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SEASONAL UNIT ROOT TESTING BASED ON THE TEMPORAL AGGREGATION OF SEASONAL CYCLES

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Abstract

The effects of systematic sampling and temporal aggregation on the seasonal cycle model (see Miron, 1993) and the seasonally integrated process (see Hylleberg et al., 1990) are discussed. The temporal aggregation theory is used to improve the sequential test for monthly seasonal unit roots of Rodrigues and Franses (2003). It is shown by simulation that the monthly sequential test has better finite sample properties than the BM test (see Beaulieu and Miron, 1993). The new test is applied to monthly US Industrial Production and, contrary to the BM test, rejects the presence of any seasonal unit root.

1 Introduction

The seasonal analysis of discrete time series is strongly influenced by the aliasing effect (see Koopmans, 1974), since the seasonal cycles with a period smaller than twice the sampling interval are not observable with its real period but with a longer one. For example, a monthly seasonal cycle with a period of 4 months turns into a quarterly cycle with a period of 12 months. This effect may lead to erroneous interpretations of the nature of seasonal cycles.

Testing for seasonal unit roots, as any seasonal analysis, is affected by the aliasing effect, and therefore when a particular seasonal unit root is not rejected, it is not possible to state whether a unit root is present at the equivalent underlying seasonal frequency or at a nonobservable one with the available data. The most serious problem concerning the aliasing effect and unit root testing, is that a zero-frequency unit

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root can be due to an underlying seasonal unit root. This zero-frequency aliasing may even lead to spurious cointegration (see Granger and Siklos, 1995).

The aliasing problem can be alleviated if time series is available at a sampling frequency where the relevant cycles are observable without aliasing. However, when seasonal unit roots are tested with high frequency time series, the presence of many seasonal cycles deteriorates the power of the HEGY test (see Rodrigues and Osborn, 1999). To deal with this problem, Rodrigues and Franses (2003) have proposed a sequential strategy for testing monthly seasonal unit roots. The idea is to use the quarterly unit root findings to reduce the set of monthly frequencies where to test for a unit root. However, the Rodrigues and Franses' links among the frequencies are not entirely correct and the method does not use the quarterly information efficiently. Therefore, we propose an improved version of Rodrigues and Franses' test based on the theoretical relation between seasonally integrated (SI) processes measured at different sampling intervals. The new test is not only more powerful than the BM test but in some occasions more robust to structural breaks at the deterministic seasonal component.

The outline of the paper is as follows. In section 2 we study the effects of systematic sampling and temporal aggregation on the seasonal cycle model and on the SI model, and discuss its implications for the seasonal unit root tests which allow to test for separate unit roots (see Hylleberg *et al.*, 1990; Beaulieu and Miron, 1993; Canova and Hansen, 1995; Caner, 1998; Taylor, 1998; Smith and Taylor, 1999 and Taylor, 2003). In section 3 we describe an improved version of the sequential approach of Rodrigues and Franses. The finite sample properties of the new method are compared with the BM tests in section 4 by means of Monte Carlo simulation. In section 5 we apply the new tools for testing monthly seasonal unit roots at the US Industrial Production (1950M1-2003M11). Finally section 6 concludes the paper. The proofs of the theorems are presented in the appendix 1, the monthly regressors for the monthly sequential test are listed in the appendix 2, and the tables with the new critical values are in the appendix 3.

A word on notation. W denotes a standard Brownian motion. t denotes months, τ denotes bimonths, T denotes quarters, and Υ denotes semesters. x denotes a time series measured at the shortest sampling interval the time series is available (disaggregated time series). X^s denotes a systematically sampled time series and Y a temporally aggregated time series. Concretely, $X_\tau^s \equiv x_{2(\tau-1)+s}$ $(s = 1, 2)$ and $Y_\tau = \Sigma_{s=1}^2 X_\tau^s$ denote the bimonthly series; $X_T^s \equiv x_{3(\tau-1)+s}$ $(s = 1, 2, 3)$ and $Y_T = \Sigma_{s=1}^3 X_T^s$ denote the quarterly series; and $X_\Upsilon^s \equiv x_{6(\tau-1)+s}$ $(s = 1, ..., 6)$ and $Y_\Upsilon = \Sigma_{s=1}^6 X_\Upsilon^s$ denote the semi-annual series.

2 Temporal Aggregation of Seasonal Time Series Models

In this section we discuss the implications of the relation of seasonal cycles at different sampling intervals for testing seasonal unit roots with systematically sampled and temporally aggregated time series. For this purpose, we first present the relation among seasonal cycles measured monthly, bimonthly, quarterly and semiannually. Next, we discuss the links between the trigonometric representation of the seasonal cycle model measured at those sampling intervals. Then, we discuss the temporal aggregation effect on the SI process, and the implications of systematic sampling and temporal aggregation for seasonal unit root testing.

2.1 Relation among seasonal cycles at different sampling intervals

Table 1 presents the relation of monthly, bimonthly, quarterly and semi-annual seasonal cycles. The cycles below the horizontal line are aliases of the observable seasonal cycles above the line. The table shows how the seasonal cycles of a disaggregated process are observed after systematic sampling. For example, a monthly seasonal cycle with a period of 4 months (at frequency $\pi/2$) is observable without aliasing as a bimonthly seasonal cycle with a period of 4 months (at frequency π), aliased as a quarterly seasonal cycle with a period of 12 months (at frequency $\pi/2$), and aliased as a semi-annual seasonal cycle with a period of 12 months (at frequency $\pi/2$). It is seen in table 1, how different seasonal cycles of a disaggregated model are observable as the same seasonal cycle at a systematically sampled process. For example a zero-frequency component at the semi-annual model can be generated by a zero-frequency component at the quarterly process or by a quarterly seasonal cycle with a period of 6 months (at frequency π).

Table 1: Relation among seasonal cycles at different sampling intervals

Monthly Interval			Bimonthly Interval			Quarterly Interval			Semi-annual Interval		
θ_k	Period	Cycles	θ_k	Period	Cycles	θ_k	Period	Cycles	θ_k	Period	Cycles
0	∞	0	0	∞	0	0	∞	0	0	∞	0
$\pi/6$	12	1	$\pi/3$	12	1	$\pi/2$	12	1	π	12	1
$\pi/3$	6	2	$2\pi/3$	6	2	π	6	2	0	∞	0
$\pi/2$	4	3	π	4	3	$\pi/2$	12	1	π	12	1
$2\pi/3$	3	4	$2\pi/3$	6	2	0	∞	0	0	∞	0
$5\pi/6$	2.4	5	$\pi/3$	12	1	$\pi/2$	12	1	π	12	1
π	2	6	0	∞	0	π	6	2	0	∞	0

* θ_k is the seasonal frequency; periods in months and cycles per year.

2.2 Temporal Aggregation of the Seasonal Cycle Model

Let us consider the temporal aggregation of a monthly time series generated by the seasonal cycle model (Miron, 1993):

$$x_t = \Sigma_{s=1}^{12} \psi_s d_{s,t} + u_t, \quad (1)$$

where ψ_s denotes the seasonal mean of x_t, $d_{s,t}$ is a monthly seasonal dummy variable and u_t is a linear process. The focus of the paper is on seasonal unit root tests, which are based not on the time-domain representation of seasonality but on the spectral representation. Therefore, it is necessary to discuss the effects of systematic sampling and temporal aggregation on the trigonometric representation of the seasonal cycle model (1) (see Hannan *et al.*, 1970):

$$x_t = \gamma_0 + \Sigma_{k=1}^{5} (\gamma_k \cos(\theta_k t) + \lambda_k \sin(\theta_k t)) + \gamma_6 \cos(\pi t) + u_t,$$

where γ_0 denotes the overall mean of x_t (the overall drift when x_t is the first differenced series), γ_k $(k = 1, ..., 6)$, λ_k $(k = 1, ..., 5)$ are the coefficients associated with the seasonal cycles with a period of $\frac{12}{k}$ months, and $\theta_k = k\pi/6$ $(k = 1, ..., 5)$ denotes the seasonal frequency.

Theorem 1 *Let x_t be a monthly process generated by*

$$\begin{aligned} x_t \; = \; & \gamma_0 + \gamma_1 \cos(\frac{\pi}{6} t) + \lambda_1 \sin(\frac{\pi}{6} t) + \gamma_2 \cos(\frac{\pi}{3} t) + \lambda_2 \sin(\frac{\pi}{3} t) + \gamma_3 \cos(\frac{\pi}{2} t) + \lambda_3 \sin(\frac{\pi}{2} t) \\ & + \gamma_4 \cos(\frac{2\pi}{3} t) + \lambda_4 \sin(\frac{2\pi}{3} t) + \gamma_5 \cos(\frac{5\pi}{6} t) + \lambda_5 \sin(\frac{5\pi}{6} t) + \gamma_6 \cos(\pi t) + u_t, \end{aligned}$$

where u_t is a linear process.

The systematically sampled bimonthly processes X_τ^s $(s = 1, 2)$ are

$$X_\tau^s = \Gamma_0^s + \Gamma_1^s \cos(\frac{\pi}{3}\tau) + \Lambda_1^s \sin(\frac{\pi}{3}\tau) + \Gamma_2^s \cos(\frac{2\pi}{3}\tau) + \Lambda_2^s \sin(\frac{2\pi}{3}\tau) + \Gamma_3^s \cos(\pi\tau) + U_\tau^s,$$

where $\Gamma_0^1 \equiv \gamma_0 - \gamma_6$, $\Gamma_0^2 \equiv \gamma_0 + \gamma_6$, $\Gamma_1^1 \equiv \sqrt{3}/2(\gamma_1 - \gamma_5) - 1/2(\lambda_1 + \lambda_5)$, $\Gamma_1^2 \equiv \gamma_1 + \gamma_5$, $\Lambda_1^1 \equiv 1/2(\gamma_1 - \gamma_5) + \sqrt{3}/2(\lambda_1 + \lambda_5)$, $\Lambda_1^2 \equiv \lambda_1 - \lambda_5$, $\Gamma_2^1 \equiv 1/2(\gamma_2 - \gamma_4) - \sqrt{3}/2(\lambda_2 + \lambda_4)$, $\Gamma_2^2 \equiv \gamma_2 + \gamma_4$, $\Lambda_2^1 \equiv \sqrt{3}/2(\gamma_2 - \gamma_4) + 1/2(\lambda_2 + \lambda_4)$, $\Lambda_2^2 \equiv \lambda_2 - \lambda_4$, $\Gamma_3^1 \equiv -\lambda_3$, $\Gamma_3^2 \equiv \gamma_3$, and $U_\tau^s = u_{2(\tau-1)+s}$.

The temporally aggregated bimonthly process Y_τ is

$$Y_\tau = \Gamma_0 + \Gamma_1 \cos(\frac{\pi}{3}\tau) + \Lambda_1 \sin(\frac{\pi}{3}\tau) + \Gamma_2 \cos(\frac{2\pi}{3}\tau) + \Lambda_2 \sin(\frac{2\pi}{3}\tau) + \Gamma_3 \cos(\pi\tau) + V_\tau,$$

where $\Gamma_0 = 2\gamma_0$, $\Gamma_1 = \frac{2+\sqrt{3}}{2}\gamma_1 + \frac{2-\sqrt{3}}{2}\gamma_5 - 1/2(\lambda_1 + \lambda_5)$, $\Lambda_1 = \frac{1}{2}(\gamma_1 - \gamma_5) + \frac{2+\sqrt{3}}{2}\lambda_1 + \frac{\sqrt{3}-2}{2}\lambda_5$, $\Gamma_2 = \frac{3}{2}\gamma_2 + \frac{1}{2}\gamma_4 - \frac{\sqrt{3}}{2}(\lambda_2 + \lambda_4)$, $\Lambda_2 = \frac{\sqrt{3}}{2}(\gamma_2 - \gamma_4) + \frac{3}{2}\lambda_2 - \frac{1}{2}\lambda_4$, $\Gamma_3 = \gamma_3 - \lambda_3$, and $V_\tau \equiv \Sigma_{s=1}^{2} U_\tau^s$.

The systematically sampled quarterly processes X_T^s $(s = 1, 2, 3)$ *are*

$$X_T^s = \Gamma_0^s + \Gamma_1^s \cos(\frac{\pi}{2}T) + \Lambda_1^s \sin(\frac{\pi}{2}T) + \Gamma_2^s \cos(\pi T) + U_T^s,$$

where $\Gamma_0^1 \equiv \gamma_0 + \frac{1}{2}(\sqrt{3}\lambda_4 - \gamma_4)$, $\Gamma_0^2 \equiv \gamma_0 - \frac{1}{2}(\gamma_4 + \sqrt{3}\lambda_4)$, $\Gamma_0^3 \equiv \gamma_0 + \gamma_4$, $\Gamma_1^1 \equiv \frac{1}{2}(\gamma_1 - \sqrt{3}\lambda_1 - 2\gamma_3 + \gamma_5 + \sqrt{3}\lambda_5)$, $\Gamma_1^2 \equiv \frac{1}{2}(\sqrt{3}\gamma_1 - \lambda_1 - 2\lambda_3 - \sqrt{3}\gamma_5 - \lambda_5)$, $\Gamma_1^3 \equiv \gamma_1 + \gamma_3 + \gamma_5$, $\Lambda_1^1 \equiv \frac{1}{2}(\sqrt{3}\gamma_1 + \lambda_1 + 2\lambda_3 - \sqrt{3}\gamma_5 + \lambda_5)$, $\Lambda_1^2 \equiv \frac{1}{2}(\gamma_1 + \sqrt{3}\lambda_1 - 2\gamma_3 + \gamma_5 - \sqrt{3}\lambda_5)$, $\Lambda_1^3 \equiv \lambda_1 - \lambda_3 + \lambda_5$, $\Gamma_2^1 \equiv \frac{1}{2}(-\gamma_2 - \sqrt{3}\lambda_2 + 2\gamma_6)$, $\Gamma_2^2 \equiv \frac{1}{2}(\gamma_2 - \sqrt{3}\lambda_2 - 2\gamma_6)$, $\Gamma_2^3 \equiv \gamma_2 + 2\gamma_6$, *and* $U_T^s \equiv u_{3(T-1)+s}$.

The temporally aggregated quarterly process Y_T *is*

$$Y_T = \Gamma_0 + \Gamma_1 \cos(\frac{\pi}{2}T) + \Lambda_1 \sin(\frac{\pi}{2}T) + \Gamma_2 \cos(\pi T) + V_T,$$

where $\Gamma_0 = 3\gamma_0$, $\Gamma_1 = \frac{3+\sqrt{3}}{2}\gamma_1 - \frac{2+\sqrt{3}}{2}\lambda_1 - \lambda_3 + \frac{3-\sqrt{3}}{2}\gamma_5 - \frac{2-\sqrt{3}}{2}\lambda_5$, $\Lambda_1 = \frac{2+\sqrt{3}}{2}\gamma_1 + \frac{3+\sqrt{3}}{2}\lambda_1 - \gamma_3 + \frac{2-\sqrt{3}}{2}\gamma_5 + \frac{3-\sqrt{3}}{2}\lambda_5$, $\Gamma_2 = \gamma_2 - \sqrt{3}\lambda_2 + 2\gamma_6$, *and* $V_T \equiv \Sigma_{s=1}^3 U_T^s$.

The systematically sampled semi-annual processes X_Υ^s $(s = 1, ..., 6)$ *are*

$$X_\Upsilon^s = \Gamma_0^s + \Gamma_1^s \cos(\pi\Upsilon) + U_\Upsilon^s,$$

where $\Gamma_0^1 \equiv 1/2(2\gamma_0 + \gamma_2 + \sqrt{3}\lambda_2 - \boldsymbol{\gamma}_4 + \sqrt{3}\lambda_4 - 2\gamma_6)$, $\Gamma_0^2 \equiv 1/2(2\gamma_0 - \gamma_2 + \sqrt{3}\lambda_2 - \boldsymbol{\gamma}_4 - \sqrt{3}\lambda_4 + 2\gamma_6)$, $\Gamma_0^3 \equiv \gamma_0 - \gamma_2 + \boldsymbol{\gamma}_4 - \gamma_6$, $\Gamma_0^4 \equiv 1/2(2\gamma_0 - \gamma_2 - \sqrt{3}\lambda_2 - \boldsymbol{\gamma}_4 + \sqrt{3}\lambda_4 + 2\gamma_6)$, $\Gamma_0^5 \equiv 1/2(2\gamma_0 + \gamma_2 - \sqrt{3}\lambda_2 + \boldsymbol{\gamma}_4 - \sqrt{3}\lambda_4 - 2\gamma_6)$, $\Gamma_0^6 \equiv \gamma_0 + \gamma_2 + \boldsymbol{\gamma}_4 + \gamma_6$, $\Gamma_1^1 \equiv 1/2(-\sqrt{3}\gamma_1 - \lambda_1 - 2\lambda_3 + \sqrt{3}\gamma_5 - \lambda_5)$, $\Gamma_1^2 \equiv 1/2(-\gamma_1 - \sqrt{3}\lambda_1 + 2\boldsymbol{\gamma}_3 - \gamma_5 + \sqrt{3}\lambda_5)$, $\Gamma_1^3 \equiv -\lambda_1 + \lambda_3 - \lambda_5$, $\Gamma_1^4 \equiv 1/2(\gamma_1 - \sqrt{3}\lambda_1 - 2\boldsymbol{\gamma}_3 + \gamma_5 + \sqrt{3}\lambda_5)$, $\Gamma_1^5 \equiv 1/2(\sqrt{3}\gamma_1 - \lambda_1 - 2\lambda_3 - \sqrt{3}\gamma_5 - \lambda_5)$, $\Gamma_1^6 \equiv \gamma_1 + \boldsymbol{\gamma}_3 + \gamma_5$ *and* $U_{6(\Upsilon-1)+s}$.

The temporally aggregated semi-annual time series Y_Υ *is*

$$Y_\Upsilon = \Gamma_0 + \Gamma_1 \cos(\pi\Upsilon) + V_\Upsilon,$$

where $\Gamma_0 = 6\gamma_0$, $\Gamma_1 = \frac{3}{2}\gamma_1 - \frac{4+3\sqrt{3}}{2}\lambda_1 - \lambda_3 + \frac{3}{2}\gamma_5 + \frac{3\sqrt{3}-4}{2}\lambda_5$, *and* $V_\Upsilon \equiv \Sigma_{s=1}^6 U_\Upsilon^s$.

Proof. See the appendix 1. ■

Corollary 2 *Let* x_T *be a quarterly process generated by*

$$x_T = \gamma_0 + \gamma_1 \cos(\frac{\pi}{2}T) + \lambda_1 \sin(\frac{\pi}{2}T) + \gamma_2 \cos(\pi T) + u_T,$$

where u_T *is a linear process.*

The systematically sampled semi-annual processes X_Υ^s $(s = 1, 2)$ *are*

$$X_\Upsilon^s = \Gamma_0^s + \Gamma_1^s \cos(\pi\Upsilon) + U_\Upsilon^s,$$

where $\Gamma_0^1 = \gamma_0 - \gamma_2$, $\Gamma_0^2 = \gamma_0 + \gamma_2$, $\Gamma_1^1 = -\lambda_1$, $\Gamma_1^2 = \gamma_1$ *and* $U_\Upsilon^s = u_{2(T-1)+s}$.

The temporally aggregated semi-annual process Y_Υ *is*

$$Y_\Upsilon = \Gamma_0 + \Gamma_1 \cos(\pi\Upsilon) + V_\Upsilon,$$

where $\Gamma_0 = 2\gamma_0$, $\Gamma_1 = \gamma_1 - \lambda_1$, *and* $V_\Upsilon \equiv \Sigma_{s=1}^2 U_\Upsilon^s$.

As seen in Theorem 1 and Corollary 2, the dependency of the 'aggregated' spectral coefficients Γ_j^s and Λ_j^s on the 'underlying' spectral coefficients γ_k and λ_k reflect the relation among seasonal cycles measured at different sampling intervals (see table 1). Let us consider the case of the quarterly seasonal model. From Corollary 2, different seasonal patterns are allowed at the quarterly model. When $\gamma_1 \neq 0$ there is an annual cycle linking the 2nd quarter with the 4th quarter, when $\lambda_1 \neq 0$ there is another annual cycle relating the 1st quarter with the 3rd quarter, and when $\gamma_2 \neq 0$ there is a semi-annual cycle. Any of the associated semi-annual processes X_Υ^1, X_Υ^2, or Y_Υ only capture part or a distorted part of the underlying seasonal component due to an aggregation bias or an information loss. The annual cycle of X_Υ^1 captures the cycle linking the 1st and 3th quarters (λ_1) while its overall mean (or drift) is affected by the aggregation bias since this parameter is a linear combination of the overall mean (or drift) of the quarterly model and the quarterly π-frequency ($\Gamma_0^1 = \gamma_0 - \gamma_2$). Analogously X_Υ^2 reflects the annual cycle linking the 2nd and the 4th quarters (γ_1) and the quarterly overall mean (or drift) is $\Gamma_0^2 = \gamma_0 + \gamma_2$. The overall mean (or drift) of the temporally aggregated series Y_Υ is only linked to the quarterly overall mean (or drift) γ_0, and therefore there is no aggregation bias or loss of information about the quarterly overall mean (or drift). However, the information about the semi-annual cycle (γ_2) is lost by the effect of the summation filter, and the annual cycle of Y_Υ is affected by the aggregation bias ($\Gamma_1 = \gamma_1 - \lambda_1$). Then, it is possible (when $\gamma_1 = \lambda_1$) that the annual cycle disappears from the model of the temporally aggregated series. All the quarterly parameters $\gamma_0, \gamma_1, \lambda_1$, and γ_2 only can be recovered with two of the semi-annual models of X_Υ^1, X_Υ^2, or Y_Υ.

2.3 Temporal Aggregation of the Seasonal Unit Root Model

Theorem 3 *Let x_t be a monthly* $\mathbf{SI}(d_0, ..., d_6)$ *process with* $d_k = 0, 1$

$$(1-L)^{d_0}(1-\sqrt{3}L+L^2)^{d_1}(1-L+L^2)^{d_2}(1+L^2)^{d_3}(1+L+L^2)^{d_4}(1+\sqrt{3}L+L^2)^{d_5}(1+L)^{d_6}x_t = u_t$$

where u_t is a stationary and invertible ARMA process.

The systematically sampled bimonthly processes are $X_\tau^s\text{~}\mathbf{SI}(d_0^*, d_1^*, d_2^*, d_3)$ $(s = 1, 2)$

$$(1 - L^2)^{d_0^*}(1 - L^2 + L^4)^{d_1^*}(1 + L^2 + L^4)^{d_2^*}(1 + L^2)^{d_3}X_\tau^s = U_\tau^s,$$

and the temporally aggregated bimonthly process is $Y_\tau\text{~}\mathbf{SI}(d_0, d_1^*, d_2^*, d_3)$

$$(1 - L^2)^{d_0}(1 - L^2 + L^4)^{d_1^*}(1 + L^2 + L^4)^{d_2^*}(1 + L^2)^{d_3}Y_\tau = V_\tau,$$

where $L^2 X_\tau^s = X_{\tau-1}^s$, $d_0^* = \max\{d_0, d_6\}$, $d_1^* = \max\{d_1, d_5\}$, *and* $d_2^* = \max\{d_2, d_4\}$.

The systematically sampled quarterly processes are $X_T^s\text{~}\mathbf{SI}(d_0^*, d_1^*, d_2^*)$ $(s = 1, 2, 3)$

$$(1 - L^3)^{d_0^*}(1 + L^6)^{d_1^*}(1 + L^3)^{d_2^*}X_T^s = U_T^s,$$

and the temporally aggregated quarterly process is $Y_T \tilde{} \mathbf{SI}(d_0, d_1^*, d_2^*)$

$$(1 - L^3)^{d_0}(1 + L^6)^{d_1^*}(1 + L^3)^{d_2^*} Y_T = V_T,$$

where $L^3 X_T^s = X_{T-1}^s$, $d_0^* = \max\{d_0, d_4\}$, $d_1^* = \max\{d_1, d_3, d_5\}$, *and* $d_2^* = \max\{d_2, d_6\}$. *The systematically sampled semi-annual processes are* $X_\Upsilon^s \tilde{} \mathbf{SI}(d_0^*, d_1^*)$ $(s = 1, ..., 6)$

$$(1 - L^6)^{d_0^*}(1 + L^6)^{d_1^*} X_\Upsilon^s = U_\Upsilon^s,$$

and the temporally aggregated semi-annual process is $Y_\Upsilon \tilde{} \mathbf{SI}(d_0, d_1^*)$

$$(1 - L^6)^{d_0}(1 + L^6)^{d_1^*} Y_\Upsilon = V_\Upsilon,$$

where $L^6 X_\Upsilon^s = X_{\Upsilon-1}^s$, $d_0^* = \max\{d_0, d_2, d_4, d_6\}$, *and* $d_1^* = \max\{d_1, d_3, d_5\}$.

Proof. See the proof and the error terms in the appendix 1. ■

As seen in theorem 3, the dependency of the aggregated orders of seasonal integration $d_1^*, ..., d_{6/m}^*$ at the bimonthly ($m = 2$), quarterly ($m = 3$), and semi-annual ($m = 6$) models is determined by the relation among seasonal cycles at different sampling intervals (see table 1). The link is exactly the same for all the systematically sampled series X^s and for the temporally aggregated series Y.[1] Specifically, the order of integration at a particular 'aggregated' frequency d_j^* depends on the order of integration of the monthly unit root that generates the same cycle and on the orders of integration affected by aliasing. For example, the $\pi/2$-frequency unit root of any quarterly model X_T^1, X_T^2, X_T^3, or Y_T is linked to the monthly unit roots at the $\pi/6$-frequency, the monthly seasonal cycle with the same period than the quarterly cycle, and the aliased frequencies $\pi/2$ and $5\pi/6$. All orders of seasonal integration are at most one because the monthly process has hidden periodicity of order m.[2]

The differential effect for X^s and Y takes place at the zero-frequency since the summation filter present at Y cancels the unit roots that aliases to the zero-frequency. In the case of the sampled series X^s the zero-frequency integration order d_0^* is linked to the monthly zero-frequency d_0 and other monthly seasonal integration orders, while the zero-frequency integration order of Y is only linked to the disaggregated zero-frequency integration order. For example, the monthly $2\pi/3$ unit root turns into a zero-frequency unit root after quarterly systematic sampling and is cancelled after quarterly temporal aggregation.[3]

The main difference between the systematic sampling of SI processes and the deterministic seasonal models is that seasonal unit roots do not allow the presence

[1] The Monte Carlo results of Granger and Siklos (1995) are in contradiction to this common effect, probably due to the poor performance of the monthly seasonal unit root test they use.

[2] A model is said to be without hidden periodicity of order m if the characteristic roots of the AR polynomial satisfy $z_j^m = z_l^m$ iff $z_j = z_l$ $\forall j, l$ (see Stram and Wei, 1986).

[3] Rodrigues and Franses (2003) find by simulation that the monthly root at $2\pi/3$ is linked to the quarterly root at the $\pi/2$-frequency which is not correct. If we raise to 3 the root $z = e^{i2\pi/3}$ becomes $z^3 = e^{i2\pi} = 1$.

of seasonal cycles at some but not all of X_T^s. For example if the monthly series contains seasonal unit roots at some or all the frequency $\pi/6$, $\pi/2$, and $5\pi/6$, the three quarterly series X_T^1, X_T^2, X_T^3 will contain a $\pi/2$-frequency unit root. In the case of a deterministic monthly seasonal cycle it is possible that for particular restrictions at the spectral coefficients of the deterministic seasonal components at frequencies $\pi/6$, $\pi/2$, and $5\pi/6$ (when $\gamma_1+\gamma_3+\gamma_5=0$ and $\lambda_1-\lambda_3+\lambda_5=0$) the quarterly series X_T^1 and X_T^2 have a seasonal cycle at frequency $\pi/2$ but not X_T^3. We discuss this issue in more detail in section 3.

The different seasonal behavior of the seasons has important implications when we consider structural breaks at the spectral coefficients. A structural break at one of the monthly spectral coefficients can affect the seasonal cycles of the quarterly series in a different way. For example, if a structural break affects λ_2, associated to the $\pi/3$-frequency of the monthly process, then only the quarterly parameters Γ_2^1 and Γ_2^2 will be affected by the break but not Γ_2^3.[4]

Let us consider the temporal aggregation of a quarterly flexible SI process.

Corollary 4 *Let x_T be a quarterly* $\mathbf{SI}(d_0,d_1,d_2)$ *process with* $d_k=0,1$

$$(1-L)^{d_0}(1+L^2)^{d_1}(1+L)^{d_2}x_T=u_T,$$

where u_t is a stationary and invertible ARMA process. The semi-annual processes are seasonally integrated, $X_\Upsilon^s\tilde{}\mathbf{SI}(d_0^*,d_1)$ $(s=1,2)$ *and* $Y_\Upsilon\tilde{}\mathbf{SI}(d_0,d_1)$

$$\begin{aligned}(1-L^2)^{d_0^*}(1+L^2)^{d_1}X_\Upsilon^s &= U_\Upsilon^s,\\ (1-L^2)^{d_0}(1+L^2)^{d_1}Y_\Upsilon &= V_\Upsilon,\end{aligned}$$

where $L^2X_\Upsilon^s=X_{\Upsilon-1}^s$, *and* $d_0^*=\max\{d_0,d_2\}$.

From corollary 4, the semi-annual processes X_Υ^s and Y_Υ are integrated at the π-frequency if the quarterly process is integrated at the $\pi/2$-frequency. The semi-annual systematically sampled processes X_Υ^s are integrated at the zero-frequency if the quarterly process is integrated at the zero or at the π frequency; and the temporally aggregated semi-annual process Y_Υ is integrated at the zero-frequency only if the quarterly process is integrated at the zero-frequency.

As discussed in the preceding lines, systematic sampling through the aliasing effect confuses different seasonal cycles when these cycles are observed at a longer sampling interval they are generated. The aliasing effect in many cases prohibits the interpretation of the observable cycles with discrete time series since the information on the underlying seasonality is lost with systematic sampling. This situation has important implications for testing seasonal unit roots. When using HEGY-type tests, the rejection of a particular unit root at the seasons X^s or at the temporally aggregated series Y implies the rejection of the associated unit roots at the disaggregated process x,

[4] This issue is discussed in more detail at section 3.

while the nonrejection of the root implies the nonrejection of at least some of the associated roots of the disaggregated process. When using CH-type tests, the rejection of deterministic seasonality at a particular frequency at X^s or Y implies the presence of some or all the associated disaggregated roots, while the nonrejection implies the absence of all the associated disaggregated roots. For example, the nonrejection of a quarterly π-frequency unit root does not guarantee that the monthly process has a unit root at the $\pi/3$ frequency, because a monthly π-frequency unit root can generate the same quarterly unit root. On the other hand, the rejection of the quarterly π-frequency unit root implies the rejection of the monthly unit roots at frequencies $\pi/3$, and π.

3 Sequential Test for Monthly Seasonal Unit Roots

The relation of seasonal cycles at different sampling intervals can be used to propose an alternative approach to test for seasonal unit roots. The precedents of this idea are Rodrigues and Franses (2003) who propose a sequential test for monthly seasonal unit roots, and Franses (1994) who proposes a cointegration approach for testing quarterly unit roots. We focus on the sequential test because its original implementation has some important limitations. Rodrigues and Franses propose a test which uses pretesting information on quarterly unit roots at the temporally aggregated series Y_T to simplify the monthly test. However, the links between the monthly and quarterly unit roots are derived by simulation and erroneously it is found a relation between the monthly $2\pi/3$-frequency unit root and the quarterly $\pi/2$-frequency unit root. In addition, the test is based on the quarterly unit root findings at Y_T, which is an inefficient use of the quarterly information contained at the monthly data.

We propose another version of the sequential test. Concretely, we suggest to use the quarterly seasonal unit root findings at the three systematically sampled time series $X_T^1 = x_{3(T-1)+1}$, $X_T^2 = x_{3(T-1)+2}$, and $X_T^3 = x_{3(T-1)+3}$ to reduce the set of monthly frequencies where to test for unit roots.

Let us consider a monthly series generated by a $\mathbf{SI}(d_0, ..., d_6)$ process ($d_k = 0, 1$ for $k = 0, ..., 6$):

$$\phi(L)\alpha(L)(x_t - \gamma_0 - \Sigma_{k=1}^5(\gamma_k \cos(\theta_k t) + \lambda_k \sin(\theta_k t)) - \gamma_6 \cos(\pi t) - \delta t) = u_t,$$

where $\theta_k = k\pi/6$, $\phi(L)$ is a lag polynomial with all its roots outside the unit circle and $\alpha(L)$ is a lag polynomial with some or all seasonal unit roots. The sequential test for monthly seasonal unit roots can be implemented in two steps:

1. Test with a HEGY-type test for the presence of quarterly seasonal unit roots at the quarterly series X_T^1, X_T^2, and X_T^3:

$$\begin{aligned} X_T^s - X_{T-4}^s &= \Gamma_0^s + \Gamma_1^s \cos(\frac{\pi}{2}T) + \Lambda_1^s \sin(\frac{\pi}{2}T) + \Gamma_2^s \cos(\pi T) + \Delta T \\ &\quad + \Pi_0 X_{0,T-1}^s + \Pi_1^{\alpha} X_{1,T-1}^{\alpha s} + \Pi_1^{\beta} X_{1,T-1}^{\beta s} + \Pi_2 X_{2,T-1}^s + U_T, \end{aligned}$$

where $X^s_{0,T} = X^s_T + X^s_{T-1} + X^s_{T-2} + X^s_{T-3}$, $X^{\alpha s}_{1,T} = -X^s_{T-1} + X^s_{T-3}$, $X^{\beta s}_{1,T} = -X^s_T + X^s_{T-2}$, and $X^s_{2,T} = -X^s_T + X^s_{T-1} - X^s_{T-2} + X^s_{T-3}$. The test statistics associated to the 0, $\pi/2$, and π frequencies are denoted t^s_0, F^s_1, and t^s_2, respectively. Given the asymptotic uncorrelation of the three semi-annual series under the null, the quarterly tests t^s_0, F^s_1, and t^s_2, are mutually independent across seasons (see Chan and Wei, 1988):

$$\Pr(t^1 \in C^1, t^2 \in C^2, t^3 \in C^3) = \Pr(t^1 \in C^1)\Pr(t^2 \in C^2)\Pr(t^3 \in C^3),$$

where the realizations of t^s fall into the region C^s of the observation space χ which is common for the three random variables.

1.1 SE1 Test (Conservative strategy)

The rejection of a particular unit root at the three quarterly series X^s_T implies the rejection of all the associated of monthly unit roots (see table 1). Otherwise, it is necessary to test monthly for the presence of the monthly unit roots. The individual level a_1 for the quarterly tests have to be determined according to a desired overall level A_1:

$$\Pr(t^1 \geqslant a_1, t^2 \geqslant a_1, t^3 \geqslant a_1) = \Pr(t^1 \geqslant a_1)\Pr(t^2 \geqslant a_1)\Pr(t^3 \geqslant a_1) = (1-a_1)^3 = 1 - A_1.$$

Then, overall levels (A_1) 1%, 5% and 10% correspond to the individual levels (a_1) 0.334%, 1.6952%, and 3.451%, respectively.[5]

1.2 SE2 Test

The rejection of a particular quarterly unit root at least at one of the quarterly series implies the rejection of the associated monthly unit roots. Otherwise, the associated monthly unit roots need to be tested. The individual level for the quarterly tests now can be obtained from

$$\Pr(t^s \geqslant a_2, t^{s+1} < a_2, t^{s+2} < a_2) = \Pr(t^s \geqslant a_2)\Pr(t^{s+1} < a_2)\Pr(t^{s+2} \geqslant a_2) = (1-a_2)a_2^2 = (1-A_2)A_2^2.$$

The overall levels (A_2) of 1%, 5% and 10% correspond in this case to individual levels (a_2) 0.01%, 0.239%, and 0.917%, respectively.[6]

The conservative sequential test SE1 can be used with small samples where size distortions or power problems may affect the HEGY test, while the SE2 test can be used with moderate or big samples where only structural breaks may affect the behavior of the HEGY test. In this case, the rejection of a particular unit root at some but not all X^s_T reveals a possible structural break at a spectral coefficient which affects the aggregated deterministic components of the quarterly series X^s_T in a different way. For example, if we assume that the levels of a monthly series x_t is generated by a seasonal cycle model, a structural break at the monthly parameter $\lambda_2 -> \lambda_2 + \delta$ will only affect the quarterly parameters $\Gamma^1_2 -> \Gamma^1_2 - \frac{\sqrt{3}}{2}\delta$ and $\Gamma^2_2 = \Gamma^2_2 - \frac{\sqrt{3}}{2}\delta$ but not Γ^3_2.

[5] The critical values for the quarterly individual tests are presented in table 9 at the appendix 3.

[6] The critical values for the quarterly individual tests of SE2 test are presented in table 10 at the appendix 3.

As a second example, a structural break of the type $\gamma_2- > \gamma_2 + \delta$, $\gamma_6- > \gamma_6 + \frac{1}{2}\delta$ will only affect the parameter $\Gamma_2^3- > \Gamma_2^3 + 2\delta$ but not Γ_2^1 or Γ_2^2.

Therefore, testing for quarterly seasonal unit roots at the three sampled series is a more robust strategy with structural breaks than testing directly at x_t for all the monthly unit roots. Obviously if the structural break affects the deterministic component of the three series the three quarterly tests will be affected. For example when there is a structural break at the overall mean $\gamma_0- > \gamma_0+\delta$, the three quarterly overall means will be affected in the same way $\Gamma_0^1- > \Gamma_0^1 + \delta$, $\Gamma_0^2- > \Gamma_0^2 + \delta$, and $\Gamma_0^3- > \Gamma_0^3 + \delta$.[7,8]

2. Test for the presence of the restricted set of monthly unit roots at the auxiliar regression:

$$\begin{aligned} &\widetilde{\phi}(L)\alpha(L)(x_t - \widetilde{\gamma}_0 - \Sigma_{k=1}^5(\widetilde{\gamma}_k \cos(\theta_k t) + \widetilde{\lambda}_k \sin(\theta_k t)) - \widetilde{\gamma}_6 \cos(\pi t) - \widetilde{\delta} t) \\ = &\sum_{k\in\{0,6\}} \pi_j \widetilde{x}_{j,t-1} + \sum_{k\in\{1,\dots,5\}} (\pi_k^\alpha \widetilde{x}_{k,t-1}^\alpha + \pi_k^\beta \widetilde{x}_{k,t}^\beta) + u_t \end{aligned}$$

where[9] $\widetilde{x}_{0,t-1} = \widetilde{S}_0(L)x_{t-1}$, $\widetilde{x}_{k,t-1}^\alpha = (\cos(k\pi/6)-L)\widetilde{S}_k(L)x_{t-1}$, $\widetilde{x}_{k,t-1}^\beta = -\sin(k\pi/6)\widetilde{S}_k(L)x_{t-1}$, $\widetilde{x}_{6,t-1} = -\widetilde{S}_6(L)x_{t-1}$, $\widetilde{S}_0(L) = \alpha(L)(1-L)^{-1}$, $\widetilde{S}_k(L) = \alpha(L)(1-2\cos(k\pi/6)L+L^2)^{-1}$, and $\widetilde{S}_6(L) = \alpha(L)(1+L)^{-1}$.

Table 2: Sequential Testing for Monthly Unit Roots

Case	Quarterly Unit Roots	Monthly Regressand ($\alpha(L)x_t$)	Monthly Regressors (k)
A	$0, \pi/2, \pi$	$\Delta_{12}x_t$	$0,1,2,3,4,5,6$
B	$0, \pi/2$	$(1-L^3+L^6-L^9)x_t$	$0,1,3,4,5$
C	$0, \pi$	$(1-L^6)x_t$	$0,2,4,6$
D	$\pi/2, \pi$	$(1+L^3+L^6+L^9)x_t$	$1,2,3,5,6$
E	0	$(1-L^3)x_t$	$0,4$
F	$\pi/2$	$(1+L^6)x_t$	$1,3,5$
G	π	$(1+L^3)x_t$	$2,6$

Due to the asymptotic uncorrelation of the regressors under the null (Chan and Wei, 1988) the separate tests for seasonal unit roots can be obtained from the following regressions:

$$\begin{aligned} \Delta\widetilde{x}_{0,t} &= \pi_0\widetilde{x}_{0,t-1} + u_t, && (2) \\ (1-2\cos\theta_k\ L + L^2)\widetilde{x}_{k,t} &= \pi_k^\alpha\widetilde{x}_{k,t-1}^\alpha + \pi_k^\alpha\widetilde{x}_{k,t-1}^\alpha + u_t,\ k=1,...5, && (3) \\ (1+L)\widetilde{x}_{6,t} &= -\pi_6\widetilde{x}_{6,t-1} + u_t, && (4) \end{aligned}$$

[7] See the links between Γ_j^s $(s = 1, 2, 3)$ and γ_k, λ_k at theorem 1.

[8] Changing seasonality can be due other reasons that a technological or an institutional change like a different seasonality depending on the phase of the business cycle, a pattern which cannot be represented by a SI process or a seasonal cycle model with structural breaks. Other classes of models like periodic autoregressive models (see Franses, 1996) or nonlinear models (see for example van Dijk et al., 2003) better represent that changing seasonality.

[9] The expressions for the regressors of all possible cases are listed in table 2.

where $\widetilde{x}_{k,t} = \widetilde{S}_k(L)x_t$. Even though the seasonal unit roots is tested separately at every seasonal frequency, the possible presence of the unattended unit roots implies that the original series is filtered from the other unit roots $\widetilde{x}_{j,t}$ $(j = 0, ..., 6)$. Filtering for unattended unit roots is shown to be not neutral for the power of the HEGY test at the zero-frequency (see Franses, 1991), because the summation filter $S_0(L)$ increases the persistence of the stationary process at the zero-frequency. This important effect of the summation filter on the behavior of the HEGY test has not been stressed that much in the literature, and we show in table 3 that a very similar situation is produced at all the seasonal frequencies. For example a similar effect when filtering for all possible unattended seasonal unit roots is produced at the frequencies $\pi/6$, $5\pi/6$, and π, a smaller effect at frequencies $\pi/3$, and $2\pi/3$, and the smallest one at the $\pi/2$-frequency. The filters applied to the monthly series x_t to obtain the regressands of the separate test for seasonal unit roots implied by restricted monthly regressions B-G have a smaller impact on the spectrum than the filters of the BM test, and therefore it is likely that when some quarterly unit roots are rejected this will device a more powerful monthly test.

Table 3: $\lim_{\omega \to \theta_k}$ of the filter functions $\left|\widetilde{S}_k(e^{i\omega})\right|^2$ for Monthly regression

Cases	0	1	2	3	4	5	6
A	144	144	48	36	48	144	144
B	36	72	–	18	12	72	–
C	36	–	12	–	12	–	36
D	–	72	12	18	–	72	36
E	9.0	–	–	–	3	–	–
F	–	36	–	9	–	36	–
G	–	–	3	–	–	–	9

For example if we do not reject ± 1 $(0, \pi)$ and reject $\pm i$ $(\pi/2)$ either at all (SE1) or at some (SE2) of the sampled quarterly series, the restricted monthly auxiliar regression, without transient dynamics or deterministic regressors, reads:

$$(1 - L^6)x_t = \widetilde{\pi}_0\widetilde{x}_{0,t-1} + \widetilde{\pi}_2\widetilde{x}^{\alpha}_{2,t-1} + \widetilde{\pi}^{\beta}_2\widetilde{x}^{\beta}_{2,t-1} + \widetilde{\pi}^{\alpha}_4\widetilde{x}^{\alpha}_{4,t-1} + \widetilde{\pi}^{\beta}_4\widetilde{x}^{\beta}_{4,t-1} + \widetilde{\pi}_6\widetilde{x}_{6,t-1} + u_t, \quad (5)$$

where $\widetilde{x}_{0,t-1}$, $\widetilde{x}_{6,t-1}$, $\widetilde{x}^{\alpha}_{2,t-1}$, $\widetilde{x}^{\beta}_{2,t-1}$, $\widetilde{x}^{\alpha}_{4,t-1}$, and $\widetilde{x}^{\beta}_{4,t-1}$ are given in the appendix 2.

The distributions of the restricted monthly t statistics $\widetilde{t}_j$ $(j = 0, 6)$ for the presence of real unit roots and the restricted F statistics $\widetilde{F}_k$ $(k = 1, ..., 5)$ for the presence of complex unit roots are (see Chan and Wei, 1988):[10]

$$\widetilde{t}_k \implies \frac{\int W_k dW_k}{\left(\int W_k^2\right)^{1/2}} \ (k{=}0{,}6),$$

$$\widetilde{F}_k \implies \frac{\left(\int W_{\alpha,k} dW_{\alpha,k} + \int W_{\beta,k} dW_{\beta,k}\right)^2}{2\left(\int W_{\alpha,k}^2 + \int W_{\beta,k}^2\right)^k} + \frac{\left(\int W_{\beta,k} dW_{\alpha,k} - \int W_{\alpha,k} dW_{\beta,k}\right)^2}{2\left(\int W_{\alpha,k}^2 + \int W_{\beta,k}^2\right)^k} \ (k{=}1{,}...{,}5),$$

where W_k, $W_{\alpha,k}$, and $W_{\beta,k}$ are standard Brownian motions.

In section 4 we evaluate the finite sample properties of the sequential approach, and in section 5 we apply this method to test for seasonal unit roots at the U.S. Industrial Production. The sequential testing approach can also be implemented by using bimonthly rather than quarterly findings on unit roots. The same method can be extended easily for testing daily or weekly unit roots. The sequential approach can be also applied to the likelihood ratio tests of Smith and Taylor (1999). We leave these issues for future research.

4 Finite Sample Properties

We compare the BM test (see Beaulieu & Miron, 1993) with the sequential tests SE1 and SE2. To do so we use the DGP $(1-r^3L^3+r^6L^6-r^9L^9)x_t = u_t$ which coincide with case B where $r = \{1, 0.95, 0.9, 0.85\}$ and u_t is a zero-mean Gaussian white noise with unit variance, $t = 1, ..., 12N$, with two sample sizes $N = 10, 20$. 15,000 replications are used to compute the empirical size and power of the tests. The overall level of the test for the monthly seasonal unit roots of interest is 5%.

Table 4 contains the empirical size and empirical power of both tests. As seen in the table the SE2 is slightly oversized, for the sample N=20 with empirical levels around 6% and 7%. On the other hand, the empirical size of the SE1 test is practically equivalent to the nominal size.

If we focus on the power, is seen how the sequential tests are more powerful than the BM test, being the SE2 the most powerful. The gains of power obtained with the SE2 with respect to the other tests are specially significative at the zero-frequency, for example when $r = 0.85$ and $N = 10$, the power of the SE2 test is 0.57 while the power of the SE1 test is 0.42 and the power of the BM test is 0.35. The bigger power of the sequential tests than the BM test can be explained by the fact that the sequential procedures apply shorter moving average filters to the monthly series than the annual summation filter applied by the BM approach (see Franses, 1991). The bigger power of the SE2 with respect to the SE1 can be explained by the same reasoning.

[10] The critical values for the restricted versions of the monthly test are presented in the appendix 3. The critical values for case A can be found in Beaulieu and Miron (1993) and Franses and Hobijn (1997).

If we look at the seasonal frequencies for the same case, the power gain obtained with the sequential test SE2 (0.90, 0.89, 0.89 and 087 at frequencies 1,3,4,5) is still relevant with respect to the BM test (0.78, 0.77, 0.77, and 0.81 at frequencies 1,3,4,5) and only slightly bigger than the power of the SE1 test (0.85, 0.85, 0.82, and 0.84 at frequencies 1,3,4,5). The gains of power at the seasonal frequencies can be explained by similar arguments than the zero-frequency. In addition the seasonal frequency are more separate from each other, and the monthly auxiliar regression of the sequential tests have smaller number of parameters than the BM test.

Table 4:Empirical size ($r=1$) and power ($r \neq 1$) of BM and Sequential Tests

$(1-r^3L^3+r^6L^6-r^9L^9)x_t = u_t,\ t=1,...,12N$								
r	Test	0	1	2	3	4	5	6
				N=10				
1	BM	0.06	0.06	–	0.05	0.05	0.05	–
1	SE1	0.05	0.05	–	0.05	0.05	0.05	–
1	SE2	0.05	0.05	–	0.05	0.05	0.05	–
0.95	BM	0.11	0.22	–	0.20	0.24	0.20	–
0.95	SE1	0.10	0.23	–	0.24	0.24	0.24	–
0.95	SE2	0.12	0.23	–	0.24	0.24	0.24	–
0.9	BM	0.23	0.56	–	0.51	0.56	0.55	–
0.9	SE1	0.25	0.63	–	0.63	0.61	0.62	–
0.9	SE2	0.33	0.65	–	0.66	0.65	0.63	–
0.85	BM	0.35	0.78	–	0.77	0.77	0.81	–
0.85	SE1	0.42	0.85	–	0.85	0.82	0.84	–
0.85	SE2	0.57	0.89	–	0.89	0.88	0.86	–
				N=20				
1	BM	0.05	0.05	–	0.05	0.05	0.05	–
1	SE1	0.05	0.05	–	0.05	0.05	0.05	–
1	SE2	0.06	0.06	–	0.06	0.07	0.06	–
0.95	BM	0.25	0.66	–	0.62	0.67	0.62	–
0.95	SE1	0.31	0.72	–	0.72	0.73	0.70	–
0.95	SE2	0.50	0.78	–	0.78	0.80	0.77	–
0.9	BM	0.63	0.99	–	0.99	0.98	0.99	–
0.9	SE1	0.94	1.00	–	1.00	1.00	1.00	–
0.9	SE2	0.99	1.00	–	1.00	1.00	1.00	–
0.85	BM	0.85	1.00	–	1.00	1.00	1.00	–
0.85	SE1	1.00	1.00	–	1.00	1.00	1.00	–
0.85	SE2	1.00	1.00	–	1.00	1.00	1.00	–

5 Empirical Application

We test for the presence of seasonal unit roots at the monthly US Industrial Production (1950M1-2003M11) (x_t), by using the BM and the sequential methods proposed in the paper.

Table 5 presents the results of the BM test, concretely the t-statistic for the monthly real unit roots and the joint F-statistic for the monthly complex unit roots. The auxiliar regression for the BM test includes seasonal dummies, a time trend, and the lag p={1,2,4,5,6,9,12,13,15,16,19,21,24,25,26,28,29,31}.[11] The monthly test does not reject the presence of the nonseasonal unit root and the seasonal unit root at the $\pi/3$-frequency, while rejects at the 5% the presence of seasonal unit roots at frequencies $\pi/2$ and π; and rejects at the 1% the presence of seasonal unit roots at frequencies $2\pi/3$, $\pi/6$, and $5\pi/6$.

Table 5: BM Test for Seasonal Unit Roots at the Monthly Time Series x_t

t_0	F_4	F_1	F_3	F_5	F_2	t_6
−1.83	21.44***	13.88***	8.17**	10.01***	2.24	−2.98**

Table 6 presents the results of the quarterly individual tests. The auxiliar regressions of the quarterly tests for seasonal unit roots at X_T^s include seasonal dummies, time trend and the lag p={8,4,-}.[12] The quarterly tests do not reject the presence of the nonseasonal unit root at all X_T^s, reject at the 1% with the joint F statistic the $\pi/2$-frequency unit root; and do not reject the π-frequency unit root at X_T^1 and X_T^2 but reject at the 1% at X_T^3. From the quarterly results we clearly reject the presence of the monthly unit roots at frequencies $\pi/6$, $\pi/2$, and $5\pi/6$; we do not reject the presence of monthly unit roots at the zero or $2\pi/3$ frequencies, and have some doubts about the presence of monthly unit roots at $\pi/3$ or π frequencies

Table 6: Test for Seasonal Unit Roots at the Quarterly Time Series X_T^s

s	t_0	F_1	t_2
1	−1.62	31.82***	−2.01
2	−2.06	37.36***	−2.06
3	−2.80	72.99***	−10.50***

We apply now the conservative sequential test SE1 (case C of table 2). In this case, the auxiliar regression contains seasonal dummies, a time trend, and the lag p={1,2,4,6,7,8,10,11,12,13,15,18,19,21,23,24,25,26, 27,30,31,32,34}. The SE1 does not reject the nonseasonal unit root and the $\pi/3$-frequency unit root while rejects at the 1% the presence of a unit root at frequencies $2\pi/3$ and π (see table 7).

[11] The augmentation of the monthly auxiliar regression is specified by eliminating from a maximum lag of 36 periods the less significative lag while the joint significance of all the eliminated lags is rejected at the 5% level.

[12] The augmentation of the quarterly auxiliar regression is determined by eliminating from a maximum lag of 12 quarters the last lag if it is nonsignificant at the 10%.

Table 7: SE1 Test for Seasonal Unit Roots at the Monthly Series x_t

$\widetilde{t}_0$	$\widetilde{F}_4$	$\widetilde{F}_1$	$\widetilde{F}_3$	$\widetilde{F}_5$	$\widetilde{F}_2$	$\widetilde{t}_6$
−1.58	15.48***	–	–	–	4.96	−4.48***

Finally, we apply the sequential test SE2 (case E of table 2). Now the auxiliar regression contains the same deterministic regressors as the BM and SE1 test and the lag p={1,3,4,6,7,9, 10,12,14,16,18,19,21,22,27,29,30,31,33,34,36}. The SE2 test does not reject the nonseasonal unit root and rejects at the 1% the presence of the $2\pi/3$-frequency unit root (see table 8).

Table 8: SE2 Test for Seasonal Unit Roots at the Monthly Series x_t

$\widetilde{t}_0$	$\widetilde{F}_4$	$\widetilde{F}_1$	$\widetilde{F}_3$	$\widetilde{F}_5$	$\widetilde{F}_2$	$\widetilde{t}_6$
−1.72	36.06***	–	–	–	–	–

To summarize the results up to this point, all the monthly tests support the presence of a nonseasonal unit root at the U.S. Industrial Production. Concerning the seasonal unit roots, even though the BM and SE1 tests do not reject the presence of a seasonal unit root at the $\pi/3$-frequency, a potential structural break may lead these monthly tests to a erroneous nonrejection. The evidence on a potential structural break is provided by a clear rejection of the π-frequency unit root at X_T^3 and the nonrejection of the same root at the other two quarterly series. This situation indicates a possible structural break at the parameter λ_2, associated to the monthly $\pi/3$-frequency and the quarterly π-frequency of the seasons X_T^1 and X_T^2. The size of the sample recommends to choose the less conservative SE2 test which is robust to this kind of structural breaks, and rejects the presence of the $\pi/3$-frequency unit root.

Given the rejection of seasonal unit roots and the nonrejection of the nonseasonal unit root, we apply the instability test of Hansen (1992) to the regression of the first differences of the monthly series Δx_t on the spectral regressors of a deterministic seasonal component:[13]

$$\Delta x_t = \gamma_0^\# + \Sigma_{k=1}^5(\gamma_k^\# \cos(\theta_k t) + \lambda_k^\# \sin(\theta_k t)) + \gamma_6^\# \cos(\pi t) + u_t,$$

The joint stability test statistic L_c =14.66 rejects the stability of the parameters of this regression, and the main responsible of such instability is the coefficient associated with the spectral coefficients $\gamma_2^\#$ associated to the $\pi/3$-frequency. This coefficient is in terms of the original parameters of the monthly auxiliar regression $\gamma_2^\# = \frac{1}{2}\gamma_2 + \frac{\sqrt{3}}{2}\lambda_2$.

Table 9: Instability Individual Test

$\gamma_0^\#$	$\gamma_1^\#$	$\lambda_1^\#$	$\gamma_2^\#$	$\lambda_2^\#$	$\gamma_3^\#$	$\lambda_3^\#$	$\gamma_4^\#$	$\lambda_4^\#$	$\gamma_5^\#$	$\lambda_5^\#$	$\gamma_6^\#$
0.33	0.55**	0.14	9.05***	0.76**	0.34	2.37***	0.17	0.19	0.80**	0.75**	0.68**

[13] This test can only be applied to a stationary regression.

The individual instability tests add additional evidence on the presence of an important structural break at the deterministic seasonal component associated to the $\pi/3$-frequency. The instability test also indicates possible structural breaks at other frequencies, but the much lower values of the statistics suggest that these breaks are not big enough to confuse the seasonal unit root tests.

6 Conclusion

This paper has discussed the effects of systematic sampling and temporal aggregation on two relevant linear seasonal models, the seasonal cycle model and the SI model. It has been shown how the sampling effects on the seasonal cycles determine the relation among the spectral coefficients of the seasonal cycle model and the relation among the seasonal unit roots of processes measured at different sampling intervals, i.e. monthly, bimonthly, quarterly, and semiannually.

The temporal aggregation theory of seasonal cycles has been used to propose an improved version of the sequential test for monthly unit roots of Rodrigues and Franses (2003). The proposed method can be considered a serious alternative to the BM test since in addition to have better finite sample properties it is more robust to structural breaks at the deterministic seasonal component.

The sequential test has been applied to the monthly US Industrial Production (1950M1-2003M11), and it has been shown how the seasonal unit root detected at the $\pi/3$-frequency by the BM test is only an artifact of the presence of an important structural break at the spectral coefficient associated to the $\pi/3$-frequency.

The sequential test can be easily extended to the likelihood test of Smith and Taylor (1999), and to testing seasonal unit roots at higher frequency data like daily or weekly time series.

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Appendix 1: Proofs of theorems.

Proof of Theorem 1. Substitute the monthly time index t by the bimonthly time index $2(\tau-1)+s$ for $s=1,2$; by the quarterly time index $3(T-1)+s$ for $s=1,2,3$; and by the semi-annual time index $6(\Upsilon-1)+s$ for $s=1,...,6$, and simplify the expressions. ■

Proof of Theorem 3. Since the relation of seasonal cycles (table 1) proves the relation among different seasonal unit roots, to demonstrate the existence of a systematically sampled and a temporally sampled SI process it is necessary to demonstrate that the corresponding aggregated error terms are invertible for any combination of seasonal unit roots at the monthly process. Then, let x_t be an univariate time series x_t generated by

$$\alpha(L)x_t = u_t, \tag{6}$$

where the S simple reciprocals of the roots of $\alpha(z)$ lie on the unit circle ($z_1 = e^{i\theta_1}$,..., $z_S = e^{i\theta_S}$) and u_t is a white noise process. Assume that some different roots $z_l = e^{i\theta_l} \neq z_j = e^{i\theta_j}$ of $\alpha(L)$ when raised to the positive integer m, are equal $z_l^m = e^{im\theta_l} = z_j^m = e^{im\theta_j}$, that is the model has hidden periodicity. Let us denote by n the number of subsets of unit roots of $\alpha(L)$ linked to a particular cycle $m\theta_j$ (j=1, ..., n), h_n is the number of unit roots in the subsets, and $h = \Sigma_1^n h_n$ is the total number of hidden unit roots. Similarly, we use the notation n^y, h_{n^y}, and h^y for the summed process with $n^y \leq n$ and $h^y \leq h$.

Equation (6) can be written in terms of the reciprocal roots of the AR polynomial:

$$\Pi_1^S(1 - e^{i\theta_j}L)x_t = u_t, \tag{7}$$

Then, let us multiply both sides of (7) by the lag polynomial $\Pi_1^S(1-e^{im\theta_j}L^m)/\Pi_1^S(1-e^{i\theta_j}L)$ to obtain the model:

$$\Pi_1^S(1 - e^{im\theta_j}L^m)x_t = \frac{\Pi_1^S(1 - e^{im\theta_j}L^m)}{\Pi_1^S(1 - e^{i\theta_j}L)}u_t. \tag{8}$$

The lags of the AR operator of model (8) are observable at a longer sampling interval m. Note that (8) is noninvertible.

Let us consider the summed process $y_t = s_m(L)x_t$, that from (6) reads:

$$\alpha(L)y_t = s_m(L)u_t. \tag{9}$$

The summation polynomial $s_m(L)$ contains $m-1$ unit roots,

$$s_m(L) = \begin{cases} (1+L)\Sigma_{j=1}^{m/2-1}(1 - e^{i\frac{2j\pi}{m}}L)(1 - e^{-i\frac{2j\pi}{m}}L), & m \text{ even} \\ \Sigma_{j=1}^{[m/2]}(1 - e^{i\frac{2j\pi}{m}}L)(1 - e^{-i\frac{2j\pi}{m}}L). & m \text{ odd} \end{cases} \tag{10}$$

such that $\alpha(L)$ and $s_m(L)$ may have $S - S^y \geq 0$ common unit roots denoted by $\beta(L)$:

$$\alpha^y(L)y_t = s_m^y(L)u_t, \tag{11}$$

where $\alpha^y(L) = \alpha(L)/\beta(L)$, and $s^y_m(L) = s_m(L)/\beta(L)$. Then let us multiply (11) by $\Pi_1^{S^y}(1 - e^{im\theta_j}L^m)/\Pi_1^{S^y}(1 - e^{i\theta_j}L)$:

$$\Pi_1^{S^y}(1 - e^{im\theta_j}L^m)y_t = \Pi_1^{S^y}\frac{1 - e^{im\theta_j}L^m}{1 - e^{i\theta_j}L}s^y_m(L)u_t, \tag{12}$$

a representation of y_t with all the AR lags multiples of m.

Let us introduce hidden periodicity of order m due to h θ_j-frequency unit roots ($2 \leq h \leq m$) such that $e^{i\theta_j} = e^{im\theta_j} = e^{i\theta^*}$ ($j = 1, ..., h$). Then the model of x_t (8) can be written

$$\Pi_1^{S-h}(1 - e^{i\theta_j}L)\Pi_{S-h+1}^{S}(1 - e^{i\theta_j}L)x_t = u_t. \tag{13}$$

Again, let us multiply (13) by $\Pi_1^{S-h}\frac{1-e^{im\theta_j}L^m}{1-e^{i\theta_j}L}\frac{(1-e^{i\theta^*}L^m)^h}{\Pi_{S-h+1}^{S}(1-e^{i\theta_j}L)}$:

$$\Pi_1^{S-h}\left(1 - e^{im\theta_j}L^m\right)(1 - e^{i\theta^*}L^m)^h x_t = \Pi_1^{S-h}\frac{1 - e^{im\theta_j}L^m}{1 - e^{i\theta_j}L}\frac{(1 - e^{i\theta^*}L^m)^h}{\Pi_{S-h+1}^{S}(1 - e^{i\theta_j}L)}u_t. \tag{14}$$

When $h < m$ the AR and MA polynomials of (14) have a common term $(1 - e^{i\theta^*}L^m)^{h-1}$:

$$\Pi_1^{S-h}\left(1 - e^{im\theta_j}L^m\right)(1 - e^{i\theta^*}L^m)x_t = \Pi_1^{S-h}\frac{1 - e^{im\theta_j}L^m}{1 - e^{i\theta_j}L}\frac{1 - e^{i\theta^*}L^m}{\Pi_{S-h+1}^{S}(1 - e^{i\theta_j}L)}u_t. \tag{15}$$

When $h = m$, $\Pi_{S-h+1}^{S}(1 - e^{i\theta_j}L) = 1 - e^{i\theta^*}L^m$ and (15) simplifies to:

$$\Pi_1^{S-h}\left(1 - e^{im\theta_j}L^m\right)(1 - e^{i\theta^*}L^m)x_t = \Pi_1^{S-h}\frac{1 - e^{im\theta_j}L^m}{1 - e^{i\theta_j}L}u_t. \tag{16}$$

We can obtain the model of y_t for the case of hidden periodicity in an analogous way than we have obtained the model of x_t.

Periodic AR model and we obtain the multivariate representation of U

The model of the systematically sampled series X without and with hidden periodicity are obtained by applying systematic sampling to (8) and (16), respectively. Then, the error terms of the SI bimonthly models are

$$\begin{aligned} U_\tau^s &= \left[\frac{1 - L^2}{(1 - L)^{d_0}(1 + L)^{d_6}}\right]^{d_0^*}\left[\frac{1 - L^2 + L^4}{(1 - \sqrt{3}L + L^2)^{d_1}(1 + \sqrt{3}L + L^2)^{d_5}}\right]^{d_1^*} \\ &\quad \left[\frac{1 + L^2 + L^4}{(1 - L + L^2)^{d_2}(1 + L + L^2)^{d_4}}\right]^{d_2^*} u_{2(T-1)+s}, \\ V_\tau &= \Sigma_{s=1}^2 U_\tau^s, \end{aligned}$$

where $d_0^* = \max\{d_0, d_6\}$, $d_1^* = \max\{d_1, d_5\}$, and $d_2^* = \max\{d_2, d_4\}$; the error terms of the SI quarterly processes are

$$\begin{aligned} U_T^s &= \left[\frac{1-L^3}{(1-L)^{d_0}(1+L+L^2)^{d_4}}\right]^{d_0^*} \left[\frac{1+L^6}{(1-\sqrt{3}L+L^2)^{d_1}(1+L^2)^{d_3}(1+\sqrt{3}L+L^2)^{d_5}}\right]^{d_1^*} \\ &\quad \left[\frac{1+L^3}{(1-L+L^2)^{d_2}(1+L)^{d_6}}\right]^{d_2^*} u_{3(T-1)+s}, \\ V_T &= \Sigma_{s=1}^3 U_T^s, \end{aligned}$$

where $d_0^* = \max\{d_0, d_4\}$, $d_1^* = \max\{d_1, d_3, d_5\}$, and $d_2^* = \max\{d_2, d_6\}$, and the error terms of the SI semi-annual processes are

$$\begin{aligned} U_\Upsilon^s &= \left[\frac{1-L^6}{(1-L)^{d_0}(1-L+L^2)^{d_2}(1+L+L^2)^{d_4}(1+L)^{d_6}}\right]^{d_0^*} \\ &\quad \left[\frac{1+L^6}{(1-\sqrt{3}L+L^2)^{d_1}(1+L^2)^{d_3}(1+\sqrt{3}L+L^2)^{d_5}}\right]^{d_1^*} u_{6(\Upsilon-1)+s}, \\ V_\Upsilon &= \Sigma_{s=1}^6 U_\Upsilon^s, \end{aligned}$$

where $d_0^* = \max\{d_0, d_2, d_4, d_6\}$, and $d_1^* = \max\{d_1, d_3, d_5\}$.

By Theorem 1 of Niemi (1984), the systematically sampled and temporally aggregated models are always invertible because the aggregate MA components U^s and V do not contain in any case the unit root component $(1 - e^{im\theta_k}L^m)$. ■

Appendix 2: Auxiliar Regressors for the Sequential Monthly Test

Case B

$$\begin{aligned} \widetilde{x}_{0,t-1} &= (L+L^2+L^3+L^7+L^8+L^9)x_t, \\ \widetilde{x}^\alpha_{1,t-1} &= (\frac{1}{2}\sqrt{3}L+\frac{1}{2}L^2-\frac{1+\sqrt{3}}{2}L^4-\frac{1+\sqrt{3}}{2}L^5-L^6+\frac{1}{2}L^7+\frac{1}{2}\sqrt{3}L^8+L^9)x_t, \\ \widetilde{x}^\beta_{1,t-1} &= (-\frac{1}{2}L-\frac{1}{2}\sqrt{3}L^2-L^3+\frac{1-\sqrt{3}}{2}L^4-\frac{1-\sqrt{3}}{2}L^5+L^6+\frac{\sqrt{3}}{2}L^7+\frac{1}{2}L^8)x_t, \\ \widetilde{x}^\alpha_{3,t-1} &= (-L^2+L^4+L^5-L^6-L^7+L^9)x_t, \\ \widetilde{x}^\beta_{3,t-1} &= (-L+L^3+L^4-L^5-L^6+L^8)x_t, \\ \widetilde{x}^\alpha_{4,t-1} &= (-\frac{1}{2}L-\frac{1}{2}L^2+L^3-\frac{1}{2}L^7-\frac{1}{2}L^8+L^9)x_t, \\ \widetilde{x}^\beta_{4,t-1} &= \frac{\sqrt{3}}{2}(L-L^2+L^7-L^8)x_t, \\ \widetilde{x}^\alpha_{5,t-1} &= (-\frac{1}{2}\sqrt{3}L+\frac{1}{2}L^2+(\frac{\sqrt{3}-1}{2})L^4+(\frac{\sqrt{3}-1}{2})L^5-L^6+\frac{1}{2}L^7-\frac{1}{2}\sqrt{3}L^8+L^9)x_t, \\ \widetilde{x}^\beta_{5,t-1} &= (-\frac{1}{2}L+\frac{\sqrt{3}}{2}L^2-L^3+(\frac{1+\sqrt{3}}{2})L^4-(\frac{1+\sqrt{3}}{2})L^5+L^6-\frac{1}{2}\sqrt{3}L^7+\frac{1}{2}L^8)x_t. \end{aligned}$$

Case C

$$\begin{aligned}
\widetilde{x}_{0,t-1} &= \left(L+L^2+L^3+L^4+L^5+L^6\right)x_t, \\
\widetilde{x}^{\alpha}_{2,t-1} &= (\frac{1}{2}L-\frac{1}{2}L^2-L^3-\frac{1}{2}L^4+\frac{1}{2}L^5+L^6)x_t, \\
\widetilde{x}^{\beta}_{2,t-1} &= -\frac{1}{2}\sqrt{3}(L+L^2-L^4-L^5)x_t, \\
\widetilde{x}^{\alpha}_{4,t-1} &= -\frac{1}{2}\sqrt{3}\left(L-L^2+L^4-L^5\right)x_t, \\
\widetilde{x}^{\beta}_{4,t-1} &= -\frac{1}{2}\sqrt{3}\left(L-L^2+L^4-L^5\right)x_t, \\
\widetilde{x}_{6,t-1} &= (-L+L^2-L^3+L^4-L^5+L^6)x_t.
\end{aligned}$$

Case D

$$\begin{aligned}
\widetilde{x}^{\alpha}_{1,t-1} &= (\frac{1}{2}\sqrt{3}L+\frac{1}{2}L^2-\frac{1-\sqrt{3}}{2}L^4+\frac{1-\sqrt{3}}{2}L^5-L^6-\frac{1}{2}L^7-\frac{1}{2}\sqrt{3}L^8-L^9)x_t, \\
\widetilde{x}^{\beta}_{1,t-1} &= -\frac{1}{2}(L+\sqrt{3}L^2+2L^3+(1+\sqrt{3})L^4+(1+\sqrt{3})L^5+2L^6+\sqrt{3}L^7+L^8)x_t, \\
\widetilde{x}^{\alpha}_{2,t-1} &= (\frac{1}{2}L-\frac{1}{2}L^2-L^3+\frac{1}{2}L^7-\frac{1}{2}L^8-L^9)x_t, \\
\widetilde{x}^{\beta}_{2,t-1} &= -\frac{1}{2}\sqrt{3}\left(L+L^2+L^7+L^8\right)x_t, \\
\widetilde{x}^{\alpha}_{3,t-1} &= \left(-L^2+L^4-L^5-L^6+L^7-L^9\right)x_t, \\
\widetilde{x}^{\beta}_{3,t-1} &= -\left(L-L^3+L^4+L^5-L^6+L^8\right)x_t \\
\widetilde{x}^{\alpha}_{4,t-1} &= (-\frac{1}{2}\sqrt{3}L+\frac{1}{2}L^2-\frac{1+\sqrt{3}}{2}L^4+\frac{1+\sqrt{3}}{2}L^5-L^6-\frac{1}{2}L^7+\frac{1}{2}\sqrt{3}L^8-L^9)x_t, \\
\widetilde{x}^{\beta}_{4,t-1} &= -\frac{1}{2}(L-\sqrt{3}L^2+2L^3+(1-\sqrt{3})L^4+(1-\sqrt{3})L^5+2L^6-\sqrt{3}L^7+L^8)x_t, \\
\widetilde{x}_{6,t-1} &= \left(-L+L^2-L^3-L^7+L^8-L^9\right)x_t.
\end{aligned}$$

Case E

$$\begin{aligned}
\widetilde{x}_{0,t-1} &= (L+L^2+L^3)x_t, \\
\widetilde{x}^{\alpha}_{4,t-1} &= (-\frac{1}{2}L-\frac{1}{2}L^2+L^3)x_t, \\
\widetilde{x}^{\beta}_{4,t-1} &= -\frac{1}{2}\sqrt{3}(L-L^2)x_t.
\end{aligned}$$

Case F

$$
\begin{aligned}
\widetilde{x}^{\alpha}_{1,t-1} &= (\frac{\sqrt{3}}{2}L + \frac{1}{2}L^2 - \frac{1}{2}L^4 - \frac{\sqrt{3}}{2}L^5 - L^6)x_t, \\
\widetilde{x}^{\beta}_{1,t-1} &= -\frac{1}{2}(L + \sqrt{3}L^2 + 2L^3 + \sqrt{3}L^4 + L^5)x_t, \\
\widetilde{x}^{\alpha}_{3,t-1} &= \left(-L^2 + L^4 - L^6\right) x_t, \\
\widetilde{x}^{\beta}_{3,t-1} &= -\left(L - L^3 + L^5\right) x_t, \\
\widetilde{x}^{\alpha}_{5,t-1} &= (-\frac{1}{2}\sqrt{3}L + \frac{1}{2}L^2 - \frac{1}{2}L^4 + \frac{1}{2}\sqrt{3}L^5 - L^6)x_t, \\
\widetilde{x}^{\beta}_{5,t-1} &= -\frac{1}{2}(L - \sqrt{3}L^2 + 2L^3 - \sqrt{3}L^4 + L^5)x_t.
\end{aligned}
$$

Case G

$$
\begin{aligned}
\widetilde{x}^{\alpha}_{2,t-1} &= (\frac{1}{2}L - \frac{1}{2}L^2 - L^3)x_t, \quad \widetilde{x}^{\beta}_{2,t-1} = -\frac{1}{2}\sqrt{3}(L + L^2)x_t, \\
\widetilde{x}_{6,t-1} &= -(L - L^2 + L^3)x_t.
\end{aligned}
$$

Appendix 3: Critical Values

Table 10: Critical values for the quarterly statistics of the sequential test SE1

		Seasonal dummies					Seasonal dummies and time trend				
	A_1	10	20	30	40	50	10	20	30	40	50
t_0	0.01	−3.80	−3.80	−3.78	−3.78	−3.81	−4.39	−4.36	−4.33	−4.25	−4.35
	0.05	−3.20	−3.21	−3.23	−3.26	−3.26	−3.78	−3.77	−3.79	−3.77	−3.82
	0.10	−2.92	−2.95	−2.97	−2.99	−3.00	−3.49	−3.50	−3.53	−3.52	−3.55
F_1	0.01	11.22	10.56	10.43	10.31	10.14	11.14	10.46	10.35	10.33	10.08
	0.05	8.54	8.08	8.20	8.25	8.08	8.36	8.08	8.19	8.19	8.06
	0.10	7.30	7.16	7.16	7.21	7.13	7.18	7.10	7.11	7.19	7.11
t_2	0.01	−3.87	−3.74	−3.75	−3.78	−3.74	−3.88	−3.75	−3.75	−3.76	−3.74
	0.05	−3.24	−3.22	−3.23	−3.25	−3.24	−3.24	−3.23	−3.23	−3.24	−3.24
	0.10	−2.94	−2.95	−2.97	−2.96	−2.99	−2.94	−2.95	−2.97	−2.96	−2.99

Note: CVs generated with $x_T - x_{T-4} = \varepsilon_T$~i.i.d.N(0,1) $T = 1, ..., 4N = \{10, 20, 30, 40, 50\}$. Overall levels ($A_1$) of 0.01, 0.05 and 0.10 correspond to individual levels of 0.00334, 0.01695, and 0.03451, respectively. To generate the critical values, 30000 Monte Carlo replications are used with GAUSS RNDN function of GAUSS for Windows NT/95 Version 3.2.38.

Table 11: Critical values for the quarterly statistics of the sequential test SE2

		Seasonal dummies					Seasonal dummies and time trend				
	A_2	10	20	30	40	50	10	20	30	40	50
t_0	0.01	−5.22	−5.03	−4.60	−4.68	−4.83	−5.82	−5.19	−5.51	−5.07	−5.14
	0.05	−3.94	−3.87	−3.87	−3.89	−3.86	−4.59	−4.43	−4.42	−4.44	−4.31
	0.10	−3.41	−3.42	−3.48	−3.45	−3.46	−4.02	−4.00	−4.00	−3.99	−3.96
F_1	0.01	19.16	16.42	15.96	14.06	15.35	19.42	16.42	16.28	14.07	15.24
	0.05	12.22	11.09	11.08	10.49	10.57	12.13	10.98	11.07	10.48	10.57
	0.10	9.71	8.93	9.22	8.96	8.85	9.58	8.90	9.20	8.95	8.81
t_2	0.01	−4.77	−4.68	−4.64	−4.61	−4.93	−4.78	−4.68	−4.72	−4.61	−4.88
	0.05	−3.90	−3.88	−3.88	−3.86	−3.82	−3.91	−3.86	−3.88	−3.86	−3.83
	0.10	−3.43	−3.45	−3.46	−3.43	−3.47	−3.43	−3.45	−3.46	−3.43	−3.46

Note: CVs generated with $x_T - x_{T-4} = \varepsilon_T$~i.i.d.N(0,1) $T = 1, ..., 4N = \{10, 20, 30, 40, 50\}$. Overall levels ($A_2$) of 0.01, 0.05 and 0.10 correspond to individual levels of 0.0001, 0.00239 and 0.00917, respectively. To generate the critical values, 30000 Monte Carlo replications are used with GAUSS RNDN function of GAUSS for Windows NT/95 Version 3.2.38.

Table 12: Critical values for the monthly statistics of the restricted case B

		Seasonal dummies					Seasonal dummies and time trend				
		10	20	30	40	50	10	20	30	40	50
$\widetilde{t}_0$	0.01	−3.21	−3.32	−3.36	−3.38	−3.37	−3.73	−3.86	−3.88	−3.90	−3.90
	0.05	−2.65	−2.77	−2.79	−2.82	−2.84	−3.17	−3.29	−3.32	−3.36	−3.36
	0.10	−2.38	−2.47	−2.50	−2.53	−2.53	−2.89	−3.01	−3.04	−3.08	−3.08
$\widetilde{F}_1$	0.01	8.14	8.39	8.35	8.73	8.70	8.17	8.38	8.34	8.70	8.70
	0.05	5.91	6.25	6.36	6.49	6.46	5.93	6.24	6.35	6.50	6.45
	0.10	4.97	5.25	5.37	5.46	5.47	4.95	5.25	5.36	5.46	5.46
$\widetilde{F}_3$	0.01	7.97	8.41	8.54	8.59	8.55	7.90	8.41	8.51	8.59	8.53
	0.05	5.91	6.22	6.36	6.50	6.55	5.87	6.21	6.35	6.49	6.54
	0.10	4.98	5.27	5.39	5.46	5.53	4.95	5.26	5.38	5.46	5.51
$\widetilde{F}_4$	0.01	7.97	8.37	8.58	8.55	8.82	7.93	8.34	8.58	8.55	8.81
	0.05	5.87	6.23	6.36	6.42	6.52	5.85	6.20	6.36	6.41	6.51
	0.10	4.95	5.25	5.38	5.43	5.48	4.92	5.24	5.37	5.42	5.48
$\widetilde{F}_5$	0.01	8.02	8.45	8.54	8.51	8.70	7.99	8.44	8.52	8.52	8.70
	0.05	5.91	6.27	6.44	6.45	6.52	5.85	6.25	6.44	6.43	6.51
	0.10	4.94	5.31	5.41	5.44	5.50	4.91	5.30	5.40	5.43	5.50

Note: CVs generated with $(1 - L^3 + L^6 - L^9)x_t = \varepsilon_t$~i.i.d.N(0,1) $t = 1, ..., 12N$={10,20,30,40,50}.

Table 13: Critical values for the monthly statistics of the restricted case C

		Seasonal dummies					Seasonal dummies and time trend				
		10	20	30	40	50	10	20	30	40	50
$\widetilde{t}_0$	0.01	−3.23	−3.33	−3.35	−3.39	−3.42	−3.75	−3.86	−3.87	−3.91	−3.89
	0.05	−2.67	−2.78	−2.78	−2.82	−2.83	−3.20	−3.31	−3.34	−3.37	−3.36
	0.10	−2.38	−2.48	−2.50	−2.53	−2.53	−2.91	−3.03	−3.05	−3.08	−3.08
$\widetilde{F}_2$	0.01	8.20	8.39	8.54	8.61	8.79	8.18	8.36	8.50	8.60	8.76
	0.05	6.05	6.32	6.38	6.50	6.49	5.99	6.31	6.36	6.49	6.48
	0.10	5.00	5.33	5.41	5.49	5.52	4.97	5.32	5.41	5.48	5.51
$\widetilde{F}_4$	0.01	8.19	8.41	8.34	8.62	8.70	8.18	8.40	8.31	8.62	8.70
	0.05	6.03	6.27	6.39	6.49	6.49	5.98	6.26	6.37	6.47	6.49
	0.10	5.03	5.29	5.44	5.47	5.46	4.98	5.28	5.43	5.46	5.45
$\widetilde{t}_6$	0.01	−3.24	−3.35	−3.36	−3.38	−3.38	−3.22	−3.35	−3.35	−3.38	−3.38
	0.05	−2.65	−2.75	−2.79	−2.82	−2.84	−2.65	−2.75	−2.79	−2.82	−2.84
	0.10	−2.37	−2.47	−2.50	−2.52	−2.52	−2.37	−2.47	−2.50	−2.52	−2.52

Note: CVs generated with $(1-L^6)x_t = \varepsilon_t$~i.i.d.N(0,1) $t=1,...,12N$={10,20,30,40,50}.

Table 14: Critical values for the monthly statistics of the restricted case D

		Seasonal dummies					Seasonal dummies and time trend				
		10	20	30	40	50	10	20	30	40	50
$\widetilde{F}_1$	0.01	8.07	8.30	8.51	8.64	8.62	8.09	8.32	8.50	8.62	8.61
	0.05	5.86	6.19	6.37	6.46	6.44	5.82	6.16	6.36	6.45	6.43
	0.10	4.92	5.26	5.38	5.47	5.46	4.88	5.24	5.36	5.46	5.45
$\widetilde{F}_2$	0.01	7.95	8.34	8.56	8.53	8.66	7.88	8.32	8.51	8.50	8.65
	0.05	5.93	6.23	6.39	6.37	6.51	5.87	6.20	6.37	6.36	6.50
	0.10	4.98	5.25	5.39	5.44	5.53	4.92	5.22	5.38	5.42	5.51
$\widetilde{F}_3$	0.01	8.01	8.33	8.52	8.61	8.65	7.92	8.27	8.48	8.59	8.63
	0.05	5.89	6.24	6.38	6.43	6.53	5.82	6.18	6.35	6.40	6.52
	0.10	4.94	5.25	5.38	5.43	5.51	4.89	5.22	5.36	5.41	5.50
$\widetilde{F}_5$	0.01	8.06	8.41	8.49	8.59	8.63	8.00	8.37	8.47	8.57	8.62
	0.05	5.87	6.26	6.38	6.37	6.45	5.80	6.23	6.35	6.36	6.43
	0.10	4.92	5.26	5.39	5.43	5.49	4.86	5.23	5.36	5.42	5.47
$\widetilde{t}_6$	0.01	−3.26	−3.32	−3.35	−3.38	−3.38	−3.24	−3.31	−3.34	−3.38	−3.37
	0.05	−2.68	−2.78	−2.79	−2.82	−2.80	−2.66	−2.77	−2.78	−2.81	−2.80
	0.10	−2.39	−2.48	−2.50	−2.54	−2.51	−2.37	−2.47	−2.49	−2.53	−2.50

Note: CVs generated with $(1+L^3+L^6+L^9)x_t = \varepsilon_t$~i.i.d.N(0,1) $t=1,...,12N$={10,20,30,40,50}.

Table 15: Critical values for the monthly statistics of the restricted case E

		Seasonal dummies					Seasonal dummies and time trend				
		10	20	30	40	50	10	20	30	40	50
	0.01	−3.26	−3.34	−3.36	−3.37	−3.45	−3.76	−3.82	−3.89	−3.92	−3.90
$\widetilde{t}_0$	0.05	−2.69	−2.78	−2.79	−2.82	−2.84	−3.21	−3.28	−3.33	−3.36	−3.38
	0.10	−2.39	−2.48	−2.50	−2.52	−2.54	−2.93	−3.01	−3.04	−3.07	−3.10
	0.01	8.31	8.48	8.71	8.63	8.71	8.25	8.46	8.71	8.61	8.71
$\widetilde{F}_4$	0.05	6.11	6.39	6.51	6.50	6.51	6.06	6.38	6.49	6.49	6.51
	0.10	5.12	5.40	5.47	5.47	5.51	5.08	5.39	5.47	5.46	5.51

Note: CVs generated with $(1 - L^3)x_t = \varepsilon_t$~i.i.d.N(0,1) $t = 1, ..., 12N$={10,20,30,40,50}.

Table 16: Critical values for the monthly statistics of the restricted case F

		Seasonal dummies					Seasonal dummies and time trend				
		10	20	30	40	50	10	20	30	40	50
	0.01	8.15	8.42	8.72	8.68	8.73	8.10	8.42	8.68	8.64	8.70
$\widetilde{F}_1$	0.05	5.99	6.29	6.47	6.46	6.50	5.95	6.29	6.45	6.45	6.49
	0.10	4.97	5.30	5.39	5.46	5.49	4.93	5.28	5.37	5.45	5.48
	0.01	8.06	8.54	8.61	8.66	8.61	7.99	8.52	8.59	8.63	8.59
$\widetilde{F}_3$	0.05	5.98	6.32	6.42	6.49	6.51	5.91	6.30	6.40	6.47	6.50
	0.10	5.02	5.30	5.42	5.45	5.51	4.95	5.27	5.40	5.44	5.50
	0.01	8.13	8.40	8.73	8.74	8.68	8.08	8.33	8.71	8.72	8.66
$\widetilde{F}_5$	0.05	6.03	6.27	6.44	6.47	6.55	5.96	6.24	6.42	6.45	6.53
	0.10	5.04	5.30	5.43	5.48	5.52	4.98	5.27	5.42	5.47	5.51

Note: CVs generated with $(1 + L^6)x_t = \varepsilon_t$~i.i.d.N(0,1) $t = 1, ..., 12N$={10,20,30,40,50}.

Table 17: Critical values for the monthly statistics of the restricted case G

		Seasonal dummies					Seasonal dummies and time trend				
		10	20	30	40	50	10	20	30	40	50
	0.01	8.50	8.57	8.62	8.60	8.59	8.43	8.50	8.58	8.58	8.58
$\widetilde{F}_2$	0.05	6.13	6.37	6.43	6.52	6.50	6.07	6.34	6.42	6.51	6.48
	0.1	5.16	5.34	5.45	5.54	5.50	5.10	5.30	5.43	5.52	5.48
	0.01	−3.22	−3.36	−3.34	−3.38	−3.37	−3.20	−3.35	−3.33	−3.37	−3.36
$\widetilde{t}_6$	0.05	−2.66	−2.76	−2.79	−2.79	−2.82	−2.64	−2.75	−2.78	−2.79	−2.81
	0.10	−2.38	−2.47	−2.51	−2.50	−2.55	−2.37	−2.46	−2.50	−2.50	−2.54

Note: CVs generated with $(1 + L^3)x_t = \varepsilon_t$~i.i.d.N(0,1) $t = 1, ..., 12N$={10,20,30,40,50}.

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