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# Skills, sunspots and cycles*

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**Abstract**

This paper explores the ability of a class of one-sector, multi-input models to generate indeterminate equilibrium paths, and endogenous cycles, without relying on factors' hoarding. The model presents a novel theoretical economic mechanism that supports sunspot-driven expansions without requiring upward sloping labor demand schedules. Its distinctive characteristic is that the skill composition of aggregate labor demand drives expansionary i.i.d. demand shocks. Next, the model explains the labor market dynamics from the supply side, while endogenizing the capital productivity response to changes in the aggregate labor demand composition. Last but not least, it is worth to mention that the model presents an effective shock propagation mechanism that operates into the labor market and across labor market segments through the cross elasticities of equilibrium labor demand and supplies. In this respect the model can be seen as quite general formulation (with or without aggregate increasing returns to scale) for analyzing labor market dynamics within a general equilibrium model with labor market segmentation.

*Journal of Economic Literature Classification Numbers*: O40, J24
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**– Comments welcome –**

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# 1 Introduction

In the last few years it has been recognized that indeterminacy of the equilibrium is a phenomenon that arises in representative-agent, infinite horizon economies if the assumption of constant returns to scale and/or perfect competition is relaxed.

The class of one-sector models with indeterminacy (e.g. Farmer [6], Farmer and Guo [5] or Benhabib and Farmer [2]) however, requires a degree of increasing returns which is too high with respect to what recent estimates suggest (see, among the others, Basu and Fernald [1], Sbordone [21], Jimenez and Marchetti [13]). The high degree of increasing returns is also responsible for an undesirable properties of this class of models: specifically, in order to have local indeterminacy, the labor demand schedule must be upward sloping (Benhabib and Farmer [2]).

Generally speaking, the economic literature proposes two classes of remedies to overcome these difficulty: the introduction of factor hoarding within one-sector models (e.g. Wen [23], Weder [22] );[1] or the explicit specification of a second sector (e.g. Benhabib and Farmer [3], Perli [19]).[2]

This paper explores the ability of a class of one-sector models to generate indeterminate equilibrium paths and endogenous cycles, without relying on factors' hoarding. In particular, we consider a one sector economy in which there exist one type of capital stock, and a finite number of heterogenous labor services, which are assumed to be heterogeneous along the skill/productivity dimension.[3] The model's formulation is quite general and it can can be applied to explain endogenous fluctuations of skilled and unskilled workers in bad and good times under indeterminacy, and to understand the relationships between aggregate productivity and the composition of labor demand.

Here is an overview of our results. **First**, the model presents a novel theoretical economic

[1]The introduction of factor hoarding can sensibly reduce the amount of externality needed for having indeterminacy. For instance, in a model with variable capacity utilization, Weder [22] shows how indeterminacy can arise by assuming low externalities coupled with factor hoarding. Analogous results can be obtained by introducing the need for firms to devote a share of labor services to the maintenance of capital stock, as in Guo and Lansing [10]. See also Kim [14] for a survey on sources of externalities.

[2]The introduction of a second sector solves this problem. Perli [19] explicitly introduce an household production sector into a model with externalities and increasing returns. He shows that cycles driven by self-fulfilling prophecies can exist with external effects in labor and capital that are sensibly smaller than previously thought. He also shows that the equilibrium labor demand of his model is well behaved, in the sense that it slopes down. A similar result (indeterminacy with low externalities) has been obtained by Benhabib and Farmer [3] in a two sector model with sector specific instead of aggregate externalities. Their model, however, may have equilibria where consumption and hours are negatively correlated when the driving force is a sunspot rather than a technology shock.

[3]Notice, however, that what matters is the heterogeneity itself, and it is possible to obtain qualitatively analogous results for different kinds of heterogeneity (i.e. distinguishing between regular and underground labor services, or between labor services spatially separated).

mechanism that supports sunspot-driven expansions without requiring upward sloping labor demand schedules;[4] its distinctive characteristic is that the skill composition of aggregate labor demand drives expansionary i.i.d. demand shocks. In addition, a casual inspection of the cyclical behavior of skilled and unskilled workers for the United States economy qualitatively supports the presented mechanism. **Second**, the model explains the labor market dynamics from the supply side, while endogenizing the capital productivity response to changes in the aggregate labor demand composition. **Last but not least**, it is worth to mention that the model presents an effective shock propagation mechanism that operates into the labor market and across labor market segments through the cross elasticities of equilibrium labor demand and supplies. In this respect the model can be seen as quite general formulation (with or without aggregate increasing returns to scale) for answering selected labor market questions within a dynamic general equilibrium model with labor market segmentation.

The paper is organized as follows. Section 2 presents the theoretical model and its equilibrium; Section 3, then, discusses the topological properties of stationary state, derives conditions for indeterminacy and explains how the theoretical mechanism of the model works. Section 4, next, calibrates the model for the U.S. economy, and studies the model response to extrinsic uncertainty *via* generalized impulse response functions. Finally, Section 5 concludes. Proofs and derivation are included in the Appendices at the end of the paper.

# 2 The Model

Assume that there exist one aggregate capital stock, and $0 < M < \infty$ types of labor services, which are applied to the existing capital stock. In this sense the labor market is said to be *segmented*. In addition, there are two classes of agents in the models: firms and households.

## 2.1 Firms

Suppose that the production technology for the homogenous good uses $M+1$ inputs: the aggregate capital stock $k_t$, and $M$ different types of labor services, denoted as $n^j$, with $j = 1, 2, ..., M$; given these preliminaries, the $i-$th firm's production function reads:

$$y_{i,t} = A_t k_{i,t}^{\alpha_0} \left[ \prod_{j=1}^{M} (n_{i,t}^{j})^{\alpha_j} \right], \text{ with } \sum_{j=0}^{M} \alpha_j = 1.$$

[4]The proposed mechanism differs from the customary one, and we consider it complementary to that one.

The quantity $A_t$ (defined below) represents an aggregate production externality

$$A_t = \underbrace{\{K_t^{\alpha_0}\}^{\omega}}_{\text{Marshallian Ext.}} \prod_{j=1}^{M} \underbrace{\left[(N_t^j)^{\alpha_j}\right]^{\eta_j}}_{\text{j-th labor Externality.}}, \quad \omega \neq \eta_\kappa \neq \eta_j, \quad \kappa \neq j,$$

where $K_t$ and the $N_t^{j\prime}s$ are the economy-wide levels of the production inputs.

The aggregate external effect has $M+1$ different sources. The first one, without loss of generality, is related to the Marshallian effect, analogous to that of standard one-sector models (e.g. Farmer and Guo [5]). The other ones act through the various types of labor services. This formulation explicitly distinguishes among each labor-specific external effect: for example, the quantity $\left[(N_t^j)^{\alpha_j}\right]^{\eta_j}$ denotes the external effect associated to the $j-th$ type of labor. Finally, the parameters $(\omega,\ \eta_j,\ j=1,2,...,M)$ are assumed to be different one from the other to exploit the distinctive characteristics that each production factor has.[5]

As firms are all identical, overall level of output for a given level of input-utilization is given by:

$$Y_t = A_t \int_i \left\{ k_{i,t}^{\alpha_0} \left[ \prod_{j=1}^{M} (n_{it}^j)^{\alpha_j} \right] \right\} di = K_t^{\alpha_0(1+\omega)} \left[ \prod_{j=1}^{M} (N_t^j)^{(1+\eta_j)\alpha_j} \right]. \tag{1}$$

Increasing returns to scale are a pure aggregate phenomenon (as equation (1) suggests), and returns to scale faced by each firm in production are constant, i.e. $\alpha_0 = 1 - \sum_{j=1}^{m} \alpha_j$. Assume, next, that each firm takes $K$, $N^1, \ldots, N^M$ as given.[6] As markets are competitive, firm's behavior is described by the $M+1$ first order conditions for the (expected) profit maximization, with respect to $k_{i,t}$, $n_{i,t}^1, ..., n_{i,t}^M$ :

$$\begin{aligned}
k_{i,t} &: \quad \alpha_0 A_t \frac{\partial y_{i.t}}{\partial k_{i,t}} = r_t \\
n_{i,t}^1 &: \quad \alpha_1 A_t \frac{\partial y_{i,t}}{\partial n_{i,t}^1} = w_t^1 \\
& \quad \vdots \\
n_{i,t}^M &: \quad \alpha_M A_t \frac{y_{i,t}}{n_{i,t}^M} = w_t^M.
\end{aligned} \tag{2}$$

[5]This formulation adds to the analysis greater generality, as it encompasses a large class of one sector economies that do not explicitly distinguish among the input specific external effects. More details are offered in the following pages.

[6]In this context the externality $A_t$ acts at pure aggregate-systemic level, as in Romer's [20] endogenous growth model.

All kinds of labor services are employed in equilibrium, due to the Cobb-Douglas production structure;[7] otherwise the model would end up into a trivial solution ($n_{i,t}^{1} = ... = n_{i,t}^{M} = 0$). Hence, all firms will employ all labor types, as in a sort of separating equilibrium, paying each labor at a different wage rate, gross of the additional disutility cost paid for getting the higher productivity. It seem indeed reasonable to imagine that relatively more productive types of labor are also more costly for the firm and for the consumer/worker. The net-of-effort wage rates will be identical (more details to come).

## 2.2 Households

Suppose that there exist a continuum of identical households, indexed with super-script $i$, uniformly distributed over the unit interval. Suppose that each household supplies $j = 1, 2, ..., M$ different types of labor $n_{i,t}^{j}$, and assume that each household is complete, in the sense that all households supply all types of labor services.

The households preferences are structured in the following way. The consumption flows $c_{i,t}$ generates $\log(c_{i,t})$ level of utility, and total labor $n_{i,t} = \sum_{j=1}^{M} n_{i,t}^{j}$ generates an overall disutility of work equal to $[D/(1+\xi)]\, n_{i,t}^{1+\xi}$. In addition each type of labor determines an idiosyncratic disutility $[B_j/(1+\psi_j)] \left(n_{i,t}^{j}\right)^{1+\psi_j}$, which captures the labor heterogeneity (or labor market segmentation). Without loss of generality, we can order the labor services along the disutility dimension as Assumption 1 suggests.

**Assumption 1** $B_1 < B_2 < \cdots < B_M$.

The quantities $[B_j/(1+\psi_j)] \left(n_{i,t}^{j}\right)^{1+\psi_j}$ are proxies for the labor-specific effort exerted by each household. Labor types with higher $B$ are assumed to be more costly for the consumers/workers. (**new!**) On the labor demand side, next, we assume that the higher the cost (i.e. the $B_j$), the higher the marginal labor productivity ($MPN(B_j)$, $j = 1, ..., M$) of the $B_j$ type of labor service.

**Assumption 2** $MPN(B_1) < MPN(B_2) < \cdots < MPN(B_M)$.

If we interpret labor heterogeneity as stemming from an un-modeled human capital stock and/or skills, the $B_j$ disutility parameters would be associated to additional effort

[7] A nested CES structure on production would allow for a more general analysis, and we should expect that the value of the elasticity of substitution (say $\sigma$) would play an interesting role. There exist, however, several difficulties to estimate this parameter because it captures substitution both within and across industries. Moreover, the majority of macro-estimates are between $\sigma = 1$ and $\sigma = 2$ (e.g. Freeman 1986), which correspond to the Cobb Douglas case. In this respect we consider our formulation a sufficiently good approximation of the actual production technology.

needed to acquired an higher education (or on-the-job training). It is like saying that each different type of labor $j$ pays a different cost for acquiring its skills or characteristics. Now, this formulation is not addressing a fully fledged "heterogeneity problem", but it is looking at a parsimonious model capable of capturing this issue.

Assuming separability, we specify the momentary household utility function as:

$$\mathcal{V}_{i,t}\left(c_{i,t}, n_{i,t}^{1}, \cdots, n_{i,t}^{M}\right) = \log c_t^i - \frac{D}{1+\xi} n_{i,t}^{1+\xi} - \sum_{j=1}^{M} \frac{B_j}{1+\psi_j} \left(n_{i,t}^{j}\right)^{1+\psi_j}.$$

Notice that the labor heterogeneity mainly comes from the supply side, the production technology being Cobb-Douglas. It is therefore important for the model that the parameters $\xi, \psi_j \neq 0$, otherwise it would be possible to reallocate labor supply across all labor types without incurring in idiosyncratic costs. In this case we should expect that all labor types would behave identically.[8]

Next, the household' feasibility constraint ensures that the sum of consumption $c_{i,t}$ and investment $i_{i,t}$ does not exceed consumers' income,

$$c_{i,t} + i_{i,t} = r_t k_{i,t} + \sum_{j=1}^{M} w_t^j n_{i,t}^j,$$

and capital stock is accumulated according to a customary state equation, i.e.

$$k_{i,t+1} = (1-\delta) k_{i,t} + i_{i,t},$$

where $\delta$ denotes a quarterly capital stock depreciation rate.

Imposing, then, a constant subjective discount rate $0 < \beta < 1$, and defining $\mu_{i,t}$ as the costate variable, we form the Lagrangean of household control problem:

$$\mathcal{L}_0^h = E_0 \sum_{t=0}^{\infty} \beta^t \mathcal{V}_{i,t} + E_0 \sum_{t=0}^{\infty} \mu_{i,t} \left( r_t k_{i,t} + \sum_{j=1}^{M} w_t^j n_{i,t}^j - c_{i,t} - i_{i,t} \right).$$

Household's optimal choice is characterized by the following necessary and sufficient conditions:

[8] An alternative formulation would be to specify production technology as a nested C.E.S. function, and to assume linear costs for each labor type (i.e. $\xi = \psi_j = 0$. In this case, however, it would not be possible to analytically derive the conditions for indeterminacy; for this reason we prefer to analyze the actual version with perfect substitutability on the technology side, and idiosyncratic costs on the supply side.

$$
\begin{aligned}
c_{i,t} &: \beta^t c_{i,t}^{-1} = \mu_{i,t} \\
n_{i,t}^1 &: \beta^t D n_{i,t}^{\xi} + \beta^t B_1 \left(n_{i,t}^1\right)^{\psi_1} = \mu_{i,t} w_t^1 \\
&\quad \vdots \\
n_{i,t}^M &: \beta^t D n_{i,t}^{\xi} + \beta^t B_M \left(n_{i,t}^M\right)^{\psi_M} = \mu_{i,t} w_t^M \\
k_{i,t+1} &: E_t \left\{\mu_{i,t+1} \left[(1-\delta) + r_{t+1}\right]\right\} = \mu_{i,t} \\
&\quad \lim_{t \to \infty} E_0 \mu_{i,t} k_{i,t} = 0
\end{aligned} \tag{3}
$$

The model collapses to the standard one sector model with aggregate increasing returns to scale (e.g. Farmer and Guo [5]) setting $M = 1$ and $\omega = \eta_1 = \eta$ into the previous equilibrium conditions.

## 2.3 Symmetric perfect foresight equilibrium

We focus on a symmetric perfect foresight equilibrium in which firms make zero profits. In equilibrium the aggregate consistency requires that $y_{i,t} = Y_t$, $k_{i,t} = K_t$, $n_{i,t}^j = N_t^j$, $c_t = C_t$, where capital letters denote aggregate equilibrium quantities[9]. An equilibrium is a sequence of prices $\left\{w_t^1, \cdots, w_t^m, r_t\right\}_{t=0}^{\infty}$ and a sequence of quantities $\left\{N_t^1, \cdots, N_t^m, K_{t+1}, C_t, \right\}_{t=0}^{\infty}$ such that firms and households solve their optimization problems and the resource constraints are satisfied. As a result, the first order conditions characterizing the equilibrium are given by:

$$
\begin{aligned}
D N_t^{\xi} + B_1 \left(N_t^1\right)^{\psi_1} &= (C_t)^{-1} \alpha_1 \frac{Y_t}{N_t^1} \\
&\quad \vdots \\
D N_t^{\xi} + B_M \left(N_t^M\right)^{\psi_M} &= (C_t)^{-1} \alpha_M \frac{Y_t}{N_t^M} \\
(C_{t+1})^{-1} \left((1-\delta) + \alpha_0 \frac{Y_{t+1}}{K_{t+1}}\right) \beta &= (C_t)^{-1} \\
K_{t+1} &= K_t^{\alpha_0(1+\omega)} \prod_{j=1}^{M} N_t^{\alpha_j(1+\eta_j)} + (1-\delta) K_t - C_t \\
\lim_{T \to \infty} (C_T)^{-1} K_T &= 0.
\end{aligned}
$$

[9]The aggregate resource constraint holds: $C_t + I_t = Y_t$.

# 3 Topological properties and endogenous cycles

## 3.1 Topological properties

To solve the model, we log-linearize the economy-wide version of first order conditions (2) and (3) around the steady state. To study how agents "animal spirits" operate into an economy with indeterminacy, production externalities and $M$ types of labor input, we arrange the system of linearized equations in a way such that consumption rather than the Lagrangian multiplier appears in the state vector.

Denoting with $S_t$ as the vector $(K_t; C_t)$, the model can be reduced to the following system of linear difference equations (where hat-variables denote percentage deviations from their steady state values):[10]

$$\hat{S}_{t+1} = \mathbf{F}\hat{S}_t + \Omega\mathcal{E}_{t+1}, \tag{4}$$

where $\mathcal{E}_{t+1}$ is a $2 \times 1$ vector of one step ahead forecasting errors satisfying $E_t\mathcal{E}_{t+1} = 0$ and $\Omega$ is a coefficient matrix. Its first element $\widehat{K}_{t+1} - E_t\widehat{K}_{t+1}$ equals zero, since $\widehat{K}_{t+1}$ is known at period $t$; denote the second element with $\tilde{\varepsilon}_c = \widehat{C}_{t+1} - E_t\widehat{C}_{t+1}$. Now, when the model has a unique equilibrium (i.e., one of the eigenvalues of $\mathbf{F}$ lies outside the unit circle), the optimal decision rule for investment does not depend on the forecasting error, $\tilde{\varepsilon}_c$.[11]

If both eigenvalues of $\mathbf{F}$ lie inside the unit circle, however, the model is indeterminate in the sense that any value of $\widehat{C}_t$ is consistent with equilibrium given $\widehat{K}_t$. Hence, the forecasting error $\tilde{\varepsilon}_c$ can play a role in determining the equilibrium level of consumption. Under indeterminacy the decision rule for consumption at time $t$ take the special form

$$\widehat{C}_t = f_{21}\widehat{K}_{t-1} + f_{22}\widehat{C}_{t-1} + \omega_2\tilde{\varepsilon}_{c,t},$$

where $f_{21}$, $f_{22}$ and $\omega_2$ are the second row elements of the matrices $\mathbf{F}$ and $\Omega$. The condition $E_t\tilde{\varepsilon}_{c,t+1} = 0$ ensures that rational agents do not make systematic mistakes in forecasting future based on current information. Since $\tilde{\varepsilon}_{c,t}$ can reflect a purely extraneous shock, it can be interpreted as shock to autonomous consumption (that is the "animal spirits hypothesis").

[10] The procedure used to obtain (4) is included in the Appendix.

[11] Specifically, in that case $\hat{c}_t$ can be solved forward under the expectation operator $E_t$ to eliminate any forecasting errors associated with future investment. Then the optimal decision rules at time $t$ depend only on the current capital stock $k_t$.

### 3.2 Local indeterminacy

Necessary and Sufficient Conditions (for local indeterminacy of the equilibrium path) are derived in Theorem 1 below. To present a neat economic interpretation it is convenient to write them in terms of elasticities and cross-elasticities of the demand schedules for capital, and for the various types of labor with respect to the $M+1$ production inputs. To ease the economic interpretation of necessary and sufficient condition for indeterminacy we will discuss a simpler case in which $\xi = 0$.

**Theorem 1** *The equilibrium of system (4) is locally indeterminate when the following necessary and sufficient (NSC) condition holds:*

$$\boldsymbol{NSC}: \max\left(\frac{1}{\beta(1-\delta)}, \frac{\underline{\mathcal{R}}}{\underline{\mathcal{R}}-1}\right) < \Phi < \frac{\overline{\mathcal{R}}}{\overline{\mathcal{R}}-1},$$

*where* $\Phi = \sum_{j=1}^{M} \frac{(1+\eta_j)\alpha_j}{1+\psi_j}$ *is the sum of ratios between cross elasticities of the (inverse) linearized labor demand functions and the corresponding elasticities of (inverse) linearized labor supply functions plus one,* $\underline{\mathcal{R}} = \frac{\delta(1-s_I)[1-\beta(1-\delta)](1-\alpha_0(1+\omega))+2[\delta\alpha_0(1+\omega)+s_I(2-\delta)]}{\delta(1-s_I)[1-\beta(1-\delta)](1-\alpha_0(1+\omega))+2[\delta\alpha_0(1+\omega)+s_I(1-\beta(1-\delta))]} > 1$, $\overline{\mathcal{R}} = \frac{\delta s_I - \delta\alpha_0(1+\omega)}{s_I[1-\beta(1-\delta)]-\delta\alpha_0(1+\omega)} > 1$, *and* $s_I = I^*/Y^*$ *denotes the (steady state) share of investment.*

**Proof.** see Appendix B. ■

Condition **NSC** is enlightening about the nature of the economic process at basis of indeterminacy in our model. Rewriting $\Phi$ in terms of elasticities of the inverse labor demand $\left(e_{j,\kappa}^{d}\right)$ and supply $\left(e_{j,\kappa}^{s}\right)$ yields $\sum_{j=1}^{M} \frac{(1+\eta_j)\alpha_j}{1+\psi_j} = \sum_{j=1,\kappa\neq j}^{M} \frac{\epsilon_{j,\kappa}^{d}}{1+\epsilon_{j,\kappa}^{s}}$.

Consider the lower bound of **NSC** first, $\left(\max\left(\frac{1}{\beta(1-\delta)}, \frac{\underline{\mathcal{R}}}{\underline{\mathcal{R}}-1}\right) < \sum_{j=1,\kappa\neq j}^{M} \frac{\epsilon_{j,\kappa}^{d}}{1+\epsilon_{j,\kappa}^{s}}\right)$. It suggests that the labor demand schedules should have a *sufficiently large* response to changes in equilibrium employment, and in the same time, the response of labor supply schedules should be *sufficiently small*.

But, at the same time, that this responses should *not be too large* as for labor demand functions *and not too small* as for labor supply functions, as the upper bound suggests that $\sum_{j=1,\kappa\neq j}^{M} \frac{\epsilon_{j,\kappa}^{d}}{1+\epsilon_{j,\kappa}^{s}} < \frac{\overline{\mathcal{R}}}{\overline{\mathcal{R}}-1}$.

Condition NSC represents a building block of the theoretical mechanism supporting self-fulfilling properties (see. Section 3.4).

In addition, the upper inequality of the Condition in Theorem 1 turns out to be particularly relevant. Corollary 1 below recast it as follows:

**Corollary 1**

$$\epsilon_{i,k}^{d} > \frac{s_I}{\delta}\left[1 + (1-\beta)(1-\delta)\sum_{j=1}^{M}\frac{\epsilon_{j,\kappa}^{d}}{1+\epsilon_j^{s}}\right], \tag{5}$$

*where $\epsilon_{i,k}^{d}$ denotes the elasticity of i-th labor demand to capital stock.*

**Proof.** Algebraic derivation. ■

Condition (5) suggests that labor demand functions should react relatively more to changes in capital stock rather than changes in labor services, and that, *ceteris paribus,*labor supply functions should be sufficiently elastic.[12] In other words, each labor demand schedule should display a large enough response to variation in capital stock for expectation to be self-fulfilled.

## 3.3 Well behaved labor demand schedules

The explicit distinction of aggregate labor into different categories, each endowed with distinctive technical and productivity characteristics yields an interesting and welcome result.

Precisely, each labor demand schedule is *well behaved*, compared to standard one-sector models where labor demand is upward sloping. A labor demand function is said to be well behaved when it slopes down over its wage domain, that is when the partial derivative of the inverse labor demand with respect the corresponding labor input is negative; re-writing the linearized labor demand functions as $\widehat{w}_t^j = \widehat{w}^j\left(\widehat{N}_t^1, \cdots, \widehat{N}_t^M\right)$, the $h-th$ labor demand schedule is said to be well behaved if:

$$\frac{\partial \widehat{w}^h}{\partial \widehat{N}_t^h} < 0,\ \widehat{N}_t^h = \left(\widehat{N}_t^1, \cdots, \widehat{N}_t^M\right)$$

To better appreciate how the labor market dynamics relates to the labor segmentation hypothesis, it is convenient to derive a set of restrictions on selected parameters ensuring that the previous inequality holds for all $h$. A natural choice is to focus on the labor shares (the $\alpha_j' s$) and the externality parameters (the $\eta_j' s$). For our production technology (equation (1)) the previous condition reads:

$$\frac{\partial \widehat{w}^h}{\partial \widehat{N}_t^h} < 0 \Leftrightarrow \eta_h < \frac{1-\alpha_h}{\alpha_h},$$

for each type of labor: $h = 1, \cdots, M$.

[12] Technically speaking, for the generic inverse demand function of labor of type $i$, $\left(\frac{\partial \widehat{w}^i}{\partial \widehat{K}}\right)^d$ should be larger than $\sum_{j=1}^{M}\frac{\left(\partial \widehat{w}^i/\partial \widehat{N}^j\right)^d}{1+\left(\partial \widehat{w}^i/\partial \widehat{N}^j\right)^s}$, which is also reduced by quantities $\frac{s_I}{\delta}$ and $(1-\delta)(1-\beta)$.

What we claim next is that the introduction of labor input heterogeneity eases the conditions for having well behaved labor demand schedules. Denote with $\eta_h^{**} = \frac{1-\alpha_h}{\alpha_h}$ the largest degrees of input-specific increasing returns to scale such that (**i**) local indeterminacy arises *and* (**ii**) labor demand schedules are well behaved. Having the individual firm production function constant returns to scale, i.e. $\alpha_0 + \sum_{j=1}^{M} \alpha_j = 1$, it is possible to rewrite each labor shares as $\alpha_h = 1 - \alpha_0 - \sum_{j \neq h}^{M} \alpha_j$; then, the previous inequality reads:

$$\frac{\partial \widehat{w}^h}{\partial \widehat{N}_t^h} < 0 \Leftrightarrow \eta_h < \frac{\alpha_0 + \sum_{j \neq h}^{M} \alpha_j}{1 - \alpha_0 - \sum_{j \neq h}^{M} \alpha_j},$$

and the threshold level equals to $\eta_h^{**} = \left(\alpha_0 + \sum_{j \neq h}^{M} \alpha_j\right) \diagup \left(1 - \alpha_0 - \sum_{j \neq h}^{M} \alpha_j\right)$. Now, if the number $M$ of labor types shrinks, the upper bound decreases for the remaining labor inputs, while reducing, by this end, the parameters' region in which the equilibrium is locally indeterminate and the labor demand schedules are well behaved at the same time.

This suggests that the model requires a relatively lower degree of returns to scale to induce local indeterminacy, compared to the standard model.[13] In addition to the novel theoretical mechanism, this in a welcome results of the model, since Farmer and Guo [5] show that for having indeterminacy they need to have a very large externality parameter. To display indeterminacy their model needs a high degree of increasing returns to scale, which equals $\eta = 0.39$, which is way above their threshold ($\eta^* = 0.23$) for having a well behaved demand schedule.

The reason why the $M-$input model ensures that demand functions slope down rests in the underlying necessary condition for indeterminacy. It is convenient to recall, for readers' convenience, that that in a one sector economy indeterminacy arises when the product between labor share ($\alpha_1$) and externality parameter ($1 + \eta_1$) divided by the labor supply elasticity is larger than unity, that is when $\frac{\alpha_1(1+\eta_1)}{1+\psi} - 1 > 0$. The sign of this inequality is dominated by the externality parameter $\eta_1$ that should be larger that a certain threshold in order to have indeterminacy. As already mentioned before, this implies that the labor demand function is positively sloped.

Now, our economy displays indeterminacy when the *sum* of the product between labor shares $\left(\alpha_j' s\right)$ and externality parameters $(1 + \eta_j)$ divided by the labor supplies elasticities $(1 + \psi_j)$ is larger than unity, i.e. $\left[\sum_{j=1}^{M} \frac{(1+\eta_j)\alpha_j}{1+\psi_j}\right] - 1 > 0$. What happens here is that the required degree of increasing returns to scale is "spread out" among a sufficiently large

[13] Notice, also that this is not the main goal of the paper, especially because the literature proposes several mechanisms capable of reducing the required degree of returns to scale for indeterminacy to figures even smaller than the actual one.

number of labor inputs. This ensures that we can have a sufficiently low labor-specific externality parameter in order to keep each labor demand schedule sloping down. The composition effect is crucial to obtain this result. A consequence is that the theoretical mechanism operating under indeterminacy differs from the standard one.

### 3.4 The model theoretical mechanism

The condition derived in Theorem 1 characterizes the economic mechanism explaining the model reaction to stochastic shocks, and particularly to an i.i.d. sunspot shock. The very idea of the "animal spirits hypothesis" is that expectations are self-fulfilled under local indeterminacy of the equilibrium path. This means that following a positive sunspot shock *today*, a rational consumer should expect a higher income *tomorrow*;[14] the self-fulfilling mechanism, generated under indeterminacy, pushes indeed the economy into an expansionary pattern. From a more technical perspective, in Farmer and Guo [5] a positive sunspot shock $\hat{\varepsilon}_t$ on the labor supply $\hat{w}_t = \widehat{C}_t + \hat{\varepsilon}_t$ shifts upward the wage $\hat{w}_t$; as the labor demand is upward sloping, this induces an increase in equilibrium labor, thus creating an expansionary push on the economy.

In our model the interaction between inputs' markets differs from the standard one, still the sunspot shock $\hat{\varepsilon}_t$ having an expansionary effect over the economy. Suppose, for the sake of simplicity, that there exist just two types of labor, skilled ($N_2$) and unskilled ($N_1$) labor services. The labor markets response following a positive sunspot shock is presented in **Figure 1**. In the diagram we assume that the skilled labor supply and demand are relatively more elastic than the unskilled counterparts, and that the skilled wage is larger than the unskilled wage, in order to be consistent with Assumptions 1 and 2.

Now, suppose the economy is at the steady state (equilibrium (1) in the figure: $N_1^*$ and $N_2^*$), and let a sunspot shock hit the economy. As expected, households are willing to have a higher consumption flow $\uparrow C$, and, at the same time, to work less. From a geometrical viewpoint, this shifts upward the two labor supplies, as in Farmer and Guo [5], but, as the labor demands are well behaved in our economy, this would induce a *reduction* in the equilibrium levels of both types of labor (equilibrium (2) in the figure: $N_1^{**}$ and $N_2^{**}$).[15] Notice, also that the skilled counterpart $N_2$ falls relatively less than the unskilled component

[14] This is represented by the forecasting error previously defined, $\tilde{\varepsilon}_{c,t}$. It can reflect a purely extraneous shock, and it can be interpreted as shock to autonomous consumption.

[15] To see this more clearly, consider the inverse (linearized) demand for type 1 labor:

$$\widehat{w}_t^1 = [(1+\omega)\alpha_0]\,\widehat{K}_t + [(1+\eta_1)\alpha_1 - 1]\,\widehat{N}_t^1 + [(1+\eta_2)\,(1-\alpha_0-\alpha_1)]\,\widehat{N}_t^2$$

i.e.a function $w^1 = L_1^D\{N^1; N^2, K, \}$, whose partial derivatives have the following signs: $\frac{\partial L_1^D}{\partial N^1} < 0$, $\frac{\partial L_1^D}{\partial N^2} > 0$. Symmetrically, the other wage - the demand for type 2 labor - equals:

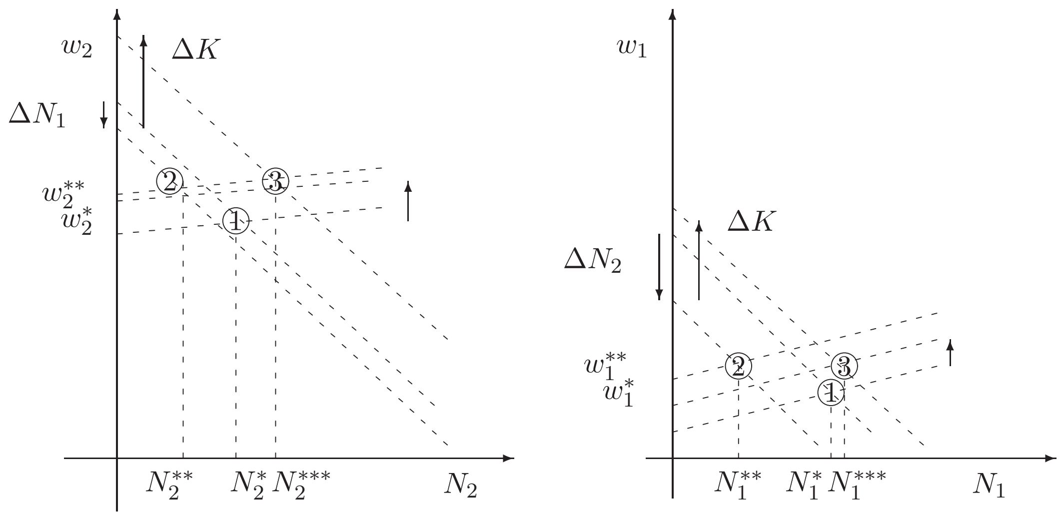


Figure 1: **Theoretical Mechanism**. Unskilled labor market: $(N_1,\ w_1)$; Skilled labor market: $(N_2,\ w_2)$; consider the steady state (1): $N_1^*$ and $N_2^*$. Skilled and unskilled labor supply schedules shift upward after an i.i.d. sunspot shock; the economy would enter into a recession ((2): $N_1^{**}$ and $N_2^{**}$) as labor demands are negatively sloped. But, in a perfect foresight equilibrium, the labor input reallocation toward the relative more skilled labor input would increase capital productivity. This triggers the capital accumulation ($\Delta K > 0$) that shifts out both labor demand schedules, driving the economy into the conjectured expansion ((3): $N_1^{***}$ and $N_2^{***}$). The cross-interaction between labor markets ($\Delta N_1$ and $\Delta N_2$) further strengthen the shifts of labor demands.

$N_1$ (formally $\Delta N_1 > \Delta N_2$), because demand and supply schedules have different slopes.

This is the crucial part of the mechanism in which the composition of aggregate labor demand changes towards a more qualified combination of hired labor services. This makes capital more productive, the interest rate increases, and households increase capital accumulation. Now, recall the economic intuition behind Corollary 1: under indeterminacy, the outward shift of labor demands (driven by an increase of aggregate capital stock) should offset the initial inward shit (triggered by the desired higher consumption), as the economy enters an expansion (equilibrium (3) in the figure: $N_1^{***}$ and $N_2^{***}$).

This suggests that, under indeterminacy, the increase in capital stock offsets the initial decrease in the labor demands, and that labor demands are shifted outward. Eventually wages (and $r$) increase, as well as equilibrium levels of capital and both labor services. The overall increase in inputs usage drives the economy into a self-fulfilling expansion.[16]

To conclude, notice that this mechanism is distinctive of this class of models with heterogenous labor services. In the standard one-sector model, indeed, an analogous increase in capital stock would work against the self fulfillment of the expansionary prophecies. That happens because of the upward sloping labor demand schedule. The increase in equilibrium capital stock would induce an inward shift in the labor demand schedule, pushing the economy into a recession.

# 4 Parametrization and dynamic responses

## 4.1 Parametrization

The model is then parameterized for the United states economy. We consider two types of labor services, skilled and unskilled, following the OECD definition (more details below). The system of equations we use to compute the dynamic equilibria of the model depends on a set of eleven parameters. **Six** pertain to household preferences, $(\psi_1, \psi_2, \xi, B_1, B_2, \beta)$, and **five** to technology (the private capital share $\alpha$, the unskilled labor share $\rho$, and the corresponding externality coefficients $\omega, \eta_1, \eta_2$, respectively).

---

$$\widehat{w}_t^2 = [(1+\omega)\alpha_0]\,\widehat{K}_t + [(1+\eta)\alpha_1]\,\widehat{N}_t^1 + [(1+\eta_2)(1-\alpha_0-\alpha_1)-1]\,\widehat{N}_t^2$$

and it is written as $w^2 = L_2^D\{N^1, N^2, K\}$ where $\frac{\partial L_2^D}{\partial N^2} < 0, \frac{\partial L_2^D}{\partial N^1} > 0$. Now, the initial fall in each sector equilibrium labor services (that is a movement along each sector demand schedule) induces a further reduction in each sector employment through an inward shift of demand schedules (that is a schedule shift, induced by a change in the other-sector equilibrium employment).

[16]It is useful to underline here that, for expositional purpose, the dynamics of labor market has been sequentially depicted. Being the model in discrete time, all events happens simultaneously within each time period.

**Skilled-unskilled labor** have been identified using OECD data for the U.S. economy[17]; according to these data, the average value (for the 1997-2000 period) of the share of total labor force with higher education (ISCED 5A6 - 5B) equals 34.03%, giving rise to a steady state ratio for $\left(\frac{N^1}{N^2}\right)^*$ of 1.94. The parameter $B_2$ is used for calibrating the ratio between unskilled and skilled workers to that value; precisely $B_2^* = 0.5$. Next, parameters $D$ and $B_1^*$ are set respectively to 0.8 and 0.2, consistently with Assumption 1. The remaining **preference parameters** $(\beta, \xi, \psi_1, \psi_2)$ are calibrated to $\beta^* = 0.984$, $\xi^* = 0.009$, $\psi_1^* = 0.7$, and $\psi_2^* = 0.01$.

**Technology parameters** $\alpha, \rho, \delta$ are calibrated as follows. The capital share $\alpha^*$ is set to 0.23, a standard calibration for a one sector economy with aggregate increasing returns (i.e. Farmer and Guo [5]). Papageorgiou [18] estimates a production function with skilled and unskilled labor components for the United States economy suggesting that the share of skilled labor $\alpha_2^*$ can be calibrated to 0.36, and the unskilled labor share $\alpha_1$ equals 0.41. Quarterly capital depreciation rate is set to $\delta = 0.025$. Next, the three, input-specific, **externality parameters** are set to $\omega^* = 0.26$, $\eta_1^* = 0.25$, $\eta_2^* = 0.72$. The overall degree of increasing returns equals 1.42.

For such parametrization, the model's attractor is a sink so that the linearized system (in reduced form and excluding the shocks) in capital and consumption is:

$$\begin{bmatrix} \widehat{K}_{t+1} \\ \widehat{C}_{t+1} \end{bmatrix} = \begin{bmatrix} 0.3098 & 2.3204 \\ -0.1346 & 1.2649 \end{bmatrix} \begin{bmatrix} \widehat{K}_t \\ \widehat{C}_t \end{bmatrix}.$$

The dynamical model has two complex conjugated eigenvalues: the two roots equals $0.7873 + 0.2903i$ and $0.78735 - 0.2903i$, thus the system's attractor is a sink.

### 4.2 Dynamic responses

The next pages presents the impulse response functions following an i.i.d. sunspot shock. **Figure 2** below includes the dynamic response to consumption and capital (upper panel), output, investment, and consumption (middle panel), skilled labor, unskilled labor and total employment (lower panel).

A sunspot shock leads to an increase in capital, consumption, equilibrium employment and production output, consistently with the theoretical mechanism detailed in section 3.4. In addition, both labor types of labor services increase, but the skilled component increases

[17] Data source: OECD [17], table 4 Labor Force Statistics by educational attaintment (for the United States). List of time series: ISCED 0/1 Series Name U17 E0 2032; ISCED 2 Series Name U17 E0 2232; ISCED 3A Series Name U17 E0 2432; ISCED 5A/6 Series Name U17 E0 2B32; ISCED 5B Series Name U17 E0 2C32.

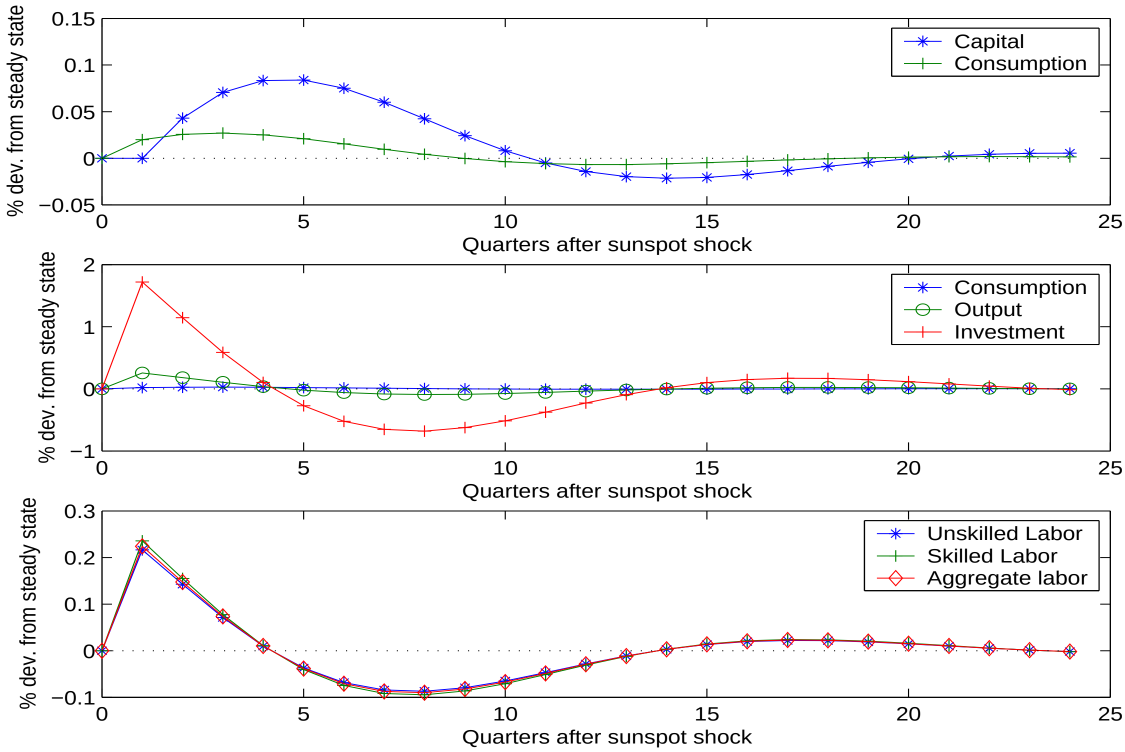


Figure 2: The figure shows the first 24 quarter response of capital and consumption (upper panel), consumption, output and investment flow (middle panel), skilled labor, unskilled labor and aggregate labor (lower panel) to an i.i.d. sunspot shock. The curves are the quarterly percentage deviations from a baseline scenario where all innovations are set to zero.

relatively more than the unskilled counterparts. This recomposition of equilibrium labor services raise the capital productivity and trigger, by this end, capital accumulation. This is confirmed by comparing the labor responses (lower panel) with the capital response (upper panel); a casual inspection suggests that the capital stock lags the increase in labor services by two/three quarters.

**Figure 3**, next, completes the picture presenting dynamic responses of prices: wage rates together with final output (upper panel) and interest rate together with investment flow.

Both wage rates increase after the sunspot shock. This is consistent with the mechanism. Recall, from Figure 1, that a sunspot shock shifts upward both labor supplies (before triggering the labor demand dynamics), raising, by this end, the equilibrium wage rate. The fact that the proposed mechanism is in place is confirmed by the fact that the economy

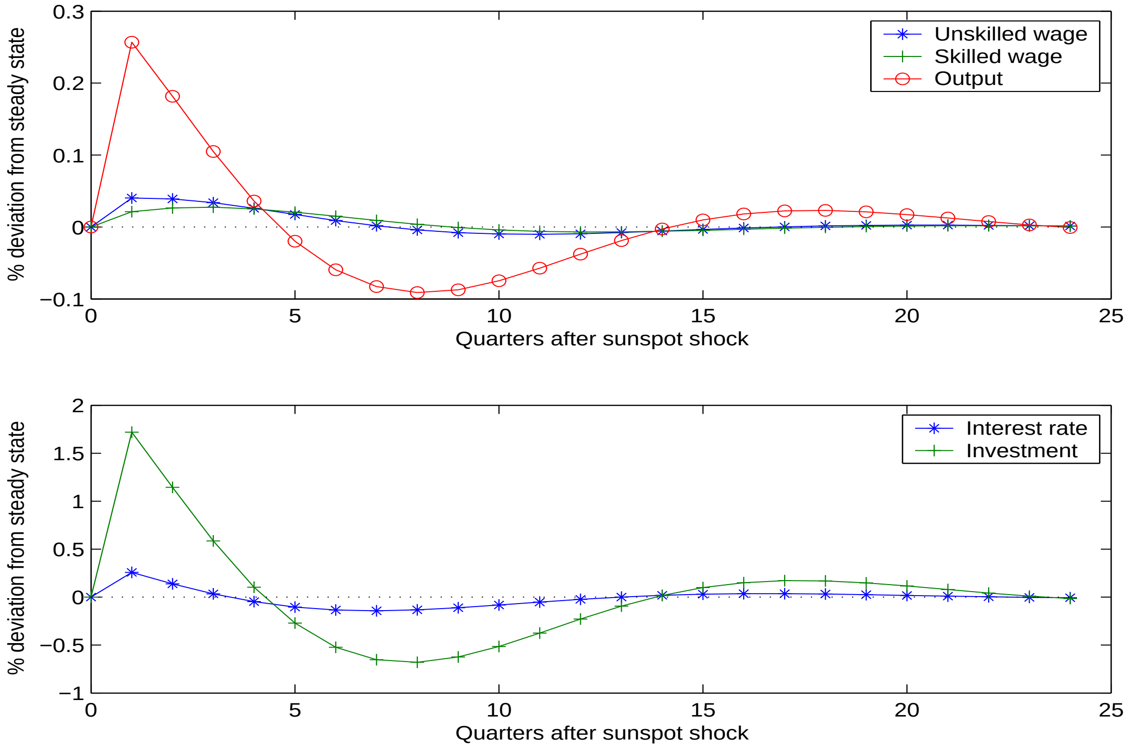


Figure 3: The figure shows the first 24 quarter response of unskilled wage, skilled wage and output (upper panel), interest rate and investment (lower panel) to an i.i.d. sunspot shock. The curves are the quarterly percentage deviations from a baseline scenario where all innovations are set to zero.

expands. Indeed, if it were not operating, the economy would enter into a recession, being the labor demand downward sloping. Interest rate increases at the impact, and then decays following a non monotonic pattern, differently from the wage rate. This is a consequence of the animal spirit hypothesis that generates the endogenous cycles.

## 4.3 Selected empirical evidence

It is finally interesting to casually inspect the cyclical behavior of skilled and unskilled labor services for the United States economy, at the business cycle frequencies. We aim to understand, looking at selected stylized facts, whether the data would support the theoretical mechanism operating in the model. We mainly focus on the contemporaneous correlation among the cyclical component of four classes of differently skilled workers and the GDP (**Table 1**). The cyclical component has been extracted using the Baxter-King band pass

Table 1: Correlation coefficients

| Skill (1) | Skill (2) | Skill (3) | Skill (4) | GDP | |
|---|---|---|---|---|---|
| 1.0000 | 0.5607 | 0.3768 | −0.5554 | −0.0752 | Skill (1) |
| | 1.0000 | 0.6104 | 0.1916 | 0.6138 | Skill (2) |
| | | 1.0000 | 0.2480 | 0.6545 | Skill (3) |
| | | | 1.0000 | 0.7999 | Skill (4) |
| | | | | 1.0000 | GDP |

Table 1: Correlation coefficients, using the observations 1995:1–2005:2 5% critical value (two-tailed) = 0.3044 for n = 42; Skill (1): less than high school diploma; Skill (2): high school graduates no college; Skill (3): some college or associate degree; Skill (4): BA degree and higher. Sources: OECD [17], table 4 Labor Force Statistics by educational attaintment (for the United States) and Authors' calculations.

filter.[18]

Table 1 suggests that the higher the skill distance among classes of workers, the more different are the cyclical properties. Specifically, the correlation between relatively more skilled workers and the GDP and among them is large and positive (i.e. correlation between workers with some college or associate degree (Skill 3) and workers with BA degree and higher Skill (4) is about 0.24; the correlation between workers with some college or associate degree (Skill 3) and the GDP is 0.65, and between workers with BA degree and higher Skill (4) and the GDP is about 0.80).

On the other hand, there exists a negative correlation between low skilled workers and the GDP (about -0.10) and a strong negative correlation between low skilled and high skilled workers (about -0.55).

In summary, it can be concluded that the data would support the existence of some recomposition of aggregate labor demand over the business cycle toward relatively more skilled workers. Of course, this section represents a casual glance at the relationships among skilled, unskilled, the GDP over the business cycle; we leave further developments to future works.

# 5 Conclusions

This paper explores the ability of a class of one-sector, multi-input models to generate indeterminate equilibrium paths, and endogenous cycles, without relying on factors' hoarding. In particular, we consider a one sector economy in which there exist one type of capital stock,

[18]The frequency bounds for the Baxter–King bandpass filter are set to 8 and 32, and the approximation order for the Baxter–King bandpass filter is set to 8.

but a finite number of heterogenous labor services, which are assumed to be heterogeneous along the skill/productivity dimension. What matters is the heterogeneity itself, and it is possible to obtain qualitatively analogous results for different kinds of heterogeneity (i.e. distinguishing between regular and underground labor services, or between labor services spatially separated). The model's formulation is quite general and it can can be applied to explain endogenous fluctuations of skilled and unskilled workers in bad and good times under indeterminacy, and to understand how labor service reallocation has an aggregate impact over the economy.

The model presents a novel theoretical economic mechanism that supports sunspot-driven expansions without requiring upward sloping labor demand schedules. The proposed mechanism differs from the customary one, and we consider it complementary to that one. Its distinctive characteristic is that the skill composition of aggregate labor demand drives expansionary i.i.d. demand shocks, and that there exists a "*composition*" effect (more details to come). A casual inspection of data for the United States economy supports the theoretical mechanism proposed in the paper.

It is finally worth to mention that the model presents an effective shock propagation mechanism that operates into the labor market and across labor market segments through the cross elasticities of equilibrium labor demand and supplies. In this respect the model can be seen as quite general formulation (with or without aggregate increasing returns to scale) for analyzing selected labor market question within a dynamic general equilibrium model with labor market segmentation.

# References


[1] Basu, S. Fernald, J. (1997) ”Returns to Scale in U.S.Production: Estimates and Implications”, *Journal of Political Economy*, 105, 249-83.

[2] Benhabib, J. Farmer, R. (1994) ”Indeterminacy and Increasing Returns”, *Journal of Economic Theory*, 63, 19-41.

[3] Benhabib, J. Farmer, R. (1996) ”Indeterminacy and Sector Specific Externalities”, *Journal of Monetary Economics*, 371, 421-443.

[4] Busato, F. Chiarini, B. (2004) ”Market and underground activities in a two-sector dynamic equilibrium model” , *Economic Theory*, 23, 831-861.

[5] Farmer, R. Guo, J. (1994) ”Real Business Cycle and the Animal Spirit Hypothesis”, J*ournal of Economic Theory*, 63, 42-72.

[6] Farmer, R. (1999) ”The macroeconomics of self-fulfilling prophecies”, Cambridge, MIT Press.

[7] Gandolfo, G. (1998) ”Economic Dynamics; $2^{nd}$ edition”,Berlin, Springer.

[8] Guckenheimer, J. Holmes P. (1983) ”Nonlinear osclillations ,dynamical systems and bifurcations of vector fields”, Berlin, Spinger-Verlag.

[9] Guo, J. Harrison, S. (2001) ”Indeterminacy with Capital Utilization and Sector-Specific Externalities,” Economics Letters, 72, 355-360.

[10] Guo, J., Lansing. K. (2004) ”Maintenance Labor and Indeterminacy under Increasing Returns to Scale”, mimeo.

[11] Hodrick, R. Prescott, E. (1980) ”Postwar U.S. Business Cycle”, Discussion Paper 451, Carnagie-Mellon University.

[12] Iooss, G. (1979) ”Bifurcations of maps and applications”, Amsterdam, North Holland.

[13] Jimenez, M. Marchetti, D. (2002) ”Interpreting the procyclical productivity of manufacturing sectors: can we really rule out external effects?”, *Applied Economics*, 34, 805-817.

[14] Kim, J. (1997) ”Three Sources of Increasinf Returns to Scale”, mimeo.

[15] Lorenz, H. W. (1993) ”Nonlinear dynamical economics and chaotic motion”, Berlin Springer-Verlag.

[16] Muir, T. (1960) A Treatise on the Theory of Determinants. New York: Dover.

[17] OECD (2004) "Statistical Compendium 2004/1", Paris, OECD.

[18] Papageorgiou, C. (2001) "Distinguishing between the effects of primary and post-primary education on economic growth", mimeo.

[19] Perli, R. (1998) "Indeterminacy, Home Production and the Business Cycle: a Calibrated Analysis", *Journal of Monetary Economics*, 41, 105-125.

[20] Romer, P. (1986) "Increasing Returns and Long Run Growth", *Journal of Political Economy*, 94, 1002-1037.

[21] Sbordone, A.. (1997) "Interpreting the procyclical productivity of manufacturing sectors: external effects or labour hoarding?", *Journal of Money, Credit and Banking*, 29, 26-45.

[22] Weder, M. (2003) "On the Plausibility of Sunspot Equilibria", *Research in Economics*, 57, 65-81.

[23] Wen, Y. (1998) "Capacity Utilization under Increasing Returns to Scale", mimeo.

# Appendix: Proof of theorem 1.

**Preliminaries.** The first step is to build a $2 \times 2$ dynamical system in the variables $\widehat{K}_t$ and $\widehat{C}_t$ from the linearized equilibrium conditions. Consider the labor demand and supply functions:

$$\begin{aligned}
\text{demand for } \widehat{K} \quad &: \quad \widehat{r}_t = [(1+\omega)\alpha_0 - 1]\,\widehat{K}_t + \sum_{j=1}^{M} [(1+\eta_j)\alpha_j]\,\widehat{N}_t^j \\
\text{demand for } \widehat{N}^h \quad &: \quad \widehat{w}_t^h = [(1+\omega)\alpha_0]\,\widehat{K}_t + \sum_{j \neq h}^{M} [(1+\eta_j)\alpha_j]\,\widehat{N}_t^j + [(1+\eta_h)\alpha_h - 1]\,\widehat{N}_t^h \\
\text{supply for } \widehat{N}^h \quad &: \quad \widehat{w}_t^h = (\psi_h)\,\widehat{N}_t^h + \widehat{C}_t
\end{aligned}$$

The difference between the wages of two different types of labor $j$ and $h$ is, from the demand side: $\widehat{w}_t^j - \widehat{w}_t^h = \widehat{N}_t^h - \widehat{N}_t^j$, and from the supply functions: $\widehat{w}_t^j - \widehat{w}_t^h = \psi_j \widehat{N}_t^j - \psi_h \widehat{N}_t^h$. By merging together the two equations, we obtain:

$$\widehat{N}_t^j = \left( \frac{1+\psi_h}{1+\psi_j} \right) \widehat{N}_t^h$$

The latter relationship holds in equilibrium for each pair of labor input:

$$\begin{aligned}
\widehat{N}_t^1 \quad &= \quad \left( \frac{1+\psi_h}{1+\psi_1} \right) \widehat{N}_t^h \\
&\vdots \\
\widehat{N}_t^M \quad &= \quad \left( \frac{1+\psi_h}{1+\psi_M} \right) \widehat{N}_t^h
\end{aligned} \tag{6}$$

The equilibrium value of $\widehat{N}_t^h$ can then be obtained by equating supply and demand and by solving for $\widehat{N}_t^h$:

$$\widehat{N}_t^h = [(1+\psi_h)(\Phi - 1)]^{-1}\,\widehat{C}_t - \left[ \frac{(1+\omega)\alpha_0}{(1+\psi_h)(\Phi - 1)} \right] \widehat{K}_t \tag{7}$$

with $\Phi = \sum_{h=1}^{M} \frac{(1+\eta_h)\alpha_h}{1+\psi_h}$. Equation (7) is a general form solution for each type of labor $h = 1, ..., M$ and by using it, the Euler equation and the budget constraint can be reduced to the following $2 \times 2$ dynamical system in $\widehat{K}$ and $\widehat{C}$:

$$\begin{bmatrix} \widehat{K}_{t+1} \\ \widehat{C}_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{s_I}J_3 & -\frac{1}{s_I}J_4 \\ \frac{J_1}{J_2 s_I}J_3 & \frac{1}{J_2} - \frac{J_1}{J_2 s_I}J_4 \end{bmatrix} \begin{bmatrix} \widehat{K}_t \\ \widehat{C}_t \end{bmatrix} \tag{8}$$

where the $J's$ are defined below. Define the set of our model parameters by $\mathbf{P}$, and a continuous mapping $\varphi(\mathbf{P})$ such that[19]: $\varphi(\mathbf{P}) : \mathbf{P} \mapsto \Re$.

$$\begin{aligned} J_1 &= [1-\beta(1-\delta)]\{\alpha_0(1+\omega)[1+\varphi(\mathbf{P})]-1\}; \\ J_2 &= 1+[1-\beta(1-\delta)]\varphi(\mathbf{P}); \\ J_3 &= \delta\alpha_0(1+\omega)(1+\varphi(\mathbf{P}))+(1-\delta)s_I; \\ J_4 &= \delta\varphi(\mathbf{P})+\delta s_C. \end{aligned}$$

where $\varphi(\mathbf{P}) = \frac{\Phi}{1-\Phi}$, $s_C = \frac{C^*}{Y^*}$, $\frac{s_I}{\delta} = \frac{K^*}{Y^*}$ and $s_C + s_I = 1$ (starred variables indicates the steady state values).

Gandolfo [7] (chapter 5) states necessary and sufficient conditions for a discrete dynamical system (like system (8)) to display local indeterminacy of the equilibrium path. In terms of our notation, they read:

$$\frac{(J_3 - s_I)(1-J_2) + J_1 J_4}{s_I J_2} > 0 \tag{9}$$

$$\frac{(J_3 + s_I)(1+J_2) - J_1 J_4}{s_I J_2} > 0 \tag{10}$$

$$\frac{s_I J_2 - J_3}{s_I J_2} > 0 \tag{11}$$

**Strategy.** To derive indeterminacy conditions in terms of the parameters of our interest we use a constructive argument, which is made of the following four steps.

- **Step 1.** Rewrite (9)-(11) in terms of $\varphi(\mathbf{P}) : \mathbf{P} \mapsto \Re$;
- **Step 2**. Define two subsets of the reals $S_1 \subset \Re$ and $S_2 \subset \Re$ ($S_1 \cap S_2 = \varnothing$) in which the model display ($S_1$) and does not display ($S_2$) indeterminacy, respectively;
- **Step 3**. Show that the subset $S_1$ has a non-empty interior, and therefore that there exist parameters' values for which the stationary state is indeterminate;
- **Step 4.** Invert the function $\varphi(\mathbf{P})$ on the subset $S_1$, and derive, by this hand, conditions on the parameters $\mathbf{P}$ for the stationary state being indeterminate.

**Step 1**. Rewrite (9)-(11) in terms of $\varphi(\mathbf{P}) : \mathbf{P} \mapsto \Re$. Algebraic manipulations yield:

[19] Actually, the function $\varphi(\mathbf{P})$ depends only on the $\alpha_h$, the $\eta_h$ and the $\psi_h$.

**(I.)** $\frac{\delta(1-s_I)[\beta(1-\delta)-1]\{(1-\alpha_0(1+\omega))(\varphi(\mathbf{P}))+1-\alpha_0(1+\omega)\}}{s_I+s_I(1-\beta(1-\delta))(\varphi(\mathbf{P}))} > 0.$

**(II.)** $\frac{\{\delta(1-s_I)[1-\beta(1-\delta)](1-\alpha(1+\omega))+2[\delta\alpha_0(1+\omega)+s_I(1-\beta(1-\delta))]\}(\varphi(\mathbf{P}))}{s_I+s_I(1-\beta(1-\delta))(\varphi(\mathbf{P}))}$

$+\frac{\delta(1-s_I)[1-\beta(1-\delta)](1-\alpha(1+\omega))+2[\delta\alpha_0(1+\omega)+s_I(2-\delta)]}{s_I+s_I(1-\beta(1-\delta))(\varphi(\mathbf{P}))} > 0.$

**(III.)** $\frac{[s_I(1-\beta(1-\delta))-\delta\alpha_0(1+\omega)](\varphi(\mathbf{P}))+\delta[s_I-\alpha_0(1+\omega)]}{s_I+s_I(1-\beta(1-\delta))(\varphi(\mathbf{P}))} > 0.$

**Step 2**. Conditions **(I.)-(III.)** define a system of inequalities, which, in turns, defines two subsets of the reals $S_1 \subset \Re$ and $S_2 \subset \Re$ ($S_1 \cap S_2 = \varnothing$) defined as follows:

- $S_1 \subset \Re$: model displays indeterminacy (all inequalities **(I.)-(III.)** are satisfied);
- $S_2 \subset \Re$: model does not display indeterminacy (at least one inequality among **(I.)-(III.)** is not satisfied).

In other words, if $\varphi(\mathbf{P}) \in S_1$ the equilibrium is indeterminate, and if $\varphi(\mathbf{P}) \in S_2$ the equilibrium is not indeterminate.

**Step 3**. Notice that the conditions **(I.)-(III.)** share the same denominator, and they are all functions of $\varphi(\mathbf{P})$. Hence they are functions $\mathcal{C}_i : \Re \mapsto graph(\varphi(\mathbf{P})) \subseteq \Re$. The zeros of these functions are the values delimiting the intervals over which the conditions are (are not) simultaneously satisfied. They are:

$$\begin{aligned}
R_I^0 &= -1 \\
R_{II}^0 &= -\frac{\delta(1-s_I)\left[1-\beta(1-\delta)\right](1-\alpha_0(1+\omega))+2\left[\delta\alpha_0(1+\omega)+s_I(2-\delta)\right]}{\delta(1-s_I)\left[1-\beta(1-\delta)\right](1-\alpha_0(1+\omega))+2\left[\delta\alpha_0(1+\omega)+s_I(1-\beta(1-\delta))\right]} \\
R_{III}^0 &= -\frac{\delta s_I-\delta\alpha_0(1+\omega)}{s_I(1-\beta(1-\delta))-\delta\alpha_0(1+\omega)} \\
R_D^0 &= -\frac{1}{1-\beta(1-\delta)}
\end{aligned}$$

where $R_D^0$ denotes the zero of the common denominator. It is convenient to rewrite the conditions **(I.)-(III.)** in terms of the values delimiting the intervals $S_1$ and $S_2$. Algebraic manipulations yield to the following necessary and sufficient condition for indeterminacy:

$$\max\left(R_D^0, R_{II}^0\right) < R_{III}^0 < R_I^0 \qquad (\bigstar)$$

The following theorem, together with the analysis completed in 'Step 4' below, shows that there exists a non-empty set of parameters for which condition ($\bigstar$) is satisfied.

**Theorem 2 (Non Empty Parameter Space for Indeterminacy)** *The model equilibrium is locally indeterminate iff the following inequalities hold:*

$$\max\left(R_D^0, R_{II}^0\right) < R_{III}^0 < R_I^0.$$

**Proof.** We prove the theorem by proving the following preliminary claims.

**Claim 3** $R_D^0 < R_I^0$ *and* $R_{II}^0 < R_I^0$.

**Proof.** The first follows directly from $1-\beta(1-\delta) < 1$. Furthermore $R_{II}^0$ is always negative (for all parameters' values) and its denominator is always smaller than its numerator, as $2-\delta > 1-\beta(1-\delta)$ (in absolute values). So it must be $R_{II}^0 < R_I^0$. ■

**Claim 4** $R_{III}^0 < R_I^0$.

**Proof.** $R_{III}^0 < R_I^0$ implies that $\frac{\delta s_I - \delta\alpha_0(1+\omega)}{s_I(1-\beta(1-\delta))-\delta\alpha_0(1+\omega)} > 1$, and we show that this can happen *if and only if* $s_I(1-\beta(1-\delta)) < \delta\alpha_0(1+\omega)$; in fact, if this is true, then $\delta s_I < \delta\alpha_0(1+\omega)$, as $1-\beta(1-\delta) > \delta$. When this is the case, the numerator of $R_{III}^0$ is always negative and greater (in absolute value) than the denominator of $R_{III}^0$ (which is also negative). Thus $\frac{\delta s_I - \delta\alpha_0(1+\omega)}{s_I(1-\beta(1-\delta))-\delta\alpha_0(1+\omega)}$ is positive and greater than one. Now it is straightforward to show that the inequality $s_I(1-\beta(1-\delta)) < \delta\alpha_0(1+\omega)$ is always true; as $s_I = \frac{\alpha_0\beta\delta}{1-\beta(1-\delta)}$, we have that $\beta\delta\alpha_0 < \delta\alpha_0(1+\omega)$ always holds. So that $R_{III}^0 < R_I^0$. ■

**Claim 5** $R_D^0 < R_{III}^0$.

**Proof.** The inequality can be recast as: $\frac{\delta s_I - \delta\alpha_0(1+\omega)}{s_I(1-\beta(1-\delta))-\delta\alpha_0(1+\omega)} < \frac{1}{1-\beta(1-\delta)}$. Note that when the term $\delta\alpha_0(1+\omega)$ (which is always $\geqq 0$) is zero, the first fraction reduces to $\frac{1}{1-\beta(1-\delta)}\delta$ which is always smaller than $\frac{1}{1-\beta(1-\delta)}$. We show that when the term $\delta\alpha_0(1+\omega)$ increases, passing from zero to positive numbers, the fraction $\frac{\delta s_I - \delta\alpha_0(1+\omega)}{s_I(1-\beta(1-\delta))-\delta\alpha_0(1+\omega)}$ monotonically decreases, so that it must always be $R_D^0 < R_{III}^0$. Consider $\delta\alpha_0(1+\omega)$ as a function of $\omega$: when $\omega$ is equal to[20] $-1$, the fraction collapses to $\frac{1}{1-\beta(1-\delta)}\delta$; when $\omega$ increases, the term $\delta\alpha_0(1+\omega)$ monotonically increases. Now $\frac{d(-R_{III})}{d\omega} = \frac{-\delta\alpha_0}{s_I(1-\beta(1-\delta))-\delta\alpha_0(1+\omega)}\left[1 - \frac{\delta s_I - \delta\alpha_0(1+\omega)}{s_I(1-\beta(1-\delta))-\delta\alpha_0(1+\omega)}\right]$. We have seen before that $-\frac{\delta s_I - \delta\alpha_0(1+\omega)}{s_I(1-\beta(1-\delta))-\delta\alpha_0(1+\omega)} < -1$; but then it is $\frac{d(-R_{III})}{d\omega} < 0$ for all the parameters' values. Thus $R_D^0 < R_{III}^0$. ■

**Claim 6** $R_{II}^0 < R_{III}^0$.

[20] Obviously $\omega < 0$ is not an interesting case in our model (it could be interpreted as a *negative* exeternality at system level), but for the sake of the argument it can be accepted just to see what is the effect on the fraction of the term $\delta\alpha_0(1+\omega)$ when the latter is arbitrary small.

**Proof.** We demonstrate this inequality by contradiction. Assume that $R^0_{III} < R^0_{II}$; given the inequalities demonstrated above, two cases are possible: either $R^0_{II} < R^0_D$, or $R^0_D < R^0_{II}$; the first one is clearly impossible, as it would imply that $R^0_{III} < R^0_{II} < R^0_D$ and we have seen that it is $R^0_D < R^0_{III}$. Next, consider the second one: $R^0_D < R^0_{III} < R^0_{II}$; in this case the situation would be the one depicted in Figure 4.A (recall that the slope of Numerator **(II.)** is always positive). In the interval $\left(R^0_{II}; R^0_I\right)$ indeterminacy is impossible, as Numerator **(III.)** $< 0$ and Denominator $> 0$; this is also true in the interval $(R_D; R_{II})$, as Numerator **(II.)** $< 0$, Denominator $> 0$, and in the regions outside the two intervals. Thus the unique ordering compatible with indeterminacy is $R^0_{II} < R^0_{III}$. ■

**Claim 7** *$R^0_{II} < R^0_D$ and $R^0_D < R^0_{II}$ are possible and compatible with an interval for $S_1$ being non-empty.*

**Proof.** The order between $R^0_{II}$ and $R^0_D$ does not affect the existence of a non-empty parameter space for indeterminacy of equilibrium, as Figure 4.B) illustrates. ■

The interval $\max\left(R^0_D, R^0_{II}\right) < R^0_{III} < R^0_I$ is thus a viable region for indeterminacy, as for all the values of $\varphi(\mathbf{P})$ falling in this region, the necessary and sufficient conditions for indeterminacy **(I.)-(III.)** are satisfied. In summary, we have demonstrated that for having indeterminacy the following inequalities must hold: $R^0_D < R^0_I$, $R^0_{II} < R^0_I$, $R^0_{III} < R^0_I$, $R^0_D < R^0_{III}$, $R^0_{II} < R^0_{III}$, $R^0_{II} < R^0_D$ or $R^0_D < R^0_{II}$. By merging all these inequalities together, the orderings compatible with an indeterminacy region turn out to be $R^0_D < R^0_{II} < R^0_{III} < R^0_I$ and/or $R^0_{II} < R^0_D < R^0_{III} < R^0_I$. This completes the proof of theorem (2). ■

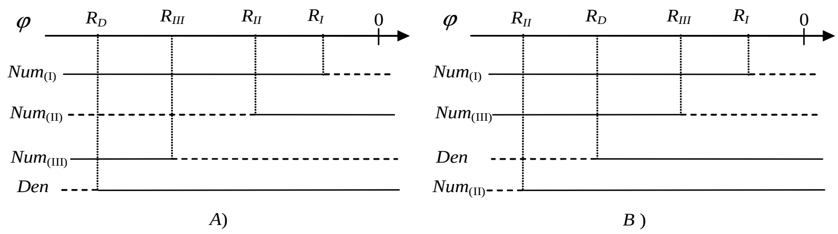


Figure 4: **Auxiliary intervals**: dotted lines represent negative values of the correspondig function of $\varphi(\mathbf{P})$, (i.e. of the three numerators of **(I.)-(III.)** and the common denumerator) while solid lines represent positive values.

<u>**Step 4**</u>. So we have two possible orderings defining the indeterminacy region; one is given by the interval $(R^0_{II}; R^0_{III})$, or:

$$-\underline{\mathcal{R}} < \varphi(\mathbf{P}) < -\overline{\mathcal{R}}, \tag{12}$$

where:

$$
\begin{aligned}
\underline{\mathcal{R}} &= \frac{\delta(1-s_I)\left[1-\beta(1-\delta)\right](1-\alpha_0(1+\omega)) + 2\left[\delta\alpha_0(1+\omega) + s_I(2-\delta)\right]}{\delta(1-s_I)\left[1-\beta(1-\delta)\right](1-\alpha_0(1+\omega)) + 2\left[\delta\alpha_0(1+\omega) + s_I(1-\beta(1-\delta))\right]} \\
\overline{\mathcal{R}} &= \frac{\delta s_I - \delta\alpha_0(1+\omega)}{s_I(1-\beta(1-\delta)) - \delta\alpha_0(1+\omega)}
\end{aligned}
$$

The other one is given by $\left(R_D^0, R_{III}^0\right)$, or:

$$
-\frac{1}{1-\beta(1-\delta)} < \varphi(\mathbf{P}) < -\overline{\mathcal{R}} \tag{13}
$$

Both the previous conditions suggests that for having indeterminacy, the ratio $\frac{\Phi}{1-\Phi} = \varphi(\mathbf{P})$ must be negative, larger (in modulus) than one and finally included between two specific values. Putting together (12) and (13) and solving for $\Phi$, the necessary and sufficient condition (NSC) in the main text can be obtained. This completes the proof of Theorem 1.

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