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Estimating the LQAC Model with I(2) Variables.

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Abstract.

This paper derives a method for estimating and testing the Linear Quadratic Adjustment Cost (LQAC) model when the target variable and some of the forcing variables follow I(2) processes. Based on a forward-looking error-correction formulation of the model it is shown how to obtain strongly consistent estimates of the structural long-run parameters and the adjustment cost parameter from both a linear and a non-linear cointegrating regression, where first-differences of the I(2) variables are included as regressors (multicointegration). Further, based on the estimated parameter values, it is shown how to test and evaluate the LQAC model using a VAR approach. In an empirical application using UK money demand data, the non-linear multicointegrating regression delivers an economically plausible estimate of the adjustment cost parameter. However, the exact restrictions implied by the LQAC model under rational expectations are strongly rejected.

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1. Introduction.

Slow adjustment of macroeconomic variables in response to shocks is often explained as a result of adjustment costs. In the literature the Linear Quadratic Adjustment Cost (LQAC) model is a widely used specification upon which the dynamic adjustment in labour input, money holdings, inventories, etc. is interpreted, see Sargent (1978) and Kennan (1979) for two early applications. Recently a number of papers have investigated how to estimate and test the LQAC model when time-series are integrated processes, see e.g. Dolado *et al*. (1991), Gregory (1994), Gregory *et al*. (1993), Engsted and Haldrup (1994, 1995), and Rossana (1995). In particular, Dolado *et al*. (1991) show how, in an Euler-equation context, the structural parameters of the model can be estimated consistently in a combination of cointegration regressions and instrumental variables regressions.

A limitation of the analysis by Dolado *et al*. is that they only consider the case with a *single* forcing, or exogenous, variable. The generalization to the more realistic case with several forcing variables (possibly integrated of different orders) is not straightforward within their framework.

In this paper we follow the route set forth by Dolado *et al*. and analyze in more detail the LQAC model when the endogenous variable is an I(2) process. This case is clearly empirically interesting. In e.g. the money demand literature the LQAC model is often used to rationalize slow adjustment in agent's nominal money holdings, see e.g. Domowitz and Hakkio (1990), Cuthbertson (1988), Cuthbertson and Taylor (1987, 1990), and Muscatelli (1988, 1989), and nominal money is frequently found to be integrated of order two, see e.g. Johansen (1992b) and Haldrup (1994). However, in contrast to Dolado *et al*. (1991) we focus our attention on the forward-looking error-correction formulation of the model. Basically this has three major advantages. First, the analysis with more than one forcing variable becomes relatively straightforward. In the case of money demand the forcing variables would naturally be nominal prices, which possibly are I(2), and real income and nominal interest rates, which are I(1). Second, instead of a two-step procedure as suggested by Dolado *et al.*, we opt for a one step procedure where the statistical properties (in terms of consistency) of parameter estimates appear to be superior compared to the two-step procedure. Thirdly, in addition to pure statistical testing, an informal evaluation of the fit of the model can be obtained along the lines suggested by Campbell and Shiller (1987) and Engsted and Haldrup (1994, 1995). An important byproduct of our approach is that if we are willing to prefix the discount-factor (which is often done in empirical studies using the LQAC model), then not only long-run structural parameters, but also the *adjustment cost* parameter, can be estimated super-consistently in a non-linear cointegration regression.[1] The cointegration properties of the data, as these are suggested by the theory, can also be tested in an unrestricted model. However, the merit of the non-linear cointegration regression is that it supplies a direct test of the model since the non-linear restriction can be tested using a likelihood ratio test with a standard asymptotic χ^2 distribution.

The plan of the paper is the following. In the next section an error-correction formulation of the LQAC model with rational expectations is derived for the case with an I(2) target variable, and a mixture of I(1) and I(2) forcing variables. In particular we emphasize the non-linear restrictions that are implied by the model and the fact that first differences of the I(2) variables need to be included in the cointegration relations. This type of cointegrated models where differenced variables are needed to obtain stationarity is frequently referred to as multi- or polynomial cointegration. Problems concerned with estimation and testing are also discussed. In section 3 it is demonstrated how, given the estimates of the structural parameters from the cointegration regression, the LQAC model can be tested and evaluated both formally and informally using a VAR approach. In section 4 the methods are applied to UK money demand data. The final section concludes.

2. Cointegration implications of the LQAC model with I(2) variables.

According to the LQAC model the economic agent chooses a sequence of the decision variable, m_t, in order to minimize the conditional expectation of the intertemporal cost function

$$L_t = \sum_{i=0}^{\infty} \beta^i [\theta(m_{t+i} - m^*_{t+i})^2 + (m_{t+i} - m_{t+i-1})^2]. \qquad (1)$$

[1] See also Engsted (1993), Clarida (1994), and Ogaki (1992) for other examples of dynamic rational expectations models where the structural parameters can be estimated in cointegration regressions.

β is the subjective discount rate; θ is the relative cost parameter; and m_t^* denotes the desired long-run level of m_t. The first order condition to this minimization problem is the Euler-equation

$$\Delta m_t = \beta E_t \Delta m_{t+1} - \theta(m_t - m_t^*)$$

which can be reparameterized as

$$\Delta^2 m_{t+1} = (\beta^{-1} - 1)\Delta m_t + \frac{\theta}{\beta}(m_t - m_t^*) + u_{t+1} \tag{2}$$

where u_{t+1} is an I(0) rational expectations error.

When m_t is I(2), equation (2) implies that, unless β=1 (i.e. the case of no discounting), m_t-m_t^* will be I(1) and cointegrated with Δm_t with cointegration parameter $\theta/(\beta\text{-}1)$. In the case with only one forcing variable, x_t, determining m_t^*, and assuming a linear relationship, we have: $m_t^*=\gamma x_t+\varepsilon_t$. This implies that the long-run parameter, γ, can be estimated super-consistently in a cointegration regression between m_t and x_t. Thereafter, the parameter $\theta/(\beta\text{-}1)$ can be estimated super-consistently in a second cointegration regression between (m_t-γx_t) and Δm_t. This strategy was suggested by Dolado *et al*. (1991). Note, however, that the second step of the procedure does not make it possible to separately identify θ and β (unless we prefix β). Dolado *et al*. therefore suggest a third step where, conditional on the second step estimate of $\theta/(\beta\text{-}1)$, β is estimated consistently in a standard instrumental variables regression.

In the following we will derive an alternative estimation strategy based on the forward-looking error-correction formulation of the model, where we allow for more than a single forcing variable determining m_t^*. We start by solving the Euler-equation using lag-operator techniques (see e.g. Sargent, 1979), whereby we obtain the following forward-looking representation

$$m_t = \lambda m_{t-1} + (1-\lambda)(1-\lambda\beta)\sum_{i=0}^{\infty}(\lambda\beta)^i E_t m_{t+i}^* \tag{3}$$

where λ is the stable root satisfying the characteristic equation βz^2-$(1+\beta+\theta)z+1=0$ implied by the Euler-equation. Let $\boldsymbol{x}_t$ be a vector of forcing variables, and partition this vector into $\boldsymbol{x}_{1t}$ containing all I(1) forcing variables, and $\boldsymbol{x}_{2t}$ containing all I(2) forcing variables. Assume that m_t^* is a linear function of $\boldsymbol{x}_t$: $m_t^*=\gamma'\boldsymbol{x}_t+\varepsilon_t=\gamma'_1\boldsymbol{x}_{1t}+\gamma'_2\boldsymbol{x}_{2t}+\varepsilon_t$. The error term ε_t reflects the idea that the econometrician's information set generally will be smaller than the economic agent's information set. Discrepancies from the forward looking model may thus be due to a large error component. Equation (3) can now be reparameterized into the following equation:

$$\Delta^2 m_t = (\lambda-1)[m_{t-1} - \gamma'\boldsymbol{x}_{t-1} - \frac{1}{\lambda-1}\Delta m_{t-1} - \frac{\gamma'_2}{1-\lambda\beta}\Delta\boldsymbol{x}_{2,t-1}] + (1-\lambda)\sum_{i=0}^{\infty}(\lambda\beta)^i E_t[\gamma'_1\Delta\boldsymbol{x}_{1,t+i} + \frac{\gamma'_2}{1-\lambda\beta}\Delta^2\boldsymbol{x}_{2,t+i}] + \mu_t \tag{4}$$

This equation is a forward-looking error-correction model, defined for I(2) variables. The double-difference of m_t, which is I(0), is a function of current and expected future values of the first-difference of the I(1) forcing variables, current and expected future values of the double-difference of the I(2) forcing variables, and an error-correction term which captures the gradual adjustment due to adjustment costs.

From the error-correction term an interesting statistical implication of the model can be seen: First differences of the I(2) variables are needed in the cointegration relationship between m_t and $\boldsymbol{x}_t$ in order to reduce the integration order from I(2) to I(0). This is an example of multi- or polynomial cointegration, see e.g. Yoo (1986), Granger and Lee (1989, 1990), Gregoir and Laroque (1994), Johansen (1992a, 1995), and Haldrup and Salmon (1995). Thus, the cointegration relation, with a non-linear relationship between the parameters, can be written as

$$m_t = \gamma'_1\boldsymbol{x}_{1t} + \gamma'_2\boldsymbol{x}_{2t} + \frac{1}{\lambda-1}\Delta m_t + \frac{\gamma'_2}{1-\lambda\beta}\Delta\boldsymbol{x}_{2t} + e_t. \tag{5}$$

Note, that from a cointegration point of view, estimation of this equation in *un*restricted form does not require that *both* $\Delta \boldsymbol{x}_{2t}$ and Δm_t be included as regressors, since m_t and $\boldsymbol{x}_{2t}$ are assumed to be I(2) with $(m_t - \gamma'_2 \boldsymbol{x}_{2t})$ being I(1). The differenced I(2) variables will thus contribute with a stationary I(0) term when entering unrestrictedly. More specifically, equation (5) can be rewritten as

$$m_t = \gamma'_1 \boldsymbol{x}_{1t} + \gamma'_2 \boldsymbol{x}_{2t} + \frac{1}{\lambda - 1}(\Delta m_t - \gamma'_2 \Delta \boldsymbol{x}_{2t}) + \gamma'_2 \left(\frac{1}{1-\lambda\beta} + \frac{1}{\lambda-1}\right)\Delta \boldsymbol{x}_{2t} + e_t \qquad (6)$$

demonstrating that it suffices to test whether the variables m_t, $\boldsymbol{x}_{1t}$, $\boldsymbol{x}_{2t}$ and $\Delta \boldsymbol{x}_{2t}$ constitute an I(0) cointegrating relation. Alternatively Δm_t may enter in place of $\Delta \boldsymbol{x}_{2t}$ implying another parametric restriction to be satisfied:

$$m_t = \gamma'_1 \boldsymbol{x}_{1t} + \gamma'_2 \boldsymbol{x}_{2t} - \frac{1}{1-\lambda\beta}(\Delta m_t - \gamma'_2 \Delta \boldsymbol{x}_{2t}) + \left(\frac{1}{\lambda-1} + \frac{1}{1-\lambda\beta}\right)\Delta m_t + e_t \, . \qquad (7)$$

This cointegration property is fully consistent with the cointegration implications dictated in (2). Existing procedures can be used to test for cointegration in this case[2], see e.g. the systems approach for I(2) systems of Johansen (1995) or the single equation approach of Haldrup (1994).

Given that cointegration is found and β is prefixed at a reasonable economic value[3], it is possible to identify the unknown parameters γ_1, γ_2, and λ. This is similar to the procedure suggested by Dolado *et al*. (1991) with the modification that all model parameters are now estimated in one step (rather than in two) and with the implication that the parameters of the I(2)-variables are estimated super-super consistently, i.e. of order $O_p(n^{-2})$, see Haldrup (1994). The two step procedure of Dolado *et al*. (1991) will only result in $O_p(n^{-1})$ consistency (super consistency) of *all* the unrestricted model parameters.

Although the above procedure is fully legitimate, it is obvious from (5), that there

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[2] Of course, it is perfectly valid to include both $\Delta \boldsymbol{x}_{2t}$ and Δm_t in an unrestricted regression. However, with respect to cointegration testing, the problem about inclusion of these variables in an unrestricted regression, is that even asymptotically, a Dickey-Fuller test of whether the regression residuals are I(1) will have a distribution which depends upon nuisance parameters. This follows from the fact that implicitly there is an I(0) relation build into the model.

[3] It is common practice in much empirical rational expectations literature to preset the discount factor, see also Gregory et al. (1993).

exists more structure in the model that can be exploited in order to obtain more efficient parameter estimates. When β is pre-fixed we demonstrate in the appendix that the non-linear least squares (NLS) estimates of the model parameters γ_1, γ_2, and λ will also be of orders $O_p(n^{-1})$, $O_p(n^{-2})$, and $O_p(n^{-1})$, respectively, given, of course, that the model is fully cointegrated[4]. Moreover, in this situation the model has an overidentifying restriction which can be naturally tested[5] by a conventional likelihood ratio test distributed as $\chi^2(1)$. However, this is not a complete test of the model since only restrictions associated with the cointegration parameters are tested. In the next section we elaborate on a complete test of the model and suggest a way of measuring the model fit.

3. VAR tests of the LQAC model.

Due to the rapid convergence of both the OLS and NLS estimates discussed in the previous section, these are robust to stationary discrepancies from the underlying model. For example, we can add a stationary shock term with arbitrary autocorrelation to the cost function (1) and still, as long as m_t and $\boldsymbol{x}_{2t}$ are I(2) and $\boldsymbol{x}_{1t}$ is I(1), the cointegration relationship among the variables remains as in (5). However, if we want to carry out a formal (and complete) statistical test of the LQAC model under rational expectations, this test becomes crucially dependent on the presence and exact nature of the shock term (and also on the error term ε_t in the long-run relationship for m_t^*). In the literature such a statistical test is often obtained by assuming particular processes for the shock and error terms and then estimating a VAR model for the underlying variables, followed by the calculation of a likelihood ratio test of the cross-equation restrictions implied by the LQAC specification under rational expectations.

However, recently a number of researchers[6] have argued that since typical inter-temporal optimizing rational expectations models (like the LQAC model) are only crude

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[4] In the appendix we discuss some technical difficulties about testing for the null of no cointegration when the model from the outset has been imposed the model restrictions.

[5] Note that such a test needs to be based on a dynamic cointegration regression that permits lags of the differenced I(1) series, and the twice differenced I(2) series in the regression to whiten the errors.

[6] See e.g. Campbell and Shiller (1987), Durlauf and Hall (1988, 1989a,b, 1994), Durlauf and Macchini (1993), and Watson (1993).

approximations with no claim to describe *all* the characteristics of the actual data, we should expect formal specification tests to reject the models, and if they do not, it is probably due to low power of the test. This does not necessarily mean, however, that the model is completely useless in describing important aspects of the data. What is needed therefore, is a metric for measuring the accuracy of the model in approximating the data even if the model is rejected by formal testing procedures.

In Engsted and Haldrup (1994) we propose such a metric for evaluating the LQAC model when the underlying variables are I(1) processes. However, the approach extends very easily to the case where some of the variables are I(2). The idea is to rewrite equation (4) such that only observable current and lagged values occur on the left-hand side, and only *un*observable expected future values of the variables (apart from the error term) occur on the right-hand side:

$$\Delta^2 m_t - (\lambda-1)[m_{t-1} - \gamma \boldsymbol{x}_{t-1} - \frac{1}{\lambda-1}\Delta m_{t-1} - \frac{\gamma'_2}{1-\lambda\beta}\Delta \boldsymbol{x}_{2,t-1}] - (1-\lambda)[\gamma'_1 \Delta \boldsymbol{x}_{1,t} + \frac{\gamma'_2}{1-\lambda\beta}\Delta^2 \boldsymbol{x}_{2,t}]$$
$$= (1-\lambda)\sum_{i=1}^{\infty}(\lambda\beta)^i E_t[\gamma'_1 \Delta \boldsymbol{x}_{1,t+i} + \frac{\gamma'_2}{1-\lambda\beta}\Delta^2 \boldsymbol{x}_{2,t+i}] + \mu_t \ . \tag{8}$$

Denote the left-hand side of this equation the spread, S_t, and define Z_t to be equal to $\gamma'_1 \Delta \boldsymbol{x}_{1t} + \gamma'_2 (1-\lambda\beta)^{-1} \Delta^2 \boldsymbol{x}_{2t}$. Then we can set up a VAR model for S_t and Z_t

$$\begin{bmatrix} Z_t \\ S_t \end{bmatrix} = \begin{bmatrix} a(L) & b(L) \\ c(L) & d(L) \end{bmatrix} \begin{bmatrix} Z_{t-1} \\ S_{t-1} \end{bmatrix} + \begin{bmatrix} e_{1,t} \\ e_{2,t} \end{bmatrix}$$

and we can test the restrictions that equation (8) imposes on the VAR parameters.

These restrictions depend on the nature of the error term in (8), which in turn is a mixture of the error term ε_t in the long-run relation for m_t^* and a possible shock term in (1). As a benchmark model, consider the case where μ_t is zero for all *t*. Hence, in the terminology of Hansen and Sargent (1991), we have an *exact* linear rational expectations model. In this case S_t is the *optimal* predictor of the present discounted value of future Z_t's which implies Granger-causality from S_t to Z_t. Further, the cross-equation restrictions turn out to be very simple: Define a limited information set H_t as consisting of current and lagged values of Z_t and S_t, and write the VAR model in first-order companion form as

$\boldsymbol{X}_t=\mathbf{A}\boldsymbol{X}_{t-1}+\varepsilon_t$, where $\boldsymbol{X}_t=[Z_t,...,Z_{t-p},S_t,...,S_{t-p}]'$ and $\mathbf{A}$ is the companion matrix of VAR parameters (see Engsted and Haldrup (1994) for details). Then, by projecting equation (8) (with $\mu_t=0\ \forall t$) onto H_t we obtain $S_t=\mathbf{g}'\boldsymbol{X}_t=(1-\lambda)\lambda\beta\mathbf{h}'\mathbf{A}(\mathbf{I}-\lambda\beta\mathbf{A})^{-1}\boldsymbol{X}_t$, where $\mathbf{g}$ and $\mathbf{h}$ are selection vectors that pick out S_t and Z_t, respectively, from the VAR. From this expression we obtain $\mathbf{g}'(\mathbf{I}-\lambda\beta\mathbf{A})=(1-\lambda)\lambda\beta\mathbf{hA}$, which is the compact form of the cross-equation restrictions. These restrictions can be tested by a likelihood ratio test, say.

Since a formal test of these restrictions is difficult to interprete *economically*, c.f. the discussion above, one might alternatively compute an unrestricted VAR forecast of the present discounted value of future Z_t's, and compare it with the actually observed values of S_t. If the exact LQAC model is true these two variables should be equal to one another, c.f. equation (8). The unrestricted VAR forecast (called the "theoretical spread" in the terminology of Campbell and Shiller, 1987) is computed from the following formula:

$$S_t^* = (1-\lambda)\lambda\beta\boldsymbol{h}'\boldsymbol{A}(I-\lambda\beta\boldsymbol{A})^{-1}\boldsymbol{X}_t \ . \tag{9}$$

By plotting S_t and S_t^* in a diagram we obtain a visual measure of the deviations from the exact LQAC model. In other words, the degree of comovement of S_t and S_t^* measures in an informal way the accuracy of the model in approximating the data.

4. An application to UK money demand.

In this section the methods presented in the previous sections will be applied to UK money demand. The LQAC specification has been widely used in empirical money demand studies using UK data, see e.g. Cuthbertson (1988), Cuthbertson and Taylor (1987, 1990), Muscatelli (1988, 1989), and Engsted and Haldrup (1995). Cuthbertson and Taylor (1990) and Engsted and Haldrup (1995) explicitly consider how to estimate and test the LQAC model when the variables appearing in the cost function (1) are I(1) processes.[7] However, at least for the dataset analyzed in this paper, nominal money should rather be considered an I(2) process, see Johansen (1992b) and Haldrup (1994). The dataset consists of

[7] Cuthbertson and Taylor (1990) recognize that nominal money and prices probably are I(2) and consequently specify the cost function in terms of *real* money, which is clearly I(1). However, as argued by Goodfriend (1990), this has the unappealing implication that there are no costs of changing nominal money balances as long as prices change by the same magnitude. Hence, in Engsted and Haldrup (1995) we also analyze the nominal money specification, but under the assumption that nominal money is I(1).

quarterly observations, spanning the period 1963:1 to 1989:2, for the following variables: The log of nominal M1 (m_t), the log of the Total Final Expenditure deflator (p_t), the log of real Total Final Expenditure (y_t), and a 3-month learning adjusted interest rate (R_t), see Hendry and Ericsson (1991) for the precise definitions of the variables. The analyses of Johansen (1992b) and Haldrup (1994) established the most likely order of integration for these variables: m_t and p_t are I(2), and y_t and R_t are I(1). Hence, in the notation of section 2, $\boldsymbol{x}_{2t}$ contains one variable, p_t, while $\boldsymbol{x}_{1t}$ contains y_t and R_t.

In table 1 we report the results from estimating various cointegration regressions, both with and without non-linear parameter restrictions imposed. The non-linear restrictions are as dictated in (5). Some of the unrestricted parameter estimates can also be found in Haldrup (1994) and the table is directly comparable with his table 2. Johansen (1992b) finds a single cointegration vector to exist amongst the variables and the inclusion of the differenced prices, in particular, may be needed in the cointegration relation. As seen in table 1, there is some evidence of cointegration in an unrestricted regression with Δm_t as regressor (regression a1). However, since we have strong priors for the absence of money illusion in the long run, we choose to prefix the p-coefficient at unity. As seen from the table, we are now only able to find cointegration if we also impose income homogeneity (regression c1). Based on this specification we found that the likelihood ratio test of the non-linear restriction implied by the LQAC model could be strongly rejected[8]. However, despite of these findings, all constrained regressions have a negative sign to Δm_t and a positive sign to Δp_t (which jointly produces a value of λ between 0 and 1). This is consistent with the LQAC model. In the preferred model with price and income homogeneity imposed, the restricted regression implies a value of λ equal to .797. This implies a value of the relative cost parameter, θ, equal to .054. Hence this indicates that agents put much more weight to costs associated with changing money balances than to costs associated with deviations from the conjectured optimal level of money balances.

Before we proceed with the VAR tests, we would like to emphasize two useful insights that can be derived from these results, and the general approach in section 2, in

[8] The likelihood ratio test was based on regressions like (c3) and (c4) but extended by two lags of the differenced variables to whiten the error term. Since the cointegration rank is found to be one for this data set, see Johansen (1992b) and the fact that the conditioning variables cannot be rejected to be weakly exogenous with respect to the long run parameters, see Haldrup(1994), a likelihood ratio test based on a single equation analysis is valid. The test value was found to be LR=18.02 which is strongly significant in a $\chi^2(1)$ distribution.

TABLE 1. *OLS and NLS estimation of cointegration parameters for UK money demand data*, 1963:1-1989:2, *n*=105 *observations.*

		m	*p*	*y*	*R*	Δm	Δp	*Implied value of* λ	*ADF*	Z_{CI}	R^2	*DW*
(*a*1)	OLS	1	.77	1.38	-3.86	-2.39	-	.942	-2.65	-4.94*	.99	.79
(*a*2)	OLS	1	.68	1.57	-2.67	-	-2.55	.954	-2.26	-2.35	.99	.29
(*a*3)	OLS	1	.73	1.50	-3.39	-2.16	-1.49	-	-	-	.99	.73
(*a*4)	NLS	1	.78	1.28	-4.14	-1.82	1.41	.451	-	-	.99	.58
(*b*1)	OLS	1	1	.56	-4.32	-2.97	-	.948	-2.16	-2.49	.74	.73
(*b*2)	OLS	1	1	.39	-4.02	-	.90	.894	-2.22	-2.18	.68	.22
(*b*3)	OLS	1	1	.56	-4.76	-3.12	1.64	-	-	-	.75	.85
(*b*4)	NLS	1	1	.54	-4.98	-2.78	2.69	.640	-	-	.75	.77
(*c*1)	OLS	1	1	1	-4.84	-5.29	-	.962	-3.40	-3.90*	.70	.17
(*c*2)	OLS	1	1	1	-3.94	-	-.18	.791	-2.05	-1.99	.47	.10
(*c*3)	OLS	1	1	1	-5.27	-5.44	1.64	-	-	-	.70	1.29
(*c*4)	NLS	1	1	1	-5.96	-4.92	4.65	.797	-	-	.68	1.11

NOTE. The table displays single-equation estimates of cointegration parameters. The non-linear restrictions imposed follow from (5). A constant was included in all regressions. ADF indicates the Dickey-Fuller cointegration test, see Engle and Granger (1987) and Haldrup (1994); in all regressions it was necessary to include 1 lag of the differenced series. Z_{CI} is the Phillips (1987) test with truncation chosen at four lags. In (*b*) and (*c*) regressions homogeneity and a unit elasticity was imposed prior to estimation. Hence the I(1) analysis applies to these cases.

relation to the existing empirical money demand literature. First, as we have shown, the LQAC approach rationalizes the inclusion of first-differences of money (money growth) and first-differences of prices (inflation) in long-run money demand equations.[9] In particular, when both money growth and inflation are included as regressors, the approach predicts a negative coefficient to money growth, and a *positive* coefficient to inflation, which may seem odd, unless we interpret it in light of the LQAC model.

Secondly, the approach also sheds some light on the puzzling feature of many money demand studies, namely that it appears to be very difficult to decide whether money, prices, income, and interest rates themselves constitute a fully cointegrated system, or whether first-differences of money or prices are needed to obtain full cointegration. For example, Johansen (1992b) is able to find, on the one hand, full cointegration between UK money, prices, income, and interest rates, and, on the other hand, that changes in prices are needed to obtain full cointegration. Of course, these two possibilities are mutually exclusive. However, if we look at equation (5) we see that since β should be close to, but strictly less than, one in reality, and γ_2 is also close to one (otherwise there would be money illusion), the coefficients to Δm_t and Δp_t are close to but strictly different from each other. This implies that in a finite sample it will be extremely difficult to discriminate the two cases from each other, such that Δm_t-[(1-λ)/(1-λß)]Δp_t will look stationary although it is not. Since near-stationarity of Δm_t-[(1-λ)/(1-λß)]Δp_t implies near-stationarity of m_t-$\gamma' \mathbf{x}_t$, this provides an explanation for the puzzling results in Johansen (1992b).

Now let's proceed with the VAR tests of the LQAC model. As described in section 3 the exact LQAC model under rational expectations implies three different kinds of implications for a VAR model for Z_t and S_t: Granger-causality from S_t to Z_t; a particular set of cross-equation restrictions; and that S_t^* (computed according to (9)) is equal to S_t. Table 2 presents the results of examining these implications for VAR models with four different lag-lengths. The results for λ=.797 are reported, i.e. the estimate found in regression (*c*4).

As seen, for all the VAR models the exact version of the LQAC model under rational expectations is very strongly rejected by the data: S_t does not significantly Granger-cause Z_t; the likelihood ratio tests imply strong rejection of the cross-equation

[9]In a similar context Engsted (1993) has shown that under hyperinflation, the rational expectations version of the Cagan money demand model implies that real balances cointegrate with money growth.

restrictions; and the degree of comovement of S_t and S_t^* is very low. In principle these results can be reconciled with the basic LQAC specification by adding shock terms to the model, but especially the comparison of S_t and S_t^* made above suggests that the variance of such shocks must be quite large. In the terminology of Durlauf and Hall (1989, 1994) a substantial "noise" component must be added to the model in order for it to match the data. Overall, the results suggest a rather limited role for the LQAC model to explain the dynamics of nominal UK money demand.

TABLE 2. *VAR tests of the LQAC model. The tests are based on* λ=.797, *i.e. from regression* (c4) *in Table* 1.

	VAR(1)	VAR(2)	VAR(3)	VAR(4)
Test for Granger-causality from S to Z.	.765	.934	.603	.678
Test of cross-equation restrictions	.000	.000	.000	.000
Correlation(S,S)*	-.262	.621	.096	.346
Var(S)/Var(S)*	.286	.063	.081	.073

NOTE: The numbers for Granger-causality tests and the test of the cross-equation restrictions are *p*-values.

5. Concluding remarks.

The LQAC model has a long tradition in empirical macroeconomics; especially the question of how best to estimate the structural parameters of the model has been a major concern in the literature. In this paper we have shown that when the target variable and some of the forcing variables are I(2) processes, then, provided that we are willing to prefix the discount factor, the long-run parameters and the adjustment cost parameter can be estimated in a strongly consistent way from both a linear and a non-linear cointegrating regression where first-differences of the I(2) variables are included in the regression. In addition, these estimates are robust to stationary deviations from the underlying LQAC model (measurement errors, cost shocks, etc). Further, based on a specific parameterization of the forward-looking error-correction formulation of the model, we have derived a number of testable implications that the exact version of the model under rational expectations imposes on a VAR model written in a particular way.

The LQAC model has been widely used to explain UK money demand, and UK nominal money and prices are often found to be I(2) processes. Consequently, we applied the methodology to a common and well-known money demand data set for the UK. The cointegration analysis delivered economically plausible estimates of the adjustment cost parameter. This is good news for the LQAC model. On the other hand, the likelihood ratio test of the non-linear restriction and the tests based on the VAR set up strongly reject the exact version of the LQAC specification under rational expectations. Although these results can be reconciled with the basic LQAC specification (and with the cointegration results) by adding a "noise" term to the model, the variance of this term must be substantial in order for the model to match the data. Hence, overall we believe that the results in this paper constitute quite strong evidence against the LQAC model as a valid framework for analyzing UK nominal money demand.

6. Appendix.

In this appendix we derive some statistical results that are relevant for a regression model like (A.1) below, which involves both I(1) and I(2) variables and with nonlinear restrictions across the parameters. To be specific, consider the regression model

$$m_t = \gamma'_1 \boldsymbol{x}_{1t} + \gamma'_2 \boldsymbol{x}_{2t} + (\lambda - 1)^{-1} \Delta m_t + \gamma'_2 (1 - \lambda\beta)^{-1} \Delta \boldsymbol{x}_{2t} + u_t \qquad \text{(A.1)}$$

where the processes $\boldsymbol{x}_{1t}$, $\boldsymbol{x}_{2t}$ and u_t are driven as

$$\Delta \boldsymbol{x}_{1t} = \boldsymbol{\varepsilon}_{1t}$$
$$\Delta^2 \boldsymbol{x}_{2t} = \boldsymbol{\varepsilon}_{2t}$$
$$\Delta^d u_t = \varepsilon_{3t}.$$

The ε_{it}-series, i=1,2,3, are all stationary I(0) error terms. The only requirements that are needed are the weak conditions for the multivariate invariance principle to be satisfied, see e.g. Park and Phillips (1989). When d=0 the variables in (A.1) are fully cointegrated whereas if d=1 implies that only the two I(2) variables of the system, m_t and $\boldsymbol{x}_{2t}$, will cointegrate into an I(1) relation, but with no further cointegration occuring amongst the variables. Throughout, we assume that the I(2) variables of the system are CI(2,1), following Engle and Granger's (1987) terminology.

PROPOSITION: *Denote by '^' the nonlinear least squares (NLS) estimates of the parameters in (A.1), and let superscript '0' indicate the true parameter values. Then, given that the requirements of the multivariate invariance principle are satisfied we have:*

For d=0, (*full cointegration*):	For d=1 (*no cointegration*):
$n(\hat{\gamma}_1-\gamma_1^0)=O_p(1)$	$(\hat{\gamma}_1-\gamma_1^0)=O_p(1)$
$n^2(\hat{\gamma}_2-\gamma_2^0)=O_p(1)$	$n(\hat{\gamma}_2-\gamma_2^0)=O_p(1)$
$n(\hat{\lambda}-\lambda^0)=O_p(1)$.	$(\hat{\lambda}-\lambda^0)=O_p(1)$.

PROOF. Since the partial sum of the error sequence $\varepsilon_t=(\varepsilon_{1t}, \varepsilon_{2t}, \varepsilon_{3t})'$ is assumed to satisfy the multivariate invariance principle it follows that

$$\boldsymbol{B}_n(r)=n^{-1/2}\sum_1^{[nr]}\boldsymbol{\varepsilon}_t \Rightarrow \boldsymbol{B}(r)\equiv(\boldsymbol{B}_1(r),\ \boldsymbol{B}_2(r),\ B_3(r))' \qquad r\in[0,1]$$

where [.] indicates the integer value of its argument and all the $\boldsymbol{B}_i(r)$´s are Brownian Motion processes defined on the unit interval. For our asymptotic results we need not specify how these are correlated for our present purpose. For instance the above results imply that $n^{-1/2}\boldsymbol{x}_{1t}\Rightarrow\boldsymbol{B}_1(r)$ and $n^{-1/2}\boldsymbol{x}_{1t}\Rightarrow\int_0^r\boldsymbol{B}_1(s)ds\equiv\bar{\boldsymbol{B}}_1(r)$.

Since the model is linear in the variables but nonlinear in the parameters we prefer to write it as

$$m_t=x_t(\alpha)+u_t, \qquad \text{with} \qquad \alpha=(\gamma'_1,\gamma'_2,\lambda)$$

The gradient of $x_t(\alpha)$ is

$$\boldsymbol{X}_t(\alpha)=\frac{\partial x_t(\alpha)}{\partial\alpha}=(\boldsymbol{x}_{1t},\ \boldsymbol{x}_{2t}+(1-\lambda\beta)^{-1}\Delta\boldsymbol{x}_{2t},\ -(\lambda-1)^{-2}\Delta m_t+\gamma'_2\beta(1-\lambda\beta)^{-2}\Delta\boldsymbol{x}_{2t})'.$$

and associated with this vector, define D_n=diag$\{n^{1/2}, n^{3/2}, n^{1/2}\}$. It follows that

$$D_n^{-1}\boldsymbol{X}_t(\alpha)\ \Rightarrow(\boldsymbol{B}_1(r),\ \bar{\boldsymbol{B}}_2(r),\ -\gamma'_2(\lambda-1)^{-2}\boldsymbol{B}_2(r)+\gamma'_2\beta(1-\lambda\beta)^{-2}\boldsymbol{B}_2(r))'.$$

Since the gradient vector has components that are I(1), I(2), and I(1), respectively, it can be deduced from Park and Phillips (1989) and Haldrup (1994) that

$$n^{-1}D_n^{-1}\sum\nolimits_1^n\boldsymbol{X}_t(\alpha)\boldsymbol{X}_t(\alpha)'D_n^{-1}=O_p(1) \tag{A.2}$$

and

$$n^{-1/2-d}D_n^{-1}\sum\nolimits_1^n\boldsymbol{X}_t(\alpha)u_t=O_p(1). \tag{A.3}$$

An intermediate result that we also shall need is that for d=0

$$n^{-1}D_n^{-1}\sum\nolimits_1^n u_t\frac{\partial\boldsymbol{X}_t(\alpha)}{\partial\alpha'}D_n^{-1}=o_p(1) \tag{A.4}$$

while for d=1 the same expression becomes $O_p(1)$. To see that this is correct, notice that the Hessian of $x_t(\alpha)$,

$$\frac{\partial\boldsymbol{X}_t(\alpha)}{\partial\alpha'}=\frac{\partial^2x_t(\alpha)}{\partial\alpha\partial\alpha'}$$

has components that are of maximal order $O_p(n^{1/2})$. (A.4) then follows trivially from the Lemmas of Park and Phillips (1989) and Haldrup (1994).

NLS seeks to minimize the sum of squares function $\sum_1^n(y_t\text{-}x_t(\alpha))^2$, so in optimum

$$\sum\nolimits_1^n(y_t-x_t(\hat{\alpha}))\boldsymbol{X}_t(\hat{\alpha})=0.$$

Expand this in a short Taylor series about α_0, the vector of true parameter values, hence giving

$$0=\sum_1^n u_t \boldsymbol{X}_t(\alpha_0)+(\hat{\alpha}-\alpha_0)'\sum_1^n [-\boldsymbol{X}_t(\alpha)\boldsymbol{X}_t(\alpha)'+(y_t-x_t(\alpha^*))\frac{\partial \boldsymbol{X}_t(\alpha^*)}{\partial\alpha'}]. \qquad \text{(A.5)}$$

α^* is determined as a convex combination of $\hat{\alpha}$ and α^0 which naturally can be different for each row of the equation as dictated by Taylors Theorem. Since the contents of the bracketed term is the contribution to the sum of squares function of observation t, the last term i (A.5) will be a positive definite matrix for α^* close to α_0. In the limit this is naturally the case when $\hat{\alpha}$ is a consistent estimator since α^* is defined as a convex combination of α_0 and $\hat{\alpha}$. Hence it follows for $n \to \infty$ that

$$(\hat{\alpha}-\alpha_0)'=[\sum_1^n \boldsymbol{X}_t(\alpha_0)\boldsymbol{X}_t(\alpha_0)'-\sum_1^n \frac{\partial \boldsymbol{X}_t(\alpha_0)}{\partial\alpha'}u_t]^{-1}[\sum_1^n \boldsymbol{X}_t'(\alpha_0)u_t]$$

By using (A.2)-(A.4) we therefore obtain that

$$n^{1/2-d}D_n(\hat{\alpha}-\alpha_0)'=$$
$$(n^{-1}D_n^{-1}\sum_1^n [\boldsymbol{X}_t(\alpha_0)\boldsymbol{X}_t(\alpha_0)'-\sum_1^n \frac{\partial \boldsymbol{X}_t(\alpha_0)}{\partial\alpha'}u_t]D_n^{-1})^{-1}\ (n^{-1/2-d}D_n^{-1}\sum_1^n \boldsymbol{X}_{t'}(\alpha_0)u_t]=O_p(1).$$

This demonstrates that the asymptotic orders displayed in the propostion are valid. Observe however, that for d=1, the non-cointegration case, λ and γ_1 are actually estimated inconsistently at the rate $O_p(1)$. This is not surprising since the same result appears in linear spurious regression models when the regressors are I(1). However, the finding is not in association with the prior assumption that $\hat{\alpha}$ is consistent, hence we have provided a counterproof of the assumption. ■

Some problems implied by testing the null of no cointegration when the model is estimated with the parameter restrictions imposed.

One way to test for the null of no cointegration with a non-linear parameter restriction imposed on the model, is to consider a regression model with d=1 under the null hypothesis (using the notation of the previous section) and calculating the NLS residuals

$$\hat{u}_t = y_t - x_t(\hat{\alpha}).$$

Testing for their order of integration can then by conducted using e.g. a standard residual based cointegration Dickey-Fuller test. This procedure is a simple modification of the Engle-Granger (1987) two-step procedure. The only difference here is that a nonlinear restriction is imposed in the estimation at the first step of the procedure. However, in so doing the implied limiting distributions will be rather different from those applying to the standard regression model which is linear in the parameters. We are not going to derive the actual distribution of the Dickey-Fuller test statistic from our present set up, but it will be instructive to see how the NLS regression residuals will be constructed and to see how these behave in the limit. Ultimately this is what we need to know in order to see how the Dickey-Fuller cointegration test statistic will behave asymptotically.

Expand the NLS residuals around $\alpha=\alpha_0$ as before in a short Taylor series expansion. This yields

$$\hat{u}_t = u_t - \boldsymbol{X}_t'(\alpha^*)(\hat{\alpha} - \alpha_0)$$

where again α^{**} is a convex combination of $\hat{\alpha}$ and α_0 in order to satisfy the relation equality. Notice that for $d=1$

$$n^{-1/2}\hat{u}_t = n^{-1/2}u_t - (\boldsymbol{X}_t'(\alpha^*)D_n^{-1})(n^{-1/2}D_n(\hat{\alpha} - \alpha_0) \Rightarrow Q(r)$$

and hence the scaled residuals will tend weakly to a Brownian motion process *Q(r)*. Notice though, that this Brownian motion will itself be made up of the Brownian motion processes $\boldsymbol{B}_1(r)$, $\boldsymbol{B}_2(r)$ and $B_3(r)$ in a complicated fashion that reflects the particular type of restrictions that are imposed on the model. The distribution of the Dickey-Fuller test for the null of no cointegration will obviously depend upon *Q(r)* and the way it is constructed.

Numerical simulation of the critical values for this case can be conducted in principle. However, the restrictions to be imposed on the regression model are rather specific for this particular problem. Furthermore, difficulties are likely to arise due to the fact that an optimizing algorithm in each stage of replication will be needed in order to simulate the null distribution. A further problem is caused by the fact that some parameters will be inconsistently estimated under the null hypothesis (c.f. the previous section). In fact, this problem was also found to be of practical relevance.

7. References

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