975:04 and 1980:01 to 1980:08 (26 observations).

[14]Note that using standard errors based on the outer-product estimate of the information matrix, the parameter estimate $\hat{\lambda}$ is not significant at the 5% significance level. Using White's (1982) quasi-maximum likelihood standard errors, however, $\hat{\lambda}$ is significant even at the .1% significance level.

Table 3: UK quarterly IP (manufacturing sector) from 1948Q1 to 2005Q4

Method			*Parameter estimates*					
			$\hat{\nu}_s$	$\hat{\nu}_f$	$\hat{\lambda}$	$\hat{\eta}$	$\hat{\mu}$	$\hat{p}$
MLE	*Approximate*	*Ia*	0.0323		0.2066	0.0271	0.0078	
			(0.0070)		(0.1115)	(0.0010)	(0.0016)	
	$\ell_N(\theta)$				609.4 (5.75)			
		Ib	0.0323		0.2066	0.0271	0.0047	
			(0.0070)		(0.1115)	(0.0010)	(0.0026)	
	$\ell_N(\theta)$				609.4 (5.75)			
		IIa	0.0339	0.0407	0.5036	0.0243	0.0085	0.5423
			(0.0033)	(0.0033)	(0.0958)	(0.0010)	(0.0011)	(0.0779)
	$\ell_N(\theta)$				616.3 (24.17)			
		IIb	0.0276	0.0461	0.5276	0.0241	0.0098	0.5578
			(0.0049)	(0.0039)	(0.1599)	(0.0012)	(0.0030)	(0.1259)
	$\ell_N(\theta)$				616.4 (24.38)			
MLE	*Exact*	*I*	0.0124		1.4694	0.0222	−0.0134	
			(0.0102)		(3.3956)	(0.0036)	(0.0274)	
	$\ell_N(\theta)$				557.1 (2.47)			
		II	0.0225	0.0371	0.6750	0.0223	−0.0041	0.7201
			(0.0049)	(0.0036)	(0.3322)	(0.0018)	(0.0045)	(0.1213)
	$\ell_N(\theta)$				561.2 (15.01)			

Notes: This table reports the ML estimates for degenerated jumps (model *I*), and binomial jump-size distribution (model *II*) using *Approximate* methods: (a) around the non-stochastic steady-state and (b) around the mean values, and the *Exact* method based on Solow residuals, respectively. Standard errors and likelihood ratio tests against the null of no jumps in parentheses. Omitting periods of oil crises in model *I*. Calibrated parameters are $(\alpha, \sigma)' = (.5, .5)$; $\Delta = 1/4$; $N = 231$ ($N = 220$), for the *Exact* method $N = 203$ ($N = 192$).

$2(\ell_N(\hat{\theta}) - \ell_N(\tilde{\theta})) = 5.75$. In contrast, using the *Exact* estimation method based on modified growth rates, the likelihood ratio test does not reject the null of no jumps. We refer this to identification problems indicated by the high (but insignificant) value for the arrival rate in combination with a small value of the jump-size and the smaller sample size (quarterly data for consumption is not available prior to 1955, thus $N = 192$). Fixing the jump-sizes, that means assuming more plausible values, $\nu_s = .03$ and $\nu_f = .04$, as from Table 3, gives more reliable estimates for the *Exact* estimation approach (cf. Table 4). Nonetheless, the estimate for the deterministic trend remains insignificant.

For model II with binomial jump-size distribution, the *Approximate* approach suggests jumps roughly every $1/.5036 = 2$ years (model IIa). With a probability of 54 percent, we observe a positive jumps of the size .034 (roughly every 3.7 years), and with probability

Table 4: UK quarterly IP (manufacturing sector) with fixed jump-sizes

Method			*Parameter estimates*					
			$\hat{\nu}_s$	$\hat{\nu}_f$	$\hat{\lambda}$	$\hat{\eta}$	$\hat{\mu}$	$\hat{p}$
MLE	*Approximate*	*Ia*	0.0300		0.2370	0.0270	0.0076	
			(0.0000)		(0.0662)	(0.0009)	(0.0013)	
	$\ell_N(\theta)$				609.3 (5.69)			
		Ib	0.0300		0.2370	0.0270	0.0043	
			(0.0000)		(0.0662)	(0.0009)	(0.0021)	
	$\ell_N(\theta)$				609.3 (5.69)			
		IIa	0.0300	0.0400	0.5913	0.0238	0.0079	0.5958
			(0.0000)	(0.0000)	(0.0791)	(0.0009)	(0.0011)	(0.0589)
	$\ell_N(\theta)$				616.2 (23.97)			
		IIb	0.0300	0.0400	0.5003	0.0243	0.0092	0.5317
			(0.0000)	(0.0000)	(0.0707)	(0.0009)	(0.0020)	(0.0654)
	$\ell_N(\theta)$				615.6 (22.89)			
MLE	*Exact*	*I*	0.0300		0.1081	0.0251	0.0016	
			(0.0000)		(0.0507)	(0.0008)	(0.0022)	
	$\ell_N(\theta)$				556.7 (1.68)			
		II	0.0300	0.0400	0.3500	0.0243	0.0004	0.5510
			(0.0000)	(0.0000)	(0.0700)	(0.0010)	(0.0023)	(0.0877)
	$\ell_N(\theta)$				561 (14.42)			

Notes: This table reports the ML estimates for degenerated jumps (model *I*), and binomial jump-size distribution (model *II*) using *Approximate* methods: (a) around the non-stochastic steady-state and (b) around the mean values, and the *Exact* method based on Solow residuals with fixed jump-sizes, respectively. Standard errors and likelihood ratio tests against the null of no jumps in parentheses. Omitting periods of oil crises in model *I*. Calibrated parameters are $(\alpha, \sigma)' = (.5, .5)$; $\Delta = 1/4$; $N = 231$ ($N = 220$), for the *Exact* method $N = 203$ ($N = 192$).

of 46 percent the complementary event of a negative jump of size .041 occurs. Using, the estimates in the *Exact* approach, we obtain smaller jump-sizes while the arrival rate is higher. As already discussed, it indicates some problems of identification. Fixing the jump size, jumps arrive every $1/.35 = 2.9$ years with probability of success at 55 percent. In other words, positive jumps again arrive at business cycle frequency roughly every 5.3 years.

Using month-to-month UK industrial production (manufacturing sector) of the period from 1968:01 to 2006:03, we also find evidence of jumps. To save space, the results are referred to the appendix (Appendix D, Figure 14, Tables 11 and 12). Note, however, that monthly UK data seems to be contaminated with outliers, thus results should be interpreted with care. Fixing the jump-sizes to plausible values as obtained from quarterly data helps

Table 5: US monthly IP (manufacturing sector) from 1960:01 to 2006:03

Method			*Parameter estimates*					
			$\hat{\nu}_s$	$\hat{\nu}_f$	$\hat{\lambda}$	$\hat{\eta}$	$\hat{\mu}$	$\hat{p}$
MLE	*Approximate*	*Ia*	0.0175		0.2335	0.0250	0.0183	
			(0.0036)		(0.1118)	(0.0003)	(0.0009)	
	$\ell_N(\theta)$				1831.0 (4.26)			
		Ib	0.0175		0.2335	0.0250	0.0164	
			(0.0036)		(0.1118)	(0.0003)	(0.0014)	
	$\ell_N(\theta)$				1831.0 (4.26)			
		IIa	0.0160	0.0227	0.8023	0.0228	0.0181	0.5058
			(0.0018)	(0.0007)	(0.1401)	(0.0004)	(0.0008)	(0.0764)
	$\ell_N(\theta)$				1894.8 (58.0)			
		IIb	0.0140	0.0240	0.8499	0.0227	0.0201	0.5120
			(0.0020)	(0.0008)	(0.1999)	(0.0004)	(0.0019)	(0.1060)
	$\ell_N(\theta)$				1895.0 (58.27)			
MLE	*Exact*	*I*	0.0209		0.2160	0.0258	0.0171	
			(0.0019)		(0.0488)	(0.0003)	(0.0014)	
	$\ell_N(\theta)$				1722.3 (8.66)			
		II	0.0170	0.0211	0.7840	0.0232	0.0189	0.4370
			(0.0016)	(0.0009)	(0.1107)	(0.0004)	(0.0015)	(0.0658)
	$\ell_N(\theta)$				1800.9 (46.16)			

Notes: This table reports the ML estimates for degenerated jumps (model *I*), and binomial jump-size distribution (model *II*) using *Approximate* methods: (a) around the non-stochastic steady-state and (b) around the mean values, and the *Exact* method based on Solow residuals, respectively. Standard errors and likelihood ratio tests against the null of no jumps in parentheses. Omitting periods of oil crises in model *I*. Calibrated parameters are $(\alpha, \sigma)' = (.5, .5)$; $\Delta = 1/12$; $N = 555$ ($N = 529$), for the *Exact* method $N = 529$ ($N = 503$).

to get more reliable results but the estimated jump frequency seems to be too high.[15]

US industrial production

For the US, we use monthly industrial production (manufacturing sector) from the Bureau of Economic Analysis from 1960:01 to 2006:03 (cf. Table 5). For the ease of interpretation, we obtain annual parameter estimates by setting $\Delta = 1/12$.

From model *I*, the *Approximate* method gives jump arrivals every $1/.2335 = 4.3$ years, causing discrete changes in the annual growth rate of .018 or 1.8 percentage points. Note that the jump-size seems substantially smaller than in the UK data. Similarly, using the

[15] When fixing the jump-size to smaller values, 'outliers' are implicitly still considered as jumps, with high probability of having occurred more than once within a period and the arrival rate being estimated too high.

Exact method, positive jumps arrive every $1/.216 = 4.6$ years of size .021 or 2.1 percentage points. Again, estimates of positive jump arrivals are again at business cycle frequency, and likelihood ratio tests in (20) clearly reject the null hypothesis of no jumps. Nevertheless, the deterministic trend captures a substantial part of the growth rate.[16]

Using model *II*, the jump arrival rate is substantially higher, with jumps occurring roughly every $1/.784 = 1.3$ years. Positive jumps of are half a percentage point smaller than negative jumps and arrive with probability 51 percent (about every 2.5 years) using *Approximate* methods or 44 percent (about every 2.9 years) in the *Exact* method, respectively. Having in mind the results of Monte Carlo experiments that estimates of the true arrival rate may be positively biased, one is inclined to restrict the jump-sizes to higher values. Such strategies would deflate the arrival rate, but any choice of plausible jump-sizes seems arbitrary.

DE industrial production

For the Germany (DE), we use monthly industrial production from the Federal Statistical Office from 1960:01 to 2003:07 (cf. Table 6).[17]

The results of model I indicate the presence of jumps as the likelihood ratio test (20) rejects the null of no jumps, $2(\ell_N(\hat{\theta}) - \ell_N(\tilde{\theta})) = 4.06$. Accordingly, we obtain that positive jumps in IP growth rates compared to the UK and the US are extremely rare but drastic events, jumping every $1/.0701 = 14.3$ years by .049 or 4.9 percentage points. However, the estimate of the noise parameter, $\hat{\eta} = .054$, is also substantially higher compared to the other countries, where $\hat{\eta} = .025$ or $\hat{\eta} = .027$, as from Tables 5 and 3, respectively.

In that view, using model II allowing for negative jumps in the data, however, we can draw a different conclusion in line with previous estimates. Now jumps are more frequent, $1/.8132 = 1.2$ years and less severe. A positive jumps of size .037 comes at the probability of 26.8 percent, while negative jumps of size .03 are more frequent with complementary probability of 71.2 percent. Intuitively, positive jumps arrive therefore on average every $1/(.8132 \times .2679) = 4.6$ years again at business cycle frequency. These results impressively demonstrate that, similar to the Monte Carlo experiments, neglecting the negative jumps comes at the cost that the Brownian noise is overrated (compare .539 to .047).

[16] Note, however, that the rate of population growth (remember that labor supply is constant in the model) has not been considered. Thus conclusions with respect to the importance of discrete innovations versus continuous type accumulation cannot be drawn from this result.

[17] Two obvious outliers (1967:10/1967:11 and 1984:05/1984:06) identified in Flaig (2005) have been removed before the estimation procedure. Thanks to Gebhard Flaig for pointing out these irregularities.

Table 6: DE monthly IP (Federal Statistical Office) from 1960:01 to 2003:07

Method			*Parameter estimates*					
			$\hat{\nu}_s$	$\hat{\nu}_f$	$\hat{\lambda}$	$\hat{\eta}$	$\hat{\mu}$	$\hat{p}$
MLE	*Approximate*	*Ia*	0.0489		0.0701	0.0539	0.0134	
			(0.0082)		(0.0319)	(0.0005)	(0.0014)	
	$\ell_N(\theta)$				1352.1 (4.06)			
		Ib	0.0489		0.0701	0.0539	0.0124	
			(0.0082)		(0.0319)	(0.0005)	(0.0017)	
	$\ell_N(\theta)$				1352.1 (4.06)			
		IIa	0.0373	0.0298	0.8132	0.0472	0.0174	0.2679
			(0.0033)	(0.0036)	(0.2322)	(0.0011)	(0.002)	(0.0697)
	$\ell_N(\theta)$				1429.3 (12.45)			
		IIb	0.0343	0.0338	0.7631	0.0473	0.0231	0.2834
			(0.0036)	(0.0034)	(0.1804)	(0.0011)	(0.0029)	(0.0749)
	$\ell_N(\theta)$				1429.2 (12.36)			

Notes: This table reports the ML estimates for degenerated jumps (model I), and binomial jump-size distribution (model II) using *Approximate* methods: (a) around the non-stochastic steady-state and (b) around the mean values, respectively. Standard errors and likelihood ratio tests against the null of no jumps in parentheses. Omitting periods of oil crises in model I. Calibrated parameters are $(\alpha, \sigma)' = (.5, .5)$; $\Delta = 1/12$; $N = 523$ ($N = 497$).

Euro area (Euro12) industrial production

Finally we study euro area (Euro12) monthly industrial production from the EABCN Real Time Database (Eurostat) from 1985:01 to 2005:05.[18] Note that estimates are based on a relatively small number of observations ($N = 244$), which as shown above could result into problems of identification. Nonetheless, our estimates are reasonable and in line with the previous results finding empirical evidence for jumps (cf. Table 7).

As from model I, positive can be observed every $1/.2001 = 5$ years causing a discrete change in the growth rate of .026 or 2.6 percentage points. Moreover, the Brownian noise component as well as the deterministic drift, $\hat{\eta} = .029$ and $\hat{\mu} = .005$ (model Ib), are about the same order of magnitude as the UK estimate (compare with Table 3).

For model II, we obtain jumps every $1/.3679 = 2.7$ years. In that, positive jumps occur at probability 63 percent, which means they should be observed every $1/(.3679 \times .63) = 4.3$ years jumping by .025 or 2.5 percentage points. The complement event of negative jumps occurs at probability of 37 percent jumping by .026 or 2.6 percentage points.

[18]Since 1999 the euro area includes 12 member countries: Austria, Belgium, Finland, France, Germany, Greece, Ireland, Italy, Luxembourg, Netherlands, Portugal, Spain.

Table 7: Euro12 monthly IP (EABCN Real Time Database) from 1985:01 to 2005:05

Method			Parameter estimates					
			$\hat{\nu}_s$	$\hat{\nu}_f$	$\hat{\lambda}$	$\hat{\eta}$	$\hat{\mu}$	$\hat{p}$
MLE	*Approximate*	*Ia*	0.0261		0.2001	0.0292	0.0064	
			(0.0051)		(0.0697)	(0.0004)	(0.0011)	
	$\ell_N(\theta)$				804.5 (5.82)			
		Ib	0.0261		0.2001	0.0292	0.0040	
			(0.0051)		(0.0697)	(0.0004)	(0.0014)	
	$\ell_N(\theta)$				804.5 (5.82)			
		IIa	0.0254	0.0262	0.3679	0.0273	0.0078	0.6300
			(0.0036)	(0.0062)	(0.0973)	(0.0006)	(0.001)	(0.1119)
	$\ell_N(\theta)$				807.3 (11.28)			
		IIb	0.0248	0.0268	0.3680	0.0273	0.0069	0.6301
			(0.0035)	(0.0064)	(0.098)	(0.0006)	(0.0015)	(0.1069)
	$\ell_N(\theta)$				807.3 (11.29)			

Notes: This table reports the ML estimates for degenerated jumps (model I), and binomial jump-size distribution (model II) using *Approximate* methods: (a) around the non-stochastic steady-state and (b) around the mean values, respectively. Standard errors and likelihood ratio tests against the null of no jumps in parentheses. Calibrated parameters are $(\alpha, \sigma)' = (.5, .5)$; $\Delta = 1/12$; $N = 244$.

5.3 Sensitivity analysis

This section provides a sensitivity analysis of empirical results with respect to the output elasticity of capital, α. Of course, α is an important parameter of the macro model, say, but as we show below not in the specific context of this analysis.

From the transition density of X_Δ in (18), we find that α is contained in ω only. In other words, using $\omega_1 = \frac{\mu}{1-\alpha} - \frac{1}{2}\eta^2$ the parameter does not play any role in determining the arrival rate, λ, the jump-sizes ν_s and ν_f, the probability of positive jumps, p, well as the variance of the Brownian noise, η, respectively, employing approximations around the non-stochastic steady state (model Ia or model IIa). Accordingly, the only parameter which would be affected is the deterministic trend, μ. If the calibrated value of α was too low, the estimate of the deterministic trend, μ, would be too high. This is the case as μ measures the drift for total factor productivity, A_t. Specified as the drift of labor augmenting technology, X_t, the parameter would be independent of α as well. Because empirically only one deterministic trend can be identified, its rather a question of interpretation than of estimation.

Using approximations around expected values (model Ib or model IIb), that means with $\omega_2 = \omega_1 + \frac{\alpha}{1-\alpha}\left(\nu\lambda - \frac{1}{2}\eta^2\right)$, all parameters estimates could be affected by α. At least in our model specification, however, Monte Carlo simulations (not shown) suggest that effects on all estimates but of μ are small (model *IIb*). For model *I*, similar to approximations around the non-stochastic steady state, α does *not* affect estimates of λ, ν, and η. This is because these parameters are already determined by higher moments independently of the mean which denotes the only moment that is affected by α (cf. Appendix B.2).

Finally, the choice of α does affect the correction needed for obtaining modified growth rates as in (16) used for *Exact* estimation methods (model *I* and model *II*). Recalling that the expected output- and consumption growth rates are identical, a wrong specification of α has primarily a level effect. A higher value of the output elasticity, α, therefore implies ceteris paribus a lower estimate of the deterministic trend, μ. Using $\omega_3 = \mu - \frac{1}{2}\eta^2$, the choice of α does not further affect the estimation results. In that, the choice of α has implications on the procedure of obtaining Solow residuals, but not on the estimation of parameters.

We can summarize our empirical study as follows. In line with growing trough cycles theories we find empirical evidence of positive jumps in output growth rates at business cycle frequency. This finding is independent from the chosen estimation strategy, thus not affected by the calibrated value for α. Moreover, because approximations around the non-stochastic steady state (model *Ia* and model *IIa*) hold for the general model, explicit solutions can be used to assess how approximate methods work for special cases. We also find that negative jumps are a salient feature of real world data, which in case they are neglected severely bias the arrival rate of positive jumps.

6 Conclusion

In the paper we develop a continuous-time neoclassical stochastic growth model where the resulting output growth rate follows an (asymmetric) jump-diffusion process. By imposing different parameter restrictions we obtain two solutions in explicit form.

Based on the solutions we obtain transition densities in closed form. Two techniques, an approximate as well as an exact method, are introduced with the consequences of the approximation error being analyzed. In an extensive Monte Carlo simulation study we demonstrate that the parameters of the data generating process can be recovered. We also show that exact methods do not necessarily improve on the accuracy of estimates for the present model specification if the sample size was small or moderate.

Taking the model directly to the data, we find empirical evidence of jumps in output growth rates for the United Kingdom, the United States, Germany as well as the euro area (Euro12). Standard likelihood ratio tests are used to test for the presence of jumps, with estimates of positive jump arrivals at business cycle frequency. Though the present approach does not differentiate between endogenous and exogenous sources of jumps in factor productivity, the existence of discrete changes in macro series seems a necessary condition for the growing through cycles literature. Having identified jumps, one should tackle the question about the endogeneity or determinants of jumps.

There are a number of interesting directions for future extensions of this work. From an empirical perspective, time-varying volatility models should be considered extending the present single factor model to more factors (Johannes 2004). Different versions of modeling the distortions such as geometric diffusion with mean reversion, arithmetic diffusion could be implemented. This would allow to have effects like hump-shaped impulse responses. From a theoretical perspective, studying effects that result into negative jumps in factor productivity or recessions are necessary to account for empirically features of the data (for example explicitly modeling demand shocks as in Chen and Funke 2007). From an econometric perspective, estimating continuous-time models in macro without the use of explicit solution remains a major goal. Using closed-form sequences of approximations to the transition density instead seems a promising avenue to follow (Aït-Sahalia 2002a).

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Appendices

Contents

A Technical appendix

A.1 Itô's formula for non-degenerated jumps

Assume the following jump process

$$dX_t = a(t, X_t)dt + b(t, X_{t-}, J_t)dq_t \tag{21}$$

or more rigourously $X_t - X_0 = \int_0^t a(s, X_s)dt + \int_0^t b(s, X_{s-}, J_s)dq_t$ where J_s denotes the jump size distribution satisfying mild regularity conditions.[19] Using the time derivative together with (21), and collecting terms, we obtain

$$d\left(\int_0^t b(s, X_{s-}, J_s)dq_t\right) = b(t, X_{t-}, J_t)dq_t. \tag{22}$$

The change of variable formula (Protter 2004, p.78) for any finite variation process, X_t, with right continuous paths, and $f(X_t)$ a function with $f_X(X_t)$ exists and is continuous, is

$$f(X_t) - f(X_0) = \int_0^t f_X dX_t + \sum_{0<s\leq t} \{f(X_s) - f(X_{s-}) - f_X \Delta X_s\}$$

where $\Delta X_s = X_s - X_{s-}$ simply denotes the jump at time s. Now insert (21),

$$\begin{aligned} f(X_t) - f(X_0) &= \int_0^t f_X a(s, X_s)ds + \int_0^t b(s, X_{s-}, J_s)f_X dq_t \\ &\quad + \sum_{0<s\leq t} \{f(X_s) - f(X_{s-}) - (X_s - X_{s-})f_X\} \end{aligned}$$

which, recalling that $X_s = X_{s-} + b(s, X_{s-}, J_s)$, equivalently can be written as

$$\begin{aligned} f(X_t) - f(X_0) &= \int_0^t f_X a(s, X_s)ds + \int_0^t b(s, X_{s-}, J_s)f_X dq_t \\ &\quad + \int_0^t \{f(X_{s-} + b(s, X_{s-}, J_s)) - f(X_{s-}) - b(s, X_{s-}, J_s)f_X\}dq_t \\ &= \int_0^t f_X a(s, X_s)ds + \int_0^t \{f(X_{s-} + b(s, X_{s-}, J_s)) - f(X_{s-})\}\, dq_t \end{aligned}$$

where we simply collected terms in the last step. Taking the time derivative using (22), we obtain the familiar change of variable formula

$$df(X_t) = f_X a(t, X_t) + \{f(X_{t-} + b(t, X_{t-}, J_t)) - f(X_{t-})\}\, dq_t.$$

Hence, the usual Itô's formula (change of variables) can be used for non-degenerated jumps.

[19] Note that stochastic functions instead of deterministic functions within the stochastic integral are widely used in the financial literature (e.g. Merton 1973). This derivation, however, shows that the change of variable formula in Sennewald (2007) also applies to the case of non-degenerated jumps.

A.2 Theorems on reducible and affine SDEs

We extend the theorems of Kloeden and Platen (1999, p.125) on diffusions for jump-diffusions with non-degenerated jump size distribution. A rigorous treatment with solutions of SDEs with Poisson processes can be found in García and Griego (1994).

Theorem A.1 (Reducible SDEs) *Consider the general non-linear jump-diffusion process with polynomial drift of degree n,*

$$dX_t = (aX_t^n + bX_t)dt + cX_t dW_t + V_t X_{t-} dq_t,$$

where W_t is a Brownian motion, q_t is a Poisson process, and V_t denotes the independent jump size distribution with constant mean and variance. The solution reads

$$X_t = \Theta_t \left(X_0^{1-n} + a(1-n) \int_0^t \Theta_s^{n-1} ds \right)^{\frac{1}{1-n}} \tag{23}$$

with

$$\Theta_s = \exp\left(\left(b - \frac{1}{2}c^2 \right) s + cW_s + \int_0^s \ln(1+V_u) dq_u \right).$$

Proof. Define auxiliary variables

$$g(t, W_t, V_t, q_t) \equiv \left(b - \frac{1}{2}c^2 \right) t + cW_t + \int_0^t \ln(1+V_s) dq_s$$

and

$$h(t) \equiv \left(X_0^{1-n} + a(1-n) \int_0^t \Theta_s^{n-1} ds \right)^{\frac{1}{1-n}},$$

such that $X_t = \exp(g_t)h_t \equiv F(g_t, h_t)$. Using the rule $d\left(\int_0^t a(s)dq_s \right) = a(t)dq_t$, we obtain

$$\begin{aligned} dg_t &= \left(b - \frac{1}{2}c^2 \right) dt + cdW_t + \ln(1+V_t)dq_t, \\ dh_t &= \left(X_0^{1-n} + a(1-n) \int_0^t \Theta_s^{n-1} ds \right)^{\frac{n}{1-n}} a\Theta_t^{n-1} dt. \end{aligned}$$

With Itô's formula (change of variables) we finally obtain the evolution of X_t as

$$\begin{aligned} dX_t &= \exp(g_t) \left(X_0^{1-n} + a(1-n) \int_0^t \Theta_s^{n-1} ds \right)^{\frac{n}{1-n}} a\Theta_t^{n-1} dt + \exp(g_t)h_t \left(b - \frac{1}{2}c^2 \right) dt \\ &\quad + \frac{1}{2}\exp(g_t)h_t c^2 dt + \exp(g_t)h_t c dW_t + \left\{ \exp\left(g_{t-} + \ln(1+V_t) \right) h_{t-} - \exp(g_{t-})h_{t-} \right\} dq_t \\ &= h_t^n a\Theta_t^n dt + \left(b - \frac{1}{2}c^2 \right) X_t dt + \frac{1}{2}X_t c^2 dt + cX_t dW_t + \left\{ (1+V_t) - 1 \right\} X_{t-} dq_t \\ &= aX_t^n dt + bX_t dt + cX_t dW_t + V_t X_{t-} dq_t. \end{aligned}$$

where we inserted $X_t = \exp(g_t)h_t$ and collected terms to obtain (23). ■

Theorem A.2 (Affine SDEs) *Consider the general affine jump-diffusion process,*

$$dX_t = (a(t)X_t + b(t))dt + c(t)dW_t + V_t dq_t$$

where W_t is a Brownian motion, q_t is a Poisson process, and V_t denotes the independent jump size distribution with constant mean and variance. The solution reads

$$X_t = \Theta_{t,t_0}\left(X_{t_0} + \int_{t_0}^{t}\Theta_{s,t_0}^{-1}b(s)ds + \int_{t_0}^{t}\Theta_{s,t_0}^{-1}c(s)dW_s + \int_{t_0}^{t}\Theta_{s,t_0}^{-1}V_s dq_s\right), \tag{24}$$

with

$$\Theta_{t,t_0} = \exp\left(\int_{t_0}^{t} a(s)ds\right).$$

Proof. Define auxiliary variables

$$g(t) \equiv \int_{t_0}^{t} a(s)ds$$

and

$$h(t, W_t, V_t, q_t) \equiv X_{t_0} + \int_{t_0}^{t}\Theta_{s,t_0}^{-1}b(s)ds + \int_{t_0}^{t}\Theta_{s,t_0}^{-1}c(s)dW_s + \int_{t_0}^{t}\Theta_{s,t_0}^{-1}V_s dq_s,$$

such that $X_t = \exp(g_t)h_t \equiv F(g_t, h_t)$, with differentials

$$\begin{aligned} dg_t &= a(t)dt, \\ dh_t &= \Theta_{t,t_0}^{-1}b(t)dt + \Theta_{t,t_0}^{-1}c(t)dW_t + \Theta_{t,t_0}^{-1}V_t dq_t. \end{aligned}$$

Using this result, we finally obtain the evolution of X_t as

$$\begin{aligned} dX_t &= \exp(g_t)\left(\Theta_{t,t_0}^{-1}b(t)dt + \Theta_{t,t_0}^{-1}c(t)dW_t\right) + \exp(g_t)\Theta_{t,t_0}^{-1}V_t dq_t + \exp(g_t)h_t a(t)dt \\ &= a(t)X_t dt + b(t)dt + c(t)dW_t + V_t dq_t, \end{aligned}$$

where we inserted $X_t = \exp(g_t)h_t$ and collected terms to obtain (24). ■

A.3 The evolution of log-output

From (9), the evolution of output inserting dK_t/dt from (2) is

$$\begin{aligned} dY_t &= (\mu + \alpha(I_t/K_t - \delta))Y_t dt + Y_t\eta dz_t + (\exp(J_t) - 1)Y_{t-}dq_t \\ &= (\mu + \alpha(Y_t/K_t - C_t/K_t - \delta))Y_t dt + Y_t\eta dz_t + (\exp(J_t) - 1)Y_{t-}dq_t \end{aligned}$$

Now define log-output, $y_t \equiv \ln Y_t$, and use Itô's formula to compute the differential dy_t,

$$\begin{aligned} dy_t &= \left(\mu + \alpha\left(Y_t/K_t - C_t/K_t - \delta\right) - \frac{1}{2}\eta^2\right)dt + \eta dz_t + J_t dq_t \\ &= \left(\mu + \alpha\left(r_t/\alpha - \delta - C_t/K_t\right) - \frac{1}{2}\eta^2\right)dt + \eta dz_t + J_t dq_t. \end{aligned} \tag{25}$$

where we inserted capital rewards, $r_t = Y_K = \alpha Y_t/K_t$, in the last step.

A.4 The evolution of the rental rate of capital

Using Itô's formula (change of variables), capital rewards, $r_t = \alpha A_t (X_t L/K_t)^{1-\alpha}$, follow

$$\begin{aligned} dr_t &= r_A dA_t + (r_t - r_{t-})\, dq_t + r_K dK_t, \\ &= r_t \mu dt + r_t \eta dz_t + (\exp(J_t) - 1) r_{t-} dq_t + r_K (I_t - \delta K_t) dt, \end{aligned}$$

Now inserting $r_K = -(1-\alpha) r_t / K_t$, and replacing $Y_t/K_t = r_t/\alpha$, we obtain

$$\begin{aligned} dr_t &= r_t \mu dt + r_t \eta dz_t + (\exp(J_t) - 1) r_{t-} dq_t - (1-\alpha)(Y_t/K_t - C_t/K_t - \delta) r_t dt, \\ &= (\mu - (1-\alpha)(r_t/\alpha - \delta - C_t/K_t))\, r_t dt + r_t \eta dz_t + (\exp(J_t) - 1) r_{t-} dq_t. \end{aligned} \tag{26}$$

A.5 The maximized Bellman equation

The value of an optimal program of (5) is defined by

$$V(a(0), A(0), X(0)) = \max_{c(t)} \{U_0\},$$

which denotes the present discounted value of utility evaluated along the optimal program. Splitting up the integral gives

$$U_0 = E_0 \int_0^{\Delta t} e^{-\rho t} u(c_t) dt + E_{\Delta t} \int_{\Delta t}^{\infty} e^{-\rho t} u(c_t) dt.$$

Following Bellman's idea, the optimal program is

$$V(a(0), A(0), X(0)) = \max_{c_0} \left\{ E_0 u(c_0) \Delta t + \frac{1}{1+\rho \Delta t} E_{\Delta t} V(a(\Delta t), A(\Delta t), X(\Delta t) \right\}.$$

Multiply by $(1 + \rho \Delta t)$ gives

$$(1+\rho\Delta t) V(a(0), A(0), X(0)) = \max_{c_0} \left\{ E_0 u(c_0) \Delta t (1+\rho\Delta t) + E_{\Delta t} V(a(\Delta t), A(\Delta t)), X(\Delta t)) \right\}.$$

Now divide by Δt and move $\frac{1}{\Delta t} V(a(0), A(0), X(0))$ to the right hand side

$$\begin{aligned} \rho V(a(0), A(0), X(0)) = \max_{c_0} \Big\{ & E_0 u(c_0)(1+\rho\Delta t) + \frac{1}{\Delta t} \Big[E_{\Delta t} V(a(\Delta t), A(\Delta t), X(\Delta t)) \\ & - V(a(0), A(0), X(0)) \Big] \Big\}. \end{aligned}$$

Letting Δt become infinitesimally small, the problem becomes

$$\rho V(a(0), A(0), X(0)) = \max_{c_0} \left\{ u(c_0) + \frac{1}{dt} E_0 dV(a(0), A(0), X(0)) \right\}.$$

Because of X_t and A_t are independent, Itô's formula (change of variables) yields

$$\begin{aligned} dV_t &= V_a da_t + V_A dA_t + \frac{1}{2} V_{AA} A_t^2 \eta^2 dt + \left(V(a_t, A_t, X_t) - V(a_t, A_t, X_{t-})\right) dq_t, \\ &= V_a((r_t - \delta)a_t + w_t - c_t)dt + V_A A\mu dt + V_A A\eta dz_t + \frac{1}{2} V_{AA} A_t^2 \eta^2 dt \\ &\quad + \left(V(a_t, A_t, X_t) - V(a_t, A_t, X_{t-})\right) dq_t. \end{aligned}$$

where we inserted (7). With $E_0(dz_t) = 0$ and $E_0(dq_t) = \lambda dt$, the Bellman equation reads

$$\begin{aligned} \rho V(a(0), A(0), X(0)) &= \max_{c_0} \Big\{ u(c_0) + V_a((r_0 - \delta)a_0 + w_0 - c_0) + V_A A_t \mu + \frac{1}{2} V_{AA} A_t^2 \eta^2 \\ &\quad + \lambda \left(E_J(V(a_0, A_0, X_0)) - V(a_0, A_0, X_{0-})\right) \Big\}, \end{aligned}$$

where $E_J V(a_t, A_t, X_t)$ denotes the expected value of the optimal program with respect to J_t, as the level of X_t immediately after a jump is $X_t = (\exp(J_t))^{\frac{1}{1-\alpha}} X_{t-}$.

The first order condition reads

$$u'(c_0) = V_a(a_0, A_0, X_0), \tag{27}$$

making consumption a function of the state variables. The maximized Bellman equation is

$$\begin{aligned} \rho V(a(0), A(0), X(0)) &= u(c(a_0)) + ((r_0 - \delta)a_0 + w_0 - c(a_0))V_a + V_A A_0 \mu + \frac{1}{2} V_{AA} A_0^2 \eta^2 \\ &\quad + \lambda \left(E_J V(a_0, A_0, X_0) - V(a_0, A_0, X_{0-})\right). \end{aligned} \tag{28}$$

A.6 Long-run growth dynamics

This section derives the (non-stochastic) long-run values to which variables tend to revert in the absence of stochastic shocks. For this we make use of the concept of balanced growth well known from the corresponding deterministic standard neoclassical model. In general, using Îto's formula (change of variables), output evolves according to

$$\begin{aligned} dY_t &= Y_A dA_t + (Y_t - Y_{t-}) dq_t + Y_K dK_t, \\ \Leftrightarrow dY_t / Y_t &= 1/A_t dA_t + (\exp(J_t) - 1) dq_t + \alpha 1/K_t dK_t, \\ &= \mu dt + \alpha dK_t / K_t + \eta dz_t + (\exp(J_t) - 1) dq_t \end{aligned}$$

which can be interpreted as the stochastic growth rate of a variable. In the absence of shocks, i.e. $dz = 0$ and $dq = 0$, the economy converges to a balanced growth path similar to the deterministic version of the model (Barro and Sala-i-Martín 2003). Along such a path $(dK_t/dt)/K_t$ is a constant, accordingly from (2), the rental rate of capital, r_t, as well as the consumption-capital ratio, C_t/K_t, converges to a constant in the absence of shocks. As a corollary, in this situation, consumption and capital stock instantaneously grow at the same exogenous rate. Moreover, the market clearing condition (8) demands that output grows at this hypothetical growth rate. Along such growth path, the variables can be differentiated with respect to time,

$$\begin{aligned} (dY_t/dt)/Y_t &= \mu + \alpha(dK_t/dt)/K_t \\ \Leftrightarrow \ (dK_t/dt)/K_t &= \frac{1}{1-\alpha}\mu \end{aligned} \tag{29}$$

where in the last step we used the fact that $(dY_t/dt)/Y_t = (dK_t/dt)/K$, which is the exogenous balanced growth rate in the deterministic setup.

A.7 Proof of Theorem 2.1

The idea of this proof is to show that together with an educated guess of the value function, both the maximized Bellman equation (28) and first order condition (27) are fulfilled. We may guess that the value function reads

$$V(a_t, A_t, X_t) = \frac{\phi^{-\sigma} a_t^{1-\sigma}}{1-\sigma} + f(A_t, X_t). \tag{30}$$

To start with we rewrite the policy function using the transformation $K_t \equiv La_t$ as

$$C_t = \phi K_t \ \Leftrightarrow \ Lc_t = \phi La_t \ \Leftrightarrow \ c_t = \phi a_t. \tag{31}$$

Using (27) together with (6), and (31), we obtain $V_a = (\phi a_t)^{-\sigma}$. Moreover, our guess in (30) implies $V_A = f_A$, $V_{AA} = f_{AA}$, $V_X = f_X$. Inserting everything into (28) gives

$$\rho\frac{\phi^{-\sigma} a_t^{1-\sigma}}{1-\sigma} + \rho f(A_t, X_t) \ = \ \frac{(\phi a_t)^{1-\sigma}}{1-\sigma} + (a_t\phi)^{-\sigma}((r_t - \delta)a_t + w_t - \phi a_t) + g(A_t, X_t).$$

where we defined $g(A_t, X_t) \equiv f_A A_t \mu + \frac{1}{2} f_{AA} A_t^2 \eta^2 + \lambda(f(A_t, X_t) - f(A_t, X_{t-}))$. Note that $f(A_t, X_t)$ contains the expectation with respect to the jump J_t. Intuitively, this explains the fact why J_t does not enter the value function as an argument.

Using factor rewards $w_t = Y_L$ and $r_t = Y_K$ we obtain

$$r_t = \alpha A_t K_t^{\alpha-1}(X_t L)^{1-\alpha}, \quad w_t = (1-\alpha)A_t K_t^{\alpha} X_t^{1-\alpha} L^{-\alpha}. \tag{32}$$

Inserting now (32) together with $K_t \equiv La_t$ gives

$$\begin{aligned} \rho\frac{\phi^{-\sigma} a_t^{1-\sigma}}{1-\sigma} + \rho f(A_t, X_t) &= \frac{(\phi a_t)^{1-\sigma}}{1-\sigma} + \left(\alpha A_t X_t^{1-\alpha} a_t^{\alpha} - \delta a_t\right)(a_t\phi)^{-\sigma} \\ &\quad +(1-\alpha)A_t X_t^{1-\alpha} a_t^{\alpha}(a_t\phi)^{-\sigma} - \phi a_t(a_t\phi)^{-\sigma} + g(A_t, X_t), \\ &= \frac{(\phi a_t)^{1-\sigma}}{1-\sigma} + A_t X_t^{1-\alpha} a_t^{\alpha}(a_t\phi)^{-\sigma} - \delta a_t(a_t\phi)^{-\sigma} - \phi a_t(a_t\phi)^{-\sigma} \\ &\quad +g(A_t, X_t). \end{aligned}$$

Using the condition $\alpha = \sigma$ with $\rho f(A_t, X_t) = A_t X_t^{1-\alpha}\phi^{-\sigma} + g(A_t, X_t)$ it becomes

$$\begin{aligned} \rho\frac{\phi^{-\sigma}a_t^{1-\sigma}}{1-\sigma} &= \frac{(\phi a_t)^{1-\sigma}}{1-\sigma} - \delta a_t^{1-\sigma}\phi^{-\sigma} - (\phi a_t)^{1-\sigma} \\ \Leftrightarrow \ \rho &= \phi - (1-\sigma)\delta - (1-\sigma)\phi. \end{aligned}$$

Finally we can solve for ϕ which is (10).

A.8 Proofs of Corollaries 2.2 and 2.3

Proof of Corollary 2.2

Inserting $C_t = \phi K_t$ from (31) in (26), we obtain

$$dr_t = (\mu - (1-\alpha)(r_t/\alpha - \delta - \phi)) r_t dt + r_t \eta dz_t + (\exp(J_t) - 1) r_{t-} dq_t.$$

We now rewrite the equation by using the condition $\alpha = \sigma$, and inserting ϕ from (10) to

$$\begin{aligned} dr_t &= \left(\mu - \frac{1-\alpha}{\alpha}(r_t - \alpha\delta - \rho - (1-\sigma)\delta)\right) r_t dt + r_t\eta dz_t + (\exp(J_t)-1) r_{t-} dq_t \\ &= \left(\mu - \frac{1-\alpha}{\alpha}(r_t - \delta - \rho)\right) r_t dt + r_t\eta dz_t + (\exp(J_t)-1) r_{t-} dq_t \\ &= \frac{1-\alpha}{\alpha} r_t \left(\frac{\alpha}{1-\alpha}\mu + \rho + \delta - r_t\right) dt + \eta r_t dz_t + (\exp(J_t)-1) r_{t-} dq_t. \end{aligned}$$

Using the definitions c_1 and c_2 we finally obtain (11).

Proof of Corollary 2.3

Inserting the policy function $C_t = \phi K_t$ into (25) gives

$$dy_t = \left(\mu + r_t - \alpha(\delta + \phi) - \frac{1}{2}\eta^2\right) dt + \eta dz_t + J_t dq_t.$$

Applying Theorem A.2 and inserting ϕ from (10) yields,

$$\begin{aligned} y_t &= y_{t_0} + \int_{t_0}^t \left(\mu + r_s - \alpha(\delta+\phi) - \frac{1}{2}\eta^2\right) ds + \eta z_t + \int_{t_0}^t J_s dq_s \\ &= y_{t_0} + (t - t_0)\left(\mu - \rho - \delta - \frac{1}{2}\eta^2\right) + \int_{t_0}^t r_s ds + \eta z_t + \int_{t_0}^t J_s dq_s, \end{aligned} \tag{33}$$

where r_s is known explicitly from its solution in (12). Thus, the growth rate per unit of time, Δ, is given by

$$y_t - y_{t-\Delta} = \Delta\left(\mu - \rho - \delta - \frac{1}{2}\eta^2\right) + \int_{t-\Delta}^t r_s ds + \eta(z_t - z_{t-\Delta}) + \int_{t-\Delta}^t J_s dq_s.$$

A.9 An alternative solution

The proofs for the following Theorem A.3, Corollary A.4, and Corollary A.5 are analogous to Section A.7, and Section A.8, respectively, and are contained in the Referees' appendix available on request.

Theorem A.3 *If $\sigma > 1$ and the condition*

$$\rho = (\alpha\sigma - 1)\delta - \sigma\mu + \frac{1}{2}(1+\sigma)\sigma\eta^2 + \lambda E_J\left((\exp(J_t))^{-\sigma}\right) - \lambda$$

is fulfilled, consumption is a constant fraction of income, $C = \vartheta Y$, where

$$\vartheta = \frac{\sigma - 1}{\sigma}. \tag{34}$$

Corollary A.4 *The rental rate of capital follows*

$$dr = c_3 r_t (c_4 - r_t)\, dt + \eta r_t dz_t + (\exp(J_t) - 1) r_{t-} dq_t \tag{35}$$

where $c_3 \equiv \frac{1-\alpha}{\alpha\sigma}$, and $c_4 \equiv \frac{\alpha\sigma}{1-\alpha}\mu + \alpha\sigma\delta$, respectively.

Similarly to Corollary 2.2 in the text, c_4 is the long-run equilibrium (or the long-run value towards the rental rate of capital tends to revert), and c_3 is the speed of reversion.

Corollary A.5 *The growth rate of output per unit of time, $g_\Delta \equiv y_t - y_{t-\Delta}$, reads*

$$g_\Delta = \Delta\left(\mu - \alpha\delta - \frac{1}{2}\eta^2\right) + 1/\sigma \int_{t-\Delta}^{t} r_s ds + \eta(z_t - z_{t-\Delta}) + \int_{t-\Delta}^{t} J_s dq_s. \tag{36}$$

A.10 ACFs of simulated growth rates

In the simulation study of realized growth rates (cf. Figure 1), we use $N = 1050$ ($\Delta = 1$) where we cut of the first 500 observations to neglect transitional effects. We set parameters $(\nu_s, \nu_f, \lambda, \eta, \mu, p)' = (.05, 0, .05, .01, 1)$ and other parameters $(\rho, \alpha, \sigma, \delta)' = (.03, \cdot, \cdot, 0)$.

Figure 4: Simulated growth rates and ACF, $\alpha = \sigma = .9$

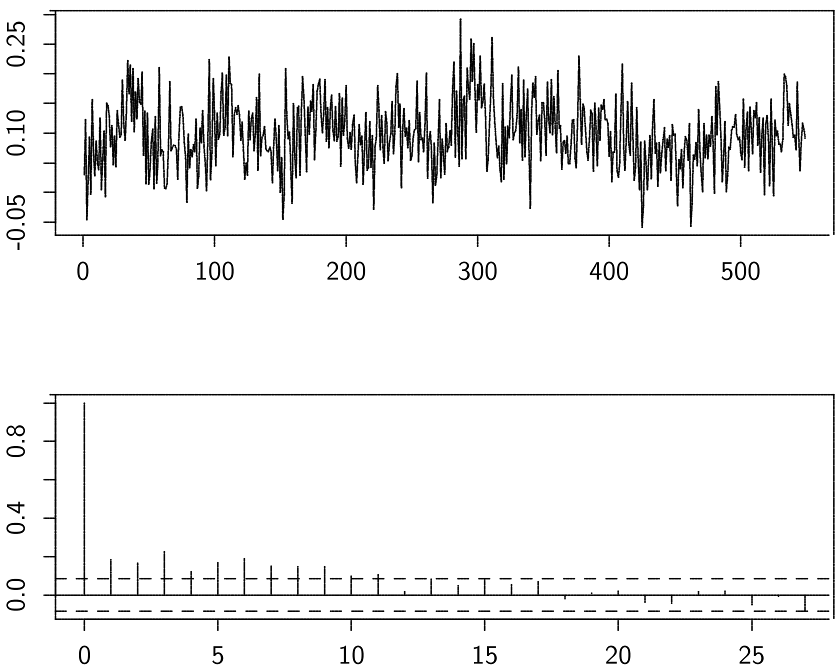
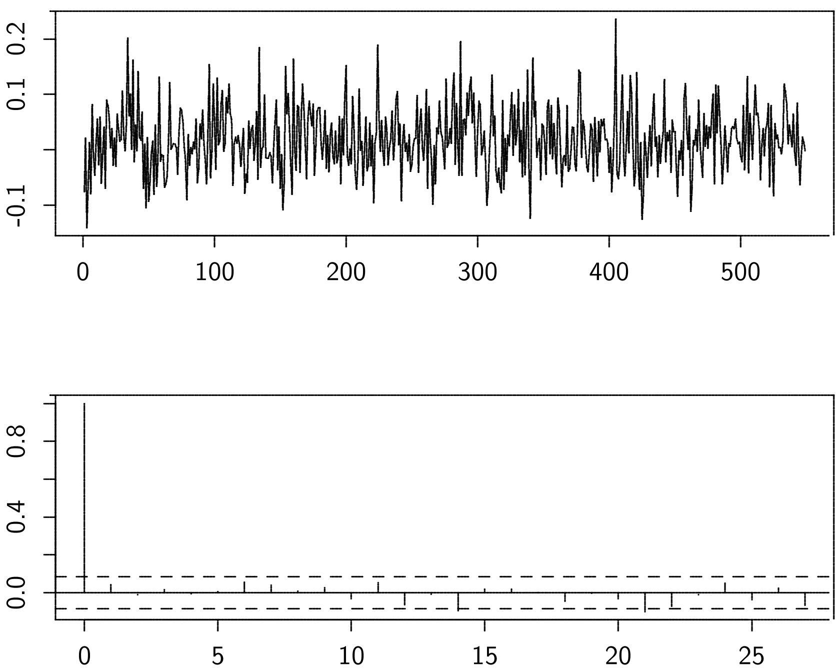


Figure 5: Simulated growth rates and ACF, $\alpha = \sigma = .1$



Notes: These figures show the ACFs of simulated growth rates with different parameters for the speed of reversion. The upper figure simulates resulting growth rates with low speed of reversion, whereas the lower figure uses a ceteris paribus analysis with parameters that imply a high speed of reversion. Simulations are based on parameters $(\nu_s, \nu_f, \lambda, \eta, \mu, p)' = (.05, 0, .05, .01, 1)$, other parameters $(\rho, \alpha, \sigma, \delta)' = (.03, \cdot, \cdot, 0)$ and use the same realizations of stochastic processes.

B Estimation strategy

B.1 Approximate growth rates

This section derives ω_1 and ω_2, respectively.

B.1.1 Approximation around the non-stochastic steady state

In the near neighborhood of the non-stochastic steady state, i.e. the value towards r_t tends to revert, $\int_{t-\Delta}^{t} r_s ds$ in (11) can well be approximated by $\Delta c_2 = \Delta \left(\frac{\alpha}{1-\alpha}\mu + \rho + \delta\right)$. Inserting this into (13) yields

$$g_\Delta \approx \Delta \left(\frac{\mu}{1-\alpha} - \frac{1}{2}\eta^2 \right) + \eta(z_t - z_{t-\Delta}) + \int_{t-\Delta}^{t} J_s dq_s. \tag{37}$$

Hence, we obtain $\omega_1 \equiv \frac{\mu}{1-\alpha} - \frac{1}{2}\eta^2$. When inserting $(dK/dt)/K = \frac{1}{1-\alpha}\mu$ of Appendix A.6 in equation (9), solving and computing the output growth rate, we obtain (37) also for the general case. Although motivated by our explicit solution, the approach can well be used to estimate the underlying parameters for $\alpha \neq \sigma$.

B.1.2 Approximation around expected values

We are now interested in obtaining the mean of the output growth rate per unit of time. Using the expectation operator with (13) yields

$$\begin{aligned} E(g_\Delta) &= \Delta \left(\mu - \rho - \delta - \frac{1}{2}\eta^2 \right) + E\int_{t-\Delta}^{t} r_s ds + E\int_{t-\Delta}^{t} J_s dq_s, \\ &= \Delta \left(\mu - \rho - \delta - \frac{1}{2}\eta^2 + \nu\lambda \right) + E\int_{t-\Delta}^{t} r_s ds. \end{aligned} \tag{38}$$

Now we make use of the fact that the capital stock can be observed because of $C_t = \phi K_t$. According to (1) logarithmic output, $y_t \equiv \ln Y_t$, is

$$\begin{aligned} y_t &= \ln A_t + \alpha \ln K_t + (1-\alpha)(\ln X_t + \ln L) \\ \Leftrightarrow \alpha \ln K_t &= y_t - \ln A_t - (1-\alpha)(\ln X_t + \ln L) \end{aligned}$$

Insert the solution $C = \phi K$ from Theorem 2.1 to obtain growth rates of consumption,

$$\begin{aligned} \alpha(\ln C_t - \ln C_{t-\Delta}) &= y_t - y_{t-\Delta} - (\ln A_t - \ln A_{t-\Delta}) - (1-\alpha)(\ln X_t - \ln X_{t-\Delta}) \\ &= -\Delta(\rho + \delta) + \int_{t-\Delta}^{t} r_s ds \end{aligned} \tag{39}$$

where we inserted the growth rates per unit of time of the solutions to the SDEs in (4) and (3), $\ln A_t - \ln A_{t-\Delta} = (\mu - \frac{1}{2}\eta^2)\Delta + \eta(z_t - z_{t-\Delta})$, and $\ln X_t - \ln X_{t-\Delta} = \frac{1}{1-\alpha}\int_{t-\Delta}^t J_s dq_s$ as well as output growth rates from (13), respectively. Defining $g_\Delta^c \equiv \ln C_t - \ln C_{t-\Delta}$,

$$\alpha E g_\Delta^c = -\Delta(\rho + \delta) + E\int_{t-\Delta}^t r_s ds$$

It follows from the economy's aggregate resource constraint (8) that consumption and investment in expectation can only growth at constant rates indefinitely if both grow at the same rate as expected output. In particular we then observe $Eg_\Delta = Eg_\Delta^c$, and using (38),

$$Eg_\Delta = \Delta\frac{1}{1-\alpha}\left(\mu - \frac{1}{2}\eta^2 + \nu\lambda\right).$$

With this result, from (38) we obtain

$$\begin{aligned} E\int_{t-\Delta}^t r_s ds &= \Delta\left(\rho + \delta + \frac{\alpha}{1-\alpha}\left(\mu - \frac{1}{2}\eta^2 + \nu\lambda\right)\right) \\ \Leftrightarrow\ E(r_t) &= c_2 + \frac{\alpha}{1-\alpha}\left(\nu\lambda - \frac{1}{2}\eta^2\right), \end{aligned} \tag{40}$$

assuming both integrals $\int_{t-\Delta}^t r_s ds$ and $E\int_{t-\Delta}^t r_s ds$ to exist (cf. Posch and Wälde 2006).[20] Intuitively, it reflects the tendency parameter c_2 adjusted for the (asymmetric) jump term as well as the Jensens-inequality term. Note that the term $\frac{\alpha}{1-\alpha}$ again refers to the speed of reversion for capital rewards. If the speed of reversion was high, the expected value would differ only slightly from its tendency parameter c_2.[21]

To summarize, $\int_{t-\Delta}^t r_s ds$ can be approximated by $\Delta\left(c_2 + \frac{\alpha}{1-\alpha}\left(\nu\lambda - \frac{1}{2}\eta^2\right)\right)$. Inserting into output growth rates (13) yields

$$\begin{aligned} g_\Delta &\approx \Delta\left(\mu - \frac{1}{2}\eta^2\right) + \Delta\left(\frac{\alpha}{1-\alpha}\left(\mu - \frac{1}{2}\eta^2 + \nu\lambda\right)\right) + \eta(z_t - z_{t-\Delta}) + \int_{t-\Delta}^t J_s dq_s \\ &= \Delta\left(\frac{1}{1-\alpha}\left(\mu - \frac{1}{2}\eta^2\right) + \frac{\alpha}{1-\alpha}\nu\lambda\right) + \eta(z_t - z_{t-\Delta}) + \int_{t-\Delta}^t J_s dq_s \end{aligned} \tag{41}$$

Hence, we can define $\omega_2 \equiv \frac{\mu}{1-\alpha} - \frac{1}{1-\alpha}\frac{1}{2}\eta^2 + \frac{\alpha}{1-\alpha}\nu\lambda$. Contrarily to the previous approach, this estimation method yields consistent parameter estimates.

[20] The assumption of r_t to be stationary seems plausible both from theoretical and empirical considerations.

[21] Similarly, using the plausibility consideration that non-zero long-run expected growth rate cannot differ for other solutions, the first moment of capital rewards in the constant savings-rate solution reads

$$\begin{aligned} E\int_{t-\Delta}^t r_s ds &= \Delta\left(\sigma\alpha\tau_a + \sigma\alpha\delta + \frac{\sigma\alpha}{1-\alpha}\mu + \varphi - \frac{\sigma\alpha}{1-\alpha}\frac{1}{2}\eta^2 + \frac{\sigma\alpha}{1-\alpha}\nu\lambda\right) \\ \Leftrightarrow\ E(r_t) &= c_6 + \frac{\sigma\alpha}{1-\alpha}\left(\nu\lambda - \frac{1}{2}\eta^2\right) \end{aligned}$$

which reflects the tendency parameter adjusted for positve jumps as well as the Jensens-inequality term. Again, the term $\frac{\sigma\alpha}{1-\alpha}$ is the inverse of the speed of reversion for the rental rate of capital. Similar results using uncertain population growth can be found in Merton (1999, Table 17.2).

B.2 Alternative estimation methods

Though the following alternative estimation approaches cannot compete in finite sample sizes with (efficient) maximum likelihood (ML) estimation as the theoretical model provides the complete specification of the probability distribution of the data, they are introduced to understand asymptotic properties of estimates.

The first four moments of the process X_Δ in (17) are (Press 1967, Aït-Sahalia 2004),

$$\begin{aligned}
m_1'(\Delta,\theta) &= \Delta(\omega+\nu\lambda),\\
m_2(\Delta,\theta) &= \Delta(\eta^2+(\nu^2+\gamma)\lambda),\\
m_3(\Delta,\theta) &= \Delta\lambda\nu(\nu^2+3\gamma),\\
m_4(\Delta,\theta) &= \Delta(\nu^4\lambda+6\nu^2\gamma\lambda+3\gamma^2\lambda)+3\Delta^2(\eta^2+(\nu^2+\gamma)\lambda)^2,
\end{aligned}$$

where m_i denotes the ith central moment. For 6 unknown parameters in θ, we need at least 6 moment conditions to fully determine the estimates. Unfortunately, it is not possible to derive explicit expressions for the binomial jump-size specification (model II). Thus, after deriving more moments we would have to apply numerical method of moments (MM), or with a specified weighting matrix using general method of moments (GMM) estimation.

For the degenerated jump-size specification (model I), we obtain explicit conditions for the parameter vector. Setting γ to zero, we can explicitly solve for the parameters as follows. Inserting m_2 and m_3 in m_4 yields

$$\nu = \frac{m_4-3m_2^2}{m_3}, \tag{42}$$

which uniquely specifies the jump-size. Using (42) with m_3, we obtain the arrival rate,

$$\lambda=\frac{m_3^4}{\Delta(m_4-3m_2^2)^3}. \tag{43}$$

For the Brownian noise, we use (42) and (43) together with m_2 and get

$$\eta^2=\frac{m_2(m_4-3m_2^2)-m_3^2}{\Delta(m_4-3m_2^2)}. \tag{44}$$

The remaining moment condition determine μ, which is contained in $\omega\in(\omega_1,\omega_2,\omega_3)$,

$$\omega = \frac{m_1'(m_4-3m_2^2)^2-m_3^3}{\Delta(m_4-3m_2^2)^2}.$$

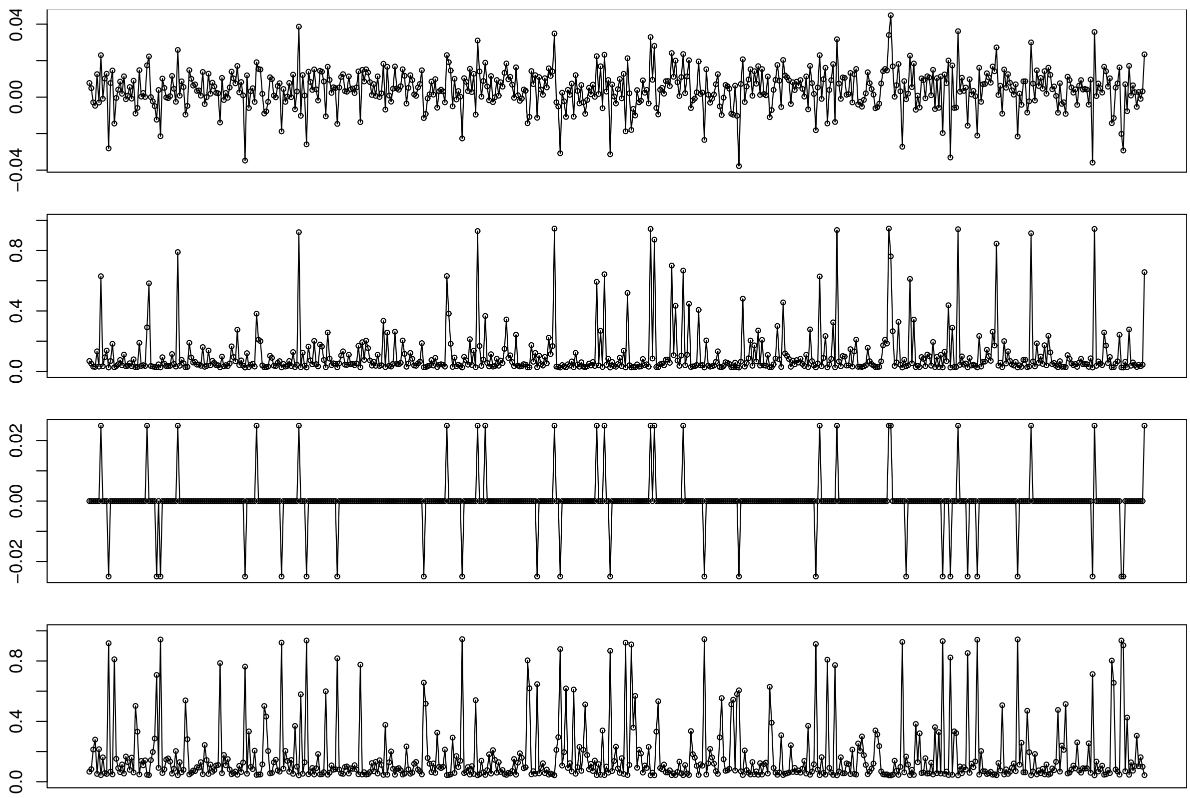
C Monte Carlo evidence

Table 8: Comparison of Monte Carlo estimates and sample parameters (.95 quantile)

Method			Parameter estimate averages					
			$\hat{\nu}_s$	$\hat{\nu}_f$	$\hat{\lambda}$	$\hat{\eta}$	$\hat{\mu}$	$\hat{p}$
Sample averages			0.025	0.025	0.7898	0.0250	0.0199	0.5005
			(0.0000)	(0.000)	(0.1218)	(0.0008)	(0.0033)	(0.0774)
MLE	*Approximate*	*Ia*	0.0258		0.2264	0.0306	0.0172	
			(0.0156)		(0.0881)	(0.0016)	(0.0039)	
		Ib	0.0239		0.2312	0.0305	0.0151	
			(0.0184)		(0.0885)	(0.0016)	(0.0053)	
		IIa	0.0251	0.0251	0.8743	0.0248	0.0198	0.4995
			(0.0034)	(0.0035)	(0.2550)	(0.0012)	(0.0036)	(0.1229)
		IIb	0.0228	0.0274	0.8683	0.0248	0.0209	0.5018
			(0.0036)	(0.0032)	(0.2453)	(0.0012)	(0.0044)	(0.1214)
MLE	*Exact*	*I*	0.0231		0.2347	0.0304	0.0153	
			(0.0193)		(0.0889)	(0.0015)	(0.0057)	
		II	0.0250	0.0250	0.8738	0.0247	0.0199	0.4992
			(0.0034)	(0.0035)	(0.2536)	(0.0012)	(0.0044)	(0.1228)

Notes: This table reports the averages over the .95 quantile of $M = 5000$ Monte Carlo simulations ordered by $\hat{\lambda}$. It compares averages of MLE for (model *I*) degenerated jumps and (model *II*) binomial jump-size distribution using *Approximate* methods: (a) around the non-stochastic steady-state and (b) around the mean values, and the *Exact* method based on Solow residuals, respectively, with sample averages. Standard deviations of estimated parameters in parentheses. Calibrated parameters are $(\alpha, \sigma)' = (.5, .5)$; $\Delta = 1/10$; $N = 550$.

Figure 6: Inferring jumps from large realized growth rates



Notes: This figures reports an arbitrary realization of a Monte Carlo sample path of growth rates (upper panel), the estimated probability of one positive jump (second panel), the jump-sizes (third panel), and the estimated probability of one negative jump (lower panel) based on the binomial jump-size distribution (model *IIa*) with approximation around the non-stochastic steady-state (compare with Table 1) ($\Delta = 1/10$; $N = 550$).

D Empirical results

D.1 Descriptive statistics

Table 9: Descriptive statistics of IP growth rates.

	UK $\Delta = 1/4$	UK $\Delta = 1/12$	US $\Delta = 1/12$	DE $\Delta = 1/12$	Euro12 $\Delta = 1/12$
Mean	0.0041	0.0006	0.0028	0.0020	0.0015
Standard dev.	0.0177	0.0150	0.0084	0.0159	0.0091
Skewness	−0.1402	−0.7204	−0.6612	−0.0419	0.1354
Excess kurtosis	1.4366	10.8683	3.2766	0.8515	1.2020
Minimum	−0.0517	−0.1008	−0.0459	−0.0482	−0.0302
Maximum	0.0626	0.0893	0.0295	0.0669	0.0331
Observations	231	459	555	523	244

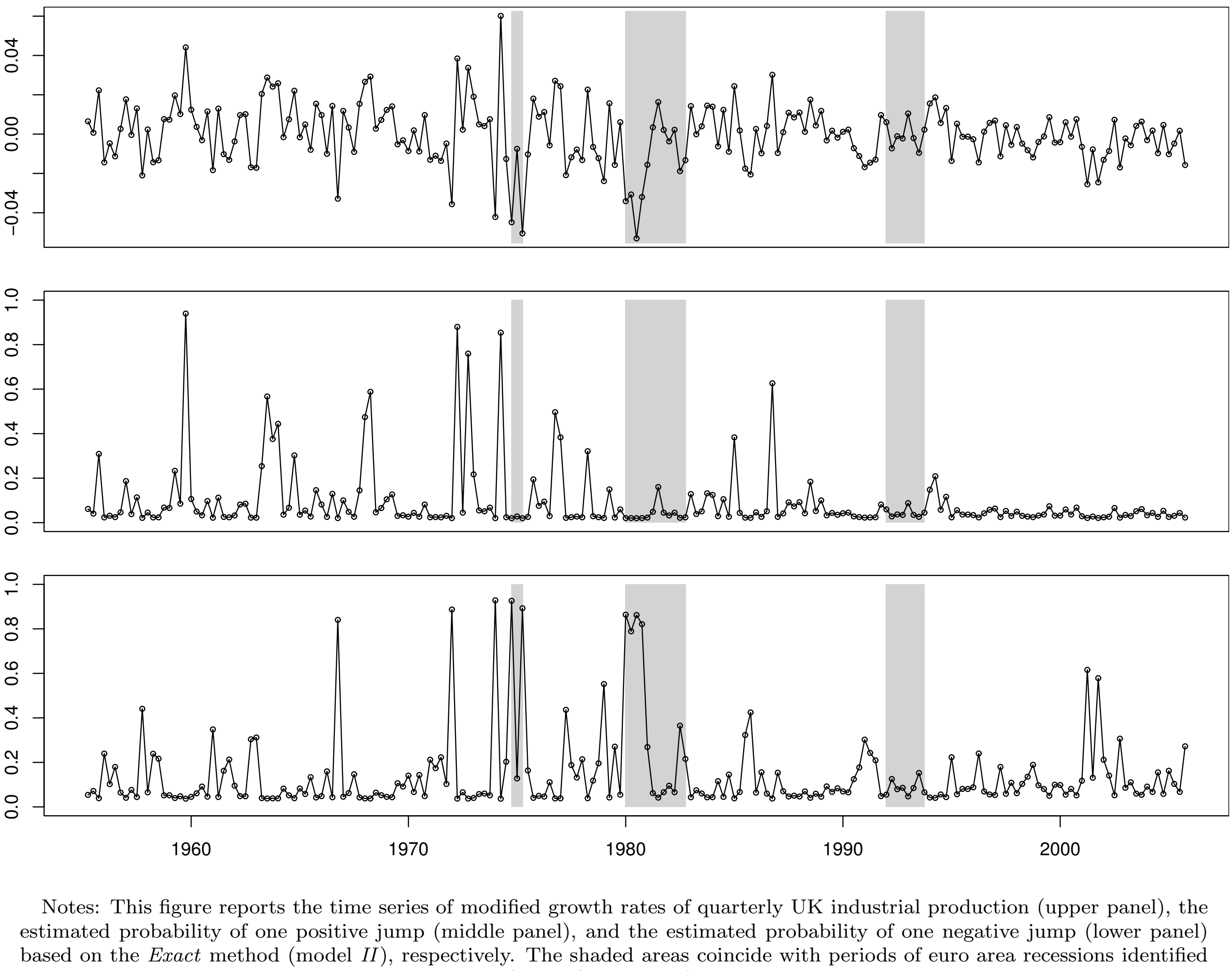
Table 10: Descriptive statistics of IP growth rates omitting oil crises.

	UK $\Delta = 1/4$	UK $\Delta = 1/12$	US $\Delta = 1/12$	DE $\Delta = 1/12$	Euro12 $\Delta = 1/12$
Mean	0.0055	0.0012	0.0034	0.0024	0.0015
Standard dev.	0.0154	0.0144	0.0076	0.0160	0.0091
Skewness	0.2776	−0.6491	−0.0142	−0.0416	0.1354
Excess kurtosis	0.7592	12.6234	0.9841	0.7945	1.2020
Minimum	−0.0377	−0.1008	−0.0233	−0.0482	−0.0302
Maximum	0.0566	0.0893	0.0295	0.0669	0.0331
Observations	220	433	529	497	244

Notes: These tables report the descriptive statistics of growth rates of industrial production (IP) per unit of time, Δ, for the United Kingdom (UK), the United States (US), Germany (DE), and the euro area (Euro12).

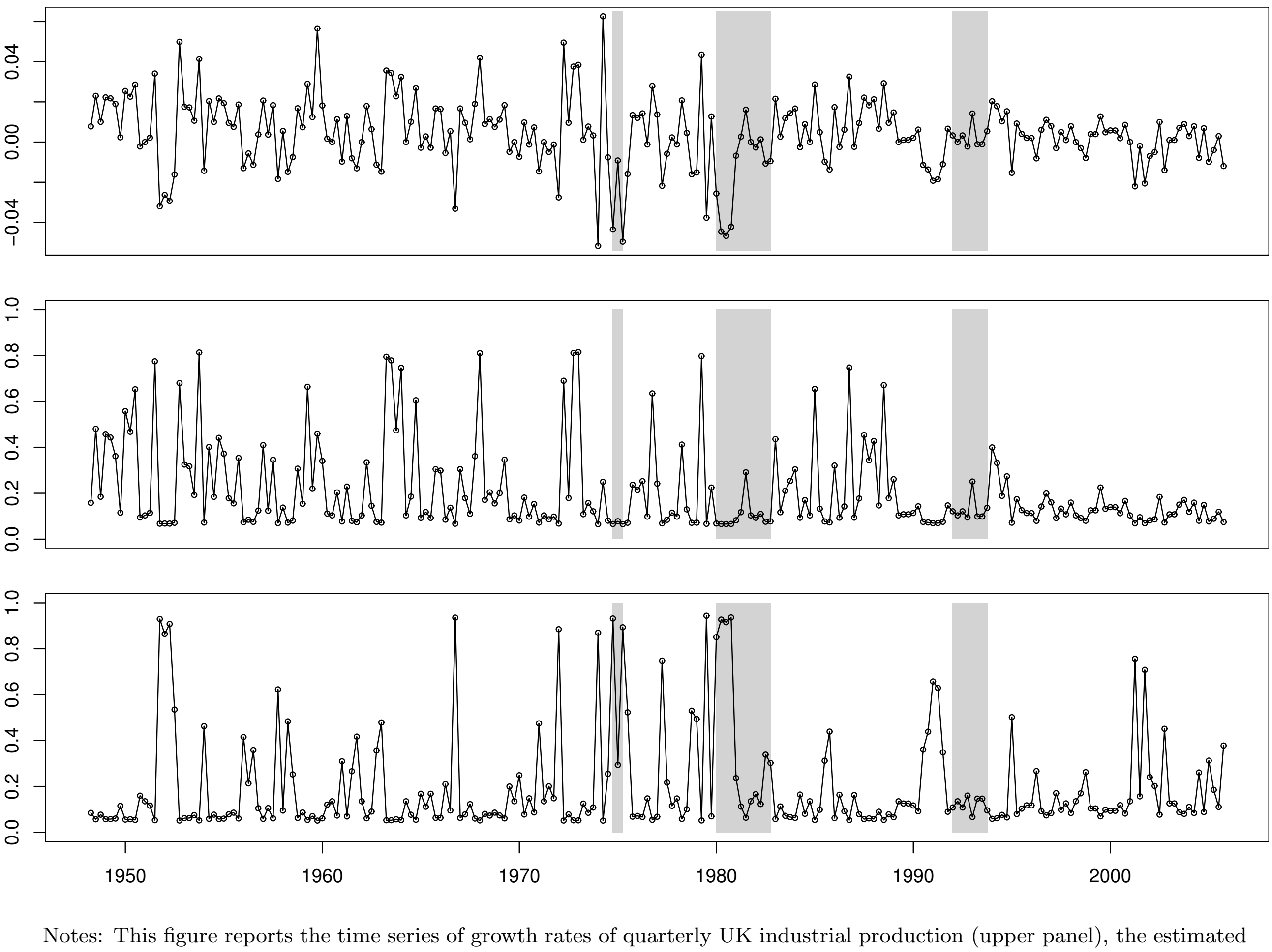
D.2 Empirical estimates

Figure 7: UK industrial production (manufacturing sector), modified quarter-to-quarter growth rates and jump probabilities (model *II*) from 1955Q1 to 2005Q4, seasonally adjusted



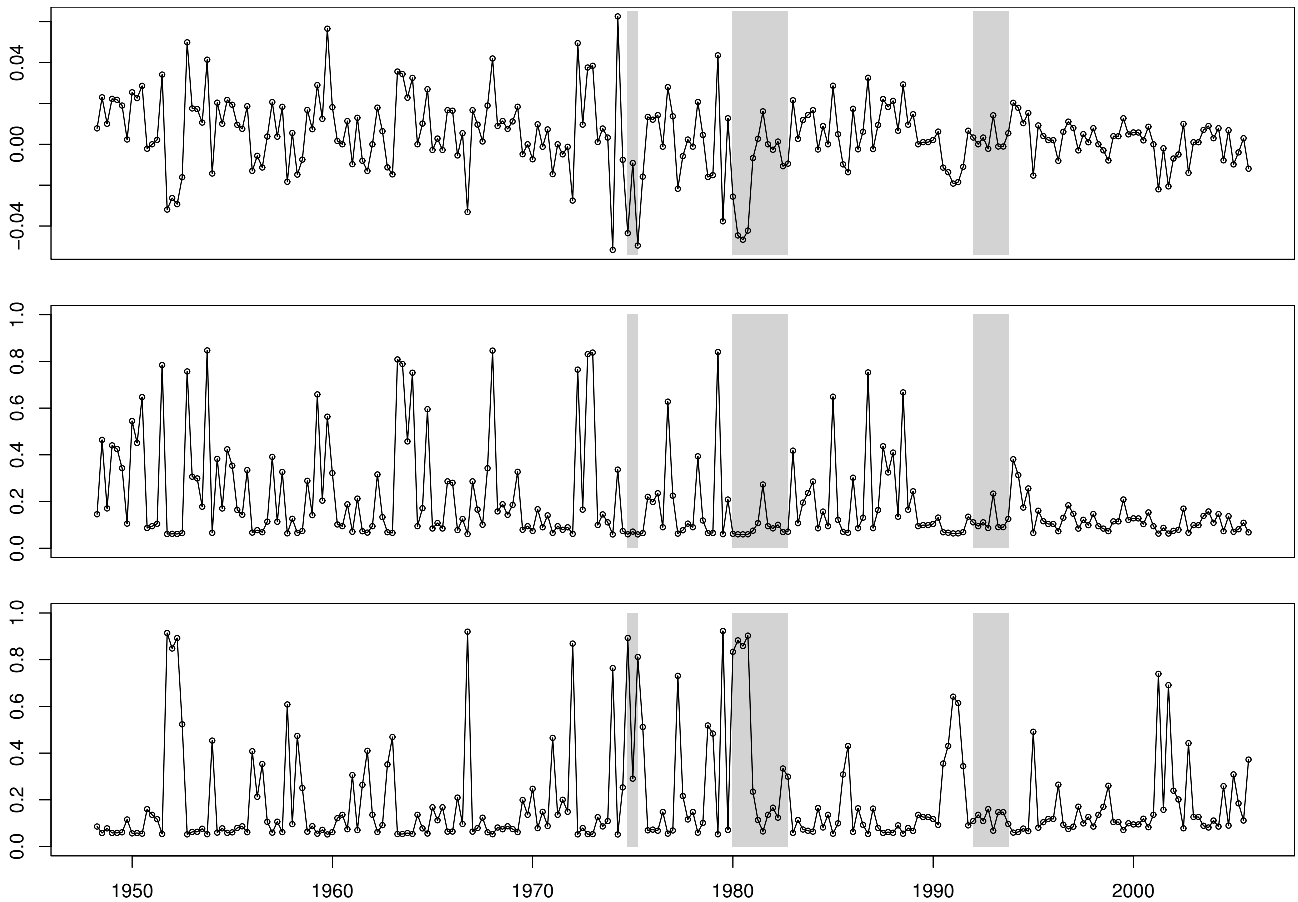
Notes: This figure reports the time series of modified growth rates of quarterly UK industrial production (upper panel), the estimated probability of one positive jump (middle panel), and the estimated probability of one negative jump (lower panel) based on the *Exact* method (model *II*), respectively. The shaded areas coincide with periods of euro area recessions identified by the CEPR's Business Cycle Dating Committee. ($\Delta = 1/4$; $N = 203$)

Figure 8: UK industrial production (manufacturing sector), quarter-to-quarter growth rates and jump probabilities (model *IIb*) from 1948Q1 to 2005Q4, seasonally adjusted



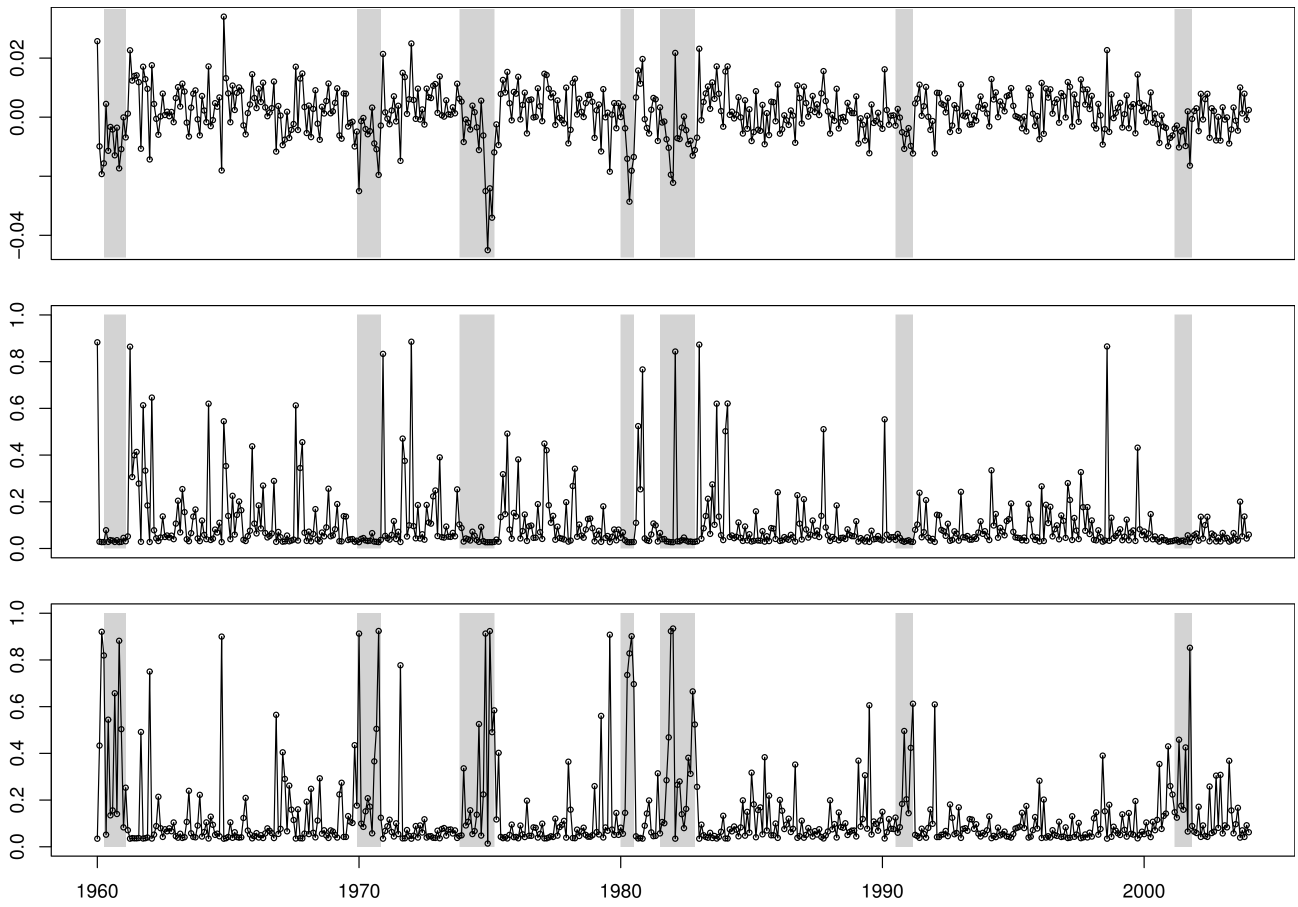
Notes: This figure reports the time series of growth rates of quarterly UK industrial production (upper panel), the estimated probability of one positive jump (middle panel), and the estimated probability of one negative jump (lower panel) based on the *Approximate* method (model *IIb*), respectively. The shaded areas coincide with periods of euro area recessions identified by the CEPR's Business Cycle Dating Committee. ($\Delta = 1/4$; $N = 231$)

Figure 9: UK industrial production (manufacturing sector), quarter-to-quarter growth rates and jump probabilities (model *IIb*) using fixed jump-sizes



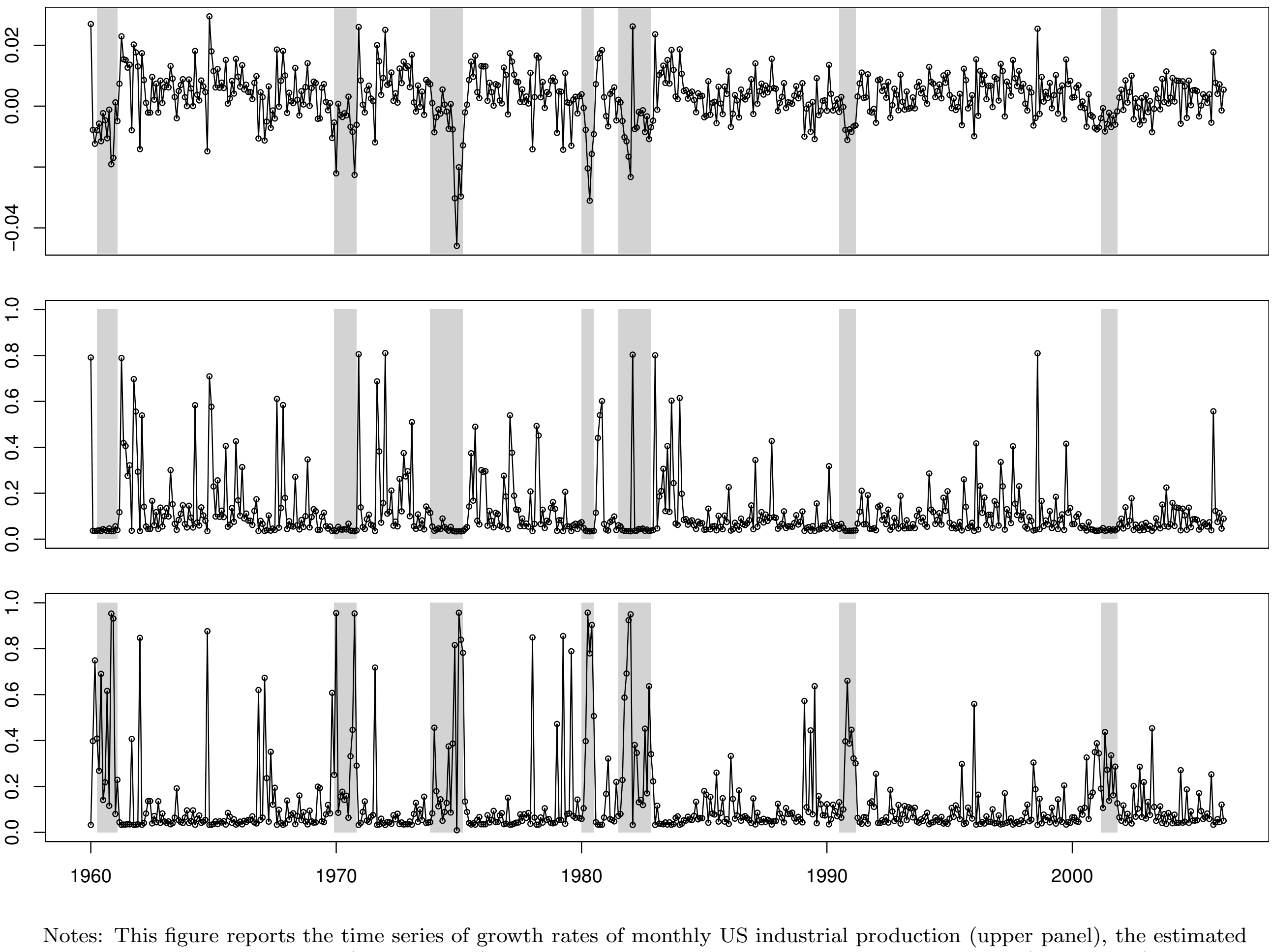
Notes: This figure reports the time series of growth rates of quarterly UK industrial production (upper panel), the estimated probability of one positive jump (middle panel), and the estimated probability of one negative jump (lower panel) based on the *Approximate* method (model *IIb*) using fixed-jump sizes (cf. Table 4), respectively. The shaded areas coincide with periods of euro area recessions identified by the CEPR's Business Cycle Dating Committee. ($\Delta = 1/4$; $N = 231$)

Figure 10: US industrial production (manufacturing sector), modified month-to-month growth rates and jump probabilities (model *II*) from 1960:01 to 2004:01, seasonally adjusted



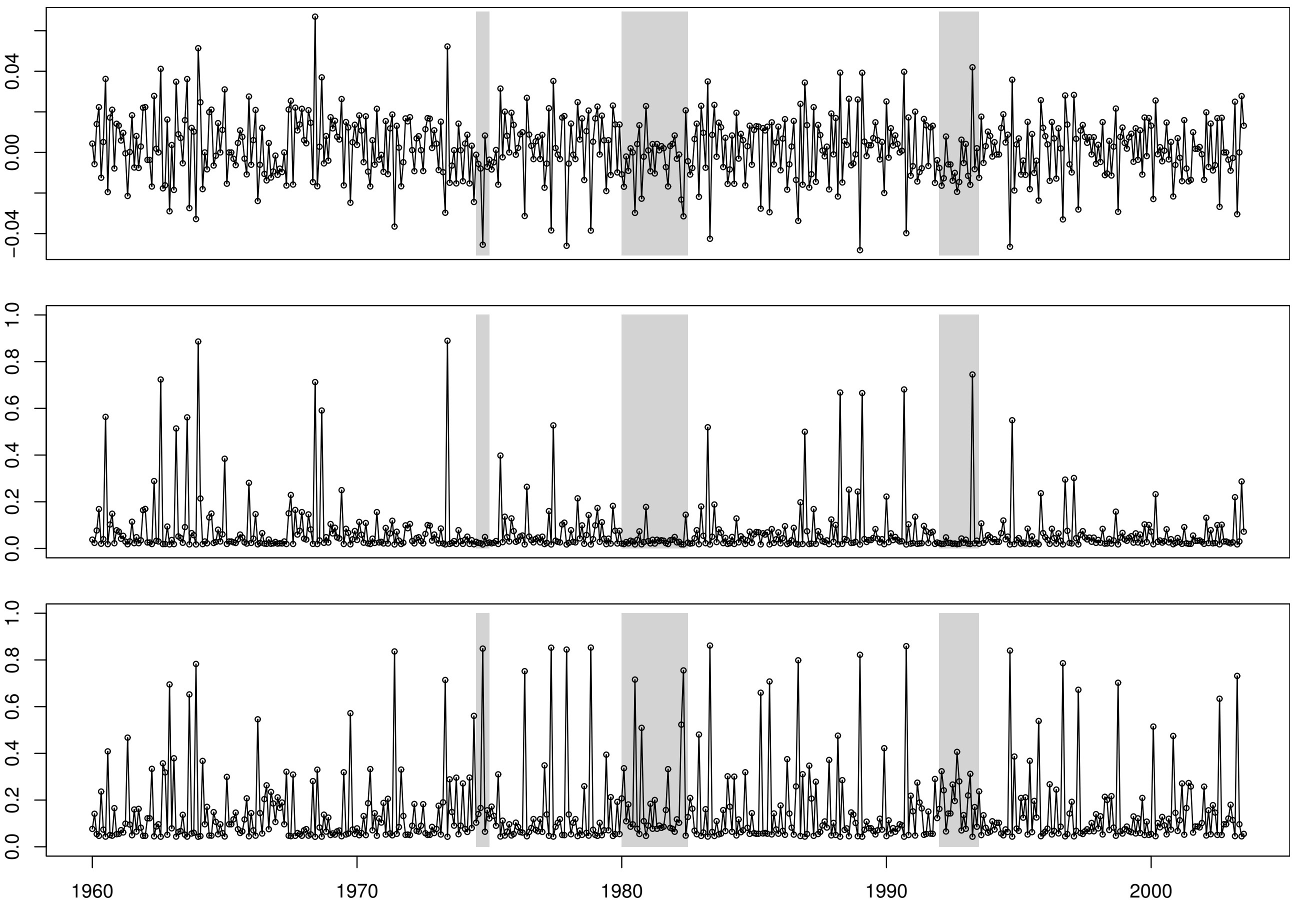
Notes: This figure reports the time series of modified growth rates of monthly US industrial production (upper panel), the estimated probability of one positive jump (middle panel), and the estimated probability of one negative jump (lower panel) based on the *Exact* method (model *II*), respectively. The shaded areas coincide with periods of US recessions identified by the NBER's Business Cycle Dating Committee. ($\Delta = 1/12$; $N = 529$)

Figure 11: US industrial production (manufacturing sector), month-to-month growth rates and jump probabilities (model *IIb*) from 1960:01 to 2006:03, seasonally adjusted



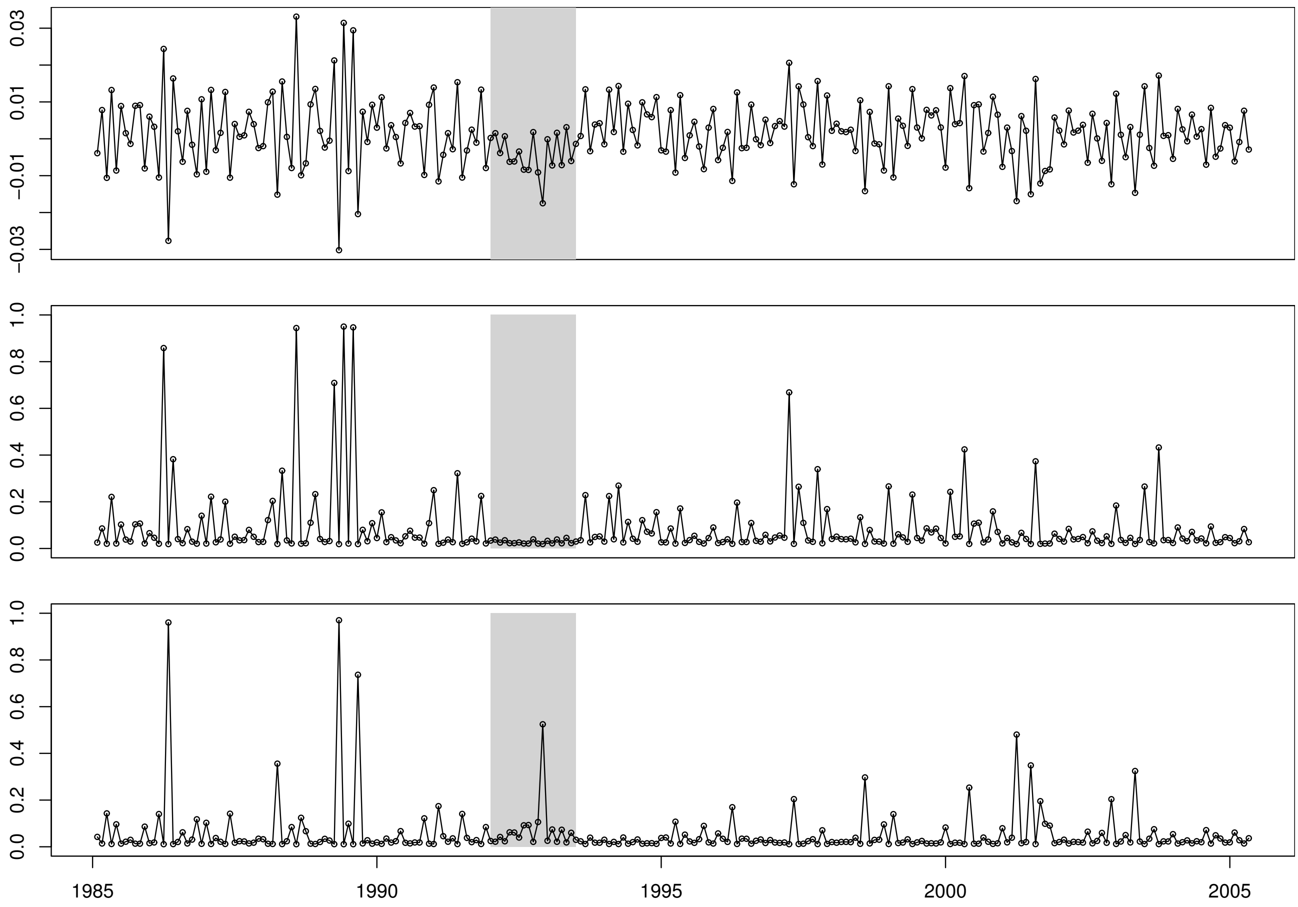
Notes: This figure reports the time series of growth rates of monthly US industrial production (upper panel), the estimated probability of one positive jump (middle panel), and the estimated probability of one negative jump (lower panel) based on the *Approximate* method (model *IIb*), respectively. The shaded areas coincide with periods of US recessions identified by the NBER's Business Cycle Dating Committee. ($\Delta = 1/12$; $N = 555$)

Figure 12: DE industrial production (Federal Statistical Office), month-to-month growth rates and jump probabilities (model *IIb*) from 1960:01 to 2003:07, seasonally adjusted



Notes: This figure reports the time series of growth rates of monthly DE industrial production (upper panel), the estimated probability of one positive jump (middle panel), and the estimated probability of one negative jump (lower panel) based on the *Approximate* method (model *IIb*), respectively. The shaded areas coincide with periods of euro area recessions identified by the CEPR's Business Cycle Dating Committee. ($\Delta = 1/12$; $N = 523$)

Figure 13: Euro12 industrial production (EABCN Real Time Database), month-to-month growth rates, jump probabilities (model *IIb*) from 1985:01 to 2005:05, seasonally adjusted



Notes: This figure reports the time series of growth rates of monthly euro area industrial production (upper panel), the estimated probability of one positive jump (middle panel), and the estimated probability of one negative jump (lower panel) based on the *Approximate* method (model *IIb*), respectively. The shaded areas coincide with periods of euro area recessions identified by the CEPR's Business Cycle Dating Committee. ($\Delta = 1/12$; $N = 244$)

Table 11: UK monthly IP (manufacturing sector) from 1968:01 to 2006:03

Method			$\hat{\nu}_s$	$\hat{\nu}_f$	$\hat{\lambda}$	$\hat{\eta}$	$\hat{\mu}$	$\hat{p}$
MLE	*Approximate*	*Ia*	0.0678		0.1005	0.0452	0.0041	
			(0.0017)		(0.0160)	(0.0002)	(0.0013)	
	$\ell_N(\theta)$				1243.9 (47.21)			
		Ib	0.0678		0.1005	0.0452	0.0012	
			(0.0017)		(0.0160)	(0.0002)	(0.0015)	
	$\ell_N(\theta)$				1243.9 (47.21)			
		IIa	0.0654	0.0535	0.3564	0.0374	0.0073	0.2988
			(0.0011)	(0.0009)	(0.0273)	(0.0003)	(0.0009)	(0.0357)
	$\ell_N(\theta)$				1360.1 (167.79)			
		IIb	0.0631	0.0553	0.3545	0.0374	0.0111	0.3003
			(0.0011)	(0.0009)	(0.0276)	(0.0003)	(0.0012)	(0.0362)
	$\ell_N(\theta)$				1360.0 (167.57)			

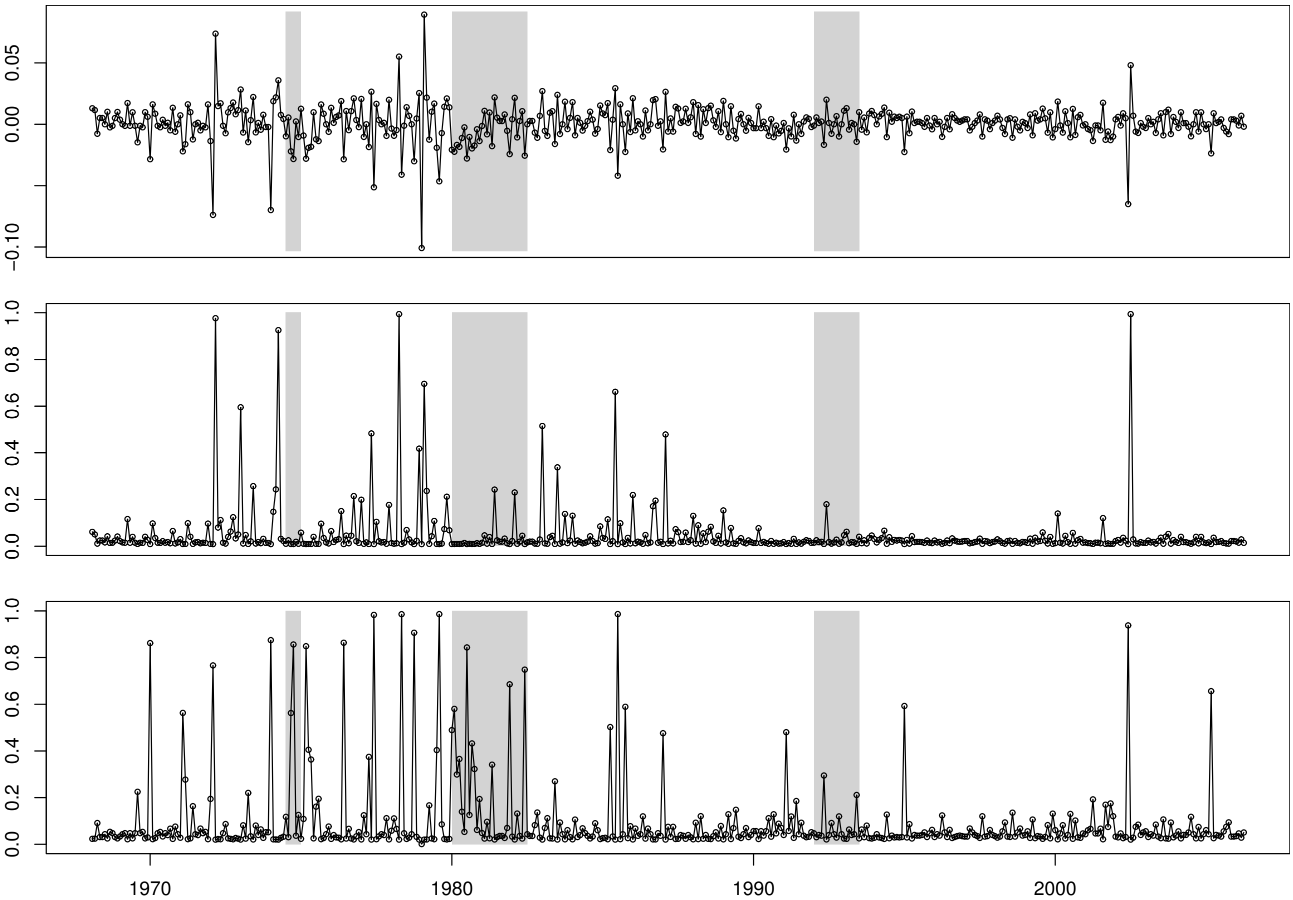
Notes: This table reports the ML estimates for degenerated jumps (model *I*), and binomial jump-size distribution (model *II*) using *Approximate* methods: (a) around the non-stochastic steady-state, and (b) around the mean values, respectively. Standard errors and likelihood ratio tests against the null of no jumps in parentheses. Omitting periods of oil crises in model *I*. Calibrated parameters are $(\alpha, \sigma) = (.5, .5)$; $\Delta = 1/12$; $N = 459$ ($N = 433$).

Table 12: UK monthly IP (manufacturing sector) with fixed jump-sizes

Method			$\hat{\nu}_s$	$\hat{\nu}_f$	$\hat{\lambda}$	$\hat{\eta}$	$\hat{\mu}$	$\hat{p}$
MLE	*Approximate*	*Ia*	0.0300		0.2339	0.0455	0.004	
			(0.0000)		(0.0254)	(0.0002)	(0.0015)	
	$\ell_N(\theta)$				1234.3 (27.93)			
		Ib	0.0300		0.2339	0.0455	0.0010	
			(0.0000)		(0.0254)	(0.0002)	(0.0016)	
	$\ell_N(\theta)$				1234.3 (27.93)			
		IIa	0.0300	0.0400	0.7779	0.0356	0.0075	0.4429
			(0.0000)	(0.0000)	(0.0464)	(0.0004)	(0.001)	(0.0274)
	$\ell_N(\theta)$				1348.2 (144.13)			
		IIb	0.0300	0.0400	1.0681	0.0342	0.0242	0.2450
			(0.0000)	(0.0000)	(0.0593)	(0.0004)	(0.0015)	(0.0194)
	$\ell_N(\theta)$				1349.8 (147.35)			

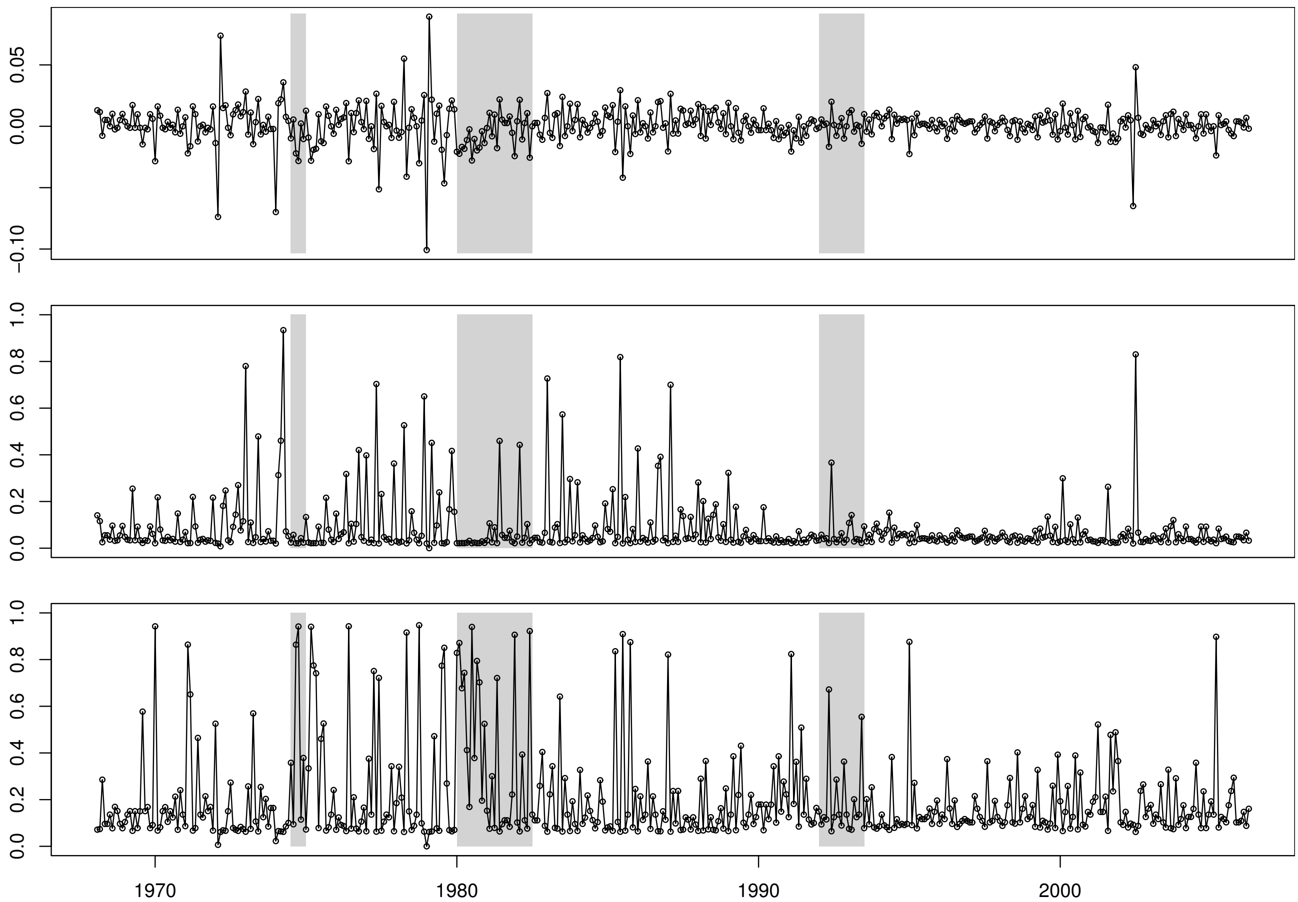
Notes: This table reports the ML estimates for degenerated jumps (model *I*), and binomial jump-size distribution (model *II*) using *Approximate* methods: (a) around the non-stochastic steady-state, and (b) around the mean values with fixed jump-sizes, respectively. Standard errors and likelihood ratio tests against the null of no jumps in parentheses. Omitting periods of oil crises in model *I*. Calibrated parameters are $(\alpha, \sigma) = (.5, .5)$; $\Delta = 1/12$; $N = 459$ ($N = 433$).

Figure 14: UK industrial production (manufacturing sector), month-to-month growth rates and jump probabilities (model *IIb*) from 1968:01 to 2006:04, seasonally adjusted



Notes: This figure reports the time series of growth rates of monthly UK industrial production (upper panel), the estimated probability of one positive jump (middle panel), and the estimated probability of one negative jump (lower panel) based on the *Approximate* method (model *IIb*), respectively. The shaded areas coincide with periods of euro area recessions identified by the CEPR's Business Cycle Dating Committee. ($\Delta = 1/12$; $N = 459$)

Figure 15: UK industrial production (manufacturing sector), month-to-month growth rates and jump probabilities (model *IIb*) with fixed jump-sizes



Notes: This figure reports the time series of growth rates of monthly UK industrial production (upper panel), the estimated probability of one positive jump (middle panel), and the estimated probability of one negative jump (lower panel) based on the *Approximate* method (model *IIb*) using fixed jump-sizes (cf. Table 12), respectively. The shaded areas coincide with periods of euro area recessions identified by the CEPR's Business Cycle Dating Committee. ($\Delta = 1/12$; $N = 459$)

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