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# Job Heterogeneity and Coordination Frictions\*

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## Abstract

We develop a new directed search model of a frictional labor market with a continuum of heterogeneous workers and firms. We estimate two versions of the model - auction and price posting - using Danish data on wages and productivities. Assuming heterogeneous workers with no comparative advantage, we find that each model gives a reasonable approximation of the statistical moments of both the wage and productivity distribution. A sensitivity analysis then draws out further implications of the theory. We explain how the feasible matchings between workers and firms changes as the worker moves up the job ladder, how the identification of assortative matching is fundamentally different in directed and undirected search models, how our theory accounts for business cycle facts related to inter-temporal changes in job offer distributions, and how our model could also be used to identify the contributions of specific versus general human capital.

Keywords: Directed Search, Auctions, Wage Posting, On-the-Job Search, Comparative Advantage, Assortative Matching, Business Cycles, Human Capital.

JEL Classification: J64; J63; E32

## 1 Introduction

The goal of this paper is to understand the determinants of the economic landscape facing a job searcher. This means that we seek an economic model that has spillovers between the workers' incentives to accept quality differentiated job opportunities and the employers' incentives for their creation. We also seek to estimate this type of model using matched employer-employee data on the wages of workers and the productivities of their various employers. This applied analysis demands an equilibrium framework that has individual job hazards exhibiting continuous dispersion in both wages and productivity.

For this purpose, we develop a directed search model with on-the-job search which is flexible, but yet parsimonious. The model features endogenous productivity and wage dispersion for ex-ante identical agents due to the existence of coordination frictions.

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We begin by considering the static version of the model. We show that the technological requirement for non-degenerate job offer distributions is that the opportunity cost of higher vacant job types increases sufficiently fast relative to the potential productivity of a worker in each job type. We also find that the workers' outside options affect the job offer distribution differently in the continuous firm type case than it does in a discrete firm type case. For example, if there are only two job types, then an increase in the outside option of a worker leads to a decrease in the number of offers from all employer types. However, in the continuous job type case, a higher outside option leads to a complete loss of job offers by lower firm types, but it has no effect on the frequency of job offers by higher type firms.

Next, we show that the model is easily extended to a dynamic environment with on-the-job search for the continuous firm type case - especially since higher outside options only affect the cutoff and do not alter the shape of the non-degenerate job offer distributions. The dynamic model generates an equilibrium unemployment rate among long-lived workers by assuming a constant job destruction rate across all jobs.

Our directed search model is flexible in several dimensions and can be used for a variety of purposes related to the structural modeling of the labor market. For example, we compare two basic theories of pricing in directed search equilibrium - auctions and posting.<sup>1</sup> The auction and posting models have equivalent implications concerning the set of firm types in the distribution of job offers, but they differ by the fact that only the auction model predicts a disperse set of wage offers for each worker type at each firm type.

We estimate by non-parametric simulated maximum likelihood both the auction and posting versions of our dynamic model using Danish data on wages and firm productivities. When we estimate the homogenous workers version of our model by only fitting the productivity distribution, we find that the auction model accounts for more of the dispersion of wages but that the mode of the wage distribution is too low. By contrast, the price posting model does a better job of explaining the mode of the wage distribution but it predicts less wage dispersion than the auction model.

One simple extension of the benchmark model that helps explain the data on wages and firm productivities is to make the workers heterogeneous. Here, we adopt a simple benchmark case of no-comparative advantage (Refer to Sattinger (1993) and Shimer (2005)). This is a special case of the model that implies no systematic sorting of different worker types across firm types. The estimation results give a reasonable fit of the wage and productivity distributions, but the nature of worker heterogeneity now depends on whether we adopt an auction or posting specification. For example, the estimated auction model implies an asymmetric workforce with a small group of low productivity workers and a larger group of high productivity workers while the estimated wage posting model spreads worker types more equally over a similar range of worker types.

In the benchmark model, the feasible matchings between workers and firms change as the worker moves up the job ladder. This feasible matching set is largest for unemployed workers and it shrinks whenever a worker accepts a job further up the job ladder. On-the-job search is important in our empirical estimation, because the rate of job separations into unemployment is small and so, given a realistic discount rate, the worker is only willing to accept a low productivity

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<sup>1</sup>Refer to Burdett, Shi and Wright (2000) and Julien, Kennes and King (2000).

job if it has a similar option value as a high productivity jobs (Hornstein, Krusell and Violante (2012)).

Our estimated model does not have sorting because we assume that there is no comparative advantage across worker types. However, this assumption is easily weakened by allowing super/sub-modular surplus functions for worker/firm matches. We also discuss the problem of how to identify sorting. This requires fundamentally different methods than that which are valid for undirected search models of on-the-job search. The salient problem of identification in the directed search equilibrium is that the entire shape of the job offer distribution is an equilibrium object that can vary across different groups of job searchers.

The analysis of business cycles represent a key challenge for any equilibrium framework that attempts to model wage and productivity dispersion (Refer to Menzio and Shi (2011)). The challenge is to explain important facts such as the observation that changes in wage dispersion are pro-cyclical (Refer to Morin (2013)). Our model explains this fact by technological drivers that cause the number of job offers of each job type to fall (and rise) in equal proportion. Moreover, in our model, any source of economic growth that does not cause a change in the dispersion of wages and productivities is equivalent to a no-comparative advantage increase in productivity. Therefore, pro-cyclical variations in wage dispersion can be interpreted as a form of intertemporal sorting where workers are more likely to be matched with good jobs during upturns.

Finally, we can also relate our analysis to the problem of human capital accumulation. The main difference between the random and directed search models of human capital accumulation is that firms in directed search equilibrium do not extend offers to workers that will be rejected in favor of their current employer. Therefore, wage increases that reduce the arrival rate of job offers can be interpreted as specific human capital in a directed search framework. We explain how a version of our model with human capital accumulation is closely related to the analysis of Bagger, Postel-Vinay, Fontaine, and Robin (2011) and why a directed search framework might be useful for identifying the different contributions of specific and general human capital.

The paper is organized as follows. We develop the static version of the model in section 2, the dynamic model with on-the-job search in sections 3, posting in section 4, and heterogeneous workers in sections 5. In section 6, we estimate the benchmark dynamic models using Danish data. A sensitivity analysis of this estimated model is then given in section 7. The final section offers concluding remarks.

## 2 The static model

Workers are allocated into a set of submarkets, where each submarket contains all workers of a similar type (Notation to express specific worker types will be given later in the analysis). Each submarket has many of these identical workers and, potentially, many firms of types  $k \in [0, \infty]$ . A type  $k$  firm produces  $y(k)$  output if matched to a worker. Each worker is endowed with one unit of labor to sell and each firm has one job opportunity that can employ one worker. The higher firm types produce more output but face increasingly higher opportunity costs,  $c(k)$ , such that  $y', c', c''$  and  $-y''$  are all positive. The worker also has an outside opportunity,  $y(\underline{k})$ , which is the worker's output if the worker is not employed by a firm. For simplicity, we normalize the

size of the worker population to one. All agents are risk neutral, expected utility maximizers.

Workers and firms play a simple three stage game. In the first stage of the game, a measure of firms of types  $k \in [0, \infty]$  enters the matching submarket, and each firm in this measure pays its associated opportunity cost,  $c(k)$ . In the second stage of the game, the newly created job opportunities are randomly assigned to worker locations. In the final stage of the game, each firm in a local market bids a wage for the worker's labor services according to the rules of a simple second price auction. The worker then either selects one of these bids or else remains unemployed. The payoff of the worker is his wage if he is hired and his outside opportunity otherwise. If the firm hires the worker, its payoff is the productivity of the job less the capital cost and the worker's wage. If the firm does not hire the worker, its payoff is minus the opportunity cost. The payoff of a firm that does not enter this submarket (perhaps to enter another submarket with another set of different workers) is normalized to zero. In this case, the firm is earning its opportunity cost. The game is solved by backwards induction.

## 2.1 Third stage

Consider first the optimal bidding strategies in the final stage of the game when the allocation of all job opportunities and workers is known. Let  $k_1$  and  $k_2$  denote a worker's best and second best job opportunity. Given competitive bidding for the worker's labor services, the worker is employed by the highest valuation firm at a wage given by

$$w(k_1, k_2) = y(k_2), \quad (1)$$

The revenue of the worker's employer is thus  $y(k_1) - y(k_2)$  and the revenue of all other firms bidding for this worker is then zero.

## 2.2 Second stage

The second stage of the game is simple random assignment. Let  $\phi(k)$  denote the mass of job types greater than  $k$ , which is determined in the first stage. The random assignment of job opportunities to workers means that the number of job opportunities of type greater than  $k$  at each worker's location is distributed Poisson with parameter  $\phi(k)$ .

In this auction model, the overall distribution of wages and productivities is determined by two order statistics:  $G_1(k)$ , which denotes the fraction of workers with a best job offer less than  $y(k)$  and,  $G_2(k)$ , which denotes the fraction of workers with a second best job offer less than  $y(k)$ . If job types are offered on a range  $k \in [\hat{k}, k^*]$ , the formulas for the first and second order statistics are as follows.

**Proposition 1.** For values of  $k_1, k_2 \in [\hat{k}, k^*]$ ,

$$G_1(k) = e^{-\phi(k)}, \text{ and} \quad (2)$$

$$G_2(k) = e^{-\phi(k)} + \phi(k) e^{-\phi(k)}. \quad (3)$$

where  $G_1(\hat{k})$  is the probability of no offer and  $G_2(\hat{k})$  is the probability that the number of offers are zero or one.

*Proof.* The probability that  $k$  is the highest offer for a worker is given by

$$\begin{aligned} G_1(k) &= \sum_{x=0}^{\infty} \left( \frac{1}{1+x} \right) \frac{e^{-\phi(k)} \phi(k)^x}{x!} \\ &= e^{-\phi(k)} \end{aligned}$$

Likewise,  $e^{-\phi(k)} + \phi(k) e^{-\phi(k)}$  is the probability that there are not two offers better than  $k$ .  $\square$

The cumulative distribution function  $G_1(k)$  is equivalent to the distribution of worker productivities, because the workers are always employed at the most productive available job. The cumulative distribution function  $G_2(k)$  is equivalent to the distribution of wages, because the worker is always paid a wage equal to his/her second highest valuation.

### 2.3 First stage: Two employer types

Before we tackle the entry decision of the continuum of job types case, it is natural to ask what happens if there are discrete firm types.<sup>2</sup> Suppose that there are three types of jobs, say  $k \in \{0, 1, 2\}$ . Let  $k = 0$  be the job type that gives the worker's outside option. In this case, the free entry conditions for good (type 2) and bad (type 1) jobs are given by

$$c(1) = (y(1) - y(0)) e^{-\phi(1)} \quad (4)$$

and

$$c(2) = (y(2) - y(0)) e^{-\phi(1)} + (y(2) - y(1)) e^{-\phi(2)} \left( 1 - e^{-(\phi(1) - \phi(2))} \right) \quad (5)$$

If both job types are offered in equilibrium (i.e.  $\phi(2) > 0$  and  $\phi(1) > \phi(2)$ ), the menu of job productivities and opportunity costs must satisfy the following 'concavity' condition. We require

$$y(2) - c(2) > y(1) - c(1)$$

and

$$\frac{y(1) - y(0)}{c(1)} > \frac{y(2) - y(0)}{c(2)}$$

Furthermore, in such an equilibrium, an increase in the outside option of the worker,  $y(0)$ , decreases both  $\phi(1)$  and  $\phi(2)$ . Therefore, a higher outside option reduces the likelihood that a worker is contacted by good and bad jobs.

### 2.4 First stage: A continuum of employer types

In an equilibrium with a continuum of job types, the entry decision for jobs of different types can be linked to a boundary condition on the lowest quality job type and a simple difference equation relating the returns of a particular job type and the returns of a job type immediately ranked above this job type. Let  $\hat{k}$  denote the lowest quality job type offered in equilibrium, such that the total mass of jobs of all types is given by  $\phi(\hat{k})$ . The following proposition relates  $\hat{k}$  and  $\phi(\hat{k})$  to the outside option of a worker,  $\underline{k}$ . We have

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<sup>2</sup>Further discussion of the two job types model is given in Julien, Kennes, and King (2006).

**Proposition 2.** *The lowest job type offered in equilibrium is given by*

$$\hat{k}(\underline{k}) = \arg \max_k (y(k) - y(\underline{k})) / c(k) \quad (6)$$

and the total mass of jobs offered in equilibrium is

$$\phi(\hat{k}) = -\log \left( \left( y(\hat{k}) - y(\underline{k}) \right) / c(\hat{k}) \right). \quad (7)$$

*Proof.* Consider the values of  $\hat{k}(\underline{k})$  and  $\phi(\hat{k})$ . Since the lowest quality job earns a positive return of  $y(\hat{k})$  if and only if there is no other firm at the local market of this worker, its expected return is given by  $(y(\hat{k}) - y(\underline{k})) e^{-\phi(\hat{k})}$  where  $\phi(\hat{k})$  is the measure of jobs greater than  $\hat{k}$ . Suppose that the total mass of jobs is some value  $\phi$ , which is less than  $\phi(\hat{k})$ . Then the total mass of job openings is

$$\phi < \max \left\{ \phi(k) \mid e^{-\phi(k)} (y(k) - y(\underline{k})) = c(k) \right\}$$

In this case,  $e^{-\phi(k)} (y(k) - y(\underline{k})) > c(k)$ , and thus the returns to entry of type  $\hat{k}(\underline{k})$  jobs imply  $\phi(\hat{k}) - \phi > 0$ , a contradiction. The total mass of jobs can also not exceed  $\phi(\hat{k})$ , because the argmax operator in equation (6) looks for the largest possible value of  $\phi(k)$  that satisfies free entry of low type jobs.  $\square$

Notice that the outside option influences the lowest job type offered to workers. A higher outside option,  $y(\underline{k})$  always raises the value of  $\hat{k}(\underline{k})$  and reduces the supply of jobs in equilibrium  $\phi(\hat{k})$ .

In this auction model, the payoff of each firm type entering the matching market is a function of its productivity and the probability that it faces a competitor of type  $k$ . Given the distribution of job types over a discrete set of job types, we have a simple expression for the payoff of a type  $k$  firm. Thus the expected return of a type  $k_n$  job in the free entry equilibrium (when  $\phi(k)$  is positive) is given by

$$c(k_n) = \sum_{i=1}^n (y(k_n) - y(k_{i-1})) e^{-\phi(k_i)} \left( 1 - e^{-(\phi(k_{i-1}) - \phi(k_i))} \right) \quad (8)$$

where  $k_1 = \hat{k}(\underline{k})$ ,  $k_0 = \underline{k}$  and  $\phi(k) = \infty$ .

Differencing the payoffs and opportunity costs of any pair of adjacent job types, we get the following difference equation

$$c(k_{i+1}) - c(k_i) = (y(k_{i+1}) - y(k_i)) e^{-\phi(k_{i+1})} \quad (9)$$

which must be satisfied for all job types offered in equilibrium. This result can be used to derive a necessary condition for the free-entry equilibrium with positive contribution by all job types in the continuous case. We have

**Proposition 3.** *If a continuum of job types on an interval  $k \in [\hat{k}, k^*]$ , then the supply of jobs above type  $k$  jobs is given by*

$$\phi(k) = \log \left( \frac{y'(k)}{c'(k)} \right) \quad (10)$$

*Proof.* Let  $\Delta = k_i - k_{i-1}$  denote the interval of successive job types for which equation (9) holds. The proposition then follows by taking the limit of equation (9) as the interval  $\Delta$  becomes small.  $\square$

Note that propositions 2 and 3 give a boundary condition for the supply of jobs and a differential equation relating the measure of jobs on any interval above this boundary point. It is also clear from equation (10) that the upper bound on job types is  $k^*$  where  $y'(k^*) = c'(k^*)$ . In this case, the supply of jobs at and above  $k^*$  must be zero.

## 2.5 Social planning solution

The results summarized in propositions 1 through 3 give only a set of necessary conditions for a particular distribution of job types in a directed search equilibrium. We have not yet used information about the second derivatives of the functions  $y(k)$  and  $c(k)$  to establish the uniqueness and existence of this equilibrium. An important step to establishing these results is the main result of Shimer (2005) and Shi (2001) that the directed search equilibrium is constrained efficient. We can then prove that the assignment, which is described in propositions (2) and (3) is the decentralized equilibrium given appropriate conditions on  $y(k)$  and  $c(k)$ . In particular, we will use the social planning problem to prove that a continuum of offer types with an upper bound job type defined by  $k^*$  is constrained efficient. The main result is stated as follows

**Proposition 4.** *If there is a continuum of job types, the social planner is maximizing surplus according to the assignment given by propositions (2) and (3) whenever*

$$\left[ \frac{-[y''(k)c'(k) - y'(k)c''(k)]}{(y'(k))^2} \right] \left[ \log \left( \frac{y'(k)}{c'(k)} \right) \right]$$

is positive over the range  $k \in [\widehat{k}, k^*]$

*Proof.* Wages are determined by the second best offer. The distribution of the second best offer is given by  $G_2(k) = e^{-\phi(k)} + \phi(k)e^{-\phi(k)}$  and the corresponding density is then

$$g_2(k) = -\phi'(k)\phi(k)e^{-\phi(k)}$$

Therefore, we need to calculate  $\phi'(k)$

$$\begin{aligned} \phi'(k) &= \frac{d \log \left( \frac{y'(k)}{c'(k)} \right)}{dk} \\ &= \frac{y''(k)}{y'(k)} - \frac{c''(k)}{c'(k)} \end{aligned}$$

Inserting  $\phi(k)$  and  $\phi'(k)$  in the equation for  $g_2(k)$  gives us

$$\begin{aligned} g_2(k) &= -\phi'(k)\phi(k)e^{-\phi(k)} \\ &= \left[ \frac{-[y''(k)c'(k) - y'(k)c''(k)]}{(y'(k))^2} \right] \left[ \log \left( \frac{y'(k)}{c'(k)} \right) \right] \end{aligned}$$



As we know that we must have that  $\phi'(k) < 0$  we have that  $y''(k)c'(k) - y'(k)c''(k) < 0$ . Naturally, the density is positive for all  $k$  until  $k = k^*$  where  $y'(k^*) = c'(k^*)$ . The expected sum of wages up to some value  $\tilde{k}$  is given by

$$\int_{\hat{k}}^{\tilde{k}} y(k) g_2(k) dk = \int_{\hat{k}}^{\tilde{k}} y(k) \left[ \frac{-[y''(k)c'(k) - y'(k)c''(k)]}{(y'(k))^2} \right] \left[ \log \left( \frac{y'(k)}{c'(k)} \right) \right] dk$$

Since all terms are positive for  $k \leq k^*$  the social planner maximizes the r.h.s. by setting  $\tilde{k} = k^*$ .  $\square$

Therefore, we find that a simple concavity condition ensures an equilibrium with a distribution of job types, which is equivalent to that described by propositions 2 and 3. For example, if  $c(k)$  is linear, the requirement is simply that  $y(k)$  is increasing and concave.

## 2.6 Expected wages at each firm

The following proposition establishes that we can order firm types (higher opportunity cost firms) by the expected wages of a common worker type employed by such firms.

**Proposition 5.** *Higher firm types pay ex ante identical workers higher expected wages than lower firm types*

*Proof.* Consider the average productivity of an ex ante identical worker over the range of jobs less than some value of job type,  $\tilde{k}$ .

$$E(y | k \leq \tilde{k}) = \frac{\int_{\hat{k}}^{\tilde{k}} y(k) g_1(k) dk}{G_1(\tilde{k})} \quad (11)$$

This expectation is increasing in  $\tilde{k}$  because higher job types are more productive. Since this expectation is also the expected wage paid by a type  $\tilde{k}$  job, we establish the monotonicity of expected wage payments by higher firm types.  $\square$

## 2.7 Joint distribution of valuations

It is straightforward to calculate the joint distribution of valuations. For example, let  $G(k_1, k_2)$  denote the fraction of workers with a best job greater than  $k_1$  and a second best job greater than  $k_2$ . We find:

**Proposition 6.** *For values of  $k_1, k_2 \in [\hat{k}, k^*]$ ,*

$$G(k_1, k_2) = (1 + \phi(k_2) - \phi(k_1)) e^{-\phi(k_2)} \quad (12)$$

*Proof.* See Appendix  $\square$

The joint distribution of first and second best offers is not needed to describe the distribution of wages in the static model. However, in a dynamic environment, the worker's per period wage will be a function of his first and second best offers. The basic idea is that the worker's second best offer describes the threat point used in setting the wage with his current employer while the

productivity of his current employer (his best offer) gives the worker's threat point when setting wages with any future employer who might be contacted by on-the-job search. The worker's current wage then balances these two concerns.

## 2.8 Equivalence with seller posting

An alternative pricing mechanism to an auction is the assumption that each seller posts a price for each type of buyer. The analysis of this case, which is considered by Shimer (2005), can be used to compare the payoffs of sellers and buyers in the auction and price posting models, and to verify that the expected payoffs of buyers and sellers in both models are equivalent. Consequently, the posted price offered to buyer  $k$  by the seller must equal the price that this buyer would expect to be pay in the auction.

## 2.9 Equivalence with buyer posting

Another pricing mechanism that can be applied to this framework is the posting of a price by each buyer. This price posting mechanism has been explored in Mortensen (2003) for a closely related model of coordination frictions - which is itself a variation on the classic models of price dispersion by Butters (1977), Burdett and Judd (1983), and Burdett and Mortensen (1998). In Mortensen's model, sellers are randomly assigned to buyers as in our model of coordination frictions, and the expected payoffs of sellers and heterogeneous buyers are also equivalent to the outcomes of the auction model.<sup>3</sup> Moreover, in the limit as the number of different buyer types gets large (as we are assuming in the continuous case), the price posted by each type of buyers will converge to a single value. Therefore, prices in the buyer posting equilibrium are equivalent to the prices in the seller posting equilibrium.

## 3 Making the model dynamic

The dynamic model is a repeated version of the static model. The workers and firms are now infinitely-lived with risk neutral preferences and a common discount factor  $\beta$ . Time is discrete. The total population of workers is normalized to one and the population of firms is determined by free entry. At the start of each period new firms can choose to enter and select a type  $k \in [0, \infty]$  job opportunity, where  $y(k)$  and  $C(k)$  denote the job's productivity and capital cost, respectively. Each worker has one unit of labor to sell and each job opportunity can employ one worker. Each worker is also endowed with an outside opportunity  $y(\underline{k})$ , which is the worker's productivity if no job opportunity is forthcoming. Once a worker is assigned a job, there is a probability  $\delta$  that the job opportunity is destroyed.

The matching game within each period is identical to the static model. The only additional elements are (i) the existence of random job separations at the beginning of the period and (ii) the possibility for additional job creation and matching opportunities in each subsequent period.

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<sup>3</sup>The payoffs of this model are developed on pages 12-14 of Mortensen (2003).

### 3.1 Equilibrium

Let  $\Lambda(k)$  denote the present value of a match between a worker and a type  $k$  firm. We assume that the function relating a job's present value to its type is well behaved. This means that the derivatives  $\Lambda'$  and  $-\Lambda''$  are positive. This assumption allows us to solve the equilibrium job offer distribution in a fashion equivalent to the static model. Using propositions 2 and 3, the supply of type  $k$  jobs directed at workers employed in jobs with type  $k_1$  employers is given by the function

$$\phi(k) = \log \left( \frac{\Lambda'(k)}{C'(k)} \right) \quad (13)$$

over the range  $[\hat{k}(k_1), k^*]$  where  $\hat{k}(k_1) = \arg \max (\Lambda(k) - \Lambda(k_1)) / C(k)$  and  $k^* = \arg \max \Lambda(k) - C(k)$ .

On-the-job search is accommodated by the fact that all new employers making job offers know the worker's current employer's type. Therefore, the worker's option to remain with his current employer is analogous to the outside option of the static model. In particular, the productivity of the worker's current employer does not affect the function  $\phi(k)$ , but rather simply changes the lower support over which the different job types are offered.

#### 3.1.1 Wages and productivity

Given that we have an immediate solution for the equilibrium allocation of job offers to workers each period as function of the present value of different matches and the associated capital costs of different job types, we can now use the workers' asset equations to derive the wages and productivity of each type of job. The worker's current second best offer is the reservation value that was used to set the wage with his current employer, while the productivity of the current employer (the best offer) gives the worker's reservation value when setting wages with any future employers contacted by on-the-job search. Therefore, the expected present value for a worker in a type  $k_1$  job with a second best offer of  $k_2$  at the start of the period is given by

$$W(k_1, k_2) = \Lambda(k_2) G_1(\hat{k}) + \Lambda(k_1) \left( G_2(\hat{k}) - G_1(\hat{k}) \right) + \int_{z=\hat{k}}^{k^*} \Lambda(z) dG_2(z) \quad (14)$$

where  $\hat{k} = \hat{k}(k_1)$ ,  $G_1(k) = e^{-\phi(k)}$  and  $G_2(k) = e^{-\phi(k)} + \phi(k) e^{-\phi(k)}$ . The first term on the right hand side of this equation captures the event that the worker gets no new offers this period; the second term is the event that the worker has a single offer, which means that he will be paid a wage equal to the total surplus associated with employment at the incumbent employer; and the final term captures the pay increases due to the possibility of multiple offers.

Given that wage contracts are determined by auction, the present value of a worker with a type  $k_1$  employer and a type  $k_2$  second best offer is  $\Lambda(k_2)$ . This means that the wage  $w(k_1, k_2)$  of a worker in this negotiation state satisfies the following asset equation:

$$\Lambda(k_2) = w(k_1, k_2) + \beta [(1 - \delta) W(k_1, k_2) + \delta W(\underline{k}, \underline{k})] \quad (15)$$

where the future stream of returns is given by  $W(k_1, k_2)$  if the worker does not suffer a job loss at the end of the period and  $W(\underline{k}, \underline{k})$  otherwise.

If  $k_1 = k_2 = k$ , then the worker effectively becomes the residual claimant of his employment contract - he owns the job. In this case, the workers earn a wage equal to the output of the firm. That is

$$w(k, k) = y(k). \quad (16)$$

### 3.1.2 Distribution of productivities

Let  $u$  denote the steady-state unemployment. In equilibrium, the flow out of unemployment,  $u \left(1 - G_1 \left(\widehat{k}(\underline{k})\right)\right)$  equals the flow into unemployment, which consists of workers laid off who do not receive a job offer, that is  $(1 - u) \delta G_1 \left(\widehat{k}(\underline{k})\right)$ . Therefore, the steady-state unemployment is given by

$$u = \frac{\delta G_1 \left(\widehat{k}(\underline{k})\right)}{1 - (1 - \delta) G_1 \left(\widehat{k}(\underline{k})\right)} \quad (17)$$

We let  $n(k)$  denote the density of workers employed in a type  $k$  job and  $N(k)$  denote the distribution of workers with job types less than  $k$  or being unemployed. Furthermore, let  $\widehat{k}^{-1}(k)$  denote the inverse of the function  $\widehat{k}(\underline{k})$ . The transition equation for the density of job types is given by

$$n'(k) = n(k) (1 - \delta) G_1 \left(\widehat{k}(k)\right) + \left[u + (1 - \delta) N \left(\widehat{k}^{-1}(k) - u\right) + \delta (1 - u)\right] g_1(k)$$

where the first term on the right hand side is the density of workers in type  $k$  jobs who stays at the type  $k$  firm whereas the second term is the density of workers changing to type  $k$  jobs. The steady-state density of workers in type  $k$  jobs is solved for by setting  $n'(k) = n(k)$  and is given by

$$n(k) = \frac{\left[\delta + (1 - \delta) N \left(\widehat{k}^{-1}(k)\right)\right] g_1(k)}{1 - (1 - \delta) G_1 \left(\widehat{k}(k)\right)} \quad (18)$$

where the end point of this differential equation is  $N \left(\widehat{k}^{-1}(k)\right) = u$ .

The distribution of productivities is then characterized by the density in equation (18) and by equation (16), which gives the productivity of a type  $k$  employer. That is

$$\Omega_p = \{y(k), n(k) | k \in [\underline{k}, k^*]\}$$

It is also worth noting that the computation of the distribution of worker productivity requires only knowledge of  $G_1(k)$ .

### 3.1.3 Distribution of wages

The final task is to characterize the steady-state joint distribution of first and second best offers. In each period, a worker employed in a type  $z \in [\underline{k}, k^*]$  job, has a joint distribution of jobs offers,  $G(k_1, k_2)$ , over the interval  $[\widehat{k}(z), k^*]$ , where  $G(k_1, k_2)$  is given by equation (11). We let  $g(k_1, k_2)$  denote the implied joint density of offers for an unemployed worker ( $z = \underline{k}$ ). Let  $x(k_1, k_2)$  denote the joint density of workers employed in a type  $k_1$  job and with a second best

opportunity of a type  $k_2$  job. The transition equation for  $x(k_1, k_2)$  is given by

$$\begin{aligned}
x'(k_1, k_2) &= x(k_1, k_2) (1 - \delta) G_1(\widehat{k}(k_1)) \\
&\quad + \left( u + (1 - \delta) \left( N(\widehat{k}^{-1}(k_2)) - u \right) + \delta(1 - u) \right) g(k_1, k_2) \\
&\quad + n(k_2) (1 - \delta) 1 \left\{ k_2 \geq \widehat{k}^{-1}(k_1) \right\} \int_{\underline{k}}^{\widehat{k}(k_2)} g(k_1, \widetilde{k}_2) d\widetilde{k}_2
\end{aligned} \tag{19}$$

where the first term on the right hand side is the quantity of agents in the  $(k_1, k_2)$  state in the previous period who do not loose their job and are not recruited to a new firm this period, the second term is the quantity of workers who move into the  $(k_1, k_2)$  state by means of getting multiple offers, and the third term is the quantity of workers who move into the  $(k_1, k_2)$  state by means of getting a single type  $k_1$  offer and having a type  $k_2$  incumbent employer. The steady-state distribution is solved by setting  $x'(k_1, k_2) = x(k_1, k_2)$ . We have that

$$\begin{aligned}
x(k_1, k_2) &= n(k_1) \frac{\left( \delta + (1 - \delta) N(\widehat{k}^{-1}(k_2)) \right) g(k_1, k_2)}{\left( \delta + (1 - \delta) N(\widehat{k}^{-1}(k_1)) \right) g_1(k_1)} \\
&\quad + n(k_2) \frac{(1 - \delta) \left\{ k_2 \geq \widehat{k}^{-1}(k_1) \right\} \int_{\underline{k}}^{\widehat{k}(k_2)} g(k_1, \widetilde{k}_2) d\widetilde{k}_2}{1 - (1 - \delta) G_1(\widehat{k}(k_1))}
\end{aligned} \tag{20}$$

The joint distribution of wages and productivities for the economy is then characterized by this density equation together with equations (15) and (16), which describes the wages and productivities of all workers as a function of their employment state,  $(k_1, k_2)$ . That is

$$\Omega_w = \{w(k_1, k_2), x(k_1, k_2) | k_1, k_2 \in [\underline{k}, k^*]\}$$

## 4 Auctions versus posting

It will prove useful for many purposes to assume that wages are posted (either by the seller or the buyers). However, if we assume that firms (or perhaps workers) post wages and commit to these wages forever, then the equilibrium will be very different from the auction model. For example, suppose that an intermediate type firm posts a wage and commits to paying the worker during the entire course of employment. This will imply very different incentives than the auction model concerning raiding by other firms. If a firm raids under auctions, then it knows it will have to pay the entire surplus of the current match while if it considers raiding under the committed posted wage then the expected wage payment will be lower. So, other things equal, the full commitment posting equilibrium will be subject to too much raiding, because the raiding firm only needs to pay the worker their current wage and not the value of their current match.

It is a straightforward to construct a modified posting equilibrium that ensures expected pay-offs are equivalent under auctions and posted prices. Here we simply assume that firms respond to any counter-offers that are associated with raiding. Therefore, the posted wage only extends a commitment to the wage offer in the period that it is made and not to subsequent periods. Of

course, under this directed equilibrium, firms will raid only if they will hire the worker. Therefore, the modified posting mechanism will imply a constant wage over the worker's career at the firm (there is no other source of wage progression such as human capital accumulation or changing outside opportunities).

In this constrained efficient equilibrium, the posted wage is equal to the expected wage under auctions. Therefore, by proposition 5, we can simply integrate over the different present values for workers employed in a type  $k_1$  firm to get the present value of being employed in this firm under posting.

$$W(k_1) = \frac{\int_{\hat{k}}^{k_1} \Lambda(z) dG_1(z)}{G_1(k_1)} \quad (21)$$

We then derive the relevant wage flows from the underlying asset equations as in the dynamic auction model. In particular,  $W(k_1, k_2)$  is now independent of  $k_2$  and equation (14) is replaced by the following equation.

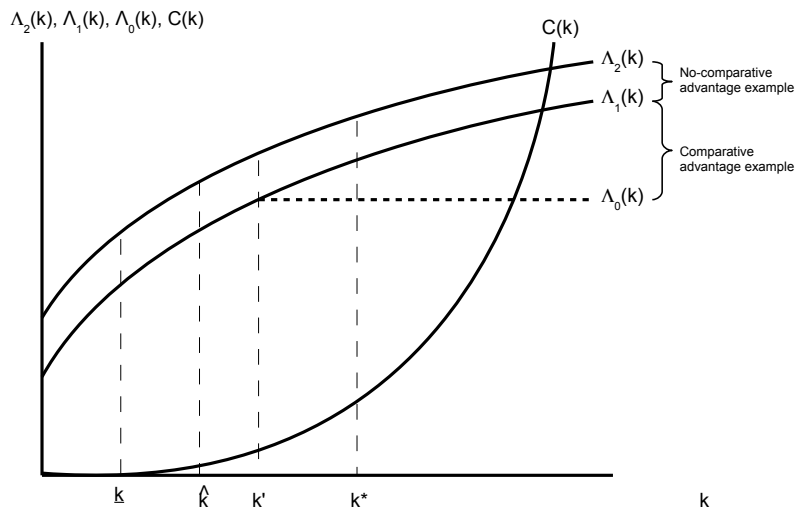
$$W(k_1) = \Lambda(k_1)G_1(\hat{k}(k_1)) + \int_{\hat{k}(k_1)}^{k^*} \Lambda(z) dG_1(z) \quad (22)$$

A key analytical advantage of the posting model is that wages depend only on the worker's job type and thus each productivity level is associated with a unique wage level. Thus there is no need to compute the joint density of first and second highest employer valuations in order to compute the distribution of wages.

## 5 Heterogeneous workers

Heterogeneous workers are easy to introduce into our environment because matching is directed. The main idea is that the opportunity costs of each type of vacant job must be the same across matching markets, but the surplus associated with a match between each worker type and each firm type can be different. Figure 1 illustrates an example of heterogeneous workers with and without comparative advantage, which we discuss in this section.

Figure 1. Examples of heterogeneous workers



## 5.1 No-comparative advantage

A useful benchmark model is to assume that no worker has comparative advantage over another worker. In this case, with different worker types  $i, j$ , the functional form of  $\Lambda_i(k)$  satisfies  $\Delta = \Lambda_i(k) - \Lambda_j(k)$  for all  $k$  where  $\Delta$  is a scalar. From propositions 2 and 3, this no comparative advantage case implies that all workers get the same frequency of offers from each type of firm even though they are of different productivity. This also means, given the common value  $\underline{k}$ , that the values of  $\hat{k}$  and  $k^*$  are the same in equilibrium. Figure 1 illustrates an example of a no-sorting equilibrium for type 1 and 2 workers such that the functions  $\Lambda_i(k)$  and  $C(k)$  satisfy the concavity requirement of proposition 4.

If there is no sorting, a higher worker type produces the same additional surplus in each job. Moreover, because the matching equilibrium is constrained efficient, the worker must always receive this same expected amount as compensation in all job types. In the static model, this means that the no sorting case is equivalent to assuming that there is no correlation between the wage for each worker type and the productivity of each firm type. In the dynamic model, the no sorting outcomes also implies no correlation between the expected discounted presented value of wages for each worker type and the firm type. Moreover, since all workers enjoy same arrival rate of offers that move them from low type firms to higher type firms, there will also be no correlation between the wages of each type of worker and firm's type. Therefore, the no sorting outcome implies a zero correlation between worker and firm fixed effects for wage equations in the form of Abowd, Kramarz and Magolis (1999).

## 5.2 Assortative matching

Assortative matching is a nonrandom matching pattern where particular groups of workers and firms match with each other more frequently than would be expected under random matching. In our model, assortative matching occurs when some group of workers has comparative advantage over another group. This means that functional forms for  $\Lambda_i(k)$  are not shifted in parallel as in the no comparative advantage case.

The example in figure 1 can also be used to show that equilibrium wages are not generally monotone in productivity if workers have comparative advantage. The type 0 worker is equally productive as the type 1 worker in job types up to  $k'$ , but produces no additional output beyond this point. All workers have the same value of home production and thus get the same total mass of offers, as is determined by proposition 2. However, the type 1 and 2 workers get job offers from employers above type  $k'$  while the type 0 worker does not. Therefore, given the formula in proposition 4, the expected value of returns of a type 0 worker in a type  $k'$  job,  $E_0(\Lambda | k \leq k')$ , is measurably higher than the expected value of returns of a type 1 worker in a similar job,  $E_1(\Lambda | k \leq k')$ , because both workers have the same distribution of offers below type  $k'$  and the low type worker has a mass of type  $k'$  offers while the higher type worker does not. If this mass of job offers is sufficiently large, then the type 0 worker also earns a higher expected value of returns in the type  $k'$  job than the type 2 worker, who is actually more productive in this job type. The implication is that we cannot generally use the wages of two workers within a firm to rank workers.<sup>4</sup>

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<sup>4</sup>Results related to the non-monotonicity of wages are used by Eeckhout and Kircher (2011) to establish that

Finally, we can note that the assumptions needed to generate overlapping matching sets in our model of comparative and absolute advantage are fundamentally different than the essential assumptions of random matching models. For example, Gautier and Teulings (2012) develop a random matching model that presumes that firm and worker types are ordered on a circle - with better matches being between workers and firms with similar positions - and that each firm type has identical opportunity costs of not being matched. In this tractable model of comparative advantage, they find that matching is assortative and that matching sets overlap. However, in the directed search equilibrium with identical opportunity costs for each employer type, each worker type will only get job offers from a firm type of the same location. So, an ordered menu of job types by opportunity costs, which is essential to our model of wage and productivity dispersion, is not essential in the theory of undirected search.

## 6 Taking the model to the data

In this section, we will estimate the model in order to understand the nature of these trade-off, and to learn how such estimation results are affected by the alternative pricing mechanisms and worker heterogeneity.

Our empirical analysis of alternative pricing mechanisms centers on the auction and price posting models of coordination frictions. Each of these two models are of basic interest. On the one hand, the auction model functions as a benchmark for an environment with the maximum amount of wage dispersion relative to dispersion of job productivities. On the other hand, the price posting model functions at the opposite extreme by minimizing the amount of wage dispersion given the equilibrium dispersion of job productivities.

Our empirical analysis of worker heterogeneity seeks to find out what amount of productivity dispersion is attributed to specific groups of workers under the price posting and auction specifications. To study this problem we compare the implications of a homogenous workers model with a benchmark model of heterogeneous workers. We are interested in these two specifications for two basic reasons. The homogenous workers model is of interest because it functions to explain wage dispersion purely by the equilibrium dispersion of job productivities. Similarly, the heterogeneous workers model (with no-comparative advantage both at home and at work) functions to explain wage dispersion by allowing for worker productivity differences while retaining the equilibrium implication that all workers are equally likely to be employed by all firms. In effect, this latter specification eliminates the additional mechanism of assortative matching. The estimated models of this section then provide a natural starting point to develop further sensitivity analysis, including the problem of assortative matching. This sensitivity analysis is developed in section 7.

### 6.1 Baseline Parameter Values

The main benchmark model has ex ante identical workers and firms, and it is described by the following parameters: (i) the discount factor  $\beta$ , (ii) the exogenous rate of job separations  $\delta$ , and (iii) the menu of opportunity costs associated with jobs of each productivity type  $\{y(k), C(k)\}$ .

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it is generally impossible to establish whether matching is positive or negative assortative using only wage data.



For the estimation we set  $M = 51$  and fix the following parameters  $\beta = 0.99$ ,  $\delta = 0.12$  and total mass of offers,  $\phi_1 = 1.187$ , so we do not use these parameters to fit the productivity and wage distributions. The latter two parameters determines the equilibrium unemployment to 5 percent. We estimate the benchmark model using both auction and price posting specifications. We also estimate the models with worker heterogeneity and no-comparative advantage.

## 6.2 Data

We seek to match the empirical observable wage and productivity distributions. For this purpose we use a Danish register-based matched employer-employee data set for the year 2007. The hourly wage rate is taken from the Integrated Database of Labor Market (IDA). This wage is calculated by Statistics Denmark by dividing the labor income by the number of hours worked. Whereas the labor income is precisely recorded in the administrative registers, the hours worked is imputed from mandatory pension payments which have four levels depending on the number of hours worked.

Our measure of firm productivity is the hourly average of value-added per full-time worker. Statistics Denmark conducts an annual survey of firms. Firms are sampled according to their size such that firms with more than 49 workers are always sampled, whereas firms with, for example, between 5 and 9 workers are only sampled with probability of 10 percent. We only use observations for persons employed by firms in the survey.<sup>5</sup>

We use the FIDA key to link the employee and employer data. We have access to a population data set of Danes aged 15 years or more. This data set has 4,465,874 observations. We only select workers aged 16-64 which reduces the sample to 3,542,311 persons. Furthermore, we only include persons with a positive wage and with a reliable wage estimate. This leaves us with 2,156,719 observations. Additionally, we exclude workers employed in the public sector which reduces the sample to 1,296,824 observations. Next, we only include observations for firms with a positive value-added per worker which reduces the sample size to 1,077,035. Next, we only include the following five industries: a) manufacturing, b) construction, c) wholesale and retail, d) transport, storage and communication, and e) real estate, renting and business activity. This gives us a sample of 1,031,374 observations. Finally, we discard all observations where the value-added is imputed by Statistics Denmark. This gives us a final data set of 651,722 workers employed by 8,236 firms.

## 6.3 Estimation

We fix the parameters for the discount rate, the overall job arrival rate for unemployed and the exogenous job destruction rate. We assume  $M = 51$  different firm types and estimate the parameters  $\phi_2, \dots, \phi_M$  and  $\Lambda_1, \dots, \Lambda_M$  by use of nonparametric simulated maximum likelihood (Refer to Fermanian and Salanie (2004)). From data we can by kernel methods estimate the density function of, for example, wages, which we denote by  $\hat{p}_h(w)$  where  $h$  is the bandwidth whereas we from our theoretical model for a given set of parameters  $\theta$  can calculate the density function  $p(w|\theta)$ . However, since we only have a discrete wage distribution for the theoretical

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<sup>5</sup>The data set draws on the same data sources as in Bagger, Christensen and Mortensen (2011) and we refer the reader to this paper for more details on the structure of the data sets used and the precise variable definitions.

model and since these discrete points are endogenous we use a smooth density function  $\tilde{p}_b(w|\theta)$  using kernel methods also for the theoretical model and where  $b$  is the bandwidth. The objective of the estimation is to minimize the distance between the two probability density functions and we use the Kullback-Leibler distance. We select  $N = 100$  points of the wage distribution, i.e.  $\bar{w}_1, \dots, \bar{w}_N$ .

$$L(\theta_N) = \frac{1}{N^2} \sum_{i=1}^N (\log \hat{p}_h(\bar{w}_i) - \log \tilde{p}_b(\bar{w}_i|\theta)) \hat{p}_h(\bar{w}_i)$$

Minimizing this function corresponds to maximizing

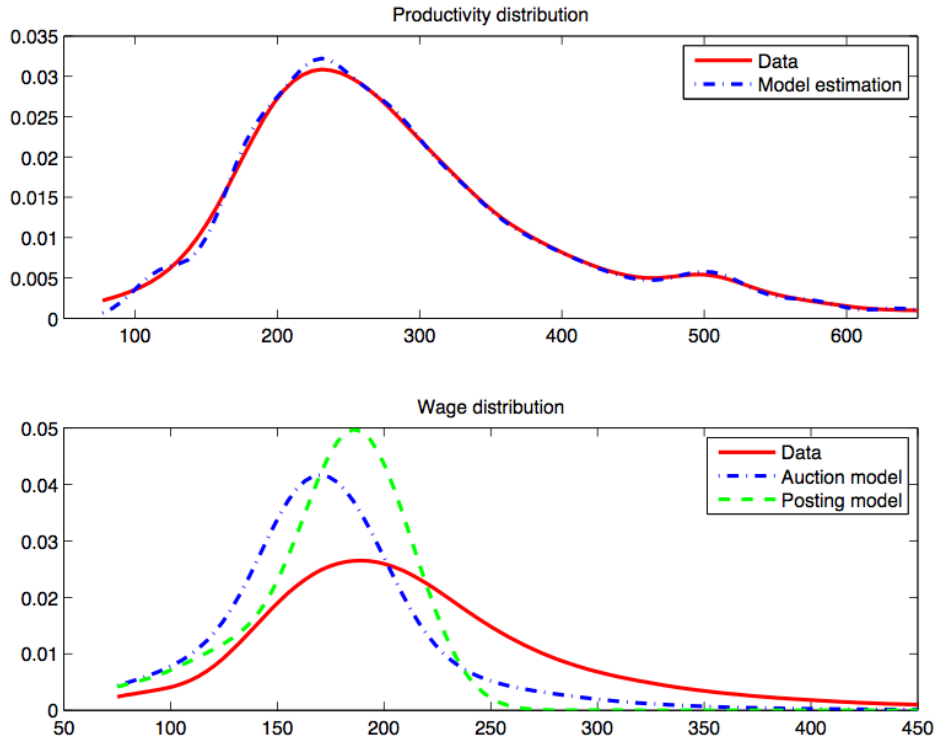
$$\tilde{L}(\theta_N) = \frac{1}{N^2} \sum_{i=1}^N \log \tilde{p}_b(\bar{w}_i|\theta) \hat{p}_h(\bar{w}_i)$$

For the estimation, we let  $\bar{w}_1, \dots, \bar{w}_{100}$ , be equally spaced in between the first percentile and the 97th percentile of the wage and productivity distributions.

## 6.4 Estimation of the benchmark model

The benchmark model with homogenous workers is first estimated using data on firm productivity. This exercise puts both the auction and price posting models on an equal footing, because each model predicts equivalent distributions of productivities given the same assumptions on the opportunity costs of jobs. The results of this estimation exercise is given in figure 2. For all figures in this section we use a bandwidth of 20.

Figure 2. The benchmark models estimated on productivities

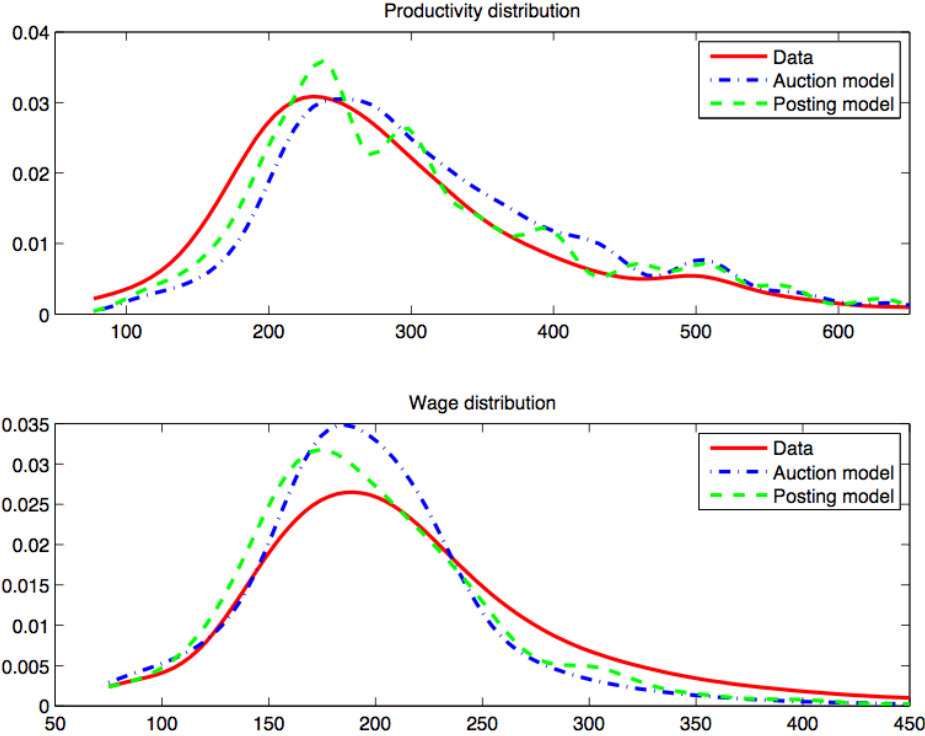


The top graph illustrates that the benchmark model is extremely flexible and can easily track the data on productivity dispersion (both models have identical implications for productivity

dispersion). Here, productivities are determined endogenously, because the number of firms entering this labor market of each productivity is determined by the equilibrium opportunities for ex post profits weighted against the associated opportunity costs of not entering this submarket in favor of another labor submarket. The second graph in figure 2 then compares the wage distributions of the two models with the actual data. We first note that the wage distribution of each model does not suffer a problem of increasing density over the support of the wage distribution. This is explained by the heterogeneity of jobs, which is determined endogenously. We also see that both models fail to explain key features of the wage distribution. Here we see the variance of the wage distribution is more realistic in the auction model but that the mode of the wage distribution is more realistic in the posting model.

Figure 3 gives the results of estimation when we try to fit both the wage and productivity distributions simultaneously, again assuming homogenous workers. In each case, the auction and price posting models are unable to closely match both wage and productivity dispersion.

Figure 3. Benchmark model estimated on wages and productivities



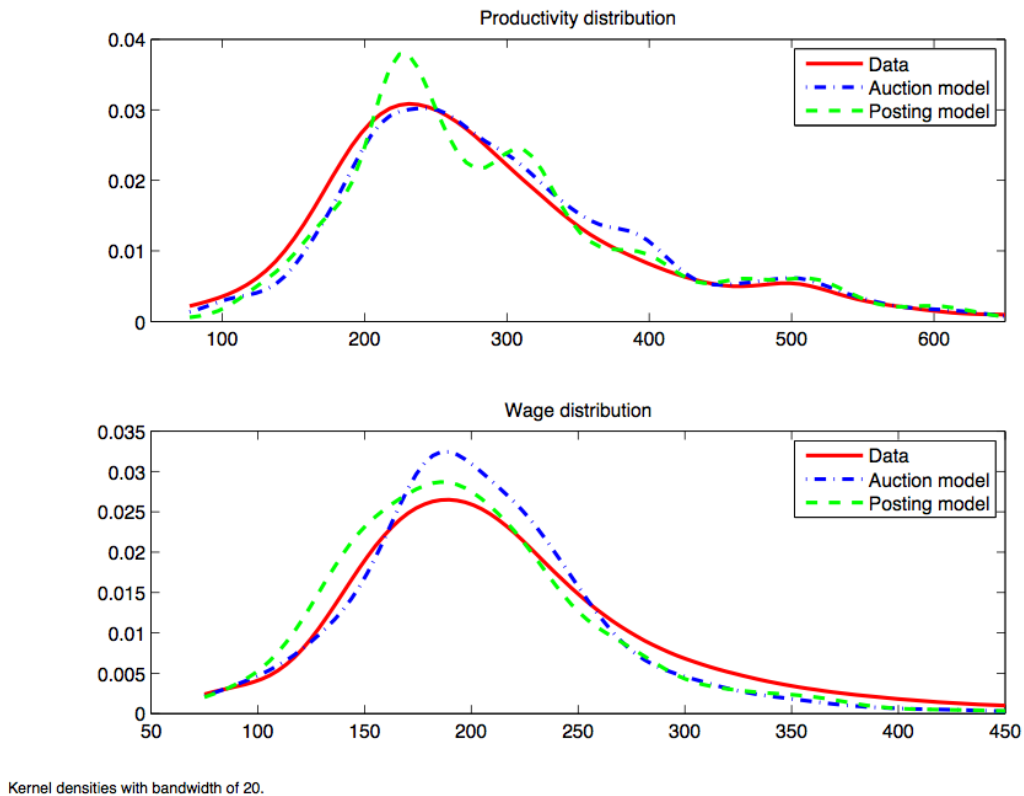
Kernel densities with bandwidth of 20.

The estimated productivity distributions are affected by how the two models in figure 2 fail to fit the wage distribution. The auction model now shifts the mode of the productivity distribution to the right to better explain the wage distribution. The posting model makes no change in the mode of the distribution but it tends to widen the tails of the productivity distribution especially concerning the dispersion of high quality jobs with productivity greater than 300 Danish Kroners per hour.

## 6.5 No-comparative advantage

The final estimation exercise is to allow for the possibility of worker heterogeneity. We adopt a version of worker heterogeneity that does not affect the job offer distribution for each type of worker. Namely, we assume that worker's productivity and outside options are scaled by a common factor. This is closely related to the no-comparative advantage case considered in Shimer (2005) with the added assumption that outside options are also scaled. This assumption serves as a useful benchmark here because it rules out sorting - where different job types approaching different workers differently. The estimation results are given below.

Figure 4. Estimated model with no-comparative advantage



These estimation results reveal that worker heterogeneity with non-comparative advantage is broadly consistent with an equilibrium explanation of wage and productivity dispersion. The estimation results also reveal that there is no clear advantage over price posting and auction models to explain wage and productivity dispersion.

It is interesting that these models use worker heterogeneity in different ways to fit the data on wage and productivity dispersion. The auction estimation essentially creates two bins of very high and very low types with a small fraction (approx. 0.1) of agents acting as low types. The posting estimation creates a third bin of intermediate types and spreads the agents evenly across these three bins. Intuitively, this allows the auction model to shift the mode of the wage distribution while also giving a modest increase in the amount of wage dispersion. In the posting model, the only needed change is to increase the dispersion of wages and so this is captured by a disperse set of worker types spread evenly over the worker type space.

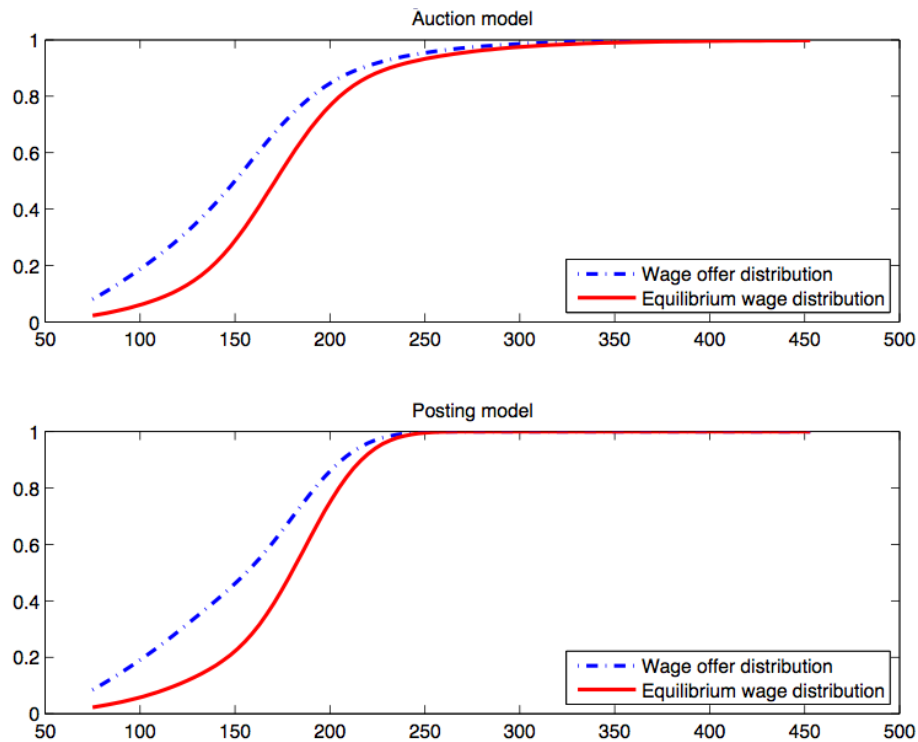
## 7 Sensitivity analysis

The empirical section revealed that our general model is sufficiently flexible give a reasonable fit of the observed wage and productivity dispersion with a parsimonious set of model parameters. The present section uses this estimated model to frame our discussion of how our model framework can be used to confront additional observations about the labor market in general equilibrium. The goal is to offer some insights into how our model might be further developed to quantify and explain observations related to (i) the job ladder, (ii) sorting, (iii) the joint behavior of wages and productivities over the business cycle, and (iv) the accumulation of general and specific human capital. This discussion is meant to clarify what is expected from the theory of directed search equilibrium and we leave estimation of these additional mechanisms to future work.

### 7.1 The job offer distribution

Figure 5 depicts the offer distribution of an unemployed worker (for both the auction and posting model) when we fit the benchmark model to exactly explain the same distribution of firm productivities (Recall the estimated model of figure 2).

Figure 5. Wage and wage offer distributions



Kernel densities for the auction model. Bandwidth used is 20.

When the dispersion of firm types in the offer distributions are identical, we find that the auction model generates much more wage dispersion among new hires even though each model is fitted to explain the same distribution of firm productivities. Most of this additional variance is attributed to wages at the top end of the wage distribution than at the bottom end. This is explained by the fact that lowest quality jobs pay very low wages in the posting model and thus

there is little difference in the variation in wages between the auction and posting models for low quality jobs. These different predictions are also consistent with related research analyzing random matching models. For example, Postel-Vinay and Robin (2002) find that the auction model is more realistic than the posting model (which is featured in Bontemps, Robin and van den Berg (2000)), because the auction model helps to explain the high variance of wages at the top end of the wage distribution given that the data features only limited variance in job productivities.

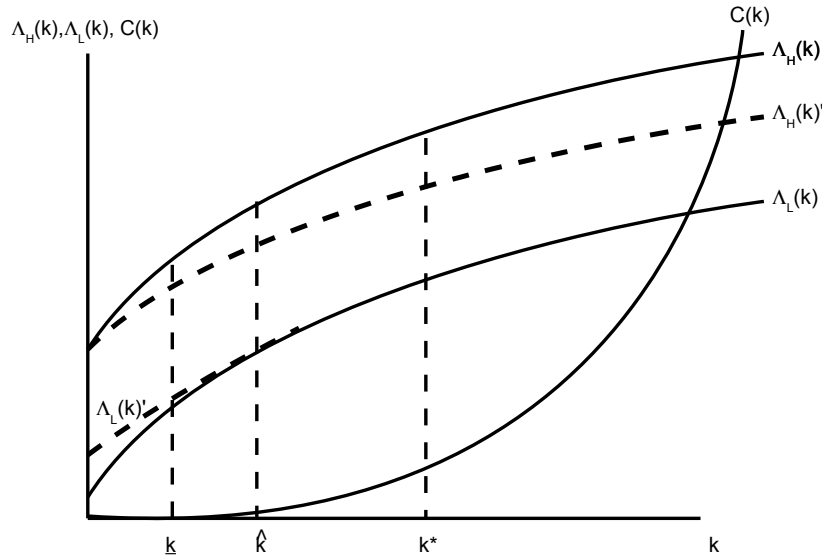
The difference between the job offer distribution of unemployed workers and the overall distribution of wages in our estimated model is explained by on-the-job search. On-the-job is important in our explanation of wage and productivity dispersion, because the workers are willing to accept a low productivity job with low exogenous rate of separations into unemployment only if there also exists future opportunities to be matched while employed (Refer to Hornstein et al (2012)). This means that we could not fit wage and productivity data in our model without on-the-job search while also maintaining realistic job exit rates into unemployment across all employment types.

Finally, the different job offer distributions of employed workers in our model are also consistent with related research that focuses on explaining the differences in job-to-job transitions. For example, Ridder and van den Berg (2003) find that higher wage/productivity jobs have low rates of job separations, which supports the idea that match specific rents can vary tremendously from one match to another. We also note that the auction model is distinguished from a posting model, because the auction model implies an independent match effect (wages can vary across similar employers) that has no bearing on the arrival of offers.

## 7.2 Assortative matching

We can use the estimated no-comparative advantage model to depict how exogenous changes might lead to assortative matching outcomes. For example, a progressive tax on the top income units (matches between high type workers and high type firms) will tend to flatten and decrease the function  $\Lambda_H(k)$  leaving the function  $\Lambda_L(k)$  unchanged. This sub-modular change in the surplus function,  $\Lambda_i(k)$ , leads to negative assortative matching (NAM), because it decreases the supply of high type jobs directed at high type workers without any effect on low type workers. Alternatively, a tax cut on low paying income units (matches between low type workers and low type jobs) will tend to flatten and increase the function  $\Lambda_L(k)$  without changing the function  $\Lambda_H(k)$ . This super-modular change in the set of surplus functions will lead to positive assortative matching (PAM), because it increases the number of low type workers in bad jobs without changing the rates of employment in such jobs by high type workers.

Figure 6. Comparative statics and assortative matching



The analysis of comparative advantage is complicated by the existence of on-the-job search, which is a basic feature of our estimated model. For example, suppose that low type workers have comparative advantage in high type jobs (i.e. has a steeper sloped schedule,  $\Lambda_i(k)$ ).<sup>6</sup> In this case, low type workers must (i) exit slowly out of unemployment, (ii) have on average lower expected starting wages, and (iii) be disproportionately matched to higher type firms. However, (i),(ii) and (iii) can also be the outcomes of our environment if high type workers have absolute advantage in all job types with comparative advantage in high type jobs, because of the non-monotonicity of wages with respect to particular job types, and the fact that good jobs are disproportionately represented in the distribution of job offers involving job to job transitions.

Bagger and Lentz (2012) develop a random matching model where high type workers search more intensively and their identification strategy is focused on the correlation between inferred skills and unemployment durations. The crucial difference of their model of undirected search equilibrium with ours, is that all workers draw (albeit with different frequencies) from a common distribution of job offers.

In order to assess whether a particular worker (or group of workers) has comparative advantage in our model, we need an ordering of workers and firms, and the slope of the function  $\Lambda_i(k)$  for each worker type. For example, suppose that we wish to compare the job opportunities of a particular worker (or a group of similar workers) with the opportunities of the worker type who populates our benchmark model in figure 2. In our analysis, each worker's schedule of match productivities,  $\Lambda_i(k)$ , is uniquely determined by his distribution of wage offers out of unemployment, because all workers draw offers from a common set of employers with identical opportunity costs. Job to job transitions are also informative of this wage offer distribution, because a worker continues to draw from an identical distribution of job offers (above a critical value) given his own type and his employer's type. This information characterizes the function  $\Lambda_i(k)$  and the slope of this function establish the nature of comparative advantage with the reference group.

<sup>6</sup>Sorting here is based on the menu of present values associated with each worker-employer match,  $\Lambda_i(k)$ , and not directly based on the firm's output (Eeckhout and Kircher 2011).

The ordering of workers into high and low types is given by the overall height of the function  $\Lambda_i(k)$  - as this determines the lifetime expected income of the worker. An ordering of firms is also straightforward to be obtained due to proposition 5.<sup>7</sup>

### 7.3 Business cycles

A directed search model also speaks to the fact that job offer distributions may vary not only across individuals but also across time. Therefore, another important application of a directed search equilibrium is to analyze labor market transitions related to business cycles (Menzio and Shi 2011). Directed search is analytically convenient because it considerably narrows the range of interactive strategies between firms and workers. Our model contributes to this line of analysis by allowing unemployed and employed workers to draw from a non-degenerate distribution of job offers at each point in time.

Since we assume search (and recruiting) is directed, it is very simple to introduce aggregate shocks into this environment. The main idea is that solution of the model depends only on the specification of the technology and not on the current distribution of workers across job types. In our dynamic analysis, this added complication simply means that the supply of job offers of a worker in a type  $k$  job is given by a modified version of equation (13). In particular, the equilibrium distribution of new jobs offered in each period must satisfy

$$\phi_t(k) = \log \left( \frac{\Lambda'_t(k)}{C'_t(k)} \right) \quad (23)$$

The goal is then to use this non-stationary version of our dynamic model to document how wage and productivity dispersion changes over time.

There are two interesting cases to consider. In the first case, we can assume that the functions  $\Lambda(k)$  and  $C(k)$  change according to the no-comparative case. This is relevant to a theory of economic growth where the amount of wage and productivity dispersion remains constant over time on the balanced growth path. The neutral outcome on the distribution of job types is illustrated by the same variations in  $\Lambda(k)$  and  $C(k)$  depicted in figure 1 with high type workers now serving to illustrate the outcome of an economic upturn. The basic idea here is to assume that we equally scale upwards the worker productivities across all jobs including home production, and then the theory will imply that workers will be employed in all job types with the same frequency, and that wages will simply increase to reflect the worker's higher productivity.

Another interesting case is perhaps more realistic for the analysis of business cycle shocks. In particular, a very different specification from the no-comparative advantage case is to assume that following a business cycle shock there is a different frequency by which all job types are created. We model this by imposing functional forms on  $\Lambda(k)$  and  $C(k)$  that generate a common increase/decrease in the number of offers across by all firms. This means that changes in technology scale  $\phi_t(k)$  for all job types  $k$  by a common factor  $\alpha$ ,

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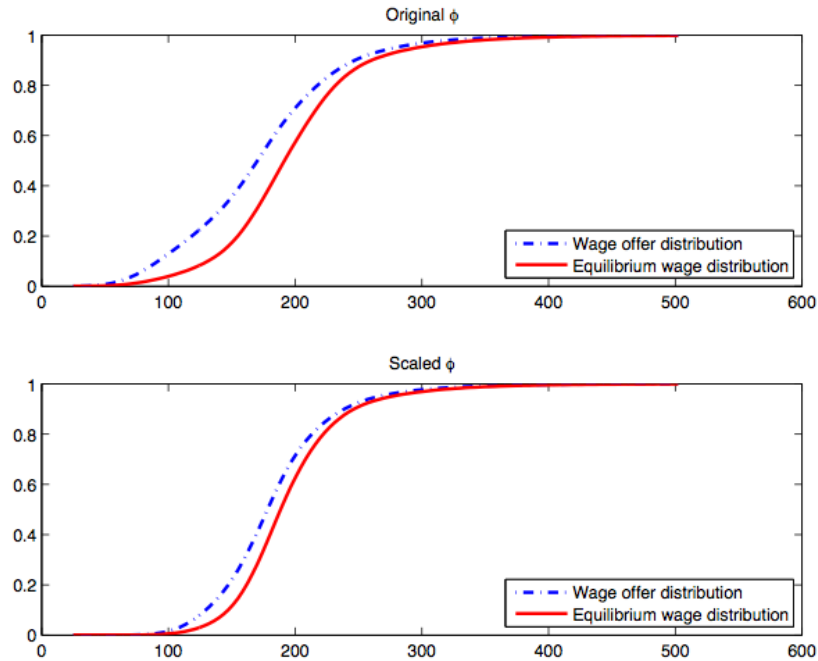
<sup>7</sup>Our estimated model presently orders firms by value added data. This is the same ordering implied by proposition 5 for no-comparative advantage, because a higher value added firm is then a higher wage firm. If there is comparative advantage, the non-monotonicity of wages with comparative advantage means that the ranking of firms must be decided by movements of similar workers across firms, and not by the productivity of groups of workers within firms. In this case, proposition 5 offers the only tool to order firm/job types.



$$\phi_{t'}(k) = \alpha \phi_t(k) = \log \left( \frac{\Lambda'_t(k; \alpha)}{C'_t(k; \alpha)} \right) \quad (24)$$

This version of the model is helpful for understanding changes in wage dispersion at high frequencies. For example, consider the estimated model that was described in figure 2. If we decrease the number of offers across all jobs by 30 percent, the unemployment increases from 5.0 percent to 8.5 percent.

Figure 7. Job offer distribution as a function of overall offer frequency,  $\alpha$



Kernel densities for the auction model. Bandwidth used is 20.

This extension of our estimated model is useful for understanding an emerging stylized fact that wage dispersion varies pro-cyclically over the business cycle (See Morin 2013).<sup>8</sup> This somewhat paradoxical finding is explained by the fact that a decrease in the frequency of job offers over all job types actually decreases wage dispersion. Given our estimated model, this is owed to the fact the high productivity jobs are extremely scarce in a downturn. Therefore, an increase in their number (along with other jobs) in an upturn will cause workers to gain a much larger income share from such jobs. This has the effect of flattening the wage offer distribution in upturns.

#### 7.4 Human capital accumulation

An important mechanism by which the opportunities of workers and firms change in the labor market over time is through the accumulation (or loss) of human capital. We now briefly discuss how specific and general human capital accumulation can be tractably analyzed in our model. We also discuss how we might identify the contributions of each mechanism to wage progression.

<sup>8</sup>Morin's (2013) model is closely related to ours. Some basic differences are that matching is not directed and that wage posting is an absolute commitment.

The accumulation of specific human capital changes the values  $(k_1, k_2)$  for a worker with their current employer. A gain of firm specific human capital implies the worker's current job type,  $k_1$ , must increase to the value of a comparably productive firm type that might hire this worker in a firm-to-firm transition. So an exogenous gain of specific human capital changes the employee's employer type to new value  $k'_1 = k_1 + \Delta$ .<sup>9</sup> The wage increase of the worker, assuming piecewise contracts, gives the worker a wage increase equal to his higher productivity. This change also gives a new value for  $k'_2 = k_2 + \Delta'$ , which represents the worker's bargaining position, such that the worker gets this wage increase in exactly the same way that would occur if the worker was raided by the higher type firm  $k'_1$ . The effects of the higher values  $(k'_1, k'_2)$  are to give the worker a higher wage and a corresponding reduction in the number of future job offers from rival employers.

The mechanics of general human capital are also straightforward to analyze. A gain in general human capital can be modeled as increase in the worker's productivity across all jobs and this has a different effect than an increase in specific human capital. In particular, if a worker's productivity increases in all jobs, then we have the same effect as the case where workers are ranked by absolute advantage but with no-comparative advantage. In this case, the worker will get also get a wage increase equal to his higher productivity. However, the worker gets no reduction in the frequency of offers, and the offers that do arrive now pay higher wages by an amount equal to the worker's greater productivity.

These results suggest that an ambitious empirical study could be developed to help identify wage gains due to search and to specific and general human capital. For example, Bagger et al (2012) develop and estimate an equilibrium job search model of worker careers, allowing for general human capital accumulation, employer heterogeneity and individual-level shocks. Our model could be readily applied to the type of data analyzed in Bagger et al (2012). The main difference is that our analysis would require specific human capital since wage increases inside the firm that reduce job exit rates would not exist otherwise. By contrast, Bagger et al (2012) do not attempt to identify specific human capital because they assume undirected search and thus employed workers may get competing offers that have the same effect.

Finally, we should conclude by noting that human capital accumulation is perhaps one of the key places to identify propagation mechanisms related to business cycles. Our analysis could be extended to integrate various sorts of propagation mechanisms within a unified framework. For example, we might wish to understand whether human capital is lost during unemployment (Pissarides 1992) or whether it is gained in particular types of jobs given particular work intensities (Rogerson and Wallenius 2009).

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<sup>9</sup>Alternatively, we might also wish to add an intensive effort margin and relate such transitions to worker investments in learning by doing. For example, Kennes and Knowles (2013) use a marriage model with 'kids' to detail how models with coordination frictions and matched specific investments are solved. The idea is that kids reflect an investment that has a specific component related to the utility of a marriage between the biological parents and a general component (which can be negative) related to how a single mother is able to form a productive marriage with another male who is not the father of the child.

## 8 Conclusions

We have presented a new model of directed search that can confront labor market facts related to stochastic non-degenerate job offer distributions in a matching market with a continuum of worker and firm types. We found that the technological requirement for non-degenerate job offer distributions is that the opportunity cost of higher vacant job types increases sufficiently fast relative to the potential productivity of a worker in each job type. We also found that an increase in the outside option of a worker leads to a complete loss of job offers by lower firm types, but it has no effect on the frequency of job offers by higher type firms. These results allowed us to easily extend the model of continuous, non-degenerate offer distributions to a dynamic environment with on-the-job search.

We also estimated the model using cross sectional wage and productivity data in an attempt to identify the opportunity costs of different job types. The estimated benchmark model of no comparative advantage offered a reasonable explanation of this data for both the auction and posting specifications. However, we also found that the estimated auction and posting models have different implications concerning the distribution of heterogeneous worker types.

A sensitivity analysis of our estimated benchmark model provided insights regarding how the theory could be used to explain additional data and how different structural mechanisms could be incorporated. We emphasized that our analysis can be used to explain various facts related to the job ladder both in and out of steady state, such as the frequency of job-to-job transitions, and wage dispersion and unemployment over the business cycle. We also offered suggestions on how to identify patterns of assortative matching and the accumulation of specific and general human capital. The main theoretical mechanism that unifies all of these problems is the existence of a common menu of opportunity costs across employer types that can be used to independently determine the unique job offer distribution of each worker type.

One useful application might be to develop the international trade version of this model. In particular, it is straightforward to label one group of workers, Canadians, and another group of workers, Americans, and to then work out the effects of different trade instruments, etc. This theory of equilibrium wage and productivity dispersion is also suggestive that there might be many opportunities to devise labor market policies that provides a more equal distribution of income without a major trade-off regarding efficiency (refer to Julien, Kennes, King and Mangin (2009)). Our theoretical model of coordination frictions in a dynamic directed search equilibrium with a continuum of buyer and seller types might also be relevant to the analysis of other markets.

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## Appendix

**Proposition. 6:** For values of  $k_1, k_2 \in [\hat{k}, k^*]$ ,

$$G(k_1, k_2) = (1 + \phi(k_2) - \phi(k_1)) e^{-\phi(k_2)}$$

*Proof.* Suppose a worker receives  $n$  job offers. The probability that  $j$  of these are below  $k_2$  and that  $n - j$  offers are less than  $k_1$  where  $\underline{k} \leq k_2 \leq k_1 \leq k^*$  is given by

$$\binom{n}{j} F(k_2)^j [1 - F(k_1)]^{n-j}$$

where  $\binom{n}{j} = \frac{n!}{(n-j)!j!}$ . Taking the negative cross-derivative of this delivers the joint density of the  $j$ 'th and  $j + 1$ 'th order statistics

$$\begin{aligned} g_{j,j+1}(k_1, k_2|n) &= - \left[ - \binom{n}{j} j F(k_2)^{j-1} (n-j) [1 - F(k_1)]^{n-j-1} f(k_1) f(k_2) \right] \\ &= \frac{n! j F(k_2)^{j-1} (n-j) [1 - F(k_1)]^{n-j-1} f(k_1) f(k_2)}{(n-j)! j!} \\ &= \frac{n! F(k_2)^{j-1} [1 - F(k_1)]^{n-j-1} f(k_1) f(k_2)}{(n-j-1)! (j-1)!} \end{aligned}$$

We are only interested in the best and second best offers, so we set  $j = n - 1$  and  $j + 1 = n$ . This gives us

$$g(k_1, k_2|n) = \frac{n! F(k_2)^{n-2} f(k_1) f(k_2)}{(n-2)!}$$

Summing over all possible number of job offers we obtain

$$\begin{aligned} g(k_1, k_2) &= \sum_{n=2}^{\infty} \frac{e^{-\phi(\underline{k})} \phi(\underline{k})^n}{n!} \frac{n! F(k_2)^{n-2} f(k_1) f(k_2)}{(n-2)!} \\ &= f(k_1) f(k_2) e^{-\phi(\underline{k})} \sum_{n=2}^{\infty} \frac{\phi(\underline{k})^n F(k_2)^{n-2}}{(n-2)!} \\ &= f(k_1) f(k_2) e^{-\phi(\underline{k}) F(k_2)} e^{-\phi(\underline{k})(1-F(k_2))} \sum_{n=2}^{\infty} \frac{\phi(\underline{k})^n F(k_2)^{n-2}}{(n-2)!} \\ &= \phi(\underline{k})^2 f(k_1) f(k_2) e^{-\phi(\underline{k})(1-F(k_2))} \sum_{n=2}^{\infty} \frac{e^{-\phi(\underline{k}) F(k_2)} (\phi(\underline{k}) F(k_2))^{n-2}}{(n-2)!} \\ &= \phi'(k_1) \phi'(k_2) e^{-\phi(k_2)} \end{aligned}$$

Integrating the density gives us the bivariate distribution function

$$\begin{aligned}
G(k_1, k_2) &= \int_0^{k_2} \int_{k_2}^{k_1} \phi'(z_1) \phi'(z_2) e^{-\phi(z_2)} dz_1 dz_2 \\
&= \int_0^{k_2} \phi'(z_2) e^{-\phi(z_2)} \int_{k_2}^{k_1} \phi'(z_1) dz_1 dz_2 \\
&= \int_0^{k_2} \phi'(z_2) e^{-\phi(z_2)} (\phi(k_1) - \phi(k_2)) dz_2 \\
&= \phi(k_1) \int_0^{k_2} \phi'(z_2) e^{-\phi(z_2)} dz_2 + \int_0^{k_2} -\phi'(z_2) \phi(k_2) e^{-\phi(z_2)} dz_2 \\
&= -\phi(k_1) \int_{-\phi(k_2)}^{-\phi(k_1)} e^u du + \int_0^{k_2} g(z_2) dz_2 \\
&= -\phi(k_1) \left( e^{-\phi(k_2)} - e^{-\phi(k_1)} + z_0 \right) + (1 + \phi(k_2)) e_0^{-\phi(k_2)} \\
&= (1 + \phi(k_2) - \phi(k_1)) e^{-\phi(k_2)} + \phi(k_1) \left( e^{-\phi(k_1)} - z_0 \right) \\
&= \phi(k_1) \int_0^{k_2} \phi'(z_2) e^{-\phi(z_2)} dz_2 + \int_0^{k_2} -\phi'(z_2) \phi(k_2) e^{-\phi(z_2)} dz_2 \\
&= G_2(k_2) - \phi(k_1) \int_{-\phi(k_2)}^{-\phi(k_1)} e^u du \\
&= G_2(k_2) - \phi(k_1) \left( e^{-\phi(k_2)} - e^{-\phi(k_1)} \right) \\
&= G_2(k_2) + \phi(k_1) \left( e^{-\phi(k_1)} - e^{-\phi(k_2)} \right)
\end{aligned}$$

where  $z_0$  is a constant. We can determine  $z_0$  by using that  $G(k_1, k_1) = G(k_1)$

$$\begin{aligned}
(1 + \phi(k_1) - \phi(k_1)) e^{-\phi(k_1)} + \phi(k_1) \left( e^{-\phi(k_1)} - z_0 \right) &= e^{-\phi(k_1)} \Leftrightarrow \\
z_0 &= e^{-\phi(k_1)}
\end{aligned}$$

Hence, we have that

$$G(k_1, k_2) = (1 + \phi(k_2) - \phi(k_1)) e^{-\phi(k_2)}$$

or alternatively  $G(k_1, k_2) = G_2(k_2) - \phi(k_1) e^{-\phi(k_2)}$ . □

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