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Coordination Frictions and Job Heterogeneity: A Discrete Time Analysis

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Coordination Frictions and Job Heterogeneity: A Discrete Time Analysis[1]

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Abstract

This paper develops and extends a dynamic, discrete time, job to worker matching model in which jobs are heterogeneous in equilibrium. The key assumptions of this economic environment are (i) matching is directed and (ii) coordination frictions lead to heterogeneous local labor markets. We derive a number of new theoretical results, which are essential for the empirical application of this type of model to matched employer-employee microdata. First, we offer a robust equilibrium concept in which there is a continuous dispersion of job productivities and wages. Second, we show that our model can be readily solved with continuous exogenous worker heterogeneity, where high type workers (high outside options and productivity) earn higher wages in high type jobs and are hired at least as frequently to the better job types as low type workers (low outside options and productivity). Third, we demonstrate that the tractability of this framework is enhanced by analyzing and proving the equivalence of 'seller auctions' and 'buyer posting'. We also prove a related result concerning the equivalence of buyer posting and seller posting when buyers differ continuously. Finally, we show that all of these results preserve the essential tractability of the baseline model with aggregate shocks. Therefore, we offer a parsimonious, general equilibrium framework in which to study the process by which the continuous dispersion of wages and productivities varies over the business cycle for a large population of workers with continuous dispersion of unobserved worker types.

1 Introduction

The goal of this paper is to provide a framework in which to understand the determinants of the economic landscape facing a job searcher. This means that we seek an economic model that has spillovers between the workers' incentives to accept quality differentiated job opportunities and the employers' incentives for their creation. We also seek to develop a framework for this analysis that is useful for empirical research. Thus we have some basic requirements for the theory. First, in order to make this work consistent with prevailing duration models, we wish to have an equilibrium framework that has individual job hazards exhibiting continuous dispersion in both wages and productivity. Second, in order to allow for the possibility of unobserved heterogeneity, we seek a framework with continuous and exogenous differences in worker types.[1] Finally, we wish to have a framework that allows for the possibility of aggregate shocks. This is necessary for any empirical analysis that seeks to study the process by which the composition of jobs changes over the cycle.

The starting point for our analysis is the model of Julien, Kennes and King (2006), which from this point on we referred to as JKK. The key assumptions of this dynamic job to worker matching model are that matching is directed, local markets are heterogenous due to coordination frictions, and time is discrete. This model generates equilibrium technology dispersion, which implies wage and productivity dispersion among otherwise similar workers. The model is solved by a simple block recursive formula and it offers a tractable analysis of business cycle fluctuations, because of the assumption of directed matching.[2]

The JKK framework suffers a number of shortcomings that potentially limit its use for empirical analysis and thus ultimately its use for the evaluation of policy. One shortcoming of this framework is the assumption that firms choose between only two types of jobs and thus wage and productivity

[1]There is a rich history of empirical research done with matching models, which puts emphasis on the importance of job productivity (firm) heterogeneity as a source of wage heterogeneity. Much of this work begins with the Burdett and Mortensen (1999) model. See Bowlus, Kiefer, Neumann (1995), Bontemps, Robin and van den Berg (2000), Christiansen et al (2005), Postal Viney and Robin (2002) for structural analysis, which builds on that framework. See also Abowd, Kramarz and Margolis (1999) for a more descriptive approach, which reaches similar conclusions.

[2]See Menzio and Shi (2008) and (2009) for generalized statements of models with these properties.

dispersion is not continuous. A second shortcoming is the fact that workers are homogenous. Here, it is not clear how to model worker heterogeneity without greatly increasing the computational complexity of this framework. A third potential shortcoming of the JKK model is that it assumes that wages are determined by auction. This assumption is obviously convenient if the number of job types is finite. However, it might seem unclear how to model wages by this method if job types vary continuously. For this reason it is of interest to investigate this problem more thoroughly. This includes the analysis of order statistics, which is necessary for auctions if there is continuum of job types, and the potential analytical value of alternative pricing assumptions such as wage posting.

The first major result of this paper is that the JKK framework can be readily extended to include a general and continuous menu of job types - i.e. two continuous functions giving the productivity of each job and their associated capital costs. In the static model, this extension yields a very simple closed form solution for the equilibrium supply of different job types. This equilibrium relationship also features the interesting result that the worker's outside option does not influence the distribution of job types above a threshold value, which is itself a function of this outside option. These results are somewhat surprising, because they are not anticipated by the discrete job type formulation of this problem, which is considered in JKK. For example, a key proposition in JKK is that workers in unemployment (low outside option) get more good job offers than workers in a bad job (high outside option).

The second major result of this paper is the development of a simple trick for the solution of the distribution of job types for the dynamic version of the model in discrete time. This method uses the same equilibrium job supply equation of the static model with the difference being that the present values of different job types are used instead of productivities when specifying the menu of technologies. This solution implicitly assumes that these present values are well behaved (i.e. a concave function of the associated capital costs) and we can then readily solve for the equilibrium unemployment rate and the distribution of worker productivities. We can also, with some additional complexity, solve for the distribution of wages. The benefit of our solution, which uses an explicit equation for the equilibrium, over the block recursive solution of JKK is that our solution method helps us avoid obvious cases where some job types might not be an equilibrium.

The third major result of this paper is that we can readily solve equilibrium wages and job assignments when workers have heterogenous outside options and abilities. This result is not immediately apparent given our trick for solving the model when worker types are homogenous, because the value functions of workers in similar job types will differ according to the worker's type. However, we show that we do not need to recalculate a new equilibrium job assignment rule and wage function as worker types change continuously. First, we show that workers with different outside options must have identical wages each period whenever they have similar first and second highest valuations within their current local labor markets. The main idea is that efficiency demands that such workers are recruited equally intensively by all job types and thus recruiters must also offer them similar wages. The only difference is that we must calculate a new job quality threshold as we raise the worker's outside option. A related result hold for differences in idiosyncratic worker productivities.[3] If some workers are more productivity in all jobs, then the wage differences of these workers in each job type will simply reflect these idiosyncratic productivity differences - wages are simply scaled by worker type. Consequently, higher job types recruit each type of worker equally intensively, because the workers are always rewarded their marginal contribution of being a better job candidate.[4]

The fourth major result is that we show that there is an equivalence between 'seller auctions' and 'buyer posting' in this environment. This result offer some promise to simplify the computation of equilibrium decisions in this model. The argument is as follows. On-the-one hand, we find that it easiest to solve for equilibrium job allocations if wages are determined by auctions. In this case, the potential employers simply need to anticipate the highest productivity competitor in each local market. Thus the payoff to firms (buyers) is given by a very simple order statistic even though firms differ continuously in equilibrium. On the other hand, the wage distribution is somewhat easier to calculate if firms post wages. In this case, the distribution of wages simply tracts the distribution of worker productivities, rather

[3] These results do not rely on complementarities in production between high type workers and high type firms. Therefore, issues of assortative matching related to Becker (1981) and the complications associated with studying these problems in a frictional economy (See Shimer and Smith 2000) are not analyzed in the present paper.

[4] This is a simple application of the Mortensen rule (Mortensen 1982).

than the more complicated joint density of first and second highest valuations. The fact that buyer wage posting and seller auctions give identical equilibrium outcomes means we can solve for allocations using the auction model and then solve for the distribution of wages for the entire population using the wage vector of the wage posting model.

We also consider the possibility of posting by sellers. This assumption does not offer an obvious solution method for our model. For example, the analysis of Shi (2006) reveals a number of technical difficulties with directed search models, where sellers post prices, coordination frictions lead to multilateral matching, and buyers are heterogenous with a finite number of types. However, we can offer the following corollary to our main equivalence result. We find that the equilibrium wage distribution generated of the buyer posting game must be equivalent to the equilibrium wage distribution of the seller posting game if and only if there is a continuum of buyer types. In particular, if auction and seller posting models give equivalent expected payoffs, then it must be the case that there is an equivalence between the allocations derived in buyer posting and seller posting models. It then follows that an environment with continuous differences in types, that the vector of wages posted by sellers in the seller posting model must be equivalent to the equilibrium wage vector in the buyer posting model. This result does not hold if there is a discrete number of buyer types.

The final result of this paper is to verify that all of the extensions and generalizations considered above do not fundamentally alter the ability our solution methods to readily compute the equilibrium if there are aggregate shocks. This means that we have a dynamic discrete time framework with continuous dispersion of worker types in which to under how the continuous dispersion of worker productivities and wages varies over the business cycle

The rest of the paper is organized as follows. In the next section, we solve a static version of the model. The following section characterizes the equilibrium of a dynamic version of this model when workers are homogenous. The subsequent sections then show how the model is solved with worker heterogeneity, wage posting by buyers, wage posting by sellers, and finally the general problem of aggregate shocks. The last section offers some concluding remarks.

2 The static model

Consider a simple economy with large numbers of risk neutral workers and firms. Each worker is endowed with one unit of labor to sell and each firm has one job opportunity that can employ one worker. The population of workers is normalized to one while the population of firms is determined by free entry. A firm that enters this economy selects a job opportunity from a continuous menu of $k \in [0, \infty]$ job opportunities. A type k job opportunity has an irreversible capital cost of $c(k)$ and it produces an output of $y(k)$ when matched with a worker, where the derivatives y', c', c'' and $-y''$ are all positive. The worker also has an outside opportunity, $y(\underline{k})$, which is the worker's output if the worker is not employed by a firm.

The workers and firms interact as follows. We assume each worker is spatially separated and each newly minted job opportunity can be assigned to only one worker location. The nature of this assignment is described by a three stage game. In the first stage of the game, new firms decide to enter the economy. If a firm enters, it selects a job opportunity type and pays its capital cost. In the second stage of the game, the newly created job opportunities are randomly assigned to worker locations. In the final stage of the game, each firm in a local market bids a wage for the worker's labor services according to the rules of a second price auction. The worker then either selects one of these bids or else remains unemployed. The payoff of the worker is their wage if they are hired and their outside opportunity otherwise. If the firm hires the worker, its payoff is the productivity of their job less the capital cost and the worker's wage. If the firm does not hire the worker, its payoff is minus the capital cost. A firm that does not enter this economy has a payoff that is normalized to zero. The timing of this 'auction game' is illustrated as follows

Timing of the auction game

——|————————————|————————————|——>

Stage 1	Stage 2	Stage 3
Firms choose job type from menu $\{y(k), c(k)\}$	Firms are randomly assigned to workers	Workers auction their services to the available local firms

2.1 Equilibrium

The equilibrium of the matching game is solved by backwards induction. Consider first the optimal bidding strategies in the final stage of the game when the allocation of all job opportunities and workers is known. Let k_1 and k_2 denote a worker's best and second best job opportunity. Given competitive bidding for the worker's labor services, the worker is employed by the highest valuation firm at a wage given by

$$w(k_1, k_2) = y(k_2), \tag{1}$$

The revenue of the worker's employer is thus $y(k_1) - y(k_2)$ and the revenues of all other firms bidding for this worker are then zero.

The second stage of the game is simple random assignment. Let $\phi(k)$ denote the number of job types greater than k, which is determined in the first stage. The random assignment of job opportunities to workers means that the number of job opportunities of type greater than k at each worker's location is distributed Poisson with parameter $\phi(k)$.

The first stage of the game is the entry decision and technology choice of the firms. The free entry equilibrium is as follows:

Proposition 1 *In the static model, the supply of type k jobs is given by*

$$\phi(k) = \log\left(\frac{y'(k)}{c'(k)}\right) \tag{2}$$

over the range $\left[\widehat{k}(\underline{k}), k^*\right]$ *where the lower bound is* $\widehat{k}(\underline{k}) = \arg\max_k (y(k) - y(\underline{k}))/c(k)$ *and the upper bound is* $k^* = \arg\max_k (y(k) - c(k))$.

Proof. See appendix. ■

Thus firms offer a continuum of different job types in equilibrium.[5]

The simplicity of this equilibrium solution is perhaps surprising, because it is not anticipated by the discrete formulation of this problem. For example, if there are only two job types, then the outside option of the worker affects the supply of good jobs above the cutoff point. In this case, Julien,

[5] The mechanism is equivalent to the problem of noisey search found in Butters (1977) and Burdett and Judd (1983). Thus Acemoglu and Shimer (2000) is a closely related model of wage and technology dispersion.

Kennes and King (2006) show that workers in bad jobs (high outside option workers) get fewer good job offers than unemployed workers (low outside option workers). We have shown that this complication does not arise when there is a continuum of job types. Here, a higher outside option raises the cutoff point by which lower quality jobs are offered, but it does not affect the distribution of higher job types.

It is also worth noting that an increase in the productivity of the worker in all jobs does not change the allocation of workers to the high type firms, because (like a change to the outside option) this change has no effect on $y^{'}(k)/c'(k)$. The only difference is that these higher productivity workers are more frequently hired by the low type firms (i.e. there is a lower value of $\widehat{k}$.).

There are important differences between a change in the outside option of the worker and a change in the workers productivity. A higher outside option raises the workers the workers job quality threshold while higher productivity lowers this threshold. The effects on wages are of also different. A higher outside option has no effect on wages while a higher idiosyncratic worker productivity level causes a wage increase in all possible job types by an amount equal to the size of this productivity improvement. It is interesting to note that we can give higher productivity workers identical job arrival rates as less productivity workers simply by raising the high productivity workers' outside option. Setup this way, the model gives similar implications as a random matching model where all workers get the same arrival rate of offers while higher productivity workers are those individuals who earn higher wages and produce more output in all jobs.

2.1.1 Wage and productivity dispersion

The distribution of wages and productivities is determined by two order statistics concerning the distribution of job offers at each worker's location: $G_1(k)$, which denotes the fraction of workers with a best job offer less than $y(k)$ and, $G_2(k)$, which denotes the fraction of workers with a second best job offer less than $y(k)$. The formulas for these distributions are as follows.

Proposition 2 *For values of* $k_1, k_2 \in \left[\widehat{k}, k^*\right]$,

$$\begin{aligned} G_1(k) &= e^{-\phi(k)}, \text{ and} &(3)\\ G_2(k) &= e^{-\phi(k)} + \phi(k)\, e^{-\phi(k)}. &(4)\end{aligned}$$

where $G_1\left(\widehat{k}\right)$ *is the probability of no offer and* $G_2\left(\widehat{k}\right)$ *is the probability of zero or one offers.*

Proof. See appendix. ■

The cumulative distribution function $G_1(k)$ is equivalent to the distribution of worker productivities, because the workers are always employed at the most productive available job. The cumulative distribution function $G_2(k)$ is equivalent to the distribution of worker wages, because the worker's is always paid a wage equal to his/her second highest valuation.

2.1.2 Joint offer distribution

Let $G(k_1, k_2)$ denote the fraction of workers with a best job greater than k_1 and a second best job greater than k_2. We find:

Proposition 3 *For values of* $k_1, k_2 \in \left[\widehat{k}, k^*\right]$,

$$G(k_1, k_2) = (1 + \phi(k_2) - \phi(k_1))\, e^{-\phi(k_2)} \tag{5}$$

Proof. See appendix. ■

The joint distribution of first and second best offers is not needed to describe the distribution of wages in the static model. However, in a dynamic environment, the worker's per period wage will be a function of their first and second best offer. The basic idea is that the worker's second best offer describes the threat point used in setting the wage with their current employer while the productivity of their current employer (their best offer) gives the worker's threat point when setting wages with any future employer who might be contacted by on-the-job search. The worker's current wage then balances these two concerns.

2.2 Example

The characterization of the equilibrium of this model is somewhat abstract in relation to the analysis of the JKK model. The main reason is that we use order statistics to describe matching outcomes with a continuum of firm types while the JKK model has a discrete number of types and thus no need for any special statistical tools. To illustrate this, we can use a simple example. Suppose that the menu facing firms, $\{y(k), c(k)\}$ is given by $\left\{2\sqrt{k}, k\right\}$ and that the outside option of the workers is zero. This technology menu is illustrated in figure 1 below.

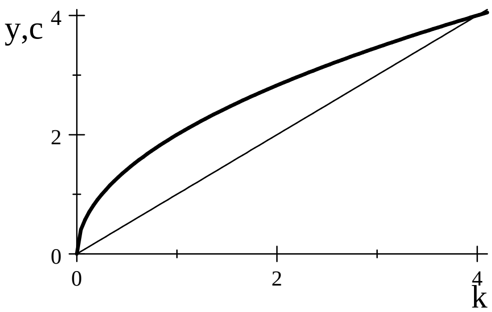

Figure 1. Technology menu

where the thick line is $y(k)$ and the thin line is $c(k)$. This example has the obvious property that we can expect a mixed strategy equilibrium over jobs types as in the JKK model. That is it satisfies the necessary and sufficient condition of JKK for a mixed strategy equilibrium with two types that (i) $y(k) - c(k) > y(k') - c(k')$ and (ii) $y(k')/c(k') > y(k)/c(k)$ for $k > k'$.

Our solution gives the mixed strategy equilibrium over all possible job types. In this example, the ratio $y'(k)/c'(k) = 1/\sqrt{k}$. Also, $\widehat{k} = 0$ and $k^* = 1$. Therefore, in equilibrium, equation (2) implies that the quantity of job types greater than k is given by $\phi(k) = \log\left(1/\sqrt{k}\right)$ We illustrate this relationship below

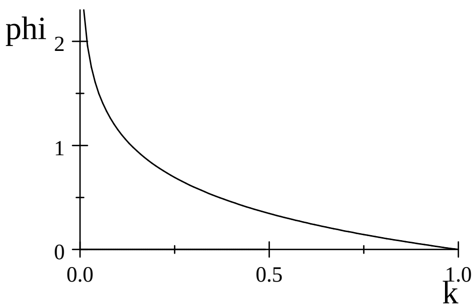

Figure 2. Quantity of type > k job vacancies

We can then plot the productivity and wage distribution, $G_1(k) = e^{-\phi(k)}$ and $G_2(k) = e^{-\phi(k)} + \phi(k) e^{-\phi(k)} = \sqrt{k} + \sqrt{k} \log\left(1/\sqrt{k}\right)$, as follows

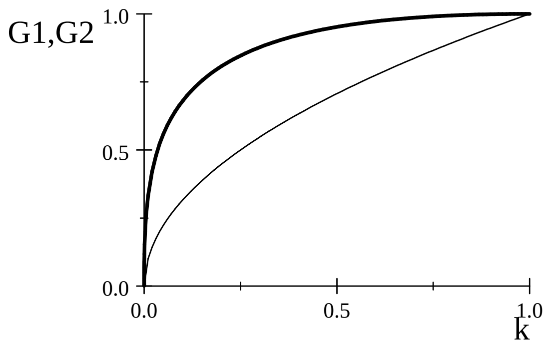

Figure 3. Wage and productivity cdfs

where the solid line is the cumulative density function of the wage distribution for employed workers and the thin line is the cumulative density function of the productivity of each of these employed workers.

This example illustrates the trade-off between job quality and probability of trade, which is central to this equilibrium with a continuous dispersion of job types. To see this, consider figure 2 where we observe that are very many of the lowest type jobs and very few of the highest quality jobs. This can be an equilibrium because figure 3 reveals that the high quality jobs are accepted with high probability while the low quality jobs are accepted with low probability. The reason that jobs are heterogenous in equilibrium is partly due to the fact that all jobs independent of their quality impose the same externality on the lower quality jobs. Thus the total number of jobs will be dictated by the returns to posting low quality jobs. However, all jobs will not be low quality. The reason for this is that the payoffs of the higher quality jobs are much more affected by the presence of other higher quality jobs that lower quality jobs.

The fact that we have an explicit analytical solution for the equilibrium distribution of job types has much greater application than the simple static model derived here. In the next section, we will show how we can extend this method of solving for the distribution of job types to a dynamic environment with auctions, homogenous workers and no aggregate shocks. Then, in subsequent sections, we will show how we can use continue to use this solution method to study even a richer array of dynamic economic environments with equilibrium firm heterogeneity, including (i) economies with exogenous variation in worker outside options, (ii) environments where firms post wages, (iii) environments where workers post wages, and (iv) environments with common aggregate shocks to the menu of technologies, $\{y(k), c(k)\}$. All of these results will rely on there being a continuum of possible firm types rather than a discrete number.

3 The dynamic model

The dynamic model is a repeated version of the static model. The workers and firms are now infinitely lived with risk neutral preferences and a common discount factor β. Time is discrete. The total population of workers is normalized to one and the population of firms is determined by free entry. At the start of each period new firms can choose to enter and select a type $k \in [0, \infty]$ job opportunity, where $y(k)$ and $C(k)$ denote the job's productivity and capital cost, respectively. We assume that the derivatives y', C', C'' and $-y''$ are all positive. Each worker has one unit of labor to sell and each job opportunity can employ one worker. Each worker is also endowed with an outside opportunity $y(\underline{k})$, which is the worker's productivity if no job opportunity is forthcoming. Once a worker is assigned a job, there is a probability δ that the job opportunity is destroyed.

The matching game within each period is identical to the static model. The only additional elements are (i) the existence of random job separations at the end of the period and (ii) the possibility for additional job creation and matching opportunities in each subsequent period.

3.1 Equilibrium

Let $\Lambda(k)$ denote the present value of a match between a worker and a type k. We assume that that the function relating a jobs present value to its type is well behaved. This means that the derivatives Λ' and $-\Lambda''$ are positive. This assumptions allows us to solve the equilibrium job offer distribution in a fashion equivalent to the static model. Using proposition 1, the supply of type k jobs directed at workers employed in jobs with type k_1 employers is given by the function

$$\phi(k) = \log\left(\frac{\Lambda'(k)}{C'(k)}\right) \tag{6}$$

over the range $\left[\widehat{k}(k_1), k^*\right]$ where $\widehat{k}(k_1) = \arg\max(\Lambda(k) - \Lambda(k_1))/C(k)$ and $k^* = \arg\max \Lambda(k) - C(k)$.

On-the-job search is accommodated by the fact that all new employers making job offers know the worker's current employer's type. Therefore, the worker's option to remain with their current employer is analogous to the outside option of the static model. In particular, the productivity of the worker's current employer does not affect the function $\phi(k)$, but rather simply changes the lower support over which the different job types are offered.

3.1.1 Wages and productivity

Given that we have an immediate solution for the equilibrium allocation of jobs to workers each period as function of the present value of different matches and the associated capital costs of different job types, we can now use the workers asset equations to derive the wages and productivity of each type of job. The worker's current second best offer is the reservation value that was used to set their wage with their current employer while the productivity of their current employer (their best offer) gives the worker's reservation value when setting wages with future potential employers contacted by on-the-job search. Therefore, the expected present value of a worker in a type k_1 job with a second best offer of k_2 at the start of the period is given by

$$W^D(k_1, k_2) = \Lambda(k_2) G_1(\widehat{k}) + \Lambda(k_1)\left(G_2(\widehat{k}) - G_1(\widehat{k})\right) + \int_{z=\widehat{k}}^{k^*} \Lambda(z)\, dG_2(z) \tag{7}$$

where $\widehat{k} = \widehat{k}(k_1)$, $G_1(k) = e^{-\phi(k)}$ and $G_2(k) = e^{-\phi(k)} + \phi(k) e^{-\phi(k)}$. The first term on the right hand side of this equation captures the event that the worker gets no new offers this period; the second term is the event that the worker has a single offer, which means that they will be paid a wage equal to the total surplus associated with their employment at the incumbent employer; and the final term capture the pay increases due to the possibility of multiple offers.

Given that wage contracts are determined by auction, the present value of a worker with a type k_1 employer and a type k_2 second best offer is $\Lambda(k_2)$. This means that the wage $w(k_1, k_2)$ of a worker in this negotiation state satisfies the following asset equation:

$$\Lambda\left(k_2\right)=w(k_1,k_2)+\beta\left[\left(1-\delta\right)W^D\left(k_1,k_2\right)+\delta W^D\left(\underline{k},\underline{k}\right)\right] \tag{8}$$

where the future stream of returns is given by $W^D\left(k_1,k_2\right)$ if the worker does not suffer a job loss at the end of the period and $W^D\left(\underline{k},\underline{k}\right)$ otherwise.

If $k_1=k_2=k$, then the worker effectively becomes the residual claimant of their employment contract - they own the job. In this case, the workers earn a wage equal to the output of the firm. That is

$$y\left(k\right)=w(k,k). \tag{9}$$

It is worth noting that the computation of the productivity of each job type requires only knowledge of $G_1\left(k\right)$ and $G_2\left(k\right)$ and not the joint density of first and second best offers.

3.1.2 Distribution of productivities

A fraction $1-\delta$ of all employed workers loose their job at the end of the period and the probability that a job searcher - those unemployed last period plus the job losers - find a job is given by $G_1(\widehat{k}\left(\underline{k}\right))$. Let $n\left(k\right)$ denote the density of workers employed in a type k job and $N\left(k\right)$ denote the density of workers with job types less than k. The transition equation for the density of job types is given by

$$n'\left(k\right)=n\left(k\right)G_1\left(\widehat{k}\left(k\right)\right)\left(1-\delta\right)+\left[\delta(1-N\left(\widehat{k}^{-1}\left(k\right)\right))+N\left(\widehat{k}^{-1}\left(k\right)\right)\right]g_1\left(k\right) \tag{10}$$

where the first term on the right hand side is the density of workers in type k jobs who have the same employer as the previous period and the second term is the density of workers in type k jobs who have a new employer. The steady state unemployment rate is given by

$$u\left(\underline{k}\right)=\frac{\delta G_1\left(\widehat{k}\left(\underline{k}\right)\right)}{1-G_1\left(\widehat{k}\left(\underline{k}\right)\right)+\delta G_1\left(\widehat{k}\left(\underline{k}\right)\right)} \tag{11}$$

and the density function of workers in type k jobs is given by

$$n(k) = \frac{\left(\delta(1 - N\left(\widehat{k}^{-1}(k)\right)) + N\left(\widehat{k}^{-1}(k)\right)\right) g^1(k)}{1 - G_1\left(\widehat{k}(k)\right) + \delta G_1\left(\widehat{k}(k)\right)} \tag{12}$$

where the end point of this differential equation $N\left(\widehat{k}^{-1}(\underline{k})\right) = u$. The distribution of productivities is then characterized by this density equation and by equation (9), which gives the productivity of a type k employer. That is

$$\Omega_p = \{y(k), n(k) | k \in [\underline{k}, k^*]\} \tag{13}$$

It is also worth noting that the computation of the distribution of worker productivity requires only knowledge of $G_1(k)$.

3.1.3 Distribution of wages

The final task is to characterize the steady state joint distribution of first and second best offers. In each period, a worker employed in a type $z \in [\underline{k}, k^*]$ job, has a joint distribution of jobs offers, $G(k_1, k_2)$, over the interval $\left[\widehat{k}(z), k^*\right]$, where $G(k_1, k_2)$ is given by equation (5). We let $g(k_1, k_2)$ denote the implied joint density of offers for an unemployed worker ($z = \underline{k}$). Let $x(k_1, k_2)$ denote the joint density of workers employed in a type k_1 job and with a second best opportunity of a type k_2 job. The transition equation for $x(k_1, k_2)$ is given by

$$\begin{aligned} x'(k_1, k_2) &= x(k_1, k_2) G_1\left(\widehat{k}(k_1)\right)(1 - \delta) \\ &\quad + \left(\delta(1 - N\left(\widehat{k}^{-1}(k_2)\right)) + N\left(\widehat{k}^{-1}(k_2)\right)\right) g(k_1, k_2) \\ &\quad + n(k_2)(1 - \delta) 1\left\{k_2 \geq \widehat{k}^{-1}(k_1)\right\} \int_{\underline{k}}^{\widehat{k}(k_2)} g\left(k_1, \widetilde{k}_2\right) d\widetilde{k}_2 \end{aligned} \tag{14}$$

where the first term on the right hand side is the quantity of agents in the k_1, k_2 state in the previous period who do not loose their job and are not recruited this period, the second term is the quantity of workers who move into the k_1, k_2 state by means of getting multiple offers, and the third term is the quantity of workers who move into the k_1, k_2 state by means of getting a single type k_1 offer and having a type k_2 incumbent employer. The steady state distribution is solved by setting $x'(k_1,k_2) = x(k_1,k_2)$. We get

$$\begin{aligned} x\left(k_1,k_2\right) &= n\left(k_1\right)\frac{\left(\delta(1-N\left(\widehat{k}^{-1}\left(k_2\right)\right))+N\left(\widehat{k}^{-1}\left(k_2\right)\right)\right)}{\left(\delta(1-N\left(\widehat{k}^{-1}\left(k_1\right)\right))+N\left(\widehat{k}^{-1}\left(k_1\right)\right)\right)}\frac{g\left(k_1,k_2\right)}{g^1\left(k_1\right)} \\ &\quad + n\left(k_2\right)\frac{(1-\delta)\left\{k_2 \geq \widehat{k}^{-1}\left(k_1\right)\right\}\int_{\underline{k}}^{\widehat{k}(k_2)} g\left(k_1,\widetilde{k}_2\right)d\widetilde{k}_2}{1-G_1\left(\widehat{k}\left(k_1\right)\right)+\delta G_1\left(\widehat{k}\left(k_1\right)\right)}. \end{aligned} \quad (15)$$

The joint distribution of wages and productivities for the economy is then characterized by this density equation together with equations (8) and (9), which describes the wages and productivities of all workers as a function of their employment state, k_1, k_2. That is

$$\Omega_w = \left\{w\left(k_1,k_2\right), x\left(k_1,k_2\right) | k_1,k_2 \in [\underline{k},k^*]\right\} \quad (16)$$

This is a much more complicated object to compute than the distribution of worker productivities. In particular this result requires us to use the joint distribution of job types, which we have characterized in proposition 3. We return to this in section 5, where we make a comparison of the complexity in calculating this wage distribution under auctions with the wage distribution under posting.

4 Heterogeneous returns to unemployment

This section extends our dynamic analysis to economic environments where workers have heterogeneous returns to unemployments (refer to Albrect and Axell 1984). Here, we derive the key result that we can solve our model for the case where these outside options vary continuous. For example, we might assume that the cumulative density function for the outside option is given by $H\left(\underline{k}\right)$ over the range $[0,k^*]$. The following two propositions characterize the equilibrium.

Proposition 4 *Suppose that two workers have different outside options (i.e. their flow returns in unemployment are $\underline{k}$ and $\underline{k}' > \underline{k}$, respectively) and the same first and second best job assignment (k_1 and k_2), then they must have*

1. *identical wages, $w\left(k_1,k_2|\underline{k}\right) = w\left(k_1,k_2|\underline{k}'\right)$, and*

2. identical recruiting, $\Phi(k|k_1,\underline{k}) = \Phi\left(k|k_1,\underline{k}'\right)$ *for all* $k \in \left[\widehat{k}, k^*\right]$

Proof. See Appendix ■

Therefore, we can solve the equilibrium distribution of posted job types each period using equation (6) for the lowest outside options workers. The wages and transition probabilities for all other workers are then identical except that these workers have a higher cut-off threshold.

Of course, as in the static model, the returns to unemployment change the job quality threshold To solve the mapping from the workers outside option into job offers, we now need only compute the function $\widehat{k}(\underline{k})$. We have

Proposition 5 *The lowest quality job* k *offered to a worker in a type* $\underline{k}$ *satisfies the following free entry condition:*

$$C(k) = e^{-\Phi(k)}\left(\frac{y(k) - w(k,\underline{k})}{1-\beta\left(1-e^{-\phi\left(\widehat{k}(k)\right)}\right)}\right) \tag{17}$$

which holds for all k*.*

Proof. See Appendix ■

Therefore, a worker with higher outside options will have a different job quality threshold than a worker with a lower outside option. However, above these thresholds, these two workers will have similar arrival rates of different job qualities and earn similar wages in these jobs.

5 Heterogeneous skills

This section extends the analysis of the dynamic model to economic environments where workers have continuous differences in productivity. Like our analysis of the static model, we assume that higher productivity gives a worker absolute advantage in all jobs. Therefore, the labor productivity of a match between a worker with productivity h_i in a match with a type k firm is given by

$$p_{ik} = h_i + y(k)$$

A change in worker productivity gives the following results.

Proposition 6 *Suppose two workers are distinguished by skill types h and h', respectively), then they must have*

1. *wage differences proportional to productivity differences,* $w(k_1, k_2|h) = w(k_1, k_2|h') - (h' - h)$, *and*

2. *identical recruiting,* $\Phi(k|k_1, h') = \Phi(k|k_1, h)$ *for all* $k \in \left[\widehat{k}, k^*\right]$

   where $\widehat{k}$ *for each worker type determined by equation (17), such that* $w(k_1, k_2|h) = w\left(k_1, k_2|h^0\right) - \left(h - h^0\right)$ *where* $w\left(k_1, k_2|h^0\right)$ *is the wage function of the homogenous worker model.*

Proof. See appendix ■

Therefore, as in the static model, a change in worker productivity does not require recomputation of wages (except to scale by productivity) and job arrival rates (except to scale by the job quality threshold).

The interaction of heterogenous outside options and heterogenous worker productivity gives our model much flexibility to explain matched employer-employee data while maintaining the assumption that jobs are determined by free entry and firms face an identical ex ante menu of technologies. In our model, an increase in the outside option increases the chance that a worker will be employed in a high type job but decreases the probability of employment. Likewise, an increase in the productivity of a worker increases the wage of the worker in all jobs but it also means that such a worker is more likely to be employed in a low type job. Consequently, to explain why a 'high' type worker is employed more frequently in high type jobs and at higher wages means then that the worker must have both higher outside options and higher skills.

6 Buyer posting

There is a relative large literature that study matching models where firms wage posting instead of interacting in local labor markets with auctions determining prices. The posting frameworks can be divided into two basic types of models. On the one hand, there exist models in which buyers post wages and the equilibrium is not degenerate because of the possibility of unintended buyer competition (either due to coordination frictions as in this model - noisy search - or due to on-the-job search. Refer to Burdett and

Judd (1984)). The Burdett-Mortensen (1999) model is a classic example of this approach applied to the labor market, where Mortensen (2003) develops a static model that is closely related to the static model in the present paper. Another type of matching model with posting is one where sellers post prices. This type of framework also leads to a non degenerate equilibrium provided that search directed[6] In this section and the following section, we consider how the equilibrium allocations and prices of our model are affected by these alternative pricing assumptions.

We begin by assuming that firms post wages. In this case, our model is one of buyer posting. The description of this game is summarized by the following time-line.

Timing of the buyer posting game

——|——————————|——————————|——>

Stage 1	Stage 2	Stage 3
Firm i choose job type from menu $\{y(k), c(k)\}$ and posts wage w_i	Firms are randomly assigned to workers	Workers select the best wage from the set of available local firms

The main difference with the auction model is that each firm within each category of job type commits to a wage. This game is somewhat difficult to solve because the mixed strategy equilibrium now consists of two mixed strategies - the technology choice and the wage choice. It is possible to prove the following result.

Proposition 7 *The allocations of jobs to workers in the auction game is equivalent to the allocation of jobs to workers in the buyer posting game*

Proof. See Appendix ■

We can use this result to generate the wage distribution of the buyer posting game as follows. We know (i) that the payoff of each type of firm will be the same if the firm posts a wage prior to its assignment to workers as

[6] (In the absence of directed search, the seller posting model has a degenerate outcome in which sellers acts as monopolists - an outcome commonly known as the Diamond paradox and attributed to Diamond (1971).

what is when there is an auction at each of these local markets. For this to be true for all firms, this result must imply (ii) that there is also equivalent probabilities across different firms of each firm type hiring a worker. This means that $e^{-\phi(k)}$ is the probability that a type k firm will hire a worker in both the auction and wage posting model. Given (i) and (ii) we can state that the wage of a type k firm under the assumption of wage posting is equivalent to the expected wage paid by a type k firm under auction. This means that we can write the posted wage by

$$w^p(k) = \int_{z=0}^{k} y(z) dG_1(z) \tag{18}$$

where $G_1(z) = e^{-\phi(z)}$. is equivalent to the auction model.

Similarly, we can solve for the wage distribution of workers in the wage posting version of the dynamic model by the fact the distribution of firm productivities is given by equation (). In the dynamic game, the analogous expression is

$$W(k) = \int_{z=0}^{k} \Lambda(z) dG_1(z) \tag{19}$$

where we can think of sellers posting present values of income rather than actual wages. Of course, we can derive the relevant wages from the underlying asset equations as we do in the dynamic auction model.

A key advantage of the posting model over the auction model comes when we wish to compute the distribution of workers in each wage state. In the auction game we need to track the first and second highest valuations. The posting model, by contrast, simply requires us to track the distribution of worker productivities. In particular, we have the following much simpler expression.

$$\Omega_w^p = \{w^p(k), n(k) | k \in [\underline{k}, k^*]\} \tag{20}$$

where $n(k)$ is equivalent to equation (12) of the auction model. This distribution of wages is easier to compute than the auction model, equation (16), because there is no need to compute the joint distribution of first and second offers.

7 Seller posting

The possibility of seller posting with directed search has been analyzed by Peters (1984), Montgommery (1991) and Burdett, Shi and Wright (2001).[7] These models are interesting for a number of reasons. First, the assumption of seller posting is often treated as synonymous with the assumption directed search, which is at the center of our analysis. If sellers post prices, they can inform buyers of their type by the wage posted. To some extent, a similar assumption can be maintained if sellers auctions with reserve prices. However, the limiting outcome of the auction model is generally to set the reserve price equal to zero (or the outside option of the lowest valuation buyer) and thus we are left with a model without reserve prices (refer to Julien, Kennes and King 2000). This means that the directed search version of the auction model offers no way to communicate type by prices - we must thereby assume that worker types are communicated by some other means in the auction model.

Another interesting feature of seller posting model is that there is considerable research devoted to two themes: (i) the equivalence of expected payoffs under seller auctions and seller price posting and (ii) the equivalence of outcomes when sellers posted prices and when outcomes are determined by a social planner. These studies typically find that seller posting models typically yield identical expected payoffs as auction models (refer Kultti 1999) and that these environments also yield constrained efficient allocations (refer to Peters 1984). We give a related comparison of economic environments where buyers post and economic environments where sellers post.

A natural question emerges as to whether we can use the seller posting approach for the case where buyers differ continuously. This game is much more complex than most problems typically studied under price posting. Instead of a single price, each seller posts a continuum of buyer type dependent prices in competition with other sellers. The seller posting game

[7] Alternative approaches to matching including some model where matching is directed use a continuous time formulation. In such environments it is somewhat difficult to specify the underlying micro environment, which leads to the friction related to the problem of finding a match. The advantage of these models is that buyers are never hetorogenous in a local market. This means that one can use the method of Moen (1997) to characterize the equilibrium. See Shi (2009) for a more recent analysis of on-the-job search, in which workers move incrementally up a job ladder as a function of their current place on this ladder.

version of our model is described by the following time-line of events.

Timing of the seller posting game

——|——————————————|——————————————|——>

Stage 1	Stage 2	Stage 3
Firms choose job type from menu $\{y(k), c(k)\}$ Each worker i post a wage schedule $w_i(k)$	Firms are directed by the set of worker wage functions $\{w_i(k)\}$	Workers select the highest available job type and are paid their posted wage

Because this game is so complicated, we do not attempt to solve it here. However, we appeal to the well known equivalence result between auctions and seller posting games of directed search. This equivalence can be traced to the competition of sellers and buyers and to the fact that the game is one of complete information - the only random element is the arbitrary moves a nature leading to coordination frictions. (If information is imperfect, there may be a trade-off, on-the-one hand, auction games are better at revealing information about buyers but poor at revealing information about sellers, on-the-other hand posting games are better at revealing information about sellers but poor at revealing information about buyers). The following result can then be established

Proposition 8 *If auctions and seller posting give identical allocations of jobs to workers, then wage dispersion under seller posting is equivalent to wage dispersion under buyer posting*

Proof. See Appendix ■

This result suggests that there is no observable difference to a full information directed search environment with continuous buyer types determined by free entry where buyers post and a similar environment where sellers post. In this case, a model such as Burdett, Shi and Wright (2001) and a model such as Burdett and Mortensen (1999) give identical outcomes. This is not generally true if the number of buyer types is finite.

Seller posting games are often treated as synonymous with competitive search (refer to Moen 1997). The advantage of these models is that buyers are always homogenous in local markets. Shi (2009) shows that a model of competitive search in continuous time can be solved and used to model many of the same economic problems, which are relevant to the present paper. In Shi's model, the continuity of time means that sellers move incrementally up a job ladder . no discrete jumps - and thus never face a complicated posting game, because the relevant buyers are always homogeneous. One difficulty with this approach is that it is hard to reconcile these models with a fully microfounded matching friction, which is usually taken to be synonymous with a problem of coordination and the possibility of unrealized investment potentials associated with not exploiting a particular opportunity first (or best). This conceptual problem is not an issue in a discrete time matching model. Here, the problem of costly irreversible investments not bearing fruit on either side of a matching market is clearly defined. We have shown that such models lead naturally to an equilibrium with a continuum of heterogenous buyers. This means that seller posting models are rendered intractable in discrete time environments.

8 Aggregate and idiosyncratic shocks

Shocks to this economic environment are crucial for two basic reasons. First, in the absence of aggregate shocks, the model is in steady state and thus there is no possibility to use this framework to model the propagation mechanisms associated with business cycles. Second, shocks are needed to understand idiosyncratic wage changes during each worker's tenure at each firm. In particular, the directedness of search implies that new job offers always lead the worker to switch jobs and thus there is no bidding up of the worker's wages while employed. In this sense, a richer model is needed to explain wage movements than the standard theory of wage and productivity dispersion with undirected search, because these model allow wage changes due to unintended counteroffers from alternative employers.

There are two basic ways to introduce shocks to this model. One method is to assume that there are changes in the technological menu of firms. This means that the output of each job type and the associated capital cost is time varying. Thus the menu is given by

$$\{y_t\left(k\right), c_t(k)\}$$

where changes in the parameters characterizing the functions $y_t(k)$ and $c_t(k)$ might follow a first order Markov process, for example. The environment remains tractable, because matching is directed. Therefore, the recruiting strategies of each firm type is independent of the distribution of employed and unemployed worker

We can also consider shocks to worker 'types'. These types of shocks are extremely tractable in our framework, because shocks to either the worker's outside option or their idiosyncratic productivity do not affect $\Lambda'(k)/C'(k)$ for the benchmark case of a worker with a constant outside option and productivity. With these shocks, there is no change in the underlying recruiting rule and wage functions for the economy as a whole. The impact of these shocks are easy to anticipate. On the one hand, a fall in the worker's idiosyncratic productivity will raise the job quality threshold (thus fewer overall offers directed at the worker) and lower the worker's wages by an amount equal to the fall in the idiosyncratic productivity. On the other hand, an increase in the worker's outside option also raises the job quality threshold and consequently fewer job offers. However, in this case the wage of each job is unchanged.

9 Conclusions

This paper makes contributions to the discrete time modeling of matching problems with coordination frictions. One potential value of our work is to demonstrate that there are no obvious barriers for structural empirical work that favors a continuous time model over a discrete time model. For example, the fact that we can now generate a continuous dispersion of technologies means that our model can be used to more fully explain observed 'residual' wage and productivity dispersion. Another feature of our analysis is that we can characterize the equilibrium with continuous worker heterogeneity where high type workers (high outside options and productivity) earn higher wages in high type jobs and are hired at least as frequently to the better job types as low type workers (low outside options and productivity). Therefore, this framework can be readily applied to microdata, where variations in outside options and productivity can take the role of unobserved heterogeneity, to explain why some workers earn consistency higher wages and be of higher productivity than other workers while also being the workers who are most frequently employed. Finally, because we can apply all of these elements to an environment with aggregate shocks, the model can be used to study cyclical data on wages and productivities and conclusions can be drawn about the underlying shocks driving the business cycle.

Another feature of our analysis is that we can nest a large class of random matching models within our environment. This requires that we simply scale up the outside options of higher productivity workers to ensure that they get the same offer distribution - i.e. the same arrival rate of quality differentiated jobs - as lower productivity workers. Consequently, we can conduct an almost equivalent empirical analysis of our model to that done by Bagger, Christensen and Mortensen (2010), which takes the assignment of diverse jobs to diverse workers as exogenous.[8] One advantage of our framework is that it provides an equilibrium explanation of the existence of quality differentiated employment opportunities as a function of a zero expected profit free entry equilibrium. Therefore, all wage and productivity variation in our model is explained either by coordination frictions or differences in worker types. A second advantage is that we can conduct a counterfactual analysis about changes in technology and policy and draw implications about the propagation of such changes in both the short, medium and long run.

A discrete time, full information formulation of job matching, which is the approach followed in this paper, may also be useful for the analysis of additional frictions than simple problems of coordination. In particular, we believe that these models might be useful for analyzing how the problem of coordination frictions interacts with the problems of irreversibility and time consistency. For example, in a marriage market, where matching is at least as difficult as in the labor market, children are the irreversible (and arguably specific) investment of a household and couples may suffer problems of time consistency related to agreements about child care and the continuation of the marriage.[9] For both of these additional frictions, there are well established and widely understood theoretical tools that permit the analysis of such problems in discrete time with full information.[10] The fact

[8] The Bagger, et al, model also has firms changing their wages as more workers are hired. Therefore, they analyze a third channel by which the wages of all workers are affected by variations in firm size. Of course, it is extremely difficult to distinguish this effect from a common change in the productivity of all workers within a firm, which would then lead to more hiring.

[9] Specific human capital is another relevant example.

[10] For example, Marcet and Marimon (2004) provide an extremely tractable analysis of

that our model fulfills these requirements implies an opportunity to borrow these methods and extend their application.[11]

Another advantage of a discrete time model of coordination frictions is that it offers a useful bridge to understanding and extending models of matching without coordination frictions. The crucial models of matching by Becker (1981) and task assignment by Rosen (1983) are essentially static models. In this case, the multilateral heterogeneous local markets of our model can be easily interpreted as an extension and generalization of these important models to environments with coordination frictions. However, it also seems of obvious importance to extend our analysis of multilateral matching to the types of problems, which are studied in these static frameworks. In particular, it would be of interest to better understand the creation of teams and the transferability of utility and tasks. Our discrete time model might be used for this purpose, because it features a continuum of job types in equilibrium and constrained efficient outcomes. Therefore, the crucial analytical properties of classic static matching models - the assignment of a continuous set of tasks and partnerships to a continuous set of heterogenous agents for the achievement of a constrained maximum of economic surplus (i.e. a stable matching with transferable utility) - is also operational in our framework.

Lastly, we wish to observe that this paper reveals some obvious analytical benefits to working with matching models with microfounded matching technologies (coordination frictions) and directed search (i.e. full information). In these environments, there exists 'equivalence' results concerning the allocative outcomes and pricing implications of alternative pricing mechanisms, and 'constrained efficiency' results concerning decentralized and centrally planned solutions of these environments.[12] This paper has shown that these properties can be extremely helpful if we are to characterize the equilibrium of more complex versions of these matching models - especially models

recursive contracts for a discrete time full information model. Our model satisfies both of these assumptions. Chatterjee et al (2008) show how problems of time consistency can be used to empirically model the problem of moral hazard concerning investments.

[11] Kennes and Knowles (2010) provide an equilibrium analysis of step families in a mariatal matching markets with irreversible investments in chidren and an own child bias. They show that this irreversible investment problem has a block recursive solution in their model of directed matching, where women without children get more proposals from unattached men than divorced women with children.

[12] Julien, Kennes and King (2008) show that these 'equivalence' and 'efficiency' results also extend to directed search models of money.

where there is a continuum of types on both sides of the market. Here, we find that the auction model most easily yields the equilibrium distribution of firm types (productivities) while the posting model yields a simpler to compute pricing distribution (wages) in the dynamic context. Other results, such as the independence of unemployment benefits on the wages of worker with otherwise similar jobs are also easier to anticipate and establish once the lesson of constrained efficiency is firmly established. In the absence of equivalence and constrained efficiency, we would have faced a much tougher road on which to establish our basic results.

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10 Appendix

Proposition 1 In the static model, the supply of type k jobs is given by

$$\phi(k) = \log\left(\frac{y'(k)}{c'(k)}\right) \tag{21}$$

over the range $\left[\widehat{k}(\underline{k}), k^*\right]$ where the lower bound is $\widehat{k}(\underline{k}) = \arg\max_k (y(k) - y(\underline{k}))/c(k)$ and the upper bound is $k^* = \arg\max_k (y(k) - c(k))$.

Proof. It suffices to establish the following facts:

Fact 1. The distribution of job types must lie inside the interval $\left[\widehat{k}(\underline{k}), k^*\right]$. Here, it is useful to consider the possibility that firms are restricted to using a single job type. The value $\widehat{k}(\underline{k})$ is job type that maximizes the number of opening. That is it is the argmax of job entry given free entry $\widehat{k}(\underline{k}) = \arg\max\left\{\phi(k) | e^{-\phi(k)}(y(k) - y(\underline{k})) = c(k)\right\}$ while k^* gives the maximum output if there is a measure zero of jobs assigned to workers, $k^* = \arg\max\left\{e^{-\phi(k)}(y(k) - y(\underline{k})) = c(k) | \phi(k) = 0\right\}$.

Fact 2. Any equilibrium assignment of jobs has jobs of type $\widehat{k}(\underline{k})$. Suppose not. Then the total number of openings is $\phi < \max\{\phi(k) | e^{-\phi(k)}$ $(y(k) - y(\underline{k})) = c(k).\}$ In this case, $e^{-\phi(k)}(y(k) - y(\underline{k})) > c(k)$, which implies entry $\widehat{k}(\underline{k})$ jobs. A contradiction

Fact 3. The equilibrium is a mixed strategy equilibrium. This follows from a result in JKK, which states that given two possible job types k and $k' > k$, both are offered if $y(k') - c(k') > y(k) - c(k)$ and $y(k)/c(k) < y(k') c(k')$.

Fact 4. In any equilibrium with multiple job types, the following expressions must hold

$$C(k) - C(k') = (y(k) - y(k'))\, e^{-\Phi(k)}$$

where $\Phi(k)$ is the total number of all openings greater than k.

Fact 5. In any equilibrium with a continuum of job types, $\phi(k) = \log\left(y'(k)/c'(k)\right)$ for all k. Follows from the limit of () given the interval between job types becomes small.

Fact 6. Define a social welfare function where $S(k)$ is the social welfare if the set of jobs are restricted to be below type k. Presently we have numerical results - to show that this is increasing over the interval $\left[\widehat{k}\left(\underline{k}\right), k^*\right]$ under the assumption that jobs are determined by the equation in fact 4. We are working on the analytical proof of this result. Since the equilibrium is constrained efficient, we know that the upper bound of job types for this social planning solution is equivalent to that óf the decentralized economy.

Fact 7. The JKK model features and equilibrium features with high and low jobs types if $y\left(k\right)-c\left(k\right) > y\left(\widehat{k}\right)-c\left(\widehat{k}\right)$ and $y\left(k^*\right)/c\left(k^*\right) < y\left(\widehat{k}\right)c\left(\widehat{k}\right)$, for $k > \widehat{k}$, which is their definition of concavity. ■

Proposition 2 For values of $k_1, k_2 \in \left[\widehat{k}, k^*\right]$,

$$G_1\left(k\right) = e^{-\phi(k)}, \text{ and} \tag{22}$$

$$G_2\left(k\right) = e^{-\phi(k)} + \phi\left(k\right)e^{-\phi(k)}. \tag{23}$$

where $G_1\left(\widehat{k}\right)$ is the probability of no offer and $G_2\left(\widehat{k}\right)$ is the probability of zero or one offers.

Proof. The quantity of workers with job type greater than k is $\phi\left(k\right)$. The probability that a worker does not receive one of these offers is $e^{-\phi(k)}$. Therefore, the probability that the worker receives an offer less than k is also $e^{-\phi(k)}$. Thus $G_1\left(k\right) = e^{-\phi(k)}$. Likewise, the probability that a worker does not receive two offers greater than k is $e^{-\phi(k)} + \phi\left(k\right)e^{-\phi(k)}$, where $e^{-\phi(k)}$ is the probability of no offers and $\phi\left(k\right)e^{-\phi(k)}$ is the probability of one offer. Therefore, the probability that the workers second highest offer is less than k is also given by $e^{-\phi(k)} + \phi\left(k\right)e^{-\phi(k)}$. Thus $G_1\left(k\right) = e^{-\phi(k)} + \phi\left(k\right)e^{-\phi(k)}$. ■

Proposition 3 For values of $k_1, k_2 \in \left[\widehat{k}, k^*\right]$,

$$G\left(k_1, k_2\right) = \left(1 + \phi\left(k_2\right) - \phi\left(k_1\right)\right)e^{-\phi(k_2)} \tag{24}$$

Proof. Suppose a worker receives n job offers. The probability that j of these are below k_2 and that $n-j$ offers are less than k_1 where $\underline{k} \leq k_2 \leq k_1 \leq k^*$ is given by ■

$$\begin{pmatrix} n \\ j \end{pmatrix} F\left(k_2\right)^j\left[1 - F\left(k_1\right)\right]^{n-j}$$

where $\begin{pmatrix} n \\ j \end{pmatrix} = \frac{n!}{(n-j)!j!}$. Taking the negative cross-derivative of this delivers the joint density of the j'th and $j+1$'th order statistics

$$g_{j,j+1}(k_1,k_2|n) = -\left[-\begin{pmatrix} n \\ j \end{pmatrix} jF(k_2)^{j-1}(n-j)[1-F(k_1)]^{n-j-1} f(k_1) f(k_2)\right]$$

$$\begin{aligned}
&= \frac{n!jF(k_2)^{j-1}(n-j)[1-F(k_1)]^{n-j-1} f(k_1) f(k_2)}{(n-j)!j!} \\
&= \frac{n!F(k_2)^{j-1}[1-F(k_1)]^{n-j-1} f(k_1) f(k_2)}{(n-j-1)!(j-1)!}
\end{aligned}$$

We are only interested in the best and second best offers, so we set $j = n-1$ and $j+1 = n$. This gives us

$$g(k_1,k_2|n) = \frac{n!F(k_2)^{n-2} f(k_1) f(k_2)}{(n-2)!}$$

Summing over all possible number of job offers we obtain

$$\begin{aligned}
g(k_1,k_2) &= \sum_{n=2}^{\infty} \frac{e^{-\phi(\underline{k})}\phi(\underline{k})^n}{n!} \frac{n!F(k_2)^{n-2} f(k_1) f(k_2)}{(n-2)!} \\
&= f(k_1) f(k_2) e^{-\phi(\underline{k})} \sum_{n=2}^{\infty} \frac{\phi(\underline{k})^n F(k_2)^{n-2}}{(n-2)!}
\end{aligned}$$

$$\begin{aligned}
&= f(k_1) f(k_2) e^{-\phi(\underline{k})F(k_2)} e^{-\phi(\underline{k})(1-F(k_2))} \sum_{n=2}^{\infty} \frac{\phi(\underline{k})^n F(k_2)^{n-2}}{(n-2)!} \\
&= \phi(\underline{k})^2 f(k_1) f(k_2) e^{-\phi(\underline{k})(1-F(k_2))} \sum_{n=2}^{\infty} \frac{e^{-\phi(\underline{k})F(k_2)} (\phi(\underline{k}) F(k_2))^{n-2}}{(n-2)!} \\
&= \phi'(k_1) \phi'(k_2) e^{-\phi(k_2)}
\end{aligned}$$

Integrating the density gives us the bivariate distribution function

$$\begin{aligned}
G(k_1,k_2) &= \int_0^{k_2} \int_{k_2}^{k_1} \phi'(z_1) \phi'(z_2) e^{-\phi(z_2)} dz_1 dz_2 \\
&= \int_0^{k_2} \phi'(z_2) e^{-\phi(z_2)} \int_{k_2}^{k_1} \phi'(z_1) dz_1 dz_2
\end{aligned}$$

$$\begin{aligned}
&= \int_0^{k_2} \phi'(z_2)\, e^{-\phi(z_2)} \left(\phi(k_1) - \phi(k_2)\right) dz_2 \\
&= \phi(k_1) \int_0^{k_2} \phi'(z_2)\, e^{-\phi(z_2)} dz_2 + \int_0^{k_2} -\phi'(z_2)\, \phi(k_2)\, e^{-\phi(z_2)} dz_2
\end{aligned}$$

$$\begin{aligned}
&= -\phi(k_1) \int_{-\phi(\underline{k})}^{-\phi(k_2)} e^u du + \int_0^{k_2} g(z_2)\, dz_2 \\
&= -\phi(k_1) \left(e^{-\phi(k_2)} - e^{-\phi(\underline{k})} + z_0\right) + (1 + \phi(k_2))\, e_0^{-\phi(k_2)}
\end{aligned}$$

$$\begin{aligned}
&= (1 + \phi(k_2) - \phi(k_1))\, e^{-\phi(k_2)} + \phi(k_1) \left(e^{-\phi(\underline{k})} - z_0\right) \\
&= \phi(k_1) \int_0^{k_2} \phi'(z_2)\, e^{-\phi(z_2)} dz_2 + \int_0^{k_2} -\phi'(z_2)\, \phi(k_2)\, e^{-\phi(z_2)} dz_2
\end{aligned}$$

$$\begin{aligned}
&= G_2(k_2) - \phi(k_1) \int_{-\phi(\underline{k})}^{-\phi(k_2)} e^u du \\
&= G_2(k_2) - \phi(k_1) \left(e^{-\phi(k_2)} - e^{-\phi(\underline{k})}\right) \\
&= G_2(k_2) + \phi(k_1) \left(e^{-\phi(\underline{k})} - e^{-\phi(k_2)}\right)
\end{aligned}$$

where z_0 is a constant. We can determine z_0 by using that $G(k_1, k_1) = G(k_1)$

$$\begin{aligned}
(1 + \phi(k_1) - \phi(k_1))\, e^{-\phi(k_1)} + \phi(k_1) \left(e^{-\phi(\underline{k})} - z_0\right) &= e^{-\phi(k_1)} \Leftrightarrow \\
z_0 &= e^{-\phi(\underline{k})}
\end{aligned}$$

Hence, we have that

$$G(k_1, k_2) = (1 + \phi(k_2) - \phi(k_1))\, e^{-\phi(k_2)}$$

or alternatively $G(k_1, k_2) = G_2(k_2) - \phi(k_1)\, e^{-\phi(k_2)}$.

Proposition 4 Suppose that two workers have different outside options (i.e. their flow returns in unemployment are $\underline{k}$ and $\underline{k}' > \underline{k}$, respectively) and the same first and second best job assignment (k_1 and k_2), then they must have

1. identical wages, $w(k_1, k_2|\underline{k}) = w\left(k_1, k_2|\underline{k}'\right)$, and

2. identical recruiting, $\Phi(k|k_1, \underline{k}) = \Phi\left(k|k_1, \underline{k}'\right)$ for all $k \in \left[\widehat{k}, k^*\right]$

Proof. We can establish

Fact 1. The frictional allocation of jobs to workers in each productivity state is directed and efficient

Fact 2. The social value of recruiting a worker in productivity state k_1 is equal despite differences if the workers are thrown into unemployment by the common shock. This is because this event is independent of any recruiting decision. Therefore, $\Phi(k|k_1, \underline{k}) = \Phi\left(k|k_1, \underline{k}'\right)$.

Fact 3. Given facts 1 and 2, firms must face the same incentives for recruiting a worker in state k_1. Therefore, they must offer the same wage in any future state k_1', k_2'. Therefore, $\Phi(k|k_1, \underline{k}) = \Phi\left(k|k_1, \underline{k}'\right)$ for all $k \in \left[\widehat{k}, k^*\right]$. ■

Proposition 5 In a stationary dynamic environment, the lowest quality job k offered to a worker in a type $\underline{k}$ satisfies the following free entry condition:

$$C(k) = e^{-\Phi(k)}\left(\frac{y(k) - w(k, \underline{k})}{1 - \beta\left(1 - e^{-\phi\left(\widehat{k}(k)\right)}\right)}\right)$$

which holds for all k.

Proof. This is simply the free entry condition for the lowest quality job type. ■

Proposition 6 Suppose two workers are distinguished by skill types h and h', respectively) and the same first and second best job assignment (k_1 and k_2), then they must have

1. wage differences proportional to productivity differences, $w(k_1, k_2|h) = w(k_1, k_2|h') - (h' - h)$, and
2. identical recruiting, $\Phi(k|k_1, h') = \Phi(k|k_1, h)$ for all $k \in \left[\widehat{k}, k^*\right]$ where $\widehat{k}$ for each worker type determined by equation (17), such that $w(k_1, k_2|h) = w\left(k_1, k_2|h^0\right) - \left(h - h^0\right)$ where $w\left(k_1, k_2|h^0\right)$ is the wage function generated in the homogenous worker model, where the supply of jobs is characterized by equation (6).

Proof. Similar to proposition 4. ■

Proposition 7 The allocations of jobs to workers in the auction game is equivalent to the allocation of jobs to workers in the buyer posting game

Proof. Fact 1.Suppose that there is only one type of job. In the static model, if firms post wages, then the wage distribution is given by

$$F(w) = \frac{1}{\phi(k)} \log\left(\frac{y(k) - y(\underline{k})}{y(k) - w}\right)$$

where the lower support is $y(\underline{k})$ and the upper support is $\overline{w} = \left(1 - e^{-\phi(k)}\right) y(k)$ The expected payoff the seller (worker) is then $\left(1 - e^{-\phi(k)} - \phi(k) e^{-\phi(k)}\right) y(k)$ and the expected payoff of the buyer is $y(k) e^{-\phi(k)}$

Fact 2. The extension of this model to a discrete number of types yields equivalent expected payoffs to the auction model. Also, higher type firms always pay higher wages. Thus, like the auction model, the offers of high type firms are always given priority by the workers. The expected payoff of firms in both models is

$$c(k) = y(k) e^{-\phi(k)}$$

where $\Phi(k) = \sum_{i=k}^{M} \phi(i)$ where $\phi(i)$ is the number of type i jobs created, and $i \in \{0, 1..., M\}$. where $y(i+1) > y(i)$.

Fact 3. It follows, in the limit, as the number of job types gets large that the expected payoffs of workers and firms in the posting and auction models must also be identical. Therefore, these models must generate identical allocations. ■

Proposition 8 If auctions and seller posting give identical allocations of jobs to workers, then wage dispersion under seller posting is equivalent to wage dispersion under buyer posting

Proof. It suffices to show the following facts:

Fact 1. Auction and buyer posting models give equivalent expected payoffs

Fact 2. Auction and seller posting models give equivalent expected payoffs

Fact 3. Given facts 1 and 2, the vector of wages posted by sellers in the seller posting model must be equivalent to the equilibrium wage vector in the buyer posting model. ■

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