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Pre-Auction Offers in Asymmetric
First-Price and Second-Price Auctions

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Abstract

We consider "must-sell" auctions with asymmetric buyers. First, we study auctions with two asymmetric buyers, where the distribution of valuations of the strong buyer is "stretched" relative to that of the weak buyer. Then, it is known that *inefficient* first-price auctions are more profitable for the seller than *efficient* second-price auctions. This is because the former *favor* the weak buyer. However, we show that the seller can do one better by augmenting the first-price auction by a pre-auction offer made exclusively to the strong buyer. Should the strong buyer reject the offer, the object is simply sold in an ordinary first-price auction. The result is driven by the fact that the unmodified first-price auction is *too favorable* to the weak buyer, and that the pre-auction offer allows some correction of this to the benefit of the seller. Secondly, we show quite generally that pre-auction offers never increase the profitability of second-price auctions, since they introduce the wrong kind of favoritism from the perspective of seller profits.

Keywords: first-price and second-price auctions, asymmetric bidders, pre-auction offers. JEL: D02, D44, D82.

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1 Introduction

A seller of a unique item is often confronted by two "problems". On the one hand, he *must sell*, and, on the other, he faces *heterogenous potential buyers* with unknown valuations of the item. In such a setting, simply posting a price may well be counterproductive as well as non-credible, since buyers will conclude that if no one takes the posted price, some kind of negotiation or auction-like mechanism will subsequently be used by the seller to allocate the item. Also, since the seller is unable to commit not to sell the item, the revenue maximizing mechanism, in which trading occurs with a probability strictly less than one, is precluded. Hence, whatever mechanism the seller tries to set up, it must ultimately involve trading with probability one. Such mechanisms are the object of study in the present paper.

In the *symmetric* independent private-values setting, it is well-know that any of the *efficient*, standard must-sell auction formats are, in fact, revenue maximizing in the class of all must-sell mechanisms (see Bulow & Klemperer (1996) and Kirkegaard (2006)). Hence, in order to raise (expected) revenue, some inefficiency must be induced through probabilistic withholding of the item.[1] Thus, if withholding is ruled out, the seller cannot improve upon the standard auction formats, when buyer valuations are unknown but drawn from the same distribution. In contrast, if the symmetry assumption is dropped, it is also well-known that the standard must-sell auction formats are no longer revenue equivalent (see Maskin & Riley (2000)). In fact, under certain conditions,[2] an *inefficient* first-price auction may revenue-dominate an *efficient* second-price format, when potential buyers are asymmetric. Since both standard auction formats are must-sell formats, the inefficiency in the first-price format is *not* related to withholding, but to mis-allocation, in the sense that the item may not be sold to the highest-valuation bidder. Hence, this type of inefficiency may work to the advantage of the seller, and this paper investigates how the seller may try to exploit this further, when potential buyers are identifiably heterogenous *ex ante*. Thus, we assume that the seller is able to identify different types of bidders, though not the actual valuation of any particular (type of) bidder.

Let us give a few of examples of what we have in mind. First, when a (local) government auctions off the rights to collect garbage or provide bus

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[1] In the symmetric setting, any standard format augmented by a suitably chosen reserve price implements the optimal transfers and trading from the perspective of seller revenue.

[2] For more on this, see below.

transportation in a certain area, it is often possible to identify whether particular bidders already provide similar services elsewhere or whether they are "greenfield" entrants. Also, bidders know that the contract must (ultimately) be offered to someone with probability one. Similarly, in a liquidation sale of an estate including artwork, silverware and antique furniture, the seller may be able to identify professional and private buyers. Again, all potential buyers may know that the estate must be liquidated. Finally, in a takeover contest, on the face of it there may be an obvious acquirer (e.g., a firm in a similar (or, complementary) line of business with which the management or the board of the target firm has close ties). That is, there may be a strong potential buyer. However, there may also be a set of alternative potential acquirers, that is, weak potential buyers in our terminology. In addition, once a takeover contest has been initiated, all potential buyers may surmise that an eventual takeover is a sure thing.

In the proposed model-setting, the seller can approach a particular potential buyer and make him a *take-it-or-leave-it* offer. However, if the offer is turned down, it is understood by all the parties that the item will subsequently be sold with probability one in some mechanism. Hence, we essentially introduce the possibility of making pre-auctions offers to particular buyers before some type of must-sell auction is staged among the asymmetric bidders. Of course, if the pre-auction offer is accepted, the trading mechanism never progresses to the auction stage.[3] We study the revenue effects of such pre-auction offers when the auction stage is comprised of either a first-price auction or a second-price auction.

In the first-price auction, we identify two conflicting forces which influence the profitability of a pre-auction offer to the *strong* buyer. On the one hand, the pre-auction offer implies that the strong buyer is more likely to win if his valuation is high, and this tends to *increase* revenue. On the other hand, if the strong buyer rejects the offer, thereby revealing his valuation is not too high, the outcome in the auction changes because the weak buyer bids less aggressively than without the pre-auction offer. This implies that a strong buyer with an intermediate valuation is also more likely to win more often, which will tend to *reduce* revenue. The first part of Section 2 below is devoted to explaining this trade-off. Incidentally, we notice that introducing a pre-auction offer may improve efficiency in the first-price auction.

However, since the analysis of asymmetric first-price auctions is notori-

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[3]Pre-auction offers and their acceptance are legally binding, and there is no default.

ously difficult,[4] we use a simple, *two-bidder* example to capture the effects at play when allowing a pre-auction offer. Specifically, both buyers have a valuation drawn from a *uniform* distribution, but the distributions are "stretched" over different supports. In this environment, the weak buyer bids more aggressively than the strong buyer in a first-price auction. Hence, the weak buyer occasionally wins when efficiency deems that he should not. Though the first-price auction is inefficient, it can also be shown that it yields higher (expected) revenue than a second-price auction. While the trade-off from introducing a pre-auction offer is general, the use of a specific example allows us to prove that the positive factor may outweigh the negative, implying that a suitably chosen pre-auction offer to the strong buyer improves revenue in the first-price setting.[5] Heuristically, this result is driven by the fact that the first-price auction is *too favorable* to the weak buyer, and that the introduction of a pre-auction offer to the strong bidder allows some correction of this to the benefit of the seller. In contrast, we show quite generally[6] that pre-auction offers can never increase the profitability of efficient second-price auctions, since they introduce the wrong kind of favoritism and, thus, the wrong kind of inefficiency from the perspective of seller profits.

The existing literature on pre-auction offers in auction-like mechanisms is scant. Bulow & Klemperer (1996, p. 189) remark that pre-auction offers are not profitable in the *symmetric* case, when a rejection of the offer is followed by a must-sell auction.[7] This is, of course, immediately relevant for the takeover contest alluded to above, when potential buyers are symmetric *ex ante*.[8] The literature on *buy-outs* in auctions is also of some relevance

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[4]In particular, the first-order conditions of bidder optimization generally give rise to a system of differential equations which eludes explicit solution, save in special cases (see Plum (1992), Lebrun (1999), Maskin & Riley (2000) and Krishna (2002, Ch. 4)).

[5]While a similar trade-off would be at work, were we to give a pre-auction offer to the weak buyer, we shall argue that this makes little sense with the type of asymmetry assumed here.

[6]That is, for an arbitrary number of asymmetric bidders and without making particular distributional assumptions.

[7]However, Ivanova-Stenzel & Kröger (2005) suggest that pre-auction offers may raise profits in the symmetric case when bidders are risk averse.

[8]Bulow & Klemperer explicitly relate their results to U.S. takeover law. There, company boards are required to show due diligence with respect to the maximization of shareholder value before entering into *exclusive negotiation* with a single potential buyer. Their main result is that effort is better spent looking for more buyers, to increase competition, rather than negotiating exclusively with one buyer. See Kirkegaard (2006) for an alternative and short proof of this result. Though related, our focus is different, in that we assume

for this paper.[9] Particularly in online auctions, sellers often stipulate a buy-out price, which will end the auction immediately, if *some* bidder accepts it. This has been motivated by risk aversion or impatience on the part of either sellers or buyers and by the increasing price paths in sequential auctions associated with multi-unit demands. On the surface, buy-out offers appear similar to pre-auction offers. However, buy-out offers are general and made to all potential buyers, whereas the pre-auction offers considered here are made exclusively to a particular potential buyer based on *ex ante* information on his type. The latter only makes sense, if the potential buyers are identifiably heterogenous, whereas the literature on buy-outs has (so far) assumed that buyers are homogenous *ex ante*.

At a more general level, this paper is related to the work of Bulow & Roberts (1989) and Bulow & Klemperer (1996), who developed the basic relationship between monopoly pricing and (optimal) auctions. In order to maximize profits, the monopolist will generally try to sell to buyers with the highest marginal revenues and only to those buyers whose marginal revenues exceed marginal cost. If marginal cost is taken to be the value for the seller of retaining an item for himself, then this immediately ties together optimal auction-reserves and discrimination between heterogenous bidders with third-degree price discrimination by a monopolist. Our results similarly trade on how adaptations of standard auction formats allow the seller to "manipulate" the marginal revenue of the marginal bidder to his own advantage.

Moreover, our work is closely related to the small number of papers that study first-price auctions with asymmetric bidders, notably Lebrun (1999), and Maskin & Riley (2000) (see also Kirkegaard (2005) and Krishna (2002, Ch. 4)). The example and the results derived on pre-auction offers in first-price auctions in this paper explicitly take as their point of departure the revenue rankings found by Maskin & Riley (2000).

The remainder is organized as follows. Section 2 first provides some fundamental intuitions on our key results.[10] Then, we turn our attention to (*two-bidder*) first-price auctions augmented by a pre-auction offer to the strong buyer. This shows how a suitably chosen pre-auction offer allows

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ex ante asymmetries between potential buyers, while exclusive pre-auction offers are made on a *take-it-or-leave-it* basis and always followed by a standard auction if rejected.

[9]See Budish & Takeyama (2001), Reynolds & Wooders (2003), Mathews (2003, 2004), Hidvégi, Wang & Whinston (2003) and Kirkegaard & Overgaard (2004).

[10]This is accomplished by drawing on the analogy between auctions and monopoly-pricing, as suggested by Bulow & Roberts (1989) and Bulow & Klemperer (1996).

a profitable correction of the outcome of the first-price mechanism.[11] Finally, we turn to second-price auctions and show, quite generally, that any pre-auction offer (irrespective of the offeree) will be self-defeating from the perspective of the seller.[12] Section 3 offers a few concluding remarks. Since little insight is gained from the details of the formal analysis, this has largely been relegated to the Appendix.

2   Good Efficiency, Bad Efficiency?

First, a thought experiment. Consider a monopolist with zero marginal costs, who is discriminating between two markets. In market i, willingness-to-pay ranges from 0 to i, $i = 1, 2$. Then, it is well understood that, under weak conditions, it is optimal to favor market 1, the "weak" market, with a lower price. This is exemplified in Fig. 1, where (inverse) demand curves are assumed to be linear (*heavy* lines are demands, while *thin* lines are marginal revenues).[13]
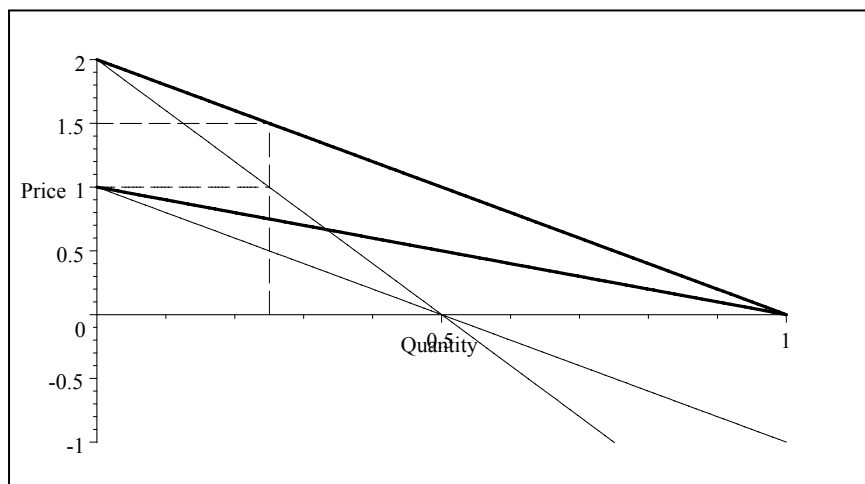

Fig.1. A weak and a strong market ("stretching")

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[11] That is, by awarding the item more often to the buyer with the high marginal revenue.

[12] Since the item tends to be awarded more often to buyers with low marginal revenues.

[13] Unless capacity is large and must be sold, a monopolist with zero marginal costs, would set prices 0.5 in the weak market and 1 in the strong market. These prices equate marginal revenue across markets and with marginal cost, and the monopolist would sell 0.5 units in each market.

Indeed, this conclusion continues to hold, even if the firm is somewhat capacity constrained. The existing capacity is spread across the two markets to equalize marginal revenue, implying that the price in the weak market must be lower. However, if the firm faces a stronger capacity constraint, it is clear that the weak market is not served at all, and that the entire capacity must be sold on the stronger market. In this case, and only in this case, will the allocation of goods be efficient. It must be pointed out that the assumption of a different least upper bound on the willingness-to-pay (vertical intercept) in the two markets is crucial for the latter result.

To summarize, the firm in question will decide to sell only on the strong market if capacity is very constrained, but will otherwise serve both markets, favoring the weak market.

Now, keeping this in mind, we turn to an auctioneer selling an indivisible good to one of two asymmetric buyers. The seller puts zero value on consuming the good himself, whereas buyers have privately known valuations. For the weak buyer, this is somewhere between 0 and 1, while the valuation of the strong buyer is between 0 and 2. Importantly, Bulow and Roberts (1989) have shown that the problem facing the auctioneer is essentially identical to the problem facing the monopolist we started out considering, with a possible capacity constraint.

If the auctioneer uses a second-price auction, the good is won by whoever values it the most. That is, the auction is efficient. In a first-price auction, however, the weak buyer will bid more aggressively than the stronger buyer, because, from his point of view, competition is stronger, see Lebrun (1999), Maskin and Riley (2000) or Krishna (2004, Ch. 4).

Fig. 2 illustrates the equilibrium bidding strategies in the first-price auction when valuations are drawn from uniform distributions corresponding to the demands in Fig. 1 (for details on the equilibrium bidding, see the Appendix).[14] In Fig. 2, the weak buyer's bidding function, $b_w(v)$, is *heavy*, while that of the strong buyer, $b_s(v)$, is *thin*. It is evident that the weak bidder does, indeed, bid more aggressively. Consequently, the weak buyer will win the auction more often than is efficient. Thus, while the efficient second-price auction can be compared to uniform pricing, the inefficient first-price auction is similar to price discrimination in favor of a weak market. This provides some intuition behind the result that the first-price auction is more profitable

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[14]This is one of the only cases in which equilibrium strategies, and equilibrium revenue, can be calculated explicitly for first-price auctions. See Plum (1992) for another example.
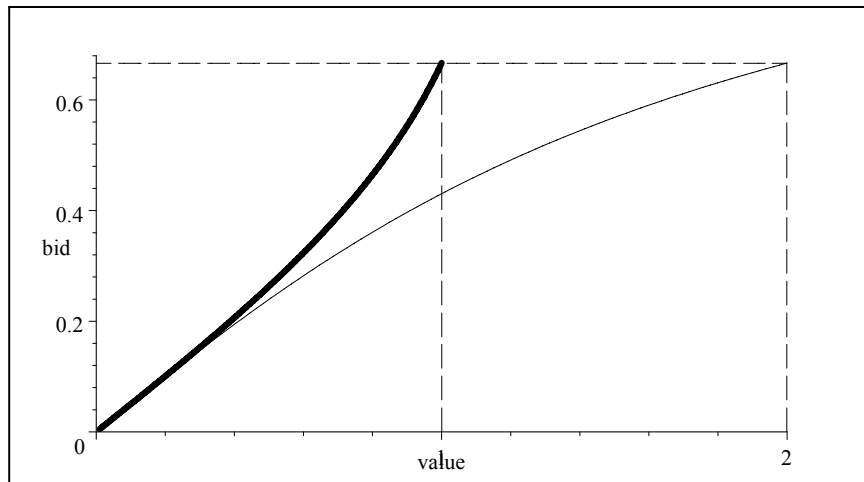
than the second-price auction.[15]

Fig. 2. Equilibrium bidding strategies

We notice that the maximum bid in the first-price auction is common to the bidders. In the example it is given by $b_w(1) = b_s(2) = \bar{b} = \frac{2}{3}$ (for details, see the Appendix). To see why this must be the case, suppose bidding strategies were increasing and that the maximum bids differed. Then, one of the bidders (the one with the higher maximum bid) would win with probability 1 for valuations above a certain threshold. But then, types of this bidder with valuation above the threshold could safely lower their bids somewhat and still win with probability 1. In a first-price auction this clearly increases bidder payoff, which contradicts that the putative strategy profile could be in equilibrium. Hence, maximum bids must coincide.

An alternative illustration of the mechanisms at work in this example can be provided by looking at the probability of winning as a function of valuation for the weak and strong bidder, respectively.[16] Denote these probabilities by $q_w(v)$ and $q_s(v)$, which we derive in the Appendix. Fig. 3 illustrates these for both the first-price and the second-price auction ($q_w(v)$ is *heavy*, and $q_s(v)$ is *thin*, while unbroken lines capture the first-price auction and broken lines

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[15] The proof of this, however, is not as straightforward. See Maskin and Riley (2000).

[16] We refer the reader to Kirkegaard (2005), who draws on abstract results from the mechanism design literature (notably, the implications of incentive compatibility for the probabilities of winning) to back out results for first-price auctions with asymmetric bidders.
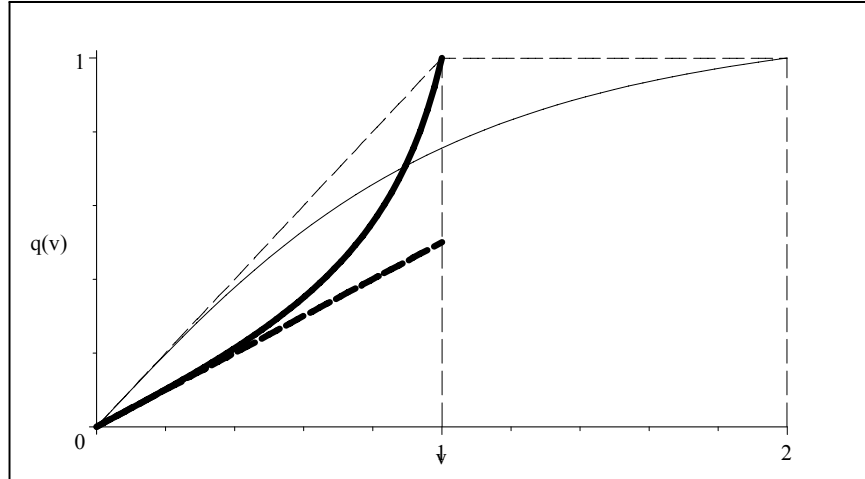
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capture the second-price auction).

Fig. 3. Probability of winning

First, consider the first-price auction. At low valuations the strong bidder
has a higher probability of winning compared to the weak bidder, while the
opposite is true at high valuations. Hence, while the weak bidder bids uni-
formly more aggressively, the fact that the distribution of the strong bidder
is "stretched" implies that this aggression is dominated by the strength of
the competition at "low" valuations. In contrast, in the second-price auction
bidders bid their values, and the strong bidder has a uniformly higher prob-
ability of winning, since he is facing weaker bidding competition. In other
words, the weak buyer is "favored" in the first-price auction, while this is not
the case in the second-price auction.

2.1 Pre-auction offers in first-price auctions

Despite the remarks above, the first-price auction is not ideal from the per-
spective of the auctioneer. The equilibrium in a first-price auction involves
buyers following strictly increasing bidding strategies, ranging from zero (or
the reserve price) to *a common maximal bid*. That is, the strong buyer with
valuation 2 would bid the same as the weak buyer with valuation 1. Clearly,
this implies that the weak buyer with valuation 1 wins more often than is
efficient. Though this may appear to be beneficial for the auctioneer, the fact
of the matter is that, while it is profitable to favor the weak buyer, it can be

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overdone. In Fig. 1, for example, the weak buyer should never be favored to such a degree that he outbids the strong buyer with a valuation above 1.5 (for which marginal revenue, or virtual valuation, of the strong buyer is 1).

To remedy this drawback, we propose a pre-auction offer to the *strong* buyer.[17] The strong buyer is given the choice of buying the good outright at the proposed price, or to reject, in which case a first-price auction is held. The offer should be so high that it appeals to the strong buyer only if his valuation is high (somewhere above 1.5). This allows "efficiency at the top" – the strong buyer wins for sure if his valuation is very high – while favoring the weak buyer when the strong buyer does not have a very high valuation.[18]

However, this mechanism has its weak spots as well. In particular, if the strong buyer rejects the pre-auction offer, the weak buyer infers that the strong buyer is not so strong after all. Therefore, the incentive to bid aggressively is diminished. While the weak buyer still bids more aggressively than the strong buyer, the difference declines. The weak buyer wins less often than without the pre-auction offer. Since the weak buyer is not favored as much, this raises the possibility that revenue decreases.

To examine the size of the two opposing forces on revenue by introducing a pre-auction offer, we assume in the following that valuations are drawn from uniform distributions over the aforementioned ranges. While most of the details are in the Appendix, we outline the analysis in the following. For any given pre-auction offer, p, it is easy to show that there is a unique *threshold equilibrium*, in which the strong buyer accepts the pre-auction offer if, and only if, his valuation exceeds some valuation, $z(p)$.[19] Thus, rather than deciding on a pre-auction offer, p, we will consider the choice of a threshold valuation, z, to target. The higher z is, the higher the pre-auction offer is, and the less likely it is that the offer is accepted. Fig. 4 depicts the expected revenue from any choice of z, $z \in [1.5, 2]$. If $z = 2$, the offer is never accepted, and the mechanism is essentially the unmodified first-price auction.

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[17] As noted in the Introduction, for the type of (targeted) pre-auction offer considered in this paper to make sense, we have to be able to identify *a priori* who in strong. Thus, the pre-auction offers considered in this paper are conceptually very different from the *buy-out prices* analysed in, e.g., Kirkegaard & Overgaard (2004). A buy-out price is more like a pre-auction offer made to all potential buyers in non-discriminatory fashion.

[18] Making a pre-auction offer to the weak buyer makes little sense, since this would introduce even more severe inefficiencies at the top.

[19] If p is rejected, the weak buyer infers that the strong buyer's valuation is below z. In the ensuing auction, the common maximal bid is exactly p.
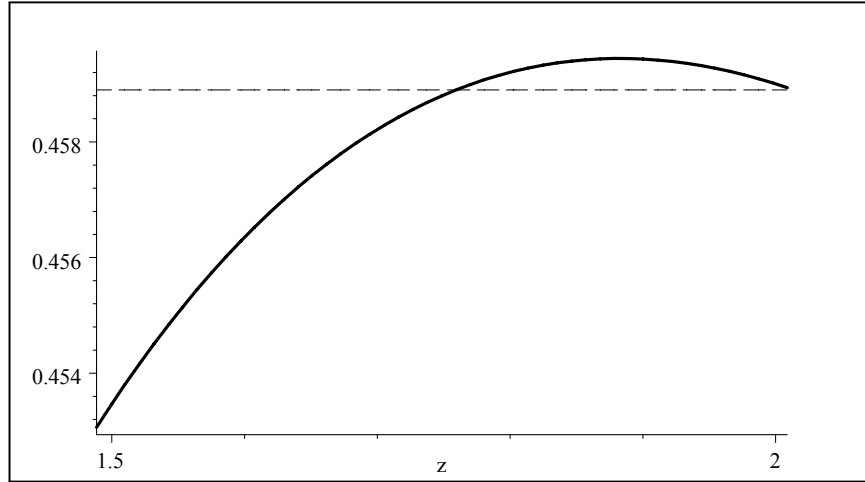
Fig. 4. Profit as a function of the threshold with z in $[1.5, 2]$

Fig. 4 reveals that the auctioneer can benefit from stipulating a pre-auction offer which is accepted with positive probability. The optimal threshold, z^*, is approximately 1.885, which is induced by a pre-auction price of $p^* \simeq 0.653$, while the probability of acceptance by the strong bidder is approximately 0.058. If the pre-auction offer is rejected, the common maximal bid in the auction is $p^* \simeq 0.653$, and Fig. 5 illustrates the bidding strategies (the dashed lines replicate the baseline case from Fig. 2 without a pre-auction offer to the strong bidder).
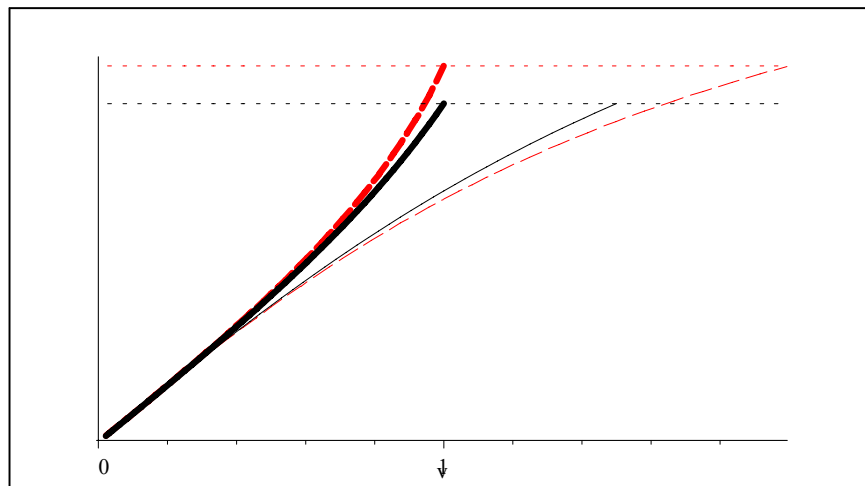

Fig. 5. Bidding in first-price auction with and without pre-auction offer

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With a pre-auction offer, we note that the weak buyer bids *less* aggressively in the auction after the strongest opponent types have been eliminated by taking the offer. This captures that, for the weak buyer, competition has become weaker, and it is expected to take less to win. In contrast, the remaining, "low" types of the strong buyer bid *more* aggressively. To see this, recall that in a first-price format, where the winner pays his bid, any bidder must weigh the decrease in payoff from winning against the increased probability of winning when the bid is raised. But here the increased density of opponent bids (due to the compression of the weak buyer's bidding interval) tilts this cost-benefit trade-off in favor of higher bids. Hence, the remaining types of the strong buyer bid more aggressively.

In addition to the fact that a pre-auction offer improves revenue, it is interesting to observe that the threshold should be strictly higher than 1.5, the point at which the strong buyer's marginal revenue (virtual valuation) enters the range of the weak buyer's marginal revenue from above. The reason is that by reducing the asymmetry too much in the auction following the offer, the weak buyer is favored too little.

2.2 Pre-auction offers in second-price auctions

Next, we show that pre-auction offers in second-price auctions will *always* decrease revenue. To illustrate, we start with another thought experiment. The monopolist depicted in Fig. 1 is deciding whether to sell his entire capacity to one particular market, rather than setting the same price in both markets (uniform pricing) to clear his capacity. The former choice allows some (extreme) discrimination between markets, whereas the latter is efficient and favors neither market. Despite this, it is easy to see that uniform pricing dominates exclusive dealing. The reason is quite simply that by combining the two markets into one (as under uniform pricing), willingness-to-pay for a given capacity is higher than if one deals only with one market.[20] For example, in Fig. 1, if capacity is 0.8, the choices are to sell everything on market 1 at a unit price of 0.2, to sell everything on market 2 at a unit price of 0.4, or to sell at a capacity clearing price of 0.8 across both markets. Uniform pricing is more profitable, as it yields a higher unit price.[21]

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[20] Notice that this argument does not rely on one market being stronger than the other.

[21] An alternative explanation might be useful. Assume the monopolist is *initially* selling his capacity of 0.8 to market 2. By reverting to uniform pricing, and a price of 0.8, he

Now, as suggested already, a second-price auction can be compared to uniform pricing, since no one is favored, and the buyer with the highest valuation wins (efficiency). Likewise, a pre-auction offer in a second-price auction is to some extent similar to exclusive dealing in a monopoly. If the buyer targeted accepts the offer, inefficiency may result because another buyer may have a higher valuation. Efficiency can be restored only by abolishing the pre-auction offer and treating all buyers the same.[22] As in the monopoly case, the efficient auction (without a pre-auction offer) is more profitable than favoring one particular buyer. A formal proof of this can be found in the Appendix. The result holds for any number of buyers, and does not require that one buyer is stronger than others.[23] Specifically, assume there are n risk neutral buyers. Buyer i draws a valuation independently from the distribution F_i on $[0, \overline{v}_i]$, $i = 1, ..., n$. F_i has no mass points, and is strictly increasing and continuously differentiable on $(0, \overline{v}_i)$. Then we can state.

Proposition 1 *A second-price auction preceded by a pre-auction offer is revenue-dominated by the straight second-price auction.*

3   Concluding Remarks

In this paper, we have considered auctions with asymmetric bidders. Then, it is well known that standard auctions are neither revenue equivalent nor necessarily efficient. Therefore, we analyzed whether pre-auction offers made exclusively to particular bidders raise revenue, and we showed that this can be the case when rejected offers are followed by a first-price auction, but never when rejected offers are followed by a second-price auction.[24] Moreover, a pre-auction offer may increase efficiency in the first price auction, whereas it will lead to a loss in efficiency in the second price auction.

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will sell only 0.6 on market 2, and the remaining 0.2 on market 1. The loss on market 2 is captured by the lost marginal revenue on 0.2 units. Since marginal revenue is below demand, this must be below 0.8 on each unit moved from market 2 to market 1. On the other hand, the gain is *on average* 0.8 for each of the units moved (since revenue on each unit is 0.8). Hence, the gain exceeds the loss.

[22]In contrast, abolishing pre-auction offers in first-price auctions does not restore efficiency.

[23]Also, the result remains valid in the presence of a reserve price (at least one below the pre-auction offer). Hence, we are not constrained to looking at "must sell" auctions.

[24]In contrast, under standard regularity and with symmetric buyers, Bulow & Klemperer (1996) show that pre-auction offers decrease revenue, *regardless* of the format.

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Appendix

First-price auctions. Assume the weak buyer's valuation is drawn from the uniform distribution over $[0,1]$, while the strong buyer's valuation is drawn from the uniform distribution over $[0,z]$, $z \geq 1$. Then, we know from the analysis of Maskin and Riley (2000) or Krishna (2002) that the highest bid is

$$\frac{z}{1+z}, \tag{A1}$$

while the bidding functions are

$$b_w(v;z) = \frac{z^2}{(z^2-1)v}\left(1 - \sqrt{1 - \frac{(z^2-1)}{z^2}v^2}\right), v \in [0,1]$$

$$b_s(v;z) = \frac{z^2}{(z^2-1)v}\left(\sqrt{1 + \frac{(z^2-1)}{z^2}v^2} - 1\right), v \in [0,z)$$

where w denotes the weak buyer, and s the strong buyer.

For example, when $z = 2$, we get $b_w(v) = \frac{2}{3v}(2 - \sqrt{4-3v^2})$ and $b_s(v) = \frac{2}{3v}(\sqrt{4+3v^2} - 2)$. These bidding functions are depicted in Fig. 2. Given these, the weak buyer with valuation v outbids the strong buyer if the strong buyer has a valuation below $\widetilde{v}$, where $\widetilde{v}$ solves $b_w(v) = b_s(\widetilde{v})$. This event has probability $q_w(v) = \frac{v}{\sqrt{4-3v^2}}$. Similarly, a strong bidder with valuation v wins with probability $q_s(v) = \frac{2v}{\sqrt{4+3v^2}}$. These winning probabilities are graphed in Fig. 3, alongside the winning probabilities for the buyers in a second-price auction.[25]

More generally, for any z, the inverse bid functions are

$$v_w(b) = \frac{2b}{1 + \left(1 - \frac{1}{z^2}\right)b^2}$$

$$v_s(b) = \frac{2b}{1 + \left(\frac{1}{z^2} - 1\right)b^2}.$$

Then, the probability that the winning bid is below b, $F_z(b)$, is the probability that both buyers have valuations below the valuation for which a bid of b would be submitted,

$$F_z(b) = \frac{2b}{1 + \left(1 - \frac{1}{z^2}\right)b^2} \times \frac{1}{z}\left(\frac{2b}{1 + \left(\frac{1}{z^2} - 1\right)b^2}\right) = \frac{-4\gamma}{z}\frac{b^2}{b^4 - \gamma} \tag{A2}$$

---

[25] In a second-price auction, a buyer wins whenever the rival has a lower valuation than himself.

where $\gamma = \frac{1}{\left(1-\frac{1}{z^2}\right)^2}$. Letting $f_z(b)$ be the density of the winning bid, expected revenue in a first-price auction with the distributions under consideration would therefore be

$$\int_0^{\overline{b}} b f_z(b) db = \overline{b} - \int_0^{\overline{b}} F_z(b) db, \tag{A3}$$

which can be computed, given (A2).

We now turn to the first-price auction with a pre-auction offer. Again, letting p denote the pre-auction offer, we claim the strong buyer accepts p if, and only if, his valuation exceeds z, where z (uniquely) solves $p = \frac{z}{1+z}$. We start by assuming the proposed strategies form an equilibrium, and confirm this by showing that there is no incentive to deviate. First, if the auction stage is reached, the beliefs, given the strategy in the first stage, is that buyers' valuations are drawn from uniform distributions over $[0,1]$ and $[0,z]$, respectively. Given this, the equilibrium bidding strategies are outlined above. If the strong buyer deviates in the first stage, rejecting p when his valuation was above z, it is easy to show that his best response in the first price auction is to submit the highest bid, $\frac{z}{1+z}$.[26] Thus, regardless of whether he accepts or rejects, he will win with probability one, and he will pay $\frac{z}{1+z}$. Hence, there is no incentive to deviate. Neither is there an incentive for the strong buyer with a valuation below z to accept p. The reason is that he can just submit a bid of $\frac{z}{1+z}$ in the second stage and win with probability one, so he will be no worse off rejecting.

In the mechanism with a pre-auction offer, the good may be sold in the first stage, at the price stipulated by the auctioneer, $\frac{z}{1+z}$, or it may be sold in the second stage, where bidding strategies have already been outlined, given general beliefs on z. Specifically, the object is sold in the first round with probability $\frac{2-z}{2}$, i.e. the probability that the strong buyer has a valuation in excess of z. With probability $\frac{z}{2}$, the strong buyer rejects the offer, in which case the weak buyer updates his beliefs, and the expected revenue will therefore be (A3). Hence, expected revenue in the new mechanism is, as a function of z,

---

[26]To see this, notice that the strong buyer with valuation v tries to maximize $(v-b)q_s(b)$, where $q_s(b)$ is the probability that the weak buyer bids below b. At any b where the first derivative is zero for a v below z, it must be strictly positive for $v > z$.

$$
\begin{aligned}
ER(z) &= \frac{2-z}{2}\overline{b} + \frac{z}{2}\left(\overline{b} - \int_0^{\overline{b}} F_z(b)db\right) = \overline{b} - \frac{z}{2}\int_0^{\overline{b}} F_z(b)db \\[2mm]
&= \overline{b} + 2\gamma \int_0^{\overline{b}} \frac{b^2}{b^4-\gamma}db = \overline{b} + 2\gamma \int_0^{\overline{b}} \frac{b^2}{(b^2+\gamma^{1/2})(b-\gamma^{1/4})(b+\gamma^{1/4})}db \\[2mm]
&= \overline{b} + 2\gamma \int_0^{\overline{b}} \left(\frac{1}{4\gamma^{1/4}\left(b-\gamma^{1/4}\right)} - \frac{1}{4\gamma^{1/4}\left(b+\gamma^{1/4}\right)} + \frac{1}{2}\frac{1}{b^2+\gamma^{1/2}}\right)db \\[2mm]
&= \overline{b} - \frac{\gamma^{3/4}}{2}\int_0^{\overline{b}} \left(\frac{1}{\left(\gamma^{1/4}-b\right)} + \frac{1}{\left(b+\gamma^{1/4}\right)} - 2\gamma^{1/4}\frac{1}{b^2+\gamma^{1/2}}\right)db \\[2mm]
&= \overline{b} - \frac{\gamma^{3/4}}{2}\left(-\left[\ln\frac{\left(\gamma^{1/4}-b\right)}{\left(b+\gamma^{1/4}\right)}\right]_0^{\overline{b}} - 2\gamma^{1/4}\left[\frac{-1}{\gamma^{1/4}}\tan^{-1}\left(\frac{\gamma^{1/4}}{b}\right)\right]_0^{\overline{b}}\right) \\[2mm]
&= \overline{b} - \frac{\gamma^{3/4}}{2}\left(\ln\left(\frac{\overline{b}+\gamma^{1/4}}{\gamma^{1/4}-\overline{b}}\right) + 2\left(\tan^{-1}\left(\frac{\gamma^{1/4}}{\overline{b}}\right) - \frac{\pi}{2}\right)\right).
\end{aligned}
$$

This function is graphed in Fig. 4.

Second-price auctions. Assume there are n risk neutral buyers. Buyer i draws a valuation independently from the distribution function F_i on $[0,\overline{v}_i]$, $i = 1,...,n$. F_i has no mass points, is strictly increasing and continuously differentiable on $(0,\overline{v}_i)$, with f_i denoting the density. If $v > \overline{v}_i$, $F_i(v) = 1$ and $f_i(v) = 0$. Without loss of generality, the buyers are ordered such that $\overline{v}_n \geq \overline{v}_{n-1} \geq ... \geq \overline{v}_1$.

We consider the possibility that the seller makes a pre-auction offer to some buyer, buyer i, say. Buyer i accepts if his valuation is at least $\widehat{v}$. Notice that if $i = n$ and $\widehat{v} \geq \overline{v}_{n-1}$ with $\overline{v}_n > \overline{v}_{n-1}$, the pre-auction offer does not change the allocation, as buyer n would win regardless, when his valuation exceeds $\overline{v}_{n-1}$. By the *Revenue Equivalence Theorem*, revenue is unaffected. Hence, we consider thresholds below $\overline{v}_{n-1}$ in the following.

Let A be the set of buyers other than buyer i who are affected by the pre-auction offer. If $j \in A$, then $\overline{v}_j \geq \widehat{v}$, meaning that there is a chance buyer j has the highest valuation, yet loses to buyer i. Let B be the set of buyers not in A (with $\overline{v}_j < \widehat{v}$) and different from i.

Finally, let $G_j(v) = \prod_{k \neq j} F_k(v)$, be the probability that buyer j with valuation v has the highest valuation.

In the following, we will use Myerson's (1981) method of writing expected revenue. Let $J_j(v) = v - \frac{1 - F_i(v)}{f_i(v)}$, denote buyer j's *virtual valuation*, the counterpart to marginal revenue in monopoly, when his valuation is v. In monopoly, revenue is the area under marginal revenue, which in auctions means that expected revenue from buyer j can be calculated as the expectation of $J_j(v)$, taking into account the probability that buyer j with valuation v wins the auction.

Hence, we can write expected revenue as

$$
\begin{aligned}
ER_i(\widehat{v}) \;=\; & \sum_{j \in A \cup \{i\}} \int_0^{\widehat{v}} J_j(v) G_j(v) f_j(v) dv + \sum_{j \in B} \int_0^{\overline{v}_j} J_j(v) G_j(v) f_j(v) dv \\
& + \int_{\widehat{v}}^{\overline{v}_i} J_i(v) f_i(v) dv + \sum_{j \in A} \int_{\widehat{v}}^{\overline{v}_j} J_j(v) F_i(\widehat{v}) \prod_{k \neq j,i} F_k(v) f_j(v) dv,
\end{aligned}
$$

since buyer j with a valuation below $\widehat{v}$ wins if he has the highest valuation, which occurs with probability $G_j(v)$, buyer i, to whom the offer is made, wins with probability one if his valuation is above $\widehat{v}$, and buyer $j \neq i$ with valuation above $\widehat{v}$ wins if he has the highest valuation and buyer i has a valuation below $\widehat{v}$.

In contrast, expected revenue in a second-price auction without a pre-auction offer is

$$
ER_{SPA} = \sum_{j=1}^n \int_0^{\overline{v}_j} J_j(v) G_j(v) f_j(v) dv.
$$

Revenue from the two auctions can now be compared,

$$
\begin{aligned}
D_i(\widehat{v}) \;\equiv\; ER_{SPA} - ER_i(\widehat{v}) = & \sum_{j \in A \cup \{i\}} \int_{\widehat{v}}^{\overline{v}_j} J_j(v) G_j(v) f_j(v) dv \\
& - \left[\int_{\widehat{v}}^{\overline{v}_i} J_i(v) f_i(v) dv + \sum_{j \in A} \int_{\widehat{v}}^{\overline{v}_j} J_j(v) F_i(\widehat{v}) \prod_{k \neq j,i} F_k(v) f_j(v) dv \right].
\end{aligned}
$$

The first term is identical to expected revenue in a second-price auction with reserve price of $\widehat{v}$ (which buyers in B never win). The second term is revenue in a mechanism where buyer i is offered the good at a price of $\widehat{v}$, and if he rejects, all other buyers are invited to a second-price auction with

a reserve price of $\widehat{v}$. Clearly, the former auction is more profitable, implying that $D_i(\widehat{v}) > 0$ as we wanted to prove.

An alternative proof starts by examining the derivative of $ER_i(\widehat{v})$,

$$ER_i'(\widehat{v}) = f_i(\widehat{v}) \left[\sum_{j \in A} \int_{\widehat{v}}^{\overline{v}_j} J_j(v) \prod_{k \neq j,i} F_k(v) f_j(v) dv - J_i(\widehat{v})(1 - G_i(\widehat{v})) \right].$$

The first term in brackets is equivalent to the expected revenue in a second-price auction with a reserve price of $\widehat{v}$ among all the buyers except buyer i. This clearly exceeds revenue from posting a price of $\widehat{v}$, which would yield expected revenue of $\widehat{v}(1 - G_i(\widehat{v}))$. Hence,

$$ER_i'(\widehat{v}) \geq f_i(\widehat{v})(1 - G_i(\widehat{v}))(\widehat{v} - J_i(\widehat{v})) = (1 - G_i(\widehat{v}))(1 - F_i(\widehat{v})) \geq 0.$$

It follows that a pre-auction offer never improves revenue in a second-price auction.[27] Notice that we have not assumed that J_j is monotonic, as is often the case in auction theory. Bulow and Klemperer (1996) argued that pre-auction offers are not profitable in a model with symmetric buyers and monotonic virtual valuations. Hence, we generalize this result in several directions.

To reveal the intuition behind the result we rearrange the derivative,

$$ER_i'(\widehat{v}) = f_i(\widehat{v})(1 - G_i(\widehat{v})) \left[\sum_{j \in A} \int_{\widehat{v}}^{\overline{v}_j} J_j(v) \prod_{k \neq j,i} F_k(v) \frac{f_j(v)}{(1 - G_i(\widehat{v}))} dv - J_i(\widehat{v}) \right].$$

If the pre-auction offer changes the allocation, it is because buyer i wins when another buyer (in A) has a higher valuation. In this event, the gain contributing to an increase in revenue is the virtual valuation of buyer i, $J_i(\widehat{v}) \leq \widehat{v}$. The loss, however, is the virtual valuation of a buyer known to have a higher valuation. This is at least equal to $\widehat{v}$. On a market where the lowest willingness-to-pay is $\widehat{v}$ a monopolist can get at least $\widehat{v}$ for each of his units (and strictly more if he faces a capacity constraint), so the average marginal revenue, or virtual valuation, is at least $\widehat{v}$. Hence, the loss from introducing a pre-auction offer exceeds the gain.

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[27] Clearly, for any i, A and B depend on $\widehat{v}$. Given A and B, however, revenue increases by increasing $\widehat{v}$ until buyers move from A to B ($\widehat{v} = \min_{j \in A} \overline{v}_j$). As we reduce A further, revenue increases. For any given i, it follows that the optimal $\widehat{v} = \overline{v}_i$, which is equivalent to no pre-auction offer.

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