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# Competing Auctions of Skills

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**Abstract**

We generalize McAfee's (1993) game of competing sellers to the case of heterogeneous sellers. In the generalized McAfee (GM) game, the equilibrium expected job offer distribution of each worker (seller) type evolves over time as a function of stochastic events. We derive a tractable method of solving the GM game. We estimate, using non-parametric methods, a close fit between a benchmark GM game and a cross-section of Danish data on productivity and unemployment. The theoretical properties of the GM game, which relate to on-the-job search, assortative matching, aggregate and match specific shocks, and the equivalence of alternative games, are also characterized.

Keywords: Auctions, assortative matching, wage dispersion, aggregate shocks, on-the-job search.

JEL Classification: J64; J63; E32

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# 1 Introduction

Peters (1984) and McAfee (1993) pioneered the use of mechanism design theory to study coordination frictions in matching markets. This research is important because it offers a game theoretic foundation for why matching markets are characterized by a matching function between buyers and sellers, which is assumed in the Nobel prize awarded work of Diamond, Mortensen and Pissarides. The value of having a theoretical foundation for the matching function is that it offers guidance for how this function is specified if there are heterogeneous buyers and sellers, or if there are changes in policy or technology that can affect this matching over time.

In this paper we study a model of the labor market with unemployment, vacancies, wage dispersion, sorting, human capital accumulation, on-the-job search and aggregate shocks. These important elements of the labor market are united by the problem of characterizing the job offer distribution of each worker at each point in time. In order to study how this job offer distribution changes over time, we specify a matching game, which is essentially identical to McAfee (1993) with the added assumption that sellers (workers) are heterogenous. Using this generalized McAfee (GM) game, we prove necessary and sufficient conditions to generate a job offer distribution with positive support over a continuum of job types above a lower bound. We also prove, in this case, that a higher outside option for a worker does not affect the worker's job offer distribution except for a shut-down effect (a truncation) of offers at the bottom of the job offer distribution, which is strictly above the worker's current productivity.

One use of the GM game is to compare its properties with the properties of alternative game theoretic foundations of the matching function. For example, we can compare the equilibrium play in a static GM game with the equilibrium play in closely related static games where sellers (or buyers) compete by strategies other than auctions. Here we find that two very different pricing models - price posting by buyers (Mortensen 2003, pp. 16-22) and price posting by sellers (Shimer 2005) - give identical expected payoffs for the players on both sides of the market. One corollary of the equivalence of expected payoffs between alternative games is that that the normative results on efficient sorting, which are developed in Shimer (2005), can also be applied to the GM game. A second corollary is that the equilibrium job offer distribution predicted in the GM game is the same as in the posting games. Therefore, our theoretical results concerning the problem of specifying the equilibrium with a continuum of possible job types and the related benefits of computation can be extended to the descriptive analysis of Shimer (2005) and related price posting models.

A fundamental empirical question about the GM game and closely related models of price posting is whether the equilibrium job offer distributions that are predicted by the theory are consistent with the sorts of job offer distributions that we observe empirically. To address this question, we apply non-parametric estimation methods and we show that a simple benchmark version of the GM game can easily account for the Danish unemployment rate of 5% and gives a close fit of the entire productivity distribution. Key to fitting the data is the assumption that workers continue to accept job offers even if they are employed. Since the computation of this realistic model of on and off-the-job search is facilitated by the analytical results regarding the impact of worker outside options, this estimation exercise also serves to illustrate the tractable numerical solution of a complete path of equilibrium job offer distributions for long lived workers in all possible employment situations.

We also consider applications of the GM game with on and off-the-job search. One application is the problem of sorting. If the matching of worker types to job types is everywhere positive/negative assortative (higher type workers are matched more/less frequently to higher type jobs than lower type workers), then we prove that the production function describing the output possibilities between higher types must also be super/sub-modular. In addition to the prediction of a super-modular production function, a positive/negative assortative matching (PAM/NAM) equilibrium of the GM game also gives a sharp prediction about the pattern of wages and productivities between workers and firms of different types. We prove that a PAM/NAM equilibrium is always characterized by a positive/negative monotone relationship between the worker's type and the difference between the firm's output and the wage offered to that worker for each firm type.

A second application of the GM game is the analysis of aggregate and idiosyncratic shocks. For the case of aggregate shocks, we consider an unbiased shock that gives a proportional increase in the productivity of all job types - a so-called no-comparative advantage transformation of productive opportunities (Shimer 2005) - and a biased shock that, in equilibrium, gives a proportional increase in the expected number of all job types in the job offer distribution. The GM model predicts that the unbiased shock causes no change in the number of jobs offered, because the wage of each worker fully absorbs the benefits of the shock. Therefore, the unbiased shock causes a parallel rightward shift of the wage offer distribution. However, using our estimated model, we find that a positive biased shock causes a flattening of the wage offer distribution. This explains why wage dispersion can increase during an upturn.

The paper is organized as follows. Sections 2 and 3 solve static and dynamic versions of the GM game. In section 4, we estimate a benchmark version of the dynamic GM game using Danish data and explore the effect of an aggregate shock and other extensions. The final section concludes.

# 2 The Generalized McAfee (GM) game

The generalized McAfee (GM) game is the extension of McAfee's (1993) game of homogenous competing sellers to the case of heterogenous competing sellers. The GM game and the McAfee (1993) game share common assumptions about the private information of buyer types, the public information of seller types, and the set of possible selling mechanisms used in each matching round.

The informational assumptions of the GM game correspond as a first approximation to the institutional features of the labor market. The public knowledge of the worker (seller) types reflects the legal requirement that workers advertise truthful resumes in accordance with the demands of potential employers. Similarly, the private knowledge of the firm (buyer) reflects the residual claim it has on its technology.[1] And so, for the remainder of our analysis of the GM game,

[1] The existence of such institutional features could be motivated by the impossibility of

we will refer to sellers as workers and buyers as firms.

## 2.1 The static model

Workers sell their labor services by a second price auction with a reserve price equal to their continuation value. Firms are the buyers in the labor market. A firm's type is private information and the firm's valuation of the worker services will depend on the firm and worker type. There are many different firm types, $k \in [0, \infty)$. A type $k$ firm produces $y(k)$ units of output if it is matched with a worker. Each worker is endowed with one unit of labor to sell and each firm has one job opportunity that can employ one worker. The higher firm types produce more output but face increasingly higher opportunity costs, $c(k)$, such that $y', c'$, and at least one of $c''$ and $-y''$ are all positive.[2] The worker's type is public information. Workers can differ by their outside opportunity, $y(\underline{k})$, which is the worker's output if the worker is not employed by a firm. For at least one value of the outside option $\underline{k} \in [0, \infty)$, there exists a positive measure of workers, $n(\underline{k})$, which we refer to as a submarket. For simplicity, we normalize the size of the worker population to one. All agents are risk neutral, expected utility maximizers.

Later, and without loss of tractability, we also allow for worker heterogeneity with respect to their skill type, which is described by their individual production schedule, $y(k, h)$ where $h$ denotes the worker's skill type. In this case, a submarket of workers will be defined by the population of workers, $n(\underline{k}, h)$, who have an outside option type, $\underline{k}$, and an individual production schedule, $y(k, h)$. When we consider homogenous workers in skills we simply suppress the notation to exclude $h$.

The competing auction model is a three stage game. In the first stage of the game, a measure of firms of types $k \in [0, \infty)$ enter each of the submarkets, $n(\underline{k})$,

efficient bargaining with two-sided private information (Myerson and Satterthwaite 1983).

[2]The common set of opportunity costs for buyers equals the set of reservation utilities for each buyer, which must be paid to each buyer type to ensure their assignment to any particular seller type (McAfee 1993). The key assumptions leading to McAfee's (1993) theorems 1 and 2 are that each seller cannot affect this set of reservation utilities and that each buyer has perfect information about the seller's type.

and each of the firms in this measure pays its associated opportunity cost, $c(k)$. In the second stage of the game, the newly created firms play a symmetric mixed strategy regarding their assignment to a worker in this submarket. In the final stage of the game, each firm assigned to a worker bids a wage for the worker's labor services according to the rules of the worker's second price auction with a reserve price equal to the value of the worker's outside option. The worker then either selects one of these bids or he consumes his outside opportunity.

The payoff of the worker is his wage if he is hired and his outside opportunity otherwise. If the firm hires the worker, its payoff is the productivity of the firm less its opportunity cost and the worker's wage. If the firm does not hire the worker, its payoff is minus the opportunity cost. The payoff of a firm that does not enter this submarket (perhaps to enter another submarket with another set of different workers) is normalized to zero. In this case, the firm is earning its opportunity cost. The equilibrium is solved by backwards induction.

## 2.2 Third stage

Consider first the optimal bidding strategies in the final stage of the game when the number of each type of firm assigned to each worker is known. Let $k_1$ and $k_2$ denote the highest and second highest firm type bidding for the worker's services. In a second price auction with reserve price equal to the worker's continuation value, $y(\underline{k})$, each firm bids his productivity. The income earned by a worker is

$$w(k_1, k_2) = \begin{cases} y(k_2) & \text{if} \quad y(k_2) \geq y(\underline{k}) \\ y(\underline{k}) & \text{if} \quad \text{otherwise} \end{cases}, \tag{1}$$

If $k_1 > \underline{k}$, then the high valuation firm hires the worker and earns $y(k_1) - w(k_1, k_2)$ and the revenue of all other firms bidding for this worker is zero. If $k_1 \leq \underline{k}$, then all firms earn zero revenue and the worker earns his continuation value, $y(\underline{k})$.

## 2.3 Second stage

The second stage of the game is simple random assignment, which is based on the assumption that all firms entering a worker submarket, $n(\underline{k})$, play symmetric mixed strategies over which worker to visit. Let $\phi(k \mid \underline{k})$ denote the relative mass of firm types greater than $k$, which enters this submarket in the first stage. The random assignment of firms to workers means that the number of firms of type greater than $k$ at each worker's location is distributed Poisson with parameter $\phi(k \mid \underline{k})$. We refer to the function $\phi(k \mid \underline{k})$ as the type $\underline{k}$ worker's ***job offer function***.

In this auction model, the overall distribution of wages and productivities is determined by two order statistics: $G_1(k)$, which denotes the fraction of workers with a best offer less than $y(k)$ and, $G_2(k)$, which denotes the fraction of workers with a second best offer less than $y(k)$. If the set of firm types making offers are in the range of firm types $k \in \left[\check{k}, k^*\right]$, the formulas for these first and second order statistics are as follows.

**Proposition 1.** *If the number of entrants of each firm type on the range of types $k \in \left[\check{k}, k^*\right]$ is given by the job offer function, $\phi(k)$, then*

$$G_1(k) = \exp(-\phi(k)), \text{ and} \tag{2}$$

$$G_2(k) = \exp(-\phi(k)) + \phi(k)\exp(-\phi(k)). \tag{3}$$

*where $G_1\left(\check{k}\right)$ is the probability of no offer and $G_2\left(\check{k}\right)$ is the probability that the number of offers are zero or one.*

*Proof.* See appendix. □

The cumulative distribution function $G_1(k)$ is equivalent to the productivity distribution, because the workers are always employed at the most productive firm type assigned to the worker. The cumulative distribution function $G_2(k)$ is equivalent to the distribution of wages, because the wage is equal to the second most productive firm type assigned to a worker.

## 2.4 First stage: Two firm types

Before we tackle the entry decision of the continuum of firm types case, it is natural to ask what happens if there are discrete firm types. Suppose that there are just three types of firms, say $k \in \{0, 1, 2\}$. Let $k = 0$ be the firm type that is equal to the worker's outside opportunity (i.e. $y(0)$). In this case, the free entry conditions for good (type 2) and bad (type 1) firms, which equate their opportunity cost to the expected returns, are given by

$$\begin{aligned} c(1) &= (y(1) - y(0)) \exp(-\phi(1 \mid 0)) && (4) \\ c(2) &= (y(2) - y(0)) \exp(-\phi(1 \mid 0)) + \\ &\quad (y(2) - y(1)) \exp(-\phi(2 \mid 0)) (1 - \exp(-(\phi(1 \mid 0) - \phi(2 \mid 0)))) && (5) \end{aligned}$$

Note the low type firm only earns an expected payoff if there is no other competing firm. The high type firm can also earn an expected payoff when there is a competitor, provided that this competitor is a low type. If both firm types are offered in equilibrium, i.e. $\phi(2 \mid 0) > 0$ and $\phi(1 \mid 0) > \phi(2 \mid 0)$, the menu of firm productivities and opportunity costs must satisfy the following 'concavity' condition. We require

$$y(2) - c(2) > y(1) - c(1)$$

and

$$\frac{y(1) - y(0)}{c(1)} > \frac{y(2) - y(0)}{c(2)}$$

Furthermore, in such an equilibrium, an increase in the outside opportunity of the worker, $y(0)$, decreases both $\phi(1 \mid 0)$ and $\phi(2 \mid 0)$. Therefore, a larger outside opportunity reduces the likelihood that a worker is contacted by both good and bad firm types.[3]

[3] Further discussion of the two firm type model with homogenous workers is given in Julien, Kennes, and King (2006a). Julien, Kennes and King (2006b) demonstrate that this equilibrium is constrained efficient.

## 2.5 First stage: A continuum of firm types

In an equilibrium with a continuum of firm types, the entry decision for firms of different types can be linked to a boundary condition on the lowest quality firm type and a simple difference equation relating the returns of a particular firm type and the returns of a firm type immediately ranked above this firm type. Let $\hat{k}\left(\underline{k}\right)$ denote the lowest quality firm type offered in equilibrium to a type $\underline{k}$ worker. The total mass of firms of all types directed at a type $\underline{k}$ worker is then $\phi\left(\hat{k}\left(\underline{k}\right)\right)$. The following proposition relates $\hat{k}$ and $\phi\left(\hat{k}\right)$ to the outside opportunity of a worker, $\underline{k}$. We have

**Proposition 2.** *The lowest entrant firm type for a type $\underline{k}$ worker is*

$$\hat{k}\left(\underline{k}\right) = \arg\max_{k} \left(\frac{y\left(k\right) - y(\underline{k})}{c\left(k\right)}\right) \tag{6}$$

*and the probability of at least one offer is given by*

$$1 - G_1\left(-\phi\left(\hat{k}\left(\underline{k}\right)\right)\right) = 1 - \exp\left(-\phi\left(\hat{k}\left(\underline{k}\right)\right)\right) \tag{7}$$

*where*

$$\phi\left(\hat{k}\left(\underline{k}\right)\right) = log\left(\frac{y\left(\hat{k}\right) - y(\underline{k})}{c\left(\hat{k}\right)}\right). \tag{8}$$

*Proof.* See appendix. □

The outside opportunity influences the lowest firm type offered to workers. A higher outside opportunity, $y\left(\underline{k}\right)$ always raises the value of $\hat{k}\left(\underline{k}\right)$ and it reduces the relative supply of firms assigned to this submarket, $\phi(\hat{k})$.

In this auction model, the payoff of each firm type entering the matching market is a function of its productivity and the probability that it faces a competitor of type $k$. Given the distribution of firm types over a discrete set of firm types, we have a simple expression for the payoff of a type $k$ firm. Thus the expected return of a type $k_i$ firm who enters the type $n(\underline{k})$ submarket in the free entry equilibrium (when $\phi(k \mid \underline{k})$ is positive) is given by

$$c(k_i) = \sum_{j=1}^{i} \left(y\left(k_i\right) - y\left(k_{j-1}\right)\right) \exp\left(-\phi\left(k_i \mid \underline{k}\right)\right) \left(1 - \exp\left(-\left(\phi\left(k_{j-1} \mid \underline{k}\right) - \phi\left(k_j \mid \underline{k}\right)\right)\right)\right) \tag{9}$$

where $k_1 = \hat{k}\left(\underline{k}\right)$, $k_0 = \underline{k}$ and $\phi\left(\underline{k} \mid \underline{k}\right) = \infty$. In the limit, if there is a continuum of firm types, the offer distribution of a type $\underline{k}$ worker is as follows. Assuming the following regularity condition, $y''\left(k\right) c'\left(k\right) - y'\left(k\right) c''\left(k\right) < 0$, we can prove the following result.

**Proposition 3.** *The job offer function of a type $\underline{k}$ worker is characterized by the function*

$$\phi\left(k \mid \underline{k}\right) = \phi\left(k\right) \equiv \log\left(\frac{y'\left(k\right)}{c'\left(k\right)}\right) \tag{10}$$

*and by a lower bound on the set of offers, $\hat{k}\left(\underline{k}\right)$.*

*Proof.* See appendix. □

Proposition 3 gives a boundary condition for the supply of the lowest firm types and a differential equation relating the measure of firms on any interval above this boundary point. It is also clear from equation (10) that the upper bound on firm types is $k^*$ where $y'\left(k^*\right) = c'\left(k^*\right)$. As in McAfee (1993), our model allows for any arbitrary distribution of buyer types in any particular market. To understand why, we simply note that any function $\phi\left(k \mid \underline{k}\right)$ is feasible given an appropriate choice of $\underline{k}$, $c\left(k\right)$ and $y\left(k\right)$. Therefore, using the distribution function for highest valuations, $G_1\left(k\right)$, any distribution of worker types is feasible with any set of heterogeneous firm types. Note that the regularity condition is simply a concavity requirement. For example, we require $c''\left(k\right) > 0$ if $y\left(k\right)$ is linear, or $y''\left(k\right) < 0$ if $c\left(k\right)$ is linear.

## 2.6 Wages and productivity

The expected wage of a worker with an outside opportunity of $y\left(\underline{k}\right)$ is derived by integrating the second highest valuation firm type over the set of possible realizations.

**Proposition 4.** *The expected wage of worker type, $\{y(k), \underline{k}\}$, is given by*

$$E(y(k_2) \mid \underline{k}) = G_2(\underline{k})\, y(\underline{k}) + \int_{\hat{k}}^{k^*} y(k)\, g_2(k)\, dk \tag{11}$$

*where*

$$g_2(k) = \left[\frac{-[y''(k)\, c'(k) - y'(k)\, c''(k)]}{(y'(k))^2}\right] \left[\log\left(\frac{y'(k)}{c'(k)}\right)\right].$$

*Proof.* See appendix □

The expected wage of each worker is a function of the worker's outside option $y(\underline{k})$ and their productivity schedule $y(k)$. A worker with a higher outside opportunity earns a higher wage, because he starts from a better position. It is also obvious that an increase in the productivity schedule to some new function, $\hat{y}(k) \geq y(k)$ for all $k$, holding firm type opportunity costs constant, implies a higher expected wage.

Consider the average productivity of ex ante identical workers over the range of firm types less than some value of firm type, $\widetilde{k}$.

$$E\left(y \mid k \leq \tilde{k}, \underline{k}\right) = \frac{\int_{\hat{k}}^{\tilde{k}} y(k)\, g_1(k)\, dk}{G_1(\tilde{k})} \tag{12}$$

This expectation is increasing in $\widetilde{k}$ because higher firm types are more productive. This expectation is also the expected wage paid by a type $\tilde{k}$ firm.

**Proposition 5.** *Higher firm types pay ex ante identical workers higher expected wages than lower firm types*

*Proof.* Straightforward differentiation of equation (12). □

Therefore, in the competing auction model, firms can be ranked by the the expected wage payments that are made to similar workers. Note that this result does not necessarily imply that more productive workers are paid higher wages at a common firm type. For example, it is straightforward to construct an example where the expected wage payments of two workers is not monotone in the worker's productivity at a particular firm.[4]

We can also calculate the joint distribution of wages and productivity. Let $G(k_1, k_2)$ denote the fraction of workers with a best offer no greater than a type $k_1$ firm and a second best offer no greater than a type $k_2$ firm. We find:

**Proposition 6.** *For values of* $k_1, k_2 \in \left[\widehat{k}, k^*\right]$, *the joint offer distribution*

$$G(k_1, k_2) = (1 + \phi(k_2) - \phi(k_1)) \exp(-\phi(k_2)) \tag{13}$$

*Proof.* See appendix. □

The joint distribution of first and second best offers is not needed to describe the distribution of wages in the static model. However, in a dynamic environment, the worker's per period wage will be a function of his first and second best offers. The basic idea is that the worker's second best offer describes the threat point used in setting the wage with his current employer while the productivity of his current employer (his best offer) gives the worker's threat point when setting wages with any future employer who might be contacted by on-the-job search. The worker's current wage then balances these two concerns.

## 2.7 Equivalence with (directed) seller price posting

If each worker sells his skills by a second price auction with a reserve price equal to his continuation value, each worker will expect bids to follow equation (1) even though the valuation of their services by each firm is private information to each firm. An alternative pricing mechanism is for the worker to post a vector of posted wages and sell to the highest valuation firm. Obviously, this requires that the worker knows who is the highest valuation firm, because this firm would like to purchase labor services at the lower wages posted for lower valuation firms. Thus the posting model assumes perfect information of buyer types. The basic timing of a price posting game developed by Shi (2001) and

[4] This non-montonicity of wages is key to the fundamental Eeckhout and Kircher (2011) result that it is generally impossible to establish whether matching is positive or negative assortative using only wage data.

Shimer (2005) is (i) each seller posts a vector of prices for each buyer type, (ii) the buyers choose a seller given this menu of prices for each buyer at each seller, and (iii) the seller allocates the good to the highest valuation buyer at the price posted by the seller.[5] The analysis of Shimer (2005) assumes a fixed number of heterogeneous buyers and sellers. However, it is straightforward modify this environment to include an entry decision of the different firm types - subject to their different opportunity costs - and to label buyers and sellers appropriately. We call this the *seller posting model*.

**Proposition 7.** *The seller posting model has identical expected payoffs for both workers and firms as the competing auction model.*

*Proof.* The expected payoff of the buyers and sellers in the Shi (2001) and Shimer (2005) seller posting models are equivalent to the expected payoffs of the buyers in the competing auction model, which are given in equation (9). For example, Shimer (2005) offers a compact formulation of this general structure of payoffs.[6] □

Shimer (2005) and Shi (2001) also prove that decentralized price posting equilibrium is constrained efficient. Therefore, we can also offer a corollary to proposition 8.

**Corollary.** *The competing auction equilibrium is constrained efficient.*

The efficiency of entry in the competing auction equilibrium is also analyzed by Julien, Kennes, and King (2000, 2006b). Albrecht, Gautier and Vromam (2013) also consider the efficiency of competing auctions for seller entry when buyers do not know the sellers' type.

## 2.8 Equivalence with (directed) buyer price posting

Mortensen (2003, pp. 16-22) develops a static buyer posting model of undirected matching with a fixed number of heterogeneous buyers and homogenous sellers.

[5] See also earlier work by Peters (1984) and Burdett, Shi and Wright (2001).

[6] In particular, see equation (3) on page 1002 of Shimer (2005). The opportunity cost of each buyer type is equivalent to its unique Lagrange multiplier.

The timing of this game is (i) each buyer chooses a price, (ii) the buyers are randomly allocated to a seller, and (iii) the seller selects the highest price offered. We can modify this game slightly by assuming that buyers first choose to enter subject to their opportunity costs. We call this *the buyer posting model.*

**Proposition 8.** *If workers are identical, the buyer posting model has identical expected payoffs for both workers and firms as the competing auction model.*

*Proof.* See appendix. □

Mortensen also finds that identical buyers post different prices. However, in the limit as the number of different buyer types gets large (as we are assuming in the continuous case), the wage posted by each type of buyers will converge to a single value where higher buyer types offer higher wages. Since auctions and posting give equivalent expected payoffs, this wage is given by the formula for $E\left(y \mid k \leq \tilde{k}, \underline{k}\right)$ in equation (12).

If sellers are not identical, the static model of Mortensen (2003) has different expected payoffs than the competing auction model of McAfee (1993). It is, however, possible to modify the Mortensen (2003) model to include submarkets. In this modified model, each submarket consists of the set of identical sellers and buyer entry is directed towards these submarkets. In this case, the expected payoffs of buyers and sellers become equivalent to the auction model with heterogeneous values of $\underline{k}$ for workers.

# 3 Making the GM game dynamic

The dynamic model is a repeated version of the static model. The workers and firms are infinitely-lived with risk neutral preferences and a common discount factor $\beta$. Time is discrete and the total population of workers is normalized to one. The workers sell their labor services by a second price auction with a reserve price equal to their continuation value. Firms are the buyers in the labor market. There are many different firm types, $k \in [0, \infty)$. A type $k$ firm produces $y(k)$ units of output each period if it is matched with a worker and the opportunity cost of a firm type is $C(k)$. Each worker has one unit of labor to sell and each firm can employ one worker. Each worker is also endowed with a common outside opportunity, $y(\underline{k})$, which is the worker's productivity while unemployed. Once a worker is assigned a firm type, there is a probability $\delta$ that this job opportunity is destroyed at the start of the next period. In this event, the displaced worker enters unemployment. In each subsequent period, each worker faces a repeated opportunity of matching with additional firms of different types. The environment is assumed to be stationary. Therefore, the preference and technological parameters, $\{y(k), \underline{k}, C(k), \delta, \beta\}$, are constant across all periods.

The competing auctions for the labor services of the workers within each period is described by a three stage game. In the first stage of the game, a measure of firms of types $k \in [0, \infty)$ enter each of the labor markets and each of the firms in this measure pays its associated opportunity cost, $C(k)$. In the second stage of the game, the newly created firms play a symmetric mixed strategy regarding their assignment to a worker employed by a type $k_1$ firm, which is $\underline{k}$ if the worker is unemployed. In the final stage of the game, each firm assigned to a worker bids a wage contract for the worker's labor services according to the rules of the worker's second price auction with a reserve price equal to the value of the worker's outside option. The worker then either selects one of these bids or he consumes his outside opportunity. If the firm's type is private information to the firm, an important assumption is that the actual employment contract of the worker is always made public information. Therefore, in each subsequent period, the crucial details of the worker's current employment situation (i.e. the worker's current employer type) is known to future bidders.[7]

## 3.1 Entry of firm types

The ***job value function*** $\Lambda(k)$ describes the combined present value of a match between a worker and a type $k$ firm. We are generally interested in an equilib-

[7]Therefore, we can maintain the key assumption of McAfee (1993) that the seller's type is always public information. We also follow McAfee (1993) in our assumption that the market is large and that each worker's choice of selling mechanism cannot influence the value of the firm's opportunities in subsequent periods.

rium where the job offer function is continuous over some range of firm types. Therefore, if a worker has an outside option in home production of $\underline{k}$ we assume that there exists some firm type $k$ such that

$$\Lambda(k) - \Lambda(\underline{k}) > C(k)$$

and that the properties of the job value function and the opportunity cost function satisfy the general regularity condition,

$$\Lambda''(k)\, C'(k) - \Lambda'(k)\, C''(k) < 0.$$

which corresponds to the concavity assumption of the static model (i.e the job value function must be concave $\Lambda''(k) < 0$ if the job opportunity costs are linear). We discuss this regularity assumption in section 3.4.

In the dynamic model, since the worker's fall back position of home production in unemployment is the same in each period, the job value function is unchanged as the worker moves up the job ladder. Therefore, applying the same logic as in proposition 3, the job offer function of a worker employed by a type $k_1$ employer is given by the function

$$\phi(k \mid k_1) = \phi(k) \equiv \log\left(\frac{\Lambda'(k)}{C'(k)}\right) \tag{14}$$

over the range of job types $\left[\widehat{k}(k_1), k^*\right]$ where the maximum job type $k^* = \arg\max \Lambda(k) - C(k)$ is characterized by the first order condition

$$\Lambda'(k^*) = C'(k^*) \tag{15}$$

and the minimum firm type in the job offer function,

$$\widehat{k}(k_1) = \arg\max_k \left(\Lambda(k) - \Lambda(k_1)\right)/C(k),$$

is characterized by the first order condition,

$$\frac{\Lambda'\left(\hat{k}\right)}{C'\left(\hat{k}\right)} = \frac{\Lambda\left(\hat{k}\right) - \Lambda\left(k_1\right)}{C\left(\hat{k}\right)}. \tag{16}$$

The lowest possible employer type is generally greater than $\underline{k}$. In this case, $\hat{k}\left(\underline{k}\right) > \underline{k}$ and $\Lambda\left(\hat{k}\left(\underline{k}\right)\right) > \Lambda\left(\underline{k}\right)$.

## 3.2 Wages and productivity

The wage contract of a worker is determined by the worker's second price auction with a reserve price equal to the worker's continuation value. The present value of a worker's contract at the start of the period is a function of the worker's second best offer at the time the wage was negotiated and the productivity of the worker's current employer. The worker's current employer gives the worker's continuation value when setting wages with any future employers contacted by on-the-job search. Therefore, at the start of each period, the expected present value of a worker in a type $k_1$ job with a second best offer $k_2$ from the time of the contract is given by

$$W\left(k_1, k_2\right) = \Lambda\left(k_2\right) G_1(\widehat{k}) + \Lambda\left(k_1\right)\left(G_2(\widehat{k}) - G_1(\widehat{k})\right) + \int_{z=\widehat{k}}^{k^*} \Lambda\left(z\right) dG_2\left(z\right) \tag{17}$$

where $\widehat{k} = \widehat{k}\left(k_1\right)$ and the formulas for $G_1\left(k\right)$ and $G_2\left(k\right)$ are given in proposition 1. The first term on the right hand side of this equation captures the event that the worker gets no new offers this period; the second term is the event that the worker has a single offer from a firm type above $\hat{k}$, which means that he will be paid a present value equal to the total job value of the incumbent employer[8]; and the final term captures the increased value due to the possibility of multiple offers.

Given that wage contracts are determined by auction, the present value of a worker with a type $k_1$ employer and a type $k_2$ second best offer is $\Lambda\left(k_2\right)$. This

[8]Note that the probability of receiving exactly one offer above $\hat{k}$ is given by $\left(1 - G_1(\widehat{k})\right) - \left(1 - G_2(\widehat{k})\right) = G_2(\widehat{k}) - G_1(\widehat{k})$.

means that the wage contract $w(k_1,k_2)$ of a worker in this negotiation state satisfies the following asset equation:

$$\Lambda(k_2) = w(k_1,k_2) + \beta\left[(1-\delta)W(k_1,k_2) + \delta W(\underline{k},\underline{k})\right] \tag{18}$$

where the present value of rewards in the next period is given by $W(\underline{k},\underline{k})$ if the worker is displaced and by $W(k_1,k_2)$ otherwise. Wage contracts are renegotiated in the face of new offers from firms. Therefore, if the worker contract at the start of the period is given by values, $k_1, k_2$, and then the worker realizes new values, $k_1', k_2'$, the wage contract is given by $w(k_1', k_2' \mid k_1, k_2) = w(k_1', k_2')$.

If $k_1 = k_2 = k$, then the worker effectively becomes the residual claimant of his employment contract - he owns the job. In this case, the worker earns a wage equal to the output of the firm. That is

$$w(k,k) = y(k). \tag{19}$$

The algorithm by which we solve the dynamic competing auction equilibrium model in section 4 works as follows. We first specify a job value function, $\Lambda(k)$, and a job opportunity cost function, $C(k)$, that satisfies the regularity assumption (i.e. concavity). We then use equations (14) through (16) to solve for the common job offer function and its lower boundary value $\hat{k}(k)$ as the worker climbs the job ladder. We then solve for worker present values, $W(k_1,k_2)$, using the equation (17). Next, we solve for the worker wages, $w(k_1,k_2)$, using equation (18). Finally, we solve for worker productivity, $y(k)$, using equation (19).[9]

## 3.3 Equilibrium distributions

Let $u$ denote the steady-state unemployment. In equilibrium, the flow out of unemployment, $u\left(1 - G_1\left(\widehat{k}(\underline{k})\right)\right)$ equals the flow into unemployment, which con-

[9]Similar as Menzio and Shi (2010) the equilibrium is block recursive as value and policy functions do not depend on the equilibrium allocation of workers across firms.

sists of workers laid off who do not receive a job offer, that is $(1-u)\,\delta G_1\left(\widehat{k}\left(\underline{k}\right)\right)$. Therefore, the steady-state unemployment is given by

$$u=\frac{\delta G_1\left(\widehat{k}\left(\underline{k}\right)\right)}{1-(1-\delta)\,G_1\left(\widehat{k}\left(\underline{k}\right)\right)} \tag{20}$$

We let $n\left(k\right)$ denote the density of workers employed in a type $k$ job and $N\left(k\right)$ denote the distribution of workers with job types less than $k$ or being unemployed. Furthermore, let $\widehat{k}^{-1}\left(k\right)$ denote the inverse of the function $\widehat{k}\left(\underline{k}\right)$. The transition equation for the density of job types is given by

$$\begin{aligned} n'\left(k\right) &= n\left(k\right)(1-\delta)\,G_1\left(\widehat{k}\left(k\right)\right)+ \\ &\quad \left[u+(1-\delta)\left(N\left(\widehat{k}^{-1}\left(k\right)\right)-u\right)+\delta\,(1-u)\right]g_1\left(k\right) \end{aligned} \tag{21}$$

where the first term on the right hand side is the density of workers in type $k$ jobs who stays at the type $k$ firm whereas the second term is the density of workers changing to type $k$ jobs. The steady-state density of workers in type $k$ jobs is solved for by setting $n'(k)=n(k)$ and is given by

$$n\left(k\right)=\frac{\left[\delta+(1-\delta)\,N\left(\widehat{k}^{-1}\left(k\right)\right)\right]g_1\left(k\right)}{1-(1-\delta)\,G_1\left(\widehat{k}\left(k\right)\right)} \tag{22}$$

where the end point of this differential equation is $N\left(\widehat{k}^{-1}\left(\underline{k}\right)\right)=u$.

The distribution of productivities is then characterized by the density in equation (22) and by equation (19), which gives the productivity of a type $k$ employer. That is

$$\Omega_y=\left\{y\left(k\right),n\left(k\right)|k\in[\underline{k},k^*]\right\}$$

It is also worth noting that the computation of the distribution of worker productivity requires only knowledge of $G_1\left(k\right)$.

The final task is to characterize the steady-state joint distribution of first and second best offers. In each period, a worker employed in a type $z \in [\underline{k}, k^*]$ job, has a joint distribution of jobs offers, $G(k_1, k_2)$, over the interval $\left[\widehat{k}(z), k^*\right]$, where $G(k_1, k_2)$ is given by equation (13). We let $g(k_1, k_2)$ denote the implied joint density of offers for an unemployed worker ($z = \underline{k}$). Let $x(k_1, k_2)$ denote the joint density of workers employed in a type $k_1$ job and with a second best opportunity of a type $k_2$ job. The transition equation for $x(k_1, k_2)$ is given by

$$\begin{aligned} x'(k_1, k_2) &= x(k_1, k_2)(1-\delta) G_1\left(\widehat{k}(k_1)\right) \\ &+ \left(u + (1-\delta)\left(N\left(\widehat{k}^{-1}(k_2)\right) - u\right) + \delta(1-u)\right) g(k_1, k_2) \\ &+ n(k_2)(1-\delta) 1\left\{\widehat{k}^{-1}(k_1) \geq k_2\right\} \int_{\underline{k}}^{\widehat{k}(k_2)} g\left(k_1, \widetilde{k}_2\right) d\widetilde{k}_2 \end{aligned} \tag{23}$$

where the first term on the right hand side is the quantity of agents in the $(k_1, k_2)$ state in the previous period who do not loose their job and are not recruited to a new firm this period, the second term is the quantity of workers who move into the $(k_1, k_2)$ state by means of getting multiple offers, and the third term is the quantity of workers who move into the $(k_1, k_2)$ state by means of getting a single type $k_1$ offer and having a type $k_2$ incumbent employer. The steady-state distribution is solved by setting $x'(k_1, k_2) = x(k_1, k_2)$. We have that

$$\begin{aligned} x(k_1, k_2) &= n(k_1)\left[\frac{\delta + (1-\delta) N\left(\widehat{k}^{-1}(k_2)\right)}{\delta + (1-\delta) N\left(\widehat{k}^{-1}(k_1)\right)}\right] \frac{g(k_1, k_2)}{g_1(k_1)} \\ &+ n(k_2)\frac{(1-\delta) 1\left\{\widehat{k}^{-1}(k_1) \geq k_2\right\} \int_{\underline{k}}^{\widehat{k}(k_2)} g\left(k_1, \widetilde{k}_2\right) d\widetilde{k}_2}{1 - (1-\delta) G_1\left(\widehat{k}(k_1)\right)} \end{aligned} \tag{24}$$

The joint equilibrium distribution of wages and productivities for the economy is then characterized by this density equation together with equations (18) and (19), which describe the wages and productivities of all workers as a function of their employment state, $(k_1, k_2)$. That is

$$\Omega_w = \left\{w(k_1, k_2), x(k_1, k_2) \,|\, k_1, k_2 \in [\underline{k}, k^*]\right\}$$

## 3.4 The regularity assumption

As argued earlier, the job value function, $\Lambda(k)$, is assumed to be positive, increasing, and concave.[10] This regularity assumption is useful for obtaining a non-degenerate solution to the equilibrium using a simple recursive algorithm. However, these properties of the job value function are also grounded in standard assumptions concerning the production function, $y(k)$.

The following proposition offers a necessary and sufficient condition for the existence of an equilibrium with positive job offers.

**Proposition 9.** *An equilibrium with positive job offers exists if and only if*

$$\frac{y(k^*) - y(\underline{k})}{1-\beta(1-\delta)} > C(k^*)$$

*where $y(\underline{k})$ is the worker's output at home and $y(k^*)$ is the output of job type $k^*$ such that $\frac{y'(k^*)}{1-\beta(1-\delta)} = C'(k^*)$.*

*Proof.* See appendix. □

Therefore, assuming that this condition is satisfied, the regularity assumption of a positive job value, such that $V(k^*) - V(\underline{k}) > C(k^*)$, is also satisfied.

Define $p^* < k^*$ as the firm type $p^* = \hat{k}^{-1}(k^*)$. Since the minimum job type cannot exceed $k^*$, a worker employed by a type $k_1 \geq p^*$ firm does not get any offers. The regularity assumption that the job value function is increasing in firm types follows from the assumption that higher firm types are more productive. This is formally derived by the following proposition.

**Proposition 10.** *If $C(k) = k$, the first derivatives of the job value function $\Lambda(k)$ and the production function $y(k)$ are related as follows.*

$$y' = \begin{cases} (1-\beta(1-\delta))\Lambda' & if \quad k \geq p^* \\ \left(1-\beta(1-\delta)\left(\frac{1}{\hat{\Lambda}'}\right)\right)\Lambda' & if \quad k < p^* \end{cases} \tag{25}$$

*where $\Lambda' \equiv \partial\Lambda(k)/\partial k$ and $\hat{\Lambda} \equiv \Lambda\left(\hat{k}(k)\right)$. Note that equation (14) implies $1/\Lambda' = \exp(-\phi(k)) \in [0,1]$.*

[10] Assuming that we adopt the normalization of a linear job opportunity cost function.

*Proof.* See appendix. □

Since the worker does not engage in on-the-job search once he is employed in a firm type greater than $p^*$, the equilibrium of the dynamic model is essentially the same as the equilibrium of the static model for firm types in the region $[p^*, k^*]$. In this case, the frequency of job offers is dependent on the incremental higher productivity of higher job types. If there is on-the-job search, the job value of a match with a firm type lower than $p^*$ is also partly derived by its value as a stepping stone employer, which *falls* as the firm becomes more productive.

Consider the relationship between the second derivatives of the worker's production function $y(k)$ and the second derivatives of the job value function $\Lambda(k)$.

**Proposition 11.** *If $C(k) = k$, the second derivatives of the job value function $\Lambda(k)$ and the production function $y(k)$ are related as follows.*

$$y'' = \left[\frac{y'}{\Lambda'}\right] \Lambda'' - \begin{cases} 0 & if \quad k \geq p^* \\ \beta(1-\delta)\frac{(\Lambda')^2}{\hat{\Lambda}-\Lambda} & if \quad k < p^* \end{cases} \tag{26}$$

*where* $y'/\Lambda'$ and $\left[\hat{\Lambda} - \Lambda\right]$ are positive.

*Proof.* See appendix. □

Therefore, concavity of the production function with respect to firm types is a necessary condition for concavity of the job value function with respect to firm types. Consequently, the observation that the worker gets offers from a range of firm types implies that the worker's production function must be concave.

We can seek out functional forms of the production function that are sufficient for the concavity of the job value function simply by matching the model to empirical data. For example, since wage and job offers out of unemployment are disperse for similar worker types, we can obviously start with the assumption of a concave job value function and then solve for the implied concave production function by matching this data. This is the approach taken in section 4 where we use the dynamic model to explain the empirical distribution of worker productivities. This method is both extremely flexible and yields plausible predictions. For example, from figure 1, we see that we are able to closely match the productivity distribution of the Danish economy.

## 3.5 Heterogenous skills

If the workers are heterogeneous in skills, a worker of skill type $h \in [0, \infty)$ produces $y(k, h)$ units of output if matched to a type $k \in [0, \infty)$ firm. A basic question is whether particular firm types are more likely to be directed at particular worker types. The answer to this question is determined by the derivative of the job value function, which then determines the job offer function, $\phi(k, h) = \log(\Lambda_1(k, h))$. The following proposition gives the relationship between the cross derivatives of the production function $y(k, h)$ and the cross derivatives of the job value function $\Lambda(k, h)$.

**Proposition 12.** *If $C(k) = k$, the cross derivatives of the job value function $\Lambda(k, h)$ and the production function $y(k, h)$ are related as follows.*

$$y_{12} \quad = \quad \Lambda_{12}\left[\frac{y_1}{\Lambda_1}\right] + \begin{cases} 0 & if \quad k \geq p^* \\ \beta(1-\delta)\frac{\Lambda_1}{\hat{\Lambda}_1}\left[\frac{\hat{\Lambda}_2 - \Lambda_2}{\hat{\Lambda} - \Lambda}\right] & if \quad k < p^* \end{cases} \tag{27}$$

*where* $y_1/\Lambda_1$, $\Lambda_1/\hat{\Lambda}_1$ and $\left[\hat{\Lambda} - \Lambda\right]$ are all positive. The terms $\Lambda_{12}$ and $\hat{\Lambda}_2 - \Lambda_2 = \int_{z=k}^{\hat{k}} \Lambda_{12}(z, h)\, dz$ are positive/negative if and only if $\Lambda(k, h)$ is super/submodular.

*Proof.* See appendix. □

Therefore, ***super-modularity*** of the production function is a necessary condition for super modularity of the job value function. Consequently, the observation that a higher type worker is more likely to be matched to a higher firm type from a pair of firm types $\{k, k'\}$ implies that the production function relating workers' output and the firms' output must also be super-modular.

Super-modularity of the production function is not a sufficient condition for super-modularity of the job value function since increasing the firm type cuts the effective job offer distribution at $\hat{k}(k)$ due to the directedness of search and the second-price auction. Therefore, super-modularity of the job value function at $k$ depends on the degree of complementarity at both $k$ and $\hat{k}(k)$. Hence, if increasing the firm type $k$ cuts job offers from firms with a high degree of complementarity, super-modularity of the production function may not be enough to guarantee a positive cross-derivative of the job value function at $k$.

A useful benchmark model of worker heterogeneity is the case where all workers face the same job offer function.

**Proposition 13.** *If type $h$ and type $h'$ worker production functions exhibit* ***no-comparative advantage*** *such that $y(h', k) = y(h, k) + A$,* where *$A > 0$ is a scalar, the wage function of the more productive worker is given by*

$$w(k_1, k_2 \mid h') = w(k_1, k_2 \mid h) + A \tag{28}$$

*and $\Lambda(h', k) = \Lambda(h, k) + \Delta$ where $A = (1 - \beta)\Delta$.*

*Proof.* See appendix □

The intuition for this result is simply that the higher worker type with no-comparative advantage is able to extract the full value of any additional surplus that he brings into a match. Therefore, a worker who is more productive in all matches is also paid a higher wage that reflects this difference.

More generally, the nature by which the firm shares output will depend on whether there is sorting. For example, consider how the ***residual firm revenue function*** - output minus the wage - is dependent on the worker type. By equations (17) through (19) we have the following formula.

$$y(k_1, h) - w(k_1, k_2, h) = \left[1 - \beta(1 - \delta) G_1\left(\hat{k}(k, h), h\right)\right] \left[\Lambda(k_1, h) - \Lambda(k_2, h)\right] \tag{29}$$

We can then derive the following proposition.

**Proposition 14.** *If the job value function is super-modular (positive sorting), then*

$$\frac{\partial\left[y\left(k_1,h\right)-w\left(k_1,k_2,h\right)\right]}{\partial h} \quad > \quad 0 \tag{30}$$

*If the job value function is sub-modular (negative sorting), then* $\frac{\partial[y(k_1,h)-w(k_1,k_2,h)]}{\partial h} < 0$*. If the job value function is modular (no sorting), then* $\frac{\partial[y(k_1,h)-w(k_1,k_2,h)]}{\partial h} = 0$.

*Proof.* See appendix ☐

If the value function is super-modular then the loss of value of search by working in a $k_1$ job compared to a $k_2$ job is smaller and then the residual firm revenue function is not constant as with the modular case in proposition 13. A higher degree of complementarity both implies that more productive workers can have bidders from a larger range of firm types (i.e. $\partial G_1\left(\hat{k}\left(k,h\right),h\right)/\partial h < 0$) and directly through the greater production potential among high productive firms. The opposite is true when the job value function is sub-modular since more productive workers face a larger loss in the value of search by working in a $k_1$ job compared to a $k_2$ job. The monotonicity of the residual firm payment function with respect to worker type demonstrates that this model places intuitive restrictions on labor market outcomes related to assortative matching outcomes. In particular, higher type firms are directed at high type workers, if and only if such workers offer a larger stream of residual payments to the firm.

## 3.6 Aggregate shocks

An ***aggregate shock*** impacts the entire population of workers by changing the technology from $\{y\left(k,h\right),C\left(k\right)\}$ to $\left\{y\left(k,h\right)',C\left(k\right)\right\}$, where $C\left(k\right)$ is the common opportunity cost of each job type. This changes the entire job ladder for each worker by effectively re-labeling each type $h$ worker into a new type $\tilde{h}$ worker such that $y(k,\tilde{h}) = y\left(k,h\right)'$. Therefore, for each employment state $\{k_1,k_2\}$, we can characterize the impact of the aggregate shock on the expected job offer distribution using the the results of propositions 12, 13 and 14. The solution of the GM game after the aggregate shock remains tractable, because the only link between workers in the population is the common opportunity cost of job creation, $C(k)$, which is taken as parametric under the assumption of free entry. As a baseline model, we can also assume that the aggregate shock has no effect on the workers employment state, $\{k_1, k_2\}$. The meaning of this assumption is that the productivity transformations for new and existing jobs are identical and that the worker's bargaining position is also transformed similarly.

To gain some further insights about aggregate shocks, we also consider two examples. The first example is a no-comparative advantage increase in the productivity of all workers across all job opportunities. In this case, the production function of each worker for each job type is scaled in proportion. Therefore, by proposition 13, the job offer function of each worker type is given by

$$\phi\left(k \mid y(k)' = y(k) + A, C(k)\right) = \phi\left(k \mid y(k), C(k)\right) \tag{31}$$

where $A$ is a scalar. The effect is that all workers transition to new job types with the same frequency as before the shock and that the wage of each worker will simply be scaled higher by an amount equal to the change in productivity, $A$. Another example of an aggregate shock is one that changes the workers' skill sets from $y(k)$ to $y(k)'$ such that the workers gets proportionally more offers from all job types. Therefore, given this new technology, the equilibrium job offer function is given by

$$\phi\left(k \mid y(k)', C(k)\right) = \alpha\phi\left(k \mid y(k), C(k)\right) \tag{32}$$

where $\alpha$ is a positive scalar. In section 4, we study the quantitative implications of such a shock by using the estimated benchmark model as the starting point.

## 3.7 Match specific shocks

A ***match specific shock*** is a shock that impacts only the worker's current match. The shock can be represented by a transformation of the worker's current job type, $k_1$, into a new job type, $\tilde{k}_1$. In particular, if the worker's productivity in a match with their current employer increases by a factor of $\lambda$, then the worker is now effectively employed in a job type $\tilde{k}_1$ that solves $\lambda y(k_1) = y(\hat{k}_1)$.

This affects the worker's current position on the job ladder without affecting the ladder itself. In this case, the expected job offer function becomes $\phi(\tilde{k}_1)$ instead of $\phi(k_1)$. Thus, a match specific shock that increases worker productivity also decreases the expected number of offers from raiding firms.

If a match specific shock causes the worker to climb up the productivity ladder from $k_1$ to $\tilde{k}_1$, then the worker's wage will generally be affected. If the shock causes no change in $k_2$, then the model predicts that the worker's wage must increase, because the higher productivity of the match shuts down outside offers, i.e. $\hat{k}\left(\tilde{k}_1\right) \geq \hat{k}(k_1)$. The model also predicts that a higher/lower value of $k_2$, holding $k_1$ constant, increases/decreases wages. Therefore, there exists innovation in the value of $k_2$ such that the match specific productivity shock has no effect on wages (or possibly causes the wage to increase proportionally to the productivity increase).

# 4 Taking the GM game to the data

Our empirical analysis evaluates the benchmark dynamic version of the GM game. We also set up a comparison with a revenue posting model (See appendix). Each of these two models are of basic interest. On the one hand, the auction model functions as a benchmark for an environment with the maximum amount of wage dispersion relative to dispersion of job productivities. On the other hand, the price posting model functions at the opposite extreme by minimizing the amount of wage dispersion given the equilibrium dispersion of job productivities.

Our empirical analysis of worker heterogeneity seeks to find out what amount of productivity dispersion is attributed to specific groups of workers under the price posting and auction specifications. To study this problem we compare the implications of a homogenous workers model with a benchmark model of heterogeneous workers. We are interested in these two specifications for two basic reasons. The homogenous workers model is of interest because it functions to explain wage dispersion purely by the equilibrium dispersion of job productivities. Similarly, the heterogeneous workers model (with no-comparative advantage both at home and at work) functions to explain wage dispersion by allowing for worker productivity differences while retaining the equilibrium implication that all workers are equally likely to be employed by all firms. In effect, this latter specification eliminates the additional mechanism of assortative matching.

## 4.1 Baseline Parameter Values

The main benchmark model has ex ante identical workers and firms, and it is described by the following parameters: (i) the discount factor $\beta$, (ii) the exogenous rate of job separations $\delta$, and (iii) the menu of opportunity costs associated with jobs of each productivity type $\{y(k), C(k), \underline{k}\}$. For the estimation we set $M = 51$ and fix the following parameters $\beta = 0.99$, $\delta = 0.12$ and total mass of offers, $\phi_1 = 1.187$, so we do not use these parameters to fit the productivity and wage distributions. The latter two parameters determines the equilibrium unemployment to 5 percent. We estimate the benchmark model using both auction and price posting specifications. We also estimate the models with worker heterogeneity and no-comparative advantage.

## 4.2 Data

We seek to match the empirical observable wage and productivity distributions. For this purpose we use a Danish register-based matched employer-employee data set for the year 2007. The hourly wage rate is taken from the Integrated Database of Labor Market (IDA). This wage is calculated by Statistics Denmark by dividing the labor income by the number of hours worked. Whereas the labor income is precisely recorded in the administrative registers, the hours worked is imputed from mandatory pension payments which have four levels depending on the number of hours worked.

Our measure of firm productivity is the hourly average of value-added per full-time worker. Statistics Denmark conducts an annual survey of firms. Firms are sampled according to their size such that firms with more than 49 workers are always sampled, whereas firms with, for example, between 5 and 9 workers are only sampled with probability of 10 percent. We only use observations for persons employed by firms in the survey.[11]

We use the FIDA key to link the employee and employer data. We have access to a population data set of Danes aged 15 years or more. This data set has $4,465,874$ observations. We only select workers aged 16-64 which reduces the sample to $3,542,311$ persons. Furthermore, we only include persons with a positive wage and with a reliable wage estimate. This leaves us with $2,156,719$ observations. Additionally, we exclude workers employed in the public sector which reduces the sample to $1,296,824$ observations. Next, we only include observations for firms with a positive value-added per worker which reduces the sample size to $1,077,035$. Next, we only include the following five industries: a) manufacturing, b) construction, c) wholesale and retail, d) transport, storage and communication, and e) real estate, renting and business activity. This gives us a sample of $1,031,374$ observations. Finally, we discard all observations where the value-added is imputed by Statistics Denmark. This gives us a final data set of $651,722$ workers employed by $8,236$ firms.

## 4.3 Estimation

We fix the parameters for the discount rate, the overall job arrival rate for unemployed and the exogenous job destruction rate. We assume $M = 51$ different firm types and estimate the parameters $\phi_2, ..., \phi_M$ and $\Lambda_1, ..., \Lambda_M$ by use of nonparametric simulated maximum likelihood (Refer to Fermanian and Salanie, 2004). From data we can by kernel methods estimate the density function of, for example, wages, which we denote by $\hat{p}_h(w)$ where $h$ is the bandwidth whereas we from our theoretical model for a given set of parameters $\theta$ can calculate the density function $p(w|\theta)$. However, since we only have a discrete wage distribution for the theoretical model and since these discrete points are endogenous we use a smooth density function $\tilde{p}_b(w|\theta)$ using kernel methods also for the theoretical model and where $b$ is the bandwidth. The objective of the estimation is to minimize the distance between the two probability density functions and

[11]The data set draws on the same data sources as in Bagger, Christensen and Mortensen (2011) and we refer the reader to this paper for more details on the structure of the data sets used and the precise variable definitions.

we use the Kullback-Leibler distance. We select $N = 100$ points of the wage distribution, i.e. $\bar{w}_1, ..., \bar{w}_N$.

$$L(\theta_N) = \frac{1}{N^2} \sum_{i=1}^{N} \left( \log \hat{p}_h(\bar{w}_i) - \log \tilde{p}_b(\bar{w}_i|\theta) \right) \hat{p}_h(\bar{w}_i) \tag{33}$$

Minimizing this function corresponds to maximizing

$$\tilde{L}(\theta_N) = \frac{1}{N^2} \sum_{i=1}^{N} \log \tilde{p}_b(\bar{w}_i|\theta) \, \hat{p}_h(\bar{w}_i) \tag{34}$$

For the estimation, we let $\bar{w}_1, ..., \bar{w}_{100}$, be equally spaced in between the first percentile and the 97th percentile of the wage and productivity distributions.

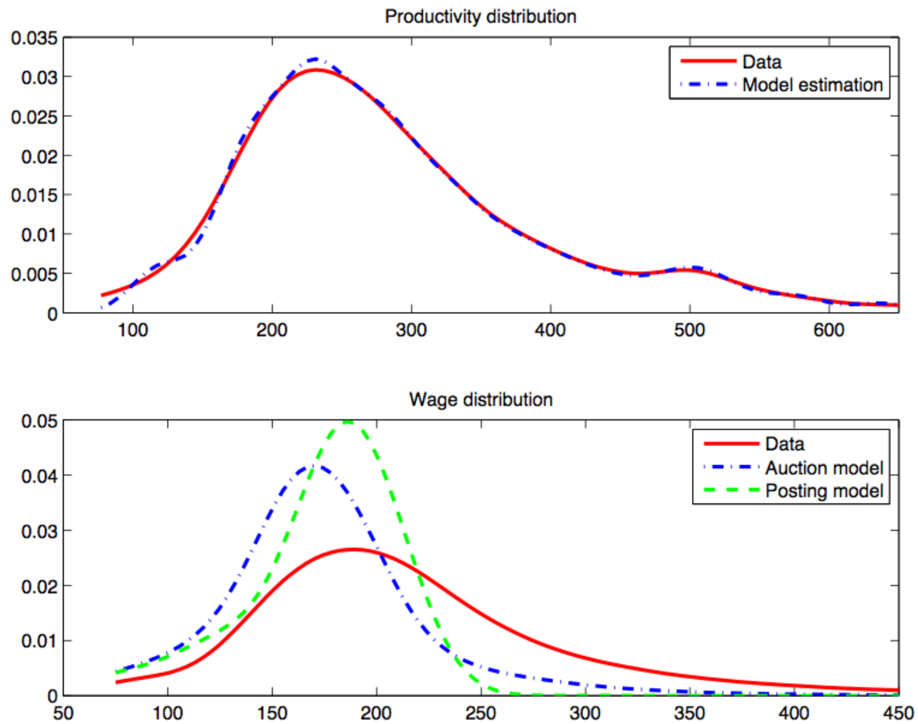
## 4.4 Comparison of the auction and posting models

Our empirical analysis of alternative pricing mechanisms centers on the auction and price posting models of coordination frictions. Each of these two models are of basic interest. On the one hand, the auction model functions as a benchmark for an environment with the maximum amount of wage dispersion relative to dispersion of job productivities. On the other hand, the price posting model functions at the opposite extreme by minimizing the amount of wage dispersion given the equilibrium dispersion of job productivities.

The benchmark model with homogenous workers is first estimated using data on firm productivity. This exercise puts both the auction and price posting models on an equal footing, because each model predicts equivalent distributions of productivities given the same assumptions on the opportunity costs of jobs. The results of this estimation exercise is given in figure 1. For, this figure and all figures in this section we use a bandwidth of 20 DKK (US\$1≅ 6 DKK).

The top graph illustrates that the benchmark model is extremely flexible and can easily track the data on productivity dispersion (both models have identical implications for productivity dispersion). Here, productivities are determined endogenously, because the number of firms entering this labor market of each productivity is determined by the equilibrium opportunities for ex post profits

Figure 1: The benchmark models estimated on productivities

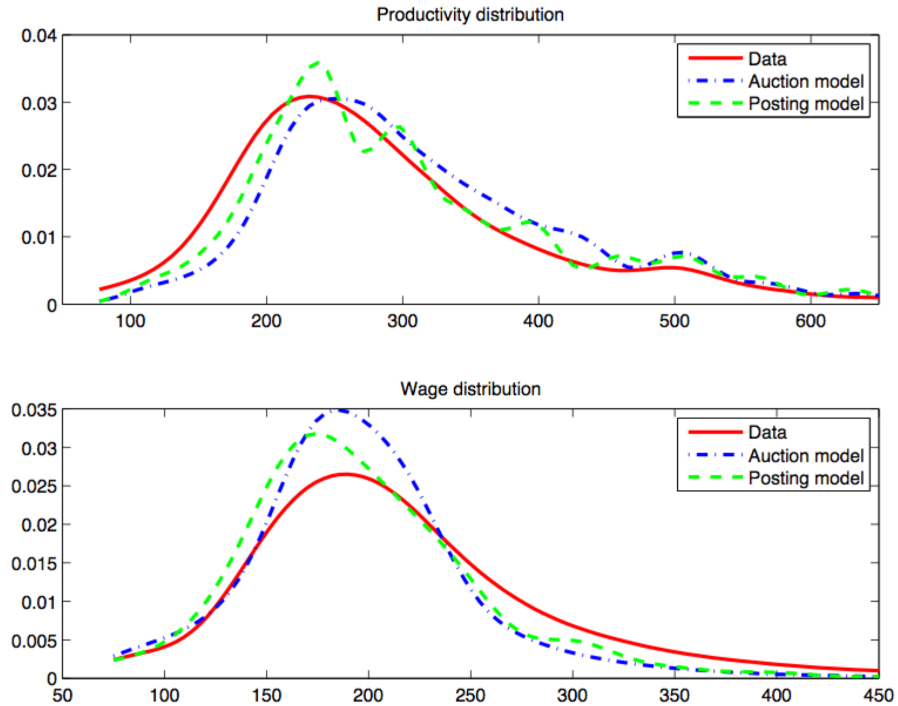



weighted against the associated opportunity costs of not entering this submarket in favor of another labor submarket. The second graph in figure 1 then compares the wage distributions of the two models with the actual data. We first note that the wage distribution of each model does not suffer a problem of increasing density over the support of the wage distribution. This is explained by the heterogeneity of jobs, which is determined endogenously. We also see that both models fail to explain key features of the wage distribution. Here we see the variance of the wage distribution is more realistic in the auction model but that the mode of the wage distribution is more realistic in the posting model.

Figure 2 gives the results of estimation when we try to fit both the wage and productivity distributions simultaneously, again assuming homogenous workers. In each case, the auction and price posting models are unable to closely match both wage and productivity dispersion.

The estimated productivity distributions are affected by how the two models in figure 1 fail to fit the wage distribution. The auction model now shifts the

Figure 2: Benchmark model estimated on wages and productivities



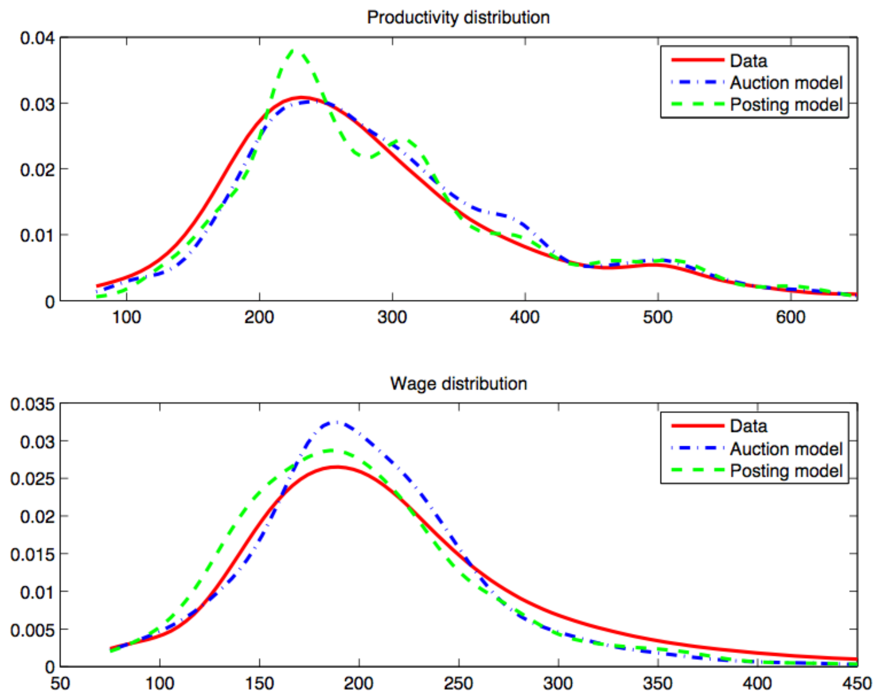

Kernel densities with bandwidth of 20.

mode of the productivity distribution to the right to better explain the wage distribution. The posting model makes no change in the mode of the distribution but it tends to widen the tails of the productivity distribution especially concerning the dispersion of high quality jobs with productivity greater than 300 DKK per hour.

## 4.5 No-comparative advantage

The final estimation exercise is to allow for the possibility of worker heterogeneity. We adopt a version of worker heterogeneity that does not affect the job offer function for each type of worker. Namely, we assume that worker's productivity and outside options are scaled by a common factor. This is closely related to the no-comparative advantage case considered in Shimer (2005) with the added assumption that outside options are also scaled. This assumption serves as a useful benchmark here because it rules out sorting - where different job types approaching different workers differently. The estimation results are given in figure 3. These estimation results reveal that worker heterogeneity with non-

Figure 3: Estimated model with no-comparative advantage




Kernel densities with bandwidth of 20.

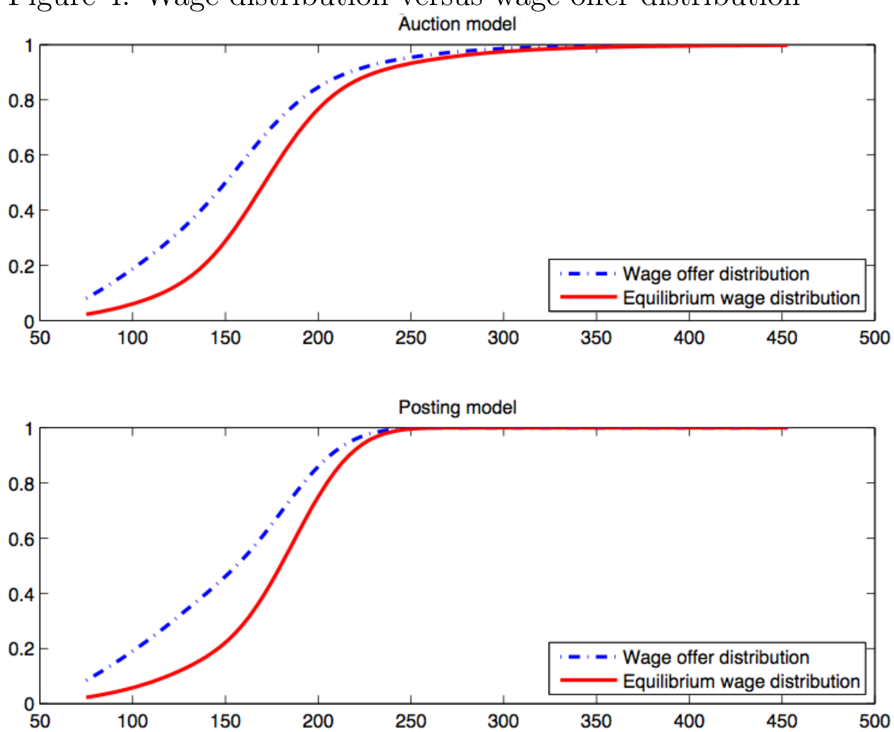
comparative advantage is broadly consistent with an equilibrium explanation of wage and productivity dispersion. The estimation results also reveal that there is no clear advantage over price posting and auction models to explain wage and productivity dispersion.

The two models use worker heterogeneity in different ways to fit the data on wage and productivity dispersion. The auction estimation essentially creates two bins of very high and very low types with a small fraction (approx. 0.1) of agents acting as low types. The posting estimation creates a third bin of intermediate types and spreads the agents evenly across these three bins. Intuitively, this allows the auction model to shift the mode of the wage distribution while also giving a modest increase in the amount of wage dispersion. In the posting model, the only needed change is to increase the dispersion of wages and so this is captured by a disperse set of worker types spread evenly over the worker type space.

## 4.6 The job offer distribution

Figure 4 depicts the offer distribution of an unemployed worker (for both the auction and posting model) when we fit the benchmark model to exactly explain the same distribution of firm productivities (Recall the estimated model of figure 1). When the dispersion of firm types in the offer distributions are identical, we

Figure 4: Wage distribution versus wage offer distribution




Kernel densities for the auction model. Bandwidth used is 20.

find that the auction model generates much more wage dispersion among new hires even though each model is fitted to explain the same distribution of firm productivities. Most of this additional variance is attributed to wages at the top end of the wage distribution than at the bottom end. This is explained by the fact that lowest quality jobs pay very low wages in the posting model and thus there is little difference in the variation in wages between the auction and posting models for low quality jobs. These different predictions are also consistent with related research analyzing random matching models. For example, Postel-Vinay and Robin (2002) find that an auction model is more realistic than a wage posting model (Bontemps, Robin and van den Berg, 2000), because the auction model helps to explain the high variance of wages at the top end of the wage distribution given that the data features only limited variance in job productivities.

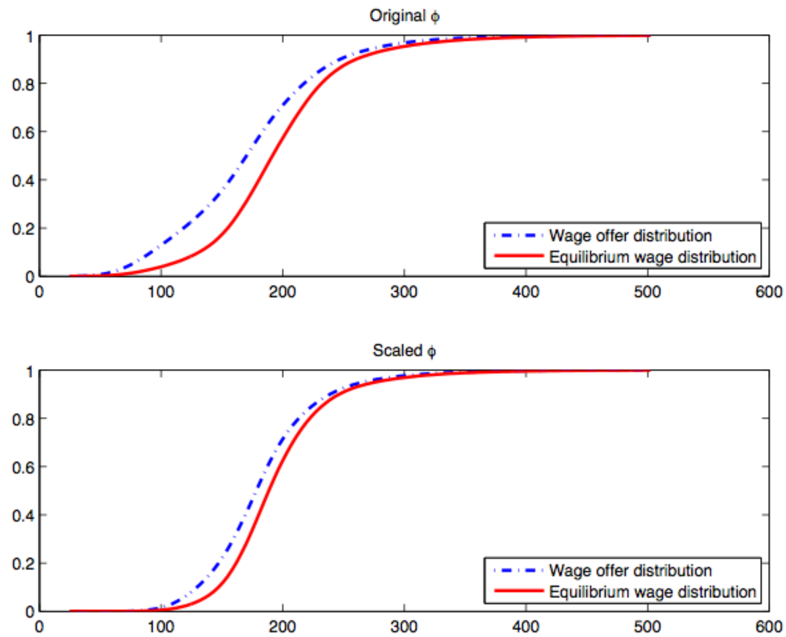
The difference between the job offer distribution of unemployed workers and the overall distribution of wages in our estimated model is explained by on-the-job search. On-the-job search is important in our explanation of wage and productivity dispersion, because the workers are willing to accept a low productivity job with low exogenous rate of separations into unemployment only if there also exists future opportunities to be matched while employed (Hornstein, Krusell, and Violante, 2012). This means that we could not fit wage and productivity data in our model without on-the-job search while also maintaining realistic job exit rates into unemployment across all employment types. We also note that the auction model is distinguished from a posting model, because the auction model implies an independent match effect (wages can vary across similar employers) that has no bearing on the arrival of offers.

## 4.7 Aggregate shocks

One example of an aggregate shock is a proportional increase in the productivity of all jobs. This shock simply causes a parallel rightward shift of the wage distribution. Another example of an aggregate shock is a change in technology that causes the expected number of job offers for each productivity level to increase proportionally. To illustrate the effect of the second type of aggregate shock, we start with the estimated model that is illustrated in figure 1. If we decrease the number of offers across all jobs by 30 percent, the unemployment increases from 5.0 percent to 8.5 percent. The effect on the job offer distribution is depicted in figure 5.

In figure 5 we can see that the second type of aggregate shock is useful for

Figure 5: Job offer distribution as a function of overall offer frequency, $\alpha$




Kernel densities for the auction model. Bandwidth used is 20.

understanding an emerging stylized fact that dispersion in wage offers varies pro-cyclically over the business cycle (See Morin, 2013).[12] The somewhat paradoxical finding is owed to the scarcity of high productivity jobs in the downturn. An increase in the number of jobs for all types increases both the chance that a worker is employed in a good job and the income share the worker gets from such jobs. The increase in the number of jobs for all types also increases the availability of lower quality jobs in the chance events that good jobs are not offered. When these gains at the top and bottom of the wage distribution are combined, the overall effect is both a rightward shift and a flattening of the wage offer distribution.

[12]Morin's (2013) model bears some relationship with ours, because she assume frictional assignment. Some basic differences are that matching is not directed and that wages are posted with an absolute commitment not to change the wage in the face of counteroffers.

# 5 Conclusions

The original and important motivation of McAfee's (1993) game of competing sellers was to explain why each seller chooses to conduct a simple auction from a larger set of mechanisms. In the present paper, we have focused on extending this game to the case of heterogeneous sellers. However, the reason we are interested in this extension is for the very different purpose of deriving a theoretical model of a labor market. It is for this reason that we have chosen to call this extension of the McAfee (1993) game: *The competing auctions of skills.*

We discovered two somewhat surprising properties of the equilibrium job offer distribution in the generalized McAfee (GM) game. The first property is that the lower support of the expected distribution of bidders is strictly above the worker's outside option. The second property is that an increase in the workers outside option increases the lower support of the distribution of bidders but does not otherwise change the distribution of bidders above this new lower support. The second property facilitates the tractability of the dynamic model with on-the-job search. The estimated benchmark model also demonstrates that the equilibrium job offer distribution is not at odds with the data for suitable parameters. Therefore, at least on the dimensions of the data under consideration, this equilibrium theory of the job offer function fits the data almost as well as an unrestricted partial equilibrium model of the job offer function.

The GM game is flexible and can be used to model various labor market phenomena. For example, consider the case of either a PAM or a NAM equilibrium, which we characterized. Now suppose that there is a *passive labor market policy* that redistributes income by changing the unemployment returns of each worker type. Using the properties of the equilibrium job offer function, we know that this policy will not change the assortativeness of the matching. In particular, the policy will only change the cutoff job point in the job offer distribution of an unemployed worker without changing the nature of the equilibrium job offer distributions of these workers as they gain employment in jobs further up the job ladder. Therefore, a passive labor market policy that redistributes income across different worker types via unemployment benefits does so without changing the qualitative features of the assignment of workers to jobs. We also know from the analysis of no-comparative advantage, that the inefficiency of the passive labor market policy could be offset by a countervailing *active labor market policy* that gives the worker a constant employment subsidy for all jobs. Our analysis explains that the specification of this transfer policy is independent of the degree of sorting in the economy.

We used the GM game to study unanticipated technology shocks that impact the productivity of different workers in different job types. Future work might also use the GM game to study how anticipated (Markovian) technology shocks impact key labor market data at both the aggregate and idiosyncratic level. This extension is important if we wish to extend our empirical analysis beyond the estimation of a simple benchmark model. The extension is also tractable for the reasons illustrated by our analysis of unanticipated shocks. We anticipate that future developments of the GM game along these lines will offer a valuable framework to study labor market phenomena.

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# Appendix

**Proposition. 1** *For values of* $k_1, k_2 \in \left[\hat{k}, k^*\right]$,

$$G_1(k) = e^{-\phi(k)}, \text{ and} \quad (35)$$

$$G_2(k) = e^{-\phi(k)} + \phi(k)\, e^{-\phi(k)}. \quad (36)$$

*where* $G_1\left(\hat{k}\right)$ *is the probability of no offer and* $G_2\left(\hat{k}\right)$ *is the probability that the number of offers are zero or one.*

*Proof.* The probability that $k$ is the highest offer for a worker is given by

$$\begin{aligned} G_1(k) &= \sum_{x=0}^{\infty}\left(\frac{1}{1+x}\right)\frac{e^{-\phi(k)}\phi(k)^x}{x!} \\ &= e^{-\phi(k)} \end{aligned}$$

Likewise, $e^{-\phi(k)} + \phi(k)\, e^{-\phi(k)}$ is the probability that no two offers are better than $k$. □

**Proposition. 2** *The lowest job type offered in equilibrium is given by*

$$\hat{k}(\underline{k}) = \arg\max_k \left(y(k) - y(\underline{k})\right)/c(k) \quad (37)$$

*and the total mass of jobs offered in equilibrium is*

$$\phi(\hat{k}) = -log\left(\left(y\left(\hat{k}\right) - y(\underline{k})\right)/c\left(\hat{k}\right)\right). \quad (38)$$

*Proof.* Consider the values of $\hat{k}(\underline{k})$ and $\phi(\hat{k})$. Since the lowest quality job earns a positive return of $y(\hat{k})$ if and only if there is no other firm at the local market of this worker, its expected return is given by $\left(y(\hat{k}) - y(\underline{k})\right)e^{-\phi(\hat{k})}$ where $\phi(\hat{k})$ is the measure of jobs greater than $\hat{k}$. Suppose that the total mass of jobs is some value $\phi$, which is less than $\phi(\hat{k})$. Then the total mass of job openings is

$$\phi < \max\left\{\phi(k)\,|e^{-\phi(k)}\left(y(k) - y(\underline{k})\right) = c(k)\right\}$$

In this case, $e^{-\phi(k)}\left(y\left(k\right)-y(\underline{k})\right)>c\left(k\right)$ , and thus the returns to entry of type $\hat{k}\left(\underline{k}\right)$ jobs imply $\phi(\hat{k})-\phi>0$, a contradiction. The total mass of jobs can also not exceed $\phi(\hat{k})$, because the argmax operator in equation (6) looks for the largest possible value of $\phi(k)$ that satisfies free entry of low type jobs. □

**Proposition. 3** *The offer function of a type $\underline{k}$ worker is characterized by the function*

$$\phi\left(k\mid\underline{k}\right)=\phi\left(k\right)\equiv\log\left(\frac{y'\left(k\right)}{c'\left(k\right)}\right)$$

*and by a lower bound on the set of offers, $\hat{k}\left(\underline{k}\right)$.*

*Proof.* Differencing the payoffs and opportunity costs of any pair of adjacent job types, we get the following difference equation

$$c(k_{j+1})-c(k_j)=\left(y(k_{j+1})-y\left(k_j\right)\right)e^{-\phi(k_{j+1})} \tag{39}$$

which must be satisfied for all job types offered in equilibrium. This result can be used to derive a necessary condition for the free-entry equilibrium with positive contribution by all job types in the continuous case. Let $\triangle=k_j-k_{j\text{-}1}$ denote the interval of successive job types for which equation (9) holds. The proposition then follows by taking the limit of equation (9) as the interval $\triangle$ becomes small. As in proposition 4, we note that the function $\phi\left(k\right)$ gives a positive density of wages over the support of this job offer distribution if

$$\left[\frac{-\left[y''\left(k\right)c'\left(k\right)-y'\left(k\right)c''\left(k\right)\right]}{\left(y'\left(k\right)\right)^2}\right]\left[\log\left(\frac{y'\left(k\right)}{c'\left(k\right)}\right)\right]<0.$$

Obviously, we require $c''\left(k\right)>0$ if $y\left(k\right)$ is linear, or $y''\left(k\right)<0$ if $c\left(k\right)$ is linear. □

**Proposition. 4** *The expected wage of worker type, $\bar{k}$, is given by*

$$E\left(y\left(k_2\right)\mid\bar{k}\right)=G_2\left(\bar{k}\right)y\left(\bar{k}\right)+\int_{\hat{k}}^{k^*}y\left(k\right)g_2\left(k\right)dk$$

*where*
$$g_2(k) = \left[\frac{-\left[y''(k)\,c'(k) - y'(k)\,c''(k)\right]}{\left(y'(k)\right)^2}\right]\left[\log\left(\frac{y'(k)}{c'(k)}\right)\right].$$

*Proof.* Wages are determined by the second best offer. The distribution of the second best offer is given by $G_2(k) = e^{-\phi(k)} + \phi(k)\,e^{-\phi(k)}$ and the corresponding density is then
$$g_2(k) = -\phi'(k)\,\phi(k)\,e^{-\phi(k)}$$

Therefore, we need to calculate $\phi'(k)$

$$\phi'(k) = \frac{d\log\left(\frac{y'(k)}{c'(k)}\right)}{dk} = \frac{y''(k)}{y'(k)} - \frac{c''(k)}{c'(k)}$$

Inserting $\phi(k)$ and $\phi'(k)$ in the equation for $g_2(k)$ gives us

$$\begin{aligned} g_2(k) &= -\phi'(k)\,\phi(k)\,e^{-\phi(k)} \\ &= \left[\frac{-\left[y''(k)\,c'(k) - y'(k)\,c''(k)\right]}{\left(y'(k)\right)^2}\right]\left[\log\left(\frac{y'(k)}{c'(k)}\right)\right] \end{aligned}$$

where this is positive everywhere on $\hat{k}$ to $k^*$ if there exists concavity of $y$ relative to $c$ . That is

$$\left[\frac{-\left[y''(k)\,c'(k) - y'(k)\,c''(k)\right]}{\left(y'(k)\right)^2}\right]\left[\log\left(\frac{y'(k)}{c'(k)}\right)\right] < 0.$$

As we know that we must have that $\phi'(k) < 0$ we have that $y''(k)\,c'(k) - y'(k)\,c''(k) < 0$. Naturally, the density is positive for all $k$ until $k = k^*$ where $y'(k^*) = c'(k^*)$. The expected sum of wages up to some value $\widetilde{k}$ is given by

$$\int_{\hat{k}}^{\tilde{k}} y(k)\,g_2(k)\,dk = \int_{\hat{k}}^{\tilde{k}} y(k)\left[\frac{-\left[y''(k)\,c'(k) - y'(k)\,c''(k)\right]}{\left(y'(k)\right)^2}\right]\left[\log\left(\frac{y'(k)}{c'(k)}\right)\right]dk$$

Since all terms are positive for $k \leq k^*$ the worker's wage is given by integrating this expression to the point where setting $\tilde{k} = k^*$. □

**Proposition. 6** *For values of* $k_1, k_2 \in \left[\widehat{k}, k^*\right]$,

$$G(k_1, k_2) = (1 + \phi(k_2) - \phi(k_1))\, e^{-\phi(k_2)}$$

*Proof.* Suppose a worker receives $n$ job offers. The probability that $j$ of these are below $k_2$ and that $n-j$ offers are less than $k_1$ where $\underline{k} \leq k_2 \leq k_1 \leq k^*$ is given by

$$\begin{pmatrix} n \\ j \end{pmatrix} F(k_2)^j \left[1 - F(k_1)\right]^{n-j}$$

where $\begin{pmatrix} n \\ j \end{pmatrix} = \frac{n!}{(n-j)!j!}$. Taking the negative cross-derivative of this delivers the joint density of the $j$'th and $j+1$'th order statistics

$$\begin{aligned}
g_{j,j+1}(k_1, k_2|n) &= -\left[-\begin{pmatrix} n \\ j \end{pmatrix} jF(k_2)^{j-1}(n-j)\left[1 - F(k_1)\right]^{n-j-1} f(k_1) f(k_2)\right] \\
&= \frac{n!jF(k_2)^{j-1}(n-j)\left[1 - F(k_1)\right]^{n-j-1} f(k_1) f(k_2)}{(n-j)!j!} \\
&= \frac{n!F(k_2)^{j-1}\left[1 - F(k_1)\right]^{n-j-1} f(k_1) f(k_2)}{(n-j-1)!(j-1)!}
\end{aligned}$$

We are only interested in the best and second best offers, so we set $j = n-1$ and $j+1 = n$. This gives us

$$g(k_1, k_2|n) = \frac{n!F(k_2)^{n-2} f(k_1) f(k_2)}{(n-2)!}$$

Summing over all possible number of job offers we obtain

$$
\begin{aligned}
g(k_1,k_2) &= \sum_{n=2}^{\infty} \frac{e^{-\phi(\underline{k})}\phi(\underline{k})^n}{n!} \frac{n! F(k_2)^{n-2} f(k_1) f(k_2)}{(n-2)!} \\
&= f(k_1) f(k_2) e^{-\phi(\underline{k})} \sum_{n=2}^{\infty} \frac{\phi(\underline{k})^n F(k_2)^{n-2}}{(n-2)!} \\
&= f(k_1) f(k_2) e^{-\phi(\underline{k})F(k_2)} e^{-\phi(\underline{k})(1-F(k_2))} \sum_{n=2}^{\infty} \frac{\phi(\underline{k})^n F(k_2)^{n-2}}{(n-2)!} \\
&= \phi(\underline{k})^2 f(k_1) f(k_2) e^{-\phi(\underline{k})(1-F(k_2))} \sum_{n=2}^{\infty} \frac{e^{-\phi(\underline{k})F(k_2)} \left(\phi(\underline{k}) F(k_2)\right)^{n-2}}{(n-2)!} \\
&= \phi'(k_1) \phi'(k_2) e^{-\phi(k_2)}
\end{aligned}
$$

Integrating the density gives us the bivariate distribution function

$$
\begin{aligned}
G(k_1,k_2) &= \int_{\underline{k}}^{k_2} \int_{k_2}^{k_1} \phi'(z_1) \phi'(z_2) e^{-\phi(z_2)} dz_1 dz_2 \\
&= \int_{\underline{k}}^{k_2} \phi'(z_2) e^{-\phi(z_2)} \int_{k_2}^{k_1} \phi'(z_1) dz_1 dz_2 \\
&= \int_{\underline{k}}^{k_2} \phi'(z_2) e^{-\phi(z_2)} \left(\phi(k_1) - \phi(k_2)\right) dz_2 \\
&= \phi(k_1) \int_{\underline{k}}^{k_2} \phi'(z_2) e^{-\phi(z_2)} dz_2 - \int_{\underline{k}}^{k_2} \phi'(z_2) \phi(k_2) e^{-\phi(z_2)} dz_2 \\
&= -\phi(k_1) \int_{\underline{k}}^{k_2} g_1(z_2) dz_2 + \int_{\underline{k}}^{k_2} g_2(z_2) dz_2 \\
&= -\phi(k_1) \left(e^{-\phi(k_2)} - e^{-\phi(\underline{k})}\right) + (1 + \phi(k_2)) e^{-\phi(k_2)} + z_0 \\
&= (1 + \phi(k_2) - \phi(k_1)) e^{-\phi(k_2)} + \phi(k_1) \left(e^{-\phi(\underline{k})}\right) + z_0
\end{aligned}
$$

where $z_0$ is a constant. We can determine $z_0$ by using that $G(k_1,k_1) = G(k_1)$

$$
\begin{aligned}
(1 + \phi(k_1) - \phi(k_1)) e^{-\phi(k_1)} + \phi(k_1) e^{-\phi(\underline{k})} + z_0 &= e^{-\phi(k_1)} \Leftrightarrow \\
z_0 &= -\phi(k_1) e^{-\phi(\underline{k})}
\end{aligned}
$$

Hence, we have that

$$G(k_1,k_2) = (1+\phi(k_2)-\phi(k_1))\, e^{-\phi(k_2)}$$

or alternatively $G(k_1,k_2) = G_2(k_2) - \phi(k_1)\, e^{-\phi(k_2)}$. $\square$

**Proposition. 8.** *If workers are identical, the buyer posting model has identical expected payoffs for both workers and firms as the competing auction model.*

*Proof.* In order to show that the expected payoffs in the buyer posting model are equivalent to the expected payoffs in the competing auction model, we need to explicitly calculate the payoffs for the buyer posting model. As in Mortensen (2003, p. 16-25), we only consider the case of two firm types where $y(2) > y(1)$. First, using equation (1.16)[13] we can write

$$\begin{aligned}
\frac{1}{\lambda}\ln\left(\frac{y(1)-\underline{y}}{y(1)-\overline{w}(1)}\right) &= q \Leftrightarrow \overline{w}(1) = y(1) - \left(y(1)-\underline{y}\right)\exp(-\lambda q) \\
\frac{1}{\lambda}\ln\left(\frac{y(2)-\overline{w}(1)}{y(2)-\overline{w}(2)}\right) &= 1-q \Leftrightarrow \overline{w}(2) = y(2) - (y(2)-\overline{w}(1))\exp(-\lambda(1-q))
\end{aligned}$$

where $q$ denotes firm 1' share of the job offers such that in terms of our notation $\lambda q = \phi(1|0) - \phi(2|0)$ and $\lambda(1-q) = \phi(2|0)$, and $\overline{w}(1)$ and $\overline{w}(2)$ denote respectively the highest wage paid in firm 1 and 2. We can write the low productive firm's payoff by (see also Mortensen's equation (1.21))

$$\begin{aligned}
\pi^*(y(1)) &= (y(1)-\overline{w}(1))\exp(-\lambda(1-q)) \\
&= \left(y(1)-\underline{y}\right)\exp(-\lambda)
\end{aligned}$$

and the high productive firm's payoff by (see also Mortensen's equation (1.22))

$$\begin{aligned}
\pi^*(y(2)) &= y(2)-\overline{w}(2) \\
&= (y(2)-\overline{w}(1))\exp(-\lambda(1-q)) \\
&= \left(y(2)-\underline{y}\right)\exp(-\lambda) + (y(2)-y(1))\exp(-\lambda(1-q))(1-\exp(-\lambda q))
\end{aligned}$$

[13] We notice that there is a small typo in Mortensen's equation (1.16) where the second equation should be equal to $1-q$.

Therefore, $\pi^* (y(1))$ is equal to the right-hand side of equation (4), whereas $\pi^* (y(2))$ is equal to the right-hand side of equation (5). With the same entry costs $c(1)$ and $c(2)$ as in the competing auction model, the entry decision is identical and, hence, the firms' mass of job offers will obviously be the same in the two models, i.e. $\lambda q = \phi(1|0) - \phi(2|0)$ and $\lambda(1-q) = \phi(2|0)$. $\square$

**Proposition. 9.** *An equilibrium with positive job offers exists if and only if*

$$\frac{y(k^*) - y(\underline{k})}{1-\beta(1-\delta)} > C(k^*)$$

*where $y(\underline{k})$ is the worker's output at home and $y(k^*)$ is the output of job type $k^*$ such that $\frac{y'(k^*)}{1-\beta(1-\delta)} = C'(k^*)$.*

*Proof.* We can write the job value function evaluated in $k^*$ as

$$\Lambda(k^*) = \frac{y(k^*)}{1-\beta(1-\delta)} + \frac{\beta\delta W(\underline{k},\underline{k})}{1-\beta(1-\delta)}$$

Define $p^* < k^*$ as the firm type $p^* = \hat{k}^{-1}(k^*)$. Since the minimum job type cannot exceed $k^*$, a worker employed by a type $k_1 \geq p^*$ firm does not get any offers. Subtracting $\Lambda(p^*)$ from $\Lambda(k^*)$ to obtain

$$\Lambda(k^*) - \Lambda(p^*) = \frac{y(k^*) - y(p^*)}{1-\beta(1-\delta)}$$

For an equilibrium with positive job offers to exist, we need that $\underline{k} \leq p^*$ which corresponds to

$$\frac{\Lambda(k^*) - \Lambda(\underline{k})}{\Lambda'(k^*)} = \frac{y(k^*) - y(\underline{k})}{1-\beta(1-\delta)} \geq C(\underline{k})$$

$\square$

**Proposition. 10.** *If $C(k) = k$, the first derivatives of the job value function $\Lambda(k)$ and the production function $y(k)$ are related as follows.$\Lambda'(k) = y'(k) + \beta(1-\delta)\frac{\Lambda'(k)}{\Lambda'(\hat{k})}$*

$$y' = \begin{cases} \Lambda' - \beta(1-\delta)\Lambda' & if \quad k \geq p^* \\ \Lambda' - \beta(1-\delta)\frac{\Lambda'}{\hat{\Lambda}'} & if \quad k < p^* \end{cases}$$

*where* $\Lambda' \equiv \partial\Lambda(k)/\partial\Lambda k$ *and* $\hat{\Lambda} \equiv \Lambda\left(\hat{k}(k)\right)$*. Note that equations (14) and (16) imply* $1/\Lambda' = \exp(-\phi(k)) \in [0,1]$ *and* $\hat{k}(k) = \frac{\hat{\Lambda}-\Lambda}{\hat{\Lambda}'} > k$*.*

*Proof.* For $k \geq p^*$ we have that the job value function for a worker employed in a type $k$ firm and with a second-best offer also being $k$ is given by

$$\Lambda(k) = y(k) + \beta(1-\delta)\Lambda(k) + \beta\delta W(\underline{k}, \underline{k})$$

Differentiating with respect to $k$ and re-arranging gives

$$y'(k) = (1-\beta(1-\delta))\Lambda'(k)$$

For $k < p^*$ differentiating the job value function for a worker employed in a type $k$ firm and with a second-best offer also being $k$ with respect to $k$ delivers

$$\begin{aligned}
\Lambda'(k) &= y'(k) + \beta(1-\delta)\left[\Lambda'(k) G_2\left(\hat{k}\right) + \left(\Lambda(k) - \Lambda\left(\hat{k}\right)\right) G_2'\left(\hat{k}\right)\hat{k}'(k)\right] \\
&= y'(k) + \beta(1-\delta)\left[\Lambda'(k) G_2\left(\hat{k}\right) - \Lambda'\left(\hat{k}\right) G_2'\left(\hat{k}\right)\hat{k}(k)\hat{k}'(k)\right] \qquad (40)
\end{aligned}$$

Differentiating equation (16) with respect to $k$, we can write that $\hat{k}(k)\hat{k}'(k) = -\frac{\Lambda'(k)}{\Lambda''\left(\hat{k}\right)}$. Besides this, we also plug in $G_2\left(\hat{k}\right) = \frac{1+\log\left(\Lambda'\left(\hat{k}\right)\right)}{\Lambda'\left(\hat{k}\right)}$ and $G_2'\left(\hat{k}\right) = -\frac{\Lambda''\left(\hat{k}\right)\log\left(\Lambda'\left(\hat{k}\right)\right)}{\left(\Lambda'\left(\hat{k}\right)\right)^2}$ in equation (40) and re-arrange to obtain

$$y'(k) = \left(1 - \beta(1-\delta)\frac{1}{\Lambda'\left(\hat{k}\right)}\right)\Lambda'(k)$$

$\square$

**Proposition. 11.** *If* $C(k) = k$*, the second derivatives of the job value function* $\Lambda(k)$ *and the production function* $y(k)$ *are related as follows.*

$$y'' = \left[\frac{y'}{\Lambda'}\right]\Lambda'' - \begin{cases} 0 & if \quad k \geq p^* \\ \beta(1-\delta)\frac{\Lambda'}{\hat{\Lambda}'}\left[\frac{\Lambda}{\hat{\Lambda}-\Lambda}\right] & if \quad k < p^* \end{cases}$$

*where* $y'/\Lambda'$, $\Lambda'/\hat{\Lambda}'$ and $\left[\hat{\Lambda} - \Lambda\right]$ are all positive.

*Proof.* Differentiation of equation (26) with respect to $k$ yields

$$y''(k) = \left(1 - \beta(1-\delta)\frac{1}{\Lambda'\left(\hat{k}\right)}\right)\Lambda''(k) + \beta(1-\delta)\frac{\Lambda''\left(\hat{k}\right)\hat{k}'(k)}{\Lambda'\left(\hat{k}\right)}\Lambda'(k)$$

Then, using equation (16) and its differentiated version, we obtain

$$y''(k) = \left(1 - \beta(1-\delta)\frac{1}{\Lambda'\left(\hat{k}\right)}\right)\Lambda''(k) - \beta(1-\delta)\frac{\left[\Lambda'(k)\right]^2}{\Lambda\left(\hat{k}\right) - \Lambda(k)}$$

□

**Proposition. 12.** *If* $C(k) = k$, *the cross derivatives of the job value function* $\Lambda(k,h)$ *and the production function* $y(k,h)$ *are related as follows.*

$$y_{12} = \Lambda_{12}\left[\frac{y_1}{\Lambda_1}\right] + \begin{cases} 0 & if \quad k \geq p^* \\ \beta(1-\delta)\frac{\Lambda_1}{\hat{\Lambda}_1}\left[\frac{\hat{\Lambda}_2 - \Lambda_2}{\hat{\Lambda} - \Lambda}\right] & if \quad k < p^* \end{cases}$$

*where* $y_1/\Lambda_1$, $\Lambda_1/\hat{\Lambda}_1$ and $\left[\hat{\Lambda} - \Lambda\right]$ are all positive. The terms $\Lambda_{12}$ and $\hat{\Lambda}_2 - \Lambda_2 = \int_{z=k}^{\hat{k}} \Lambda_{12}(z,h)\,dz$ are positive if and only if $\Lambda(k,h)$ is super-modular.

*Proof.* The result is straightforward to derive for $k \geq p^*$, so here we only consider the case where $k < p^*$. Write equation (25) as

$$y_1'(k,h) = \left(1 - \beta(1-\delta)\frac{1}{\Lambda_1'\left(\hat{k},h\right)}\right)\Lambda_1'(k,h)$$

and differentiate with respect to $h$ to obtain

$$\begin{aligned} y''_{12}(k,h) &= \left(1-\beta(1-\delta)\frac{1}{\Lambda'_1\left(\hat{k},h\right)}\right)\Lambda''_{12}(k,h) \\ &\quad +\beta(1-\delta)\frac{\Lambda'_1(k,h)\left[\Lambda''_{12}\left(\hat{k},h\right)+\Lambda''_{11}\left(\hat{k},h\right)\hat{k}'_2(k,h)\right]}{\left(\Lambda'_1\left(\hat{k},h\right)\right)^2} \end{aligned}$$

$$\begin{aligned} &= \left(1-\beta(1-\delta)\frac{1}{\Lambda'_1\left(\hat{k},h\right)}\right)\Lambda''_{12}(k,h) \\ &\quad +\beta(1-\delta)\frac{\Lambda'_1(k,h)\left[\Lambda''_{12}\left(\hat{k},h\right)+\Lambda''_{11}\left(\hat{k},h\right)\hat{k}'_2(k,h)\right]}{\left(\Lambda'_1\left(\hat{k},h\right)\right)^2} \end{aligned}$$

Using equation (16) differentiated with respect to $h$ and equation (25) , we can write

$$y''_{12}(k,h)=\left[\frac{y'_1(k,h)}{\Lambda'_1(k,h)}\right]\Lambda''_{12}(k,h)+\beta(1-\delta)\frac{\Lambda'_1(k,h)}{\Lambda'_1\left(\hat{k},h\right)}\frac{\Lambda'_2\left(\hat{k},h\right)-\Lambda'_2(k,h)}{\Lambda\left(\hat{k},h\right)-\Lambda(k,h)}$$

where all terms on the right hand side are positive whenever $\Lambda(k,h)$ is supermodular. Moreover, if $\Lambda(k,h)$ is sub-modular then so is $y(k,h)$. □

**Proposition. 13.** *If a worker has absolute but no-comparative advantage to another worker, then the two workers have identical job offer functions and wage of the more productive worker is given by*

$$w\left(k_1,k_2 \mid y(k)'\right)=w\left(k_1,k_2 \mid y(k)\right)+A$$

*where $A>0$ is scalar equal to the productivity difference between these two workers.*

*Proof.* Consider two workers of type $h$ and $h'$ and assume that $\Lambda(k,h')=$

$\Lambda(k,h)+\Delta$ for all $k$ and where $\Delta$ is a constant. Since the slope of the job value functions for the two different worker types is identical, $G_1(k,h') = G_1(k,h)$, $G_2(k,h') = G_2(k,h)$, $\hat{k}(k,h') = \hat{k}(k,h)$, and $k^*(h) = k^*(h')$. Next, consider the difference in expected present values of the workers in a type $k_1$ job and a second-best offer of $k_2$

$$\begin{aligned} W(k_1,k_2,h') - W(k_1,k_2,h) &= [\Lambda(k_2,h') - \Lambda(k_2,h)]\, G_1\left(\hat{k}\right) \\ &\quad + [\Lambda(k_1,h') - \Lambda(k_1,h)]\left[G_2\left(\hat{k}\right) - G_1\left(\hat{k}\right)\right] \\ &\quad + \int_{z=\hat{k}}^{k^*} [\Lambda(z,h') - \Lambda(z,h)]\, dG_2(z) \\ &= \Delta \end{aligned}$$

Subtracting the job value function for worker type $h$ from worker type $h'$ delivers

$$\begin{aligned} \Delta &= w(k_1,k_2,h') - w(k_1,k_2,h) \\ &\quad + \beta\begin{bmatrix} (1-\delta)(W(k_1,k_2,h') - W_B(k_1,k_2,h)) \\ +\delta\left(W_A(\underline{k},\underline{k},h') - W_B(\underline{k},\underline{k},h)\right) \end{bmatrix} \Leftrightarrow \\ w(k_1,k_2,h') &= w(k_1,k_2,h) + (1-\beta)\Delta \end{aligned}$$

Clearly, this also implies that $y(k_1,h') = y(k_1,h) + (1-\beta)\Delta$. □

**Proposition. 14.** *If the job value function is super-modular (positive assortative matching), then*

$$\frac{\partial\left[y(k_1,h) - w(k_1,k_2,h)\right]}{\partial h} > 0$$

*Proof.* The residual firm payment function is

$$y(k_1,h) - w(k_1,k_2,h) = \left[1 - \beta(1-\delta)\, G_1\left(\hat{k}(k,h), h\right)\right][\Lambda(k_1,h) - \Lambda(k_2,h)]$$

Differentiate by $h$

$$\begin{aligned}
\frac{\partial\left[y\left(k_1,h\right)-w\left(k_1,k_2,h\right)\right]}{\partial h} &= -\beta\left(1-\delta\right)\left[\Lambda\left(k_1,h\right)-\Lambda\left(k_2,h\right)\right]\frac{\partial G_1\left(\hat{k}\left(k,h\right),h\right)}{\partial h} \\
&\quad+\left[1-\beta\left(1-\delta\right)G_1\left(\hat{k}\left(k,h\right),h\right)\right]\left[\Lambda_2'\left(k_1,h\right)-\Lambda_2'\left(k_2,h\right)\right] \\
&= -\beta\left(1-\delta\right)\left[\Lambda\left(k_1,h\right)-\Lambda\left(k_2,h\right)\right]\frac{\partial G_1\left(\hat{k}\left(k,h\right),h\right)}{\partial h} \\
&\quad+\left[1-\beta\left(1-\delta\right)G_1\left(\hat{k}\left(k,h\right),h\right)\right]\int_{k_2}^{k_1}\Lambda_{12}''\left(z,h\right)dz
\end{aligned}$$

Therefore, if $\frac{\partial G_1\left(\hat{k}(k,h),h\right)}{\partial h}<0$ and the job value function is super-modular, then $\frac{\partial[y(k_1,h)-w(k_1,k_2,h)]}{\partial h}>0$. The formula

$$G_1\left(\hat{k},h\right) = \exp\left(-\phi\left(\hat{k},h\right)\right)=\frac{1}{\Lambda_1'\left(\hat{k},h\right)}$$

$$\frac{\partial G_1\left(\hat{k},h\right)}{\partial h}=\frac{-\left[\Lambda_{12}''\left(\hat{k},h\right)+\Lambda_{11}''\left(\hat{k},h\right)\hat{k}_2'\left(k,h\right)\right]}{\left[\Lambda_1'\left(\hat{k},h\right)\right]^2}$$

To sign the numerator, we differentiate equation (16), i.e. $\hat{k}\left(k,h\right)\Lambda_1'\left(\hat{k},h\right)=\Lambda\left(\hat{k},h\right)-\Lambda\left(k,h\right)$, with respect to $h$. This delivers

$$\begin{aligned}
&\hat{k}_2'\left(k,h\right)\Lambda_1'\left(\hat{k},h\right)+\hat{k}\left(k,h\right)\Lambda_{11}''\left(\hat{k},h\right)\hat{k}_2'\left(k,h\right)+\hat{k}\left(k,h\right)\Lambda_{12}''\left(\hat{k},h\right) \\
=\;&\Lambda_2'\left(\hat{k},h\right)+\Lambda_1'\left(\hat{k},h\right)\hat{k}_2'\left(k,h\right)-\Lambda_2'\left(k,h\right)\Leftrightarrow \\
&\hat{k}_2'\left(k,h\right)\Lambda_1'\left(\hat{k},h\right)+\hat{k}\left(k,h\right)\left[\Lambda_{12}''\left(\hat{k},h\right)+\Lambda_{11}''\left(\hat{k},h\right)\hat{k}_2'\left(k,h\right)\right] \\
=\;&\Lambda_2'\left(\hat{k},h\right)-\Lambda_2'\left(k,h\right)+\Lambda_1'\left(\hat{k},h\right)\hat{k}_2'\left(k,h\right)\Leftrightarrow
\end{aligned}$$

$$
\begin{aligned}
\hat{k}(k,h)\left[\Lambda_{12}''\left(\hat{k},h\right)+\Lambda_{11}''\left(\hat{k},h\right)\hat{k}_2'(k,h)\right] &= \Lambda_2'\left(\hat{k},h\right)-\Lambda_2'(k,h)\Leftrightarrow \\
\Lambda_{12}''\left(\hat{k},h\right)+\Lambda_{11}''\left(\hat{k},h\right)\hat{k}_2'(k,h) &= \frac{\Lambda_2'\left(\hat{k},h\right)-\Lambda_2'(k,h)}{\hat{k}(k,h)}\Leftrightarrow \\
\Lambda_{12}''\left(\hat{k},h\right)+\Lambda_{11}''\left(\hat{k},h\right)\hat{k}_2'(k,h) &= \frac{\int_{z=k}^{\hat{k}}\Lambda_{12}''(z,h)\,dz}{\Lambda\left(\hat{k},h\right)-\Lambda(k,h)}\Lambda_1'\left(\hat{k},h\right)
\end{aligned}
$$

Since the right hand side is positive whenever the job value function is super-modular, then $\frac{\partial G_1\left(\hat{k},h\right)}{\partial h}<0$. Therefore, super-modularity of the job value function implies that the residual firm payment function is increasing in worker type. This is reversed if the job value function is sub-modular. □

## Appendix to section 4: Revenue equivalent posting model

In estimation section, we estimate a wage posting model that gives workers and firms equal expected payoffs as the competing auction model. We describe here how the wages are computed for this model. We construct the expected revenue equivalent posting model for the general specification of parameters, $\{y(k),C(k),\underline{k},\delta,\beta\}$, as follows. We begin by asserting that the worker has the same job offer function as in the auction model. This means that $k^*(posting)=k^*$ and $\phi(k\mid posting)=\phi(k)$. Therefore, the various order statistics for job offers are the same as the auction model. For example, $G_1(k\mid posting)=G_1(k)$. We can also assume that the job value function is the same in the auction and posting model, $\Lambda(k\mid posting)=\Lambda(k)$. We then apply proposition 5 to derive an expected value for a worker in a type $k_1$ job that is equivalent to the auction equilibrium. Therefore, the value of a worker employed in a type $k$ firm is given by

$$
Z(k)=\frac{\int_{\hat{k}(k)}^{k}\Lambda(z)dG_1(z)}{G_1(k)} \tag{41}
$$

Similarly, we modify equation (17) to derive the worker's expected value of search, $S(k)$, given that the worker is always rewarded the same wage $w(k)$ at each firm type

$$S(k) = Z(k)G_1(k) + \int_{\widehat{k}(k)}^{k^*} Z(z)dG_1(z) \tag{42}$$

Finally, we can calculate the wage payment to the worker using the following equation

$$Z(k) = w(k) + \beta\left[(1-\delta)S(k) + \delta S(\underline{k})\right] \tag{43}$$

where $Z(k)$ analogous to equation (18). The key difference of this model and the competing auction model is that the worker earns a unique wage at each job type. Therefore, in the calculation of the the overall distribution of wages, we simply use the overall distribution of firm types $n(k)$ of the auction model in order to determine the frequency of different wages in the overall distribution of workers.

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