# SCHOOL OF ECONOMICS AND MANAGEMENT


## ECONOMICS WORKING PAPER 2006-13


# Rules of Normalisation and their Importance for Interpretation of Systems of Optimal Taxation


Knud Jørgen Munk



## UNIVERSITY OF AARHUS

BUILDING 1322 - 8000 AARHUS C - DENMARK ☎ +45 8942 1133

# Rules of Normalisation and their Importance for Interpretation of Systems of Optimal Taxation

Knud J. Munk†

University of Aarhus

This Version: December 2006

## Abstract

The adoption of proper rules of normalisation is in general considered a trivial problem which deserves little attention. Possibly for that very reason errors in normalisation have resulted in flawed interpretations of the conditions for optimal commodity taxation. We state based on an explicit representation of the general equilibrium conditions the rules of normalisation in standard optimal tax models. This allows us to provide an intuitive explanation of what determines the optimal tax system. Finally, we review a number of examples where lack of precision with respect to normalisation in otherwise important contributions to the literature on optimal taxation has given rise to misinterpretations of of analytical results.

**Keywords:**
Public economics, optimal taxation, normalisation rules, p-complements, q-complements, distance function.

**JEL classification codes**:
H2

**Acknowledgements:**

The initial work for this article has been made during a visit at CES, Catholic University of Leuven 1994-1997 and during visits at EPRU, Institute of Economics, University of Copenhagen 1997-1999. The encouragement and the resources provided at both places for this work are gratefully acknowledged. I also would like to thank Bjørn Sandvik who has shared his work on the subject with me and provided inspiration for the final version.

† Department of Economics, University of Aarhus, Building 1322, DK 8000 Aarhus C. E-mail: kmunk@econ.au.dk

# 1 Introduction

Ambiguity with respect to the meaning of the assumption that “labour cannot be taxed” has resulted in misinterpretation of the Ramsey conditions for optimal taxation. The purpose of this paper is threefold: 1) To state the rules of normalisation based on an explicit representation of the general equilibrium assumptions underlying optimal tax analysis, 2) on this basis to provide an intuitive explanation of what determines the structure of optimal taxation, and 3) to draw attention to instances where eminent economists due to errors of normalisation have given misleading, if not outright wrong interpretations of analytical results. We make no claim to originality concerning the results presented in this paper; the contribution is within a unified framework to present the right answers and discard the wrong ones.

# 2 Rules of normalisation in general equilibrium tax models

Consider a competitive economy with N production sectors, H households labelled h∈(1,..,H) and a government labelled G. The primary factors are labelled j∈(1,..,M), the produced commodities i∈(M+1,..,M+N) and the production sectors k∈(1,..,N). We denote the index set of households, *H*, of primary factors, *F*, of produced commodities, *C*, of production sectors, *A*= *C*, and define $FC \equiv F \cup C$

Outputs are $Y_k$, k∈ $C \equiv$(1,..,N), and the corresponding input vectors $\mathbf{v}^k \equiv \left(v_1^k,..., v_{\mathrm{M}}^k\right)$, k∈*C*. Household consumption is $\mathbf{c}^h \equiv (c_1^h,...,c_{\mathrm{M+N}}^k)$, h∈*H*, household endowment $\boldsymbol{\omega}^h \equiv (0,.,\omega_{M+1}^h,...,\omega_{\mathrm{M+N}}^k)$, h∈*H*, and household net trade $\mathbf{x}^h \equiv (x_1^h,..., x_{\mathrm{M+N}}^k) = (\mathbf{c}^h - \boldsymbol{\omega}^h)$, h∈*H*.

The government can levy commodity taxes, $\mathbf{t} \equiv \left(t_1,...,t_{\mathrm{M+N}}\right)$. The rates of tax on the value of the household’s endowment is $\tau^1$ and on its profit income is $\tau^2$, respectively.

We make two alternative assumptions with respect to household commodity taxes

> ***A1:.*** *Consumption* $\mathbf{c}^h \equiv (c_1^h,...,c_{\mathrm{M+N}}^k)$, *h*∈*H, can be taxed*
>
> ***A2:*** *Only market transactions* $\mathbf{x}^h \equiv (x_1^h,..., x_{\mathrm{M+N}}^k)$, *h*∈ *H, can be taxed*

Tax rates are defined relative to producer prices, $\mathbf{p} \equiv \left(p_1,...,p_{\mathrm{M+N}}\right)$. A *proportional tax system* is defined as a $\mathbf{t} \equiv \left(t_1,...,t_{\mathrm{M+N}}\right)$ where $T_{\mathrm{i}} \equiv \dfrac{p_{\mathrm{i}} + t_{\mathrm{i}}}{p_{\mathrm{i}}} = T$, *i*∈ *FC.*

We specify the assumptions about utility maximisation by households, by profit maximisation by production sectors, by material balance for primary and produced commodities, and by government budget balance, which constitute the general equilibrium conditions defining the set of feasible tax systems:

***1) Utility maximisation***

With respect to the conditions for utility maximisation we make two alternative assumptions:

Suppose ***A1*** *(consumption taxable)* holds and $\tau^2 = 1$ (*no untaxed profit*), then the utility maximisation assumption may be expressed by

$$M^h\left(\mathbf{q}, u^h\right) = (1-\tau^1)\sum_{i\in F} p_i \omega_i \quad (1)$$

$$c_i^h = M_i^h\left(\mathbf{q}, u^h\right) \quad i\in FC,\ h\in H \quad (2)$$

Suppose ***A2*** *(only net trade taxable)* holds and $\tau^1 = 0$ *(no tax on the endowment)*, then utility maximisation assumption may be expressed by

$$M^h\left(\mathbf{q}, u^h\right) = (1-\tau^2)\pi^h \quad h\in H \quad (3)$$

$$x_i^h = M_i^h\left(\mathbf{q}, u^h\right) - \omega_i^h \quad i\in FC,\ h\in H \quad (4)$$

where $M^h\left(\mathbf{q}, u^h\right)$ is the full income function, and $\pi^h$ the profit income of the $h^{th}$ household.

***2) Profit maximisation***

We can state the profit maximisation assumptions as follows

$$\Pi_{Y_k}^k\left(\mathbf{p}, Y_k\right) = p_k - C_{Y_k}^k\left(\mathbf{p}, Y_k\right) = 0 \quad k\in A \quad (5)$$

$$v_j^k = C_j^k\left(\mathbf{p}, Y^k\right), \quad j\in FC,\ k\in A \quad (6)$$

where $\Pi^k\left(\mathbf{p}, Y_k\right)$ is the output contingent profit function and $C^k\left(\mathbf{p}, Y_k\right)$ the cost function of the $k^{th}$ production sectors

***3) Material balance***

$$Y_i = \sum_{j\in A} v_i^j + \sum_{h\in H} x_i^h + x_i^G \quad i\in C \quad (7)$$

$$0 = \sum_{j\in A} v_i^j + \sum_{h\in H} x_i^h + x_i^G \quad i\in F \quad (8)$$

***4) Government budget balance***

$$\sum_{h\in H}\sum_{i\in FC} t_i x_i^h + \tau^1 \sum_{h\in H}\sum_{i\in F} p_i \omega_i^h + \sum_{h\in H} \tau^2 \pi^h - \sum_{i\in FC} p_i x_i^G \qquad (9)$$

By substitution, and by applying Walras' law to eliminate the government's budget constraint, we consolidate these conditions for the two alternative assumptions about household behaviour.

First, suppose ***A1*** holds *(consumption taxable)* and that $\tau^2 = 1$ *(no untaxed profit)*, then a feasible tax system must satisfy the following conditions

$$M^h\left(\mathbf{q},u^h\right) = (1-\tau^1)\sum_{i\in F} p_i \omega_i \qquad h\in H \quad (10)$$

where $\theta_j^h$ indicates the share of the $h^{th}$ household in the profit of the $j^{th}$ production sector.

$$Y_i = \sum_{j\in A} c_i^j\left(\mathbf{p}\right)Y_j + \sum_{h\in H} M_i^h\left(\mathbf{q},u^h\right) - \omega_i^h + x_i^G \qquad i\in C \ (11)$$

$$0 = \sum_{j\in A} c_i^j\left(\mathbf{p}\right)Y_j + \sum_{h\in H} c_i^h\left(\mathbf{q},u^h\right) - \omega_i^h + x_i^G \qquad i\in F \ (12)$$

$$p_j = C_j^j\left(\mathbf{p},Y_j\right) \qquad j\in C \ (13)$$

Second, suppose ***A2*** *(only net trade taxable)* holds, and $\tau^1 = 0$, then a feasible tax system must satisfy

$$M^h\left(\mathbf{q},u^h\right) = (1-\tau^2)\sum_{j\in C} \theta_j^h \Pi^j\left(\mathbf{p},Y_j\right) \qquad h\in H \ (14)$$

where $\theta_j^h$ indicates the share of the $h^{th}$ household in the profit of the $j^{th}$ production sector, and Equations (11), (12), (13), as above.

Analysis of these equilibrium conditions using the homogeneity properties of expenditure and profit functions, results in the following two propositions (see Munk 1978, 1980, 1982):

***Proposition 1:*** *Suppose **A1** holds (consumption taxable) and* $\tau^2 = 1$*(no untaxed profit) then*

*(i)* *Either a producer price or a consumer price may be fixed without loss of generality.*

*(ii)* *If the government's requirement can be financed by an endowment tax at rate* $\bar{\tau}^1$ *without the use of commodity taxes, i.e. with* $\bar{\mathbf{q}} = \bar{\mathbf{p}}$, *then it can also be financed without an endowment tax by commodity taxes with*

   *a. consumer price* $\mathbf{q} = 1/(1-\tau^1)\bar{\mathbf{q}}$, *or*

   *b. producer prices* $\mathbf{p} = (1-\tau^1)\bar{\mathbf{p}}$.

*i.e. by a proportional tax system,* $T = 1/(1-\tau^1)$, *i*∈ *FC.*

***Proposition 2:*** *Suppose* ***A1*** *(only net trade taxable)* holds and $\tau^1 = 0$ *(no tax on the endowment), then*

*(i)* *Either a producer price or a consumer price may be fixed without loss of generality*

*(ii)* *If the government's requirement can be financed by an endowment tax at rate* $\tau^2$ *without the use of commodity taxes, i.e. with* $\bar{\mathbf{q}} = \bar{\mathbf{p}}$ *, then it can also be financed without an endowment tax by commodity taxes with*

*a.* *consumer price* $\mathbf{q} = 1/(1-\tau^2)\bar{\mathbf{q}}$ *, or*

*b.* *producer prices* $\mathbf{p} = (1-\tau^2)\bar{\mathbf{p}}$ *.*

*i.e. by a proportional tax system,* $T = 1/(1-\tau^2)$ *,* $i \in$ *FC.*

*(iii)* *If there is no untaxed profit, a proportional tax structure will generate no tax revenue.*

A great number of optimal tax models are based on the following assumptions

***A3****: Constant returns to scale production structure*

***A4****: Household only have an initial endowment of one primary factor, denoted 0*

Suppose ***A1*** (*consumption is taxable*) and ***A3-A4*** apply, then retaining the government's budget constraint, and by Walras' law now deleting the material balance constraint, the general equilibrium conditions for feasible tax solution may be expressed as

$$M^h\left(\mathbf{q}, u^h\right) = (1-\tau^1)\omega_0 p_0 \qquad h \in H \quad (15)$$

$$\sum_{i \in C} t_i M_i^h(\mathbf{q}, u) + t_0 M_0^h(\mathbf{q}, u) + \tau^1 \omega_0 p_0 - G = 0 \quad (16)$$

From (15) and (16) and *Proposition 1, part ii,* it follows

***Proposition 3:*** *If* ***A1*** *(consumption taxable)* and ***A3-A4*** *hold then*

*(i)* *The first best allocation can be achieved either by a tax on the endowment at* $\tau^{1*} = G/\omega_0 p_0$ *or by a proportional tax system* $T = 1/(1-\tau^{1*})$ *,* $i \in$ *FC*

*(ii)* *The first best cannot be achieved by commodity taxation if one commodity cannot be taxed.*

*(iii)* *A rule of normalisation which assumes one commodity untaxed therefore involves a loss of generality.*

Suppose ***A2*** (*consumption is taxable*) and ***A3-A4*** apply, then a the general equilibrium conditions for feasible tax solution may be expressed as

$$M^h\left(\mathbf{q}, u^h\right) = \omega_0 q_0 \qquad h \in H \quad (17)$$

$$\sum_{i \in C} t_i M_i(\mathbf{q}, u) + t_0\left(M_0^h(\mathbf{q}, u) - \omega_0^h\right) - G = 0 \quad (18)$$

By the homogeneity of expenditure functions, it follows from (17) and (18)

***Proposition 4:*** Suppose ***A2*** (*net trade is taxable*) and ***A3-A4*** apply and $\tau^1 = 0$ *(no tax on the endowment), then*

*i. A proportional tax system does not generate any tax revenue*

*ii. Assuming one commodity untaxed may be adopted as a rule of normalisation without any loss of a loss of generality*

If we supplement Equations (15) and (16) with the additional conditions that $q_0 = p_0$, we define the same set of feasible tax systems as defined by Equations (17) and (18). We thus have

***Proposition 5:*** *Suppose **A3-A4** hold and $\tau^1 = 0$ (no tax on the endowment), then the set of feasible tax instruments which satisfy the equilibrium conditions given **A1** (that consumption can be taxed) with the additional assumption that $q_0 = p_0$, is the same as that which satisfy the equilibrium conditions given **A2** (net trade is taxable).*

# 3 The basic efficiency trade-off

The standard optimal tax model (see Diamond and Mirrlees 1971), makes the assumption that since information about household consumption and factor endowment is private the government is only able to base taxation on market transactions, i.e. ***A2*** *(only net trade taxable)* and $\tau^1 = 0$. In the following, we will rely on these assumptions and in addition on ***A3-A4.*** Furthermore in order to focus on efficiency considerations we assume

***A5****: The household sector consist of one representative household*

Since the (net) expenditure function my be defined as $E(\mathbf{q},u) \equiv M(\mathbf{q},u) - \omega_0 p_0$ the government's maximisation problem may be expressed as to maximise $u$ subject to (17) and (18), i.e.

$$\begin{aligned} &\underset{\{q_i, i\in \mathrm{FC}\}, u}{Max} \; u \text{ s.t.} \\ &E(\mathbf{q},u) = 0, \\ &\sum_{i\in \mathrm{FC}} t_i E_i(\mathbf{q},u) - G = 0 \end{aligned} \tag{19}$$

The corresponding Lagrangian expression is

$$\mathfrak{L} = u + \mu\big(0 - E(\mathbf{q},u)\big) + \lambda\left(\sum_{i\in \mathrm{FC}} t_i E_i(\mathbf{q},u) - \sum_{i\in \mathrm{FC}} p_i x_i^G\right) \tag{20}$$

The first order conditions for an optimal solution with respect to $u$ and $\mathbf{q} \equiv (q_0, q_1, .., q_N)$, respectively, are

$$\frac{\partial \mathcal{L}}{\partial u} = 1 - \mu E_u(\mathbf{q}, u) + \lambda \sum_{i=FC} t_i E_{iu}(\mathbf{q}, u) = 0 \qquad h \in H \quad (21)$$

$$\frac{\partial \mathcal{L}}{\partial q_k} = -\mu E_k(\mathbf{q}, u) + \lambda \left( \sum_{i \in FC} t_i E_{ik}(\mathbf{q}, u) + E_k(\mathbf{q}, u) \right) = 0, \qquad k \in FC \quad (22)$$

From (21) we obtain dividing by $E_u$ and reordering

$$\mu = \frac{\partial V}{\partial I}(\mathbf{q}, I) + \lambda \sum_{i \in FC} t_i \frac{\partial x_i}{\partial I}(\mathbf{q}, I)$$

where $V(\mathbf{q}, I)$ is the indirect utility function and $x_i(\mathbf{q}, I)$ is the ordinary demand function for commodity *i*. The multiplier $\mu$ thus represents the social value of transferring one unit of the endowment from the household to the government.

Assuming the primary factor is *"labour"* and adopting as rule of normalisation that *"labour is not taxed"*, from (22) we obtain directly the well-known Ramsey conditions from which an optimal tax system can be derived

$$\sum_{i \in C} t_i E_{ki}(\mathbf{q}, u) = -\theta E_k(\mathbf{q}, u) \qquad k \in C \quad (23)$$

where $E_k(\mathbf{q}, u) \equiv \frac{\partial E}{\partial q_k}(\mathbf{q}, u) = x_k(\mathbf{q}, u)$, $E_{ki} = E_{ik} \equiv \frac{\partial^2 E}{\partial q_i \, \partial q_k}(\mathbf{q}, u)$ and $\theta \equiv (\lambda - \mu) / \lambda$.

By the concavity of the expenditure function $\theta > 0$ if $G > 0$.

The standard interpretation of the Ramsey conditions is that the optimal tax system involves equi-proportional reduction in compensated demand (Mirrlees 1971). However, this result has not provided much intuitive insight into what determines the optimal tax system.

In the case of an economy with only two produced commodities, say 1 and 2, we may from (23) derive the following optimal tax formula (see Harberger 1971)

$$\frac{t_1}{p_1} \bigg/ \frac{t_2}{p_2} = \left( \frac{\sigma_{00} + \sigma_{02}}{\sigma_{00} + \sigma_{01}} \right) \quad (24)$$

The interpretation of this is the Corlett and Hague (1953) rule which says that those commodities "most complementary with leisure" should be taxed at the highest rate.

However, the question is, if a rather simplistic argument comparing two alternative, but equivalent, formulations of the government's maximisation problem, Equations (19) and (25), does not provide even greater insight about the structure of optimal commodity taxation.

The formulation of government's maximisation problem based on (15) and (16),

$$\begin{aligned} &\underset{\{q_i, i\in \mathrm{FC}\},u}{Max}\ u \ \text{ s.t.} \\ &M\left(\mathbf{q},u\right) = \omega_0 p_0 \\ &\sum_{i\in F\mathrm{C}} t_i\, M_i\ (\mathbf{q},u)\ -G = 0 \\ &q_0 = p_0 \end{aligned} \tag{25}$$

is by *Proposition 5* equivalent to that based on (17) and (18), i.e. the standard formulation represented by (19). *Proposition 3, part i* says that the solution to the government's maximisation problem is first best without the last constraint, $q_0 = p_0$. The leads to

***Proposition 6:*** *Substituting distortionary taxes for non-distortionary taxes maintaining constrained optimisation*

*i. Reduces social welfare*

*ii. Involve a substitution away from the consumption of produced commodities towards the consumption of leisure as the terms of trade of leisure must improve in the process.*

It follows from *Proposition 6* that starting from a proportional tax system based only on the produced commodities, $T = 1/(1-\tau^1)$, $i\in C$, i.e. an allocation where the technical rate of transformation in consumption between produced commodities is the same as the technical rate of transformation in production, it is in general possible to encourage the supply of labour and increase social welfare by increasing the tax on the commodities which are highly complementary to leisure and by reducing those on commodities which are less so. However, doing so creates a distortion by making the marginal rate of transformation in consumption differ from the marginal rate of transformation in production for the produced commodities. This reasoning leads to the following

**Conjecture:** *The optimal tax system represents a compromise between achieving two objectives*

1) *The objective of not distorting the technical rate of transformation in consumption between produced commodities,* $x_i$ *, i* $\in$*C, (Objective 1), and*
2) *The objective of maintaining the supply of labour (Objective 2).*

*The optimal tax structure is achieved at the point where the marginal gain in terms of encouragement of the supply of labour corresponds to the marginal loss in terms of distortion of the pattern of consumption of the produced commodities. At this point the commodities which are most complementary to the household consumption of the primary factor will be taxed at the highest rate.*

## 4 Clarification of key concepts

Prior to subjecting the *Conjecture* to rigorous analysis, it is helpful to clarify the terminology with respect to two key concepts used in the interpretation of the Ramsey conditions (23). We will make precise the meaning of two terms: that of *"labour untaxed"* and, second, of *"complementary with leisure"*.

***"Labour Untaxed"***

It is customary in optimal tax models to assume "*labour untaxed*" although assuming the rate of tax on the net trade in any commodity as fixed will do as well and be more consistent with empirical facts. However, in assuming labour untaxed it is important to distinguish between the assumptions that *"the supply of labour to the market cannot be taxed"*, and that *"the household consumption of labour ("leisure") cannot be taxed"*. The former involves no loss of generality (*Propositions 4, part ii)*, whereas the latter does imply a loss of generality as it implies restrictions on the set of feasible tax instruments (*Propositions 3, part ii)*.

***"Complementary to leisure"***

The concept of complementarity may be defined in two different ways. The first definition is based on the well-known Allen (or Allen-Uzawa) elasticity of substitution $\sigma_{ij}$. This is defined as

$$\sigma_{ij} \equiv \frac{M(\mathbf{q},u) M_{ij}(\mathbf{q},u)}{M_i(\mathbf{q},u) M_j(\mathbf{q},u)} \qquad i, j \in FC$$

where $M(\mathbf{q},u)$ is the full income expenditure function.

The second definition is based on the less well-known *Antonelli elasticity of complementarity*, $\rho_{ij}$, which is defined as

$$\rho_{ij} \equiv \frac{D(\mathbf{c},u) D_{ij}(\mathbf{c},u)}{D_i(\mathbf{c},u) D_j(\mathbf{c},u)} \qquad i, j \in FC$$

where $D(\mathbf{c},u)$ is the distance function (Deaton 1979).

***Definition 1**:* Two commodities *i* and *j* are *net p-complements (substitutes)* for a given level of utility if an increase in the price of the $j^{th}$ commodity increases (decreases) the quantity consumed of the $i^{th}$ commodity (keeping the prices of all other commodities constant), i.e. if $\sigma_{ij} \geq 0 (< 0)(i \neq j)$.

***Definition 2:*** Two commodities *i* and *j* are *net q-complements (substitutes)* if, for a given level of utility, an increase in the consumption of the $j^{th}$ commodity increases (decreases) the marginal valuation of the $i^{\text{th}}$ commodity keeping the consumption of all other commodities constant, i.e. if $\rho_{ij} \geq 0\left(<0\right)(i \neq j)$.

The two concepts are linked since the distance function may be defined from the full income function. Let $\tilde{\mathbf{q}}$ be the vector of normalised commodity prices whose elements are $\tilde{q}_i = q_i / M$ , i=0, …,N, where $M$ is full income. By the homogeneity of degree 1 of $M\left(\mathbf{q},u\right)$ we have

$$M\left(\tilde{\mathbf{q}},u\right) = MM\left(\mathbf{q},u\right) \tag{26}$$

and thus

$$\left\{\tilde{M}_{ij}\right\} \equiv \left\{\frac{\partial M_i\left(\tilde{\mathbf{q}},u\right)}{\partial \tilde{q}_j}\right\} = 1/M\left\{M_{ij}\right\} \tag{27}$$

The distance function may then be defined as

$$D\left(\mathbf{c},u\right) \equiv \min_{\tilde{\mathbf{q}}}\left\{\tilde{\mathbf{q}}\mathbf{c} : M\left(\tilde{\mathbf{q}},u\right) \geq 1\right\} \tag{28}$$

The distance function gives the maximum amount by which the consumption vector must be deflated or inflated to reach the indifference curve associated with $u$ . The distance function is increasing, linear homogenous, and concave with respect to $\mathbf{x}$ , and decreasing in *u*.

Application of Shepard's lemma to the distance function yields the system of compensated *inverse demand function*s, $a_i\left(\mathbf{x},u\right)$, $i \in FC$

$$\tilde{q}_i = \frac{\partial D\left(\mathbf{c},u\right)}{\partial x_i} \equiv a_i\left(\mathbf{c},u\right) \qquad\qquad i \in FC \tag{29}$$

Inverse demands measure the marginal evaluation of the consumption of commodities by households. Linear homogeneity of the distance function implies that $D_i\left(\mathbf{x},u\right) \equiv a_i\left(\mathbf{x},u\right)$ is homogenous of degree zero in $\mathbf{x}$, and the concavity implies that the matrix $\left\{D_{ij}\right\} \equiv \left\{\frac{\partial a_i\left(\mathbf{c},u\right)}{\partial c_j}\right\}$ is negative and symmetric. The inverse demand functions thus possess similar properties as the ordinary compensated demand functions $M_i^h\left(\mathbf{q},u^h\right)$, $i \in FC$ .

# 5 Derivation of optimal tax formulae

As $\mathbf{M_{qq}} = \mathbf{E_{qq}}$ and $\mathbf{E_q} = \mathbf{c} - \boldsymbol{\omega}$, (23) may in matrix notation be written

$$\mathbf{M_{qq}t} = -\theta\left(\mathbf{c} - \boldsymbol{\omega}\right) \tag{30}$$

Pre-multiplying by $\mathbf{D_{qq}}$ and using that $\tilde{\mathbf{M}}_{\mathbf{qq}} = 1/M\ \mathbf{M_{qq}}$ (see (27)) we have

$$\mathbf{D_{qq}M_{qq}t} = -\left(\theta/\mathrm{M}\right)\left(\mathbf{c} - \boldsymbol{\omega}\right) \tag{31}$$

The Antonelli matrix $\left\{D_{ij}\right\}$ is the generalised inverse of the Slutsky matrix $\left\{\tilde{M}_{ij}\right\}$, i.e.

$$\mathbf{D_{qq}}\tilde{\mathbf{M}}_{\mathbf{qq}} = I - \tilde{\mathbf{q}}\mathbf{c}' \tag{32}$$

We therefore have

$$\left(I - \tilde{\mathbf{q}}\mathbf{c}'\right)\mathbf{t} = -\left(\theta/\mathrm{M}\right)\left(\mathbf{c} - \boldsymbol{\omega}\right) \tag{33}$$

$$\mathbf{t} = \tilde{\mathbf{q}}\mathbf{c}'\mathbf{t} - \left(\theta/\mathrm{M}\right)\mathbf{D_{qq}}\left(\mathbf{c} - \boldsymbol{\omega}\right) \tag{34}$$

Since $\mathbf{D_{qq}c} = 0$

$$\mathbf{t} = \tilde{\mathbf{q}}\mathbf{c}'\mathbf{t} + \left(\theta/\mathrm{M}\right)\mathbf{D}\boldsymbol{\omega} \tag{35}$$

The government's budget constraint is $\left(\mathbf{c} - \boldsymbol{\omega}\right)'\mathbf{t} = G$ which may therefore be rewritten (35) as

$$\mathbf{t} = \tilde{\mathbf{q}}\left(G + \boldsymbol{\omega}'\mathbf{t}\right) + \left(\theta/\mathrm{M}\right)\mathbf{D}\boldsymbol{\omega} \tag{36}$$

Multiplying by $\boldsymbol{\omega}$ we have

$$\mathbf{t}\boldsymbol{\omega} = \tilde{\mathbf{q}}\left(G + \boldsymbol{\omega}'\mathbf{t}\right)\boldsymbol{\omega} + \left(\theta/\mathrm{M}\right)\boldsymbol{\omega}'\mathbf{D}\boldsymbol{\omega} \tag{37}$$

Assuming as a matter of normalisation that labour is untaxed, i.e. $t_0 = 0$, and thus $\boldsymbol{\omega}'\mathbf{t} = 0$ the necessary conditions for an optimal tax structure (36), may be written as

$$\mathbf{t}' = G\tilde{\mathbf{q}}'\boldsymbol{\omega} + \left(\theta/\mathrm{M}\right)\boldsymbol{\omega}'\mathbf{D}\boldsymbol{\omega} \tag{38}$$

The household's budget constraint is $\tilde{\mathbf{q}}'\boldsymbol{\omega} = 1$, thus

$$G = -(\theta/\mathrm{M})\boldsymbol{\omega}'\mathbf{D}\boldsymbol{\omega} \tag{39}$$

$$\theta/M = -\frac{G}{\boldsymbol{\omega}'\mathbf{D}\boldsymbol{\omega}} \tag{40}$$

As $\boldsymbol{\omega} = (\omega_0, 0, , 0) =$

$$\theta/M = \frac{G}{\omega_0 D_{00}(\mathbf{c}, u)\omega_0} \tag{41}$$

$$\omega_0\, \theta/M = \frac{G}{D_{00}(\mathbf{c}, u)\omega_0} \tag{42}$$

Substituting in (38) by (42) we have

$$t_k = G\tilde{q}_k + \frac{G}{D_{00}\omega_0} D_{0k} \qquad k \in C \tag{43}$$

and rewriting using that M= $q_0\omega_0$

$$t_k = \tilde{q}_k \left( G + \frac{D_{0k}/\tilde{q}_0\tilde{q}_k}{D_{00}/\tilde{q}_0\tilde{q}_0} \right) \qquad k \in C \tag{44}$$

Substituting using the definitions of $\rho_{ij}$, we have

$$\frac{t_k}{q_k} = \frac{G}{M}\left(1 + \frac{\rho_{k0}}{\rho_{00}}\right) \qquad k \in C \tag{45}$$

or

$$\frac{t_j}{p_j} \Big/ \frac{t_k}{p_k} = \left( \frac{\rho_{00} + \rho_{j0}}{\rho_{00} + \rho_{k0}} \right) \qquad k, j \in C \tag{46}$$

Assuming that all $-\rho_{00} > \rho_{j0} > 0$ are positive, it follows that the commodity which is most *q*-complementary with leisure will be taxed at the highest rates (see Deaton 1981). Note that the smaller is $\rho_{00} < 0$, the greater is the concavity of the indifference surface and hence the curvature of the indifference curves for given value of $c_0$. Moreover, the smaller is $\rho_{00} < 0$, the greater the role of *Objective 1* in determining the optimal tax rates. Note also that the ranking of the tax rates is independent of the rule of normalisation adopted.

We are thus able to convey an intuitive, yet rigorous explanation of what determines the optimal tax system by the following proposition

***Proposition 7:*** *The optimal tax system represent a compromise between two objectives: First, that of not distorting the pattern of consumption of the produced commodities (Objective 1), represented by* $\rho_{00}$*, and second, that of discouraging the untaxed consumption of the primary factor (Objective 2), represented by* $\rho_{j0} - \rho_{k0}$.

We have thus confirmed and made precise the *Conjecture* formulated in Section 3. It has become clear that the appropriate concept of complementarity in relation to the interpretation of optimal tax formulae is capture by the Antonelli elasticities. This is not generally recognised in the literature.

# 6 The optimal tax structure with more than one primary factor

Explaining the optimal tax system based on *Objective 1 and Objective 2* also provides an intuitive understanding of the Diamond and Mirrlees Production Efficiency Theorem. This theorem states that when optimal lump-sum transfers are not feasible, productive efficiency is still desirable if all market transaction can be taxed optimally, although Pareto efficiency is not possible. In other words, although it is not possible to equate the subjective rate of transformation in consumption with the objective rate of transformation in production, it is still desirable to maintain the same rate of transformation in all production sectors (contrary to what one might expect based on the Theorem of the second-best). The intuition behind this result is that distorting the rates of transformation between production sectors does not provide any increased leverage with respect to alleviating the distortion of the pattern of consumption of the produced commodities (*Objective 1*) or discouraging the untaxed use of the primary factors (*Objective 2*) as long as consumer prices can be fixed independently of producer prices. On the other hand, when restrictions are imposed on the government's choice of optimal tax rates, influencing producer prices may be a way to change consumer prices, and the production efficiency theorem no longer applies (see Dasgupta & Stiglitz 1971 and Munk 1978 and 1980, 1998).

# 7 Examples from the literature

We now provide examples from the literature on optimal taxation to illustrate how ambiguity with respect to the key concepts and assumptions has resulted in misinterpretation of analytical results.

Early contributions to the development of the modern theory of optimal taxation published in prestigious journals such as American Economic Review, Review of Economic Studies and Journal of Public Economics were marred by errors with respect to normalisation. This in no way denies the considerable intellectual achievements of these early contributions; however the focus here is on flaws which have had repercussions in academic work to the present day.

In a one household economy, a set of commodity taxes which raise the price of all goods by the same proportion has the same effect as a lump-sum tax. Based on this observation, Dixit (1970) concluded that if a lump sum tax is not feasible but all commodities can be taxed, then the optimal system involves a uniform tax on all goods in the sense of all net trade. Assuming that the government's revenue requirement is lower than untaxed private profit, this is correct. Sandmo (1974, pp. 704f) is, however, also correct in pointing out that Dixit was inconsistent in assuming both constant producer prices *and* positive unearned income. Yet Sandmo carries the argument too far when he claims that such a tax system is meaningless. Furthermore, is incorrect when he makes the opposite assertion, namely that in the model employed by Dixit (1970) assuming one commodity untaxed involves a loss in generality (Sandmo 1974, p. 702). In an economy with non-linear production, one commodity *can* without loss of generality be assumed untaxed if 1) there are constant returns to scale, 2) if the profit is taxed at 100%, or 3) if production takes place under government control such that all profit occur to the government.

There is a close relationship between optimal taxation and optimal public sector pricing. However, the differences have not always been fully appreciated. Baumol and Bradford (1970) use "all commodities" in the sense of all marketed commodities, as Dixit. When they claim that a proportional tax structure cannot be optimal when all commodities can be taxed, they must therefore - as Sandmo - implicitly assume that the households receive no profit income. This is indeed a natural assumption in the context of public sector pricing, if all production is assumed to take place in the public sector. Public sector production corresponds to private sector production with a 100% profit tax. Lerner (1970) claims that that they implicitly assume constant marginal costs, and that results would change if they took private sector production into account. Baumol and Bradford (1972) acknowledge this, but it is in fact not quite correct. The production structure does not affect the optimal tax structure when the household receives no profit income, whether this is because the profit is taxed at 100% or occur to the government directly.

For the same reason, Stiglitz and Dasgupta (1971) were mistaken when they, in a model with untaxed profit, claim that assuming one commodity untaxed is just a normalisation rule (see Munk 1980). Contrary to the interpretation by Stiglitz and Dasgupta (1971), the optimal producer prices when all commodities cannot be taxed therefore do not only represents the objective of taxing pure profit. They also represent the objective of indirectly taxing leisure.

Dixit (1975) makes the mistake of assuming leisure as untaxed as a matter of normalisation in a model where the household's endowment and/or its profit income may be taxed directly. With respect to assuming labour as untaxed in the presence of untaxed profit the error was rectified in Dixit and Munk (1977) and with respect to assuming the household's endowment untaxed in Munk (2006).

The misconception of Dixit (1975 with respect to the implication of normalisation was not corrected in Atkinson and Stiglitz "Lectures on Public Economics", (see p. 385), which 20 years after its publication is still the most authoritative source in the field. This may explain why the misconception has persisted in the literature until this day and has been manifested in a recent influential text book on Public Economics by Myles (1995). Myles (1995) for example writes (quoted from Sandvik 2003):

> "It has been shown that in an economy with constant returns to scale consumer and producer prices can be normalised separately and that standard procedure is to make one good the numeraire and set the consumer and producer prices equal. This normalisation also has the effect of setting the tax on that good to zero. In particular, the zero tax is just a result of the normalisation rule. In particular, the zero tax carries no implication about the nature of the good nor about the ability to tax that good. This follows since the good with zero tax can be chosen arbitrarily from the set of available goods.
>
> Unfortunately, this reasoning has not been as clearly appreciated in some literature, it has been inferred from this that, since leisure cannot be measures in the same way as purchase of other commodities can, the zero tax on leisure is a restriction on the permissible tax system bought about by an inability to tax leisure. In addition the further inference is usually made that the optimal tax system aims to overcome this missing tax on the leisure by taxing goods complementary to leisure.
>
> Particular examples of this is found in the Corlett and Hague (1953) by 'taxing those goods complementary with leisure, one is to some extent taxing leisure itself' (p. 26) and Layard and Walters (1978) 'the theory of second best tells us that if we cannot tax leisure we can do better than by taxing all goods. Equi-proportionally' (p. 184). Many other instances of similar statements could easily be given. This of course is a false interpretation. When real restrictions upon the permissible range of tax instruments are introduced the result obtained are affected. A number of such restrictions are considered in Munk (1980) where it is shown that the resulting optimal tax structure is sensitive to the precise restrictions imposed"

However, from the previous analysis it is clear that Myles' criticism of Corlett and Hague and Layard and Walters is totally unjustified [1].

Furthermore in the literature (see for example Myles 1995 p124) the focus in the interpretation of Corlett and Hague rule has been on how the optimal tax structure

---

[1] Although tax rates change under renormalization, the relative size of consumer prices, and hence the ranking of the tax rates remain the same; if one commodity is taxed at a higher rate than another commodity for one optimal tax vector based on a particular rule of normalisation, then this will also be the case for all optimal tax vectors: If $t = q - p$, where $t_0 = 0$, is the vector of taxes before renormalization, and $\tilde{t} = kq - p$, where $k > 0$, is the vector of taxes after normalisation, then $\tilde{t}_i > \tilde{t}_j$ if $t_i > t_j$.

contributes to achieving *Objective 2*, viz. that the commodity which is most complementary with the household's untaxed use of the primary factor should be taxed at relative highest rate to encourage the supply of the primary factor. However, the optimal tax structure depends, as we have seen, on both indicators. Only by having this in mind is it possible to understand the optimal tax system as a compromise between *Objective 1* and *Objective 2*.

Deaton (1981, p1256) also fails to recognise the special role of labour as the single commodity of which the household consume part of its initial endowment for the interpretation of the Corlett-Hague result. He writes

> [The Corlett.- Hague rule] calls for a higher tax on the taxed good that is a complement to the numeraire. This generates the somewhat misleading explanation that we “cannot” tax good zero, so we minimise distortions by taxing more heavily its compliments. Recall that the choice of untaxed good is arbitrary and that [The Corlett.- Hague rule] applies to any of the three goods. … Note the special place occupied by good 0, leisure: … the asymmetry is due to the numeraire role of labour (or leisure). Since leisure is untaxed, government revenue is implicitly measures in labour units so that by taxing complements either the revenue good, taxation is rendered easier. In general the government will presumably whish to purchase goods other than labour and this would leads to a different tax rule. For example, a king who must pay a tribute of oxen to a neighbour conqueror would do well to levy relatively high taxes on goods complementary with oxen.

Stern (1986) correctly identifies the special role of labour in standard optimal tax models as the only commodity where the household has an initial endowment and thus is able consume part of it without being subjected to taxation. However, he is wrong in criticising Deaton for using the full income expenditure function in deriving the optimal tax structure. As we have demonstrated in this paper, it is possible to use the full income expenditure function for this purpose imposing the condition that the consumption of leisure cannot be taxed as a constraint rather than as a normalisation rule.

Finally, Auerbach (1987) argues that one should use leisure as numeraire as this gives the most intuitive interpretation. He thus fails to recognise the distinction between the behavioural assumption that the consumption of leisure cannot be taxed and the that the supply of labour to the market cannot be taxed. The latter may be considered as a rule of normalisation involving no loss of generality, while the former may not.

# 8 Concluding remarks

This paper has reconsidered the question of normalisation in the context of optimal tax theory. We have shown that much light can be thrown on this issue by considering in detail the equilibrium conditions defining the set of feasible tax systems. We have confirmed the conjecture by several authors of Corlett and Hague’s result that commodities which are complementarity to leisure not only in the two commodity case but also in general should be taxed at a relative high rate, and made precise the appropriate concept of complementarity. As these authors have noted but denied by others, the result is not an artefact of normalisation, but is invariant under renormalization. We have furthermore drawn attention to ambiguity with respect to the meaning of the assumption of "*labour being untaxed*". These ambiguities have given rise to much confusion. A clear distinction should be made between *"market use of time"* and *"non-market use of time"*. Clarifying the issue of normalisation has helped to provide an intuitive explanation of what determines the optimal tax system as a trade-off between two objectives, on the one hand not to distort the pattern of consumption of produced commodities, and on the other hand to stimulate the supply of labour. Finally, we have in prominent contributions to the literature on optimal taxation highlighted ambiguous interpretation due to errors in normalisation.

## References


Auerbach, Alan (1987). “The Theory of Excess Burden and Optimal Taxation”, in A. Auerbach and M. Felstein (eds), Handbook of Public Economics, Amsterdam, North Holland, 61-128

Atkinson, A. B. and J. Stiglitz (1980), *Lectures on Public Economics*, McGraw-Hill

Baumol, W. J. and D. F. Bradford (1970), “Optimal Departures from Marginal Costs Pricing”, *American Economic Review*, 60, 265-283

Baumol, W. J. and D. F. Bradford (1972), “Optimal Taxes and Pricing: Reply” *American Economic Review*, 62

Corlett, W. J. and D. C. Hague (1953), “Complementarity and excess burden of taxation”, *Review of Economic Studies*, 21, 21-30

Deaton A. (1979): “The D Function in Consumer Behaviour with Application to Index Numbers and Optimal Taxation”, Review of Economics Studies 21, 21-30

Deaton A. (1981) “Optimal taxes and the structure of preferences”, *Econometrica* 49, 1245-1260

Diamond, P. A. and J. A. Mirrlees, 1971 “Optimal taxation and public production: I and II, *American Economic Review*, 60, 223-240

Dixit, A. (1970), “On the optimum structure of commodity taxes”, *American Economic Review*, 60, 295-301

Dixit, A. (1975) "Welfare effect of tax and price changes", *Journal of Public Economics,* 4, 103-123

Dixit, A. and K. J. Munk (1977) "Welfare effect of tax and price changes. A Correction", *Journal of Public Economics*, 8, 103-107

Harberger, A.C., (1971), “Three basic postulates for applied economics: An interpretative essay, *Journal of Economic Literature* 9, 785-97

Layard and Walters (1978), *“Microeconomic Theory”*, McGraw-Hill Education

Lerner, A. P. (1970), “On the Optimum Structure of Commodity Taxes”, *American Economic Review*, 60, 284-294

Mirrlees, J.A. (1971), “Optimal tax theory: a synthesis”, *Journal of Public Economics,* 6, 327-358

Munk, K. J. (1978) "Optimal Taxation and Pure Profit", *Scandinavian Journal of Economics,* 80, 1-19

Munk, K. J. (1980), "Optimal taxation with some non-taxable commodities", *Review of Economic Studies,* 47, 755-765

Munk, K. J. (1982) "On the difference between Optimal Public sector pricing and Optimal Taxation", *Recherches Economique de Louvain,* 48, 177-190

Munk, K. J. (1998),"Should Governments Create Productive Inefficiency", Working Paper 1998-20, EPRU, University of Copenhagen,

Munk, K. J. (2006),"Welfare effects of tax and price changes and the CES-UT utility function", Working Paper, University of Aarhus

Myles, G. R. (1995) "*Public Economics"*, Cambridge University Press

Sandmo, A. (1974), "A Note on the Structures of Optimal Taxation", *American Economic Review,* 64, 701-706

Sandmo, A (1976): Optimal Taxation: An Introduction to the Literature, Journal of Public Economics, 76, 37-54

Sandvik B. (2003), "Optimal taxation and normalisations", CESifo Working Paper Series No. 1025

Stern N. (1986): A note on commodity taxation. The choice of variable and the Slutsky, Hessian and Antonelli matrices (SHAM), Review of Economics Studies 53, 293-299

Stiglitz J. E. and P. Dasgupta (1971), "Differential taxation, Public Goods, and Economic Efficiency", *Revenue of Economic Studies,* 151-174

**Economics Working Paper**

2006-1 Niels Haldrup and Andreu Sansó: A Note on the Vogelsang Test for Additive Outliers.

2006-2 Charlotte Christiansen, Juanna Schröter Joensen and Helena Skyt Nielsen: The Risk-Return Trade-Off in Human Capital Investment.

2006-3 Gunnar Bårdsen and Niels Haldrup: A Gaussian IV estimator of cointegrating relations.

2006-4 Svend Hylleberg: Seasonal Adjustment.

2006-5 Kristin J. Kleinjans and Jinkook Lee: The link between individual expectations and savings: Do nursing home expectations matter?

2006-6 Jakob Roland Munch, Michael Rosholm and Michael Svarer: Home Ownership, Job Duration, and Wages.

2006-7 Francesco Busato and Enrico Marchetti: Skills, sunspots and cycles.

2006-8 Peter Sandholt Jensen and Allan H. Würtz: On determining the importance of a regressor with small and undersized samples.

2006-9 Michael Svarer: Working Late: Do Workplace Sex Ratios Affect Partnership Formation and Dissolution?

2006-10 Helena Skyt Nielsen and Michael Svarer: Educational Homogamy: Preferences or Opportunities?

2006-11 Juanna Schrøter Joensen and Helena Skyt Nielsen: Is there a Causal effect of High School Math on Labor Market Outcomes?

2006-12 Knud Jørgen Munk: On the Rationale for the Use of Border Taxes in Developing Countries.

2006-13 Knud Jørgen Munk: Rules of Normalisation and their Importance for Interpretation of Systems of Optimal Taxation.