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Multi-Unit Demand

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INSTITUT FOR ØKONOMI

AFDELING FOR NATIONALØKONOMI - AARHUS UNIVERSITET - BYGNING 350
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8000 AARHUS C - DENMARK ☎ +45 89 42 11 33 - TELEFAX +45 86 13 63 34

Buy-Out Prices in Online Auctions: Multi-Unit Demand

René Kirkegaard and Per Baltzer Overgaard
Department of Economics, University of Aarhus
DK-8000 Aarhus C, Denmark*

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Abstract

On many online auction sites it is now possible for a seller to augment his auction with a maximum or buy-out price. The use of this instrument has been justified in "one-shot" auctions by appeal to impatience or risk aversion. Here we offer additional justification by observing that trading on internet auctions is not of a "one-shot" nature, but that market participants expect more transactions in the future. This has important implications when bidders desire multiple objects. Specifically, it is shown that an early seller has an incentive to introduce a buy-out price, if similar products are offered later on by other sellers. The buy-out price will increase revenue in the current auction, but revenue in future auctions will decrease, as will the sum of revenues. In contrast, if a single seller owns multiple units, overall revenue will increase, if buyers anticipate the use of buy-out prices in the future by this seller. In both cases, an optimally chosen buy-out price introduces potential inefficiencies in the allocation.

*E-mail: rkirkegaard@econ.au.dk and povergaard@econ.au.dk. Revised versions will be available at:
http://www.econ.au.dk/vip_htm/povergaard/pbohome/pbohome.html

1 Introduction

The presence of buy-out prices[1] in online auctions has thus far been explained by focusing on a *single* auction and assuming that individuals exhibit either *risk aversion* or *impatience*.[2] In this paper we take a somewhat broader view of auction markets, realizing, in particular, that buyers and sellers alike are aware of the fact that new products will be offered on the market in the future. This will tend to depress revenue in today's auctions, as buyers know that close substitutes will be offered tomorrow. In this *dynamic* environment we will show that there are at least two reasons to introduce buy-out prices, even if agents are patient and risk neutral.

Buy-out prices or maximum prices in online auctions were noted by Lucking-Reiley (2000) in his empirical overview of auction activities on the Internet. Since (sell) auctions are ostensibly staged to illicit high prices in situations where markets are thin and sellers are short on information about the willingness-to-pay of potential buyers, such buy-out prices may appear surprising. In fact, Lucking-Reiley explicitly posed this as a challenge to theorists. In addition, he quoted evidence to suggest that the *exercise* of posted buy-out options is not uncommon in online auctions.[3]

Reynolds and Wooders (2002) provide some additional information on the *frequency of buy-out prices* in *Yahoo!* and *eBay* auctions, though, *not* on the frequency with which the option was *exercised* by some bidder. The categories sampled on March 27, 2002, were automobiles, clothing, DVD players, VCR's, digital cameras and TV sets. A total of 1.248 auctioned items were sampled from *Yahoo!*, of which 842 had a buy-out price posted by the seller (roughly, 66%). In similar fashion, 31.142 auctioned items were sampled from *eBay*, of which 12.480 had a buy-out price posted by the seller (roughly,

[1]Alternatively, this is often referred to as *buy prices* or *maximum prices*. In offline settings, this phenomenon also has a certain affinity with "$\$xx$ or best offer", where it is, presumably, implicit that, if someone makes an offer of $\$xx$, then the trade is finalized immediately, while if someone makes a lower offer initially, then the seller will wait a while to see if a better offer comes along. Also, a buy-out price has a certain similarity with a massive *jump bid* intended to end an auction quickly.

[2]See, Budish and Takeyama (2001), Mathews (2002) and Reynolds and Wooders (2002). In future work we hope to return to the use of buy-out prices in auctions where sellers try to respond to possible *bidder collusion*.

[3]He quotes the case of *LabX* (a lab equipment auction site), where buy-out options are exercised by some bidder in 10% of the cases where they appear. Hence, buy-out prices do more than just attract attention.

40%). There is some variation across the categories of goods sampled, but the frequency of buy-out prices never drops below 25% in the sample. Hence, in these categories, at least, the appearance of buy-out prices is very frequent.

For *eBay*, Mathews (2002) presents some numbers on the *frequency with which buy-out options are exercised* when offered.[4] For two categories of games (racing and sports) for *Sony PS2*, Mathews reports that on January 29 - 30, 2001, 210 items were on offer. A buy-out option was available on 124 items (59%), and it was exercised 34 times (27% of the times it was offered). So, at least in these categories, the exercise frequency is high.

Formally, we analyze *eBay*'s version of a buy-out price, termed the *Buy It Now* price. Here is how the *Buy It Now* price roughly works from the seller's viewpoint:[5] "If a buyer is willing to meet your *Buy It Now* price before the first bid comes in, your item sells instantly and your auction ends. Or, if a bid comes in first, the *Buy It Now* option disappears. Then your auction proceeds normally." Hence, in *eBay* auctions, the buy-out price is temporary.[6]

Throughout this paper we assume that potential buyers or bidders have *multi-unit demands*, with diminishing marginal utility. With two objects for sale and at least two bidders, it has been shown by Black and de Meza (1992) that auction revenue will increase over time and that the auction outcome is efficient under these assumptions. In particular, in a sequence of second-price or English auctions, the seller offering his good today will not earn as much as a competing seller offering a similar good tomorrow, that is, prices are increasing.[7]

However, for the case with *two individual sellers*, we show that the first seller can always increase his revenue by introducing a buy-out price. The

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[4] He also presents aggregate numbers on the frequency with which buy-out prices are offered at *eBay*. The reported range around 40% is roughly in line with the numbers reported for specific categories by Reynolds and Wooders (2002).

[5] For more details on the *eBay* version and other versions of a buy-out price, see e.g. Lucking-Reiley (2000), Budish and Takeyama (2001), Mathews (2002) and Reynolds and Wooders (2002).

[6] For more details on the *Buy It Now* feature in *eBay* auctions the reader should consult pages.ebay.com/help/sell/bin.html. *eBay* introduced this feature in January 2001.

[7] In fact, Black and de Meza (1992) were interested in what some have referred to as *The Declining Price Anomaly.* Therefore, they went on to consider an option of the following kind: the winner of the first item is given the option of buying the second item at the same price. This, apparently, is observed in certain multi-unit auctions, and it is enough to lead to a declining price path.

revenue to the second seller is adversely affected, as is overall revenue. An optimally chosen buy-out price in the first auction also introduces *inefficiency*, in the sense that *a bidder who should have won no object wins one.* Our analysis is partial in the following sense. We consider a sequence of two second-price (or English) auctions, allowing the first seller the possibility of introducing a buy-out price without giving the second seller the opportunity to respond in kind. Thus, we essentially show that an auction market without buy-out prices is unstable, in the sense that current sellers will try to force the auction site to (at least temporarily) allow buy-out prices.

Next, we consider the consequences of buy-out prices for a *single seller* intending to sell two objects. We show that this seller can increase his total expected revenue by augmenting the second auction with a buy-out price, which depends on the outcome of the first auction. The buy-out price should be set fairly low, thus allowing the winner of the first auction a disproportionately large chance of winning the second auction as well. Hence, the sequence of auctions is *inefficient*, in the sense that *one buyer may win two objects when efficiency dictates he should only win one.* In this case overall revenue will increase. The reason is the same as that which induces a monopolist to offer quantity discounts that are detrimental to efficiency: buyers with high demand contribute with higher marginal revenue on two objects than buyers with low demand do on one object.

The rest of the paper is organized as follows. In Section 2 we set up a simple model and present the results for the bench-mark case where a sequence of two second-price auctions is staged. Then, Section 3 shows that the first seller among a pack of competing sellers can increase his lot by offering a buy-out price. Section 4 then examines the use of buy-out prices by a single seller offering more than one good. Section 5 contains a few concluding remarks. A selection of proofs is in the Appendix.

2 Model and Bench-Mark

In this section we first set up the model and then derive results for the bench-mark case where a sequence of two second-price auctions is staged.

2.1 Model

We assume that two objects are offered for sale sequentially,[8] and that there are two potential buyers on the market. Hence, the number of objects coincide with the number of buyers, this number being equal to two in order to make the analysis manageable. Each buyer i, $i = 1, 2$, is characterized by a type, v_i, drawn from a continuously differentiable distribution function, $F(v_i)$, without mass points. We assume that $v_i \in [\underline{v}, \overline{v}]$. The value to bidder i of the first unit purchased is v_i, while the value of the second unit is kv_i, $0 < k < 1$. Hence, each bidder desires both units, but individual demands are downward sloping.

2.2 Two straight second-price auctions

To keep the analysis simple, we have ignored reserve prices in the auctions. In this setting, Black and de Meza (1992) were the first[9] to solve for equilibrium strategies in a sequence of two second-price (or English) auctions, under more general assumptions than those considered here.[10] Applied to our set of assumptions, they find the following.

Proposition 1 (Black and de Meza (1992)) *When there are two a priori symmetric agents in the game, the unique symmetric equilibrium is for agent i to bid kv_i in stage one, and bid v_i in stage two if stage one was lost, and kv_i otherwise. The equilibrium outcome is efficient.*

Thus, in the last round, a bidder simply bids his valuation of the remaining object. This, however, depends on whether the bidder won or lost the first object. In the first round, each bidder bids k times his valuation for the first item won. Hence, the first object is sold for a price equal to k times the lowest valuation, while the second object is sold for a price equal to the

[8] The two objects are considered homogenous by the bidders, or they are simply two units of the same good.

[9] See also Katzman (1999).

[10] Black and de Meza explicitly consider sealed-bid auctions, while they also have an informal discussion of English auctions. Throughout our formal analysis, we restrict attention to a setting with two bidders, in which case second-price and English auctions are equivalent. With more than two bidders this equivalence may break down. In the informal discussion immediately below, we comment on some key properties of second-price, sealed-bid auctions with arbitrary numbers of bidders.

maximum of k times the highest valuation and the lowest valuation. From this, it follows immediately that the revenue of the first auction is lower than the revenue of the second.

To see what is going on here, let us start by making a few general remarks on second-price, sealed-bid auctions in the independent, private values case with n bidders. We first note that in case of *symmetric*, *increasing* bidding strategies, the fine details of any bidder's bid function are only consequential if there happens to be a competing bidder who has a valuation very close to that of the bidder in question. Hence, in equilibrium a bidder's strategy is pinned down by an indifference relation: the bidder should be indifferent between winning and losing, if his toughest competitor is identical to himself. To proceed, let us take the perspective of bidder i and label his rivals j, $j = 1, 2, ..., n-1$. Now, i's competitors have random valuations of the first item denoted Y_i with associated order-statistics $Y_{[1]} \geq Y_{[2]} \geq .... \geq Y_{[n-1]}$. Let i be male and all the rivals female.

In a *one-shot*, second-price auction bidder i essentially bids what he expects it to take to win the item, *if* he is the "top dog" - the high-valuation bidder - *and* there is someone like him among the rivals. The relevant indifference relation can be written as

$$\overbrace{v_i - b(E(Y_{[1]} \mid Y_{[1]} = v_i))}^{\text{just winning}} = \overbrace{0}^{\text{just losing}}$$

However, $E(Y_{[1]} \mid Y_{[1]} = v_i) = v_i$, and the optimal bid of i is given by

$$b(v_i) = E(Y_{[1]} \mid Y_{[1]} = v_i) = v_i$$

Thus, we obtain the familiar result that it is optimal for bidder i to bid his valuation.

In a *sequence of two* second-price auctions things are a little more complicated. Consider the last round first. If i won the first item, his valuation of the second item is $v_i^2 = kv_i$. Then, in the last round, bidder i's indifference relation is predicated on $Y_{[1]} = kv_i$ (the toughest competitor is like him *at this stage*). Thus, we can write

$$\overbrace{v_i^2 - b^2(E(Y_{[1]} \mid Y_{[1]} = v_i^2))}^{\text{just winning second}} = \overbrace{0}^{\text{just losing second}}$$

where $b^2(\cdot)$ denotes the second-round bid. Substituting for v_i^2 and noting that $E(Y_{[1]} \mid Y_{[1]} = kv_i) = kv_i$, we obtain

$$b^2(v_i^2) = b^2(kv_i) = E(Y_{[1]} \mid Y_{[1]} = kv_i) = kv_i$$

Similarly, if i lost the first item, his valuation of the second item is $v_i^2 = v_i$. Then, in the last round, bidder i's indifference relation is predicated on $\max\{kY_{[1]}, Y_{[2]}\} = v_i$ (the toughest competitor is like him *at this stage*). We can write this as

$$\overbrace{v_i^2 - b^2(E(\max\{kY_{[1]}, Y_{[2]}\} \mid \max\{kY_{[1]}, Y_{[2]}\} = v_i^2))}^{\text{just winning second}} = \overbrace{0}^{\text{just losing second}}$$

and we obtain

$$b^2(v_i^2) = b^2(v_i) = E(\max\{kY_{[1]}, Y_{[2]}\} \mid \max\{kY_{[1]}, Y_{[2]}\} = v_i) = v_i$$

The upshot is that bidder i should bid kv_i in the last round if he won the first and v_i if he lost. This is just bidding one's value in the last round.

More interestingly, consider the first round. We note that if i is the "top dog" and there is someone like i in the pack of rivals, then they each win one item in equilibrium.[11] Hence, optimal bidding by i in the first round is derived from an indifference between winning the first and the second item, which (using the results already derived) we can write as

$$\overbrace{[v_i - \underbrace{b^1(v_i)}_{b^1(Y_{[1]}) \text{ with } Y_{[1]} = v_i}] + 0}^{\text{just winning first and losing second}} = \overbrace{0 + [v_i - E(\max\{kY_{[1]}, Y_{[2]}\} \mid Y_{[1]} = v_i)]}^{\text{just losing first and winning second}}$$

Thus, in the first auction, bidder i should bid what he expects to have to pay to win the second, if he just loses the first. That is, optimal bidding in the first round is captured by

$$b^1(v_i) = E(\max\{kY_{[1]}, Y_{[2]}\} \mid Y_{[1]} = v_i) = E(\max\{kv_i, Y_{[2]}\} \mid Y_{[1]} = v_i)$$

In the general case with n bidders, we conclude that bidder i should bid the expectation of the *maximum* of k times his strongest rival's valuation of the

[11] When strategies are symmetric and increasing, the first auction is won if the toughest rival has a lower valuation, and lost if the toughest rival has a higher valuation. If the toughest rival has the same valuation as the agent himself, there is a tie, and the winner of the first auction is determined by chance. We argue that the agent must be indifferent between winning and losing the first auction in this case. Assume, to the contrary, that the agent prefers to win (lose) against an identical, strongest rival. Then, the agent should bid more (less) aggressively at the outset to win (lose) with probability one (rather than one half). This implies that the original strategies are not in equilibrium, unless the indifference condition is satisfied.

first item and his second strongest rival's valuation of the first item *predicated on the strongest rival being identical to himself.*

Finally, let us specialize to the two-bidder case. When $n = 2$, $Y_{[2]}$ is zero by construction, and the optimal bid of i reduces to

$$b^1(v_i) = E(\max\{kY_{[1]}, Y_{[2]}\} \mid Y_{[1]} = v_i) = E(\max\{kY_{[1]}, 0\} \mid Y_{[1]} = v_i) = kv_i$$

as stated in the proposition above.

Our next result characterizes the expected revenues associated with the equilibium strategies in Proposition 1.

Lemma 1 *In two straight second-price auctions with two bidders, the expected revenues in the first and second auctions are, respectively,*

$$ER_1^{SSP} = k \int_{\underline{v}}^{\overline{v}} 2x(1 - F(x))f(x)dx \tag{1}$$

and

$$\begin{aligned} ER_2^{SSP} &= \int_{\underline{v}}^{\max\{\underline{v}, k\overline{v}\}} 2x(1 - F(\frac{x}{k}))f(x)dx \\ &\quad + k \int_{\underline{v}}^{\overline{v}} 2x(F(x) - F(\max\{\underline{v}, kx\}))f(x)dx \end{aligned} \tag{2}$$

Proof. In the first auction, players bid k times their valuation, and the price is equal to the lowest bid. Hence, expected revenue is k times the expected value of the second highest valuation, which is just (1).

In the second auction there are two possible outcomes, depending on whether the same or different bidders win the two objects. The first term in (2) captures the possibility that the winner of the first object is also the winner of the second. Since the loser of the first auction bids his valuation, x say, and the winner bids k times her valuation, the price is precisely x when one player has valuation x and the other has a valuation that exceeds x/k.

However, it is also possible that the runner up in the first auction becomes the winner of the second, and this is the second term in (2). If the winner of the first auction has valuation x, her bid will be kx in the second auction. Hence, the price is kx in the second auction when one agent has type x, and the other agent has a type that is lower than x, yet sufficiently high that the bid submitted by this player exceeds kx. ■

From this we note that $ER_1^{SSP} \to 0$ and $ER_2^{SSP} \to 0$ as $k \to 0$. This, however, is just a special version of Weber's (1983) result that a sequence of second-price (or English) auctions where bidders have *unit demands* yields the same expected revenue to all sellers. With only two bidders and two items for sale, the equilibrium revenue is zero to both sellers. It is impossible to extract rent from buyers when there is no excess demand, recalling our assumption of no reserve prices.[12]

Similarly, we note that $ER_1^{SSP} \to \int_{\underline{v}}^{\overline{v}} 2x(1-F(x))f(x)dx = E(v_{[2]})$ and $ER_2^{SSP} \to \int_{\underline{v}}^{\overline{v}} 2x(1-F(x))f(x)dx = E(v_{[2]})$ as $k \to 1$. $E(v_{[2]})$ is just the expectation of the lowest of the two independent randoms draws from $F(v)$. When $k = 1$, *individual demands are horizontal*, and the behavior in the second auction is independent of the outcome of the first auction. The high valuation bidder will win both objects at a price of $v_{[2]}$, and revenue is the same in both auctions.

Example: The uniform case ($v \in [0,1]$)

To give a flavor of the results, let us consider the uniform case with $v \in [0,1]$, that is, $\underline{v} = 0$ and $\overline{v} = 1$. Thus, $f(v) = 1$ and $F(v) = v$. In this case, the expected revenues in the two auctions reduce to

$$ER_1^{SSP} = k \int_0^1 2x(1-x)dx = \frac{1}{3}k$$

and

$$\begin{aligned} ER_2^{SSP} &= \int_0^1 2x(1-\frac{x}{k})dx + k \int_{\underline{v}}^{\overline{v}} 2x(x-kx))dx \\ &= \frac{1}{3}k + \frac{1}{3}k(1-k) = ER_1^{SSP} + \frac{1}{3}k(1-k) \end{aligned}$$

We plot these expected revenues against k in Fig. 1, where ER_2^{SSP} is the *heavy* line, while ER_1^{SSP} is *thin*.

[12] Our general argument above for the n bidder case captures further aspects of Weber's results. With $k = 0$, bidding in both the first and the second auction is ultimately based purely on the expected *second* highest value among a bidder's rivals, thus, on the *third order* statistic $v_{[3]}$ of the n random valuations. From this it follows that expected revenue is the same in the two auctions when $k = 0$ (unit demands).

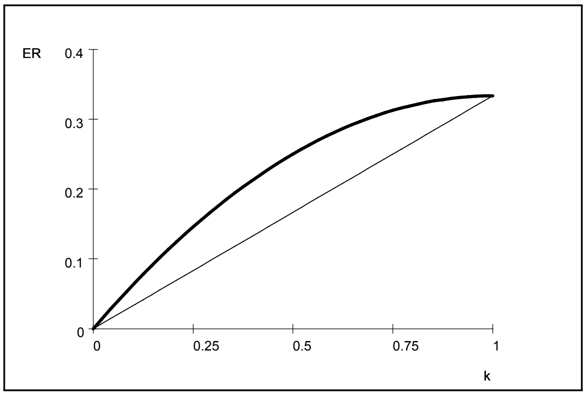

Fig. 1: Two straight second-price auctions

The *ratio* between expected revenues in the first and second auction, $RR\ (SSP) = \frac{ER_1^{SSP}}{ER_2^{SSP}} = \frac{1}{2-k}$, is illustrated in Fig. 2. Note the discontinuity at $k = 0$. When $k = 0$, both sellers earn nothing, that is, the same. However, when k is small, but strictly positive, we observe that the winner of the first auction is very unlikely also to be the winner of the second auction. Hence, the expected revenue in the first auction is k times (the expected value of) the second highest valuation, while the expected reveue in the second auction is *approximately* k times (the expected value of) the highest valuation. For the uniform case considered here, the ratio between the expected value of the highest (2/3) and the expected value of the second highest valuation (1/3) is exactly 1/2.

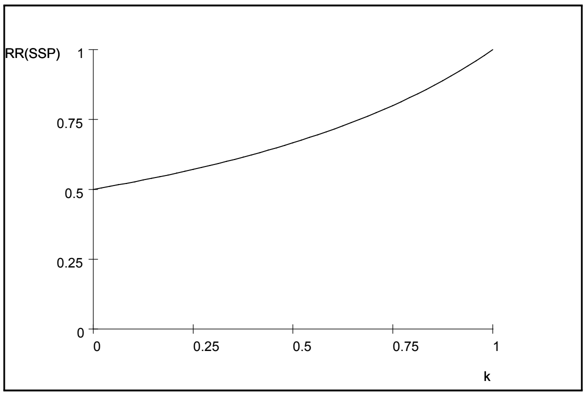

Fig. 2: The revenue-ratio in two SSP auctions

From this example it is immediate that the difference in expected revenues is significant unless k is close to one (demands are near-horizontal). For example, if $k = \frac{2}{3}$, then $ER_1^{SSP} = \frac{2}{9}$ and $ER_2^{SSP} = \frac{8}{27}$, and it follows that the (expected) first-auction revenue is only *75%* of the second-auction revenue. (*End of example*)

Given the increasing path of revenues over two straight second-price auctions, it is clear that the first of *two independent sellers* has an incentive to change the auction format.[13] In this paper we shall *first* restrict attention to the possible role of a buy-out price in the first auction when two independent sellers are selling identical objects. The first seller is interested in shifting revenues from the second to the first auction, while we shall also be interested in the consequences for efficiency and total revenue when the buy-out price is set optimally by the first seller. *Subsequently*, we turn to the case where there is a *single seller* who attempts to sell two identical objects in a sequence of auctions. Absent discounting (impatience), this seller is only interested in total expected revenue from the two auctions, while he is indifferent as to whether renenues are increasing or decreasing over the sequence. We show, however, that a suitably chosen buy-out price in the second auction, depending on the outcome of the first auction, can increase the total expected revenue of a single seller at the potential expense of efficiency.

To ease the exposition, we make the following assumption in the remainder of the paper.

Assumption 1. $k\overline{v} > \underline{v}$

Essentially, this means that *a priori* there is uncertainty as to whether an efficient mechanism would allocate both objects to the same buyer or one object to each potential buyer. Hence, it is entirely possible that bidder i's valuation of a second object exceeds bidder j's valuation the first object, $kv_i > v_j$. Economically, this is the most interesting and challenging case. We could alternatively refer to this as the case with *overlap*. In the alternative, *non-overlap* case, $k\overline{v} < \underline{v}$, any efficient mechanism would allocate one object to each potential buyer. In this case, a bidder who has already won one

[13] That is, short of moving to the last spot if possible. If selling-time is an endogenous variable, the two symmetric sellers might conceivably be involved in a war of attrition to become the last seller. This, however, is not the topic of this paper, and seller positions in the auction sequence are assumed to be exogenous.

object ceases to be an effective competitor for the second.[14]

Given Assumption 1, we note that (2) can be written as

$$
\begin{aligned}
ER_2^{SSP} &= \int_{\underline{v}}^{k\overline{v}} 2x(1-F(\frac{x}{k}))f(x)dx + k\int_{\underline{v}}^{\frac{\underline{v}}{k}} 2xF(x)f(x)dx \\
&\quad +k\int_{\frac{\underline{v}}{k}}^{\overline{v}} 2x(F(x)-F(kx))f(x)dx \qquad (3)
\end{aligned}
$$

Below, two types of inefficiency will be identified. First, a mechanism may allocate one object to a bidder who would have received no object in an efficient mechanism. As we shall see this will be a feature of the mechanism for the case with two independent sellers where the first seller sets an optimal buy-out price. Likewise, a mechanism may allocate both objects to a bidder who would only have received one object in an efficient mechanism. This will arise in the case where a single seller sets a buy-out price in the second auction which depends on the outcome of the first auction.

3 Competing Sellers

We now turn to the case where two different sellers each own one object initially. We assume that the two objects are offered sequentially, and that there are two potential buyers on the market. We allow the first seller to stipulate a buy-out price of the *eBay*-variety (*Buy It Now*) and, thus, consider the following augmented game:

1 Seller 1 announces a buy-out price, B. At this stage bidders can submit a bid of B or refrain from bidding. The object is sold at the price B if at least one bidder bids B. If both bidders bid B, one bidder is picked at random to win. If no one bids B, a normal second-price auction is staged. The price can exceed B in this event.

2 Seller 2 auctions off the second item, using a second-price auction.

[14] Thus, Assumption 1 is pretty innocuous. However, it allows us to economize on notation in the formal analysis below. For completeness, we have included Appendix B, which shows that all the results in Section 3 below hold with minor modifications when Assumption 1 is not met. The interested reader should consult Appendix B when the results in Section 3 have been derived.

Thus, in stage 1 of this game, the bidders *first* have to decide whether to take the buy-out price B or leave it. If one or more bidders take the buy-out price, the first auction ends, and the winner pays B. If no one takes the buy-out price, the first stage continues to a standard second-price auction. The second stage simply consists of a standard second-price auction.

We first derive the relationship between the level of B and the valuations of bidders who will take this buy-out price. Then we look at the relationship between the buy-out price and the expected revenues to the two sellers, including how they are ranked. Finally, we determine the optimal buy-out price for the first seller. Recall that Assumption 1 is assumed to hold throughout.

3.1 General results

We will look for a symmetric equilibrium in this augmented game in which bidders with valuations above some level $\widehat{v}$ take the buy-out price B in stage 1, while bidders with valuations below $\widehat{v}$ do not. In the augmented game, it is clear that if no bidder takes B, then it is common knowledge in equilibrium that both bidders have a type below $\widehat{v}$. That is, beliefs are symmetric, and the logic of Proposition 1 (Black and de Meza (1992)) applies to the remainder of stage 1 and to stage 2. Hence, in stage 1 bids will be kv_i, where $v_i < \widehat{v}$, $i = 1, 2$. Further, regardless of how the good is sold in stage 1, it is well known that the bid in stage 2 will be kv_i if bidder i won the first auction, and v_i otherwise. In the following we suppress the subscript when this can be done without confusion.

In the equilibrium of the augmented game, a given value of B will induce a set $[\widehat{v}, \overline{v}]$ of bidder types to take the buy-out price B in stage 1. Changing B will change $\widehat{v}$. Hence, we can determine which $\widehat{v}$ to target, and chose B accordingly. Thus, we write $B(\widehat{v})$ as the value of B that induces bidder types above $\widehat{v}$ to take B in a symmetric equilibrium. This allows us to state the following result.

Proposition 2 *Let $m(\widehat{v}) = \min\{\overline{v}, \frac{\widehat{v}}{k}\}$, and let $B(\widehat{v})$ be defined by*

$$B(\widehat{v})(1 + F(\widehat{v})) = \widehat{v}(1 - F(m(\widehat{v}))) + \int_{\underline{v}}^{m(\widehat{v})} kxf(x)dx + \int_{\underline{v}}^{\widehat{v}} kxf(x)dx \quad (4)$$

Then, it is an equilibrium for bidders with $v \in [\widehat{v}, \overline{v}]$ to bid $B(\widehat{v})$ in stage 1 and for bidders with $v \in [\underline{v}, \widehat{v})$ not to.

Proof. See Appendix A. ■

It is easily seen that $B(\overline{v}) = kE(v)$. In addition, $B(\cdot)$ may not be monotonic, implying that for a given value of B, there could be multiple symmetric equilibria. As shown next, for any distribution and $k \in (0,1)$, the first seller can strictly increase his revenue by offering a buy price that will be accepted with positive probability. But first, we return briefly to the example.

Example: The uniform case ($v \in [0,1]$)

To provide some perspective on the relationship between the buy-out price, B, and the critical valuation, $\widehat{v}$, let us first reconsider the example above. In this case (4) can be written as

$$B(\widehat{v})(1+\widehat{v}) = \begin{cases} k\left(\int_0^1 xdx + \int_0^{\widehat{v}} xdx\right) & \widehat{v} \geq k \\ k\left(\int_0^1 xdx + \int_0^{\widehat{v}} xdx\right) - \int_{\frac{\widehat{v}}{k}}^1 (kx - \widehat{v})dx & \widehat{v} \leq k \end{cases}$$

which implies that

$$B(\widehat{v}) = \begin{cases} \frac{k}{2(1+\widehat{v})}\left(1+\widehat{v}^2\right) & \widehat{v} \geq k \\ \frac{k}{2(1+\widehat{v})}\left((1+\widehat{v}^2) - (1-\frac{\widehat{v}}{k})^2\right) & \widehat{v} \leq k \end{cases}$$

From this we note that $B(\widehat{v}) < \frac{k}{2}$, so that whatever cut-off valuation $\widehat{v} \in [\underline{v}, \overline{v}] = [0,1]$ we try to implement, the implied buy-out price will always be less than k times the *unconditional* expectation of the value of the first unit won. In the special case referred to above where $k = \frac{2}{3}$, $B(\widehat{v})$ reduces to

$$B(\widehat{v}) = \begin{cases} \frac{1+\widehat{v}^2}{3(1+\widehat{v})} & \widehat{v} \geq \frac{2}{3} \\ \frac{(12-5\widehat{v})\widehat{v}}{12(1+\widehat{v})} & \widehat{v} \leq \frac{2}{3} \end{cases}$$

Hence, if we want to implement a cut-off valuation of $\widehat{v} = \frac{3}{4} > \frac{2}{3} = k$, the buy-out price must be set as

$$B(\frac{3}{4}) = \frac{1+\left(\frac{3}{4}\right)^2}{3(1+\left(\frac{3}{4}\right))} = \frac{25}{84} \approx 0.30$$

Similarly, if we want to implement a cut-off valuation of $\widehat{v} = \frac{1}{2} < \frac{2}{3} = k$, the buy-out price must be set as

$$B(\frac{1}{2}) = \frac{(12 - 5\cdot\left(\frac{1}{2}\right))\left(\frac{1}{2}\right)}{12(1+\left(\frac{1}{2}\right))} = \frac{19}{72} \approx 0.26$$

(*End of example*)

Returning to the general case, we can state the following result on the expected revenues in the two stages given some $B(\widehat{v})$.

Proposition 3 *The expected revenue in the first auction is*

$$\begin{aligned} ER_1(\widehat{v}) &= k(1-F(\widehat{v}))\left(\frac{\widehat{v}}{k}(1-F(m(\widehat{v}))) + \int_{\widehat{v}}^{m(\widehat{v})} xf(x)dx\right) \\ &\quad + k\int_{\underline{v}}^{\widehat{v}} 2x(1-F(x))f(x)dx \end{aligned} \tag{5}$$

while the expected revenue in the second auction is

$$\begin{aligned} ER_2(\widehat{v}) &= \int_{\underline{v}}^{km(\widehat{v})} 2x(1-F(\frac{x}{k}))f(x)dx + \int_{km(\widehat{v})}^{k\overline{v}} x(1-F(\frac{x}{k}))f(x)dx. \\ &\quad + k\int_{\underline{v}}^{\widehat{v}} 2x\left(F(x) - F(\max\{\underline{v}, kx\})\right)f(x)dx \\ &\quad + k\int_{\widehat{v}}^{m(\widehat{v})} 2x\left(F(\widehat{v}) - F(\max\{\underline{v}, kx\})\right)f(x)dx \\ &\quad + k\int_{\widehat{v}}^{m(\widehat{v})} x(1-F(\widehat{v}))f(x)dx + k\int_{m(\widehat{v})}^{\overline{v}} x(1-F(kx))f(x)dx \end{aligned} \tag{6}$$

Proof. For (5) see below, and for (6) see below and Appendix A. ■

We sketch the main arguments. First, consider the expected revenues in the *first auction*. When at least one of the bidders has a valuation of at least $\widehat{v}$, the buy-out price is taken and the first seller receives $B(\widehat{v})$. This event has a probability $1-F^2(\widehat{v})$. In contrast, if both bidders have valuations less than $\widehat{v}$ (i.e., $\max\{v_i, v_j\} < \widehat{v}$), the buy-out price is not taken, and the first stage continues to a second-price auction where each bidder bids kv_i according to Proposition 1. Thus, the first seller receives k times $\min\{v_i, v_j\}$. This event has a probability $F^2(\widehat{v})$. We conclude that the expected revenue to the first seller given $B(\widehat{v})$ can be written as

$$ER_1(\widehat{v}) = (1-F^2(\widehat{v})) \times B(\widehat{v}) + F^2(\widehat{v}) \times kE(\min\{v_i, v_j\} \mid \max\{v_i, v_j\} < \widehat{v})$$

However, $E(\min\{v_i, v_j\} \mid \max\{v_i, v_j\} < \widehat{v})$ is just the expected value of the second-order statistic, $v_{[2]}$, given that the first-order statistic, $v_{[1]}$, is less than $\widehat{v}$. Denote the density of $v_{[2]}$ given $v_{[1]} < \widehat{v}$ by $h^*(v)$. Then $h^*(v) = \frac{2f(v)(F(\widehat{v})-F(v))}{F^2(\widehat{v})}$ and we can write

$$\begin{aligned} E(\min\{v_i, v_j\} \quad | \quad \max\{v_i, v_j\} < \widehat{v}) &= \int_{\underline{v}}^{\widehat{v}} vh^*(v)dv \\ &= \frac{1}{F^2(\widehat{v})} \int_{\underline{v}}^{\widehat{v}} 2v(F(\widehat{v}) - F(v))f(v)dv \end{aligned}$$

Hence, expected revenue in the first auction given $B(\widehat{v})$ (or simply $\widehat{v}$) can be written as

$$\begin{aligned} ER_1(\widehat{v}) &= (1 - F^2(\widehat{v})) \times B(\widehat{v}) + k \int_{\underline{v}}^{\widehat{v}} 2v(F(\widehat{v}) - F(v))f(v)dv \\ &= [B(\widehat{v})(1 + F(\widehat{v}))](1 - F(\widehat{v})) + k \int_{\underline{v}}^{\widehat{v}} 2v(F(\widehat{v}) - F(v))f(v)dv \end{aligned}$$

Inserting $B(\widehat{v})(1 + F(\widehat{v}))$ from Proposition 1, we can write this as (5).

The derivation of the expected revenue in the *second auction* is slightly more complicated, and we relegate the formal derivation of (6) to Appendix A. However, in the second auction, the object will be bought either by the winner of the first auction, or by the loser.

The first and second term in (6) capture revenue in the former case. Assuming that the loser of stage 1 has valuation x, and bids x in stage 2, he will lose the second auction if the other bidder's bid exceeds x, which requires that the rival has a valuation which is at least x/k. The *first* term in (6) then accounts for the possibility that one bidder has a valuation below $\widehat{v}$ (and thus does not accept B) and also below $\overline{v}/k$ (implying the existence of a bidder type which has a higher marginal revenue on both units), and that the other bidder has a very high valuation, allowing him to win both auctions. The *second* term in (6) describes the case where both bidders accepted B, but that the (random) loser has a valuation which is low relative to the winner. This exhausts the possibilities that the winner is the same in both stages.

The remaining terms in (6) are relevant if the winner of stage 1 loses stage 2. Assuming this bidder has a valuation of x, say, the price in the second auction will then be equal to the bid from this bidder, namely kx. The *third* term in (6) is for cases where the winner of stage 1 did not accept the buy-out price, and where the other bidder (who must have a lower valuation) submits a bid higher than kx in stage 2. The *fourth* term in (6) is relevant when the winner of the first auction bid B, but was the only one to do so. Furthermore, the *fifth* term in (6) is for cases where both bidders bid B, but where the (random) winner of stage 1 loses stage 2 because his valuation is so small that he is certain to lose stage two given the fact that the other bidder has a valuation higher than $\widehat{v}$. Finally, the *sixth* term in (6) applies when both bidders bid B, and the (random) winner of stage 1 has a valuation which is low relative to the loser, allowing the latter to win stage 2. This exhausts the possibilities that the loser of stage 1 wins stage 2.

Example: The uniform case ($v \in [0,1]$)

In the uniform example, (5) and (6) reduce to

$$ER_1(\widehat{v}) = \begin{cases} \frac{k}{6}\left(3 - 3\widehat{v} + 3\widehat{v}^2 - \widehat{v}^3\right) & \widehat{v} \geq k \\ \frac{1}{6k}\left(6k\widehat{v} - 3(1+2k-k^2)\widehat{v}^2 + (3-k^2)\widehat{v}^3\right) & \widehat{v} \leq k \end{cases}$$

and

$$ER_2(\widehat{v}) = \begin{cases} \frac{k}{6}\left(3 - 2k + 3\widehat{v} - 3\widehat{v}^2 + \widehat{v}^3\right) & \widehat{v} \geq k \\ \frac{1}{6k}\left((3-k)k^2 + 3k(1-k)\widehat{v}^2 - (1-k^2)\widehat{v}^3\right) & \widehat{v} \leq k \end{cases}$$

For the special case considered above, where $k = \frac{2}{3}$, the expected revenues can be written as

$$ER_1(\widehat{v}) = \begin{cases} \frac{1}{9}\left(3 - 3\widehat{v} + 3\widehat{v}^2 - \widehat{v}^3\right) & \widehat{v} \geq \frac{2}{3} \\ \frac{1}{36}\left(36\widehat{v} - 51\widehat{v}^2 + 23\widehat{v}^3\right) & \widehat{v} \leq \frac{2}{3} \end{cases}$$

and

$$ER_2(\widehat{v}) = \begin{cases} \frac{1}{27}\left(5 + 9\widehat{v} - 9\widehat{v}^2 + 3\widehat{v}^3\right) & \widehat{v} \geq \frac{2}{3} \\ \frac{1}{108}\left(28 + 18\widehat{v}^2 - 15\widehat{v}^3\right) & \widehat{v} \leq \frac{2}{3} \end{cases}$$

Hence, if the buy-out price has been chosen to implement the cut-off valuation $\widehat{v} = \frac{3}{4} > \frac{2}{3} = k$, that is, $B \approx 0.3$, the expected revenues are $ER_1(\frac{3}{4}) = \frac{43}{192} \approx 0.22$ and $ER_2(\frac{3}{4}) = \frac{509}{1728} \approx 0.29$, and the ratio of expected revenues is $\frac{ER_1(\frac{3}{4})}{ER_2(\frac{3}{4})} = \frac{387}{509} \approx 0.76$.

Similarly, if the buy-out price has been chosen to implement the cut-off valuation $\widehat{v} = \frac{1}{2} < \frac{2}{3} = k$, that is $B \approx 0.26$, the expected revenues are $ER_1(\frac{1}{2}) = \frac{65}{288} \approx 0.23$ and $ER_2(\frac{1}{2}) = \frac{245}{864} \approx 0.28$, and the ratio of expected revenues is $\frac{ER_1(\frac{1}{2})}{ER_2(\frac{1}{2})} = \frac{39}{49} \approx 0.80$.

When pitted against the first auction revenues in two straight second-price auctions, we notice how the first seller can raise his revenue by introducing a buy-out price. In the next subsection we determine the optimal level of the cut-off valuation and, hence, the buy-out price to see when the buy-out price can have a significant effect on the revenues. (*End of example*)

To end this subsection we can state two more results.

Lemma 2 (Monotonicity) *(i)* $ER_2(\widehat{v})$ *is strictly increasing for* $\widehat{v} \in [\underline{v}, \overline{v})$. *(ii)* $ER_1(\widehat{v}) + ER_2(\widehat{v})$ *is strictly increasing for* $\widehat{v} \in [\underline{v}, k\overline{v})$*, and constant for* $\widehat{v} \in [k\overline{v}, \overline{v}]$*.*

Proof. See Appendix A. ■

The fact that $ER_2(\widehat{v})$ is increasing can easily be understood by the following two observations. First, if the first auction is won by the bidder with the lowest valuation (because both bidders bid the buy-out price B, and the low-valuation bidder is randomly picked as winner of the first object), the revenue to the second seller will be very low, indeed, namely k times the second highest valuation. Secondly, the larger the cut-off valuation $\widehat{v}$, the lower is the probability that the first auction is won by the bidder with the lowest valuation. Hence, as $\widehat{v}$ increases, it becomes increasingly unlikely that the buy-out price in first auction changes the identity of its winner and, therefore, the price in the second auction.

The second part of (ii) in Lemma 2 can be explained by appeal to the *Revenue Equivalence Theorem*, which states that two mechanisms that result in the same allocation must also give rise to the same overall revenue.[15] Now, the buy-out price changes the identity of the winner of the first auction *only if* both bidders accept the buy-out price *and* the random winner happens to be the low-valuation bidder. Assuming they both accept the buy-out price, we note that if the buy-out price is such that $\widehat{v} \in [k\overline{v}, \overline{v}]$, the (random) loser of the first auction must necessarily win the second. To see this, we note that the valuation of the first-auction loser and, hence, his bid in the second auction must be at least $\widehat{v}$. This, in turn, exceeds the rival bid in the second auction which is at most $k\overline{v}$. Thus, when both bidders have valuations above $\widehat{v}$, with $\widehat{v} \in [k\overline{v}, \overline{v}]$, each bidder will win precisely one unit. However, the same is true if there is no buy-out price. If both bidder have valuations in the interval $[k\overline{v}, \overline{v}]$, the bidder with the highest valuation wins

[15] See e.g. Klemperer (1999).

the first auction, and the other bidder wins the second. In conclusion, when $\widehat{v} \in [k\overline{v}, \overline{v}]$ the buy-out price might change the order in which bidders win, but not the final allocation. Consequently, overall revenue is the same with and without a buy-out price.

In contrast, for low values of $\widehat{v}$, $\widehat{v} \in [\underline{v}, k\overline{v})$, the presence of a buy-out price might change the final allocation and therefore also overall revenue. In the next subsection we discuss the consequences of this in greater detail.

Proposition 4 (Increasing prices) *$ER_2(\widehat{v}) > ER_1(\widehat{v})$, $\forall \widehat{v} \in [\underline{v}, \overline{v}]$.*

Proof. See Appendix A. ■

As remarked in relation to Proposition 1 (Black and de Meza), revenue is strictly increasing over the auction sequence when there is no buy-out price. Indeed, revenue increases with probability one in the case without a buy-out price. However, the result in Proposition 4 is only for *expected* revenues. It is entirely possible that actual, observed revenues decrease when there is a strictly positive buy-out price. For example, if one bidder has a valuation $\widehat{v} > 0$ and the other $\underline{v} = 0$, revenue in stage 1 is $B(\widehat{v}) > 0$, while revenue in stage 2 is 0. The upshot of Proposition 4 is that the first seller can increase expected revenue by introducing a buy-out price, *but* will *not* be able "to level the playing field".

3.2 The optimal buy-out price

Now, we move on to determine the *optimal* buy-out price from the perspective of the first seller. Our main result can be stated as follows.

Proposition 5 *(i) For $k < 1$ the optimal value of $\widehat{v}$ is strictly lower than $k\overline{v}$. Consequently, the sequence of auctions is inefficient when the first seller chooses the buy-out price optimally. (ii) For $k = 1$, $\widehat{v} = \overline{v}$ is optimal.*

Proof. See Appendix A. ■

This result follows more or less directly from Lemma 2. Since the sum of revenues is the same for all $\widehat{v} \in [k\overline{v}, \overline{v}]$, and revenue to the second seller is globally, strictly increasing, it follows that $\widehat{v} = k\overline{v}$ dominates all higher cut-off values from the perspective of the first seller. Further, at $\widehat{v} = k\overline{v}$ the derivative of $ER_1(\widehat{v})$ is strictly negative, and it always pays for the first seller to lower the cut-off valuation below $\widehat{v}$ by a suitable choice of the buy-out price B. The consequences for efficiency are immediate: It pays for the first seller to set the buy-out price, B, at such a level that the final allocation is inefficient with strictly positive probability. The optimal first-auction buy-out price is set such that the low-valuation bidder wins the first object with positive probability when he would have won no object in an efficient mechanism.

In the special case where $k = 1$, the behavior in the second auction is independent of the outcome of the first auction. Therefore, stage 1 is essentially equivalent to a one-shot auction. Thus, the last part of Proposition 5 shows that buy-out prices lower revenue in such auctions when buyers are risk neutral.[16]

Example: The uniform case ($v \in [0, 1]$)

To provide some perspective on the last proposition, we reconsider the uniform case. From Proposition 5 we know that $\widehat{v} < k\overline{v} = k$, for any $k \in (0, 1)$. The expected revenue to the first seller when $\widehat{v} < k$ is given by

$$ER_1(\widehat{v}) = \frac{1}{6k}((3 - k^2)\widehat{v}^3 - 3(1 + 2k - k^2)\widehat{v}^2 + 6k\widehat{v})$$

while the expected revenue to the second seller is

$$ER_2(\widehat{v}) = \frac{1}{6k}(-(1 - k^2)\widehat{v}^3 + 3k(1 - k)\widehat{v}^2 + (3 - k)k^2)$$

Maximizing $ER_1(\widehat{v})$ with respect to $\widehat{v}$ gives the optimal cut-off valuation from the perspective of the first seller

$$v^* = \frac{1 + 2k - k^2}{3 - k^2} - \frac{\left((1 + 2k - k^2)^2 - 2k(3 - k^2)\right)^{1/2}}{3 - k^2} < k = k\overline{v}$$

and the associated, optimal buy-out price, $B(v^*)$ is given by

$$B(v^*) = \frac{k}{2(1 + v^*)} \left((1 + (v^*)^2) - (1 - \frac{v^*}{k})^2 \right)$$

[16] For specific distributions, this result has already been noted by Budish and Takeyama (2001), Mathews (2002) and Reynolds and Wooders (2002). We show that this is a general property whenever the distribution function is continuously differentiable. Thus, the generality of our argument also reveals that "ironing of marginal revenue" cannot explain the use of buy-out prices in this case (for more on this, see below).

We can substitute v^* into the revenue expressions, and Fig. 3 illustrates how $ER_1(v^*)$ (*thin*) and $ER_2(v^*)$ (*heavy*) vary with k.

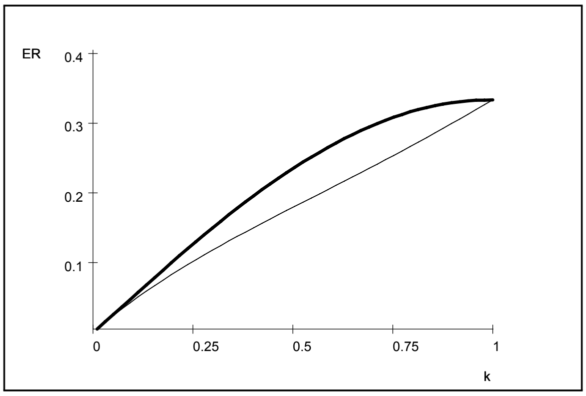

Fig. 3: Revenues in auction with buy-out

The ratio between the expected revenues given an optimally chosen buy-out price, $RR(BO) = \frac{ER_1(v^*)}{ER_2(v^*)}$, is illustrated in the following figure

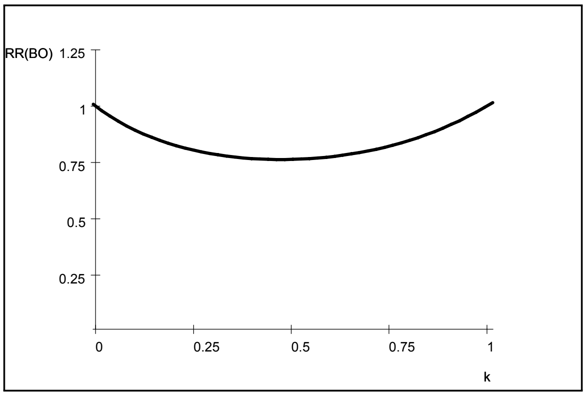

Fig. 4: Revenue Ratio in Auction with Buy-out

We can compare with the case of two straight second-price auctions illustrated in Fig. 1 and Fig. 2. In Fig. 5 we merge the information in Fig. 1 and Fig. 3. The dashed lines are for two straight second-price auctions, while the solid lines are for the case where the first seller chooses the buy-out price to implement v^*.

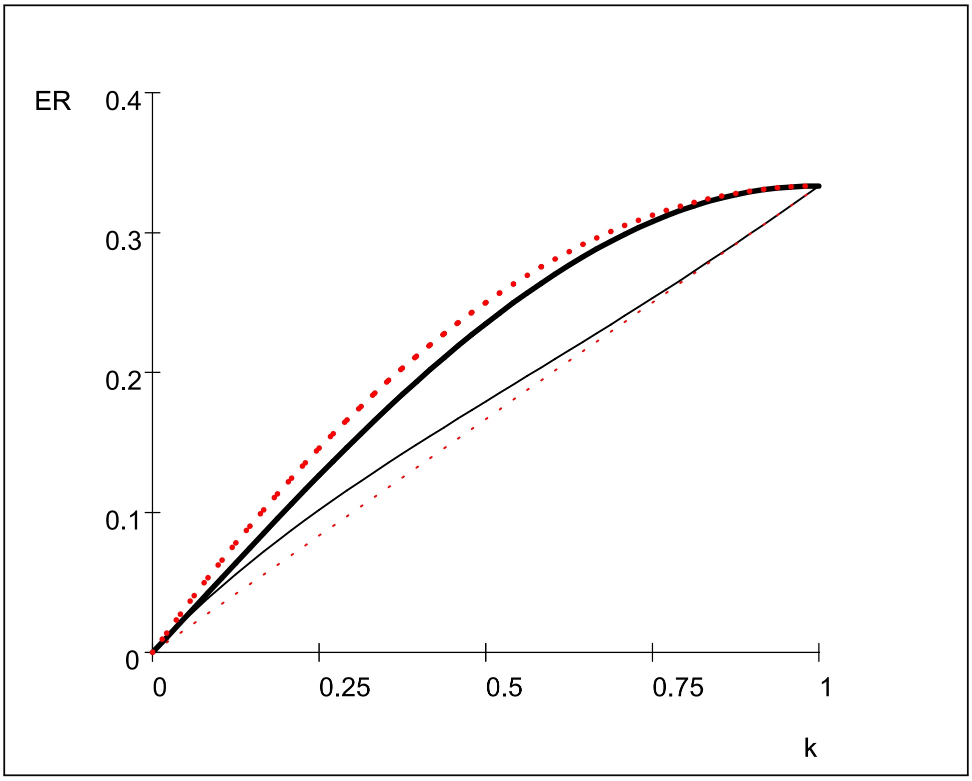

Fig. 5: Comparison of auction revenues

Fig. 6 merges the information from Fig. 2 and Fig. 4, and the *thin* line is for two straight second-price auctions, while the *heavy* line is associated with an optimal buy-out price.

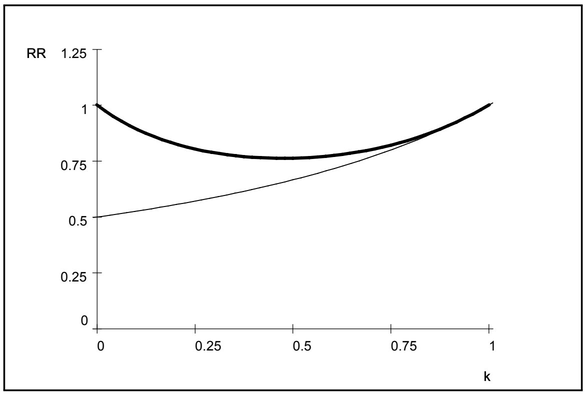

Fig. 6: Revenue Ratios

Finally, in Fig. 7 we plot the percentage gain to the first seller from an optimally chosen buy-out compared to the straight second-price auction, $G(BO) = \frac{ER_1(v^*) - ER_1^{SSP}}{ER_1^{SSP}}$.

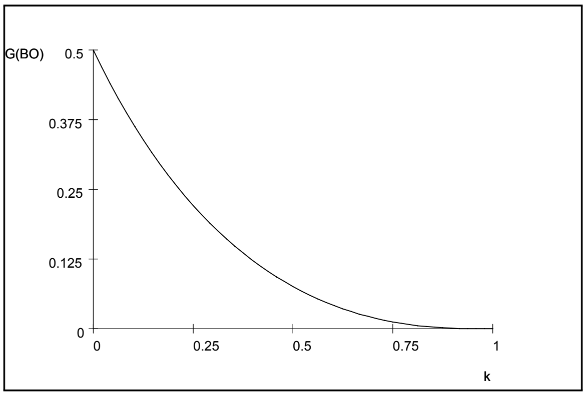

Fig. 7: Percentage gain from buy-out price

The last three figures essentially illustrate that the value from the perspective of the first seller of introducing a buy-out price is substantial when the individual demand functions are relatively steep (k small). When demands are steep, and there are only two bidders, the competition for the first object will be weak. It follows that the first seller has a strong incentive to try to improve his position in this case by introducing a suitably chosen buy-out price. The following table captures central features of the example in an alternative way.

k	ER_1^{SSP}	v^*	$B(v^*)$	$ER_1(v^*)$	$G(BO)$
0.01	0.00333	0.00995	0.00495	0.00495	0.49
0.10	0.03333	0.09549	0.04597	0.04558	0.37
0.25	0.08333	0.22618	0.10623	0.10176	0.22
0.50	0.16667	0.43308	0.20404	0.17931	0.08
0.75	0.25000	0.66667	0.32222	0.25309	0.01

Recall that in this example revenue equivalence and efficiency is lost when $\widehat{v}$ is set below $k = k\overline{v}$. Hence, a comparison of the first and third column is indicative of the inefficiency when $\widehat{v}$ is set optimally. For example, when $k = k\overline{v} = 0.5$ the optimal $\widehat{v}$ is aproximately 0.43, which implies that there is a small, but "non-trivial", probability that the final allocation is inefficient. (*End of example*)

In the next section, we assume that the two objects are owned by a *single* seller and show that a buy-out price in the *last* auction is beneficial to this seller. Before proceeding, however, it is of some value to examine more closely why overall revenue declines when a buy-out price is offered by the *first* of *two* sellers.

As mentioned, the *Revenue Equivalence Theorem* reveals that if two mechanisms yield the same allocation, expected revenue in the two mechanisms must also be the same. Since the outcome of the bench-mark model is efficient, it follows that introducing a buy-out price changes revenue if and only if[17] the resulting allocation is inefficient.

For instance, introducing a buy-out price in the first auction produces the following kind of inefficiency: an agent may win one item when he would have won none without the buy-out price. In the next section, a buy-out price in the second auction will be shown to cause another type of inefficiency: an agent may win two units, when he would have won exactly one without a buy-out price. In the latter case, an agent who would have won one unit in an efficient auction risks not winning one at all, and in this sense the type of inefficiency studied in the next section is the opposite of that studied in this section.

[17] This assumes that an agent of type $\underline{v}$ is indifferent between the two mechanisms. We will return to this point momentarily.

To understand the consequences of these different kinds of inefficiencies, it is useful to exploit the similarities between monopoly pricing and auctions[18]. When a monopolist faces agents with multi-unit demands, it is well known that the optimal pricing schedule generally involves quantity discounts. These discounts enable the monopolist to sell several units to agents with high marginal revenue on all units, without at the same time selling to agents with low marginal revenue on some units. Whether agents have unit or multi-unit demands, it is well understood that the key ingredient in the monopolist's optimization problem is marginal revenue.

Now, the expression for what amounts to marginal revenue of a bidder with valuation v in an auction is

$$J(v) = v - \frac{1 - F(v)}{f(v)}$$

for the first unit, and it can easily be shown that marginal revenue is $kJ(v)$ for the second unit[19]. The expected revenue to the seller is then

$$E\left[\sum_{i=1}^{2}\left(q_i^1(v_1,v_2)J(v_i) + q_i^2(v_1,v_2)kJ(v_i)\right)\right] - 2EU(\underline{v},\underline{v}) \tag{7}$$

where $q_i^j(v_1,v_2)$ is the probability that agent i wins at least j units, given that the two agents are of type v_1 and v_2, respectively. The last term is the expected rent obtained by an agent of type $\underline{v}$ in the mechanism. (7) is the counterpart of the revenue for a monopolist, who earns the area under the marginal revenue curve.

Clearly, if $EU(\underline{v},\underline{v})$ is the same in two different mechanisms, and if these mechanisms implement the same allocation, (i.e., the same $q_i^j(v_1,v_2)$), expected revenue must be the same. This is the *Revenue Equivalence Theorem*.

We are now equipped to provide an alternative proof of why overall revenue declines when a buy-out price is optimally chosen by the first seller.

[18]These similarities were first pointed out by Bulow and Roberts (1989) for auctions with unit demand, see also Bulow and Klemperer (1996) and Klemperer (1999). Maskin and Riley (1989) draw parallels between auctions with multi-unit demand and non-linear pricing. For more on the latter, see also Kirkegaard (2002).

[19]For a derivation of $J(v)$, see Myerson (1981) or Bulow and Roberts (1989). Since willingness-to-pay for a second unit is k times that for the first unit, it is unsurprising that marginal revenue of the second unit is k times marginal revenue of the first unit, see Kirkegaard (2002).

Given that $k\overline{v} > \widehat{v} > \underline{v}$, the allocation changes as a consequence of the buy-out price if the winner of stage 1 would not have won a unit at all in the efficient allocation. If the winner of stage 1 has valuation v this happens if $v < k\overline{v}$, and the rival bidder has valuation $x \in (\frac{v}{k}, \overline{v})$. In this case it can be shown that

$$\int_{\frac{v}{k}}^{\overline{v}} kJ(x)\frac{f(x)}{1-F(\frac{v}{k})}dx - J(v) = v - J(v) > 0$$

That is, given the event that the allocation has changed, the marginal revenue lost (which in expectation is the first term[20]) exceeds the marginal revenue gained. Hence, overall revenue decreases as the first term in (7) consequently declines, and the second term is unchanged. It is not profitable to allow an agent to win *one* unit too often, compared to the efficient allocation.

We have already argued that when $k = 1$, stage 1 is equivalent to a one-shot auction. In one-shot auctions, revenue is clearly maximized by allocating the object to the agent with the highest marginal revenue. When the agent with the highest valuation is also the agent with the highest $J(v)$, i.e. when $J(v)$ is increasing in v, this is accomplished with an efficient mechanism. However, when $J(v)$ is not monotonic, it is impossible to always give the object to the agent with the highest marginal revenue. The reason is that the auctioneer must respect the incentive compatibility constraint when designing his mechanism. To satisfy this constraint, it is necessary that the probability of winning the object is non-decreasing in the valuation.

In the cases where $J(v)$ is non-monotonic, the rules of the optimal mechanism[21] ensures that the probability of winning is constant over a subset of valuations. That is, agents with different valuations have the same probability of winning, and therefore contributes marginally the same to revenue. Hence, the optimal mechanism is said to "iron" the marginal revenue curve.

[20] That this is equal to v can be understood by the following argument. First, if bidder 1 has valuation v and faces bidder 2 with valuation $x \in (\frac{v}{k}, \overline{v})$, then the exact value of x does not influence the allocation in either auction format. Second, imagine the buy-out price has been introduced, and that bidder 1 won the first auction, but it is known that $x \in (\frac{v}{k}, \overline{v})$. Third, imagine the seller wants to revert the allocation back to the original allocation, and he therefore asks bidder 2 to pay an amount, p, to get the good. Whether or not bidder 2 accepts or rejects, he will win the second auction. Hence, if he accepts p his marginal gain is kx. Thus, if $p = v$, bidder 2 is willing to accept p regardless of x. However, if $p > v$, there is positive probability that bidder 2 will not accept. Consequently, the seller can extract precisely v from bidder 2 to ignore the buy-out price.

[21] See Myerson (1981) or Bulow and Roberts (1989).

Now, we observe that the buy-out price is a crude way of ironing the marginal revenue curve, since all agents with valuation above $\widehat{v}$ has the same probability of winning in a one-shot auction. It is crude because the interval on which marginal revenue is ironed in an optimal mechanism is always interior, whereas the buy-out price also bundles valuations close to and including $\overline{v}$ with lower valuations.

Since buy-out prices offer some (excessive) ironing, it is perhaps not obvious whether or not buy-out prices can increase revenue when $J(v)$ is nonmonotonic and $k = 1$. However, our model is sufficiently general to encompass these situations, and we can therefore conclude that buy-out prices are counterproductive even when some ironing is called for, precisely because the ironing is too crude.[22]

We conclude, quite generally, that overall revenue is adversely affected by the buy-out price, if the inefficiency is of the form that an agent wins one unit more often than is efficient. In the next section, however, we show that it is possible to increase revenue by introducing another form of inefficiency.

4 One Seller

In the following, we assume that the same seller owns both objects, and that they are sold sequentially. Above we established that total revenue decreases if a buy-out price is offered in the first auction, because an undesirable kind of inefficiency was generated. However, in the following we show that a buy-out price in the second auction produces a different type of inefficiency, one which is desirable for the seller. To this end, we consider the following augmented game:

1 The first object is sold using a second-price auction. The closing price is observed.

2 The seller announces a buy-out price, B, for the second object. The object is sold at the price B if at least one bidder bids B. If both

[22] As an aside, we note that we are not aware of any papers on auctions (or monopoly) showing that "ironing" may be counterproductive, if it is too crude in the sense of this paper. Among the related papers the model of Budish and Takeyama (2001) is discrete, while Reynolds and Wooders (2002) assume uniformly distributed valuations. Ironing is not an issue in either of these specifications. Mathews (2002) also assumes uniform distributions, but he remarks that his results holds for any distribution, though without referring to ironing.

bidders bid B, one bidder is picked at random to win. If no one bids B, a normal second-price auction is staged. The price can exceed B in this event.

In line with much of the literature on mechanism design, we will accord the seller a powerful ability to pre-commit to a particular auction design. To illustrate, suppose the first auction is conducted, and the closing price is observed. Hence, if bidding strategies in the first auction are *strictly increasing*, the valuation of the loser, v, is revealed. Contingent on this v, a buy-out price for the second auction, $B(v)$, is set. We assume throughout, and this is where commitment matters, that the relation between v and B, that is, $B(v)$, is firmly understood by bidders at the outset. Thus, the seller can credibly announce $B(v)$ before the first auction.[23]

Given this set-up, our basic argument can be outlined as follows. Assume that the bidding strategy in the first auction is strictly increasing, and that the closing price, p, is observed. Since the latter is determined by the bidding strategy of the runner-up, the valuation of this agent, v, can be deduced. Then, in the second stage, a buy-out price is offered, which is contingent on v. Assuming that the buy-out price, B, is close to v, it is not desirable for the loser of stage 1 to accept it. However, if B is lower than v, the winner of stage 1 will accept it, if his willingness-to-pay exceeds the buy-out price. The reason is that if he does not, a second-price auction ensues, in which he knows the loser of stage 1 will be willing to compete for the object until the price reaches $v > B$. Consequently, the winner of stage 1 also wins stage 2 if his valuation is at least B, although he would win less often in an efficient auction, namely when his valuation is above v.

To see why this might increase revenue, observe that it is common for a monopolist to offer quantity discounts. These discounts introduce the same kind of inefficiency as that described above. If p is the price of one unit and $p+B$ the price of two units, an agent may be willing to pay more than B for one unit, but less than p. In this case, he will obviously buy nothing. On the other hand, a buyer willing to pay at least p for one unit and an additional B for a second unit will purchase two units. Clearly, it would be efficient for these two buyers to share the two units. By introducing the inefficiency, however, the monopolist is able to sell to the agent with highest marginal revenue on the incremental unit.

[23] For more on this, see below.

To close the argument, we need to understand why this kind of inefficiency favors agents with high marginal revenue. The first observation is that $kJ(\overline{v}) > J(k\overline{v})$, implying that the agent with the highest possible valuation should win two units, even when faced with a competitor with valuation slightly higher than $k\overline{v}$, and even though this is inefficient.

Hence, inefficiency "at the top" is always desirable from the point of view of revenue generation. Often, however, inefficiency is also desirable at all other levels. Assume for the rest of the section that the following monotonicity condition is satisfied.

Assumption 2. $\frac{1-F(v)}{f(v)}$ is decreasing in v.

This increasing *hazard rate*[24] condition implies (but is not implied by) an increasing $J(\cdot)$ function (i.e., decreasing marginal revenues in the more familiar context). A consequence of this is that, for $kv \geq \underline{v}$,

$$k\frac{1-F(v)}{f(v)} < \frac{1-F(kv)}{f(kv)} \Longleftrightarrow kJ(v) > J(kv)$$

such that a bidder with valuation v should win two units when faced by a rival with a valuation in a neighborhood of kv.

Thus, the seller would like to design an auction such that a bidder with valuation $v \in [\frac{\underline{v}}{k}, \overline{v}]$ wins two units when faced by a rival with valuation close to kv, i.e. he wins two units more often than is efficient. As argued in the beginning of this section, this can be accomplished by using a buy-out price in the second auction.

To elaborate, if v is the revealed valuation of the stage 1 loser, we consider a commonly known function $B(v)$ which gives the resulting buy-out price in stage 2. That is, $B(v)$ is known before the first auction commences. The buy-out price is assumed to satisfy $B(v) \leq v$ for all v. We will then look for a discriminating equilibrium, defined as follows.

Definition 1 *A discriminating equilibrium consists of a symmetric bidding strategy in stage 1, which is strictly increasing in bidder valuation, and the following strategy in stage 2. Given that $B(v)$ is the buy-out price in stage 2, the winner of stage 1 bids $B(v)$ in stage 2 if and only if his marginal utility*

[24]The hazard rate is $h(v) = \frac{f(v)}{1-F(v)}$. An increasing hazard rate is equivalent to log-concavity of $1 - F(v)$. See Bagnoli and Bergstrom (1989) for an extensive treatment of log-concave distributions.

of the second unit exceeds $B(v)$, while the loser of stage 1 never bids $B(v)$. Bidder i, $i = 1, 2$, bids his marginal utility in stage 2, if the buy-out price was not accepted by anyone.

Inspection of Definition 1 reveals that the existence of a discriminating equilibrium necessitates that $\underline{v} > 0$. To see this, assume to the contrary that $\underline{v} = 0$, and consider the incentives of a bidder with a valuation slightly above 0. By following the equilibrium strategy, it is very unlikely that such a bidder will win either auction. Rather, it is preferable for such an agent to bid 0 in the first auction, and then accept the buy-out price (of zero) in the second auction. Since the competing agent will also want to buy the good in the second auction at the buy-out price, the low-valuation agent wins the second auction with a significant probability of 0.5

On the other hand, if $\underline{v} > 0$ and $B(v)$ is close to v, an agent with a valuation close to $\underline{v}$ prefers not to accept B if he lost stage 1, even if he deviated in the first auction. The reason is that there is a mass of types for which $kv < B$, implying that the low valuation agent wins the second auction with significant probability and pays significantly less than his own valuation when following the equilibrium strategy. This is preferable to accepting the buy-out price and winning with an even larger probability, provided that the buy-out price is large relative to the valuation.

These arguments capture the key qualitative difference between cases with $\underline{v} = 0$ and $\underline{v} > 0$. When $\underline{v} = 0$, a bidder with valuation $\underline{v}$ does not contribute to the competition for any of the units since $\underline{v} < kv$, $\forall v > \underline{v}$.[25] In contrast, when $\underline{v} > 0$, even a bidder with the lowest possible valuation, $\underline{v}$, contributes to the competition, since there is a range of v such that $kv < \underline{v}$.[26]

If the first auction was won by bidder 1, say, the winner of the second stage changes as a consequence of the buy-out price if and only if $B(v_2) < kv_1 < v_2$. In this case, bidder 1 also wins the second item, resulting in the desired inefficiency. The seller seeks to construct a $B(v)$ function which has the following properties.

Assumption 3. Define $B(v)$ on $[\underline{v}, \overline{v}]$, and assume that

[25] A bidder with valuation $\underline{v}$ could never be expected to win even in the competition for the second item. This is easily checked against the results of Section 2.

[26] In this case, a bidder with valuation $\underline{v}$ could reasonably win the second unit. Again, this can easily be checked against Section 2.

(*i*) $B(v) \in (kv, v)$, $\forall v \in (\underline{v}, k\overline{v})$, $B(v) = v$ otherwise[27]. $B(v)$ is everywhere continuous, and it is continuously differentiable with $0 < B'(v) < \infty$, $\forall v \in [\underline{v}, \overline{v}] \backslash \{k\overline{v}\}$

(*ii*) $kJ(x) > J(v)$, $\forall x \geq \frac{1}{k} B(v)$

(*iii*) The function $b(v)$ is strictly increasing[28], where

$$b(v) = \begin{cases} kv + (v - B(v)) \frac{f(\frac{1}{k} B(v))}{f(v)} \frac{1}{k} B'(v) & \text{for } v \in [\underline{v}, k\overline{v}) \\ kv & \text{for } v \in [k\overline{v}, \overline{v}] \end{cases}$$

We will show below that the function $b(v)$ is the bidding strategy in stage 1 of a discriminating equilibrium. If the loser of stage 1 is revealed to be of type $v \in [\underline{v}, k\overline{v})$, the buy-out price in stage 2 is $B(v) < k\overline{v}$, and it is accepted with strictly positive probability. However, if the loser is of a higher type, $B(v)$ exceeds $k\overline{v}$, and there is therefore zero probability that the winner of stage 1 accepts it. Note that the ability to precommit to the auction design is formally important, as the design is not time consistent. Once stage 2 is reached, it is no longer in the seller's interest to offer the buy-out price, since this will decrease revenue in stage 2.

Before stating the result of this section, we observe that the second term of (7) is unchanged. This is because a bidder with valuation $\underline{v}$ will lose stage 1 in both auction formats, and since $B(\underline{v}) = \underline{v}$ the presence of the buy-out price in stage 2 will not affect the probability of such a bidder winning (which will be $F(\frac{\underline{v}}{k})$) or the price paid in that event[29]. Furthermore, while we argued that inefficiency at the top is always desirable, we explicitly assumed that $B(k\overline{v}) = k\overline{v}$, implying that there is no inefficiency at the top. This part of Assumption 3 is made solely to simplify the proof of the following Proposition.

Proposition 6 (*i*) *Any discriminating equilibrium satisfying Assumption 3 is strictly revenue superior to the equilibrium of a sequence of second-price*

[27] If the agent who loses stage 1 deviated to an action that is not played by any type in equilibrium, this is taken to signal that $v = \underline{v}$.

[28] $b(v)$ is continuous, given part (*i*) of Assumption 3.

[29] In fact, this is why this assumption has been imposed. We could let $B(\underline{v}) < \underline{v}$, which would decrease the second term in (7) and hence increase revenue further. However, we seek the stronger result that it is the change in allocation (i.e. the inefficiency) that drives revenue up. Thus, we keep the second term the same over the two auction formats.

auctions with no buy-out price. (*ii*) *A discriminating equilibrium satisfying Assumption 3 exists whenever* $\underline{v} \geq kE(v)$. *In such an equilibrium, the bidding strategy in stage 1 is given by* $b(v)$.

Proof. (*i*) The proof is based on inspection of (7). As mentioned above, the second term is unchanged. However, the first term in (7) is higher when the buy-out price is introduced. To see this, observe that if the allocation changes, the winner of stage 1 must have a type that exceeds $\frac{B}{k}$. By the second part of Assumption 3, the marginal revenue of the second unit to this bidder is higher than the marginal revenue of the first unit to the losing bidder. Hence, for every realization of (v_1, v_2), the term inside the expectations operator in (7) is no lower, but possible higher, than without the buy price. For a proof of the second part of the proposition, see Appendix A. ■

The condition in the second part of Proposition 6 is required to eliminate any incentive to bid low in stage 1, and then bid B in stage 2 if stage 1 was lost. As can be seen by the second part of the Proposition, the presence of a buy-out price in stage two increases bids in stage 1, since $b(v) \geq kv$. Thus, revenue in stage 1 increases. The sum of revenues in the two stages also increases, despite the fact that revenue in stage 2 decreases.

We also observe that Assumption 1 implies that k is not too small, while the condition in (ii) of the proposition implies that k is not too great either, that is, $k \in \left(\frac{\underline{v}}{\overline{v}}, \frac{\underline{v}}{E(v)}\right]$.[30] As an example, the assumptions are satisfied for the uniform distribution on $[1, 2]$ with $k \in (\frac{1}{2}, \frac{2}{3})$.

Finally, we note that the conclusion that a discriminating auction (second-round buy-out) may increase overall revenue of the seller is related to a further result in Black and de Meza (1992). They show, by an example, that an option *offered to the first-round winner* of buying the second object at the first-round price may increase overall revenue above the level of two straight second-price auctions. Despite the one-sided nature of the option suggested by Black and de Meza it, presumably, trades on the same type of inefficiency as in this section. That is, the winner of the first round wins more often than is efficient.

[30] Note that these are sufficient conditions.

5 Concluding Remarks

In this paper we sought to explain the use of buy-out prices by observing that online auction markets are dynamic, with players knowing that goods not presently on the market are likely to be offered in the future. It was shown that there is an incentive for current sellers to offer a buy-out price that is accepted with positive probability. Furthermore, we showed that a sophisticated seller with several units can increase the sum of revenues by introducing a buy-out price in later auctions which is contingent on the outcome of earlier auctions.

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Appendix A

Proof of Proposition 2. Consider a bidder with valuation $v \geq \widehat{v}$. By bidding B in stage 1, his expected payoff in the two stages is

$$\begin{aligned} EU(B,v) &= \int_{\underline{v}}^{\widehat{v}} (v-B)f(x)dx + \int_{\underline{v}}^{\min\{\widehat{v},\max\{\underline{v},kv\}\}} (kv-x)f(x)dx \\ &+ \int_{\widehat{v}}^{\overline{v}} \frac{1}{2}(v-B)f(x)dx + \int_{\widehat{v}}^{\min\{\overline{v},\frac{v}{k}\}} \frac{1}{2}(v-kx)f(x)dx \\ &+ \int_{\widehat{v}}^{\max\{\widehat{v},kv\}} \frac{1}{2}(kv-x)f(x)dx \end{aligned}$$

where the five terms capture all the possible outcomes as follows. *First*, the bidder wins stage 1 at a price of B with probability one, if the competitor refrains from accepting B, i.e. has valuation below $\widehat{v}$. *Second*, with probability one, the bidder wins stage 2 at a price equal to the valuation of his rival, if this rival did not accept B in stage 1(she has a valuation below $\widehat{v}$), and if her bid, or valuation, (which exceeds $\underline{v}$) is at most kv. *Third*, the first auction is won with probability 0.5 if the opponent also bids B, i.e. if she has a valuation above $\widehat{v}$. *Fourth*, if the player lost stage 1 because the other player also bid B, the second stage is won at a price equal to the rival's bid if this bid is not too high. *Finally*, if both players bid B in stage 1 and the player in question won, we deduce that the competitor's valuation is at least $\widehat{v}$, implying that the second auction is also won if the rival's valuation is nevertheless so low that the winner of stage 1 will submit a higher bid than the loser.

If the bidder, instead, does not bid B, the first unit will be sold at a second-price auction, if the buy-out price is not accepted by the rival either. The best response in this subgame is easily shown to be to outbid the other bidder (the bidder in question is willing to bid kv, whereas the other bidder is known to be willing to bid at most $k\widehat{v}$, if she did not bid B right away). Hence, by not bidding B, expected payoff is

$$EU(NB,v) = \int_{\underline{v}}^{\widehat{v}} (v-kx)f(x)dx + \int_{\underline{v}}^{\min\{\widehat{v},\max\{\underline{v},kv\}\}} (kv-x)f(x)dx + \int_{\widehat{v}}^{\min\{\overline{v},\frac{v}{k}\}} (v-kx)f(x)dx$$

Letting $B(\widehat{v})$ be the buy-out price at which type $\widehat{v}$ is indifferent between these two strategies yields (4). In general, for $v \geq \widehat{v}$,

$$\begin{aligned} & EU(B,v) - EU(NB,v) \\ = & \int_{\underline{v}}^{\widehat{v}} (kx - B) f(x) dx + \int_{\widehat{v}}^{\overline{v}} \frac{1}{2}(v - B) f(x) dx \\ & - \int_{\widehat{v}}^{\min\{\overline{v}, \frac{v}{k}\}} \frac{1}{2}(v - kx) f(x) dx + \int_{\widehat{v}}^{\max\{\widehat{v}, kv\}} \frac{1}{2}(kv - x) f(x) dx \end{aligned} \tag{8}$$

the derivative of which is

$$\frac{1}{2}\left[1 - F(\widehat{v}) - \left(F(\min\{\overline{v}, \frac{v}{k}\} - F(\widehat{v})\right) + k\left(F(\max\{\widehat{v}, kv\}) - F(\widehat{v})\right)\right] \geq 0$$

Since $EU(B,\widehat{v}) - EU(NB,\widehat{v}) = 0$ by construction, it follows that $EU(B,v) - EU(NB,v) \geq 0$ for all $v \geq \widehat{v}$. Hence, players with high valuations have no incentive to deviate from the equilibrium strategy.

For agents of type $v < \widehat{v}$, the equilibrium strategy of not bidding B followed by bidding kv in stage 1 if the opponent did not bid B either, yields the following

$$\begin{aligned} EU(NB,v) = & \int_{\underline{v}}^{v} (v - kx) f(x) dx + \int_{\underline{v}}^{\max\{\underline{v}, kv\}} (kv - x) f(x) dx \\ & + \int_{v}^{\min\{\overline{v}, \frac{v}{k}\}} (v - kx) f(x) dx \end{aligned}$$

By bidding B, the expected payoff is

$$\begin{aligned} EU(B,v) = & \int_{\underline{v}}^{\widehat{v}} (v - B) f(x) dx + \int_{\underline{v}}^{\max\{\underline{v}, kv\}} (kv - x) f(x) dx \\ & + \int_{\widehat{v}}^{\overline{v}} \frac{1}{2}(v - B) f(x) dx + \int_{\widehat{v}}^{\max\{\widehat{v}, \min\{\overline{v}, \frac{v}{k}\}\}} \frac{1}{2}(v - kx) f(x) dx \end{aligned}$$

We observe that

$$\begin{aligned} & EU(NB,v) - EU(B,v) \\ = & \int_{\underline{v}}^{v} (v - kx) f(x) dx + \int_{v}^{\min\{\overline{v}, \frac{v}{k}\}} (v - kx) f(x) dx - \int_{\underline{v}}^{\widehat{v}} (v - B) f(x) dx \\ & - \int_{\widehat{v}}^{\overline{v}} \frac{1}{2}(v - B) f(x) dx - \int_{\widehat{v}}^{\max\{\widehat{v}, \min\{\overline{v}, \frac{v}{k}\}\}} \frac{1}{2}(v - kx) f(x) dx \end{aligned} \tag{9}$$

and that this is equal to the negative of (8) when $v = \widehat{v}$, i.e. the expression is equal to zero in this case. The derivative of (9) is

$$F(\min\{\overline{v}, \frac{v}{k}\}) - \frac{1}{2}\left(1 + F(\max\{\widehat{v}, \min\{\overline{v}, \frac{v}{k}\}\}\right) < 0$$

implying that $EU(NB, v) - EU(B, v) > 0$ for all $v < \widehat{v}$. Thus, low valuation bidders have no incentive to deviate either. This completes the proof of Proposition 2. ■

Proof of Proposition 3. If $EP_2(v|\widehat{v})$ denotes the expected payment in stage 2 of a bidder with valuation v when the cut-off valuation is $\widehat{v}$, the expected revenue in stage 2 is

$$ER_2(\widehat{v}) = 2\int_{\underline{v}}^{\overline{v}} EP_2(v|\widehat{v})f(v)dv$$

From the expressions of expected payoff given in the proof of Proposition 2, it follows that

$$EP_2(v|\widehat{v}) = \int_{\underline{v}}^{\max\{\underline{v},kv\}} xf(x)dx + \int_{v}^{\min\{\overline{v},\frac{v}{k}\}} kxf(x)dx$$

for $v < \widehat{v}$, and

$$\begin{aligned} EP_2(v|\widehat{v}) &= \int_{\underline{v}}^{\min\{\widehat{v},\max\{\underline{v},kv\}\}} xf(x)dx + \int_{\widehat{v}}^{\min\{\overline{v},\frac{v}{k}\}} \frac{1}{2}kxf(x)dx \\ &\quad + \int_{\widehat{v}}^{\max\{\widehat{v},kv\}} \frac{1}{2}xf(x)dx \end{aligned}$$

otherwise. Hence,

$$\begin{aligned} ER_2(\widehat{v}) &= 2\int_{\underline{v}}^{\widehat{v}} (\int_{\underline{v}}^{\max\{\underline{v},kv\}} xf(x)dx + \int_{v}^{\min\{\overline{v},\frac{v}{k}\}} kxf(x)dx)f(v)dv \\ &\quad +2\int_{\widehat{v}}^{\overline{v}} (\int_{\underline{v}}^{\min\{\widehat{v},\max\{\underline{v},kv\}\}} xf(x)dx + \int_{\widehat{v}}^{\min\{\overline{v},\frac{v}{k}\}} \frac{1}{2}kxf(x)dx \\ &\quad + \int_{\widehat{v}}^{\max\{\widehat{v},kv\}} \frac{1}{2}xf(x)dx)f(v)dv \end{aligned}$$

The next step is to change the order of integration of each of the five terms. The first term,

$$T^1 = 2\int_{\underline{v}}^{\widehat{v}} \int_{\underline{v}}^{\max\{\underline{v},kv\}} xf(x)f(v)dxdv$$

is obviously zero if $\underline{v} \geq k\widehat{v}$. Otherwise, it is straightforward to change the order of integration to get

$$T^1_{\underline{v}<k\widehat{v}} = 2\int_{\underline{v}}^{k\widehat{v}} \int_{\frac{x}{k}}^{\widehat{v}} xf(x)f(v)dvdx$$

Consequently, for any $\widehat{v}$,

$$\begin{aligned} T^1 &= 2\int_{\underline{v}}^{\max\{\underline{v},k\widehat{v}\}} \int_{\frac{x}{k}}^{\widehat{v}} xf(x)f(v)dvdx \\ &= 2\int_{\underline{v}}^{\max\{\underline{v},k\widehat{v}\}} xf(x)(F(\widehat{v}) - F(\frac{x}{k}))dx \end{aligned}$$

Turning to the second term,

$$\begin{aligned} T^2 &= 2\int_{\underline{v}}^{\widehat{v}} \int_{v}^{\min\{\overline{v},\frac{v}{k}\}} kxf(x)f(v)dxdv \\ &= 2\int_{\underline{v}}^{\min\{k\overline{v},\widehat{v}\}} \int_{v}^{\frac{v}{k}} kxf(x)f(v)dxdv + 2\int_{\min\{k\overline{v},\widehat{v}\}}^{\widehat{v}} \int_{v}^{\overline{v}} kxf(x)f(v)dxdv \end{aligned}$$

where the last term is zero if $\widehat{v} < k\overline{v}$. In this case, changing the order of integration yields

$$T^2_{\widehat{v}<k\overline{v}} = 2\int_{\underline{v}}^{\frac{\underline{v}}{k}} \int_{\underline{v}}^{\min\{x,\widehat{v}\}} kxf(x)f(v)dvdx + 2\int_{\frac{\underline{v}}{k}}^{\frac{\widehat{v}}{k}} \int_{kx}^{\min\{x,\widehat{v}\}} kxf(x)f(v)dvdx$$

while for $\widehat{v} \geq k\overline{v}$,

$$T^2_{\widehat{v}\geq k\overline{v}} = 2\int_{\underline{v}}^{\frac{\underline{v}}{k}} \int_{\underline{v}}^{\min\{x,\widehat{v}\}} kxf(x)f(v)dvdx + 2\int_{\frac{\underline{v}}{k}}^{\overline{v}} \int_{kx}^{\min\{x,\widehat{v}\}} kxf(x)f(v)dvdx$$

It follows that we can write this term, for all v, as

$$
\begin{aligned}
T^2 &= 2\int_{\underline{v}}^{\frac{v}{k}}\int_{\underline{v}}^{\min\{x,\widehat{v}\}} kxf(x)f(v)dvdx \\
&\quad +2\int_{\frac{v}{k}}^{m(\widehat{v})}\int_{kx}^{\min\{x,\widehat{v}\}} kxf(x)f(v)dvdx \\
&= 2\int_{\underline{v}}^{m(\widehat{v})}\int_{\max\{\underline{v},kx\}}^{\min\{x,\widehat{v}\}} kxf(x)f(v)dvdx \\
&= 2\int_{\underline{v}}^{\widehat{v}} kxf(x)(F(x)-F(\max\{\underline{v},kx\}))dx \\
&\quad +2\int_{\widehat{v}}^{m(\widehat{v})} kxf(x)(F(\widehat{v})-F(\max\{\underline{v},kx\}))dx
\end{aligned}
$$

Changing the order of integration of the third term, we find that

$$
\begin{aligned}
T^3 &= 2\int_{\underline{v}}^{\max\{\underline{v},k\widehat{v}\}}\int_{\widehat{v}}^{\overline{v}} xf(x)f(v)dvdx + 2\int_{\max\{\underline{v},k\widehat{v}\}}^{\min\{\widehat{v},k\overline{v}\}}\int_{\frac{x}{k}}^{\overline{v}} xf(x)f(v)dvdx \\
&= 2\int_{\underline{v}}^{\max\{\underline{v},k\widehat{v}\}} xf(x)(1-F(\widehat{v}))dx + 2\int_{\max\{\underline{v},k\widehat{v}\}}^{\min\{\widehat{v},k\overline{v}\}} xf(x)(1-F(\frac{x}{k}))dx
\end{aligned}
$$

The fourth term can be rewritten as

$$
\begin{aligned}
T^4 &= \int_{\widehat{v}}^{m(\widehat{v})}\int_{\widehat{v}}^{\overline{v}} kxf(x)f(v)dvdx + \int_{m(\widehat{v})}^{\overline{v}}\int_{kx}^{\overline{v}} kxf(x)f(v)dvdx \\
&= \int_{\widehat{v}}^{m(\widehat{v})} kxf(x)(1-F(\widehat{v}))dx + \int_{m(\widehat{v})}^{\overline{v}} kxf(x)(1-F(kx))dx
\end{aligned}
$$

while the fifth and final term is equal to

$$
\begin{aligned}
T^5 &= \int_{\min\{\widehat{v},k\overline{v}\}}^{k\overline{v}}\int_{\frac{x}{k}}^{\overline{v}} xf(x)f(v)dvdx \\
&= \int_{\min\{\widehat{v},k\overline{v}\}}^{k\overline{v}} xf(x)(1-F(\frac{x}{k}))dx
\end{aligned}
$$

Summing and rearranging the five terms and noting that $\min\{\widehat{v},k\overline{v}\} = km(\widehat{v})$ produce (6). This ends the proof of Proposition 3. ■

Proof of Lemma 2. (*i*) The function $m(\widehat{v})$ is differentiable everywhere except at $\widehat{v} = k\overline{v}$. Hence, for all $\widehat{v} \neq k\overline{v}$, the derivative of (6) is

$$\begin{aligned} ER_2'(\widehat{v}) &= km(\widehat{v})f(km(\widehat{v}))(1-F(m(\widehat{v})))m'(\widehat{v})k \\ &+\int_{\widehat{v}}^{m(\widehat{v})} kxf(x)dxf(\widehat{v}) - k\widehat{v}f(\widehat{v})(1-F(\widehat{v})) \\ &+2km(\widehat{v})f(m(\widehat{v}))(F(\widehat{v})-F(km(\widehat{v})))m'(\widehat{v}) \end{aligned}$$

Since the last term is always zero, the derivative can be written as

$$ER_2'(\widehat{v}) = \begin{cases} f(\widehat{v})\left((\widehat{v}-k\widehat{v})(1-F(\frac{\widehat{v}}{k})) + \int_{\widehat{v}}^{\frac{\widehat{v}}{k}} k(x-\widehat{v})f(x)dx\right) & \widehat{v} < k\overline{v} \\ f(\widehat{v})\int_{\widehat{v}}^{\overline{v}} k(x-\widehat{v})f(x)dx & \widehat{v} > k\overline{v} \end{cases} \quad (10)$$

which is strictly positive for all $\widehat{v} < \overline{v}$. Note also that when $\widehat{v}$ converges to $k\overline{v}$, $ER_2'(\widehat{v})$ converges to the same from the left and the right. That is, $ER_2'(\widehat{v})$ is continuously differentiable, and strictly increasing.

(*ii*) Again, the function $m(\widehat{v})$ is differentiable everywhere except at $\widehat{v} = k\overline{v}$. Thus, for all $\widehat{v} \neq k\overline{v}$, the derivative of (5) is

$$\begin{aligned} ER_1'(\widehat{v}) &= -f(\widehat{v})\left(\widehat{v}(1-F(m(\widehat{v}))) + \int_{\widehat{v}}^{m(\widehat{v})} kxf(x)dx\right) \\ &+(1-F(\widehat{v}))\left(1-F(m(\widehat{v})) + k\widehat{v}f(\widehat{v})\right) \\ &+(1-F(\widehat{v}))f(m(\widehat{v}))m'(\widehat{v})(km(\widehat{v})-\widehat{v}) \end{aligned}$$

Once more, the last term is always zero. Rewriting yields

$$\begin{aligned} ER_1'(\widehat{v}) &= -f(\widehat{v})\int_{\widehat{v}}^{m(\widehat{v})} k(x-\widehat{v})f(x)dx \\ &-f(\widehat{v})(1-F(m(\widehat{v})))\left(\widehat{v} - \frac{1-F(\widehat{v})}{f(\widehat{v})} - k\widehat{v}\right) \end{aligned}$$

or

$$ER_1'(\widehat{v}) = \begin{cases} -f(\widehat{v})\left(\int_{\widehat{v}}^{\frac{\widehat{v}}{k}} k(x-\widehat{v})f(x)dx + (1-F(\frac{\widehat{v}}{k}))\left(\widehat{v} - \frac{1-F(\widehat{v})}{f(\widehat{v})} - k\widehat{v}\right)\right) & \widehat{v} < k\overline{v} \\ -f(\widehat{v})\int_{\widehat{v}}^{\overline{v}} k(x-\widehat{v})f(x)dx & \widehat{v} > k\overline{v} \end{cases} \quad (11)$$

As before, when $\widehat{v}$ converges to $k\overline{v}$, $ER_1'(\widehat{v})$ converges to the same from the left and from the right, and it follows that $ER_1(\widehat{v})$ is continuously differentiable. From (10) and (11), we conclude that the derivative of $ER_1(\widehat{v}) + ER_2(\widehat{v})$ is

$$ER_1'(\widehat{v}) + ER_2'(\widehat{v}) = \begin{cases} (1 - F(\frac{1}{k}\widehat{v}))(1 - F(\widehat{v})) > 0 & \widehat{v} < k\overline{v} \\ 0 & \widehat{v} \geq k\overline{v} \end{cases}$$

This completes the proof of Lemma 2. ■

Proof of Proposition 4. Assuming that $\widehat{v} < k\overline{v}$, (5) and (6) imply

$$\begin{aligned}
&ER_2(\widehat{v}) - ER_1(\widehat{v}) \\
&= 2\int_{\underline{v}}^{\widehat{v}} xf(x)(1 - F(\frac{x}{k}))dx + 2\int_{\underline{v}}^{\widehat{v}} kxf(x)\left(F(x) - F(\max\{\underline{v}, kx\})\right)dx \\
&+ 2\int_{\widehat{v}}^{\frac{\widehat{v}}{k}} kxf(x)\left(F(\widehat{v}) - F(\max\{\underline{v}, kx\})\right)dx - 2\int_{\underline{v}}^{\widehat{v}} kxf(x)(1 - F(x))dx \\
&+ \int_{\frac{\widehat{v}}{k}}^{\overline{v}} kxf(x)(1 - F(kx))dx + \int_{\widehat{v}}^{k\overline{v}} xf(x)(1 - F(\frac{x}{k}))dx \\
&\qquad - (1 - F(\widehat{v}))\widehat{v}(1 - F(\frac{\widehat{v}}{k}))
\end{aligned}$$

Alternatively, we can write this as

$$ER_2(\widehat{v}) - ER_1(\widehat{v}) = A(\widehat{v}) + B(\widehat{v})$$

where

$$\begin{aligned}
&A(\widehat{v}) \\
= \; &2\int_{\underline{v}}^{\widehat{v}} xf(x)(1 - F(\frac{x}{k}))dx + 2\int_{\underline{v}}^{\widehat{v}} kxf(x)\left(F(x) - F(\max\{\underline{v}, kx\})\right)dx \\
&+2\int_{\widehat{v}}^{\frac{\widehat{v}}{k}} kxf(x)\left(F(\widehat{v}) - F(\max\{\underline{v}, kx\})\right)dx - 2\int_{\underline{v}}^{\widehat{v}} kxf(x)(1 - F(x))dx
\end{aligned}$$

and

$$\begin{aligned}
&B(\widehat{v}) \\
= \; &\int_{\frac{\widehat{v}}{k}}^{\overline{v}} kxf(x)(1 - F(kx))dx + \int_{\widehat{v}}^{k\overline{v}} xf(x)(1 - F(\frac{x}{k}))dx \\
&-(1 - F(\widehat{v}))\widehat{v}(1 - F(\frac{\widehat{v}}{k}))
\end{aligned}$$

Observing that $A(\underline{v}) = 0$ and

$$A'(\widehat{v}) = 2f(\widehat{v})\left((\widehat{v} - k\widehat{v})(1 - F(\frac{\widehat{v}}{k})) + \int_{\widehat{v}}^{\frac{\widehat{v}}{k}} k(x - \widehat{v})f(x)dx\right) = 2ER_2'(\widehat{v}) > 0$$

we conclude that $A(\widehat{v}) > 0$, for all $\widehat{v} \in (\underline{v}, k\overline{v}]$. Furthermore, $B(k\overline{v}) = 0$ and

$$B'(\widehat{v}) = -(1 - F(\widehat{v}))(1 - F(\frac{\widehat{v}}{k})) = -(ER_1'(\widehat{v}) + ER_2'(\widehat{v})) < 0$$

implies that $B(\widehat{v}) > 0$, for all $\widehat{v} \in [\underline{v}, k\overline{v})$. It follows that $ER_2(\widehat{v}) - ER_1(\widehat{v}) > 0$, for all $\widehat{v} \in [\underline{v}, k\overline{v}]$. Finally, Lemma 2 ensures that $ER_2(\widehat{v}) - ER_1(\widehat{v}) > 0$ on $\widehat{v} \in (k\overline{v}, \overline{v}]$ as well, since $ER_2(\widehat{v})$ increases and $ER_1(\widehat{v})$ decreases on this interval. This ends the proof of Proposition 4. ■

Proof of Proposition 5. In the proof of Lemma 2 it was established that $ER_1(\widehat{v})$ is continuously differentiable. From (11) we see specifically that

$$ER_1'(\widehat{v}) = -f(\widehat{v}) \int_{\widehat{v}}^{\overline{v}} k(x - \widehat{v})f(x)dx, \text{ for } \widehat{v} \in [k\overline{v}, \overline{v}]$$

Clearly, this is negative, and strictly so for all $\widehat{v} \in [k\overline{v}, \overline{v})$. It follows that the optimal value of $\widehat{v}$ must be strictly lower than $k\overline{v}$. The sequence of auctions is inefficient since a bidder with valuation $\widehat{v} < k\overline{v}$ faced by a rival with valuation $\overline{v}$ wins stage 1 with probability 0.5. The efficient outcome in this case is for the bidder with valuation $\overline{v}$ to win both.

However, when $k = 1$, (11) reduces to $(1 - F(\widehat{v}))^2 \geq 0$. It follows that when $k = 1$, the optimal value of $\widehat{v}$ is $\overline{v}$. This completes the proof of Proposition 5. ■

Proof of Proposition 6. To prove the second part of Proposition 6, we start with the following preliminary remarks.

(i) First observe that the assumption $\underline{v} \geq kE(v)$ implies

$$\int_{z}^{\overline{v}} (z - kx)f(x)dx \geq 0, \forall z \in [\underline{v}, k\overline{v}] \tag{12}$$

To see this, note that the derivative with respect to z is

$$f(z)\left[-(1-k)z + \frac{1 - F(z)}{f(z)}\right]$$

where the term is square brackets is decreasing in z (by Assumption 2). Hence, once the slope of (12) becomes negative, it remains negative. Consequently, (12) is minimized at one of the end-points. Clearly, (12) is positive at $z = k\overline{v}$, and $\underline{v} \geq kE(v)$ ensures that it is non-negative at $\underline{v}$.

(*ii*) It is easily seen to be a dominant strategy to bid marginal utility in stage 2, if the buy-out price was not accepted. Consider a bidder with valution z, who played his equilibrium strategy in stage 1, but lost. Then, the buy-out price in stage 2 is $B(z)$. To have a discriminating equilibrium, we require that

$$\int_{z}^{\frac{B(z)}{k}} (z-kx)f(x)dx \geq (z-B(z))[F(\frac{B(z)}{k})-F(z)+\frac{1}{2}(1-F(\frac{B(z)}{k}))] \quad (13)$$

In other words, the bidder should prefer rejecting the buy-out price to accepting it. Notice that the right-hand-side can be made arbitrarily small (and the left-hand-side strictly positive) by letting $B(z) \to z$, implying that there exists $B(\cdot)$ functions such that (13) is indeed satisfied.

(*iii*) Let $b(v)$ be the candidate for the equilibrium bidding strategy in stage 1, and assume it is strictly increasing. Since the buy-out price is at least $\underline{v}$, it is convenient to define $B^{-1}(x) = \underline{v}$ if $x \leq \underline{v}$. Then, if a bidder with valuation v decides to bid $b(z)$ in stage 1, expected payoff is

$$\begin{aligned} EU(z,v) &= \int_{\underline{v}}^{z} (v - b(x))f(x)dx + \int_{\underline{v}}^{\min\{B^{-1}(kv),z\}} (kv - B(x))f(x)dx \\ &+ \max\left\{ \int_{z}^{\max\{z,\min\{\frac{B(z)}{k},\frac{v}{k},\overline{v}\}\}} (v - kx)f(x)dx, \right. \qquad (14) \\ &\left. (v - B(z))[F(\min\{\frac{B(z)}{k},\overline{v}\}) - F(z) + \frac{1}{2}(1 - F(\min\{\frac{B(z)}{k},\overline{v}\}))] \right\} \end{aligned}$$

The first term adresses the possibility that the first auction is won. If the bidder won stage 1, it is optimal to accept the buy-out price in stage 2 if and only if it is lower than kv, and this is the second term. However, if stage 1 is lost, the bidder may or may not prefer rejecting $B(z)$ to accepting it. Given that the rival follows the equilibrium strategy, this is captured by the third term. We can now show why it is necessary that $b(v)$ takes the form described in Assumption 3.

Consider first $v < k\overline{v}$, and examine the properties of (14) for z close to v.

Given (13) is satisfied, and $kv < B(z) < v$ with $z \approx v$, (14) becomes

$$\begin{aligned} EU(z,v) &= \int_{\underline{v}}^{z} (v - b(x))f(x)dx + \int_{\underline{v}}^{B^{-1}(kv)} (kv - B(x))f(x)dx \\ &+ \int_{z}^{\frac{B(z)}{k}} (v - kx)f(x)dx \end{aligned}$$

The first order condition is then satisfied if and only if $b(v)$ is as stated in Assumption 3. Observe that as $B(v) \longrightarrow v$, $b(v) \longrightarrow kv$.

When $v, z > k\overline{v}$, $B(z) = z$, implying that (13) is satisfied. Then, for all $z > k\overline{v}$, (14) becomes

$$\begin{aligned} EU(z,v) &= \int_{\underline{v}}^{k\overline{v}} (v - b(x))f(x)dx + \int_{k\overline{v}}^{z} (v - b(x))f(x)dx \\ &+ \int_{\underline{v}}^{kv} (kv - B(x))f(x)dx + \int_{z}^{\overline{v}} (v - kx)f(x)dx \end{aligned}$$

Clearly, this is independent of z if $b(x) = kx$ for all $x \geq k\overline{v}$, implying there is no incentive to bid $b(z)$ rather than $b(v)$.

We have now shown there is no incentive to make small, local deviations. In the following we rule out sizeable deviations as well. Recall that we let v denote the valuation of the bidder, whereas z denotes the valuation the bidder pretends to have by bidding $b(z)$. Assume, for now, that $b(v)$ is strictly increasing.

(a) $B(z) \geq v$. We have already shown that if $v \geq k\overline{v}$, then it does not pay to deviate to a $z = B(z) > v$. Hence, we concentrate on $v < k\overline{v}$, and observe that it is a dominant strategy in stage 2 not to accept $B(z)$ if stage 1 was lost. Thus,

$$\begin{aligned} EU(z,v) &= \int_{\underline{v}}^{z} (v - b(x))f(x)dx + \int_{\underline{v}}^{B^{-1}(kv)} (kv - B(x))f(x)dx \\ &+ \int_{z}^{\max\{z, \frac{v}{k}\}} (v - kx)f(x)dx \end{aligned}$$

The derivative w.r.t. z is

$$EU'_z(z,v) = \begin{cases} (v - kz - (b(z) - kz))f(z) \leq 0 & \text{if } z > \frac{v}{k} \\ (kz - b(z))f(z) \leq 0 & \text{if } z < \frac{v}{k} \end{cases}$$

implying that this type of deviation is unprofitable, since it is preferable to lower z from its level of $z \geq B^{-1}(v) \geq v$.

(b) $z > v \geq B(z)$. This is possible only if $z, v \in (\underline{v}, k\overline{v})$. If a bidder with valuation v loses stage 1 with a bid of $b(z)$, he will elect not to accept $B(z)$ in stage 2 if

$$\int_z^{\frac{B(z)}{k}} (v-kx)f(x)dx \geq (v-B(z))[F(\frac{B(z)}{k})-F(z)+\frac{1}{2}(1-F(\frac{B(z)}{k}))] \quad (15)$$

Assuming that $B(z)$ is sufficiently close to z to satisfy (13), and noting that the right-hand-side of (15) increases faster in v than the left-hand-side, it follows that the inequality remains satisfied for any $v < z$. The bidder is better of not accepting $B(z)$ in stage 2 if stage 1 was lost. Hence, expected payoff is

$$\begin{aligned} EU(z,v) &= \int_{\underline{v}}^{z} (v-b(x))f(x)dx + \int_{\underline{v}}^{B^{-1}(kv)} (kv - B(x))f(x)dx \\ &+ \int_z^{\frac{B(z)}{k}} (v-kx)f(x)dx \end{aligned}$$

and the derivative is

$$\begin{aligned} EU'_z(z,v) &= (v-b(z))f(z) + (v-B(z))f(\frac{B(z)}{k})\frac{B'(z)}{k} - (v-kz)f(z) \\ &= f(z)\left[(v-B(z))\frac{f(\frac{B(z)}{k})}{f(z)}\frac{B'(z)}{k} + kz - b(z)\right] \\ &= (v-z)f(\frac{B(z)}{k})\frac{B'(z)}{k} < 0 \end{aligned}$$

Thus, this type of deviation is unprofitable too, as it pays to lower z from its high level.

(c) $v \geq z \geq B(z)$. If stage 1 was lost, the bidder can choose to either accept or reject $B(z)$ in stage 2.

$(c1)$ $v \geq z \geq B(z)$, *reject* $B(z)$ *if stage 1 was lost.* Expected payoff is

$$\begin{aligned} EU(z,v) &= \int_{\underline{v}}^{z} (v-b(x))f(x)dx + \int_{\underline{v}}^{\min\{B^{-1}(kv),z\}} (kv - B(x))f(x)dx \\ &+ \int_z^{\min\{\frac{B(z)}{k},\overline{v}\}} (v-kx)f(x)dx \end{aligned}$$

Since the second term is non-decreasing in z, the derivative w.r.t. z can be bounded below,

$$EU'_z(z,v) \geq \begin{cases} 0 & \text{if } B(z) \geq k\overline{v} \\ (v-z)f(\frac{B(z)}{k})\frac{B'(z)}{k} \geq 0 & \text{if } B(z) < k\overline{v} \end{cases}$$

as $b(z) = kz$ when $B(z) \geq k\overline{v}$. Hence, this type of deviation is not profitable either.

(c2) $v \geq z \geq B(z)$, *accept* $B(z)$ *if stage 1 was lost*. We observe that if $B(z) \geq k\overline{v}$, the winner of stage 1 will not accept $B(z)$. Then, the loser of stage 1 should not accept $B(z)$ either. Since $v \geq B(z) \geq k\overline{v}$, the loser of stage 1 is certain to win a second-price auction, and pay less than $B(z)$. Hence, in order for it to be a sensible strategy to accept $B(z)$, we must as a minimum require that $B(z) < k\overline{v}$, or $z < k\overline{v}$. Hence,

$$\begin{aligned} EU(z,v) &= \int_{\underline{v}}^{z}(v-b(x))f(x)dx + \int_{\underline{v}}^{\min\{B^{-1}(kv),z\}}(kv-B(x))f(x)dx \\ &\quad +(v-B(z))[F(\frac{B(z)}{k}) - F(z) + \frac{1}{2}(1-F(\frac{B(z)}{k}))] \end{aligned}$$

and it follows that

$$\begin{aligned} EU(v,v) - EU(z,v) &\geq D(z,v) \\ &= \int_{z}^{v}(v-b(x))f(x)dx + \int_{v}^{\min\{\frac{B(v)}{k},\overline{v}\}}(v-kx)f(x)dx \\ &\quad -(v-B(z))[F(\frac{B(z)}{k}) - F(z) + \frac{1}{2}(1-F(\frac{B(z)}{k}))] \end{aligned}$$

Now, if $v \geq k\overline{v}$, the facts that $B(v) \geq k\overline{v}$ and $b(x) = kx$ for $x \geq k\overline{v}$ imply

$$D(z,v) = \int_{z}^{\overline{v}}(B(z)-b(x))f(x)dx + (v-B(z))\frac{1}{2}(1-F(\frac{B(z)}{k}))$$

As the last term is positive and the first converges to (12) for $B(x) \longrightarrow x$, $D(z,v) > 0$ for $B(\cdot)$ functions that are close to the 45 degree line.

Finally, if $v < k\overline{v}$, $D(z,v)$ is positive for $z = v$, by (13), i.e. by the fact that it is optimal to reject the buy-out price in equilibrium. We wish to show this is also the case for $v > z$ and we thus differentiate w.r.t. v to get

$$D'_v(z,v) = F(\frac{B(v)}{k}) - \frac{1}{2}(1+F(\frac{B(z)}{k}))$$

Observing that $D'_v(z,z) < 0$, $D'_v(z,k\overline{v}) > 0$ and $D''_{vv}(z,v) > 0$, it follows that the minimum of $D(z,v)$ over $v \in (z, k\overline{v})$ is interior, and satisfies

$$F(\frac{B(v)}{k}) = \frac{1}{2}(1 + F(\frac{B(z)}{k}))$$

Hence, we conclude that

$$\begin{aligned} D(z,v) &= v[F(\frac{B(v)}{k}) - F(z) - (\frac{1}{2}(1 + F(\frac{B(z)}{k})) - F(z))] \\ &- \int_z^v b(x)f(x)dx - \int_v^{\frac{B(v)}{k}} kxf(x)dx + B(z)[\frac{1}{2}(1 + F(\frac{B(z)}{k})) - F(z)] \\ &\geq B(z)[F(\frac{B(v)}{k}) - F(z)] - \int_z^v b(x)f(x)dx - \int_v^{\frac{B(v)}{k}} kxf(x)dx \\ &= \int_z^v (B(z) - b(x))f(x)dx + \int_v^{\frac{B(v)}{k}} (B(z) - kx)f(x)dx \\ &> \int_z^{k\overline{v}} (B(z) - b(x))f(x)dx + \int_{k\overline{v}}^{\overline{v}} (B(z) - kx)f(x)dx \end{aligned}$$

where the last inequality follows from the fact the function preceding it is decreasing in v, and $v < k\overline{v}$. As $B(x) \longrightarrow x$, this converges to (12).

Hence, we conclude that if (12) is satisfied, there exists a $B(\cdot)$ function close to the 45 degree line, for which there is no incentive to deviate, regardless of the bidder's valuation.

It remains only to verify that $b(v)$ is strictly increasing. However, it is clear that for $B(v) \to v$ (with $B'(v) < \infty$) this must be the case as $b(v) \to kv$. Since $kJ(v) > J(kv)$, it follows that the second part of Assumption 3 is satisfied as well, for $B(v) \to v$. This completes the proof of Proposition 6. ■

Appendix B

In this appendix we show that all results of Section 3 hold with minor modifications when Assumption 1 is not met.

Observe first that Proposition 2 and $ER_1(\widehat{v})$ in Proposition 3 hold even when Assumption 1 is not satisfied. Consequently, the derivative of $ER_1(\widehat{v})$ is

$$\begin{aligned} ER_1'(\widehat{v}) &= -f(\widehat{v})\int_{\widehat{v}}^{m(\widehat{v})} k(x-\widehat{v})f(x)dx \\ &\quad -f(\widehat{v})(1-F(m(\widehat{v})))\left(\widehat{v}-\frac{1-F(\widehat{v})}{f(\widehat{v})}-k\widehat{v}\right) \\ &= -f(\widehat{v})\int_{\widehat{v}}^{\overline{v}} k(x-\widehat{v})f(x)dx \leq 0 \end{aligned}$$

since $m(\widehat{v}) = \overline{v}$. This immediately implies that the optimal value of $\widehat{v}$ is $\underline{v}$, and the buy-out price is thus accepted with probability 1.

Furthermore, since $k\overline{v} \leq \underline{v}$, it is clear that whoever loses stage 1 will win stage 2 with probability 1, regardless of $\widehat{v}$. Hence, by the *Revenue Equivalence Theorem*, overall revenue is the same[31] regardless of $\widehat{v}$. Since $ER_1(\widehat{v})$ is decreasing in $\widehat{v}$, it follows that $ER_2(\widehat{v})$ is increasing in $\widehat{v}$ (the equivalent of Lemma 2).

In addition, since the optimal value of $\widehat{v}$ is $\underline{v}$, the highest possible revenue to the first seller is $ER_1(\underline{v}) = B(\underline{v}) = kE(v)$. In stage 2, the loser of stage 1 will win. Defining $v_{(j)}$ as the $j'th$ highest valuation, the expected revenue is $ER_2(\underline{v}) = 0.5kE(v_{(1)}) + 0.5kE(v_{(2)})$, since any given player wins stage 1 with probability 0.5. This can be rewritten as

$$\begin{aligned} ER_2(\underline{v}) &= 0.5kE(v_{(1)}) + 0.5kE(v_{(2)}) \\ &= 0.5kE(v_{(1)}) + 0.5k\left(2E(v) - E(v_{(1)})\right) \\ &= kE(v) \\ &= ER_1(\underline{v}) \end{aligned}$$

Hence, in what seller 1 considers optimum, he earns the same as seller 2. Since the sum of revenues is constant, it follows that for any $\widehat{v} > \underline{v}$, seller 1 will be worse off than seller 2, and we have the equivalent of Proposition 4.

[31] It is easily seen that an agent of type $\underline{v}$ is indifferent between the auction formats.

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