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Implementation by Sortition in Nonexclusive Information Economies

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Abstract

We study the Bayesian implementation problem in economies that are divided into groups consisting of two or more individuals with the same information. Our results cover problems like that of allocating public funds among states, regulating activities causing externalities among firms, locating public facilities in neighborhoods, electing candidates from multiple districts etc. Instead of the standard communication protocol of direct democracy whereby the planner consults all individuals, we analyze sortition schemes whereby the planner consults only a subset of the individuals, called senators, who are selected via some kleroterion (i.e., a lottery machine) p. In general environments, under mild "economic" assumptions on preferences, we show that every social choice function (SCF) that is implementable by direct democracy is also p-implementable if p always selects two or more individuals from each group and the selection process does not partition any group into "disconnected" subgroups (in the sense that individuals belonging to different subgroups are never selected together). In quasilinear environments satisfying a generic condition on individuals' beliefs, every SCF can be implemented by a simple and economically meaningful mechanism in which the kleroterion selects a predesignated group leader and one other randomly chosen individual from each group.

Keywords: Bayesian implementation; Nonexclusive information; p-Implementation; Sortition

JEL: C72; D02; D82

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1 Introduction

In late 2008, the U.S. Senate Banking, Housing, and Urban Affairs Committee held a couple of hearings on the question of providing bailouts to the three big U.S. automakers, G.M., Ford, and Chrysler. The CEOs of these three companies testified in front of the committee regarding the financial health of their respective companies. Presumably, several other employees – e.g., the CFO, Board of Directors etc. – must be as aware of the financial condition of their respective companies as the CEOs. Such nonexclusive information environments where information pertinent to a collective decision is distributed among distinct groups of individuals are quite common. As other examples, consider the problem of allocating spectrum licenses for wireless, television or radio services among competing firms or the problem of regulating a negative externality caused by a firm on another firm. In both of these cases, the relevant information (viz., valuation of the spectrum, costs and benefits of regulating the externality) is known to more than one individual at each firm.

Postlewaite and Schmeidler (1986), and subsequently Palfrey and Srivastava (1987), study the question of designing mechanisms to implement "desirable" allocations in economies with incomplete but nonexclusive information.[1] These papers and more generally the literature on mechanism design model mechanisms as a communication protocol between the social planner and the concerned individuals. A distinctive feature of this communication protocol is that *all individuals* transmit their messages to the planner, who eventually implements an outcome using a pre-specified rule after consulting the transmitted messages. We refer to such communication protocols as direct democratic mechanisms. In contrast to the literature's almost exclusive focus on direct democratic mechanisms, communication in the real world seldom involves consulting the opinions of each and every concerned individual – perhaps out of consideration for time and cost. For instance, the U.S. Senate Banking, Housing, and Urban Affairs Committee hearings mentioned above comprised of testimonies by a select set of six to eight witnesses.[2] A natural question then is, whether there are alternative mechanisms that replicate the outcome of direct democratic mechanisms but in which the planner consults the opinions of only a small subset of the individuals? Existence or not of such alternatives has broad implications for the organization of decision-making bodies within institutions.

We first explored the above question in Saran and Tumennasan (2013) while restricting

[1] Palfrey and Srivastava (1989), Mookherjee and Reichelstein (1990), and Jackson (1991) study Bayesian implementation in more general environments where individuals may have exclusive information.

[2] In addition to the the CEOs of the three big U.S. automakers, testimonials were offered by the president of a workers union, president of an automotive retailers association, president of an automotive supplier, and an expert witness.

to complete information environments.[3] In this paper, we extend our study to incomplete but nonexclusive information environments. In our model, there are distinct groups, each comprising of two or more individuals. All individuals within a group have complete information regarding a group-specific parameter but individuals belonging to different groups do not know each other's information. The planner's objective is to (fully) implement in Bayesian Nash equilbrium a social choice function (SCF) that is "responsive" to the parameters of all groups. We consider sortition schemes as alternatives to direct democratic mechanisms, whereby a subset of the individuals – we refer to these as senators – are selected via a kleroterion (i.e., a lottery machine) p. The planner consults the messages of only the selected senators before implementing the outcome. An opinion poll that surveys a randomly sampled set of individuals is a prime example of sortition. Our model includes direct democratic mechanisms as a special case when the planner uses the kleroterion p_D which selects all individuals with probability 1.[4]

We show that three conditions are necessary for any SCF to be p-implementable (i.e., implementable by a mechanism that uses kleroterion p to select the senators). First, as the SCF is responsive to the information of all groups, the kleroterion p must not ignore any group in the sense that each selected senate must have at least one individual from every group. Second, the SCF must be p-incentive compatible, which is a set of incentive constraints that are necessary to ensure that no individual has an incentive to lie about her information in equilibrium. However, as ours is a nonexclusive information environment, these incentive constraints are satisfied by all SCFs if either (a) the kleroterion p always selects three or more individuals from each group or (b) p always selects two or more individuals from each group and every group has a state-independent common worse alternative. Third, the SCF must be Bayesian p-monotonic, which is necessary to ensure that any situation where individuals within every group coordinate on a lie to undermine the SCF is not an equilibrium.

In "economic" environments with three or more groups, p-incentive compatibility and Bayesian p-monotonicity are also sufficient for p-implementation as long as the kleroterion p always selects two or more senators from each group and it does not partition any group into "disconnected" subgroups (in the sense that individuals belonging to different subgroups are

[3]See Footnote 16 where we discuss how the current work offers a new result for the complete information case.

[4] We cannot extend our analysis to exclusive information environments. If each individual possesses exclusive information and the SCF is responsive to that information, then the planner cannot implement the SCF without knowing every individual's information. Thus, in exclusive information environments, responsive SCFs cannot be implemented using sortition schemes that select a strict subset of the individuals. Nevertheless, there might be other communication protocols that are able to replicate the outcomes of direct democratic mechanisms in such situations. Renou and Tomala (2012), for instance, show how the planner can *partially* implement incentive compatible SCFs in certain exclusive information environments using encryption techniques over sufficiently connected communication networks.

never selected together). This result in particular implies that when each of the groups has three or more individuals or a state-independent common worse alternative, then Bayesian p_D-monotonicity is both necessary and sufficient for p_D-implementation (i.e., implementation using direct democratic mechanisms) in economic environments. Bayesian p_D-monotonicity is equivalent to the Bayesian monotonicity condition in Palfrey and Srivastava (1987) who show that the latter condition is necessary and sufficient for Bayesian implementation in nonexclusive information economies with three or more individuals (see also Postlewaite and Schmeidler, 1986). Our necessary and sufficient conditions for p-implementation thus generalize previously known results for Bayesian implementation in nonexclusive information economies by allowing for a much larger class of mechanisms.

We then apply our sufficiency result to answer our motivating question in the affirmative. In particular, we identify multiple kleroteria that under mild "economic" conditions implement any SCF that is implementable by direct democratic mechanisms but in which the planner consults only the messages of two or more individuals from each group. For instance, in economic environments with three or more groups and a state-independent common worse alternative for each group, any SCF that is implementable by direct democratic mechanisms is also implementable by mechanisms that use any of the following kleroteria: (a) randomly sampling two individuals from each group, (b) selecting a predesignated group leader and one other randomly chosen individual from each group, (c) selecting two oligarchs from each group with an arbitrarily small chance of referendum, and (d) when individuals in all groups can be identified with their locations, then randomly selecting two neighbors from each group.

Our general sufficiency result, however, relies on a canonical type mechanism which uses the integer game (see Jackson (1992) for a criticism of such canonical constructions). This mechanism is unbounded and best responses do not always exist. In quasilinear environments, when individuals' beliefs satisfy "no consistent coordinated deception", we show that any SCF is implementable by a simple and economically meaningful mechanism in which the kleroterion p selects a predesignated group leader and one other randomly chosen individual from each group. Unlike the canonical mechanisms, the mechanism constructed here is compact with nonempty best responses, and asks an individual to report her information and offer a contingent asset to the planner. No consistent coordinated deception is a generic condition on individuals' beliefs, and in quasilinear environments, it implies that every SCF satisfies Bayesian p-monotonicity for the kleroterion p mentioned above. Basically, our no consistent coordinated deception condition is the counterpart of no-consistent-deception condition of Matsushima (1993) for exclusive information environments.[5] If we replace no consistent coor-

[5]Matsushima (1993) proves that if individuals' beliefs satisfy no-consistent-deception in quasilinear en-

dinated deception with the stronger assumption that there is an individual whose conditional beliefs never place the same probability on any two distinct states – which is a generic condition –, then in quasilinear environments, any SCF can be implemented using the kleroterion of oligarchy in which the planner always consults with two or more designated individuals from each group (we illustrate this result in Section 1.1 below). Such oligarchies are common in practice. For instance, each state in the U.S. is represented by two senators in the U.S. Senate. Under any proportional parliamentary system with multiple election districts, two or more individuals represent each district. Our results thus suggest that oligarchy can be a suitable alternative to direct democracy in a large class of environments.

We next present a motivating example and then proceed to lay out our formal model in Section 2. We present the necessary conditions for p-implementation in Section 3. The sufficiency result is proved in Section 4. We study p-implementation in quasilinear environments in Section 5 and conclude in Section 6. The longer proofs are collected in the Appendix.

1.1 Motivating example

Consider the case of the U.S. automobile industry bailout mentioned above. To keep things simple, suppose there are two firms 1 and 2. The financial condition of each firm could be either strong or weak. Firm x's financial condition is known to two or more of its employees. Denote this set of employees by N_x. Conditional on the financial condition of firm x, its employees N_x receive signals regarding the financial condition of the other firm. Let's assume a simple signal process. If firm x is in a strong financial condition, then any i in N_x receives the signal that firm $x' \neq x$ is in a strong financial condition with probability 0.60 and is in a weak financial condition with probability 0.40. If firm x is in a weak financial condition, then any i in N_x receives the signal that firm $x' \neq x$ is in a strong financial condition with probability 0.25 and is in a weak financial condition with probability 0.75.

The government wants to bailout a firm (by offering interest free loans, subsidies etc.) if and only if the firm is in a weak financial condition. A state of the world θ specifies the

vironments with exclusive information, then all strictly incentive compatible SCFs satisfy Bayesian monotonicity. He concludes that in such environments, strict incentive compatibility is sufficient for Bayesian implementation. Palfrey and Srivastava (1993) show that any incentive compatible SCF can be implemented in quasilinear environments with exclusive information when individuals' beliefs are type independent and satisfy no-consistent-deception. Somewhat like our mechanism, their mechanism also asks each individual to submit a type report and a transfer rule which is contingent on the type reports of everyone else. However, unlike our mechanism, their mechanism is not compact. Duggan and Roberts (2001) provide a simple mechanism to implement the efficient allocation of pollution when firms hold exclusive information and their beliefs satisfy a condition that is stronger than no-consistent-deception. In their mechanism, each firm reports its own characteristic and the probability distribution of its neighboring firm's characteristic. We can apply our results for quasilinear environments to Duggan and Roberts' setting while assuming that two or more individuals at each firm know the firm-specific characteristic.

financial conditions of the two firms. We thus have four possible states of the world and the following SCF:

		Firm 2	
		Weak	Strong
Firm 1	Weak	*Bailout both firms*	*Bailout only Firm 1*
	Strong	*Bailout only Firm 2*	*No bailouts*

The government does not know the true state. It could ask the employees to report the financial conditions of their respective firms but to make the problem interesting, let's suppose that the interests of the employees at these firms are at odds with that of the government. Specifically, suppose that individuals' utilities are quasilinear in money, viz., $v_i(a, \theta) - t_i$, where $a \in \{$*Bailout both firms*, *Bailout only firm 1*, *Bailout only firm 2*, *No bailouts*$\}$ and t_i is the net monetary transfer received by individual i from the government. Irrespective of the state, any employee strictly prefers that her firm receives the bailout, i.e., if a is such that individual i's firm receives the bailout whereas a' is such that individual i's firm does not receive the bailout, then $v_i(a, \theta) > v_i(a', \theta)$ for all θ.

In this scenario, consider any mechanism in which the government consults with only one employee – the CEO – of each firm as in the Senate Banking, Housing, and Urban Affairs Committee hearings. That is, the government asks the two CEOs to submit their respective reports, and after going through the two reports, decides whether or not to bailout each firm. Any such mechanism will be unable to achieve the government's goal because even if a firm's financial condition is strong, its CEO has an incentive to mimic a weak firm – formally, reporting a weak financial condition is strictly dominant for every CEO in the direct revelation mechanism.

At the other extreme are direct democratic mechanisms in which the government consults all employees at the two firms. That is, the government asks all employees to submit their respective reports, and after going through all the reports, decides whether or not to bailout each firm. Although there are direct democratic mechanisms that (fully) implement the government's goal, our point here is to show that the government can implement its goal also by consulting only a subset of the employees at each firm.

To see this, suppose the government asks two employees at each firm – the CEO and CFO – to submit their reports. The CEO's report has two components: First, she has to report whether her firm is in a strong or weak financial condition. Second, she has to either accept or reject the contingent asset τ_1 offered by the government. The contingent asset τ_1 pays to the CEO the following monetary amounts contingent on the reports of the two CFOs:

τ_1		CFO of Firm 2	
		Reports Strong	Reports Weak
CFO of Firm 1	Reports Strong	2	-4
	Reports Weak	-4	1

The CFO's report too has two components: First, she has to report whether her firm is in a strong or weak financial condition. Second, she has to either accept or reject the contingent asset τ_2 offered by the government. The contingent asset τ_2 pays to the CFO the following monetary amounts contingent on the reports of the two CEOs:

τ_2		CEO of Firm 2	
		Reports Strong	Reports Weak
CEO of Firm 1	Reports Strong	-1	1
	Reports Weak	1	-2

Let t^* be the amount of monetary fine such that irrespective of the state, the individual paying the fine prefers the outcome in which her firm does not receive the bailout to the outcome in which her firm receives the bailout but she pays the fine, i.e., if a is such that individual i's firm receives the bailout whereas a' is such that individual i's firm does not receive the bailout, then $v_i(a, \theta) < v_i(a', \theta) - t^*$ for all θ and i.

The government determines the outcome according to the following rules:

1. Bailout a firm if and only if its CFO reports that the firm is weak.

2. Impose the fine t^* on the CEO of firm x if and only if both she and the CFO of firm x report the same financial condition but either the CEO or CFO of firm $x' \neq x$ accepts the contingent asset offered by the government.

3. Impose the fine t^* on the CFO of firm x if and only if she and the CEO of firm x report different financial conditions.

In addition, the government fulfills its commitment with respect to the contingent asset.

Let's argue that the strategy profile in which the four individuals truthfully report the financial condition of their respective firms and reject the contingent assets is a Bayesian Nash equilibrium of the mechanism. Firstly, none of the CFOs would like to claim a different financial condition in her report because then the financial condition in the CEO's report will differ from that in the CFO's report, leading to a fine of t^* on the CFO. Secondly, given that the CEOs are reporting the true financial conditions, none of the CFOs has the incentive to accept the contingent asset τ_2. Thirdly, the bailout decision will not change if any one of the CEOs changes the claimed financial condition in her report. Finally, given that the CFOs are reporting the true financial conditions, none of the CEOs has the incentive to accept the contingent asset τ_1. Note that the equilibrium outcome is equal to the SCF without any monetary transfers.

We now argue that the mechanism fully implements the SCF because there are no other (pure-strategy) Bayesian Nash equilibria in the mechanism. It cannot be that the financial conditions reported by the CEO and CFO of firm x diverge in equilibrium. After all, the CFO would prefer to report the same financial condition as the CEO than pay the fine t^*. Thus both the CEO and CFO of each firm report the same financial condition in any Bayesian Nash equilibrium. Then it cannot be that either the CEO or CFO of any firm x' accepts the contingent asset offered by the government in equilibrium. Such a situation leads to the fine of t^* on the CEO of firm $x \neq x'$, which she can avoid by reporting a different financial condition than the CFO of firm x. We will therefore be done if we show that any strategy profile in which the CEO and CFO of any firm coordinate on an untruthful report regarding their firm's financial condition and the four individuals reject the contingent assets cannot be an equilibrium. This is because in any such strategy profile, at least one of the individuals will have an incentive to accept the contingent asset offered by the government. For instance, consider the strategy profile in which the CEO and CFO of both firms misreport the financial condition of their firm as weak irrespective of whether the firm is actually weak or strong. In this case, both the CEOs have the incentive to accept the contingent asset τ_1 as it pays them 1 for sure. Similar arguments work for other cases.

In the above mechanism, the kleroterion is such that the government consults with two designated individuals from each firm. In Remark 5.4, we give a sufficient condition for such a kleroterion to implement any SCF in quasilinear environments. In general environments, however, the planner might not be able to implement the SCF by consulting a *fixed* subset of individuals from each group. Nevertheless, as we will see later, the planner can implement the SCF by introducing randomness in the selection process.

2 Model

There are a finite number of individuals N. Each individual belongs to one of the finite number (≥ 2) of distinct groups X.[6] Let N_x be the set of individuals that belong to group $x \in X$. We thus assume that (i) $N_x \neq \emptyset$ for all $x \in X$, (ii) $\cup_{x \in X} N_x = N$, and (iii) $N_x \cap N_y = \emptyset$ for any $x \neq y$.

All individuals in group x have complete information regarding the realization of a certain

[6] If there is a single group, then our model boils down to one of complete information. See Saran and Tumennasan (2013) for a detailed analysis of this case. Also see Footnote 16.

parameter $\theta_x \in \Theta_x$, where we assume that Θ_x is finite. By a state θ we mean a profile of realizations $(\theta_x)_{x\in X}$. Let $\Theta = \prod_{x\in X}\Theta_x$ be the set of states, which we assume has at least two elements.

For each individual i, we will let $\omega(i) \in X$ denote her group and refer to a realization $\theta_{\omega(i)} \in \Theta_{\omega(i)}$ as her *type*. Let $\Theta_{-\omega(i)} = \prod_{x\neq\omega(i)}\Theta_x$ with a typical element denoted by $\theta_{-\omega(i)}$.

Let $\pi_i(\theta_{-\omega(i)}|\theta_{\omega(i)})$ be individual i's conditional belief regarding the realization of $\theta_{-\omega(i)}$ when her type is $\theta_{\omega(i)}$. We assume that $\pi_i(\cdot|\theta_{\omega(i)})$ has full support over $\Theta_{-\omega(i)}$.[7]

We will restrict attention to environments with nonexclusive information (Postlewaite and Schmeidler, 1986). In our model, the environment has *nonexclusive information* if there are at least two individuals in each group.[8]

Let A be the set of possible alternatives with a being a typical element. We let ΔA denote the set of lotteries over A (i.e., probability distributions with finite support). For any $l \in \Delta A$, let $Supp(l)$ denote its support and $l(a)$ denote the probability assigned by l to alternative a. We write a for both the alternative $a \in A$ and the lottery that assigns probability 1 to a.

An *allocation* is a function $\alpha : \Theta \to \Delta A$. Let $\mathcal{A}$ be the set of allocations. We let a denote a constant allocation that selects alternative a in each state.

A *social choice function* (SCF) is a deterministic allocation $f : \Theta \to A$.[9] We say that an SCF f is *responsive to all groups* if f responds to the information of each group. Formally, for all $x \in X$, there exist θ_x, θ'_x, and θ_{-x} such that $f(\theta_x, \theta_{-x}) \neq f(\theta'_x, \theta_{-x})$. Throughout the paper, we will consider only those SCFs that are responsive to all groups.

Individual i's preferences are captured by a Bernoulli utility function $u_i : A \times \Theta \to \Re$, which is extended over lotteries using expected utility. Thus for any lottery $l \in \Delta A$, we have $u_i(l,\theta) = \sum_{a\in Supp(l)} l(a)u_i(a,\theta)$. Then the utility of type $\theta_{\omega(i)}$ of individual i from allocation α equals

$$U_i(\alpha|\theta_{\omega(i)}) = \sum_{\theta_{-\omega(i)}\in\Theta_{-\omega(i)}} \pi_i(\theta_{-\omega(i)}|\theta_{\omega(i)})u_i\big(\alpha(\theta_{\omega(i)}, \theta_{-\omega(i)}), (\theta_{\omega(i)}, \theta_{-\omega(i)})\big).$$

[7] One can relax this assumption but then the necessary and sufficient conditions for implementation must be true for some SCF which is equal to the SCF under consideration on those states that occur with positive probability – assuming that everyone agrees on the states that occur with positive probability.

[8] In contrast, if each group consist of only a single individual, then we have the standard *exclusive information* environment. For the reasons mentioned in Footnote 4, we do not pursue the case of exclusive information any further.

[9] Our necessary and sufficient conditions for p-implementation easily generalize to social choice sets. A social choice set is a nonempty set of SCFs.

2.1 Economic environment

For any pair of allocations α and α' and $\Theta' \subseteq \Theta$ define their *splicing* $\alpha/_{\Theta'}\alpha'$ as an allocation such that $\alpha/_{\Theta'}\alpha'(\theta) = \alpha(\theta), \forall\theta \in \Theta'$, and $\alpha/_{\Theta'}\alpha'(\theta) = \alpha'(\theta)$ otherwise.

Definition 2.1. The environment has *within groups differences* if for all allocations α, states θ, groups x, and distinct individuals i and i' belonging to group x, there exists an alternative a_j for at least one individual $j \in \{i, i'\}$ such that

$$U_j(a_j/_{\Theta'}\alpha|\theta_x) > U_j(\alpha|\theta_x), \forall\Theta' \subseteq \Theta \text{ such that } \theta \in \Theta'.$$[10]

Thus for any allocation, state, and pair of individuals i and i' belonging to the same group, at least one of these individuals would prefer to change the allocation at any subset of states containing that state. Simply put, when there are within groups differences, then at any state, it is impossible to simultaneously satiate any two individuals from the same group.

Definition 2.2. The environment has *between groups differences* if for all allocations α, states θ, and individuals i and i' belonging to different groups, i.e., $\omega(i) \neq \omega(i')$, there exists an alternative a_j for at least one individual $j \in \{i, i'\}$ such that

$$U_j(a_j/_{\Theta'}\alpha|\theta_{\omega(j)}) > U_j(\alpha|\theta_{\omega(j)}), \forall\Theta' \subseteq \Theta \text{ such that } \theta \in \Theta'.$$

Thus for any allocation, state, and pair of individuals i and i' belonging to the different groups, at least one of these individuals would prefer to change the allocation at any subset of states containing that state. Simply put, when there are between groups differences, then at any state, it is impossible to simultaneously satiate any two individuals belonging to different groups.

We say that the environment is *economic* if it has either within or between groups differences.

The are several different definitions of the "economic environment" in the literature. The most commonly used definition requires that no SCF can simultaneously satiate any two

[10] A slightly stronger though perhaps more intuitive definition is that for all allocations α, groups x, distinct individuals i and i' belonging to group x, and parameter values θ_x, there exists an alternative a_j for at least one individual $j \in \{i, i'\}$ such that $U_j(a_j/_{\Theta'}\alpha|\theta_x) > U_j(\alpha|\theta_x), \forall\Theta' \subseteq \{\theta_x\} \times \Theta_{-x}$. This statement says that if we pick any two distinct individuals i and i' from group x whose type is θ_x, then at least one of them would prefer to change the allocation α at any subset of states that are consistent with her type. We can similarly strengthen the definition of between groups differences. Nevertheless, we choose to present the weaker definitions to make it easy for the reader to compare our definition of the economic environment with the one given in Jackson (1991). Jackson (1991) refers to an environment as economic if for any allocation and state, there exist two individuals who would prefer to change the allocation at any subset of states containing that state. Our definition of the economic environment is stronger than Jackson's definition.

individuals (see, for example, Serrano, 2004). Our definition of the economic environment is weaker as we allow for the possibility that the interests of the individuals within a group are perfectly aligned or that the interests of the individuals belonging to different groups are not at odds with each other. Although this difference may seem negligible, there are some interesting applications that are covered under our definition but not under the common alternative.

Between groups differences but not within groups differences: Smog and pollution produced in the Midwestern states of the U.S. are carried downwind to the Eastern states. In April 2014, the U.S. Supreme Court ruled that the Environmental Protection Agency has authority to require the Midwestern states to reduce harmful emissions that drift to the East Coast and mid-Atlantic under the Clean Air Act. In this case, one can argue that individuals share a common interest with those from the same state but not with the ones from a different state. More generally, we can have between groups differences – but not necessarily within groups differences – whenever each group (country, state, city or firm) generates a negative externality on other groups. As another example, consider the problem of allocating public funds among states/municipalities/firms. Here again, each individual within a group would like to get as many funds allocated to her own group, which of course is detrimental to the interests of the individuals in other groups.

Within groups differences but not between groups differences: Consider, for example, the problem of locating a single public facility in each neighborhood of a city. Every individual in a neighborhood has a distinct location and would like to minimize the distance between her own location and that of the public facility in her neighborhood. In this scenario, we have within groups differences – since no two individuals within a neighborhood can be simultaneously satiated – but not between groups differences – since individuals only care of the location of the public facility in their own neighborhood. As another example, consider the case where there are multiple districts and in each district, there is a group of candidates who care both for the issues and being elected to office. Here again, we have within groups differences but not necessarily between groups differences.

2.2 Mechanism

The social planner does not have any information regarding the state. She constructs a mechanism in order to implement the SCF. Usually, a mechanism is designed as a direct communication scheme between all individuals and the social planner. We generalize this construction by adding a kleroterion that selects a *senate* (i.e., a nonempty subset of individuals) which exclusively communicates with the social planner. Let $\mathcal{N} = 2^N \setminus \emptyset$ be the set of all possible senates.

A *kleroterion* p is a function $p : \mathcal{N} \to [0,1]$ such that $p(S) \geq 0, \forall S \in \mathcal{N}$, and $\sum_{S \in \mathcal{N}} p(S) = 1$. Let $\mathcal{N}(p) = \{S \in \mathcal{N} : p(S) > 0\}$ be the set of senates that are selected with a positive probability by the kleroterion p. For each i, let $\mathcal{N}(i,p) = \{S \in \mathcal{N}(p) : i \in S\}$ be the set of senates in which i is selected.

We use the term *referendum* for the event in which all individuals N are selected as senators. Let p_D denote the kleroterion in which the probability of referendum is equal to 1. We refer to p_D as *direct democracy.*

A *mechanism* is a triplet $\Gamma = ((M_i)_{i \in N}, p, (g^S)_{S \in \mathcal{N}(p)})$ such that:

- M_i is the set of messages that individual i can announce.[11]
- $p \in \mathcal{P}$ is the kleroterion used for selecting the senators. If $S \in \mathcal{N}(p)$ is the selected senate, then the messages of all individuals in S, $(m_i)_{i \in S}$, are transmitted to the planner while the messages of all individuals $j \notin S$ are ignored.
- $g^S : \prod_{i \in S} M_i \to A$ is the outcome function conditional on the selection of the senate $S \in \mathcal{N}(p)$. Note that the outcome function g^S is deterministic.

Remark 2.3. Here we are assuming that at first, each individual commits to a message, and then the planner selects the senators using the kleroterion p. After the selection of the senators, the messages they had committed to earlier get transmitted to the planner. It is not necessary to assume this timing, i.e., that individuals commit to messages before they are selected. Instead, the planner could first select the senators using p while keeping their identities secret from each other – which usually happens in an opinion poll –, and then ask the selected senators to send their messages. Since the selected senators do not know each other's identity at the time of choosing their messages but they do know that the selection was made using p, this game has exactly the same strategic form, and hence the same Bayesian Nash equilibria as does the game generated by the mechanism when individuals first commit to their messages.

For any message profile $m \in \prod_{i \in N} M_i$ and senate S, let $m^S = (m_i)_{i \in S}$. In mechanism Γ, any message profile m generates a lottery $l[m] \in \Delta A$ such that the probability of selecting alternative a equals $\sum_{S \in \mathcal{N}(p): g^S(m^S) = a} p(S)$.

A *strategy* of any individual in mechanism Γ specifies a message for each type of that individual. Precisely, a strategy is a function $\sigma_i : \Theta_{\omega(i)} \to M_i$.[12] We use m_i both for the

[11] Individual i's message will not affect the outcome if she has a zero probability of being selected as a senator. M_i is redundant in this case and we can remove it from the definition of the mechanism. However, we do not thus modify the definition since it will add more notation.

[12] We are thus restricting to pure strategies.

message $m_i \in M_i$ and the constant strategy that selects the same message m_i for all $\theta_{\omega(i)}$. In mechanism Γ, any strategy profile σ generates an allocation $\alpha[\sigma]$ as follows:

$$\alpha[\sigma](\theta) = l[(\sigma_i(\theta_{\omega(i)}))_{i \in N}], \forall \theta \in \Theta.$$

A strategy profile σ is an *Bayesian Nash equilibrium* of the mechanism Γ if for all $i \in N$, $\theta_{\omega(i)} \in \Theta_{\omega(i)}$, and $m_i \in M_i$, we have

$$U_i(\alpha[\sigma]|\theta_{\omega(i)}) \geq U_i(\alpha[m_i, \sigma_{-i}]|\theta_{\omega(i)}).$$

Let $E(\Gamma)$ denote the set of Bayesian Nash equilibria of the mechanism Γ.

2.3 p-Implementation

We are now ready to define our notion of implementation.

Definition 2.4. SCF f is *p-implementable* if there is a mechanism $\Gamma = ((M_i)_{i \in N}, p, (g^S)_{S \in \mathcal{N}(p)})$ such that

$$\{\alpha[\sigma] : \sigma \in E(\Gamma)\} = \{f\}.$$

The above definition imposes two requirements: (i) Each Bayesian Nash equilibrium σ of the mechanism Γ is such that $\alpha[\sigma] = f$. Since SCF f is deterministic, $\alpha[\sigma] = f$ is equivalent to requiring that all senates $S \in \mathcal{N}(P)$ implement the same alternative $\alpha[\sigma](\theta) = f(\theta)$ in state θ. (ii) There exists a Bayesian Nash equilibrium σ such that $\alpha[\sigma](\theta) = f(\theta), \forall \theta$.

3 Necessary Conditions

We now present the necessary conditions for p-implementation.

3.1 Cannot ignore any group

Since f is responsive to all groups, it is easy to see that if f is p-implementable, then all senates that are selected with a positive probability must have at least one senator from each group. We note this necessary condition in the following observation:

Observation. *If f is p-implementable, then p does not ignore any group, i.e.,* $|S \cap N_x| \geq 1, \forall S \in \mathcal{N}(p)$ *and* $x \in X$.

3.2 Bayesian p-monotonicity

A *coordinated deception for group x* is a function $\beta_x : \Theta_x \to \Theta_x$. A coordinated deception profile is $\beta = (\beta_x)_{x \in X}$. For any allocation α and coordinated deception profile β, let $\alpha \circ \beta \in \mathcal{A}$ be such that $\alpha \circ \beta(\theta) = \alpha\big((\beta_x(\theta_x))_{x \in X}\big), \forall \theta$.

Let $h : \mathcal{N}(p) \times \Theta \to A$ be any function that maps each senate in $\mathcal{N}(p)$ and state to an alternative. Given function h, define the allocation $\alpha[p, h]$ such that $\alpha[p, h](\theta)$ is the lottery that selects each $a \in A$ with probability $\sum_{S \in \mathcal{N}(p): h(S,\theta)=a} p(S)$.

For each individual i and SCF f, let

$$\mathcal{H}_i(p, f) = \{h : \mathcal{N}(p) \times \Theta \to A : h(S, \theta) = f(\theta), \forall S \notin \mathcal{N}(i, p) \text{ and } \theta \in \Theta\}.$$

To understand $\mathcal{H}_i(p, f)$, suppose f is p-implementable by some mechanism. If individual i unilaterally deviates from her equilibrium strategy, then she will be unable to change the outcome in all those senates S in which i is not selected. Thus the resulting allocation in each senate after individual i's deviation will be given by some function in $\mathcal{H}_i(p, f)$.

Definition 3.1. SCF f satisfies *Bayesian p-monotonicity* if for any coordinated deception profile β, whenever $f \circ \beta \neq f$, there exist $i \in N$, $\theta_{\omega(i)} \in \Theta_{\omega(i)}$ and $h \in \mathcal{H}_i(p, f)$ such that

$$U_i(\alpha[p, h] \circ \beta | \theta_{\omega(i)}) > U_i(f \circ \beta | \theta_{\omega(i)}) \text{ and } U_i(f | \beta_{\omega(i)}(\theta_{\omega(i)})) \geq U_i(\alpha[p, h] | \beta_{\omega(i)}(\theta_{\omega(i)})). \quad (1)$$

To understand Bayesian p-monotonicity, consider any Bayesian Nash equilibrium σ of the mechanism that p-implements f. If instead of playing their equilibrium strategies, individuals coordinate on a deception profile β to play $\sigma \circ \beta$ but the resulting allocation $f \circ \beta$ is different from f, then $\sigma \circ \beta$ cannot be a Bayesian Nash equilibrium of the mechanism. Hence some type $\theta_{\omega(i)}$ of some individual i must prefer to deviate from $\sigma \circ \beta$ by announcing a message m_i that is different from $\sigma_i(\beta_{\omega(i)}(\theta_{\omega(i)}))$. But the same deviation, i.e., announcing m_i instead of $\sigma_i(\beta_{\omega(i)}(\theta_{\omega(i)}))$, is available to type $\beta_{\omega(i)}(\theta_{\omega(i)})$ in equilibrium σ, and hence should not be improving for type $\beta_{\omega(i)}(\theta_{\omega(i)})$ in equilibrium. So we have two requirements on individual i's preferences, one off-the-equilibrium at $\sigma \circ \beta$ and the other on-the-equilibrium at σ. The two inequalities in (1) formally state these two requirements. We thus have the following result.

Proposition 3.2. *If f is p-implementable, then f is Bayesian p-monotonic.*

Recall that direct democracy p_D is the kleroterion in which referendum occurs with probability 1. Bayesian p_D-monotonicity is equivalent to the Bayesian monotonicity condition of Palfrey and Srivastava (1987) when the latter is applied to our setting. [13] It is thus worth

[13] Note that $\mathcal{H}_i(p_D, f)$ is equal to the set of all SCFs since $\mathcal{N}(p_D) = \{N\}$. Thus if f is Bayesian p_D-

comparing Bayesian p-monotonicity for an arbitrary p with Bayesian p_D-monotonicity.

On the one hand, if p selects a strict subset $S \subset N$ with probability 1, then Bayesian p-monotonicity is more demanding than Bayesian p_D-monotonicity. This is because for each deception that undermines the SCF, we must find an individual in S – rather than in N – whose preferences satisfy (1). On the other hand, if p is such that each individual is selected as a senator with a positive probability, then Bayesian p-monotonicity is less demanding than Bayesian p_D-monotonicity. Intuitively, individual deviations under direct democracy p_D generate deterministic allocations since the outcome function g^N is deterministic. Hence, if a deception undermines the SCF, then under p_D we search for the preference change given in (1) over the space of deterministic allocations. In contrast, if p selects individual i in more than one senate, then deviations by individual i can generate a random allocation. Thus when p selects each individual with a positive probability, then our search to satisfy (1) is not necessarily restricted to the set of deterministic allocations.[14]

3.3 p-Incentive compatibility

For each x and $k \geq 1$, let $\mathcal{S}_x^k$ denote the set of senates $S \in \mathcal{N}(p)$ such that there are exactly k individuals from group x in senate S.

For each individual i and $k \geq 1$, let p_i^k denote the probability of selecting a senate that includes individual i and $k-1$ other individuals from individual i's group, i.e., a senate in $\mathcal{S}^k_{\omega(i)} \cap \mathcal{N}(i,p)$. Let p_i^0 be the probability of selecting a senate that does not include individual i.

For any θ_x, let f_{θ_x} be the SCF such that $f_{\theta_x}(\theta') = f(\theta_x, \theta'_{-x})$ for all $\theta' \in \Theta$.

Definition 3.3. An SCF f satisfies *p-incentive compatibility* if

(i) for all i, there exists an allocation γ_i, and

(ii) for all $x \in X$, distinct $i, i' \in N_x$, distinct $\theta_x^i, \theta_x^{i'} \in \Theta_x$, and $S \in \mathcal{S}_x^2$ such that $i, i' \in S$, there exists a function $\lambda[S, \{\theta_x^i, \theta_x^{i'}\}] : \Theta_{-x} \to A$

monotonic, then for any deception β such that $f \circ \beta \neq f$, there exists $i \in N$, $\theta_{\omega(i)} \in \Theta_{\omega(i)}$ and an SCF $\hat{f}$ such that $U_i(\hat{f} \circ \beta|\theta_{\omega(i)}) > U_i(f \circ \beta|\theta_{\omega(i)})$ and $U_i(f|\beta_{\omega(i)}(\theta_{\omega(i)})) \geq U_i(\hat{f}|\beta_{\omega(i)}(\theta_{\omega(i)}))$. This is exactly the Bayesian monotonicity condition of Palfrey and Srivastava (1987) applied to our setting. The Bayesian monotonicity condition of Palfrey and Srivastava (1987) is weaker than the Bayesian monotonicity condition of Jackson (1991).

[14]To see this formally, suppose f is Bayesian p_D-monotonic and deception β is such that $f \circ \beta \neq f$. Using the equivalence established in Footnote 13, we obtain that there exists $i \in N$, $\theta_{\omega(i)} \in \Theta_{\omega(i)}$ and an SCF $\hat{f}$ such that $U_i(\hat{f} \circ \beta|\theta_{\omega(i)}) > U_i(f \circ \beta|\theta_{\omega(i)})$ and $U_i(f|\beta_{\omega(i)}(\theta_{\omega(i)})) \geq U_i(\hat{f}|\beta_{\omega(i)}(\theta_{\omega(i)}))$. If p selects individual i with a positive probability, then (1) is satisfied for $h \in \mathcal{H}_i(p,f)$ such that $h(S,\cdot) = \hat{f}(\cdot), \forall S \in \mathcal{N}(i,p)$. Hence, f is also Bayesian p-monotonic for all p that select each individual with a positive probability.

such that for all individuals j and $\theta_{\omega(j)}$, we have[15]

$$U_j(f|\theta_{\omega(j)}) \geq \frac{p_j^1}{1-p_j^0} U_j(f_{\theta^j_{\omega(j)}}|\theta_{\omega(j)}) + \sum_{S \in S^2_{\omega(j)} \cap \mathcal{N}(j,p)} U_j(\lambda[S, \{\theta^j_{\omega(j)}, \theta_{\omega(j)}\}]|\theta_{\omega(j)}) \frac{p(S)}{1-p_j^0}$$
$$+ \frac{1-p_j^0-p_j^1-p_j^2}{1-p_j^0} U_j(\gamma_j|\theta_{\omega(j)}), \forall \theta^j_{\omega(j)} \neq \theta_{\omega(j)}. \quad (2)$$

Although the above condition looks complicated, the idea behind it is quite simple. p-Incentive compatibility is necessary to ensure that types do not imitate each other in equilibrium. To see this, consider any Bayesian Nash equilibrium σ of the mechanism that p-implements f. If type $\theta_{\omega(j)}$ of individual j imitates type $\theta^j_{\omega(j)}$ of individual j by announcing $\sigma_j(\theta^j_{\omega(j)})$ instead of $\sigma_j(\theta_{\omega(j)})$, then the outcome under any senate S depends on whether j is in S or not; and if j is in S, then the outcome depends on how many other senators in S are from j's group.

If j is not in S, then the outcome under S is still $f(\theta)$ for all θ.

If j is the only senator in S from her group, then the planner cannot distinguish between this deviation by type $\theta_{\omega(j)}$ and the equilibrium message of type $\theta^j_{\omega(i)}$. As senators from other groups are following their equilibrium strategies, their messages correspond to some profile $\theta_{-\omega(j)}$. Hence, the outcome under S will be $f(\theta^j_{\omega(j)}, \theta_{-\omega(j)})$.

If j is one of the two senators in S from her group, then the planner receives contradictory messages from these two senators. Individual j announces the equilibrium message of type $\theta^j_{\omega(j)}$ while the other senator from j's group – who is of the same type as j, i.e., $\theta_{\omega(j)}$ – announces the equilibrium message of her type $\theta_{\omega(j)}$. Hence, the planner cannot identify the deviator. As senators from other groups are following their equilibrium strategies, their messages correspond to some profile $\theta_{-\omega(j)}$. Let $\lambda[S, \{\theta^j_{\omega(j)}, \theta_{\omega(j)}\}](\theta_{-\omega(j)})$ be the outcome under S in this case.

If j is one of the three or more senators in S from her group, then the planner can identify that j has deviated because all other senators from j's group continue to announce their equilibrium messages that correspond to their types being $\theta_{\omega(j)}$. As senators from other groups are following their equilibrium strategies, their messages correspond to some profile $\theta_{-\omega(j)}$. Among all the alternatives that the planner implements when faced with such a situation, pick the worst for type $\theta_{\omega(j)}$ of individual j, which defines $\gamma_j(\theta_{\omega(j)}, \theta_{-\omega(j)})$.

[15]The function $\lambda[S, \{\theta^i_x, \theta^{i'}_x\}]$ depends on the set $\{\theta^i_x, \theta^{i'}_x\}$; so the order of θ^i_x and $\theta^{i'}_x$ does not matter. Also, as $\lambda[S, \{\theta^j_{\omega(j)}, \theta_{\omega(j)}\}]$ is not an allocation, here we abuse our notation slightly by letting $U_j(\lambda[S, \{\theta^j_{\omega(j)}, \theta_{\omega(j)}\}]|\theta_{\omega(j)}) = \sum_{\theta_{-\omega(j)} \in \Theta_{-\omega(j)}} \pi_j(\theta_{-\omega(j)}|\theta_{\omega(j)}) u_j\big(\lambda[S, \{\theta^j_{\omega(j)}, \theta_{\omega(j)}\}](\theta_{-\omega(j)}), (\theta_{\omega(j)}, \theta_{-\omega(j)})\big)$.

Now it is easy to see that if type $\theta_{\omega(j)}$ of individual j does not gain by imitating type $\theta^j_{\omega(j)}$, then (2) must be satisfied. Hence, we have the following result.

Proposition 3.4. *If f is p-implementable, then f satisfies p-incentive compatibility.*

If p always selects three or more individuals from each group – so each group must have at least three individuals – , then every SCF satisfies p-incentive compatibility. Indeed, in this case, we can trivially satisfy (2) by setting $\gamma_i(\theta) = f(\theta)$ for all θ. If p always selects two or more individuals from each group, then all SCFs will satisfy p-incentive compatibility whenever each group x has a *state-independent common worse alternative* $\bar{a}_x$, i.e., $u_i(\bar{a}_x, \theta) \leq u_i(f(\theta), \theta)$ for all θ and $i \in N_x$. In this case, we can satisfy (2) by setting $\lambda[S, \{\theta^i_x, \theta^{i'}_x\}](\theta_{-x}) = \bar{a}_x$ for all θ_{-x}, and $\gamma_i(\theta) = f(\theta)$ for all θ. However, if p puts positive probability on a senate that comprises of exactly one individual from some group, then p-incentive compatibility could restrict the set of p-implementable SCFs even when all groups have state-independent common worst alternatives. Take for instance the extreme case when p always selects exactly one individual from each group. In this case, p-incentive compatibility is equivalent to requiring truth-telling to be a Bayesian Nash equilibrium in the direct mechanism $((M_i)_{i\in N}, p, (g^S)_{S\in\mathcal{N}(p)})$ where $M_i = \Theta_{\omega(i)}$ for all i and $g^S = f$ for all S.

We now compare p_D-incentive compatibility with p-incentive compatibility for an arbitrary p. Above, we identified two cases in which all SCFs satisfy p_D-incentive compatibility:

Case (i): Each group has at least three individuals. In this case, all SCFs also satisfy p-incentive compatibility if p always selects three or more individuals from each group. However, if p puts a positive probability on a senate with at most two senators from some group, then as the following example shows, there are some SCFs that do not satisfy p-incentive compatibility.

Example 3.5. There are six individuals $N = \{i_1, i_2, \cdots, i_6\}$ and two groups, x_1 and x_2. Group x_1 consists of the first three individuals while x_2 consists of the remaining three individuals. The set of alternatives is $A = \{a, b, c\}$. Furthermore, $\Theta_{x_1} = \{\theta_{x_1}\}$ and $\Theta_{x_2} = \{\theta_{x_2}, \theta'_{x_2}, \theta''_{x_2}\}$. The SCF f is such that $f(\theta_{x_1}, \theta_{x_2}) = a$, $f(\theta_{x_1}, \theta'_{x_2}) = b$, $f(\theta_{x_1}, \theta''_{x_2}) = c$.

Suppose the Bernoulli utility of individual i_4 is such that a is the worst alternative for i_4 in state $(\theta_{x_1}, \theta_{x_2})$, i.e., $u_{i_4}(a, (\theta_{x_1}, \theta_{x_2})) < \min\{u_{i_4}(b, (\theta_{x_1}, \theta_{x_2})), u_{i_4}(c, (\theta_{x_1}, \theta_{x_2}))\}$. Similarly, suppose that b is the worst alternative for i_5 in state $(\theta_{x_1}, \theta'_{x_2})$ whereas c is the worst alternative for i_6 in state $(\theta_{x_1}, \theta''_{x_2})$.

We claim that if the kleroterion p puts a positive probability on a senate with one or two senators from group x_2, then f is not p-incentive compatible. Suppose otherwise, and let S be a senate that is selected with a positive probability under p which includes one or two individuals from x_2. First, observe that (2) is not satisfied if S has only one senator from x_2. For instance, if S includes only i_4 from group x_2, then as f prescribes the worst alternative for i_4 in state $(\theta_{x_1}, \theta_{x_2})$, type θ_{x_2} of individual i_4 has an incentive to lie and claim θ'_{x_2} or θ''_{x_2}. Consequently, S must have two individuals from x_2. Suppose S includes only $\{i_4, i_5\}$ from x_2. Then because a is the worst alternative for i_4 in state $(\theta_{x_1}, \theta_{x_2})$, to satisfy (2) for type θ_{x_2} of i_4, we must have $\lambda[S, \{\theta^{i_4}_{x_2}, \theta^{i_5}_{x_2}\}](\theta_{x_1}) = a$ when $\theta^{i_4}_{x_2} = \theta'_{x_2}$ and $\theta^{i_5}_{x_2} = \theta_{x_2}$. At the same time, because b is the worst alternative for i_5 in state $(\theta_{x_1}, \theta'_{x_2})$, to satisfy (2) for type θ'_{x_2} of i_5, it must be that $\lambda[S, \{\theta^{i_4}_{x_2}, \theta^{i_5}_{x_2}\}](\theta_{x_1}) = b$ when $\theta^{i_4}_{x_2} = \theta'_{x_2}$ and $\theta^{i_5}_{x_2} = \theta_{x_2}$; a contradiction. Consequently, $S \cap N_{x_2} \neq \{i_4, i_5\}$. Similar arguments show that $S \cap N_{x_2} \neq \{i_4, i_6\}$ and $S \cap N_{x_2} \neq \{i_5, i_6\}$. Thus, f is not p-incentive compatible. $\diamond$

Case (ii): Each group has a state-independent common worse alternative. In this case, all SCFs also satisfy p-incentive compatibility if p always selects two or more individuals from each group. However, if p puts a positive probability on a senate with exactly one senator from some group, then there might be SCFs that do not satisfy p-incentive compatibility.

Thus, in either case, p_D-incentive compatibility is (weakly) more permissive than p-incentive compatibility for those p that always select two or more senators from each group. However, this relation can be reversed if the environment falls outside of the above two cases; specifically, when some group has only two individuals but does not have a state-independent common worse alternative. We next provide an example of such a reversal.

Example 3.6. There are six individuals $N = \{i_1, i_2, \cdots, i_6\}$ and two groups, x_1 and x_2. Group x_1 consists of the first four individuals while x_2 consists of the remaining two individuals. The set of alternatives is $A = \{a, b, c, d\}$. Furthermore, $\Theta_{x_1} = \{\theta_{x_1}\}$ and $\Theta_{x_2} = \{\theta_{x_2}, \theta'_{x_2}\}$. The SCF f is such that $f(\theta_{x_1}, \theta_{x_2}) = a$ and $f(\theta_{x_1}, \theta'_{x_2}) = b$.

The Bernoulli utilities of individuals i_5 and i_6 are given as follows:

$$
\begin{aligned}
u_{i_5}(a, (\theta_{x_1}, \theta_{x_2})) &= 3, & u_{i_6}(a, (\theta_{x_1}, \theta'_{x_2})) &= 5\\
u_{i_5}(b, (\theta_{x_1}, \theta_{x_2})) &= 5, & u_{i_6}(b, (\theta_{x_1}, \theta'_{x_2})) &= 3\\
u_{i_5}(c, (\theta_{x_1}, \theta_{x_2})) &= 4, & u_{i_6}(c, (\theta_{x_1}, \theta'_{x_2})) &= 1\\
u_{i_5}(d, (\theta_{x_1}, \theta_{x_2})) &= 1, & u_{i_6}(d, (\theta_{x_1}, \theta'_{x_2})) &= 4
\end{aligned}
$$

If the Bernoulli utilities are not specified above, then they are 0.

We claim that f is not p_D-incentive compatible. Observe that if (2) is satisfied for type θ_{x_2} of agent i_5, then we need $\lambda[N, \{\theta^{i_5}_{x_2}, \theta^{i_6}_{x_2}\}](\theta_{x_1})$, where $\theta^{i_5}_{x_2} = \theta'_{x_2}$ and $\theta^{i_6}_{x_2} = \theta_{x_2}$, to be either a or d. Then (2) is not satisfied for type θ'_{x_2} of agent i_6.

Consider the kleroterion p which selects each of $S_1 = \{i_1, i_2, i_3, i_5, i_6\}$ and $S_2 = \{i_1, i_2, i_4, i_5, i_6\}$ with probability 0.5. For $\theta_{x_2}^{i_5} = \theta'_{x_2}$ and $\theta_{x_2}^{i_6} = \theta_{x_2}$, define $\lambda[S_1, \{\theta_{x_2}^{i_5}, \theta_{x_2}^{i_6}\}](\theta_{x_1}) = c$ and $\lambda[S_2, \{\theta_{x_2}^{i_5}, \theta_{x_2}^{i_6}\}](\theta_{x_1}) = d$. Then it is easy to see that (2) is satisfied. Hence, f is p-incentive compatible. ◇

We conclude from the above discussion and examples that p-incentive compatibility for those p that always select two or more senators from each group is not necessarily more demanding than p_D-incentive compatibility.

Finally, we point out that some SCFs do not satisfy p-incentive compatibility for any p. For this to happen, however, it must be that there is a group with only two individuals and this group does not have a state-independent common worse alternative.

Example 3.7. Here we modify Example 3.6 by altering the Bernoulli utilities of individuals i_5 and i_6 only in the following cases:

$$u_{i_5}(d, (\theta_{x_1}, \theta_{x_2})) = 2.5, \qquad\qquad u_{i_6}(c, (\theta_{x_1}, \theta'_{x_2})) = 2.5.$$

We claim that if p does not ignore group x_2, then f violates p-incentive compatibility. Suppose not, i.e., p does not ignore group x_2 but f is p-incentive compatible. For each alternative $e \in A$, we define $\mathcal{S}_{x_2}^2(e) = \{S : \{i_5, i_6\} \subset S \,\&\, \lambda[S, \{\theta_{x_2}^{i_5}, \theta_{x_2}^{i_6}\}](\theta_{x_1}) = e\}$ where $\theta_{x_2}^{i_5} = \theta'_{x_2}$ and $\theta_{x_2}^{i_6} = \theta_{x_2}$. Set $p^2(e) = \sum_{S \in \mathcal{S}^2(e)} p(S)$. Because p does not ignore group x_2, it must be that $p_{i_5}^0 = p_{i_6}^1$ and $p_{i_6}^0 = p_{i_5}^1$. Consequently, (2) for type θ_{x_2} of i_5 and for type θ'_{x_2} of i_6 are respectively equivalent to the following:

$$3 \geq \frac{1}{1 - p_{i_6}^1} \left(p_{i_5}^1 5 + p^2(a)3 + p^2(b)5 + p^2(c)4 + p^2(d)2.5 \right)$$
$$3 \geq \frac{1}{1 - p_{i_5}^1} \left(p_{i_6}^1 5 + p^2(a)5 + p^2(b)3 + p^2(c)2.5 + p^2(d)4 \right).$$

Now multiply the first inequality by $1 - p_{i_6}^1$ and the second inequality by $1 - p_{i_5}^1$. Then add the resulting two inequalities to obtain

$$6 \geq 8p_{i_5}^1 + 8p_{i_6}^1 + 8p^2(a) + 8p^2(b) + 6.5p^2(c) + 6.5p^2(d).$$

Clearly, the above inequality cannot be true because $p_{i_5}^1 + p_{i_6}^1 + p^2(a) + p^2(b) + p^2(c) + p^2(d) = 1$.

4 Sufficient Conditions

We now present sufficient conditions for p-implementation.

Definition 4.1. For each p and x, define an *undirected graph* $G(p, x)$ such that each individual in N_x is a vertex of the graph and there exists an edge between two distinct vertices i and j if and only if there exists a senate $S \in \mathcal{N}(p)$ such that $i, j \in S$.

The graph $G(p, x)$ is *connected* if there exists an undirected path from any vertex to any other distinct vertex of the graph.

If $G(p, x)$ is connected, then that means that for each pair of distinct individuals i and j in group x, we can find a sequence of individuals $(i_1, \ldots, i_K)$ starting with $i_1 = i$ and ending with $i_K = j$ such that each adjacent pair (i_k, i_{k+1}) in the sequence is selected together as senators with a positive probability. In contrast, if the graph $G(p, x)$ is not connected, then the connected components of $G(p, x)$ define a partition of N_x such that p never selects a senate that includes individuals from two distinct members of this partition. We now provide examples of kleroteria that generate either connected or disconnected graphs.

Example 4.2 (*Kleroteria generating disconnected graphs*)**.** Consider the following examples:

Oligrachy: By oligarchy we mean a kleroterion that selects a strict subset $S \subset N$ with probability 1. Since S is a strict subset of N, there exists at least one group x such that $N_x \setminus S \neq \emptyset$. Each individual in $N_x \setminus S$ is isolated, i.e., not connected to any other individual in group x. All individuals in $S \cap N_x$ are connected to each other but there are no edges between individuals in $S \cap N_X$ and $N_x \setminus S$.

Randomly sampling one individual from each group: If the kleroterion randomly selects exactly one individual from each group, then $G(p, x)$ has no edges for all groups x. ⋄

Example 4.3 (*Kleroteria generating connected graphs*)**.** Consider the following examples:

Direct democracy: As the direct democracy p_D leads to a referendum (i.e., the event in which N is selected) with probability 1, the graph $G(p, x)$ is a complete graph (i.e., there is an edge connecting each pair of distinct vertices) for all x.

Oligrachic democracy: By oligarchic democracy we mean a kleroterion that selects a strict subset $S \subset N$ with probability $1 - \epsilon > 0$ and leads to a referendum with probability $\epsilon > 0$. Here again, $G(p, x)$ is a complete graph for all x.

Randomly sampling two or more individuals from each group: If the kleroterion randomly selects two or more individuals from each group, then again $G(p, x)$ is a complete graph for all x.

Group leader and one other randomly chosen individual from each group: Suppose the kleroterion selects two individuals from each group as follows: One of the two individuals – the group leader – is fixed while the other is chosen randomly. In this case, the graph $G(p, x)$ is a star graph for all x. Thus the group leader is connected by an edge to each of the remaining individuals in the group, who in turn are connected only to the group leader.

Neighbors on a line: Suppose individuals in each group are located at distinct points on a line and the kleroterion randomly selects two neighbors (i.e., individuals who are adjacent to each other) from each group. In this case, the graph $G(p, x)$ is a linear graph for all x in which each individual is connected to her two neighbors. $\diamond$

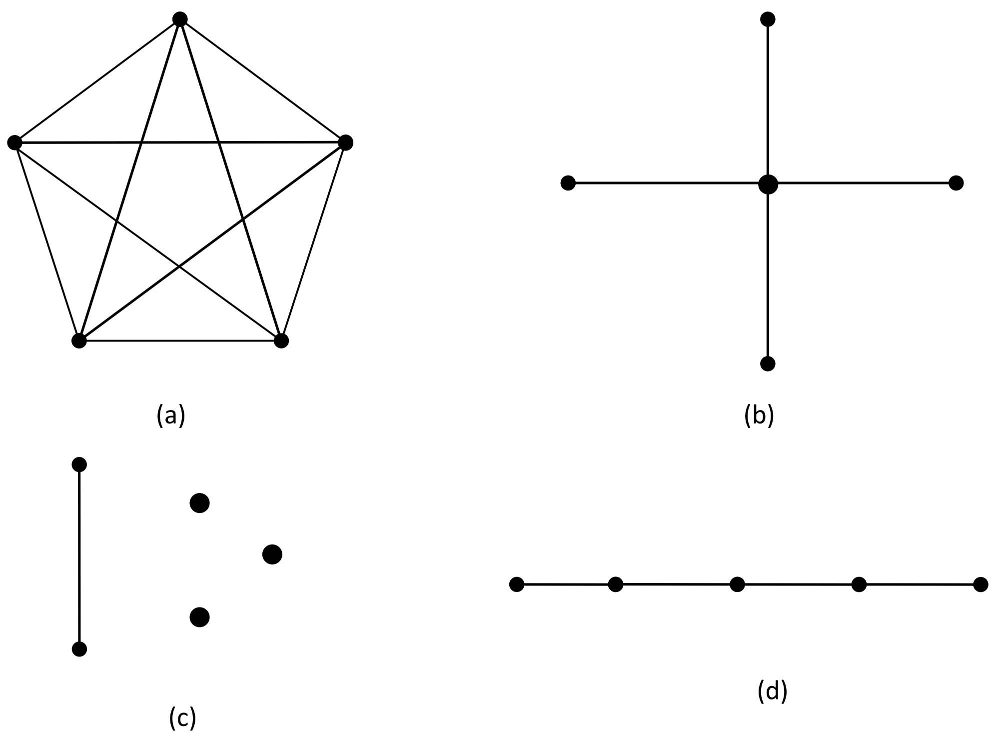

Figure 1: Examples of graph $G(p, x)$ when group x has five members. The complete graph in (a) is generated by direct democracy, oligarchic democracy, and randomly sampling two or more individuals from each group. The star graph in (b) is generated by selecting the group leader and one other randomly chosen individual from each group. The disconnected graph in (c) is generated by oligarchy with two oligarchs from group x. Finally, the linear graph in (d) is generated by randomly selecting neighbors on a line.

Kleroteria that generate connected graphs are significant for our sufficiency result which is presented next.

Proposition 4.4. *Suppose the environment is economic, there are three or more groups, and p satisfies the following two conditions:*

(i) p always selects two or more individuals from each group, i.e., $|S \cap N_x| \geq 2, \forall S \in \mathcal{N}(p)$ and $x \in X$.

(ii) The graph $G(p, x)$ is connected for all $x \in X$.

Then any SCF f that satisfies p-incentive compatibility and Bayesian p-monotonicity is p-implementable.

The mechanism that we construct to prove the above result is similar to the usual canonical mechanisms used in the implementation literature. However, there are some novel elements. As usual, each individual is asked to report her type in addition to other components. The planner knows that all individuals belonging to the same group have the same type. Thus if the planner encounters a senate in which for all groups, there is agreement in the type reports of the senators belonging that group, then the planner implements the outcome according to the SCF f. However, when the planner encounters a senate that consists of only two senators i' and j' from group x' who submit different type reports $\theta_{x'}^{i'} \neq \theta_{x'}^{j'}$ while there is agreement in the type reports of the senators from other groups, then the planner implements the outcome according to $\lambda[S, \{\theta_{x'}^{i'}, \theta_{x'}^{j'}\}]$. Finally, if the planner encounters a senate that consists of three or more senators from group x' and the type report of exactly one senator i' from group x' differs from the common type report $\theta_{x'}$ of other senators from group x' while there is agreement in the type reports of the senators from other groups, then the planner implements the outcome according to $\gamma_{i'}(\theta_{x'}, \cdot)$. As f is p-incentive compatible, this construction ensures that it is an equilibrium for all individuals to submit truthful type reports.

There are two problems that the planner might face with the above construction.

First, it might be that there is agreement in the type reports when either senate S or S' is selected but the reported state in S differs from the reported state in S'. As the planner only looks at the messages of the selected senators, there is no way for the planner to find out this discrepancy between the reported state in S and S'. The assumption that $G(p, x)$ is connected for all x helps the planner tackle this problem. If the reported state in S differs from the reported state in S', then under the assumption that $G(p, x)$ is connected for all x, there will surely be some senate S'' in which the type reports of the senators do not agree. By using the standard integer game construction, the planner can ensure that this situation is not an equilibrium in economic environments.

Second, it might be that all individuals play according to a coordinated deception profile β that generates the allocation $f \circ \beta \neq f$. To tackle this case, each individual i is also allowed to announce a function $\mathbf{y}_i : \mathcal{N}(i, p) \times \Theta \to A$. Whenever the type reports of all senators in the selected S agree with state θ but i is the only senator in S who announces $\mathbf{y}_i$ such that $\mathbf{y}_i(S', \theta') \neq f(\theta')$ for some senate S' containing i and state θ', then the planner (a) assumes that individual i is of type $\theta_{\omega(i)}$, which is the commonly reported type for group $\omega(i)$ in senate S, and (b) implements the alternative $\mathbf{y}_i(S, \theta)$ recommended by individual i if and only if following i's recommendation $\mathbf{y}_i(S', \theta')$ instead of $f(\theta')$ in all senates S' containing i and states θ' does not make type $\theta_{\omega(i)}$ of individual i better off. This rule, combined with Bayesian p-monotonicity, ensures that any coordinated deception profile β such that $f \circ \beta \neq f$ is not an equilibrium.

Remark 4.5. When there are only two groups, then we can obtain the result in Proposition 4.4 if we make the stronger assumption that either p always selects three or more individuals from each group or the environment has within groups differences (see Footnote 22).[16]

It follows from Proposition 4.4 that when there are three or more groups, implementation by direct democracy, i.e., p_D-implementation can be achieved for all SCFs that satisfy p_D-incentive compatibility and Bayesian p_D-monotonicity. Now consider any kleroterion p that always selects two or more senators from each group such that $G(p, x)$ is connected for all groups x.

As $G(p, x)$ is connected for all x, the kleroterion p is such that each individual is selected as a senator with a positive probability. Under this condition, as discussed in Section 3.2, Bayesian p-monotonicity is weaker than Bayesian p_D-monotonicity.

We argued in Section 3.3 that if p selects two or more senators from each group, then p-incentive compatibility is neither necessarily weaker nor necessarily stronger than p_D-incentive compatibility. However, we identified two cases in which all SCFs satisfy p_D-incentive compatibility: Case (i) Each group has at least three individuals and Case (ii) Each group has a state-independent common worse alternative. In Case (i), all SCFs also

[16] If there is only one group, then the environment will be one of complete information. In this case, Bayesian p-monotonicity is weaker than Maskin monotonicity (Maskin, 1999) as the former allows the planner to search for "preference reversals" over the space of lotteries. Also, recall that p-incentive compatibility is trivially satisfied when p always selects three or more individuals from each group. Finally, if there is only one group, then our economic environment assumption is equivalent to within groups differences. Thus when there is only one group x, then using similar arguments as in the proof of Proposition 4.4, we can show the following result: If the environment is economic, p always selects three or more individuals, and $G(p, x)$ is connected, then any Maskin monotonic SCF is p-implementable. This result is not comparable to the one in Saran and Tumennasan (2013) where we study the complete information case. The main result of Saran and Tumennasan (2013) is that if the environment is "economic", p always selects three or more individuals and selects every quartet of individuals with positive probability, then any Maskin monotonic SCF is p-implementable. On the one hand, the economic environment assumption in this paper is stronger than the one in Saran and Tumennasan (2013). On the other hand, the assumption that $G(p, x)$ is connected is weaker than the assumption that p selects every quartet of individuals with positive probability. Thus we now have a new result for p-implementation in complete information environments.

satisfy p-incentive compatibility for any p that always selects *three or more* individuals from each group whereas in Case (ii), all SCFs also satisfy p-incentive compatibility for any p that always selects *two or more* individuals from each group.

The above arguments thus imply the following corollaries:

Corollary 4.6. *Suppose the environment is economic, there are three or more groups, and each group has three or more individuals. If f is p_D-implementable, then f is p-implementable by any p that always selects three or more individuals from each group such that $G(p, x)$ is connected for all x.*

Thus, under the conditions of the above corollary, any SCF that is implementable by direct democracy is also implementable by any of the following kleroteria: (a) Oligarchic democracy which has three oligarchs from each group – note that ϵ can be arbitrarily small – and (b) randomly sampling three individuals from each group.

Corollary 4.7. *Suppose the environment is economic, there are three or more groups, and each group has a state-independent common worse alternative. If f is p_D-implementable, then f is p-implementable by any p that always selects two or more individuals from each group such that $G(p, x)$ is connected for all x.*

Thus, under the conditions of the above corollary, any SCF that is implementable by direct democracy is also implementable by any of the following kleroteria: (a) Oligarchic democracy which has two oligarchs from each group – again, ϵ can be arbitrarily small –, (b) randomly sampling two individuals from each group, (c) selecting the group leader and one other randomly chosen individual from each group, and (d) randomly selecting neighbors on a line.

Remark 4.8. We have analyzed the question of what SCFs can be implemented once the communication protocol or kleroterion is fixed. In contrast, if the planner can choose the kleroterion, then an SCF will be implementable if and only if it is p-implementable for some kleroterion p. Hence, the SCFs that can be implemented under the new notion must satisfy Bayesian p-monotonicity and p-incentive compatibility for some kleroterion p. Moreover, a sufficient condition for this implementation notion is the existence of a kleroterion p satisfying the conditions specified in Theorem 4.4. Thus both necessary and sufficient conditions become more permissive if the planner has freedom to choose any kleroterion.

5 Quasilinear environments

We now focus on quasilinear environments, which are a special case of economic environments. Unlike the canonical mechanism used to prove Proposition 4.4, here we present a simple mechanism that does not rely on integer games and yet implements the SCF when individuals' beliefs satisfy a generic condition called no consistent coordinated deception and the kleroterion selects the group leader and one other randomly chosen individual from each group.

Specifically, now the set of alternatives $A = Y \times \Re^N$, where y is the public decision and $t \in \Re^N$ is a profile of individual monetary transfers. Individuals' utilities are quasilinear in money, $u_i((y,t),\theta) = v_i(y,\theta) - t_i$.

Consider any coordinated deception profile β and type θ_x of individual i from group x. Define the probability distribution $\pi_i^{\beta_{-x}}(\cdot|\theta_x)$ as follows:

$$\pi_i^{\beta_{-x}}(\theta_{-x}|\theta_x) = \sum_{\theta'_{-x}:\beta_{-x}(\theta'_{-x})=\theta_{-x}} \pi_i(\theta'_{-x}|\theta_x), \forall \theta_{-x} \in \Theta_{-x}.$$

Thus $\pi_i^{\beta_{-x}}(\theta_{-x}|\theta_x)$ is the probability assigned by type θ_x of individual i to observing the report θ_{-x} when individuals in other groups coordinate on deceptions β_{-x}. In contrast, the probability distribution $\pi_i(\cdot|\beta_x(\theta_x))$ specifies for each $\theta_{-x} \in \Theta_{-x}$ the probability $\pi_i(\theta_{-x}|\beta_x(\theta_x))$ assigned by type $\beta_x(\theta_x)$ of individual i to observing the report θ_{-x} when individuals in other groups are truthful.

Definition 5.1. A coordinated deception profile β is *consistent* if $\beta(\hat{\theta}) \neq \hat{\theta}$ for some $\hat{\theta}$ and the two probability distributions $\pi_i^{\beta_{-\omega(i)}}(\cdot|\theta_{\omega(i)})$ and $\pi_i(\cdot|\beta_{\omega(i)}(\theta_{\omega(i)}))$ are equal to each other for all types $\theta_{\omega(i)}$ of all individuals i

Individuals' beliefs satisfy *no consistent coordinated deception* if there does not exist a coordinated deception profile that is consistent. This means that if β is such that $\beta(\hat{\theta}) \neq \hat{\theta}$ for some $\hat{\theta}$ then there must exist an individual i of some type $\theta_{\omega(i)}$ such that the two probability distributions $\pi_i^{\beta_{-\omega(i)}}(\cdot|\theta_{\omega(i)})$ and $\pi_i(\cdot|\beta_{\omega(i)}(\theta_{\omega(i)}))$ are not equal, i.e., for some $\theta_{-\omega(i)}$

$$\pi_i^{\beta_{-\omega(i)}}(\theta_{-\omega(i)}|\theta_{\omega(i)}) \neq \pi_i(\theta_{-\omega(i)}|\beta_{\omega(i)}(\theta_{\omega(i)})).$$

We have assumed that all individual's beliefs have full support, i.e., $\pi_i(\theta_{-\omega(i)}|\theta_{\omega(i)}) > 0$ for all $\theta_{-\omega(i)} \in \Theta_{-\omega(i)}$. Thus if the coordinated deception profile β is not a permutation of Θ (in other words, β is not a bijection), then β is not consistent. To see this, suppose $\hat{\theta}$ is not in the range of β. Then there exists an x such that $\hat{\theta}_{-x}$ is not in the range of β_{-x}. Now any type θ_x of any individual i in group x is such that $\pi_i(\cdot|\beta_x(\theta_x))$ has full support whereas $\pi_i^{\beta_{-x}}(\cdot|\theta_x)$ does not.

Thus the assumption of no consistent coordinated deception imposes a constraint only on the set of coordinated deceptions β that are permutations of Θ. To gain further insight into this condition, consider any i from group x and coordinated deception profile β that is a permutation of Θ and such that $\beta(\hat{\theta}) \neq \hat{\theta}$ for some $\hat{\theta}$. Let Π_i be the matrix of individual i's beliefs with one row for each $\theta_x \in \Theta_x$ and one column for each $\theta_{-x} \in \Theta_{-x}$. The entry in row θ_x and column θ_{-x} equals $\pi_i(\theta_{-x}|\theta_x)$. Now, permute the columns of Π_i by β_{-x}, i.e., switch each column θ_{-x} with column $\beta_{-x}(\theta_{-x})$ to obtain matrix $\Pi_i\beta_{-x}$. Next, permute the rows of $\Pi_i\beta_{-x}$ by β_x^{-1}, i.e., switch each row θ_x with row $\beta_x^{-1}(\theta_x)$ (here, β_x^{-1} is the inverse of β_x) to obtain $\beta_x^{-1}\Pi_i\beta_{-x}$. If the result is such that $\Pi_i \neq \beta_x^{-1}\Pi_i\beta_{-x}$, then β is not consistent. Based on this discussion, we have a simple sufficient condition: If there exists an individual i such that all entries in Π_i are unequal, then there is no consistent coordinated deception. This sufficient condition holds generically over the space of individual's beliefs.

In a quasilinear environment, an SCF is a pair of functions $(\mathbf{y}, \mathbf{t})$, where $\mathbf{y} : \theta \to Y$ and $\mathbf{t} : \theta \to \Re^N$. Given the SCF, for each individual i, fix a sufficiently high monetary fine t_i^* such that

$$t_i^* > \max_{\theta,\theta' \in \Theta} \left(v_i(\mathbf{y}(\theta'), \theta) - \mathbf{t}_i(\theta')\right) - \min_{\theta,\theta' \in \Theta} \left(v_i(\mathbf{y}(\theta'), \theta) - \mathbf{t}_i(\theta')\right).$$

Proposition 5.2. *Suppose we have a quasilinear environment in which individuals' beliefs satisfy no consistent coordinated deception. Then any SCF is p-implementable, where p is defined as follows:*

For each group x, designate any one of the individuals in x as the group leader. Under p, each senate comprises of all the group leaders and one other randomly chosen individual from each group.[17]

We next present the proof of the above proposition as the constructed mechanism and arguments are novel.

Proof. Consider the SCF $(\mathbf{y}, \mathbf{t})$. The mechanism is as follows. Pick any $\delta > 0$. Each individual i in group x reports a parameter value θ_x^i and offers the planner a contingent asset $\tau_x^i : \Theta_{-x} \to [-\delta, \delta]$ which pays $\tau_x^i(\theta_{-x})$ to the planner when i is selected with other individuals who report θ_{-x}.[18] Thus, individual i's message space is $M_i = \Theta_{\omega(i)} \times \mathcal{T}_{\omega(i)}$, where $\mathcal{T}_{\omega(i)}$ is a set of contingent assets $\tau_{\omega(i)} : \Theta_{-\omega(i)} \to [-\delta, \delta]$. Notice that the message space is compact since the returns on contingent assets are bounded by δ. This is significant because best responses always exist in our mechanism, which is in a stark contrast to the canonical mechanisms used in much of the implementation literature.

Pick an $S \in \mathcal{N}(p)$. Let 1_x be the designated leader of group x and $j_x \neq 1_x$ be the randomly chosen individual from group x in senate S. The outcome function g^S is given as

[17] Note that this result holds even when there are two groups.

[18] This offer is not always accepted by the planner, which will become clear when we discuss the rules of the mechanism.

follows (We number the groups from 1 to X. For $x = 1$, let $x - 1$ denote group X whereas for $x = X$, let $x + 1$ denote group 1).

Rule 1: This rule is applied when there is no discrepancy in the reported group-specific parameter values, i.e., if $m_i = (\theta_x, \tau_x^i)$ for all $i \in S \cap N_x$ and all x. In this case, the planner implements the public decision $\mathbf{y}((\theta_x)_{x \in X})$ with transfers equal to $\mathbf{t}_i((\theta_x)_{x \in X}) + \phi_i^S$ for each individual i, where the additional transfer ϕ_i^S is defined as follows.

If $i \notin S$, then $\phi_i^S = 0$.

If $i \in S$, then ϕ_i^S is depends on whether i is her group's leader or not. Specifically, for all $x \in X$:

$$\phi_{1_x}^S = \begin{cases} t_{1_x}^*, & \text{if } \exists i \in S \cap N_{x-1} \text{ such that } \mathbb{E}_{\pi_i(\cdot|\theta_{x-1})} \tau_{x-1}^i > 0 \\ 0, & \text{if } \mathbb{E}_{\pi_i(\cdot|\theta_{x-1})} \tau_{x-1}^i \leq 0, \forall i \in S \cap N_{x-1}, \\ & \text{and } \mathbb{E}_{\pi_{1_x}(\cdot|\theta_x)} \tau_x^{1_x} \leq 0 \\ \tau_x^{1_x}(\theta_{-x}), & \text{if } \mathbb{E}_{\pi_i(\cdot|\theta_{x-1})} \tau_{x-1}^i \leq 0, \forall i \in S \cap N_{x-1}, \\ & \text{and } \mathbb{E}_{\pi_{1_x}(\cdot|\theta_x)} \tau_x^{1_x} > 0. \end{cases}$$

$$\phi_{j_x}^S = \begin{cases} 0, & \text{if } \mathbb{E}_{\pi_{j_x}(\cdot|\theta_x)} \tau_x^{j_x} \leq 0 \\ \tau_x^{j_x}(\theta_{-x}), & \text{if } \mathbb{E}_{\pi_{j_x}(\cdot|\theta_x)} \tau_x^{j_x} > 0. \end{cases}$$

Thus the additional transfers of the senators in S are decided as follows. First, for all $i \in S$, the planner calculates whether she can obtain a positive expected return on the contingent asset offered by senator i, where the expectation is taken with respect to the belief of i conditional on her type being equal to the commonly reported parameter value $\theta_{\omega(i)}$. Next, the planner imposes the monetary fine of $t_{1_x}^*$ on the leader of group x whenever some senator from the adjacent group $x - 1$ offers a contingent asset that pays a positive expected return to the planner. In contrast, if the planner cannot obtain positive expected returns on any of the contingent assets offered by the senators from group $x - 1$, then the additional transfer of the group leader 1_x depends on whether her own contingent asset pays a positive expected return to the planner or not: In the former case, the additional transfer is equal to the promised return $\tau_x^{1_x}(\theta_{-x})$ whereas in the later case, the additional transfer is 0. Finally, the additional transfer of the randomly chosen individual j_x simply depends on whether her own contingent asset pays a positive expected return to the planner or not: In the former case, the additional transfer is equal to the promised return $\tau_x^{j_x}(\theta_{-x})$ whereas in the later case, the additional transfer is 0.

Rule 2: This rule is applied when there exists a discrepancy in the reported group-specific parameter values. For each group x, the planner accepts the parameter values reported by the randomly chosen individual j_x, and hence implements $\mathbf{y}((\theta_x^{j_x})_{x \in X})$ with transfers equal to $\mathbf{t}_i((\theta_x^{j_x})_{x\in X})) + \phi_i^S$ for each individual i, where now ϕ_i^S is defined as follows.

If $i \notin S$, then $\phi_i^S = 0$.

If $i \in S$, then ϕ_i^S is depends on whether i is her group's leader or not. Specifically, for all $x \in X$:

$$\phi_{1_x}^S = \begin{cases} 0, & \text{if } \exists i \in S \cap N_{x-1} \text{ such that } \mathbb{E}_{\pi_i(\cdot|\theta_{x-1}^{j_{x-1}})}\tau_{x-1}^i > 0 \\ 0, & \text{if } \mathbb{E}_{\pi_i(\cdot|\theta_{x-1}^{j_{x-1}})}\tau_{x-1}^i \le 0, \forall i \in S \cap N_{x-1}, \\ & \text{and } \mathbb{E}_{\pi_{1_x}(\cdot|\theta_x^{j_x})}\tau_x^{1_x} \le 0 \\ \tau_x^{1_x}((\theta_{x'}^{j_{x'}})_{x'\neq x}), & \text{if } \mathbb{E}_{\pi_i(\cdot|\theta_{x-1}^{j_{x-1}})}\tau_{x-1}^i \le 0, \forall i \in S \cap N_{x-1}, \\ & \text{and } \mathbb{E}_{\pi_{1_x}(\cdot|\theta_x^{j_x})}\tau_x^{1_x} > 0. \end{cases}$$

$$\phi_{j_x}^S = \begin{cases} t_{j_x}^*, & \text{if } \theta_x^{1_x} \neq \theta_x^{j_x} \\ 0, & \text{if } \theta_x^{1_x} = \theta_x^{j_x}. \end{cases}$$

Thus, compared to Rule 1, there are three changes in the additional transfers under Rule 2. First, to calculate the expected return on the contingent asset offered by senator i, the planner takes the expectation with respect to the belief of i conditional on her type being equal to the parameter value $\theta_{\omega(i)}^{j_{\omega(i)}}$ that is reported by $j_{\omega(i)}$. Second, the leader of group x does not pay the monetary fine whenever some senator from the adjacent group $x-1$ offers a contingent asset that pays a positive expected return to the planner. Third, the additional transfer of the randomly chosen individual j_x now depends on whether there is a discrepancy in the parameter values reported for group x or not: In the former case, j_x has to pay the monetary fine $t_{j_x}^*$ whereas in the later case, the additional transfer is 0.

Step 1: We argue that any strategy profile $\sigma_i(\theta_{\omega(i)}) = (\theta_{\omega(i)}, \tau_{\omega(i)}^i)$ with $\mathbb{E}_{\pi_i(\cdot|\theta_{\omega(i)})}\tau_{\omega(i)}^i \le 0$ is an equilibrium. Pick any group x.

First, consider individual 1_x of type θ_x. Suppose she deviates to $(\hat{\theta}_x, \hat{\tau}_x^{1_x})$. Since all other individuals in group x – who are of the same type as 1_x – continue to report θ_x, the planner will determine that the expected return of $\hat{\tau}_x^{1_x}$ equals $\mathbb{E}_{\pi_{1_x}(\cdot|\theta_x)}\hat{\tau}_x^{1_x}$. If $\mathbb{E}_{\pi_{1_x}(\cdot|\theta_x)}\hat{\tau}_x^{1_x} \le 0$, then irrespective of whether $\hat{\theta}_x = \theta_x$ or not, there is no change in individual 1_x's expected payoff since in all selected senates and for all θ_{-x}, the planner continues to implement $(\mathbf{y}(\theta_x, \theta_{-x}), \mathbf{t}(\theta_x, \theta_{-x}))$ without imposing any additional monetary transfer on 1_x. If $\mathbb{E}_{\pi_{1_x}(\cdot|\theta_x)}\hat{\tau}_x^{1_x} > 0$, then irrespective of whether $\hat{\theta}_x = \theta_x$ or not, this deviation only ends up increasing individual 1_x's expected monetary transfer to the planner by $\mathbb{E}_{\pi_{1_x}(\cdot|\theta_x)}\hat{\tau}_x^{1_x}$.

Second, consider individual $j \in N_x$ of type θ_x, where $j \neq 1_x$. Suppose she deviates to $(\hat{\theta}_x, \hat{\tau}_x^{1_x})$. Note that the group leader 1_x – who is of the same type as j – continues to report θ_x. Now, if $\hat{\theta} = \theta_x$ and $\mathbb{E}_{\pi_j(\cdot|\theta_x)}\hat{\tau}_x^j \le 0$, then this deviation does not change the outcome of the mechanism. If $\hat{\theta} = \theta_x$ but $\mathbb{E}_{\pi_j(\cdot|\theta_x)}\hat{\tau}_x^j > 0$, then whenever j is selected, this deviation only ends up increasing individual j's expected monetary transfer to the planner by $\mathbb{E}_{\pi_j(\cdot|\theta_x)}\hat{\tau}_x^j$. As the outcome does not change in those senates in which j is not selected, individual j will be worse off after this deviation. If $\theta'_x \neq \theta_x$, then irrespective of $\hat{\tau}_x^j$, in all senates in which j is selected and for all θ_{-x}, the planner implements $(\mathbf{y}(\theta'_x, \theta_{-x}), \mathbf{t}(\theta'_x, \theta_{-x}))$ and imposes the additional monetary fine of t_j^* on j. As the outcome does not change in those senates in which j is not selected, it follows from the definition of t_j^* that individual j will be worse off after this deviation.

Pick any equilibrium σ. We can write σ as follows. For all $x \in X$, $i \in N_x$, and $\theta_x \in \Theta_x$

$$\sigma_i(\theta_x) = (\sigma_i^1(\theta_x), \tau_x^{i,\theta_x}).$$

Step 2: In every equilibrium σ, all individuals in group x of type θ_x report the same parameter value, i.e., $\sigma_i^1(\theta_x) = \sigma_j^1(\theta_x), \forall i, j \in N_x$ and $x \in X$. If not, then there exist an individual $j \neq 1_x$ such that type θ_x of individual j reports θ_x^j whereas type θ_x of individual 1_x reports $\theta_x^{1_x} \neq \theta_x^j$. In those senates in which individuals 1_x and j are selected together from group x, individual j pays the monetary fine of t_j^*. By definition of t_j^*, type θ_x of individual j will be better off reporting $\theta_x^{1_x}$ and offering a contingent asset $\hat{\tau}_x^{j,\theta_x}$ such that $\mathbb{E}_{\pi_j(\cdot|\theta_x^{1_x})}\hat{\tau}_x^{j,\theta_x} \leq 0$.

It follows from Step 2 that in every equilibrium, all individuals in group x use a coordinated deception β_x. Thus we can write any equilibrium strategy σ as follows. There exists a coordinated deception profile β such that for all $x \in X$, $i \in N_x$, and $\theta_x \in \Theta_x$,

$$\sigma_i(\theta_x) = (\beta_x(\theta_x), \tau_x^{i,\theta_x}).$$

As a result, the outcome falls under Rule 1 in all senates and all states.

Step 3: In every equilibrium σ, all individuals offer contingent assets that pay nonpositive expected returns to the planner, i.e., $\mathbb{E}_{\pi_i(\cdot|\beta_x(\theta_x))}\tau_x^{i,\theta_x} \leq 0$ for all $x \in X$, $i \in N_x$, and $\theta_x \in \Theta_x$. Suppose not, i.e., there exists type θ'_x of individual i who offers τ_x^{i,θ'_x} with $\mathbb{E}_{\pi_i(\cdot|\beta_x(\theta'_x))}\tau_x^{i,\theta'_x} > 0$. As σ is such that there is no discrepancy in the reported group-specific parameter values, in any senate S such that $i \in S$ and state (θ'_x, θ_{-x}), the individual 1_{x+1} pays the monetary fine $t^*_{1_{x+1}}$ to the planner. Pick any type, say θ'_{x+1}, of individual 1_{x+1}. If she were to deviate by reporting $\hat{\theta}_{x+1} \neq \beta_x(\theta'_{x+1})$, then for all $\theta_{-(x+1)}$, Rule 2 will be used in every senate. As all $j \neq 1_{x+1}$ in group $x+1$ – who are of the same type as 1_{x+1} – continue to report θ'_{x+1}, the only change after this deviation is that individual 1_{x+1} will never have to pay the monetary fine of $t^*_{1_{x+1}}$. Thus type θ'_{x+1} of individual 1_{x+1} will be better off after the deviation.

It follows from Step 3 that any equilibrium strategy σ is as follows. There exists a coordinated deception profile β such that for all $x \in X$, $i \in N_x$, and $\theta_x \in \Theta_x$,

$$\sigma_i(\theta_x) = (\beta_x(\theta_x), \tau_x^{i,\theta_x}) \text{ with } \mathbb{E}_{\pi_i(\cdot|\beta_x(\theta_x))}\tau_x^{i,\theta_x} \leq 0.$$

Thus in each senate, the planner implements $(\mathbf{y}(\beta(\theta)), \mathbf{t}(\beta(\theta)))$ without any additional monetary transfers. So we will be done if we argue that $\beta(\theta) = \theta$ for all θ, which is the final step in this proof.

Step 4: Suppose there exists an equilibrium σ such that individuals are reporting according to the deception profile β where $\beta(\hat{\theta}) \neq \hat{\theta}$ for some $\hat{\theta}$. Due to no consistent coordinated deception, there exists an individual i of some type θ_x such that $\pi_i^{\beta_{-x}}(\cdot|\theta_x)$ and $\pi_i(\cdot|\beta_x(\theta_x))$ are two distinct probability distributions over Θ_{-x}. Hence, we can find a contingent asset $\hat{\tau}_x$ such that $\mathbb{E}_{\pi_i(\cdot|\beta_x(\theta_x))}\hat{\tau}_x > 0 > \mathbb{E}_{\pi_i^{\beta_{-x}}(\cdot|\theta_x)}\hat{\tau}_x$. If individual i of type θ_x were to deviate by offering the contingent asset $\hat{\tau}_x$ without changing her type report $\beta_x(\theta_x)$, then in all senates in which i is selected and for all θ_{-x}, the planner will continue to implement $(\mathbf{y}(\beta(\theta_x, \theta_{-x})), \mathbf{t}(\beta(\theta_x, \theta_{-x})))$ but now individual i will additionally pay $\hat{\tau}_x(\beta_{-x}(\theta'_{-x}))$ to the planner. The outcome will not change in those senates in which i is not selected. As $\mathbb{E}_{\pi_i^{\beta_{-x}}(\cdot|\theta_x)}\hat{\tau}_x < 0$, individual i will be better off after this deviation.[19] □

Remark 5.3. *Off-the-equilibrium monetary transfers*: In the constructed mechanism, the planner may end up with either a surplus or a deficit off-the-equilibrium because it maybe that total additional transfers $\sum_{i \in N} \phi_i^S = \sum_{i \in S} \phi_i^S \neq 0$. This can be fixed easily: if $\sum_{i \in S} \phi_i^S \neq 0$, then $-\sum_{i \in S} \phi_i^S$ can be either collected from or distributed among the individuals not in S as long as there is at least one such individual. Observe that this adjustment alters only the additional transfers of the individuals who are not in the selected senate. Given that any individual's message affects her payoff only if she is selected as a senator, the adjustment we are considering does not alter the incentives of the individuals. [20]

Remark 5.4. *Implementation by oligarchy*: Suppose there exists an individual i whose beliefs satisfy the generic condition that all entries in Π_i are unequal. Now the planner does

[19] Notice that this argument basically shows that every SCF satisfies Bayesian p-monotonicity for the kleroterion p mentioned in the proposition.

[20] Using the same idea, the planner can balance the budget in terms of total transfers as long as the SCF itself is budget balanced, i.e., $\sum_{i \in N} \mathbf{t}_i(\theta) = 0$ for all $\theta \in \Theta$. Notice that if $\sum_{i \in N} \mathbf{t}_i(\theta) \neq 0$, then the budget cannot be balanced even in equilibrium. There are of course interesting cases where the budget is not balanced in terms of total transfers. For instance, suppose that $\mathbf{y}$ is the quantity of a public good and $\mathbf{t}$ is the tuple of payments that the individuals make for the provision of the public good. In this case, it is reasonable to assume that for each θ, the SCF consists of the desirable quantity of the public good $\mathbf{y}(\theta)$ and payments $\mathbf{t}(\theta)$ such that the total payments $\sum_{i \in N} \mathbf{t}_i(\theta)$ equals the cost of providing $\mathbf{y}(\theta)$. By using the modified mechanism, the planner guarantees that the total payments equals the cost of providing the public good both on and off the equilibrium.

not have to randomly select the second individual from each group. Indeed, in this case, any SCF is p_S-implementable (i.e., implementable by selecting the subset S with probability 1) as long as $i \in S$ and S has exactly two individuals from each group. The mechanism and proof of this claim are the same as above. The only change will be in *Step 4*, where now individual i will have the incentive to deviate for all coordinated deception profiles β such that $\beta(\hat{\theta}) \neq \hat{\theta}$ for some $\hat{\theta}$. Consequently, in quasilinear environments with nonexclusive information, oligarchies comprising of two senators from each group almost always implement any social goal.[21] This positive result for implementation by oligarchies is line with the existence of legislative bodies around the world in which two or more individuals represent each state/district. We however note two caveats. First, Bayesian p_S-monotonicity is in general more demanding than Bayesian p_D-monotonicity when S is a strict subset of N. Thus, the positive result for implementation by oligarchies need not carry over to more general environments. Second, oligarchies usually fail to implement in complete information environments (Saran and Tumennasan, 2013). Indeed, no consistent coordinated deception does not hold in complete information environments (see Matsushima, 1993).

6 Conclusion

There are many important situations in which multiple individuals have access to the same information. In such nonexclusive information environments, we have identified alternative communication protocols that consult with only a small number of individuals and yet attain the same social goals as under direct democracy. If monetary transfers are available, then we have shown that all social goals can almost always be achieved under the commonly used system of oligarchy.

Appendix

Proof of Proposition 3.2 Suppose mechanism Γ p-implements f. Let β be such that $f \circ \beta \neq f$. Pick $\sigma \in E(\Gamma)$. We must have $\alpha[\sigma] = f$. For each i, let $\sigma_i \circ \beta_{\omega(i)}$ denote the strategy such that

$$\sigma_i \circ \beta_{\omega(i)}(\theta'_{\omega(i)}) = \sigma_i\big(\beta_{\omega(i)}(\theta'_{\omega(i)})\big), \forall \theta'_{\omega(i)} \in \Theta_{\omega(i)}.$$

[21] This result is also valid for oligarchies with more than two senators per group.

Let $\sigma \circ \beta = (\sigma_i \circ \beta_{\omega(i)})_{i \in N}$ and $\sigma_{-i} \circ \beta = (\sigma_j \circ \beta_{\omega(j)})_{j \neq i}$. Then $\alpha[\sigma \circ \beta] = f \circ \beta$. Since $f \circ \beta \neq f$, it must be that $\sigma \circ \beta \notin E(\Gamma)$. So there exist i, $\theta_{\omega(i)}$, and m_i such that

$$U_i(\alpha[m_i, \sigma_{-i} \circ \beta]|\theta_{\omega(i)}) > U_i(\alpha[\sigma \circ \beta]|\theta_{\omega(i)}) = U_i(f \circ \beta|\theta_{\omega(i)}).$$

However, since $\sigma \in E(\Gamma)$, it must be that for type $\beta_{\omega(i)}(\theta_{\omega(i)})$

$$U_i(f|\beta_{\omega(i)}(\theta_{\omega(i)})) = U_i(\alpha[\sigma]|\beta_{\omega(i)}(\theta_{\omega(i)})) \geq U_i(\alpha[m_i, \sigma_{-i}]|\beta_{\omega(i)}(\theta_{\omega(i)})).$$

Define $h : \mathcal{N}(p) \times \Theta \to A$ as $h(S, \theta) = g^S((m_i, (\sigma_j(\theta_{\omega(j)}))_{j \neq i})^S), \forall S \in \mathcal{N}(p)$ and θ. Clearly, $h \in \mathcal{H}_i(p, f)$. Moreover, $\alpha[p, h] = \alpha[m_i, \sigma_{-i}]$ and $\alpha[p, h] \circ \beta = \alpha[m_i, \sigma_{-i} \circ \beta]$, which concludes the argument. □

Proof of Proposition 3.4: Suppose mechanism Γ p-implements f. Fix $\sigma \in E(\Gamma)$. We have $\alpha[\sigma] = f$.

Pick any i. If $\mathcal{N}(i, p) \setminus \mathcal{S}^1_{\omega(i)} \cup \mathcal{S}^2_{\omega(i)} = \emptyset$, then let γ_i be any arbitrary allocation. If not, then for any state θ and senate $S \in \mathcal{N}(i, p) \setminus \mathcal{S}^1_{\omega(i)} \cup \mathcal{S}^2_{\omega(i)}$, consider the set of alternatives

$$A_i(S, \theta) = \{g^S(\sigma_i(\theta'_{\omega(i)}), (\sigma_j(\theta_{\omega(j)}))_{j \in S \setminus \{i\}}) : \theta'_{\omega(i)} \in \Theta_{\omega(i)}\}.$$

Notice that $g^S(\sigma_i(\theta'_{\omega(i)}), (\sigma_j(\theta_{\omega(j)}))_{j \in S \setminus \{i\}})$ is the alternative that is implemented under S in state θ when all individuals $j \neq i$ in S follow their equilibrium strategies whereas individual i sends the message $\sigma_i(\theta'_{\omega(i)})$. Now define the allocation γ_i as follows:

$$\gamma_i(\theta) \in \arg \min_{a \in A_i(S,\theta), S \in \mathcal{N}(i,p) \setminus \mathcal{S}^1_{\omega(i)} \cup \mathcal{S}^2_{\omega(i)}} u_i(a, \theta), \forall \theta.$$

Pick any x, distinct $i, i' \in N_x$, distinct $\theta^i_x, \theta^{i'}_x \in \Theta_x$, and $S \in \mathcal{S}^2_x$ such that $i, i' \in S$. Define $\lambda[S, \{\theta^i_x, \theta^{i'}_x\}]$ as follows:

$$\lambda[S, \{\theta^i_x, \theta^{i'}_x\}](\theta_{-x}) = g^S(\sigma_i(\theta^i_x), \sigma_{i'}(\theta^{i'}_x), (\sigma_j(\theta_{\omega(j)}))_{j \in S \setminus \{i, i'\}}), \forall \theta_{-x}.$$

Note that $\lambda[S, \{\theta^i_x, \theta^{i'}_x\}]$ is well defined because p does not ignore any group.

Now consider individual j of type $\theta_{\omega(j)}$. Suppose she deviates to $\sigma_j(\theta^j_{\omega(j)})$, where $\theta^j_{\omega(j)} \neq \theta_{\omega(j)}$. This deviation should not be improving as σ is an equilibrium. Pick any senate $S \in \mathcal{N}(j, p)$.

Suppose j is the only senator in S from her group. Then for all $\theta_{-\omega(j)}$, the outcome under senate S will be $g^S(\sigma_j(\theta^j_{\omega(j)}), (\sigma_i(\theta_{\omega(i)}))_{i \in S \setminus \{j\}}) = f(\theta^j_{\omega(j)}, \theta_{-\omega(j)})$.

Suppose j and j' are the only two senators in S from j's group. Since j' is also of type $\theta_{\omega(j)}$, for all $\theta_{-\omega(j)}$, the outcome under senate S will be $g^S(\sigma_j(\theta^j_{\omega(j)}), \sigma_{j'}(\theta_{\omega(j)}), (\sigma_i(\theta_{\omega(i)}))_{i\in S\setminus\{j,j'\}}) = \lambda[S, \{\theta^j_{\omega(j)}, \theta_{\omega(j)}\}](\theta_{-\omega(j)})$.

Finally, suppose $S \in \mathcal{N}(j,p) \setminus \mathcal{S}^1_{\omega(j)} \cup \mathcal{S}^2_{\omega(j)}$. Since every $j' \in S \cap N_{\omega(j)}$ is also of type $\theta_{\omega(j)}$, for all $\theta_{-\omega(j)}$, the outcome under senate S will be $g^S(\sigma_j(\theta^j_{\omega(j)}), (\sigma_i(\theta_{\omega(i)}))_{i\in S\setminus\{j\}})$, which is weakly better than $\gamma_i(\theta_{\omega(j)}, \theta_{-\omega(j)})$.

Therefore, if the deviation is not improving, then (2) must hold. □

Proof of Proposition 4.4: For any individual i, let $\mathbf{Y}_i$ be the set of all functions $\mathbf{y}_i : \mathcal{N}(i,p) \times \Theta \to A$. Let $\mathbf{f}_i \in \mathbf{Y}_i$ be such that $\mathbf{f}_i(S,\theta) = f(\theta), \forall S \in \mathcal{N}(i,p)$ and $\theta \in \Theta$.

For any individual i, let $\mathbf{A}_i$ be the set of all functions $\mathbf{a}_i : \mathcal{N}(i,p) \to A$ and $\mathbf{Z}_i$ be the set of all functions $\mathbf{z}_i : \mathcal{N}(i,p) \to \mathbb{Z}_+$, where $\mathbb{Z}_+$ is the set of all nonnegative integers.

Let $M_i = \Theta_{\omega(i)} \times \mathbf{Y}_i \times \mathbf{A}_i \times \mathbf{Z}_i$ for each i. Define g^S as follows for each $S \in \mathcal{N}(p)$:

Rule 1: If $m_i = (\theta_x, \mathbf{f}_i, \mathbf{a}_i, \mathbf{z}_i)$ with $\mathbf{z}_i(S) = 0$ for all $i \in S \cap N_x$ and $x \in X$, then

$$g^S((m_i)_{i\in S}) = f((\theta_x)_{x\in X}).$$

Rule 2.1: Suppose S is such that $S \in \mathcal{S}^2_{x'}$. If $m_i = (\theta_x, \mathbf{f}_i, \mathbf{a}_i, \mathbf{z}_i)$ with $\mathbf{z}_i(S) = 0$ for all $i \in S \cap N_x$ and $x \neq x'$ but for $i', j' \in S \cap N_{x'}$, we have $m_{i'} = (\theta^{i'}_{x'}, \mathbf{f}_{i'}, \mathbf{a}_{i'}, \mathbf{z}_{i'})$ with $\mathbf{z}_{i'}(S) = 0$ and $m_{j'} = (\theta^{j'}_{x'}, \mathbf{y}_{j'}, \mathbf{a}_{j'}, \mathbf{z}_{j'})$ where $\theta^{i'}_{x'} \neq \theta^{j'}_{x'}$, then

$$g^S((m_i)_{i\in S}) = \lambda[S, \{\theta^{i'}_{x'}, \theta^{j'}_{x'}\}]\big((\theta_x)_{x\neq x'}\big).$$

Rule 2.2: Suppose S is such that $S \in \mathcal{S}^2_{x'}$. If $m_i = (\theta_x, \mathbf{f}_i, \mathbf{a}_i, \mathbf{z}_i)$ with $\mathbf{z}_i(S) = 0$ for all $i \in S \cap N_x$ and $x \neq x'$ but for $i', j' \in S \cap N_{x'}$, we have $m_{i'} = (\theta_{x'}, \mathbf{f}_{i'}, \mathbf{a}_{i'}, \mathbf{z}_i)$ with $\mathbf{z}_{i'}(S) = 0$ and $m_{j'} = (\theta_{x'}, \mathbf{y}_{j'}, \mathbf{a}_{j'}, \mathbf{z}_{j'})$ where either $\mathbf{y}_{j'} \neq \mathbf{f}_{j'}$ or $\mathbf{z}_{j'}(S) \neq 0$, then

$$g^S((m_i)_{i\in S}) = \begin{cases} \mathbf{y}_{j'}(S, (\theta_x)_{x\in X}), & \text{if } U_{j'}(f|\theta_{x'}) \geq U_{j'}(\alpha[p,h]|\theta_{x'}) \\ f((\theta_x)_{x\in X}), & \text{otherwise,} \end{cases}$$

where $h : \mathcal{N}(p) \times \Theta \to A$ is such that for all S' and θ

$$h(S',\theta) = \begin{cases} \mathbf{y}_{j'}(S',\theta), & \text{if } j' \in S' \\ f(\theta), & \text{otherwise.} \end{cases}$$

Rule 3.1: Suppose S is such that $S \in \mathcal{S}^k_{x'}$, where $k \geq 3$. If $m_i = (\theta_x, \mathbf{f}_i, \mathbf{a}_i, \mathbf{z}_i)$ with $\mathbf{z}_i(S) = 0$ for all $i \in S \cap N_x$ and $x \neq x'$ but $j' \in S \cap N_{x'}$ is such that $m_{i'} = (\theta_{x'}, \mathbf{f}_{i'}, \mathbf{a}_{i'}, \mathbf{z}_i)$ with $\mathbf{z}_{i'}(S) = 0$

for all $i' \in S \cap N_{x'} \setminus \{j'\}$ and $m_{j'} = (\theta_{x'}^{j'}, \mathbf{y}_{j'}, \mathbf{a}_{j'}, \mathbf{z}_{j'})$ where $\theta_{x'}^{j'} \neq \theta_{x'}$, then

$$g^S((m_i)_{i\in S}) = \gamma_{j'}((\theta_x)_{x\in X}).$$

Rule 3.2: Suppose S is such that $S \in \mathcal{S}_{x'}^k$, where $k \geq 3$. If $m_i = (\theta_x, \mathbf{f}_i, \mathbf{a}_i, \mathbf{z}_i)$ with $\mathbf{z}_i(S) = 0$ for all $i \in S \cap N_x$ and $x \neq x'$ but $j' \in S \cap N_{x'}$ is such that $m_{i'} = (\theta_{x'}, \mathbf{f}_{i'}, \mathbf{a}_{i'}, \mathbf{z}_i)$ with $\mathbf{z}_{i'}(S) = 0$ for all $i' \in S \cap N_{x'} \setminus \{j'\}$ and $m_{j'} = (\theta_{x'}, \mathbf{y}_{j'}, \mathbf{a}_{j'}, \mathbf{z}_{j'})$ where either $\mathbf{y}_{j'} \neq \mathbf{f}_{j'}$ or $\mathbf{z}_{j'}(S) \neq 0$, then

$$g^S((m_i)_{i\in S}) = \begin{cases} \mathbf{y}_{j'}(S, (\theta_x)_{x\in X}), & \text{if } U_{j'}(f|\theta_{x'}) \geq U_{j'}(\alpha[p,h]|\theta_{x'}) \\ f((\theta_x)_{x\in X}), & \text{otherwise,} \end{cases}$$

where $h : \mathcal{N}(p) \times \Theta \to A$ is such that for all S' and θ

$$h(S', \theta) = \begin{cases} \mathbf{y}_{j'}(S', \theta), & \text{if } j' \in S \\ f(\theta), & \text{otherwise.} \end{cases}$$

Rule 4: In all other cases, let i^* be the player in S with the lowest index amongst those who announce the highest integer $\max_{i\in S} \mathbf{z}_i(S)$, and define $g^S((m_i)_{i\in S}) = \mathbf{a}_{i^*}(S)$.

Step 1: We argue that the strategy profile $\sigma_i(\theta_{\omega(i)}) = (\theta_{\omega(i)}, \mathbf{f}_i, \mathbf{a}_i, \mathbf{z}_i)$, where $\mathbf{z}_i(S) = 0, \forall S$, and $\mathbf{a}_i$ is any arbitrary element of $\mathbf{A}_i$, is an equilibrium. Pick any individual i of type $\theta_{\omega(i)}$. There are two possible cases:

First, suppose she were to deviate to $(\theta_{\omega(i)}^i, \mathbf{y}_i', \mathbf{a}_i', \mathbf{z}_i')$, where $\theta_{\omega(i)}^i \neq \theta_{\omega(i)}$.

- In any senate $S \in \mathcal{N}(i,p)$ such that $j \neq i$ is the only other individual in S from i's group, the outcome will be $\lambda[S, \{\theta_{\omega(i)}^i, \theta_{\omega(i)}\}](\theta_{-\omega(i)})$ for all $\theta_{-\omega(i)}$.
- In any senate $S \in \mathcal{N}(i,p)$ such that there are two other individuals in S from i's group, the outcome will be $\gamma_i(\theta_{\omega(i)}, \theta_{-\omega(i)})$ for all $\theta_{-\omega(i)}$.

Since f is p-incentive compatible, conditional on being selected as a senator, player i expects a payoff of at most $U_i(f|\theta_{\omega(i)})$ after her deviation. Of course, conditional on not being selected as a senator, player i expects the payoff $U_i(f|\theta_{\omega(i)})$ irrespective of her strategy. Hence, this deviation does not increase player i's expected payoff.

Second, suppose she were to deviate to $(\theta_{\omega(i)}, \mathbf{y}_i', \mathbf{a}_i', \mathbf{z}_i')$, where either $\mathbf{y}_i' \neq \mathbf{f}_i$ or $\mathbf{z}_i'(S) \neq 0$ for some S. Let $h : \mathcal{N}(p) \times \Theta \to A$ be such that $h(S,\theta) = \mathbf{y}_i'(S,\theta)$ if $i \in S$, and $h(S,\theta) = f(\theta)$ otherwise.

On the one hand, if $U_i(\alpha[p,h]|\theta_{\omega(i)}) > U_i(f|\theta_{\omega(i)})$, then the outcome remains $f(\theta_{\omega(i)}, \theta_{-\omega(i)})$ for all $\theta_{-\omega(i)}$ and S, which is not improving. On the other hand, if $U_i(f|\theta_{\omega(i)}) \geq U_i(\alpha[p,h]|\theta_{\omega(i)})$, then for all $\theta_{-\omega(i)}$, the outcome will be $\mathbf{y}'_i(S, (\theta_{\omega(i)}, \theta_{-\omega(i)}))$ if $i \in S$ whereas the outcome will remain $f(\theta_{\omega(i)}, \theta_{-\omega(i)})$ otherwise. Hence, in this case, the deviation will generate $\alpha[p, h]$, which is not improving.

Step 2: Suppose there exists an equilibrium σ such that in some state θ and senate S, the outcome falls under Rule 2.1, 2.2, 3.1 or 3.2. Then there exist an $x' \in X$ and individuals $i', j' \in S \cap N_{x'}$ such that $\sigma_{i'}(\theta_{x'}) = (\theta_{x'}^{i'}, \mathbf{f}_{i'}, \mathbf{a}_{i'}, \mathbf{z}_{i'})$ with $\mathbf{z}_{i'}(S) = 0$, $\sigma_{j'}(\theta_{x'}) = (\theta_{x'}^{j'}, \mathbf{y}_{j'}, \mathbf{a}_{j'}, \mathbf{z}_{j'})$, and one of the following is true: $\theta_{x'}^{i'} \neq \theta_{x'}^{j'}$, $\mathbf{y}_{j'} \neq \mathbf{f}_{j'}$ or $\mathbf{z}_{j'}(S) \neq 0$.

Let α be the allocation generated conditional on the selection of senate S, i.e., $\alpha(\theta') = g^S((\sigma_i(\theta'_{\omega(i)}))_{i \in S}), \forall \theta'$. Since the environment is economic, either there are within or between groups differences.

If there are between groups differences, then the fact that there are three or more groups implies that there exists an individual $i \in S$ belonging to some group $x \neq x'$ and a constant allocation a_i such that $U_i(a_i/_{\Theta'}\alpha|\theta_x) > U_i(\alpha|\theta_x), \forall \Theta' \subseteq \Theta$ such that $\theta \in \Theta'$.[22] Then let type θ_x of individual i deviate by announcing $\mathbf{a}'_i(S) = a_i$ and $\mathbf{z}'_i(S)$ that is greater than the highest integer announced by any type of any individual in strategy σ while keeping the rest of her message the same. After this deviation, the outcome under senate S in any state (θ_x, θ'_{-x}) is as follows:

- If Rule 1 was followed under S in state (θ_x, θ'_{-x}), then Rule 2.2 or 3.2 will be followed after the deviation. But this will not change the outcome since type θ_x of player i has not changed the first and second components of her message.

- If any rule except Rule 1 was followed under S in state (θ_x, θ'_{-x}), then Rule 4 will be followed after the deviation. (To see this, note that before the deviation by type θ_x of individual i, there was no “discrepancy” between her message and message of any other individual from group x in senate S – whose type is also θ_x; otherwise, the additional “discrepancy” between the messages of individuals i' and j' from group x' in state θ would have lead to the application of Rule 4 in senate S, which was not the case.) Thus, a_i will be implemented. In particular, this will be the case in state θ.

As a result, conditional on selection of senate S, type θ_x of player i will be better off after the deviation. Since this change in strategy does not change the outcome in any senate other than S, it follows that this is an improving deviation for type θ_x of player i, a contradiction.

[22] If there are only two groups, then it is possible that $S \in \mathcal{S}^2_{x'}$ and all individuals in group $x \neq x'$ are satiated at α whereas the two individuals i' and j' from group x' cannot force Rule 4 by unilateral deviations, which happens starting at Rule 2.1 when $\sigma_i(\theta_{x'}) = (\theta^i_{x'}, \mathbf{f}_i, \mathbf{a}_i, \mathbf{z}_i)$ with $\mathbf{z}_i(S) = 0$ for all $i \in \{i', j'\}$ and $\theta^{i'}_{x'} \neq \theta^{j'}_{x'}$. This issue can be avoided if either p always selects three or more senators from each group or the environment has within groups differences.

If there are within groups differences, then the fact that there are at least two individuals from each group in S implies that we can find an individual $i \in S$ belonging to some group $x \neq x'$ and a constant allocation a_i such that $U_i(a_i/_{\Theta'}\alpha|\theta_x) > U_i(\alpha|\theta_x), \forall \Theta' \subseteq \Theta$ such that $\theta \in \Theta'$. Using the same argument as in the previous case, we can argue that type θ_x of individual i has an improving deviation, a contradiction.

Step 3: Suppose there exists an equilibrium σ such that in some state θ and senate S, the outcome falls under Rule 4. Since every individual in S can maintain the application of Rule 4 through appropriate deviations, we can argue like in the previous case that irrespective of whether there are within or between groups differences, at least one of the individuals in S has an improving deviation, a contradiction.

Step 4: Finally, suppose there exists an equilibrium σ such that in all states, Rule 1 is used under all senates. Pick any individual i in group x. Her strategy σ_i must be such that for all θ_x,

$$\sigma_i(\theta_x) = (\sigma_i^1(\theta_x), \mathbf{f}_i, \mathbf{a}_i^{\theta_x}, \mathbf{z}_i^{\theta_x})$$

with $\mathbf{z}_i^{\theta_x}(S) = 0$ for all $S \in \mathcal{N}(i,p)$ – if this is not satisfied, then Rule 1 cannot be used when i has type θ_x and she is selected.

Consider any two individuals i and j belonging to group x. It must be that $\sigma_i^1(\theta_x) = \sigma_j^1(\theta_x)$ for all θ_x. Suppose this is not the case for θ'_x. Then Rule 1 cannot be used when θ'_x types of individuals i and j are selected together. If individuals i and j are never selected together, then the fact that $G(p,x)$ is connected implies that there must be two individuals i' and j' in group x of type θ'_x such that they are selected together and $\sigma_{i'}^1(\theta'_x) \neq \sigma_{j'}^1(\theta'_x)$, which again contradicts the application of Rule 1 in all senates and states. Thus there exist a coordinated deception profile β such that for all $x \in X$, $i \in N_x$, and $\theta_x \in \Theta_x$

$$\sigma_i(\theta_x) = (\beta_x(\theta_x), \mathbf{f}_i, \mathbf{a}_i^{\theta_x}, \mathbf{z}_i^{\theta_x})$$

with $\mathbf{z}_i^{\theta_x}(S) = 0$ for all $S \in \mathcal{N}(i,p)$.

If $f \circ \beta \neq f$, then we are done. If not, by Bayesian p-monotonicity, there exist $i \in N$, $\theta'_{\omega(i)} \in \Theta_{\omega(i)}$ and $h \in \mathcal{H}_i(p,f)$ such that

$$U_i(\alpha[p,h] \circ \beta|\theta'_{\omega(i)}) > U_i(f \circ \beta|\theta'_{\omega(i)}) \text{ and } U_i(f|\beta_{\omega(i)}(\theta'_{\omega(i)})) \geq U_i(\alpha[p,h]|\beta_{\omega(i)}(\theta'_{\omega(i)})).$$

Let type $\theta'_{\omega(i)}$ of player i deviate by announcing $\mathbf{y}_i$ such that $\mathbf{y}_i(S,\theta) = h(S,\theta), \forall \theta \in \Theta$ and $S \in \mathcal{N}(i,p)$. Then either Rule 2.2 or 3.2 will be used in any senate $S \in \mathcal{N}(i,p)$ with alternative $\mathbf{y}_i(S, \beta(\theta'_{\omega(i)}, \theta_{-\omega(i)})) = h(S, \beta(\theta'_{\omega(i)}, \theta_{-\omega(i)}))$ being implemented for all $\theta_{-\omega(i)}$. If $S \notin \mathcal{N}(i,p)$, then alternative $f(\beta(\theta'_{\omega(i)}, \theta_{-\omega(i)}))$ is implemented for all $\theta_{-\omega(i)}$. Thus we have found an improving deviation, a contradiction. □

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