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Estimating Multi-Equational LQAC Models with I(1) Variables: a VAR Approach

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Abstract.
This paper extends the existing literature on linear quadratic adjustment cost (LQAC) models under rational expectations to the inferential issues arising when: (i) agents optimise with respect to a vector of endogenous variables; (ii) the behavioural equations stemming from the agent's optimisation problem are specified as 'exact' rational expectations models; (iii) the stochastic processes involved are integrated of order one. We discuss estimation both in a 'limited-information' and in a 'full-information' framework. In the first case we show that consistent estimation of the structural parameters may be achieved by focusing on the open-form solution to the model, and implementing existing procedures. In the second case, by focusing on the unique and stable forward-looking solution to the model, we propose a likelihood-based inferential procedure in time domain. The key assumption is that agents form expectations through a cointegrated vector autoregression (CVAR) system representing the joint data generation process for both the endogenous and exogenous variables.

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suggestions. All views expressed are my own and I am solely responsible for all errors.

1. Introduction.

Considerable attention has been recently devoted to the econometric analysis of intertemporal behaviour under adjustment costs and uncertainty. The standard tool representative of this kind of models is the linear quadratic adjustment cost model (LQAC) under rational expectations. The model stems from intertemporal optimising behaviour of agents subject to quadratic adjustment costs. In particular, agents are assumed to minimise the expected discounted present value of an infinite period quadratic cost function, conditional on the information currently available. The objective function generally measures both the cost of being away from a desired target depending on a set of exogenous (forcing) variables, and the cost of adjustment. Even though quadratic costs are unlikely to apply in reality, intertemporal LQAC models are widely used for empirical research due to their capability to yield linear behavioural rules approximating sluggish dynamics observed in economic phenomena. General properties, solutions, stability requirements and estimation issues when agents choose a single variable in discrete time may be found, *inter alia*, in Kennan (1979), Sargent (1979), Hansen and Sargent (1980, 1982). Representation and estimation when the forcing variables of the system are generated by integrated processes are respectively discussed in Nickell (1985), Dolado *et al.* (1991), Gregory *et al.* (1993), West (1995), and Engsted and Haldrup (1994, 1996a, 1996b).

The analysis of LQAC models can be extended to situations where the intertemporal optimisation involves a vector of variables rather than a scalar, as in Hansen and Sargent (1981). Extending the representation and inference in this direction it is in general possible to derive structural dynamic models characterising the interrelated adjustment of many economic phenomena. The implementation of multi-equation linear quadratic adjustment cost models, henceforth MLQAC, may be found in Lucas (1967), Treadway (1971), Eichembaum (1984), Nickell (1984), Weissemberger (1986)[1]. However, so far no attention has explicitly devoted to the inferential issues arising in MLQAC models when the variables being modelled are generated by integrated processes. The purpose of the present paper is to fill this gap.

We assume that the dynamic behaviour of economic agents stems from the minimisation of the expected discounted value of a quadratic objective function, which incorporates a stochastic multivariate equilibrium target. The cost function measures both the cost of being away from the desired target, and the cost of adjustment, conditional on the available information. The multivariate target is specified as a structural system of equations linking the desired values of the vector of the endogenous variables to the expected evolution of a set of exogenous variables. Following Hansen and Sargent (1991), the model is specified as an ‘exact’ rational expectations model.

Firstly we discuss the solutions to the agent’s optimisation problem characterising the MLQAC model. We show that the open-form solution reads as a system of interrelated Euler equations, while the unique and stable (quasi)closed-form solution can be derived by suitably restricting the parameters of the objective function and reads as the forward-looking behavioural rule of the agent. Secondly we dealt with inferential issues. It is well known that estimation of econometric models involving rational expectations can be carried out either in a ‘limited-information’ context through the ‘errors-in-variables method’, or in a ‘full-information’ context, through the ‘substitution method’, see e.g. Wickens (1982). We consider both approaches.

As regards limited-information methods, we focus on the open-form solution to the MLQAC model. By generalising the procedures set out in Dolado *et al.* (1991) and Gregory *et al.* (1993), the only information exploited is the order of integration of the variables. We show that estimation can be carried out by a two-step procedure involving instrumental variables and existing procedures. Specifically, in the first-step the model fits into the framework described by Park and Phillips (1989), Phillips and Hansen (1991) and Phillips (1995), while in the second-step the model can be easily fitted into the framework described by Cumby *et al.* (1983).

As regards full-information methods, we focus on the forward-looking solution to the MLQAC model and set out a likelihood-based inferential procedure in time domain. By implementing and adapting the ideas in Baillie (1989) and Johansen and Swensen (1994), the key assumption is that agents form expectations by means of a cointegrated vector autoregressive (CVAR) system with Gaussian errors, representing the joint data generation process of both the endogenous and exogenous variables. Using the CVAR to compute the expectations, we derive a dynamic system in the predetermined variables incorporating all the cross-equations non-linear constraints implied by the rational expectations hypothesis. We show that issues such as identification, estimation and testing of the rational expectations restrictions, can be dealt with by means of the associated likelihood function. Actually, due to the cointegrating nature of the model, the inferential procedure is divided into two steps, where in the first step the parameters involved are the ones attached to the long-run equilibrium target pursued by the agent, while in the second step the ones involved regard the short-run dynamics of the model.

The plan of the paper is the following. In section 2 we introduce the agent's optimisation problem characterising the MLQAC model, and discuss its solutions and properties. Section 3 briefly deals with estimation issues in a 'limited-information' context, through the 'errors-in-variables' method. Attention is devoted on the open form-solution to the model, and the only information exploited is the order of integration of the process generating the variables. Section 4 focuses on the unique and stable forward-looking solution to the MLQAC model. The purpose is to propose a 'full-information' likelihood-based inferential procedure in time domain. Subsection 4.1 introduces the expectations generating system as a CVAR model with Gaussian errors for the observable variables. Using the CVAR to compute the expectations appearing in the structural model, in subsection 4.2 we derive a dynamic model in the predetermined variables, incorporating all the cross-equations non-linear constraints implied by the rational expectations hypothesis. Section 5 deals with identification and FIML estimation issues of such model. The analysis is based on a two-step procedure outlined in subsections 5.1 and 5.2. Section 6 completes the analysis by testing the rational expectations restrictions implied by the MLQAC model. Likelihood-ratio tests are proposed as the 'natural' solution to the problem. Section 7 contains some concluding remarks, while the technical details of the discussion may be found in the Appendix A and B.

2. The MLQAC model and its representations.

We consider a stylised intertemporal linear quadratic model in which an economic agent is faced with the task of taking a sequence of decisions at each time period t under uncertainty. The underlying probability space is (Ξ, A, P), where P is the probability measure of the agent. All the stochastic processes involved in the discussion are defined on (Ξ, A, P), are denoted by the symbol {.} and are assumed to have finite second-order moments. Decisions imply the simultaneous choices of m observable endogenous variables, denoted by $\{y_t\}$, and such choices depend on the values assumed by q observable exogenous (forcing) variables, generated by the ($q\times1$)

process $\{x_t\}$. Agents do not control $\{x_t\}$ but their intertemporal plans depend on the expected values of $\{x_t\}$. The agent's information set is given by the sigma-field Φ_t, $\Phi_{t-1}\subset\Phi_t\subset A$, and $E_t\cdot\equiv E\{\cdot\,|\,\Phi_t\}$ is the expectations operator conditional on Φ_t. We assume that $E\{\cdot\,|\,\Phi_t\}$ is defined for every process of the model and that $\Phi_t=\sigma(y_t', y_{t-1}', ..., x_t', x_{t-1}',...)$.

The optimal sequence $\{y_t\}$ is chosen to minimise the expected value of the quadratic objective-function:

$$\min_{\{y_t\}} E_t \sum_{j=0}^{\infty}\delta^j \left\{\left(y_{t+s}-y^*_{t+s}\right)'\Theta_1\left(y_{t+s}-y^*_{t+s}\right)+\Delta y_{t+s}'\Theta_2\Delta y_{t+s}\right\} \tag{2.1}$$

where Δ is the difference operator, $\Delta y_t = (y_t - y_{t-1})$, δ is the (scalar) agent's discount factor, $0<\delta<1$, Θ_1 and Θ_2 are $(m\times m)$ matrices and y_t^* denotes the $(m\times 1)$ equilibrium target pursued by the agent. Θ_1 and Θ_2 are restricted to be positive definite and symmetric, while the evolution of y_t^* is related to x_t by the system of equations:

$$y_t^* = \Gamma\, x_{t/t-1} + \Gamma_0 \tag{2.2}$$

where $x_{t/t-1}$ is the expected value of x_t made at time t-1: $x_{t/t-1}= E\{x_t\,|\,\Phi_{t-1}\}$, Γ is a matrix of structural parameters of dimensions $(m\times q)$, and Γ_0 is a $(m\times 1)$ vector of constants.

Problem (2.1) satisfies the Certainty Equivalence Theorem (see, for example, Caines, 1988) and the first-order necessary conditions consist of a set of Euler equations and associated transversality conditions. The Euler equations read:

$$\delta\, E_t\, \Delta y_{t+1} - \Delta y_t - \Theta\,(y_t - y_t^*) = 0_{m\times 1} \tag{2.3}$$

that is a second-order matrix difference equation where $\Theta=\Theta_2^{-1}\Theta_1$. System (2.3) represents the open-form solution to the MLQAC model and Θ is positive definite and in general non symmetric. The economic interpretation of the coefficients in Θ will be discussed in the examples below. The set of transversality conditions associated to (2.3) may be expressed in the form (see Appendix A):

$$\lim_{T\to\infty} \delta^T \; E_t \; [\Delta y_{t+T} + \Theta \; (y_{t+T} - y_{t+T}^*)] = 0_{m\times 1} \qquad (2.4)$$

and since these are necessary for the convergence of the infinite sum in (2.1), (2.4) ensures the existence of the solution to the economic agent's problem. As we show in the Appendix A, the (quasi)closed-form solution to (2.1) that satisfies (2.3) and (2.4) is given by the rule:

$$y_t = \Lambda y_{t-1} + \sum_{j=0}^{\infty} (\delta \Lambda)^j (I_m - \delta \Lambda)(I_m - \Lambda)\Gamma\, E_t x_{t+j} + (I_m - \Lambda)\Gamma_0 \qquad (2.5)$$

where I_m is the $(m\times m)$ identity matrix, and Λ is the stable and unique solution to the matrix equation:

$$\delta \; \Lambda^2 - [(\delta+1)I_m + \Theta] \; \Lambda + I_m = 0_{m\times m} \qquad (2.6)$$

Notice that (2.6) represents the link between the parameters of (2.3) and the parameters of the forward-looking solution to the intertemporal MLQAC model (2.5). Following Hansen and Sargent (1991), both (2.3) and (2.5) can be interpreted as 'exact' rational expectations models, in that there are no expectations in (2.3) and (2.5) involving stochastic processes unobservable to the econometrician[2]. Notice that (2.5) can also be formulated in the error-correction format (see Appendix A):

$$\Delta y_t = (\Lambda - I_m)(y_{t-1} - \Gamma x_{t-1}) + \sum_{j=0}^{\infty} (\delta \Lambda)^j (I_m - \Lambda)\Gamma\, E_t \Delta x_{t+j} + (I_m - \Lambda)\Gamma_0 \qquad (2.7)$$

which should not be interpreted as a standard error-correction model where adjustment is entirely the result of discrepancies from equilibrium in the past. To prove this, let us write (2.7) as:

$$\left(y_t - \Gamma x_t\right) = \Lambda\left(y_{t-1} - \Gamma x_{t-1}\right) - \Lambda\,\Gamma\Delta x_t + \sum_{j=1}^{\infty}\left(\mathrm{d}\,\Lambda\right)^j\left(I_m - \Lambda\right)\Gamma\,E_t\Delta x_{t+j} \tag{2.8}$$

so that it is evident that the disequilibrium is both the result of a feed-forward mechanism due to expectations about unknown future variables, and feed-back effects through lagged disequilibrium and changes in the exogenous variables. Below we report two clarifying examples in order to shed some light on the features of the model.

Example 1. When m=1 we have a scalar LQAC model. That is, y_t is a scalar, Θ_1 = c>0, Θ_2 = d>0, and system (2.3) collapses to an Euler equation of the form:

$$\delta\ E_t\ \Delta y_{t+1} - \Delta y_t - \theta(y_t - y_t^*) = 0 \tag{e.1}$$

where θ=(c/d)>0 measures the relative importance of disequilibrium and adjustment costs. The stochastic target is $y_t^* = \gamma' x_{t/t-1} + \gamma_0$ and x_t is a (q×1) vector of exogenous variables. System (2.7) collapses to:

$$\Delta y_t = (\lambda - 1)\left[y_{t-1} - \gamma' x_{t-1}\right] + (1-\lambda)\sum_{j=0}^{\infty}(\delta\,\lambda)^j\,E_t\,\gamma'\Delta x_{t+s} + \qquad (1-\lambda)\gamma_0 \tag{e.2}$$

where now λ is the stable root satisfying the characteristic equation: $\delta\lambda^2 - (\delta+1+\theta)\lambda + 1 = 0$. Estimation issues of (e.1)-(e.2) in the presence of integrated processes may be found in Dolado *et al.* (1991), Gregory *et al.* (1993) and Engsted and Haldrup (1994, 1996a, 1996b). Note that (e.1) and (e.2) and their variants have been extensively used in empirical research[3] .

Example 2. When m=2 we have a two-dimensional MLQAC, that is: $y_t = (y_{1t}, y_{2t})'$, and

$\Theta_1=\begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{bmatrix}$, $\Theta_2=\begin{bmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{bmatrix}$. If for instance q=2, $x_t = (x_{1t}, x_{2t})'$ and:

$$y_t^*=\begin{pmatrix} y_{1t}^* \\ y_{2t}^* \end{pmatrix}=\begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} x_{t/t-1} + \begin{bmatrix} \gamma_{01} \\ \gamma_{02} \end{bmatrix}$$

System (2.3) now reads:

$$\delta\, E_t\, \Delta y_{1t+1} - \Delta y_{1t} - \theta_{11}(y_{1t} - y_{1t}^*) - \theta_{12}(y_{2t} - y_{2t}^*) = 0 \qquad \text{(e.3a)}$$

$$\delta\, E_t\, \Delta y_{2t+1} - \Delta y_{2t} - \theta_{22}(y_{2t} - y_{2t}^*) - \theta_{21}(y_{1t} - y_{1t}^*) = 0 \qquad \text{(e.3b)}$$

where $\Theta=\Theta_2^{-1}\Theta_1=\begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix}$. The computation of the elements in Θ shows that these are measures of the relative importance of disequilibrium, adjustment and cross-adjustment costs. As regards (2.5) or (2.7) it is sufficient to notice that the coefficients in $\Lambda=\begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix}$ must be interpreted as 'partial-adjustment' parameters in a multivariate framework.

♦

Systems (2.3) and (2.5) (or (2.7)) form the basis form estimation techniques of the MLQAC model. While in (2.3) expectations involve the future one-period endogenous variables, in (2.7) expectations involve the future infinite values of the forcing variables. It is well known that estimation of econometric model involving rational expectations can be carried out either in a 'limited-information' context through the 'errors-in-variables method', or in a 'full-information' context, through the 'substitution method', see e.g. Wickens (1982). 'Limited-information' methods do not require a detailed specification of the mechanism by which agents form expectations. Expectations are indeed replaced by the realised (observed) values of the variables creating measurement errors. Estimation does not generally achieve full efficiency. On the other hand, ‘full-information’ methods require the specification of the mechanism by which agents form expectations. The joint estimation of the structural model and the mechanism generating expectations, while exploiting the cross-equation constraints imposed by the rational expectations restrictions, forms the basis of the fully asymptotic efficient method. These methods are traditionally considered computationally burdensome[4].

In the present paper we consider both approaches under the assumption that the variables of the MLQAC model are I(1). In section 3 we show that ‘limited-information’ methods can be easily applied to system (2.3) by implementing existing procedures. In sections 4, 5 and 6 we devote attention to the forward-looking system (2.5) and by specifying the expectations generating mechanism we propose a ‘full-information’ likelihood-based procedure in time domain.

3. Estimation in a ‘limited-information’ framework.

We examine whether an estimation procedure can be found which makes use of the second-order matrix difference equation (2.3), disregarding the specification on the mechanism by which agents forecast future values of the variables. We only assume that $\{y_t\}$ is I(1), and $\{x_t\}$ is I(1) and not cointegrated. Let us consider the following orthogonal decomposition:

$$x_t = E_{t-1}x_t + \varphi_t \qquad ; \qquad E_t\varphi_{t+1} = 0_{q\times 1} \tag{3.1a}$$

$$\Delta y_{t+1} = E_t \Delta y_{t+1} + \eta_{t+1} \qquad ; \qquad E_t\eta_{t+1} = 0_{m\times 1} \tag{3.1b}$$

where $\{\varphi_t, \Phi_t\}$ and $\{\eta_t, \Phi_t\}$ are respectively martingale difference sequences on (Ξ, A, P) and can be interpreted as ‘rational expectations’ errors. By using (3.1), the expected values in (2.3) can be substituted by the observed values, yielding the estimating system of equations:

$$\Delta y_t = \delta \Delta y_{t+1} - \Theta(y_t - \Gamma^* x_t^*) + \upsilon_t \tag{3.2}$$

where $\Gamma^*=\left[\Gamma \vdots \Gamma_0\right]$, $x_t^*=(x_t', 1)'$, $u_t = (\Theta v_t - \delta h_{t+1})$, and $v_t = -\Gamma j_t$ with $E(x_t v_t')\neq 0_{q\times m}$. It is easy to show that in (3.2) the error term u_t is such that:

$$E\{u_t \mid \Phi_t\}\neq 0_{m\times 1} \qquad (3.3a)$$

$$E\{u_t \mid \Phi_{t-1}\}= 0_{m\times 1} \qquad (3.3b)$$

If we further assume that the agent's 'rational expectations' errors are correlated, i.e. $\Sigma_{vh}= E(v_t h_t')\neq 0_{m\times m}$, then $\{u_t\}$ is an autocorrelated process of order one[5]. The structural parameters of interest are (Γ^*, δ, Θ) and possibly the coefficients of the matrix $\Sigma_u= E(u_t u_t')$. Since in many empirical studies it is common finding that the estimate of the discount factor appears to be rather imprecise, we impose an over-identification restriction on (3.2), by prefixing δ [6]. In order to ensure consistency between the order of integration of the variables and the structure of (3.2) we need the hypothesis:

$$\{(y_t - \Gamma^* x_t^*)\} \equiv \{e_{1t}\} \text{ is } I(0) \qquad (3.4)$$

which implies that the stochastic processes $\{y_t\}$ and $\{x_t\}$ are cointegrated, with cointegration matrix Γ, and constant Γ_0.

One of the purposes of the present paper is to show that consistent estimation of (Γ^*, Θ, Σ_υ) in this 'limited-information' context can be achieved by a two-step procedure where Γ^* is estimated separately from (Θ, Σ_υ). Indeed note that under the assumptions above, the partial system:

$$y_t = \Gamma x_t + \Gamma_0 + e_{1t} \qquad (3.5)$$

$$\Delta x_t = e_{2t} \qquad (3.6)$$

where the e_{it}, i=1,2, are all stationary I(0) processes, is such that $\{e_t\}$, $e_t=(e_{1t}', e_{2t}')'$, satisfies a multivariate invariance principle, see e.g. Park and Phillips (1989). Thus (3.5)-(3.6) fits into the theoretical framework described in Park and Phillips (1989) and Phillips and Hansen (1990), so that a 'first-step' and super-consistent estimate of Γ can be achieved by OLS or IV[7]. Clearly, when dealing with IV we assume that an I(1) process $\{w_t\}$ of dimension $(h\times 1)$, $h\geq q$, is available, such that $\Delta w_t = e_{3t}$, and $\{e_t\}$, $e_t=(e_{1t}', e_{2t}', e_{3t}')'$, is I(0) [8]. The main consequences of the super-consistency result is that Γ can be replaced in (3.2) by its estimator, $\hat{\Gamma}$, as if $\hat{\Gamma}$ were the true value of Γ, and $\{(y_t - \hat{\Gamma} x_t)\}$ can be treated as an I(0) process. Therefore the 'second-step' system:

$$\Delta y_t = \delta \Delta y_{t+1} - \Theta(y_t - \hat{\Gamma} x_t) + \varsigma + u_t \qquad (3.7)$$

where $\varsigma = -\Theta\Gamma_0$, is a model involving only I(0) variables where the only quantities to be estimated in (3.7) under (3.3) are the adjustment matrix Θ and the covariance matrix $\Sigma_u = E(u_t u_t')$. Note that the substitution of Γ by $\hat{\Gamma}$ only affects the asymptotics in (3.7) through terms of $o_p(T^{-1})$.

Actually, due to the correlation between the regressors and the error term, the asymptotic distribution of the OLS as well as of the IV estimator of Γ in (3.5)-(3.6) is affected by the presence of a 'second-order' bias reflecting on inference in finite samples. A procedure to eliminate this bias without fully specifying the process generating $\{e_t\}$ is suggested in Phillips and Hansen (1990) and Phillips (1995). The idea is to modify the OLS and IV estimators of Γ^* for the effects of simultaneity and serial correlation by means of a consistent and non parametric estimate of the 'long-run' covariance matrix associated to (3.5)-(3.6). This is given by the expression:

$$\Omega = \lim_{T\to\infty} E\left[T^{-1}\left(\sum_1^T e_t\right)\left(\sum_1^T e_t\right)'\right]$$

and its non-parametric estimation is discussed with details in Phillips (1995). The 'fully-modified' least squares (FM-OLS) and the 'fully-modified' instrumental variables (FM-IV) estimators of Γ achieve the same asymptotic efficiency of systems maximum likelihood, which on the contrary require a detailed specification of the process generating $\{e_t\}$. Testing general hypothesis on Γ can be carried out by means of Wald test statistics formed with the FM-OLS or FM-IV estimators, and these statistics are asymptotically distributed as χ^2.

As regards the 'second-step' estimation of $(\Theta, \Sigma_\upsilon)$ in (3.7), it is in general possible to handle the correlation between the error term and the regressors by exploiting the orthogonality condition (3.3b) and implementing the principles of the generalised method of moments (GMM), set out in Hansen (1982). It is quite clear that in (3.7) GMM estimation of $(\Theta, \Sigma_\upsilon)$ through the orthogonality condition (3.3b) specialises in IV or GIV, where observable instruments are drawn from Φ_{t-1}. Two issues should be mentioned here. First, a straightforward application of IV or GIV does not take into account the serial correlation of $\{\upsilon_t\}$, implying lack of efficiency in inference. The problem can be relaxed by applying the procedures set up in Cumby *et al.* (1983) and Hayashi and Sims (1983). These are instrumental variables methods taking into account the possible serial correlation of $\{\upsilon_t\}$[9], and achieving efficiency within a class of GMM estimators exploiting the orthogonality condition (3.3b). Furthermore, though these methods have been proposed for single-equation rational expectations models, they can be easily generalised to the multi-equational framework. Note also that consistent and semi-positive definite estimation of Σ_υ can be achieved by applying the formula in Newey and West (1987). Second, it is not actually clear whether the estimate of $(\Theta, \Sigma_\upsilon)$ is indeed consistent. This is because when applying instrumental variables methods, the cross-moment matrix of instruments and regressors in (3.7) cannot be established to be non-singular without explicitly solving the model, see e.g. Binder and Pesaran (1995, subsection 3.1) [10].

4. Estimation in a 'full-information' framework: a VAR approach.

The purpose of this section and the ones below is to propose a 'full-information' likelihood-based inferential procedure in time domain for the MLQAC model introduced in section 2. Specifically, we focus on the exact model (2.5), that is on the agent's forward-looking behavioural rule so that the structural parameters of interest are $(\Gamma, \Gamma_0, \delta, \Lambda)$. In order to apply full-information methods, we need to specify the system used by agents to compute 'model-based' expectations. This is given by a cointegrated vector autoregressive model (CVAR) introduced in sub-section 4.1 below. Then maximum likelihood methods can be applied taking account of the cross-equations restrictions linking the coefficients of the agent's decision rule to those of the CVAR expectations generating system, as outlined in sub-section 4.2 and section 5.

4.1 Expectations generating mechanism.

Grouping both the endogenous and exogenous variables in the vector $z_t=(y_t', x_t')'$ of dimensions $(p\times 1)$, $p=(m+q)$, we assume that the process generating $\{z_t\}$ is given by the Gaussian VAR(k):

$$z_t = C_1 z_{t-1} + \ldots + C_k z_{t-k} + \mu + \varepsilon_t \quad ; \quad t=1,\ldots,T \quad ; \quad \{\varepsilon_t\}\sim iidN\,(0_{p\times 1}, \Sigma_\varepsilon) \qquad (4.1)$$

where k is supposed to be known, $z_{-k+1},\ldots, z_0$ are fixed, $C_1,\ldots, C_k$ are matrices of dimensions $(p\times p)$, $\mu=(\mu_y', \mu_x')'$ is a $(p\times 1)$ constant and $\Sigma_\varepsilon=\begin{bmatrix}\Sigma_{\varepsilon y} & \Sigma_{\varepsilon yx}\\ \Sigma_{\varepsilon xy} & \Sigma_{\varepsilon x}\end{bmatrix}$. Note that by not imposing the proper exclusion restrictions on C_{yi}, $i=1,2,\ldots,k$, in $C_1=\begin{bmatrix}C_{y1}\\ C_{x1}\end{bmatrix}$, ..., $C_k=\begin{bmatrix}C_{yk}\\ C_{xk}\end{bmatrix}$, we allow for Granger causality of $\{y_t\}$ with respect of $\{x_t\}$. The condition $\Sigma_{\varepsilon xy}\neq 0_{q\times m}$ prevents $\{x_t\}$ to be strictly exogenous in (4.1). From (4.1) it is possible to derive the error-correction representation of the VAR:

$$\Delta z_t = \Pi z_{t-1} + \Pi_2 \Delta z_{t-1} + \ldots + \Pi_k \Delta z_{t-k+1} + \mu + \varepsilon_t \qquad (4.2)$$

where $\Pi=(C_1+\ldots+C_k - I_p)$ and $\Pi_i= -(C_i+\ldots+C_k)$, $i=2,\ldots,k$.

The fundamental hypothesis of the present paper is that the process $\{z_t\}$ is CI(1,1). It is thus well known that whether in (4.2):

$$\Pi=\alpha\beta' \quad ; \quad \mathrm{rank}(\alpha_\perp' \Psi\beta_\perp) = (p-r) \qquad (4.3)$$

where α and β are ($p \times r$) matrices of full column rank, $0<r<p$, $\alpha_\perp$ and $\beta_\perp$ are the $p \times (p-r)$ orthogonal complement of α and β, and $\Psi=(I_p - \Pi_2 - ... - \Pi_k)$. The r columns of β are the cointegrating vectors of the system in the sense of Engle and Granger (1987), and the constant term μ may be decomposed into two parts, $\alpha\kappa_1$, where $\kappa_1=(\alpha'\alpha)^{-1}\alpha'\mu$ is of dimensions ($r \times 1$), which contributed to the intercept in the cointegrating equations, and $\alpha_\perp\kappa_2$, where $\kappa_2=(\alpha_\perp'\alpha_\perp)^{-1}\alpha_\perp'\mu$ is of $(p-r)\times 1$ which determines a linear trend.

The econometric analysis of the model (4.2)-(4.3) has been extensively treated, *inter alia*, in Johansen (1991) and Johansen and Juselius (1990). It can be proved that, properly normalised, the distribution of the maximum likelihood (ML) of β converges at the rate T^{-1} towards a mixed Gaussian distribution, while the ML estimators of the parameters in (α, Π_2, ..., Π_k, Σ_e) converge at a rate $T^{-1/2}$, and the asymptotic distribution is a multivariate Gaussian. Also the ML estimator of μ is consistent, but its asymptotic distribution proves to be more complex with respect to the Gaussian, see e.g. Johansen (1995, chap. 13, Theorem 13.6). For the purposes of our analysis it is important to point out that that since the asymptotic covariance matrix of the estimator of β and of (α, Π_2, ..., Π_k, μ, Σ_e) is block diagonal, inference may be carried out separately, as in a two-stage procedure where in the second step the model to be estimated is the stationary error-correction system:

$$\Delta z_t = \alpha\bar{\beta}'z_{t-1} + \Pi_2\Delta z_{t-1} + ... + \Pi_k\Delta z_{t-k+1} + \mu + e_t \quad ; \quad \{e_t\}\sim \text{iidN}\,(0_{p\times 1}, \Sigma_e) \tag{4.4}$$

where $\bar{\beta}$ denotes the first-step ML estimate of β and can be treated as a known quantity. We shall return on the subject in the next sections.

4.2 The solution for the expectations and the structural model.

Expectations in the ‘exact’ model (2.5) involve the future infinite values of the forcing variables as unobservable predetermined variables at time t. Adapting the ideas in Baillie (1989) and Johansen and Swensen (1994), the purpose in this section is to use (4.1)-(4.3) to compute these unobservable expectations as function of observable predetermined variables at time t. It is then possible to generate a structural model incorporating all the restrictions implied by the rational expectation hypothesis, as well as the parameters of interest (Γ, Γ_0, δ, Λ).

Firstly, it is easy to show that using z_t, (2.5) can be written as:

$$\sum_{j=1}^{\infty} (\delta\Lambda)^{j-1} E_t \mathrm{M}_1{}' z_{t+j} + \mathrm{M}_0{}' z_t + \mathrm{M}_{-1}{}' z_{t-1} + \mathrm{M} = 0_{m\times 1}$$

(4.5)

where the ($p\times m$) matrices M_1, M_0, M_{-1} and M are defined as follows:

$$\mathrm{M}_1{}' = [0_{m\times m} \; \mathrm{M} \delta\Lambda\,(I_m - \delta\Lambda)\,(I_m - \Lambda)\,\Gamma\,]$$
$$\mathrm{M}_0{}' = [\,-I_m \; \mathrm{M}\,(I_m - \delta\Lambda)\,(I_m - \Lambda)\,\Gamma\,]$$
$$\mathrm{M}_{-1}{}' = [\Lambda \; \mathrm{M} 0_{m\times q}\,]$$
$$\mathrm{M} = (I_m - \Lambda)\,\Gamma_0$$

To prove this, note that

$$\sum_{j=1}^{\infty} (\delta\Lambda)^{j-1} E_t \mathrm{M}_1{}' z_{t+j} = \sum_{j=1}^{\infty} (\delta\Lambda)^{j-1} [0_{m\times m} \; \mathrm{M} \delta\Lambda\,(I_m - \delta\Lambda)\,(I_m - \Lambda)\,\Gamma\,]\, E_t z_{t+j}$$

$$= \sum_{j=1}^{\infty} (\delta\Lambda)^{j} (I_m - \delta\Lambda)\,(I_m - \Lambda)\,\Gamma\, E_t x_{t+j}\,;$$

$$\mathrm{M}_0{}' z_t = [\,-I_m \; \mathrm{M}\,(I_m - \delta\Lambda)\,(I_m - \Lambda)\,\Gamma\,]\begin{pmatrix} y_t \\ x_t \end{pmatrix} = -\,y_t + (I_m - \delta\Lambda)\,(I_m - \Lambda)\,\Gamma\, E_t x_t\,;$$

$$\mathrm{M}_{-1}{}' z_{t-1} = [\Lambda \; \mathrm{M} 0_{m\times q}\,]\begin{pmatrix} y_{t-1} \\ x_{t-1} \end{pmatrix} = \Lambda y_{t-1}.$$

Using iterated conditional expectations in an expression similar to (4.5) at time $t+1$, multiplying by ($\delta\,\Lambda$) and subtracting from (4.5) yields the alternative relation:

$$(\mathrm{M}_1{}' - \delta\Lambda\;\mathrm{M}_0{}')\; E_t z_{t+1} + (\mathrm{M}_0{}' - \delta\Lambda\;\mathrm{M}_{-1}{}')z_t + \mathrm{M}_{-1}{}' z_{t-1} + (I_m - \delta\Lambda)\mathrm{M} = 0_{m\times 1}$$
(4.6)

where the infinite term summation has been eliminated.

Secondly, computing $E\{z_{t+1} \mid \Phi_t\}$ from $(4.1)^{11}$ and pre-multiplying by $(M_1' - \delta\Lambda M_0')$, we get:

$$(M_1' - \delta\Lambda\, M_0')\, E_t z_{t+1} = (M_1' - \delta\Lambda\, M_0')\, C_1 z_t + \ldots + (M_1' - \delta\Lambda\, M_0')\, C_k\, z_{t+1-k} + (M_1' - \delta\Lambda\, M_0')\, \mu$$

and given that $z_t \neq 0_{p\times 1}$ a.s. for all t, substituting into (4.6) yields the following set of constraints on the parameters of the VAR (4.1):

$$(M_1' - \delta\,\Lambda\, M_0')\, C_1 + (M_0' - \delta\Lambda\, M_{-1}') = 0_{m\times p}$$
$$(M_1' - \delta\,\Lambda\, M_0')\, C_2 + M_{-1}' = 0_{m\times p}$$
$$(M_1' - \delta\,\Lambda\, M_0')\, C_i = 0_{m\times p} \;; \quad i=3, \ldots, k$$
$$(M_1' - \delta\,\Lambda\, M_0')\, \mu + (I_m - \delta\Lambda)\, M = 0_{m\times 1}$$

Since $(C_1 + \ldots + C_k - I_p) = \Pi$, $\Pi_i = -(C_i + \ldots + C_k)$, $i=2,\ldots k$, and

$$-((M_1' - \delta\,\Lambda M_0') + (M_0' - \delta\,\Lambda M_{-1}') + M_{-1}') = [\,(I_m - \delta\,\Lambda)(I_m - \Lambda)\,\mathrm{M} - (I_m - \delta\,\Lambda)(I_m - \Lambda)\,\Gamma\,]$$

$$(M_1' - \delta\,\Lambda M_0') = [\,\delta\,\Lambda\;\mathrm{M}\,0_{m\times q}\,]$$

by straightforward algebraic manipulations, the restrictions above can also be expressed as implicit conditions on the parameters of the model (4.2)-(4.3):

$$[\,\delta\;\Lambda\;\mathrm{M}\,0_{m\times q}\,]\,\alpha\beta' = [-\,(I_m - \delta\,\Lambda)(I_m - \Lambda)\;\mathrm{M}\,(I_m - \delta\,\Lambda)(I_m - \Lambda)\;\Gamma\,] \qquad (4.7)$$

$$[\,\delta\;\Lambda\;\mathrm{M}\,0_{m\times q}\,]\;\Pi_2 = [\,-\Lambda\;\mathrm{M}\,0_{m\times q}\,] \qquad (4.8)$$

$$[\,\delta\;\Lambda\;\mathrm{M}\,0_{m\times q}\,]\;\Pi_i = 0_{m\times p} \;; \quad i=3,\ldots,k \qquad (4.9)$$

$$[\,\delta\;\Lambda\;\mathrm{M}\,0_{m\times q}\,]\;\mu = -(I_m - \delta\Lambda)(I_m - \Lambda)\;\Gamma_0 \qquad (4.10)$$

The $(m\times m)$ matrix $(I_m - \delta\,\Lambda)(I_m - \Lambda)$ is non-singular and (4.7) can be decomposed into (see e.g. Johansen and Juselius, 1990):

$$[I_m \;\vdots\; -\Gamma\;]' \in \mathrm{sp}(\beta) \quad ; \quad r \geq m \tag{4.11}$$

and

$$[\;\delta\,\Lambda \;\vdots\; 0_{m\times q}\;]\;\alpha = -\,(I_m - \delta\,\Lambda)(I_m - \Lambda)\;[\;I_m \;\vdots\; -\Gamma\;]\;\beta(\beta'\beta)^{-1} \tag{4.12}$$

where (4.12) is obtained from (4.7) by postmultiplying by the full rank matrix $\beta(\beta'\beta)^{-1}$.

It is thus evident that the restrictions on the CVAR (4.2)-(4.3) implied by the rational expectation hypothesis can be separated into constraints on the long-run, given by (4.11), and constraints on the short-run dynamics given by (4.8)-(4.10) and (4.12). Partitioning $(\alpha, \Pi_2, \ldots, \Pi_k)$ conformably with z_t, (4.12) and (4.8)-(4.10) can be also expressed respectively as:

$$\alpha = \begin{bmatrix} -(\delta\Lambda)^{-1}(I_m - \delta\Lambda)(I_m - \Lambda)\omega \\ \alpha_x \end{bmatrix} \tag{4.13a}$$

$$\Pi_2 = \begin{bmatrix} \delta^{-1} I_m \;\vdots\; 0_{m\times q} \\ \Pi_{x2} \end{bmatrix} \tag{4.13b}$$

$$\Pi_i = \begin{bmatrix} 0_{m\times p} \\ \Pi_{xi} \end{bmatrix} \quad ; \quad i=3, \ldots, k \tag{4.13c}$$

$$\mu = \begin{bmatrix} -(\delta\Lambda)^{-1}(I_m - \delta\Lambda)(I_m - \Lambda)\Gamma_0 \\ \mu_x \end{bmatrix} \tag{4.14}$$

where $\omega = [I_m \;\vdots\; \Gamma]\beta(\beta'\beta)^{-1}$ is a full rank matrix of dimensions $(m\times r)$, and the submatrices in $\alpha=\begin{bmatrix}\alpha_y\\ \alpha_x\end{bmatrix}$ and $\Pi_i=\begin{bmatrix}\Pi_{yi}\\ \Pi_{xi}\end{bmatrix}$, i=2,...k, are of suitable dimensions. It is however convenient to refer to (4.13)-(4.14) compactly as

$$\mathrm{x} = \mathrm{g}(\mathrm{y}) \qquad (4.15)$$

where $\mathrm{x}=(\mathrm{vec}(\alpha)', \mathrm{vec}(\Pi_2)', ..., \mathrm{vec}(\Pi_k)', \mu')'$ is of dimensions $(n\times 1)$, $n=p(r+p(k-1)+1)$, $\mathrm{y}=(\Gamma_0', \delta, \mathrm{vec}(\Lambda)', \mathrm{vec}(\alpha_x)', \mathrm{vec}(\Pi_{x2})', ..., \mathrm{vec}(\Pi_{xk})', \mu_x')'$ is of dimensions $(s\times 1)$, $s=(m+1+m^2+q(r+p(\mathrm{k}-1)+1))$, $(s<n)$, and $\mathrm{g}: \mathrm{R}^s\rightarrow\mathrm{R}^n$ is a function with continuous partial derivatives.

Substituting (4.13)-(4.14) into (4.2), the resulting equations are:

$$\Delta y_t = -(\delta\Lambda)^{-1}(I_m-\delta\Lambda)(I_m-\Lambda)\omega\beta' z_{t-1}+\delta^{-1}\Delta y_{t-1}-(\delta\Lambda)^{-1}(I_m-\delta\Lambda)(I_m-\Lambda)\Gamma_0 + \varepsilon_{yt} \qquad (4.16a)$$

$$\Delta x_t = \alpha_x\beta' z_{t-1} + \Pi_{x2}\Delta z_{t-1} + ... + \Pi_{xk}\Delta z_{t-k+1} + \mu_x + \varepsilon_{xt} \qquad (4.16b)$$

where β is restricted as in (4.11). System (4.16) is highly non-linear, involves only predetermined observable variables and incorporates all the parametric restrictions implied by the intertemporal MLQAC model. From (4.16b) it is furthermore evident that it allows for Granger-causality of $\{\Delta y_t\}$ with respect to $\{\Delta x_t\}$.

5. Identification and estimation of the structural parameters.

System (4.16) is the estimable structural form associated to the forward-looking solution to the MLQAC model provided the condition ensuring local identification of (Γ, Γ_0, δ, Λ) are satisfied. If such parameters are (at least) locally identified, the maximisation of the log-likelihood function of (4.16) provides FIML estimates. However, since (4.11) and (4.13)-(4.14) separate respectively into restrictions on the long-run, and restrictions on the short-run parameters of the CVAR (4.2)-(4.3), identification and estimation issues can be dealt with by a two-step procedure. Indeed in the first-step super-consistent and fully efficient estimation of Γ can be achieved into the framework of (4.2)-(4.3), by suitably identifying the structure of the cointegration space. Part of the information needed is provided by (4.11) according to which the cointegration space must contain at least the long-run equilibrium target pursued by the economic agent. In the second step the suitably identified β can be replaced in (4.16) by its ML restricted estimate, $\hat{\beta}$, while ω can be replaced by $\hat{\omega} = \left[I_m \vdots \hat{\Gamma} \right] \hat{\beta}(\hat{\beta}'\hat{\beta})^{-1}$[12]. As proved in Phillips(1991) and Johansen (1991), these substitution only affect the asymptotics in (4.16) through terms of $o_P(T^{-1})$. Provided that the restrictions ensuring local identification of (Γ_0, δ, Λ) are also satisfied, consistent and efficient estimation can be achieved by maximising numerically the associated log-likelihood function. Note that in contrast to the slightly different approaches set out in Engsted and Haldrup (1994, 1996a) and Ripatti (1997) with respect to the case m=1, our procedure allows $\{\Delta y_t\}$ to Granger-cause the forcing variables $\{\Delta x_t\}$, and $\{\Delta x_t\}$ not to be weakly exogenous with respect to the long-run parameters. In other words, the forcing variables are not restricted to be strongly-exogenous. The technical details of the procedure are sketched in the subsections below.

5.1 First step.

Without imposing some a priori restrictions, the cointegrating vectors in (4.2)-(4.3) are only identified up to a non-singular linear transformation, since for any non-singular ($r\times r$) matrix K, $\alpha K'^{-1}$ and βK give the same value of Π and (α, β) and ($\alpha K'^{-1}$, βK) are observationally equivalent. A necessary condition for identification of β is that a number f, where $f \geq r^2$, of a priori restrictions (included normalisation) are imposed on its columns[13]. Let us focus on (4.11). Whether $r=m$, that is the cointegration rank of the CVAR equals the number of long-run equilibrium targets pursued by agents, (4.11) is satisfied by restricting β as:

$$\beta = \begin{pmatrix} I_m \\ -\Gamma' \end{pmatrix}$$

(5.1a)

In this case the ($m\times 1$) process $\{\beta' z_t\} \equiv \{(y_t - \Gamma x_t)\}$ is I(0) and reproduces, a part from the constant, the long-run equilibrium target (2.2) of the MLQAC model. The quantity

ω in (4.13a) becomes: $\omega=[I_m \vdots \Gamma]\beta(\beta'\beta)^{-1}=I_m$, and provided relevant economic theory does not predict homogeneity and/or cross-equation restrictions on Γ, β is just-identified ($f = r^2 = m^2$).

In the more general case, $r \geq m$, (4.11) can be satisfied by restricting β as:

$$\beta = \begin{pmatrix} I_m & \vdots & \beta_0 \\ -\Gamma' & & \end{pmatrix} \qquad (5.1b)$$

with β_0 a $p\times(r - m)$ matrix to be suitably identified[14]. Now the cointegration space contains not only the structural parameters of (2.2) (a part from the constant), but also $(r - m)$ 'additional' long-run relationships via β_0. In general we have no 'theory-lead' information to identify β_0, because relevant economic theory concerns the determination of the equilibrium target pursued by agents, and typically does not explain whether and how the forcing variables cointegrate. Nevertheless, in many cases it is not difficult to formulate 'reasonable' hypotheses about the structure of β_0. Consider, for instance, the situations where the vector of the forcing variables x_t contains short-term and long-term interest rates which are well known to move together in the long-run.

Example 3. Let us consider $z_t=(y_{1t}, y_{2t}, x_{1t}, x_{2t})'$, with $m=2$ and $q=2$. The (unrestricted) equilibrium target of the MLQAC is given by system (e.3) of example 2. It is clear that if the cointegration rank of the CVAR system for z_t is $r=2=m$, then (5.1a) becomes:

$$\beta' = \begin{bmatrix} 1 & 0 & -\gamma_{11} & -\gamma_{12} \\ 0 & 1 & -\gamma_{21} & -\gamma_{22} \end{bmatrix}$$

and if the parameters (γ_{11}, γ_{12}, γ_{21}, γ_{22}) are not restricted, β is exactly identified. Relevant economic theory about the determination of y_t* could suggest, however, constraints on (γ_{11}, γ_{12}, γ_{21}, γ_{22}). For instance, the constraint: $\gamma_{21}=\gamma_{12}$ would introduce one testable over-identification restriction on β. Let us now consider the case $r=3$. The 'additional' cointegration relationship could involve, for instance, only the forcing variables of the system, via

$$x_{1t} = \phi_{12} x_{2t}$$

so that (5.1b) would read:

$$\beta\,' = \begin{bmatrix} 1 & 0 & -\gamma_{11} & -\gamma_{12} \\ 0 & 1 & -\gamma_{21} & -\gamma_{22} \\ 0 & 0 & 1 & -\phi_{12} \end{bmatrix}$$

with $\beta_0\,' = (0,\ 0,\ 1,\ -\phi_{12})$. Observe that now one additional restriction on each of the first two rows of $\beta\,'$ is required in order to achieve just identification, for example, $\gamma_{11}=1$, and $\gamma_{21}=\gamma_{12}$.

♦

The estimation of the cointegrating relations in the context of (4.2)-(4.3) under the constraint (5.1) may be found in Johansen (1995, chap 5 and 7) and Pesaran and Shin (1994). When the cointegration rank of the system is consistent with (4.11) and the possible over-identifying restrictions on Γ predicted by relevant economic theory are not rejected by the data, it is then possible to consider the second step of the analysis.

5.2 Second step.

We focus on the I(0) system (4.16), where β is replaced by $\tilde{\beta} = \left(\begin{array}{c|c} I_m & \hat{\beta}_0 \\ -\hat{\Gamma}' & \end{array} \right)$, β_0 is supposed to be suitably identified, and ω is replaced by $\hat{\omega} = \left[I_m \;\vdots\; \hat{\Gamma} \right] \hat{\beta} (\hat{\beta}' \hat{\beta})^{-1}$. Clearly, if in (4.11) $r=m$, then $\omega = I_m$ and β will be replaced by $\hat{\beta} = \begin{pmatrix} I_m \\ -\Gamma' \end{pmatrix}$. The purpose now is to discuss the identification of $(\Gamma_0,\ \delta,\ \Lambda)$, and explore, if the case, whether it is possible to propose a FIML procedure. Observe that whenever $(\delta,\ \Lambda)$ are identified, the ML estimate of the matrix Θ in (3.7) can be derived from (2.6), yielding: $\hat{\Theta} = -(I_m - \hat{\delta}\,\hat{\Lambda})(I_m - \hat{\Lambda})\hat{\Lambda}^{-1}$.

For β and ω fixed as above, we denote by $logL(\psi)$ the Gaussian log-likelihood function of (4.16) concentrated with respect to Σ_{ε}. Following Rothemberg (1971, Theorem 1), under the usual 'regularity conditions', ψ will be locally identifiable if and only if the ($s\times s$) information matrix: $R(\psi)=E\left(\frac{\partial \log L(\psi)}{\partial \psi}\frac{\partial \log L(\psi)}{\partial \psi'}\right)$ is non-singular[15]. From (4.15) we can write:

$$\underset{(1\times s)}{\frac{\partial logL(\psi)}{\partial \psi'}} = \underset{(1\times n)}{\frac{\partial logL(\xi)}{\partial \xi'}}\;\underset{(n\times s)}{\frac{\partial g(\psi)}{\partial \psi'}} \tag{5.2}$$

with $\frac{\partial g(\psi)}{\partial \psi'}=J(\psi)=\begin{bmatrix} \frac{\partial g_1(\psi)}{\partial \psi_1} & \cdots & \frac{\partial g_1(\psi)}{\partial \psi_s} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_n(\psi)}{\partial \psi_1} & & \frac{\partial g_n(\psi)}{\partial \psi_s} \end{bmatrix}$ the ($n\times s$) Jacobian matrix of $g(\cdot)$, and $q(\xi)=\frac{\partial\, logL(\xi)}{\partial\, \xi'}$ the score of the concentrated log-likelihood function of (4.4). Using (5.2) it is then possible to show that:

$$R(\psi) = \left(\frac{\partial\, g(\psi)}{\partial\, \psi'}\right)' R(\xi)\, \frac{\partial\, g(\psi)}{\partial\, \psi'} \tag{5.3}$$

where $R(\xi)=E\left(\frac{\partial \log L(\xi)}{\partial \xi}\frac{\partial \log L(\xi)}{\partial \xi'}\right)$ is ($n\times n$) and non-singular. The structure of the information matrix in (5.3) suggests that the local identifiability of ψ depends on the rank of the Jacobian matrix, $J(\psi)$, so that the number of restrictions required to identify ψ is given by the rank deficiency of $J(\psi)$. The computation of $J(\psi)$ and the determination of its rank is dealt with in the Appendix B, leading to the following proposition:

Proposition 1.

(a) If in (3.1) $k \geq 2$, then rank($J(\psi)$)=s and the parameters in ψ are locally identified;

(b) if in (3.1) k=1, then rank($J(\psi)$)=(s–1) and one identification restriction is required on (δ, Λ);

Proof: see Appendix B.

From part (b) of the proposition it follows that whether the order of the VAR (4.1) is k=1, the 'natural' solution to achieve identification in MLQAC models is to pre-fix δ to a plausible economic value[16]. This results encompass the one in Gregory *et al.* (1993), where identification issues are discussed with respect to the assumptions: (i) m=1; (ii) the process generating $\{x_t\}$ is assumed to be strictly exogenous with respect to $\{y_t\}$.

Explicit derivation of the first-order necessary conditions to the FIML estimation of ψ may be obtained by maximising the likelihood function of (4.16). We first write (4.16) compactly as

$$\Delta z_t = \Pi(\psi)\, u_t + \varepsilon_t \qquad ; \qquad \{\varepsilon_t\} \sim \text{iidN}\ (0_{p\times 1},\ \Sigma_\varepsilon) \tag{5.4}$$

where

$$\Pi(\psi)=\begin{bmatrix} (\delta\Lambda)^{-1}(I_m-\delta\Lambda)(I_m-\Lambda)\hat{\omega} & \left[\delta^{-1}I_m \vdots 0_{m\times q}\right] & \cdots & 0_{p\times p} & (\delta\Lambda)^{-1}(I_m-\delta\Lambda)(I_m-\Lambda)\Gamma_0 \\ \alpha_x & \Pi_{x2} & \cdots & \Pi_{xk} & \mu_x \end{bmatrix} \quad ; \quad u_t = \begin{bmatrix} \hat{\beta}' z_{t-1} \\ \Delta z_{t-1} \\ \vdots \\ \Delta z_{t-k+1} \\ 1 \end{bmatrix}$$

and the vector ψ is supposed to be of dimension ($s^*\times 1$), with $s^*=(m+1+m^2+q(r+p(k-1)))$ if $k \geq 2$, and $s^*=(s-1)$ with δ pre-fixed if k=1. The estimation of models similar to (5.4) is dealt with in Sargan (1972) and Wallis (1980 pp. 64). The associated log-likelihood function can be written as:

$$log\ L(\psi,\ \Sigma_\varepsilon) = \text{cost} - (T/2)\log\ \left|\Sigma_\varepsilon\right| - (1/2)\text{tr}\{\ \Sigma_\varepsilon^{-1}\ [\Delta Z - U\ \Pi(\psi)']'[\Delta Z -$$

$U\Pi$(y)’] }

with $\Delta Z'$ = [Δz_1M... MΔz_T], and U ’= [u_1 M... Mu_T]. Concentrating $logL$(y , Σ_e) with respect to Σ_e, the first-order conditions for y read:

$$\frac{\partial\, log\, L(\psi)}{\partial\psi_i} = -\,\mathrm{tr}\{\ \Sigma_e^{-1}(y)\ [\Delta Z'U - \Pi(y)\ U'\ U]\ \frac{\partial\Pi(\psi)'}{\partial\psi_i}\ \} = 0 \qquad i=1,2,\ ...,\ s^* \tag{5.5}$$

where Σ_e(y) = $T^{-1}[\Delta Z - U\,\Pi(y)']'\ [\Delta Z - U\Pi(y)']$, and ($\partial\Pi$(y) / ∂y) is defined as in Dwyer (1967, pp. 608). Experience in solving FIML problems such as (5.5) through a variety of numerical optimisation procedures may be found, *inter alia*, in Sargan and Sylwestrowicz (1976) and Hendry (1995, Appendix A5).

6. Testing the rational expectations hypothesis.

The purpose in this section is to test restrictions implied by the rational expectations hypothesis subsumed in the intertemporal MLQAC model. That is, the purpose is to test the set of constraints on the CVAR (4.2)-(4.3), derived in sub-section 4.2 by computing the unknown expectations in the multivariate forward-looking model (2.5). It is proved in Johansen (1991, Appendix C) that stacking the parameters of (4.2)-(4.3) in the vector ζ=(vec(β)’, π’)’, where π contains the parameters not relating to the cointegration space, the likelihood ratio (LR) test statistic Q for a simple hypothesis on ζ can be approximated by:

$$-2\log Q \approx T\begin{bmatrix}\hat{\pi}-\pi \\ vec(\hat{\beta})-vec(\beta)\end{bmatrix}'\begin{bmatrix}q_{11} & q_{12} \\ q_{21} & q_{22}\end{bmatrix}\begin{bmatrix}\hat{\pi}-\pi \\ vec(\hat{\beta})-vec(\beta)\end{bmatrix}$$

$$\approx \qquad T^{1/2}\left(\hat{\pi}-\pi\right)' \qquad q_{11}T^{1/2}\left(\hat{\pi}-\pi\right) \tag{6.1}$$

$$+\ T^{1/2(\tau+1)}\left(vec(\hat{\beta})-vec(\beta)\right)'\ (T^{-\tau}\ q_{22})\ T^{1/2(\tau+1)}\left(vec(\hat{\beta})-vec(\beta)\right)$$

where τ is a positive constant and q_{11}=(1/T)[∂^2logL(ζ)/$\partial\pi\partial\pi$’], q_{12}=(1/T)[∂^2logL(ζ)/$\partial\pi\partial$vec(β)’], q_{21}=(1/T)[∂^2logL(ζ)/∂vec(β)$\partial\pi$’], q_{22}=(1/T)[∂^2logL(ζ)/∂vec(β)∂vec(β)’]. The LR test statistic decomposes into a test statistics for vec(β), and an independent test statistics for π, and vice versa. This means that in accordance with the estimation procedure of section 5, the rational expectations restrictions (4.11) and (4.13)-(4.14) can be tested separately, according to the following steps:

A) check whether $r \geq m$, i.e. if the number of long-run equilibrium relationships subsumed in the CVAR is consistent with the number of equilibrium targets pursued by agents;

B) if $r \geq m$ test the possible 'theory-lead' over-identification long-run restrictions on β;

C) if $r \geq m$ and the possible over-identification restrictions on β are not rejected, test the restrictions (4.13)-(4.14), i.e. the 'smooth hypothesis' on the short-run dynamics (4.15).

Observe that steps A and B refer to the consistency between the long-run features of the observed time series and the economic structure of the equilibrium targets pursued by agents. The determination of the number of long-run equilibrium relationships in the context of (4.2)-(4.3) is discussed, *inter alia,* in Johansen (1991), while a general approach to testing linear and possible non-linear over-identification restrictions on β in the context of (4.2)-(4.3) is dealt with in Pesaran and Shin (1994). It is there shown that the LR test statistics for these restrictions is distributed as a χ^2 with degrees of freedom given by the number of over-identification restrictions.

Finally, step C refers to the restrictions implied by the intertemporal MLQAC model on the structure of the adjustment dynamics of the mechanism generating expectations. However, for the sake of simplicity in the following we shall disregard the restrictions on the constant term given by (4.14). This is because the asymptotic distribution of the ML estimator of μ is not Gaussian and inference proves to be complex[17]. Accordingly, henceforth attention will be devoted only to (4.13), i.e. to the rational expectations restrictions involving (α, Π_2, ..., Π_k). Partitioning the vectors x and y defined in section 4 as $\mathsf{x}=(\mathsf{x}_1', \mathsf{x}_2')'$ and $\mathsf{y}=(\mathsf{y}_1', \mathsf{y}_2')'$, with $\mathsf{x}_2=\mu$ and $\mathsf{y}_2=(\Gamma_0', \mu_x')'$ both of dimension ($p\times1$), the smooth hypothesis (4.15) can be decomposed as:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = g\left(\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = \begin{pmatrix} g_1(y_1) \\ g_2(y_2) \end{pmatrix}$$

(6.2)

with g_1: $R^{s\text{-}p} \rightarrow R^{n\text{-}p}$ and g_2: $R^p \rightarrow R^p$, and where the sub-hypothesis:

$$x_1 = g_1(y_1)$$

(6.3)

represents a compact description of the constraints in (4.10). Note that the FIML estimation of x_1 can be easily achieved from the Gaussian likelihood function of the multivariate regression model (4.4), concentrated with respect to x_2=μ:

$$logL(x_1,\ \Sigma_e) = \text{cost} - (T/2)\log|\Sigma_e| - (1/2)\text{tr}\{S_e^{-1} \sum_{t=1}^{T} (e_t - \bar{e})(e_t - \bar{e})'\}$$

(6.4)

where the bar denotes average. On the other hand, the FIML estimation of y_1 can be obtained from the restricted model (4.16) (or equivalently (5.4)) by following the maximising procedure set out in subsection 5.2; the only difference needed is the concentration of the associated log-likelihood function with respect to the constant term:

$$logL(y_1, \Sigma_e) = \text{cost} - (T/2)\log|\Sigma_e| - (1/2)\ \text{tr}\{\Sigma_e^{-1} \sum_{t=1}^{T}[(z_t - \bar{z}) - \Pi(y_1)(u_t - \bar{u})][(z_t - \bar{z}) - \Pi(y_1)(u_t - \bar{u})]'\} \qquad (6.5)$$

A LR test statistic for (6.3) is then two times the difference between the unrestricted log-likelihood (6.4) and the restricted log-likelihood (6.5). Under the null hypothesis that the rational expectations restrictions (6.3) are true, the asymptotic distribution of the test statistics will be c^2 with degrees of freedom equal to the number of restrictions being tested (n–s*). Alternatively one could test (6.3) by Wald-type tests as in Revankar (1980).

7. Conclusions.

In this paper attention has been devoted to the class of LQAC models under rational expectations. We have dealt with the inferential issues arising under the following assumption: (i) agents choose a set of endogenous variables, leading to the class of MLQAC models; (ii) the structural equations stemming from the agent's optimisation problem are specified as 'exact' rational expectations models; (iii) the process generating the observable variables of the system is I(1). Both 'limited-information' and 'full-information' methods have been proposed.

As regards the 'limited-information' framework, by focusing on the open-form solution to the agent's optimisation problem, we have show that the estimation of the structural parameters of the MLQAC model can be carried out by simply implementing existing procedures. Specifically, we have shown that under suitable assumptions on the order of integration of the variables, estimation can be set out by a two-step procedure. In the first-step the model can be specified such that it fits into the framework described by Phillips and Hansen (1990), and the parameters involved regard the long-run equilibrium target pursued by agents. In the second-step the short-run adjustment parameters of the model can be estimated by instrumental variables, implementing existing procedures on rational expectations econometric models. However, due to the 'limited-information' context, the cross-moment matrix of instruments and regressors can not be established to be non-singular when applying instrumental variables methods. The 'relevance-condition' has to treated as a maintained hypothesis and thus it is not actually clear whether in the second-step the parameters associated to the short-run dynamics of the model can be estimated consistently.

As regards 'full-information' methods, point (iii) has been modelled by assuming that the I(1) 'stochastic environment' faced by agents was described by a CVAR system for joint the endogenous and exogenous variables. Focusing on the forward-looking solution to the agent's optimisation problem, and using the CVAR to compute the unknown expectations, we have proposed a likelihood-based inferential procedure in time domain. After discussing the conditions ensuring local identification of the structural parameters of interest, a two-step FIML procedure has been adopted. Indeed, due to the cointegrating nature of the model, the parameters associated to the long-run target of the model can be estimated super-consistently and efficiently, separately from the parameters associated to the dynamic adjustment. The short-run dynamics in turn can be estimated consistently and efficiently in a separate framework. The choice to divide the inferential procedure into two-steps has been adopted for the sake of simplicity: it is clear, however, that the likelihood function of the MLQAC model should be maximised in just one-step. The restrictions implied by the rational expectations hypothesis, except the ones involving the constant term of the CVAR, can be tested by likelihood-ratio statistics involving only χ^2 tables. The proposed procedure easily accommodates situations where also the forcing variables are subject to long-run equilibrium restrictions, does not require the assumption of strict exogeneity and allows for Granger causality of the endogenous variables with respect to the forcing ones.

Appendix A.

In this appendix we discuss the solutions to the MLQAC model. By applying the Certainty Equivalence Theorem, we can first regard problem (2.1) as a deterministic one. After differentiating the objective function with respect to y_{t+j}, it is possible to replace all future random variables by their (conditional) expectations. For j=0 we obtain system (2.3) in the text. The set of necessary transversality conditions are:

$$\lim_{T\to\infty} \delta^{T} E_t \Delta y_{t+T} = 0_{m\times 1} \tag{A.1}$$

but using (2.3) and the law of iterated conditional expectations, it is possible to express (A.1) as (2.4) in the text. The aim now is to derive the closed-form solution that satisfies both (2.3) and (2.4). Discussion follows the route of Nickell (1984, Appendix, section I), see also Corollary 1 in Binder and Pesaran, (1995, p.159). Let us write system (2.3) using the backward operator B as:

$$\{\delta I_m B^{-1} - G + I_m B\} E_t y_t = -\Theta E_t y_t^* \tag{A.2}$$

where $B^{-1}E_t y_t = E_t y_{t+1}$, $E_t y_t = y_t$, $E_t y_t^* = y_t^*$, and $G=[(\delta+1) I_m + \Theta]$. Observe that the matrix G has eigenvalues $(1+\delta+d_i)$ i=1,2, ..., m. To prove this, by the *Schur decomposition* (see Magnus, 1988, Chapter 1), $\Theta=HDH^{-1}$, where D is an upper triangular matrix whose diagonal elements (d_i) i=1,2, ..., m are the eigenvalues of Θ. It is now evident that

$$G = [(\delta+1)I_m+\Theta] = H\,[(\delta+1)I_m+D]\,H^{-1}$$

where $[(\delta+1)I_m+D]$ is an upper triangular matrix with diagonal elements $(1+\delta+d_i)$ i=1,2,..., m. Now let us decompose the left hand side of (A.2) as:

$$-A\,(I_m - \delta^{1/2} V B^{-1})\,(I_m - \delta^{-1/2} V B) \tag{A.3}$$

with A and V ($m\times m$) matrices. Comparing the coefficients of (A.2) and (A.3) implies:

$$A + A\,V^2 = G$$

$$A\,V = \delta^{1/2}\, I_m$$

and eliminating A yields the matrix equation for V:

$$\delta^{1/2}\, V^2 - GV + \delta^{1/2}\, I_m = 0_{m\times m} \qquad \text{(A.4)}$$

The eigenvalues, λ_v, of a generic solution V to (A.4), satisfy the determinantal equation (see Gantmacher, 1959, Chapter 8, Theorem 4):

$$\det\{\delta^{1/2}\, I_m\, \lambda_v^2 - G\lambda_v + \delta^{1/2}\, I_m\} = 0 \qquad \text{(A.5)}$$

whose degree is 2*m*. Observe that $\lambda_v = 0$ can not be a solution to (A.5). Moreover the structure of the matrices in the equation (A.5) implies that if the generic eigenvalue λ_v is a root, then so is λ_v^{-1}, and it is ruled out the possibility of roots on the unit circle. Indeed by the *Schur decomposition*:

$$\det\{\delta^{1/2} I_m \lambda_v^2 - G\,\lambda_v + \delta^{1/2} I_m\} = \det\{\delta^{1/2} I_m \lambda_v^2 - [(\delta+1) I_m + D]\,\lambda_v + \delta^{1/2} I_m\}$$
$$= \prod_{i=1}^{m} (\delta^{1/2}\lambda_v^2 - (1+\delta+d_i)\lambda_v + \delta^{1/2}) = 0.$$

and each scalar equation $\delta^{1/2}\lambda_v^2 - (1+\delta+d_i)\lambda_v + \delta^{1/2} = 0$, $i=1,2,\ldots,m$, is such that one roots falls inside the unit circle and the other root falls outside the unit circle. Accordingly all *m* pairs of roots to (A.5) will be such that one roots falls inside the unit circle, and the other root falls outside the unit circle and there exists a unique and stable matrix solving (A.4) and this will be referred to as Λ^*. Using (A.3) and substituting V with Λ^*, system (A.2) now reads as:

$$-A\ (I_m - \delta^{1/2}\ \Lambda^* B^{-1})\ (I_m - \delta^{-1/2}\ \Lambda^* B)\ E_t y_t = -\ \Theta E_t y_t^* \qquad \text{(A.6)}$$

As in the static (non stochastic) long-run equilibrium $y_t = y_t^* = y^*$, it follows:

$$-A\,(I_m - \delta^{1/2}\Lambda^*)\,(I_m - \delta^{-1/2}\Lambda^*)y^* = -\Theta y^* \Rightarrow (I_m - \delta^{1/2}\Lambda^*)\,(I_m - \delta^{-1/2}\Lambda^*) = A^{-1}\Theta \qquad (A.7)$$

and from (A.6) we can write:

$$(I_m - \delta^{-1/2}\Lambda^* B)y_t = (I_m - \delta^{1/2}\Lambda^* B^{-1})^{-1} A^{-1}\Theta\; E_t y_t^* \qquad (A.8)$$

Considering the expansion: $(I_m - \delta^{1/2}\Lambda^* B^{-1})^{-1} = \sum_{j=0}^{\infty}\left(\delta^{1/2}\Lambda^*\right)^j B^j$ and using (A.7) we obtain:

$$y_t - \delta^{-1/2}\Lambda^* y_{t-1} = \sum_{j=0}^{\infty}\left(\delta^{1/2}\Lambda^*\right)^j\left(I_m - \delta^{1/2}\Lambda^*\right)\left(I_m - \delta^{-1/2}\Lambda^*\right)E_t y_{t+j}^* \qquad (A.9)$$

Finally, using (2.2), defining the matrix $\Lambda = \delta^{-1/2}\Lambda^*$ and substituting, (A.9) reads as (2.5) in the text, while (2.6) is obtained from (A.4) by the position $V=\Lambda^*=\delta^{1/2}\Lambda$. From (2.5) it is now possible to derive the error-correction representation (2.7). This can be accomplished by the following algorithm: (1) use the expression: $\sum_{j=0}^{\infty}(\delta\Lambda)^j(I_m - \delta\Lambda) = \sum_{j=0}^{\infty}(\delta\Lambda)^j - \sum_{j=0}^{\infty}(\delta\Lambda)^{j+1}$; (2) add the quantity $(-y_{t-1})$ to both sides; (3) add the quantity $\pm(I_m - \Lambda)\Gamma x_{t-1}$ to the right hand side; (4) rearrange the terms.

Appendix B.

The purpose in this section is to prove the Proposition 1 in sub-section 5.2. This is accomplished by deriving the structure of the Jacobian matrix $J(\psi) = \dfrac{\partial g(\psi)}{\partial \psi'}$ and studying its column-rank properties. Observe that $\dfrac{\partial g(\psi)}{\partial \psi'} = \dfrac{\partial \xi}{\partial \psi'} = \dfrac{\partial(vec(\alpha)', \mathrm{vec}(\Pi_2)', \ldots \mu')'}{\partial(\Gamma_0', \delta, vec(\Lambda)', \ldots, vec(\Pi_{xk})', \mu_x')'}$, so that the Jacobian exhibits the block structure:

$J(\psi)=$

$$\begin{bmatrix} [\partial vec(\alpha)/\partial\Gamma_0']_{pr\times m} & [\partial \mathrm{vec}(\alpha)/\partial\delta]_{pr\times 1} & [\partial \mathrm{vec}(\alpha)/\partial \mathrm{vec}(\Lambda)']_{pr\times m^2} & \cdots & [\partial \mathrm{vec}(\alpha)/\partial\mu_x']_{pr\times q} \\ [\partial vec(\Pi_2)/\partial\Gamma_0']_{p^2\times m} & [\partial \mathrm{vec}(\Pi_2)/\partial\delta]_{p^2\times 1} & [\partial \mathrm{vec}(\Pi_2)/\partial \mathrm{vec}(\Lambda)']_{p^2\times m^2} & \cdots & [\partial \mathrm{vec}(\Pi_2)/\partial\mu_x']_{p^2\times q} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ [\partial\mu/\partial\Gamma_0']_{p\times m} & [\partial\mu/\partial\delta]_{p\times 1} & [\partial\mu/\partial vec(\Lambda)']_{p\times m^2} & \cdots & [\partial\mu/\partial\mu_x']_{p\times q} \end{bmatrix}$$

Applying the vec operator to both sides of (4.13)-(4.14), it is possible to compute the derivatives in $J(\psi)$. It is easy to recognise that:

$$J(\psi)=\begin{bmatrix} 0 & A_1 & A_2 & A_3 & 0 & 0 & \dots & 0 \\ 0 & A_4 & 0 & 0 & A_5 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & A_6 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_7 & A_8 & A_9 & 0 & 0 & 0 & 0 & A_* \end{bmatrix}$$

(B.2)

so that $J(\psi)$ will be of full column-rank if the two block-matrices:

$$L=\begin{bmatrix} A_1 & A_2 & A_3 \\ A_4 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ A_8 & A_9 & 0 \end{bmatrix} \qquad ; \qquad L^*=\begin{bmatrix} A_5 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_* \end{bmatrix}$$

are of full-column rank, and A_7 and A_* are linearly independent. We first focus on L^*. By the symbol K_m we denote the commutation matrix of order m as defined in Magnus, 1988, Chapter 3. The algebra we use here and the properties of the commutation matrices are described in Magnus, 1988, p. 35. Since

$$A_5=\frac{\partial\, \mathrm{vec}(\Pi_2)}{\partial\, \mathrm{vec}(\Pi_{x2})'}=K_{p^2}\frac{\partial\, vec(\Pi_2')}{\partial\, vec(\Pi_{x2})'}=K_{p^2}\begin{bmatrix} \partial\, vec\begin{pmatrix}\delta^{-1}I_m \\ 0\end{pmatrix}/\partial\, vec(\Pi_{x2})' \\ \partial\, vec(\Pi_{x2}')/\partial\, vec(\Pi_{x2})' \end{bmatrix}=$$

$$K_{p^2}\begin{bmatrix} 0_{mp\times qp} \\ K_{qp} \end{bmatrix}$$

$$A_6 = \frac{\partial\, vec(\Pi_3\,)}{\partial\, vec(\Pi_{x3}\,)'} = K_{p^2}\,\frac{\partial\, vec(\Pi_3'\,)}{\partial\, vec(\Pi_{x3}\,)'} = K_{p^2}\begin{bmatrix} 0_{mp\times qp} \\ K_{qp}\end{bmatrix}$$

$\vdots \qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$

$$A_* = \frac{\partial\mu}{\partial\mu_x'} = \begin{bmatrix} \partial vec\left\{(\delta\Lambda)^{-1}(I_m - \delta\Lambda)(I_m-\Lambda)\Gamma_0\right\}/\partial\mu_x' \\ \partial\mu_x/\partial\mu_x' \end{bmatrix} = \begin{bmatrix} 0_{m\times q} \\ I_q \end{bmatrix}$$

have full column-rank, it is possible to recognise that L^* has full column-rank $q(p(k-1)+1)$. Second, let us focus on the sub-matrix L. In this case:

$$A_1 = \frac{\partial\, vec(\alpha)}{\partial\,\delta} = K_{pr}\,\frac{\partial\, vec(\alpha')}{\partial\,\delta} = K_{pr}\begin{bmatrix} \partial\, vec\left\{\left[-(\delta\Lambda)^{-1}(I_m-\delta\Lambda)(I_m-\Lambda)\omega\right]\right\}/\partial\,\delta \\ \partial\, vec(\alpha_x')/\partial\delta \end{bmatrix}$$

$$= K_{pr}\begin{bmatrix} \delta^{-2}(I_m\otimes\omega')vec(\Lambda^{-1}-I_m) \\ 0_{qr\times 1}\end{bmatrix};$$

$$A_2 = \frac{\partial\, vec(\alpha)}{\partial\, vec(\Lambda)'} = K_{pr}\,\frac{\partial\, vec(\alpha')}{\partial\, vec(\Lambda)'} = K_{pr}\begin{bmatrix} \delta^{-1}(I_m\otimes\omega')\left[(\Lambda^{-1}\otimes\Lambda'^{-1}) - \delta\, I_{m^2}\right]K_{m^2} \\ 0_{qr\times m^2}\end{bmatrix};$$

$$A_3 = \frac{\partial\, vec(\alpha)}{\partial\, vec(\alpha_x)'} = K_{pr}\,\frac{\partial\, vec(\alpha')}{\partial\, vec(\alpha_x)'} = K_{pr}\begin{bmatrix} 0_{mr\times qr} \\ K_{qr}\end{bmatrix};$$

$$A_4 = \frac{\partial\, vec(\Pi_2\,)}{\partial\,\delta} = K_{p^2}\,\frac{\partial\, vec(\Pi_2')}{\partial\,\delta} = K_{p^2}\begin{bmatrix} \delta^{-2} vec\begin{pmatrix} I_m \\ 0_{q\times m}\end{pmatrix} \\ 0_{qp\times 1}\end{bmatrix};$$

$$A_8 = \frac{\partial\,\mu}{\partial\,\delta} = \begin{bmatrix} \partial\, vec\left\{-(\delta\Lambda)^{-1}(I_m-\delta\Lambda)(I_m-\Lambda)\Gamma_0\right\}/\partial\delta \\ \partial\,\mu_x/\partial\delta\end{bmatrix} = \begin{bmatrix} \delta^{-2}(\Gamma_0'\otimes I_m)vec(\Lambda^{-1}-I_m) \\ 0_{q\times 1}\end{bmatrix};$$

$$A_9 = \frac{\partial\, vec(\mu)}{\partial\, vec(\Lambda)'} = \begin{bmatrix} \partial\, vec\left\{-(\delta\Lambda)^{-1}(I_m-\delta\Lambda)(I_m-\Lambda)\Gamma_0\right\}/\partial\, vec(\Lambda)' \\ \partial\, vec(\mu_x)/\partial\, vec(\Lambda)'\end{bmatrix}$$

$$= \begin{bmatrix} \delta^{-1}(\Gamma_0'\otimes I_m)\left[(\Lambda^{-1}\otimes\Lambda^{-1\prime}) - \delta I_{m^2}\right]K_{m^2} \\ 0_{q\times m^2}\end{bmatrix};$$

It is evident that A_3 and A_4 are of full column-rank. We now establish the rank of A_2 and A_9. These are of full column rank m^2. To prove this it is sufficient to show that the $(m^2\times m^2)$ matrix $\left[(\Lambda^{-1}\otimes\Lambda^{-1\prime}) - \delta\, I_{m^2}\right]$ is of full column-rank. By the *Schur decomposition* (see Magnus, 1988, Chapter 1): $\Lambda^{-1} = PG^{-1}P^{-1}$, where G^{-1} is an upper triangular matrix whose diagonal elements: $\{\lambda_{-1}, \lambda_{-2}, ..., \lambda_{-m}\}$ are the eigenvalues of Λ^{-1}, therefore:

$$(\Lambda^{-1} \otimes \Lambda^{-1\prime}) = (P \otimes P^{-1\prime})(G^{-1} \otimes G^{-1})(P^{-1} \otimes P\ ')$$

Calling $F = (P \otimes P^{-1\prime})$:

$$\left[(\Lambda^{-1} \otimes \Lambda^{-1\prime}) - \delta I_{m^2}\right] = F\left[(G^{-1} \otimes G^{-1\prime}) - \delta I_{m^2}\right]F^{-1}$$

Also $\left[(G^{-1} \otimes G^{-1\prime}) - \delta I_{m^2}\right]$ is an upper triangular matrix and it will have rank m^2 if

$$\lambda_{-i}\lambda_{-j} \neq \delta \quad \text{for } i,j=1,2,...m^2$$

As the eigenvalues of Λ are in modules less than 1 (see Appendix A), it follows that the eigenvalues of Λ^{-1} will be in modules greater than 1, and their product can not be equal to δ. Thus $\det\left[(\Lambda^{-1} \otimes \Lambda^{-1\prime}) - \delta I_{m^2}\right] \neq 0$ and A_2 is of full column rank m^2, that means that L has column-rank $1+m^2+qr$. Finally,

$$A_7 = \frac{\partial \mu}{\partial \Gamma_0'} = \begin{bmatrix} \partial\left\{-(\delta\Lambda)^{-1}(I_m - \delta\Lambda)(I_m - \Lambda)\Gamma_0\right\}/\partial\Gamma_0' \\ \partial \mu_x / \partial\Gamma_0' \end{bmatrix} =$$

$$\begin{bmatrix} -(\delta\Lambda)^{-1}(I_m - \delta\Lambda)(I_m - \Lambda) \\ 0_{q\times m} \end{bmatrix};$$

so A_7 is linearly independent from A_*. Summarising: rank($J(\psi)$)$=m+1+m^2+q(r+p(k-1)+1) = s$ which proves part (a) of Proposition 1. When $k=1$ in (4.1), then $\xi=(vec(\alpha)', \mu')'$, $\psi=(\Gamma_0', \delta, vec(\Lambda)', vec(\alpha_x)', \mu_x')'$ and $s=(m+1+m^2+qr+q)$. In this case $J(\psi)$ collapses to

$$J(\psi) = \begin{bmatrix} 0 & A_1 & A_2 & A_3 & 0 \\ A_7 & A_8 & A_9 & 0 & A_* \end{bmatrix}$$

and since $A_1 = \frac{\partial\, vec(\alpha)}{\partial\delta}$ ($pr\times 1$) belongs to the space spanned by $A_2 = \frac{\partial\, vec(\alpha)}{\partial\, vec(\Lambda)'}$ ($pr\times m^2$), and $A_8 = \frac{\partial\mu}{\partial\delta}$ ($p\times 1$) belongs to the space spanned by $A_9 = \frac{\partial\, vec(\mu)}{\partial\, vec(\Lambda)'}$ ($p\times m^2$), then rank(J(ψ))=(s–1). It is therefore evident that at least one identification restriction is required on (δ, Λ). This proves part (b) of Proposition 1 and completes the proof.

♦

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Note

[1] The models in Lucas (1967) and Treadway (1971) are in continuous time, while the ones in Eichembaum (1984), Nickell (1984), Weissemberger (1986) and Binder and Pesaran (1995) are in discrete time.

[2] Following exactly the definition in Hansen and Sargent (1991, pp.45): '... *models in which there is an exact*

linear relation across forecasts of future values of one set of variables and current and past values of some other set of variables. They key requirement is that all of the variables entering this relation must be observed by the econometrician'. Actually, it is common practice in the literature on LQAC model, to specify disturbance processes reflecting the phenomenon that in forecasting the future, private agents use larger information sets than the econometrician can consider because of data limitations, see, *inter alia*,. Gregory *et al.* (1993) and Engsted and Haldrup (1994). Anyway, as pointed out in Hansen and Sargent (1991, pp.46), this interpretation of error terms can be accommodated within the class of 'exact' rational expectations models.

[3] In the money-demand framework, see e.g. Cuthbertson and Taylor (1987, 1990), Dutkovsky and Foote (1988), Muscatelli (1988, 1989), Domowitz and Hakkio (1990), Bagliano and Favero (1992), and Engsted and Haldrup(1996). See also footnote 4. In models of price adjustment, see e.g. Rotemberg (1982), and Price (1992). In the demand for labour see e.g. Sargent (1978), Nickell (1984, 1987), and Engsted and Haldrup (1994). In the consumer-demand theory see Weissemberger (1986) and the references therein.

[4] Notice that a remarkable comparison of limited and full information estimation methods of rational expectations models for money demand may be found in Ripatti (1997).

[5] It is indeed easy to show that $E(u_t u_{t-k}')=0_{m\times m}$ when $k\geq 2$. Actually it can be also proved that $\{u_t\}$ has the structure of a multivariate MA(1) process.

[6] We shall return on this subject in the sections below where identification of the parameters of the MLQAC model is discussed with details.

[7] Note that in the present context (3.5)-(3.6) can not be interpreted as a conditional multivariate regression model of y_t given x_t.

[8] Since the variables involved are I(1), the 'relevance' condition required by IV estimation holds irrespective of the properties of I(1) instruments, see e.g. Phillips and Hansen (1990, pp. 104).

[9] The procedure set up by Hayashi and Sims is based on the additional assumption of conditional homoscedasticity of the error term with respect to the set of instruments.

[10] See Phillips (1989) for the inferential consequences arising in models where the cross-moment matrix of instruments and regressors is singular. It is there stated, pp. 224, that: "... *In such models, where two-step procedures and instrumental variables are routinely used, partial identification occurs because of instruments failures. That is, the instruments fail to satisfy what might be called the relevance condition. This condition requires that the asymptotic correlation matrix between the instruments and the regressors be of full rank. If the instruments fail, then the model is only partly identified and conventional asymptotics break down.*"

[11] It is sufficient to write equation (4.1) with respect to time (t+1) and take the terms on the right-hand side.

[12] As it will be shown in sub-section 5.1, whenever in (4.11) $r=m$, then $\omega=I_m$, and no substitution for ω is actually required.

[13] general discussion about identification in cointegration analysis may be found in Pesaran and Shin (1994), Johansen (1995, chap. 5 and 7) and Boswijk (1996).

[14] Now identification of the cointegration space requires at least (r-m) restrictions on each row of Γ, and at least r restrictions (included normalisation) on each column of β_0. Observe that following the argument in Dengsoe *et al.* (1995), it is clear that when $r>m$, the process $\{x_t\}$ can not be weakly exogenous with respect to β in (4.2)-(4.3).

[15] We are implicitly assuming that ψ is a 'regular point' of $R(\psi)$, see e.g. Rothemberg (1971).

[16] For instance Gregory *et al.* (1993) point out that the intertemporal discount factor should generally fall in the range 0.96-0.99. Note that part (b) of the proposition also provides a rationale for the choice adopted in section 3, where δ has been pre-fixed on a priori grounds. Anyway, δ is actually locally identified, it is not possible, in the 'limited-information' framework depicted in section 3, to test the over-identification restriction implied by this choice.

[17] See sub-section 4.1.

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