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Unfunded pensions and endogenous labor supply

Torben M. Andersen and Joydeep Bhattacharya

School of Economics and Management
Aarhus University
Bartholins Allé 10, Building 1322
DK-8000 Aarhus C - Denmark
Phone +45 8942 1610
Mail: oekonomi@econ.au.dk
Web: www.econ.au.dk

Unfunded pensions and endogenous labor supply

Torben M. Andersen[*]
University of Aarhus, Denmark
CEPR, CESifo, and IZA

Joydeep Bhattacharya[†]
Iowa State University, USA

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Abstract. A classic result in dynamic public economics, dating back to Aaron (1966) and Samuelson (1975), states that there is no welfare rationale for PAYG pensions in a dynamically-efficient neoclassical economy with exogenous labor supply. This paper argues that this result, under the fairly-mild restriction that the old be no less risk-averse than the young, extends to a neoclassical economy with endogenous labor supply.

Keywords: pay-as-you-go, social security, endogenous labor supply, dynamic efficiency
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[*] *Corresponding author*: Torben M. Andersen, Department of Economics, Building 322, University of Aarhus, 8000 Aarhus C, Denmark; Phone: +45 8942 1609; Fax: +45 8613 6334; E-mail: tandersen@econ.au.dk

[†] Joydeep Bhattacharya, Department of Economics, Iowa State University, Ames IA 50011-1070, USA. Phone: (515) 294 5886, Fax: (515) 294 0221; E-mail: joydeep@iastate.edu

1. Introduction

Governments in about a hundred and fifty countries around the world offer some kind of old-age pension (social security) to their citizens. Most of these pension programs have a substantial unfunded, pay-as-you-go (PAYG) component: the working young are taxed and the proceeds are used to finance a transfer (pension) to the existing retired elderly (defined benefit). Even though many of these programs have been around for nearly a hundred years and routinely absorb 5-15% of G.D.P, the rationale for their very existence continues to be hotly debated – see Blake (2006) for a detailed discussion.

Among academic economists, this debate starts with a classic, justly-venerated result discussed originally by Aaron (1966) and Samuelson (1975). Consider a two-period overlapping generations model with neoclassical production (also known as the Diamond (1965) model) where the young supply their labor inelastically, the old are retired, and there is no population growth. Suppose there is a government that finances a fixed payment (benefit/pension) to each old agent by collecting a lump-sum tax from each young agent. Aaron (1966) and Samuelson (1975) showed that the introduction of such a pension system can improve the stationary welfare of all[1] two-period lived agents if and only if the economy is initially dynamically *inefficient* – the net return on capital is less than zero, the "biological interest rate" (Samuelson, 1958). The actual result, as exposited by textbooks such as Blanchard and Fischer (1989), is a bit more nuanced in that use is made of Samuelson's correspondence principle to require that the initial stationary state, the starting point of the comparison, must be dynamically stable. At a stable steady state, a small increase in the benefit level reduces private capital formation. Such crowding out of private capital is justifiable on welfare grounds if and only if the economy was overaccumulating capital (dynamically inefficient) to begin with.

The Aaron-Samuelson result provides a simple, potentially verifiable condition that served to narrow the debate: a PAYG pension system is socially desirable if and only if the net return on capital is negative. Later work by Abel, Mankiw, Summers and Zeckhauser (1989), and more recently Barbie, Hagedorn, and Kaul (2004), suggested that most developed economies, such as the U.S., are most likely dynamically *efficient*; by implication, a PAYG system in such countries is not desirable at least from the standpoint

[1] It is a standard result that introduction of a PAYG social security scheme bestow the first generation old in the system with a "gift" which in turns has a positive welfare effect. These transitional gains have to be weighted against the more permanent consequences of the scheme. In the following we follow the literature and consider only the latter, i.e. the analysis is confined to steady-state effects.

of simple lifecycle models.

This paper generalizes the environment studied by Samuelson (1975) and Blanchard and Fischer (1989) in two ways: a) it allows for endogenous/elastic labor supply, and b) it permits a distortionary tax on young labor income. We are then able to show the following: under a fairly-mild restriction that the old be no less risk-averse than the young, the Aaron-Samuelson result survives these generalizations. Aside from this, we provide a complete analysis of the issue; in fact, all of the existing results in the literature are successfully unified. Moreover, we are able to sift through the vast complexity of the comparative statics exercises and provide a deeper understanding of all the forces at work.

The current paper is closest in spirit to pioneering work by Breyer and Straub (1993), discussed in Blake (2006). Their primary focus is on the following issue. Suppose a comparison of the net return to capital with the net biological interest rate reveals that the PAYG system in place is undesirable. Would abolition of such a system (and replacement by a fully funded system) lead to an intergenerational Pareto improvement considering the fact that the young alive would have paid into the system but would not get anything in return? Breyer and Straub (1993) prove that a necessary condition for such improvement is if, in the process, labor supply is distorted.

They go on to ask: are such "static distortions" to labor supply enough to justify a transition to the fully funded system? It is here that their focus is aligned with ours as their question may be re-interpreted as indirect interest in the Aaron-Samuelson result for economies with endogenous labor supply. To answer this question, they focus only on steady states and consider the steady-state welfare of a representative two-period lived agent. Additionally, they study a PAYG system in which the contribution by the young is in the form of a distortionary payroll tax on young labor income, and the benefit to the pensioners is actuarially and intergenerationally fair: it is tied to one's labor supply (defined contributions). In this setting, they show that, in a dynamically efficient economy, a sufficient condition for steady-state welfare to fall with the payroll tax rate is that private capital does not increase with the payroll tax rate.

It is instructive to directly compare our work to Breyer and Straub (1993). We study a PAYG system with a fixed common benefit level (defined benefits), i.e., the scheme is not actuarially fair. This permits more direct comparison with the classic statements of the Aaron-Samuelson result. We go further than them because we integrate insights regarding stability of steady states from Nourry (2001). Specifically, we prove that if the initial steady state is dynamically efficient and saddle-point stable, a sufficient condition for our version of the PAYG system to be welfare-undesirable is that the capital-labor ratio does not increase with the benefit level. Breyer and Straub (1993) are largely silent on the important issue: when does their sufficient condition hold? In comparison, we show that if the old are no less risk-averse than the young, our sufficient condition holds under fairly reasonable restrictions. The upshot is that we can definitively claim to have shown that the Aaron-Samuelson result, under a mild condition, extends to economies with endogenous labor supply, something that Breyer and Straub (1993) cannot.

The plan for the rest of the paper is as follows. Section 2 outlines the environment of the model, a generalization of Diamond (1965) to endogenous labor supply. Section 3 describes the perfect-foresight competitive equilibria and their comparative static properties; it also derives a condition for saddle-path stability of a steady state. In Section 4, we derive our main results, while Section 5 concludes. Proofs of all major results are to be found in the appendices.

2. THE MODEL

We consider a textbook version of the overlapping-generations model with production due to Diamond (1965), augmented to allow for endogenous labor supply. More specifically, we study an economy consisting of an infinite sequence of two-period lived overlapping generations, an initial old generation, and an infinitely-lived government. Let $t = 1, 2, ...$ index time. At each date t, a new generation comprised of a continuum of measure one of identical members appears. Each agent is endowed with one unit of labor when young and is retired when old. In addition, the initial old agents are endowed with $K_1 > 0$ units of capital.

There is a single final good produced using a standard neoclassical production function $F(K_t, L_t)$ displaying constant returns, where K_t denotes the capital input and L_t denotes the labor input at t. Let $k_t \equiv K_t/L_t$ denote the capital-labor ratio (capital per young agent). Then, output per young agent at time t may be expressed as $f(k_t)$ where $f(k_t) \equiv F(K_t/L_t, 1)$ is the intensive production function. We assume that $f' > 0 > f''$, and that the usual Inada conditions hold. The final good can either be consumed in the period it is produced, or it can be stored to yield capital the following period. For reasons of analytical tractability, capital is assumed to depreciate 100% between periods.

Let c^t_{1t} (c^t_{2t+1}) denote the consumption of the final good at date t (date $t+1$) by a representative young (old) agent born at t. Let L_t denote the labor supply at date t by a young agent. All such agents have preferences representable by the time-separable utility function

$$(1) \quad U(c_{1t}^t, c_{2t+1}^t, L_t) \equiv u\left(c_{1t}^t\right) + v\left(c_{2t+1}^t\right) - d\left(L_t\right)$$

where u and v are strictly increasing and strictly concave in their arguments and d is strictly increasing and strictly convex in its argument. We assume the standard limit conditions (see Nourry, 2001, Assumption 1, for example) that would ultimately preclude corner optima.

Young agents supply labor in competitive labor markets, earning a wage of w_t at time t, where

$$(2) \quad w_t \equiv w(k_t) = f(k_t) - k_t f'(k_t)$$

and $w'(k_t) > 0$. In addition, capital is traded in competitive capital markets, and earns a gross real return of R_{t+1} between t and $t+1$, where

$$(3) \quad R_{t+1} \equiv R\left(k_{t+1}\right) = f'(k_{t+1})$$

with $R'\left(k_{t+1}\right) < 0$.

The government runs a standard pay-as-you-go (PAYG) pension system in each period. It levies an appropriate proportional income tax (τ) on each young agent so as to finance a lump-sum transfer of $B \geq 0$ to each of the current old. The government budget constraint is

$$(4) \quad \tau_t w_t L_t = B$$

The benefit level is taken to be the policy variable (defined benefit scheme), and hence, the tax rate is calculated from (4). Since population size does not grow, the net rate of return on the PAYG scheme is zero.[2]

Each young agent, born at date $t \geq 1$, chooses how much to consume in each period of

[2] Breyer and Straub (1993) consider three different formulations of the PAYG scheme: a) fixed contribution rate τ but pensions are tied to past labor supply, b) time-varying contribution rate, and c) lump-sum contributions and benefits.

life and how much labor to supply when young, by maximizing $U(c_{1t}^t, c_{2t+1}^t, L_t)$ subject to

$$c_{1t}^t = (1-\tau_t)\, w_t L_t - S_t \tag{5}$$

$$c_{2t+1}^t = R_{t+1} S_t + B \tag{6}$$

along with $c_{1t}^t \geq 0$, $c_{2t+1}^t \geq 0$ and $L_t \in [0,1]$; here S denotes saving. The initial old agents face the following budget constraint: $c_{20} \leq R_1 k_1 + B$. Thus, at date 1, the initial old agents receive capital income $R_1 k_1$ in addition to their pension and they consume it all.

The optimization problem of young agents can be re-stated as

$$\max_{L_t \in [0,1],\, S_t \in (0,(1-\tau) w_t L_t)} u\left((1-\tau_t)\, w_t L_t - S_t\right) + v\left(R_{t+1} S_t + B\right) - d\left(L_t\right).$$

The first order conditions to the agent's problem, assuming interior solutions, are given by

$$\Gamma_L \equiv u'(\cdot)\,(1-\tau_t)\, w_t - d'(\cdot) = 0 \tag{7}$$

$$\Gamma_S \equiv -u'(\cdot) + R_{t+1} v'(\cdot) = 0. \tag{8}$$

For future use, note

$$\begin{aligned}
\Gamma_{LL} &= u''(\cdot)\left((1-\tau_t)\, w_t\right)^2 - d''(\cdot) < 0 \\
\Gamma_{LS} &= -u''(\cdot)\,(1-\tau_t)\, w_t > 0 \\
\Gamma_{SS} &= u''(\cdot) + R_{t+1}^2 v''(\cdot) < 0 \\
\Gamma_{SL} &= -u''(\cdot)\,(1-\tau_t)\, w_t > 0.
\end{aligned}$$

The second order conditions to the agent's problem are satisfied if $\Gamma_{LL} < 0$ (which is true) and if $D \equiv \Gamma_{SS}\Gamma_{LL} - \Gamma_{SL}\Gamma_{LS}$ is assumed to be positive. Assuming $D > 0$ is the same as assuming the following:

Assumption A1

$$\left(u''(\cdot) + R^2 v''(\cdot)\right)\left(u''(\cdot)\left((1-\tau_t)\, w\right)^2 - d''(\cdot)\right) - \left(u''(\cdot)\,(1-\tau_t)\, w\right)^2 > 0. \tag{9}$$

In Appendix A, we show that a young agent's labor supply (partial equilibrium) can be written as

$$(10)\quad L_t = L(\tau_t, B, w_t, R_{t+1})$$

where

$$(11)\quad \frac{\partial L_t}{\partial \tau_t} \lesseqgtr 0, \frac{\partial L_t}{\partial B} < 0, \frac{\partial L_t}{\partial w_t} \lesseqgtr 0, \frac{\partial L_t}{\partial R_{t+1}} \lesseqgtr 0$$

and where sign $\frac{\partial L_t}{\partial \tau} = -$ sign $\frac{\partial L_t}{\partial w_t}$. Note that the standard condition from static models that the substitution effect dominates the income effect $(u'(\cdot) + u''(\cdot)(1-\tau)w > 0)$ is a sufficient condition that $\frac{\partial L_t}{\partial \tau} < 0$ and hence $\frac{\partial L_t}{\partial w_t} > 0$.

Similarly, in Appendix A, we show that the optimal savings function is uniquely summarized by

$$(12)\quad S_t = S(\tau_t, B, w_t, R_{t+1})$$

where

$$(13)\quad \frac{\partial S_t}{\partial \tau_t} \lesseqgtr 0, \frac{\partial S_t}{\partial B} < 0, \frac{\partial S_t}{\partial w_t} > 0, \frac{\partial S_t}{\partial R_{t+1}} \lesseqgtr 0$$

and where sign $\frac{\partial S_t}{\partial \tau} = -$ sign $\frac{\partial S_t}{\partial w_t}$. Moreover, as shown in Appendix A,

$$(14)\quad \text{sign } \frac{dL_t}{dR_{t+1}} = \text{sign } \frac{dS_t}{dR_{t+1}}$$

3. Equilibria

Using the government budget constraint (implying that $\tau_t = B/w_t L_t$), we have that the equilibrium employment level and capital stock can be written

$$(15)\quad \widehat{L}(B, w_t, R_{t+1}) \equiv L(B/w_t L_t, B, w_t, R_{t+1})$$
$$(16)\quad \widehat{S}(B, w_t, R_{t+1}) \equiv S(B/w_t L_t, B, w_t, R_{t+1})$$

Henceforth, we assume (as is standard):

Assumption A2 i) $\frac{\partial \widehat{S}_t}{\partial R_{t+1}} \geq 0$, i.e., private saving is non-decreasing in its return[3], and ii)

[3] A sufficient condition for this is $v''(\cdot)R^2 + v'(\cdot) > 0$.

$\frac{\partial \widehat{L}_t}{\partial w_t} \geq 0$, i.e. employment is non-decreasing in the wage rate.

Note that $\frac{\partial \widehat{S}_t}{\partial R_{t+1}} > 0$ implies that $\frac{\partial L_t}{\partial R_{t+1}} > 0$.

Since the aggregate saving of the young at any date becomes the start-of-period capital for the next date, we have

(17) $K_{t+1} = S_t$

where S_t is defined in eq. (12) above. Since each young agent supplies L_t units of labor, we have

(18) $k_{t+1}L_{t+1} = S_t$.

Perfect foresight competitive equilibria are sequences $\{k_t\}_{t=2}^{\infty}$ that satisfy (18), given the initial $k_1 > 0$, and (15), (16), (4), (2) and (3). Specifically, they are dynamic sequences $\{k_t\}_{t=2}^{\infty}$ that satisfy

(19) $k_{t+1}\ \widehat{L}\left(B, w\left(k_{t+1}\right), R\left(k_{t+2}\right)\right) = \widehat{S}(\tau, B, w\left(k_t\right), R\left(k_{t+1}\right))$

Steady state equilibria are time-invariant sequences, k, that satisfy (19). In general, as discussed in Nourry (2001), Cazzavillan and Pintus (2004), and Nourry and Vendetti (2006), conditions for the existence and uniqueness of a steady-state equilibrium in the Diamond (1965) model with endogenous labor supply, are fairly involved and somewhat unintuitive; more so, when $B > 0$. For our purposes, it suffices to assume that a steady-state solution to (19) exists.

Also for future reference, note that a steady-state equilibrium, k, will be called dynamically efficient if $R(k) > 1$, dynamically inefficient if $R(k) < 1$, and the golden rule if $R(k) = 1$.

3.1. Stability analysis. As discussed by Blanchard and Fischer (1989), Samuelson's "correspondence principle" suggested a tight link between the stability properties of a steady state and its comparative static properties. Since the ultimate goal of the paper is to establish a particular comparative static property, we start off by studying the stability properties of a steady state. To that end, we linearize (19) around a steady state to get

$$\widehat{L}\widetilde{k}_{t+1} + k\left[\widehat{L}_w w_k \widetilde{k}_{t+1} + \ \widehat{L}_R R_k \widetilde{k}_{t+2}\right] = \widehat{S}_w w_k \widetilde{k}_t + \widehat{S}_R R_k \widetilde{k}_{t+1},$$

where the tilde over k denotes deviation from its steady state value, and where X_j denotes the derivative of the function X with respect to the variable j (and $j \neq t+n, n = 0, 1, 2$). Using $w_k = -kR_k$, and reorganizing the previous equation, we get

$$k\ \widehat{L}_R R_k \widetilde{k}_{t+2} + \left[\left(\widehat{L} - k\ \widehat{L}_w k R_k \right) - \widehat{S}_R R_k \right] \widetilde{k}_{t+1} + \widehat{S}_w k R_k \widetilde{k}_t = 0$$

and finally,

$$\widetilde{k}_{t+2} + A_1\, \widetilde{k}_{t+1} + A_0\, \widetilde{k}_t = 0,$$

where

$$A_1 \equiv \frac{\left(\widehat{L} - k \widehat{L}_w k R_k \right) - \widehat{S}_R R_k}{k\ \widehat{L}_R R_k} < 0, \text{ and}$$

$$A_0 \equiv \frac{\widehat{S}_w}{\widehat{L}_R} > 0,$$

The sign conditions follow from (11), (14), and (13).

Lemma 1. *A steady state, k, is saddle-point stable if*

$$\text{(20)} \quad \left[\widehat{L} + \left[\widehat{L}_R - k\ \widehat{L}_w \right] k R_k - \widehat{S}_R R_k + \widehat{S}_w k R_k \right] > 0$$

holds.

This extends the result in Nourry (2001) to the case with a tax-financed pension scheme (defined benefits).

3.2. Comparative statics – equilibrium responses. In a steady-state equilibrium, the aggregate capital stock is given by

$$\text{(21)} \quad \widehat{K}(B) \equiv \widehat{S}\left(B,\, w\left(\frac{\widehat{K}(B)}{\widehat{L}(B)} \right),\, R\left(\frac{\widehat{K}(B)}{\widehat{L}(B)} \right) \right),$$

and equilibrium aggregate employment by

$$\text{(22)} \quad \widehat{L}(B) \equiv \widehat{L}\left(B,\, w\left(\frac{\widehat{K}(B)}{\widehat{L}(B)} \right),\, R\left(\frac{\widehat{K}(B)}{\widehat{L}(B)} \right) \right),$$

and $\widehat{k} \equiv \widehat{K}/\widehat{L}$.

We now proceed to determine the properties of equilibrium optimal savings and labor supply. Note these cannot be read off the individual responses – (11)-(**??**) discussed above – since those had not taken the government budget constraint into account. Using (4), we can re-write (7) and (8) in steady states as

$$(23)\quad u'\left(w\widehat{L}-B-\widehat{S}\right)\cdot\left[w-\frac{B}{\widehat{L}}\right]-d'\left(\widehat{L}\right)=0$$

and

$$(24)\quad -u'\left(w\widehat{L}-B-\widehat{S}\right)+R\cdot v'\left(R\widehat{S}+B\right)=0$$

Hence, totally differentiating (23)-(24) yields

$$\begin{aligned}
&u''(\cdot)\left[\widehat{L}dw+wd\widehat{L}-dB-d\widehat{S}\right]\left[w-\frac{B}{\widehat{L}}\right]+u'(\cdot)\left[dw-\frac{1}{\widehat{L}}dB+\frac{B}{\widehat{L}^2}d\widehat{L}\right]-d''(\cdot)\,d\widehat{L}=0\\
&-u''(\cdot)\left[Ldw+wd\widehat{L}-dB-d\widehat{S}\right]+v'(\cdot)\,dR+Rv''(\cdot)\left[\widehat{S}dR+Rd\widehat{S}+dB\right]=0.
\end{aligned}$$

Eventually, we will require knowledge of these equilibrium responses locally near $B=0$. To that end, evaluating these expressions for $B=0$ yields

$$(25)\quad \left[u''(\cdot)\,w^2-d''(\cdot)\right]d\widehat{L}+\left[-u''(\cdot)\,w\right]d\widehat{S}=\left[-u''(\cdot)\,w\widehat{L}-u'(\cdot)\right]dw+\left[u''(\cdot)\,w+u'(\cdot)\,\frac{1}{\widehat{L}}\right]dB$$

$$(26)\quad \left[-u''(\cdot)\,w\right]d\widehat{L}+\left[u''(\cdot)+R^2v''(\cdot)\right]d\widehat{S}=u''(\cdot)\,\widehat{L}dw-\left[u''(\cdot)+Rv''(\cdot)\right]dB+\left[-v'(\cdot)-Rv''(\cdot)\,\widehat{S}\right]dR$$

From here, the different equilibrium responses are easily computed using Cramer's rule. For example, the response of an increase in the wage rate (for fixed R and $B=0$) on equilibrium labor supply is given by

$$\begin{aligned}
\frac{\partial\widehat{L}}{\partial w} &= \frac{\begin{vmatrix} -u''(\cdot)\,w\widehat{L}-u'(\cdot) & -u''(\cdot)\,w \\ u''(\cdot)\widehat{L} & u''(\cdot)+v''(\cdot)\,R^2 \end{vmatrix}}{\begin{vmatrix} u''(\cdot)\,w^2-d''(\cdot) & -u''(\cdot)\,w \\ -u''(\cdot)\,w & u''(\cdot)+v''(\cdot)\,R^2 \end{vmatrix}}\\
&= \frac{-\left[u''(\cdot)\,w\widehat{L}+u'(\cdot)\right]\left[u''(\cdot)+v''(\cdot)\,R^2\right]+\left(u''(\cdot)\right)^2 w\widehat{L}}{\left[u''(\cdot)\,w^2-d''(\cdot)\right]\left[u''(\cdot)+v''(\cdot)\,R^2\right]-\left[u''(\cdot)\,w\right]^2}.
\end{aligned}$$

Notice that the second order condition for the individual optimization problem (evaluated at $B = \tau = 0$) ensures

$$\left[u''(\cdot)\,w^2 - d''(\cdot)\right]\left[u''(\cdot) + v''(\cdot)\,R^2\right] - \left[u''(\cdot)\,w\right]^2 > 0$$

implying, for example, that the denominator of $\frac{\partial \widehat{L}}{\partial w}$ above is positive, evaluated locally around $B = 0$. Moreover note that $\frac{\partial \widehat{L}}{\partial x}\mid_{B=0} = \frac{\partial L}{\partial x}$ for $x = w, R$.

We collect these equilibrium responses in the following lemma.

Lemma 2. *a) Define*

$$\Phi_v \equiv \frac{c_2 v''(\cdot)}{v'(\cdot)}\mid_{B=0}\,; \quad \Phi_u \equiv \frac{c_1 u''(\cdot)}{u'(\cdot)}\mid_{B=0} \tag{27}$$

Then,

$$sign\frac{\partial \widehat{L}}{\partial R} = sign\frac{\partial \widehat{S}}{\partial R} = sign\ \left[-v'(\cdot) - R\widehat{S}v''(\cdot)\right]_{B=0} = sign\ \left[-1 + \Phi_v\right]$$

Hence,

$$\Phi_v \leq 1 \Longleftrightarrow sign\ \frac{\partial \widehat{L}}{\partial R} = sign\ \frac{\partial \widehat{S}}{\partial R} \geq 0 \tag{28}$$

b)

$$\begin{aligned} \frac{\partial \widehat{S}}{\partial B} &= \frac{d''(\cdot)\left[u''(\cdot) + Rv''(\cdot)\right] - Rv''(\cdot)\left[u''(\cdot)\,w^2 + wRu'(\cdot)\,\frac{1}{\widehat{L}}\right] + wu'(\cdot)\,\frac{1}{\widehat{L}}\left[u''(\cdot) + v''(\cdot)\,R^2\right]}{\left[u''(\cdot)\,w^2 - d''(\cdot)\right]\left[u''(\cdot) + v''(\cdot)\,R^2\right] - \left[u''(\cdot)\,w\right]\left[u''(\cdot)\,w\right]} \\ &= \frac{d''(\cdot)\left[u''(\cdot) + Rv''(\cdot)\right] - Rv''(\cdot)\,u''(\cdot)\,w^2 + wu'(\cdot)\,\frac{1}{\widehat{L}}u''(\cdot)}{\left[u''(\cdot)\,w^2 - d''(\cdot)\right]\left[u''(\cdot) + v''(\cdot)\,R^2\right] - \left[u''(\cdot)\,w\right]\left[u''(\cdot)\,w\right]} < 0, \end{aligned} \tag{29}$$

and

c) if $R > 1$

$$\frac{\partial \widehat{L}}{\partial B} = \frac{u'(\cdot)\,\frac{1}{\widehat{L}}\left[u''(\cdot) + v''(\cdot)\,R^2\right] + v''(\cdot)\,u''(\cdot)\,wR\,(R-1)}{\left[u''(\cdot)\,w^2 - d''(\cdot)\right]\left[u''(\cdot) + v''(\cdot)\,R^2\right] - \left[u''(\cdot)\,w\right]\left[u''(\cdot)\,w\right]} \tag{30}$$

is of ambiguous sign.

4. WELFARE EFFECTS OF PUBLIC PENSIONS

The government is assumed to be utilitarian. It determines the benefit level B by maximizing the individual lifetime utility in a steady state. The government's objective function, by use of (4), can be written as

$$U(B) = u\left(w\left(\frac{\widehat{K}(B)}{\widehat{L}(B)}\right)\widehat{L}(B) - B - \widehat{K}(B)\right) + v\left(R\left(\frac{\widehat{K}(B)}{\widehat{L}(B)}\right)\widehat{K}(B) + B\right) - d\left(\widehat{L}(B)\right)$$

where the general equilibrium employment and capital stock are given by (22) and (21) respectively. A marginal change in the benefit level brings about a change in agent welfare by an amount:

$$U'(B) = u'(\cdot)\left[w\frac{\partial\widehat{L}}{\partial B} + w'\left(\frac{\widehat{K}}{\widehat{L}}\right)\widehat{L}\frac{\partial\left(\frac{\widehat{K}}{\widehat{L}}\right)}{\partial B} - 1 - \frac{\partial\widehat{K}}{\partial B}\right] + v'(\cdot)\left[R'\left(\frac{\widehat{K}}{\widehat{L}}\right)K\frac{\partial\frac{\widehat{K}}{\widehat{L}}}{\partial B} + R\frac{\partial\widehat{K}}{\partial B} + 1\right] - d'\left(\widehat{L}\right)\frac{\partial\widehat{L}}{\partial B}.$$

Using the first order conditions (7) and (8) as well as the government budget constraint (4), the optimality condition can be written more compactly as

$$U'(B) = v'(\cdot)\left[(1-R) + (R-1)\ w'\left(\widehat{k}\right)\widehat{L}\ \frac{\partial\widehat{k}}{\partial B} + R\tau w\left(\widehat{k}\right)\ \frac{\partial\widehat{L}}{\partial B}\right], \tag{31}$$

where $\widehat{k}$ is the general-equilibrium stationary value for the capital-labor ratio.

Broadly speaking, introducing a unfunded pension or raising the promised benefit level infinitesimally has three effects: (i) savings-effect, (ii) capital-labor substitution effect, and (iii) labor supply effect. The savings effect is the most standard and well-known (since Aaron, 1966). If the initial equilibrium is characterized by $R > 1$ (dynamic efficiency), it follows that an increase in benefits lowers utility since it shifts savings out of capital (with gross return R) into the PAYG-pension with a gross return of 1. The capital-labor substitution effect emerges because the composition of income according to its source, wages or capital, matters – it matters because wage income is earned when young and capital income when old. If $R = 1$, this composition does not matter. However, if $R > 1$, there is, on the margin, a gain from shifting to wage from capital income since the former can be invested with a return factor $R > 1$. In that case, if a benefit increase leads to an increase in the capital-labor ratio ($\frac{\partial\widehat{k}}{\partial B} > 0$), then such a benefit increase is welfare improving. The opposite holds if $\frac{\partial \widehat{k}}{\partial B} < 0$. Finally, the labor supply effect arises because the labor supply decision is distorted by the income tax – indeed employment is inefficiently low. In this case, if a benefit increase raises employment, the effect on welfare is positive. It is readily apparent from (31) that the issue of whether public pensions are desirable is not settled simply by a comparison of R with the biological interest rate.

The upshot is that very little can be said definitively about the desirability of public pensions.[4] Additional information on this issue is obtained by analyzing it near the golden rule.

Proposition 1. *When $R = 1$, i.e., the economy is initially at the golden rule, increasing the size of public pensions is not welfare neutral because of the distortion to labor supply. If a lump-sum tax is available, then at the golden rule, the optimal size of the public pension is zero.*

A version of this last result is also discussed in Blake (2006), chapter 4.

The strategy in the following is to consider the marginal value of introducing a small public pension if there is none to begin with. It follows from (31), for $B = \tau = 0$,

$$(32)\quad U'(0) = v'(\cdot)(1 - R)\left[1 - w'\left(\widehat{k}\right)\widehat{L}\frac{\partial \widehat{k}}{\partial B}\right].$$

It is immediate from (32) that if $R > 1$, a small increase in the size of the pension starting from a size of zero is *not* welfare enhancing if it is the case that

$$(33)\quad \frac{\partial \widehat{k}}{\partial B} < 0$$

holds. In other words, in a dynamically-efficient economy, a necessary condition for $U'(0) < 0$ is that $1 - w'\left(\widehat{k}\right)\widehat{L}\frac{\partial \widehat{k}}{\partial B} > 0$; a sufficient condition is that $\frac{\partial \widehat{k}}{\partial B} < 0$. Put differently, there would be *no* welfare rationale for introducing a PAYG system in a dynamically-efficient economy if $\frac{\partial \widehat{k}}{\partial B} < 0$, i.e., the pension system *crowds out the equilibrium capital-labor ratio.*

[4] In the case of exogenous labor supply, L is fixed and $\partial K/\partial B < 0$ holds at a stable steady state (see Blanchard and Fischer, 1989). Hence

$$\text{sign } U'(B) = \text{sign } (R - 1)$$

a restatement of the classic Aaron (1966) condition.

Lemma 3. *a)*

$$\frac{\partial \widehat{k}}{\partial B} = \frac{\widehat{S}_B - \widehat{k}\ \widehat{L}_B}{\widehat{L} - \left[\widehat{S}_w w_k + \widehat{S}_R R_k\right] + \widehat{k}\ \left[\widehat{L}_w w_k + \ \widehat{L}_R R_k\right]}$$

b) If k is saddle-point stable, i.e., (20) holds,

$$sign\ \frac{\partial \widehat{k}}{\partial B} = sign\ \left[\widehat{S}_B - \widehat{k}\ \widehat{L}_B\right]$$

where S_B and L_B are computed from (29)-(30).

The upshot is that use of the stability condition allows us to state a sufficient condition ruling out a welfare case for a public pension in terms of the direct savings and labor supply responses to the public pension, i.e.,

(34) $\widehat{S}_B - k\ \widehat{L}_B < 0$

The thrust of models with exogenous labor supply is that there is no case for public pensions if private savings (capital) is crowded out ($\widehat{S}_B < 0$). With endogenous labor supply, that is simply not enough since it is possible that labor supply decreases $\left(\widehat{L}_B < 0\right)$ to such an extent that the capital-labor ratio increases $\left(\frac{\partial \widehat{k}}{\partial B} > 0\right)$ even though the capital stock falls $(\widehat{S}_B < 0)$. Hence, whether there is a case or not for public pensions is not straightforwardly settled by the crowding out of savings (capital), as in the exogenous labor case. Matters are further complicated because, as is clear from (30), the sign of $\widehat{L}_B$ is ambiguous.

The key question is how the income tax affects equilibrium savings and labor supply. This is amply evident from the next result.

Proposition 2. *For a dynamically-efficient economy, there is no welfare case for introducing a PAYG pension financed by a lump-sum tax.*

The proof is in the appendix and proceeds by proving that under a lump-sum tax financing scheme, $\widehat{S}_B < 0$ and $\widehat{L}_B > 0$ – reducing capital and raising labor supply, and hence, the capital-labor ratio falls. Accordingly, from (33), there would be no welfare case for a unfunded public pension in this case.

More generally, though, as is well known, an income tax releases both a substitution and income effect, and if the former dominates (sufficient condition: $u'(.) + u''(.)w > 0$)

)it follows that labor supply decreases with a tax increase, cf (11), and this goes in the direction of increasing the capital-labor ratio. The distortion arising via the substitution effect thus makes the welfare effect of the public pension more complicated.

Corollary 1. *(Samuelson, 1975) For a dynamically-efficient economy, if labor supply is perfectly inelastic, there is no welfare case for introducing a PAYG pension.*

Since $\widehat{S}_B < 0$ holds, cf (29), it follows that $\widehat{L}_B = 0$ (perfectly-inelastic labor supply) ensures that the sufficient condition (34) holds; and this proves the corollary. This is the well-known result (Aaron, 1966 and Samuelson, 1975) from models with exogenous labor supply. By implication if the labor supply is "sufficiently" inelastic, the sufficient condition (34) holds. Heuristically, it may be conjectured that the sufficient condition is violated if, somehow, the substitution effect can be made sufficiently strong, i.e., $\widehat{L}_B$ is made sufficiently negative.

Lemma 4.

$$\begin{aligned}\widehat{L}_B &= \widehat{L}_B\,|_{lump\ sum\ tax} + \frac{u'(\cdot)\frac{1}{L}\left[u''(\cdot) + v''(\cdot)R^2\right]}{D} < \widehat{L}_B\,|_{lump\ sum\ tax} \\ \widehat{S}_B &= \widehat{S}_B\,|_{lump\ sum\ tax} + \frac{u'(\cdot)\frac{1}{L}\left[u''(\cdot)w\right]}{D} < \widehat{S}_B\,|_{lump\ sum\ tax}\end{aligned}$$

As Lemma 4 shows, a larger $u'(\cdot)$ indeed implies a stronger substitution effect reducing $\widehat{L}_B$. However, at the same time, $\widehat{S}_B$ falls, making it harder to evaluate the overall effect on (34). Using Lemma 4, we can write (34) as

$$\widehat{S}_B - k\widehat{L}_B = \underbrace{\left[\widehat{S}_B\,|_{\text{lump sum tax}} - k\widehat{L}_B\,|_{\text{lump sum tax}}\right]}_{<0}$$
$$+ \underbrace{\frac{u'(\cdot)\frac{1}{L}}{D}\left[u''(\cdot)\,(w-k) - k\,v''(\cdot)R^2\right]}_{\lesseqgtr 0} \lesseqgtr 0$$

where $D > 0$. It turns out that (34) holds under a mild condition on relative risk aversion.

Proposition 3. *In a dynamically-efficient economy, there is no case for introducing a PAYG pension scheme if $\Phi_v \geq \Phi_u$ (as defined in (27), i.e., the Arrow-Pratt measure of relative risk aversion for the old is no less than that of the young.*

The upshot of the above analysis is the following. For a dynamically-efficient neoclassical economy, there is no welfare rationale for introducing a PAYG pension system if the old are at least as risk averse as the young. Of course, both Φ_v and Φ_u are, in general, endogenous variables and hence, the result in Proposition 3 is to be understood as saying: if the equilibrium under study displays the property that the (endogenous) Arrow-Pratt measure of relative risk aversion for the old is no less than that of the young, then in a dynamically-efficient neoclassical economy, there is no welfare rationale for introducing a PAYG pension system. We close with a version of this result for the commonly-used CRRA form of time-separable utility.

Corollary 2. *Suppose* $U(c_1, c_2) = \frac{1}{1-\sigma} c_1^{1-\sigma} + \beta \frac{1}{1-\gamma} c_2^{1-\gamma}$, $\sigma > 0$, $\gamma > 0, \beta > 0$. *Then, in a dynamically-efficient economy, there is no case for introducing a PAYG pension scheme if* $\gamma \geq \sigma$.

5. Concluding remarks

It is a classical result that there is no welfare case for a PAYG pension scheme in a dynamically efficient economy. This result is based on models with exogenous labor supply. Whether endogenous labor supply goes in the direction of supporting or weakening the case for PAYG pension scheme is an open question in the literature. In this paper we have shown that the generalization to the case with endogenous labor supply and pensions financed by a linear income tax is not trivial. Asking whether there is a welfare gain from introducing a PAYG pension we showed by use of the stability condition that this depends on whether the capital-labor ratio increases or decreases. While savings unambiguously decrease and thus goes in the direction of lowering the capital-labor ratio, labor supply and hence employment may also decrease precisely because the tax is distortionary. Potentially the labor supply response may be stronger than the savings response so as to violate the sufficient condition for there to be no welfare case for a PAYG pension. We show, however, for a standard utility specification used in the literature, the sufficient condition holds, that is, even with endogenous labor supply there is no welfare case for a PAYG pension scheme.

Given the prevalence of PAYG pension schemes across countries, which in other respects have chosen very different institutional social arrangements, it remains a puzzle to understand and explain these schemes.

Appendix

A. Individual saving and labor supply

Using the first order conditions (7) and (8), it is easy to check that

$$\begin{aligned}
&\Gamma_{L\tau} = -u''(\cdot)(1-\tau)w^2L - u'(\cdot)w \lesseqqgtr 0\\
&\Gamma_{LB} = 0\\
&\Gamma_{Lw} = u''(\cdot)(1-\tau)^2 wL + u'(\cdot)(1-\tau) \lesseqqgtr 0\\
&\Gamma_{LR} = 0\\
&\Gamma_{S\tau} = u''(\cdot)wL < 0\\
&\Gamma_{SB} = Rv''(\cdot) < 0\\
&\Gamma_{Sw} = -u''(\cdot)(1-\tau)L > 0\\
&\Gamma_{SR} = R^2v''(\cdot) + v'(\cdot) \lesseqqgtr 0.
\end{aligned}$$

Note that the time index has been suppressed to simplify.

To figure out how optimal labor supply and saving responds to the tax rate and the pension, we have from (7) and (8) that

$$\begin{bmatrix} \Gamma_{SS} & \Gamma_{SL} \\ \Gamma_{LS} & \Gamma_{LL} \end{bmatrix} \begin{bmatrix} dS \\ dL \end{bmatrix} = \begin{bmatrix} -\Gamma_{SB}\,dB - \Gamma_{S\tau}d\,\tau - \Gamma_{Sw}dw - \Gamma_{SR}\,dR \\ -\Gamma_{LB}\,dB - \Gamma_{L\tau}d\,\tau - \Gamma_{Lw}dw - \Gamma_{LR}\,dR \end{bmatrix}$$

from where it follows

$$\frac{dL}{dB} = \frac{\begin{vmatrix} \Gamma_{SS} & -\Gamma_{SB} \\ \Gamma_{LS} & -\Gamma_{LB} \end{vmatrix}}{D} = \frac{-\Gamma_{SS}\Gamma_{LB} + \Gamma_{LS}\Gamma_{SB}}{D} = \frac{\Gamma_{LS}\Gamma_{SB}}{D} < 0$$

$$\frac{dL}{d\tau} = \frac{\begin{vmatrix} \Gamma_{SS} & -\Gamma_{S\tau} \\ \Gamma_{LS} & -\Gamma_{L\tau} \end{vmatrix}}{D} = \frac{-\Gamma_{SS}\Gamma_{L\tau} + \Gamma_{LS}\Gamma_{S\tau}}{D} \lesseqqgtr 0$$

$$\frac{dL}{dw} = \frac{\begin{vmatrix} \Gamma_{SS} & -\Gamma_{Sw} \\ \Gamma_{LS} & -\Gamma_{Lw} \end{vmatrix}}{D} = \frac{-\Gamma_{SS}\Gamma_{Lw} + \Gamma_{LS}\Gamma_{Sw}}{D} \lesseqqgtr 0$$

$$\frac{dL}{dR}=\frac{\begin{vmatrix}\Gamma_{SS} & -\Gamma_{SR}\\ \Gamma_{LS} & -\Gamma_{LR}\end{vmatrix}}{D}=\frac{\Gamma_{LS}\Gamma_{SR}}{D}\lesseqgtr 0$$

where $D\equiv\Gamma_{SS}\Gamma_{LL}-\Gamma_{SL}\Gamma_{LS}>0$. Also,

$$\frac{dS}{dB}=\frac{\begin{vmatrix}-\Gamma_{SB} & \Gamma_{SL}\\ -\Gamma_{LB} & \Gamma_{LL}\end{vmatrix}}{D}=\frac{-\Gamma_{SB}\Gamma_{LL}+\Gamma_{SL}\Gamma_{LB}}{D}=\frac{-\Gamma_{SB}\Gamma_{LL}}{D}<0$$

$$\frac{dS}{d\tau}=\frac{\begin{vmatrix}-\Gamma_{S\tau} & \Gamma_{SL}\\ -\Gamma_{L\tau} & \Gamma_{LL}\end{vmatrix}}{D}=\frac{-\Gamma_{S\tau}\Gamma_{LL}+\Gamma_{SL}\Gamma_{L\tau}}{D}$$

$$\frac{dS}{dw}=\frac{\begin{vmatrix}-\Gamma_{Sw} & \Gamma_{SL}\\ -\Gamma_{Lw} & \Gamma_{LL}\end{vmatrix}}{D}=\frac{-\Gamma_{Sw}\Gamma_{LL}+\Gamma_{SL}\Gamma_{Lw}}{D}\lesseqgtr 0$$

$$\frac{dS}{dR}=\frac{\begin{vmatrix}-\Gamma_{SR} & \Gamma_{SL}\\ -\Gamma_{LR} & \Gamma_{LL}\end{vmatrix}}{D}=\frac{-\Gamma_{SR}\Gamma_{LL}+\Gamma_{SL}\Gamma_{LR}}{D}=\frac{-\Gamma_{SR}\Gamma_{LL}}{D}\lesseqgtr 0.$$

First note that, since $\Gamma_{LS}>0$, and $\Gamma_{LL}<0$,

$$\operatorname{sign}\frac{dL}{dR}=\operatorname{sign}\frac{dS}{dR}$$

and each is positive if $\Gamma_{SR}>0$. Also, since $\Gamma_{SS}>0$ and $\Gamma_{LS}\Gamma_{S\tau}<0$, it follows that $\Gamma_{L\tau}<0$ is a necessary condition for $\frac{\partial L}{\partial\tau}>0$ and that $\Gamma_{L\tau}<0$ requires that income effects dominate substitution effects in individual labor supply.

1. $\frac{dS}{d\tau}$: Since $D>0$, the sign of $\frac{dS}{d\tau}$ is the same as the sign of $(-\Gamma_{S\tau}\Gamma_{LL}+\Gamma_{SL}\Gamma_{L\tau})$ which reduces to

$$-u''(\cdot)\,wL\left[u''(\cdot)\left((1-\tau)\,w\right)^2-d''(\cdot)\right]+\left[-u''(\cdot)\,(1-\tau)\,w\right]\left[-u''(\cdot)\,(1-\tau)\,w^2L-u'(\cdot)\,w\right]$$

   and further simplifies to

$$u''(\cdot)\,w\left[Ld''(\cdot)+(1-\tau)\,wu'(\cdot)\right]<0.$$

2. $\frac{dL}{d\tau}$: Since $D>0$, the sign of $\frac{dS}{d\tau}$ is the same as the sign of $-\Gamma_{SS}\Gamma_{L\tau}+\Gamma_{LS}\Gamma_{S\tau}$ which

reduces to

$$-\left[u''(\cdot)+R^2v''(\cdot)\right]\left[-u''(\cdot)(1-\tau)w^2L-u'(\cdot)w\right]+\left[-u''(\cdot)(1-\tau)w\right]\left[u''(\cdot)wL\right]$$

and further reduces to

$$\begin{aligned} &= w\left[u''(\cdot)u'(\cdot)+R^2v''(\cdot)u'(\cdot)+R^2Lw(1-\tau)u''(\cdot)v''(\cdot)\right] \\ &= w\left[u''(\cdot)u'(\cdot)+R^2v''(\cdot)u'(\cdot)\left\{1+\frac{(1-\tau)wLu''(\cdot)}{u'(\cdot)}\right\}\right] \end{aligned}$$

3. $\frac{dS}{dw}$: The sign of $\frac{dS}{dw}$ is the same as the sign of $-\Gamma_{Sw}\Gamma_{LL}+\Gamma_{SL}\Gamma_{Lw}$ which reduces to

$$-\left[-u''(\cdot)(1-\tau)L\right]\left[u''(\cdot)((1-\tau)w)^2-d''(L)\right]+\left[-u''(\cdot)(1-\tau)w\right]\left[u''(\cdot)(1-\tau)^2wL+u'(\cdot)(1-\tau)\right]$$

and further to

$$u''(\cdot)(1-\tau)\left[wu'(\cdot)(1-\tau)-d''(\cdot)L\right]$$

4. $\frac{dL}{dw}$: The sign of $\frac{dL}{dw}$ is the same as the sign of $-\Gamma_{SS}\Gamma_{Lw}+\Gamma_{LS}\Gamma_{Sw}$ which reduces to

$$-\left(u''(\cdot)+R^2v''(\cdot)\right)\left(u''(\cdot)(1-\tau)^2wL+u'(\cdot)(1-\tau)\right)+\left(-u''(\cdot)(1-\tau)w\right)\left(-u''(\cdot)(1-\tau)L\right)$$

and further to

$$(1-\tau)\left[R^2wLu''(\cdot)v''(\cdot)(1-\tau)-u'(\cdot)\left(R^2v''(\cdot)+u''(\cdot)\right)\right]>0$$

B. PROOF OF LEMMA 2

Using (25)-(26), we get

$$\frac{\partial L}{\partial w} = \frac{\begin{vmatrix} -\left[u''(\cdot)\,wL + u'(\cdot)\right] & -u''(\cdot)\,w \\ u''(\cdot)\,L & \left[u''(\cdot) + v''(\cdot)\,R^2\right] \end{vmatrix}}{A}$$

$$= \frac{-\left[u''(\cdot)\,wL + u'(\cdot)\right]\left[u''(\cdot) + v''(\cdot)\,R^2\right] + \left(u''(\cdot)\right)^2 wL}{A}$$

$$\frac{\partial S}{\partial w} = \frac{\begin{vmatrix} u''(\cdot)\,w^2 - d''(\cdot) & -\left[u''(\cdot)\,wL + u'(\cdot)\right] \\ -u''()w & u''()L \end{vmatrix}}{\begin{vmatrix} u''(\cdot)\,w^2 - d''(\cdot) & -u''(\cdot)\,w \\ -u''(\cdot)\,w & u''(\cdot) + v''(\cdot)\,R^2 \end{vmatrix}}$$

$$= \frac{\left[u''(\cdot)\,w^2 - d''(\cdot)\right]\left[u''(\cdot)\,L\right] - \left[u''(\cdot)\,wL + u'(\cdot)\right]u''(\cdot)\,w}{A}$$

where

$$A \equiv \left[u''(\cdot)\,w^2 - d''(\cdot)\right]\left[u''(\cdot) + v''(\cdot)\,R^2\right] - \left[u''(\cdot)\,w\right]^2.$$

Notice that the second order condition for the individual optimization problem (evaluated for $B = \tau = 0$) ensures $A > 0$.

The rate of return responses are as follows:

$$\frac{\partial L_t}{\partial R} = \frac{u''(\cdot)\,w\left[-v'(\cdot) - Rv''(\cdot)\,S\right]}{A}$$

$$\frac{\partial S_t}{\partial R} = \frac{\left[-v'(\cdot) - Rv''(\cdot)\,S\right]\left[u''(\cdot)\,w^2 - d''(\cdot)\right]}{A}$$

from where it is clear that

$$\text{sign}\frac{\partial L}{\partial R} = \text{sign}\ \frac{\partial S}{\partial R} = \text{sign}\ \left[-v'(\cdot) - Rv''(\cdot)\,S\right].$$

Finally, the benefit responses are as follows:

$$\frac{\partial L}{\partial B}=\frac{\begin{vmatrix} u''(\cdot)\,w+u'(\cdot)\,\frac{1}{L} & -u''(\cdot)\,w \\ -\left[u''(\cdot)+Rv''(\cdot)\right] & \left[u''(\cdot)+v''(\cdot)\,R^2\right] \end{vmatrix}}{A}$$
$$=\frac{\left[u''(\cdot)\,w+u'(\cdot)\,\frac{1}{L}\right]\left[u''(\cdot)+v''(\cdot)\,R^2\right]-\left[u''(\cdot)\,w\right]\left[u''(\cdot)+Rv''(\cdot)\right]}{A}$$
$$\frac{\partial S}{\partial B}=\frac{\begin{vmatrix} u''(\cdot)\,w^2-d''(\cdot) & u''(\cdot)\,w+u'(\cdot)\,\frac{1}{L} \\ -u''(\cdot)\,w & -\left[u''(\cdot)+Rv''(\cdot)\right] \end{vmatrix}}{A}$$
$$=\frac{-\left[u''(\cdot)\,w^2-d''(\cdot)\right]\left[u''(\cdot)+Rv''(\cdot)\right]+\left[u''(\cdot)\,w+u'(\cdot)\,\frac{1}{L}\right]\left[u''(\cdot)\,w\right]}{A}$$

The latter expressions are easily reduced to

$$\frac{\partial L}{\partial B}=\frac{u'(\cdot)\,\frac{1}{L}\left[u''(\cdot)+v''(\cdot)\,R^2\right]+v''(\cdot)\,u''(\cdot)\,wR\left[R-1\right]}{A}$$

and

$$\frac{\partial S}{\partial B}=\frac{d''(\cdot)\left[u''(\cdot)+Rv''(\cdot)\right]-Rv''(\cdot)\left[u''(\cdot)\,w^2+wRu'(\cdot)\,\frac{1}{L}\right]+wu'(\cdot)\,\frac{1}{L}\left[u''(\cdot)+v''(\cdot)\,R^2\right]}{A}$$

which are of ambiguous sign.

C. PROOF OF LEMMA 1

The characteristic polynomial is given by $p(\lambda)\equiv\lambda^2+\lambda A_1+A_0$. Stability (saddlepoint) requires one characteristic root to satisfy $|\lambda_1|<1$ and the other, $|\lambda_2|>1$. Evaluated for $\lambda=1$, we get

$$p(1)=1+A_1+A_0=1+\frac{\left[\left[\widehat{L}-k\,\widehat{L}_w kR_k\right]-S_R R_k\right]}{k\,\widehat{L}_R R_k}+\frac{S_w}{\widehat{L}_R}$$

which, after routine simplification yields,

$$p(1)=\frac{1}{k\,\widehat{L}_R R_k}\left[\widehat{L}+\left[\,\widehat{L}_R-k\,\widehat{L}_w\right]kR_k-S_R R_k+S_w kR_k\right].$$

Notice $p(-1) = 1 - A_1 + A_0 > 0$. Then, saddle path stability requires $p(1) < 0$ or that

$$1 + A_1 + A_0 = \frac{1}{k\ \widehat{L}_R R_k}\left[\widehat{L} + \left[\widehat{L}_R - k\ \widehat{L}_w\right] kR_k - S_R R_k + S_w k R_k\right] < 0.$$

Since $\widehat{L}_R > 0$ is assumed (see 14), saddle path stability requires

$$\left[\widehat{L} + \left[\widehat{L}_R - k\ \widehat{L}_w\right] kR_k - S_R R_k + S_w k R_k\right] > 0$$

hold.

D. Proof of Lemma 3

Noting that $k \equiv \widehat{K}/\widehat{L}$, it follows that

$$\frac{\partial k}{\partial B} = \frac{k}{\widehat{K}}\frac{\partial \widehat{K}}{\partial B} - \frac{k}{\widehat{L}}\frac{\partial \widehat{L}}{\partial B}$$

and

$$\widehat{L}\frac{\partial k}{\partial B} = \left[\frac{\partial \widehat{K}}{\partial B} - k\frac{\partial \widehat{L}}{\partial B}\right].$$

Since $\widehat{K} \equiv S(B, w(k), R(k))$, we have

$$\frac{\partial \widehat{K}}{\partial B} = S_B + [S_w w_k + S_R R_k]\frac{\partial k}{\partial B}$$

and since $\widehat{L} \equiv \widehat{L}(B, w(k), R(k))$, we have

$$\frac{\partial \widehat{L}}{\partial B} = \widehat{L}_B + \left[\ \widehat{L}_w w_k +\ \widehat{L}_R R_k\right]\frac{\partial k}{\partial B}.$$

Then,

$$\widehat{L}\frac{\partial k}{\partial B} = \frac{\partial \widehat{K}}{\partial B} - k\frac{\partial \widehat{L}}{\partial B} = S_B + [S_w w_k + S_R R_k]\frac{\partial k}{\partial B} - k\left(\widehat{L}_B + \left[\ \widehat{L}_w w_k +\ \widehat{L}_R R_k\right]\right)\frac{\partial k}{\partial B}$$
$$\Leftrightarrow \frac{\partial k}{\partial B} = \frac{S_B - k\ \widehat{L}_B}{\widehat{L} - [S_w w_k + S_R R_k] + k\left[\widehat{L}_w w_k +\ \widehat{L}_R R_k\right]}.$$

From (20), we have

$$\widehat{L}+\left[\widehat{L}_R-k\ \widehat{L}_w\right]kR_k-S_RR_k+S_wkR_k>0.$$

Using $w_k=-kR_k$, we can rewrite the previous condition as

$$\widehat{L}-S_RR_k-S_ww_k-\ \widehat{L}_Rw_k+kL_ww_k>0\Rightarrow\widehat{L}-(S_RR_k+S_ww_k)+\ k\left[\widehat{L}_RR_k+\widehat{L}_ww_k\right]>0$$

the same as the denominator of $\frac{\partial k}{\partial B}$. It follows that at a saddle-point stable k,

$$\text{sign}\ \left[\frac{\partial k}{\partial B}\right]=\text{sign}\ \left[S_B-kL_B\right]$$

E. PROOF OF PROPOSITION 2

If the pension is financed by a lump-sum tax levied on the young, the first order condition to the individual optimization problem can be written as

$$\begin{aligned}&u'\left(w_t\ \widehat{L}_t-B-\widehat{S}_t\right)[w_t]-d'\left(\ \widehat{L}_t\right)=0\\&-u'\left(w_t\ \widehat{L}_t-B-\widehat{S}_t\right)+R_{t+1}v'\left(R_{t+1}\widehat{S}_t+B\right)=0.\end{aligned}$$

Note that the first order conditions are evaluated in equilibrium making use of $T=B$. Hence, totally differentiating yields

$$\begin{aligned}&u''(\cdot)\left[\ \widehat{L}dw+wd\widehat{L}-dB-dS\right]w_t+u'(\cdot)dw-d''(\cdot)d\widehat{L}=0\\&-u''(\cdot)\left[\ \widehat{L}dw+wd\widehat{L}-dB-d\widehat{S}\right]+dR_{t+1}v'(\cdot)+Rv''(\cdot)\left[\widehat{S}dR+Rd\widehat{S}+dB\right]=0\end{aligned}$$

Evaluating these expressions for $B=0$ yields

$$\begin{aligned}&\left[u''(\cdot)w^2-d''(\cdot)\right]d\widehat{L}+\left[-u''(\cdot)w\right]d\widehat{S}=\left[-u''(\cdot)w\widehat{L}\right]dw+\left[u''(\cdot)w\right]dB\\&\left[-u''(\cdot)w\right]d\widehat{L}+\left[u''(\cdot)+R^2v''(\cdot)\right]d\widehat{S}=u''(\cdot)\widehat{L}dw-\left[u''(\cdot)+Rv''(\cdot)\right]dB+\left[-v'(\cdot)-Rv''(\cdot)\widehat{S}\right]dR.\end{aligned}$$

Then,

$$\frac{d\widehat{L}_t}{dB}\mid_{\text{lump sum}}=\frac{\begin{vmatrix} u''(\cdot)w & -u''(\cdot)w \\ -\left[u''(\cdot)+Rv''(\cdot)\right] & \left[u''(\cdot)+v''(\cdot)R^2\right] \end{vmatrix}}{\begin{vmatrix} u''(\cdot)w^2-d''(\cdot) & -u''(\cdot)w \\ -u''(\cdot)w & u''(\cdot)+v''(\cdot)R^2 \end{vmatrix}}$$

which may be simplified to yield

$$\frac{d\widehat{L}_t}{dB}\mid_{\text{lump sum}}=\frac{u''(\cdot)wv''(\cdot)R\left(R-1\right)}{\left[u''(\cdot)w^2-d''(\cdot)\right]\left[u''(\cdot)+v''(\cdot)R^2\right]-\left[u''(\cdot)w\right]\left[u''(\cdot)w\right]}>0.$$

Similarly,

$$\frac{d\widehat{S}_t}{dB}\mid_{\text{lump sum}}=\frac{\begin{vmatrix} u''(\cdot)w^2-d''(\cdot) & u''(\cdot)w \\ -u''(\cdot)w & -\left[u''(\cdot)+Rv''(\cdot)\right] \end{vmatrix}}{\begin{vmatrix} u''(\cdot)w^2-d''(\cdot) & -u''\left(\cdot\right)w \\ -u''(\cdot)w & u''\left(\cdot\right)+v''\left(\cdot\right)R^2 \end{vmatrix}}$$

may be simplified to yield

$$\frac{d\widehat{S}_t}{dB}\mid_{\text{lump sum}}=\frac{d''(\cdot)\left[u''(\cdot)+Rv''(\cdot)\right]-u''(\cdot)w^2Rv''(\cdot)}{\left[u''(\cdot)w^2-d''(\cdot)\right]\left[u''(\cdot)+v''(\cdot)R^2\right]-\left[u''(\cdot)w\right]\left[u''(\cdot)w\right]}<0$$

Hence, under lump sum taxation we have

$$\frac{d\widehat{L}_t}{dB}\mid_{\text{lump sum}}>0,\quad \frac{d\widehat{S}_t}{dB}\mid_{\text{lump sum}}<0.$$

F. PROOF OF LEMMA 4

Using (29)-(30), and the expressions derived in the proof of Proposition 2, we have

$$\begin{aligned}\frac{\partial\widehat{L}}{\partial B} &= \frac{\partial\widehat{L}}{\partial B}\mid_{\text{lump sum}}+\frac{u'(\cdot)\frac{1}{\widehat{L}}\left[u''(\cdot)+v''(\cdot)R^2\right]}{D}<\frac{\partial\widehat{L}}{\partial B}\mid_{\text{lump sum}}\\ \frac{\partial\widehat{S}}{\partial B} &= \frac{\partial\widehat{S}}{\partial B}\mid_{\text{lump sum}}+\frac{u'(\cdot)\frac{1}{\widehat{L}}\left[u''(\cdot)w\right]}{D}<\frac{\partial\widehat{S}}{\partial B}\mid_{\text{lump sum}}.\end{aligned}$$

G. PROOF OF PROPOSITION 3

For $\widehat{S}_B - k\widehat{L}_B < 0$ to hold, where

$$\widehat{S}_B - k\widehat{L}_B = \underbrace{\left[\widehat{S}_B \mid_{\text{lump sum tax}} - k\widehat{L}_B \mid_{\text{lump sum tax}}\right]}_{<0} + \frac{u'(\cdot)\frac{1}{L}}{D}\left[u''(\cdot)\,(w-k) - k\,v''(\cdot)R^2\right]$$

it is sufficient that

$$\left[u''(\cdot)\,(w-k) - k\,v''(\cdot)R^2\right] \leq 0$$

hold. Using the definitions in (27), we have that

$$\begin{aligned}\left[u''(\cdot)\,(w-k) - k\,v''(\cdot)R^2\right] &= u'(c_1)\left[\frac{u''(c_1)}{u'(c_1)}\frac{c_1}{c_1}(w-k) - k\frac{v''(c_2)}{v'(c_2)}\frac{c_2}{c_2}R\right] \\ &= u'(c_1)\left[\Phi_u\frac{(w-k)}{c_1} - k\Phi_v\frac{R}{c_2}\right].\end{aligned}$$

Using $c_1 = (w-k)L$ and $c_2 = kRL$, we have

$$\begin{aligned}\left[u''(\cdot)\,(w-k) - k\,v''(\cdot)R^2\right] &= u'(c_1)\left[\Phi_u\frac{(w-k)}{(w-k)L} - k\Phi_v\frac{R}{kRL}\right] \\ &= \frac{u'(c_1)}{L}\left(\Phi_u - \Phi_v\right) < 0 \text{ if } \Phi_v \leq \Phi_u.\end{aligned}$$

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