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Multivariate Lagrange Multiplier Tests for Fractional Integration

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Abstract

We introduce a multivariate Lagrange Multiplier (LM) test for fractional integration. We derive and analyze the LM statistic and show that it is asymptotically chi-squared distributed under local alternatives, and that, under Gaussianity, the LM test is asymptotically efficient against local alternatives. It is shown that the regression variant in Breitung & Hassler (2002, Journal of Econometrics 110, 167-185) is not equivalent to the LM test in the multivariate case, although it is in the univariate case. A generalization of the LM test that explicitly allows for different integration orders for each variate is also introduced. The finite sample properties of the LM test are evaluated and compared to the Breitung & Hassler (2002) test by Monte Carlo experiments. An application to multivariate time series of real interest rates for six countries is offered, demonstrating that more clear-cut evidence can be drawn from multivariate tests compared to conducting several univariate tests.

JEL Classification: C32

Keywords: Asymptotic Local Power; Efficient Test; Fractional Integration; Lagrange Multiplier Test; Multivariate Fractional Unit Root; Nonstationarity

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1 Introduction

In this paper we introduce multivariate Lagrange Multiplier (LM) tests (or efficient score tests) for fractional integration. Multivariate procedures are important since most applied work concerns multiple time series, either stationary or nonstationary. Tests for fractional integration have been examined previously by Robinson (1991, 1994), Agiakloglou & Newbold (1994), and Tanaka (1999), among others, in a univariate framework, and recently by Breitung & Hassler (2002) in the multivariate case. The objective is to test if an observed K-vector time series y_t is integrated of order d, denoted $I(d)$, against the hypothesis that it is $I(d+\theta)$ for $\theta \neq 0$. By differencing the observed time series, this is equivalent to testing if $x_t = (1-L)^d y_t$ is $I(0)$ against $I(\theta)$.

With no multivariate tests available for testing the order of fractional integration, researchers interested in multiple time series have been forced to apply univariate tests to each element of the multiple time series. This procedure is not only cumbersome, but ignores potentially important correlations between the elements of the multiple time series, which could lead to increased power of a multivariate test. Hence, the purpose of the present paper is to introduce LM tests that apply to the multivariate case, with the usual computational motivation for the LM principle. The proposed multivariate tests in the present paper in many ways parallels the ones by Choi & Ahn (1999) and Nyblom & Harvey (2000), who propose stationarity tests, i.e. tests of $I(0)$ against $I(1)$, for multiple time series, and our work can thus also be seen as a generalization of their work with the important difference that our test is directed against different (i.e. fractional) alternatives.

The tests proposed in this paper are intended primarily for preliminary data analysis. For instance, when testing the null of stationarity or $I(0)$-ness (against fractional alternatives), non-rejection would allow standard methods to be employed for conducting, e.g., causality, structural VAR, or impulse response analyses. More generally, the tests may indicate the transformation of the data that would be required in order to make the data suitable for said analyses.

Suppose we observe $\{y_t, t = 1, ..., n\}$ generated by

$$(1-L)^{d+\theta} y_t = e_t \mathbb{I}(t \geq 1), \quad t = 0, \pm 1, \pm 2, ..., \tag{1}$$

where $\mathbb{I}(\cdot)$ denotes the indicator function and e_t is $I(0)$, i.e. is covariance stationary and has spectral density that is bounded and bounded away from zero at the origin. The process y_t generated by (1) is well defined for all d, and is sometimes called a multivariate type II fractionally integrated process, see Marinucci & Robinson (1999). Deterministic terms could be added to (1), allowing for a non-zero mean and trend or deterministic seasonal behavior, see section 3.1. In section 3.2 we consider the extension to different values of d and θ for each component of y_t in (1).

For the moment, we let the errors e_t be independently and identically distributed with mean zero and positive definite covariance matrix Σ, $i.i.d.(0, \Sigma)$. In section 3.3 we relax this assumption, and let e_t follow a stationary vector autoregressive process of order p, VAR(p). Note that positive definiteness of Σ rules out cointegration among the components of y_t.

We assume that d is specified *a priori* and wish to test the hypothesis

$$H_0 : \theta = 0 \tag{2}$$

against the alternative $H_1 : \theta \neq 0$. For instance, the unit root hypothesis and the hypothesis of joint stationarity (or more precisely, weak dependence) of y_t are given by (1) and (2) with $d = 1$ and $d = 0$, respectively.

Robinson (1994) and Tanaka (1999) consider testing (2) in the univariate model, i.e. (1) with $K = 1$. Robinson (1994) shows that the frequency domain LM test statistic has a chi-squared limiting distribution under the null, and is asymptotically efficient against local alternatives, $\theta = \delta/\sqrt{n}$, under Gaussianity. Tanaka (1999) shows that the time domain LM test statistic has a normal or chi-squared limiting distribution, and is asymptotically most powerful among all the invariant tests against local alternatives under Gaussianity. In a simulation study, Tanaka (1999) also demonstrates the finite sample superiority of the time domain test over Robinson's (1994) frequency domain test. Breitung & Hassler (2002) suggest a regression variant of Tanaka's (1999) LM test similar to the Dickey-Fuller test, see also Dolado, Gonzalo & Mayoral (2002). Breitung & Hassler (2002) also suggest a multivariate version, which generalizes to a trace test for the cointegrating rank, along the lines of the Johansen (1988) test, and show that their multivariate test has a limiting chi-squared distribution, where the degrees of freedom depend only on the cointegrating rank under the null.

We show that the equivalence of the LM test and the regression based test of Breitung & Hassler (2002) fails to hold in the multivariate case. We derive the LM test statistic for the hypothesis (2) in the time domain, with the usual computational advantage of estimation under the null. Thus, no multivariate fractionally integrated model needs to be estimated, and in fact the test is based on computationally simple moment matrices, see (4) and (7) below. Desirable distributional properties and optimality properties of the test are established. In particular, the test statistic is asymptotically chi-squared distributed under local alternatives, where the degrees of freedom equal the number of restrictions tested, and under Gaussianity, it is asymptotically efficient against local alternatives.

Furthermore, the LM test is shown to be consistent against fractionally cointegrated alternatives, i.e. alternatives where the integration order of some linear combination of the observed variates is lower than the hypothesized value. Thus, the test could be employed as a test of non-cointegration against the alternative of cointegration. An extension of the LM test statistic that explicitly allows for different integration orders (both different d and different θ) for each variate in the vector time series y_t is also introduced, and its asymptotic properties examined.

In a simulation study we examine the properties of the LM test in finite samples and compare with the Breitung & Hassler (2002) test. We find that the LM test compares favorably with the Breitung & Hassler (2002) test, and in particular that the LM test has higher finite sample power than the Breitung & Hassler (2002) test in the noncointegrated model.

We apply our tests to a multivariate time series of real interest rates for six major industrialized countries previously examined by Kugler & Neusser (1993) and Choi & Ahn (1999). Kugler & Neusser (1993) apply univariate unit root tests to each element of the multiple time series which mainly reject the null of a unit root, and Choi & Ahn (1999) apply their multivariate stationarity test (i.e. test of $I(0)$ against $I(1)$) and find no evidence against the null hypothesis. Our objective is to test if the real interest rates are $I(0)$ against fractional alternatives, and the evidence we obtain from the multivariate tests is more clear-cut than the evidence from applying univariate tests to each element of the multiple time series.

The rest of the paper is laid out as follows. Next, we derive and analyze the multivariate LM test in the basic model with only one integration order common to all the variates. In section 3 we consider generalizations of the basic model allowing deterministic terms, different values of d and θ for each variate, and short-run dynamics. Section 4 presents the results of the simulation study, and section 5 presents the empirical application. Section 6 offers some concluding remarks. Proofs are collected in the appendix.

2 Multivariate LM Test

The Gaussian log-likelihood function of the model in (1) is

$$L(\theta, \Sigma) = -\frac{n}{2} \ln (2\pi |\Sigma|) - \frac{1}{2} \sum_{t=1}^{n} (1-L)^{d+\theta} y_t' \Sigma^{-1} (1-L)^{d+\theta} y_t, \tag{3}$$

and hence the score is, see also Tanaka (1999) and Breitung & Hassler (2002),

$$\begin{aligned} \left. \frac{\partial L(\theta, \Sigma)}{\partial \theta} \right|_{\theta=0, \Sigma=\hat{\Sigma}} &= -\sum_{t=1}^{n} \left(\ln (1-L) x_t' \right) \hat{\Sigma}^{-1} x_t \\ &= \operatorname{tr}\left(\hat{\Sigma}^{-1} S_{10} \right), \end{aligned} \tag{4}$$

where $x_t = (1-L)^d y_t$, $S_{10} = \sum_{t=2}^{n} x_{t-1}^* x_t'$, $x_{t-1}^* = \sum_{j=1}^{t-1} j^{-1} x_{t-j}$, and $\hat{\Sigma} = n^{-1} \sum_{t=1}^{n} x_t x_t'$ is a consistent estimate of $\Sigma = E(e_t e_t')$ under the null. When $K = 1$, i.e. when the observed time series is univariate, the score in (4), normalized by $\sqrt{n}$, reduces to Tanaka's (1999) univariate time domain score statistic, $s_n = \sqrt{n} \sum_{j=1}^{n-1} j^{-1} \rho(j)$, where $\rho(j)$ is the j'th order sample autocorrelation of x_t. Our multivariate score (4) is similar to Choi & Ahn's (1999, p. 47) SBDH statistic and Nyblom & Harvey's (2000, p. 179) LBI statistic for testing $I(0)$ against $I(1)$ in multiple time series. The difference is that we introduce the j^{-1} weights in the calculation of x_{t-1}^*, where Choi & Ahn (1999) and Nyblom & Harvey (2000) use unweighted partial sums.

Breitung & Hassler (2002) consider the test statistic

$$\Lambda_0(d) = \operatorname{tr}\left(\hat{\Sigma}^{-1} S_{10}' S_{11}^{-1} S_{10} \right), \tag{5}$$

where $S_{11} = \sum_{t=2}^{n} x_{t-1}^* x_{t-1}^{*\prime}$, and show that $\Lambda_0(d) \to_d \chi^2_{K^2}$ under the null (2). However, since $\operatorname{tr}(AB) \neq \operatorname{tr}(A) \operatorname{tr}(B)$ in general, (5) is not equivalent to the multivariate LM test of

(2), as demonstrated for the univariate test by Breitung & Hassler (2002). Instead, (5) is a regression variant along the lines of the Dickey-Fuller test and the fractional Dickey-Fuller test, see Dolado et al. (2002). Indeed, the main aim of Breitung & Hassler (2002) is to construct a fractional trace statistic similar to Johansen (1988), just as the Dickey-Fuller test generalizes to Johansen's (1988) trace statistic. In particular, (5) can be rewritten as a sum of eigenvalues, $\Lambda_0(d) = \sum_{j=1}^{K} \lambda_j$, where λ_j turns out to be the test statistic for $\phi_j = 0$ in

$$(v_j' x_t) = \phi_j' x_{t-1}^* + e_t$$

and v_j is the eigenvector corresponding to λ_j. Thus, K^2 restrictions are being tested ($\phi_j = 0, j = 1, ..., K$) instead of one restriction as in (2), which explains the K^2 degrees of freedom in the asymptotic distribution of $\Lambda_0(d)$. Consequently, the test statistic (5) is not the LM test statistic for testing the hypothesis (2).

The multivariate LM test statistic for testing (2) is, e.g. Amemiya (1985, p. 142),

$$LM = \left.\frac{\partial L(\eta)}{\partial \eta'}\right|_{\theta=0,\Sigma=\hat{\Sigma}} \left[-\left.\frac{\partial^2 L(\eta)}{\partial \eta \partial \eta'}\right|_{\theta=0,\Sigma=\hat{\Sigma}} \right]^{-1} \left.\frac{\partial L(\eta)}{\partial \eta}\right|_{\theta=0,\Sigma=\hat{\Sigma}}, \tag{6}$$

where $\eta = ((\text{vec}\,\Sigma)', \theta')'$. The relevant block of the Hessian matrix in (6) is

$$\begin{aligned} -\left.\frac{\partial^2 L(\theta,\Sigma)}{\partial \theta^2}\right|_{\theta=0,\Sigma=\hat{\Sigma}} &= \sum_{t=1}^{n} \left(\ln(1-L)\,x_t'\right) \hat{\Sigma}^{-1} \left(\ln(1-L)\,x_t\right) \\ &\quad + \frac{1}{2} \sum_{t=1}^{n} \left(x_t' \hat{\Sigma}^{-1} \left(\ln^2(1-L)\,x_t\right) + \left(\ln^2(1-L)\,x_t'\right) \hat{\Sigma}^{-1} x_t \right) \\ &= \text{tr}\left(\hat{\Sigma}^{-1} M_{11}\right), \end{aligned}$$

defining $M_{11} = S_{11} + \frac{1}{2}(S_{20} + S_{20}')$, $S_{20} = \sum_{t=1}^{n} x_{t-2}^{**} x_t'$, and $x_{t-2}^{**} = \sum_{j=1}^{t-2} j^{-1} x_{t-j-1}^*$. Thus, we find that

$$LM = \frac{\text{tr}(\hat{\Sigma}^{-1} S_{10})^2}{\text{tr}(\hat{\Sigma}^{-1} M_{11})}. \tag{7}$$

In the following theorem we present the limiting distribution of the test statistic under alternatives local to the null, $H_{1n}: \theta = \delta/\sqrt{n}$, where δ is a fixed scalar.

Theorem 1 *Under $\theta = \delta/\sqrt{n}$, the LM test statistic (7) is asymptotically distributed as $\chi_1^2\left(\mathcal{I}\delta^2\right)$, where*

$$\mathcal{I} = -E_0 \frac{1}{n} \left.\frac{\partial^2 L(\theta,\Sigma)}{\partial \theta^2}\right|_{\theta=0,\Sigma=\hat{\Sigma}} = \frac{\pi^2 K}{6}. \tag{8}$$

Under the additional assumption of Gaussianity, the test is asymptotically efficient against local alternatives.

Thus, the LM test is chi-squared with one degree of freedom under the null, which is expected since only one restriction is being tested. In contrast, the test (5) has K^2 degrees of freedom. More generally, standard statistical results apply in the present fractional model, unlike in the multivariate unit root and stationarity tests nested in autoregressive models, e.g. Phillips & Durlauf (1986), Choi & Ahn (1999), and Nyblom & Harvey (2000).

Note that Theorem 1 continues to hold if the Fisher information matrix (8) is substituted for the Hessian matrix. However, in simulation experiments not reported here, it was found that the LM test defined in (6) has superior finite sample properties, especially in the presence of short-run dynamics. In addition, when allowance is made for short-run dynamics, the calculation of the Fisher information matrices, see (16) and (17) below, can be quite complicated. Thus, we maintain the definition of the LM test in terms of the Hessian matrix as in (6).

Next, as in Choi & Ahn (1999), we use the fact that the LM test is invariant to non-singular linear transformations, i.e. transformations of the type $\bar{x}_t = Dx_t$ for D non-singular, to show that the test is consistent against fractionally cointegrated alternatives. Following Breitung & Hassler (2002), we say that y_t is fractionally cointegrated, denoted $CI\,(d,b)$, if y_t is $I\,(d)$ and there exists $K \times r$ and $K \times (K-r)$ linearly independent matrices β and γ of full rank such that

$$\begin{aligned}\gamma' y_t &\sim I\,(d)\,,\\ \beta' y_t &\sim I\,(d-b)\,,\end{aligned}$$

where it is assumed that the fractional integration order d is given, but $b > 0$ is unknown. That is, the maintained hypothesis is that y_t is $I\,(d)$, but it is now assumed that there exists some linear combination of y_t, which is integrated of a lower order. We also assume that $u_t = (\gamma, (1-L)^{-b}\,\beta)' x_t$ is $i.i.d.\,(0,\Sigma)$. The following corollary shows that our multivariate LM test (7) is consistent under the $CI(d,b)$ alternative.

Corollary 2 *The LM test statistic (7) is $O_p\,(n)$ under the alternative that y_t is $CI\,(d,b)$.*

3 Extensions of the Model

3.1 Deterministic Terms

We allow for deterministic terms in the data generating process following Robinson (1994). Suppose we observe the K-vector time series $\left\{y_t^0, t = 1, 2, ..., n\right\}$, generated by the linear model

$$y_t^0 = \beta z_t + y_t, \tag{9}$$

where z_t is a q-vector of purely deterministic components and y_t is an unobserved K-dimensional component generated by (1).

Two leading cases for the deterministic terms are $z_t = 1$ and $z_t = (1, t)'$, which yield the models $y_{kt}^0 = \beta_{k0} + y_{kt}$ and $y_{kt}^0 = \beta_{k0} + \beta_{k1} t + y_{kt}$, respectively, but other terms like seasonal dummies or polynomial trends can also be accommodated. As in Definition 2 of Robinson (1994), it is only required that $\sum_{t=1}^{n} \tilde{z}_t \tilde{z}_t{}'$ is positive definite for n sufficiently large, where $\tilde{z}_t = (1 - L)^d z_t$. It follows from Robinson (1994) that β can be estimated by least squares regression of $(1 - L)^d y_t^0$ on $\tilde{z}_t$, yielding the estimate $\tilde{\beta}$. The test statistic is then based on the residuals $\tilde{y}_t = y_t^0 - \tilde{\beta} z_t$.

Note that we assume the deterministic terms appear in the generating mechanism of the observed variate y_t^0, instead of x_t as in Breitung & Hassler (2002). This follows the approach of Robinson (1994), and is more natural for interpretation of z_t when d is nonintegral. Consider the simple case with $z_t = 1$ and $0 < d < 1/2$. In our setup, y_t^0 is then a stationary long memory process around a non-zero mean. However, in the setup of Breitung & Hassler (2002), y_t^0 would be a stationary long memory process around the fractional deterministic trend $(1 - L)^{-d} \mathbb{I}(t \geq 1)$.

3.2 Different θ for Each Variate

Suppose the generating mechanism (1) is modified to

$$(1 - L)^{d_k + \theta_k} y_{kt} = e_{kt} \mathbb{I}(t \geq 1), \quad k = 1, ..., K, \quad t = 0, \pm 1, \pm 2, ..., \tag{10}$$

such that $\theta = (\theta_1, ..., \theta_K)'$ is now a K-vector. Redefining the log-likelihood accordingly and denoting it $L_K(\theta, \Sigma)$ (subscript K denoting different θ for each variate), the score is now given by

$$\left.\frac{\partial L_K(\theta,\Sigma)}{\partial\theta}\right|_{\theta=0,\Sigma=\hat{\Sigma}} = -\sum_{t=1}^{n}\operatorname{diag}\left(\ln\left(1-L\right)x_t\right)\hat{\Sigma}^{-1}x_t$$
$$= \sum_{t=1}^{n} J_K'(x_{t-1}^{*}\otimes\hat{\Sigma}^{-1}x_t)$$
$$= J_K'\operatorname{vec}\left(\hat{\Sigma}^{-1}S_{10}'\right) \tag{11}$$

by use of $\operatorname{vec}(ABC)=(C'\otimes A)\operatorname{vec}B$ and property 1 of Lemma 1. We denote by $\operatorname{diag}(a)$ the diagonal matrix having the vector a on the diagonal, and the matrix J_K is defined in Lemma 1. As in the previous section, the score (11) reduces to the univariate score when $K=1$.

The relevant block of the Hessian matrix in (6) is

$$-\left.\frac{\partial^2 L_K(\theta,\Sigma)}{\partial\theta\partial\theta'}\right|_{\theta=0,\Sigma=\hat{\Sigma}} = \sum_{t=1}^{n}\operatorname{diag}\left(\ln\left(1-L\right)x_t\right)\hat{\Sigma}^{-1}\operatorname{diag}\left(\ln\left(1-L\right)x_t\right)$$
$$+\sum_{t=1}^{n}J_K'(I_K\otimes\hat{\Sigma}^{-1}x_t)\operatorname{diag}\left(\ln^2\left(1-L\right)x_t\right)$$
$$= \sum_{t=1}^{n}\operatorname{diag}\left(x_{t-1}^{*}\right)\hat{\Sigma}^{-1}\operatorname{diag}\left(x_{t-1}^{*}\right)+\sum_{t=1}^{n}\operatorname{diag}(\hat{\Sigma}^{-1}x_t)\operatorname{diag}\left(x_{t-2}^{**}\right)$$
$$= S_{11}\odot\hat{\Sigma}^{-1}+(\hat{\Sigma}^{-1}S_{20}')\odot I_K,$$

using property 3 of Lemma 1. Here, $\odot$ denotes the Hadamard product, see the appendix or Magnus & Neudecker (1999). We thus form the LM test statistic

$$LM_K=\operatorname{vec}(\hat{\Sigma}^{-1}S_{10}')'J_K\left(S_{11}\odot\hat{\Sigma}^{-1}+(\hat{\Sigma}^{-1}S_{20}')\odot I_K\right)^{-1}J_K'\operatorname{vec}(\hat{\Sigma}^{-1}S_{10}'). \tag{12}$$

The asymptotic distribution of the test statistic under local alternatives, $H_{1n}:\theta=\delta/\sqrt{n}$, where δ is now a fixed K-vector, is given by the following theorem.

Theorem 3 *Under $\theta=\delta/\sqrt{n}$, δ a fixed K-vector, the LM test statistic (12) is asymptotically distributed as $\chi_K^2\left(\delta'\mathcal{I}_K\delta\right)$, where*

$$\mathcal{I}_K=-E_0\frac{1}{n}\left.\frac{\partial^2 L_K(\theta,\Sigma)}{\partial\theta\partial\theta'}\right|_{\theta=0,\Sigma=\hat{\Sigma}}=\frac{\pi^2}{6}\Sigma\odot\Sigma^{-1}.$$

Under the additional assumption of Gaussianity, the test is asymptotically efficient against local alternatives.

From Theorem 3 it is worth noting once more that, in the more general model considered in this section, the degrees of freedom still equal the number of restrictions tested, K.

3.3 Short-run Dynamics

In this section we allow for short-run dynamics following Tanaka (1999) and Breitung & Hassler (2002). In particular, suppose e_t is generated according to the vector autoregressive process

$$A(L)\,e_t = \varepsilon_t, \quad t = 0, \pm 1, \pm 2, ..., \tag{13}$$

where ε_t satisfies the assumptions of e_t before. Here, $A(z)$ is a matrix polynomial of order p, and such that e_t is a stationary VAR(p) process and $A(1)$ has full rank. The parameters of $A(z)$ are gathered in the K^2p-vector $a = \text{vec}(A_1, ..., A_p)$, and we also define $\phi = \left(\theta', a'\right)'$.

We construct the test statistics based on the prewhitened series, i.e. we use the residuals from the regression

$$e_t = \hat{A}_1 e_{t-1} + ... + \hat{A}_p e_{t-p} + \hat{\varepsilon}_t, \quad t = 1, ..., n,$$

and define $\hat{\varepsilon}^*_{t-1} = \sum_{j=1}^{t-1} j^{-1}\hat{\varepsilon}_{t-j}$, $\hat{\varepsilon}^{**}_{t-2} = \sum_{j=1}^{t-1} j^{-1}\hat{\varepsilon}^*_{t-j-1}$, and $X_{t-1} = \left(x'_{t-1}, ..., x'_{t-p}\right)'$. The test statistics (7) and (12) are now defined in terms of $\hat{\Sigma} = n^{-1}\sum_{t=1}^n \hat{\varepsilon}_t\hat{\varepsilon}'_t$, $\hat{S}_{10} = \sum_{t=2}^n \hat{\varepsilon}^*_{t-1}\hat{\varepsilon}'_t$, $\hat{S}_{11} = \sum_{t=2}^n \hat{\varepsilon}^*_{t-1}\hat{\varepsilon}^{*\prime}_{t-1}$, $\hat{S}_{20} = \sum_{t=2}^n \hat{\varepsilon}^{**}_{t-2}\hat{\varepsilon}'_t$, $\hat{S}_{x1} = \sum_{t=2}^n X_{t-1}\hat{\varepsilon}^{*\prime}_{t-1}$, $S_{xx} = \sum_{t=2}^n X_{t-1}X'_{t-1}$, and the Hessian matrices

$$-\left.\frac{\partial^2 L(\theta, a, \Sigma)}{\partial\phi\partial\phi'}\right|_{\theta=0, a=\hat{a}, \Sigma=\hat{\Sigma}} = \begin{bmatrix} \text{tr}(\hat{\Sigma}^{-1}\hat{M}_{11}) & \text{vec}(\hat{S}_{x1})' \\ \text{vec}\,\hat{S}_{x1} & S_{xx}\otimes\hat{\Sigma}^{-1} \end{bmatrix},$$

$$-\left.\frac{\partial^2 L_K(\theta, a, \Sigma)}{\partial\phi\partial\phi'}\right|_{\theta=0, a=\hat{a}, \Sigma=\hat{\Sigma}} = \begin{bmatrix} \hat{S}_{11}\odot\hat{\Sigma}^{-1} + (\hat{\Sigma}^{-1}\hat{S}'_{20})\odot I_K & J'_K(\hat{S}'_{x1}\otimes I_K) \\ (\hat{S}_{x1}\otimes I_K)J_K & S_{xx}\otimes\hat{\Sigma}^{-1} \end{bmatrix}.$$

Applying the partitioned matrix inverse formula, the test statistics are

$$LM = \frac{\text{tr}(\hat{\Sigma}^{-1}S_{10})^2}{\text{tr}(\hat{\Sigma}^{-1}(\hat{M}_{11} - \hat{S}_{x1}S_{xx}^{-1}\hat{S}_{x1}))}, \tag{14}$$

$$LM_K = \text{vec}(\hat{\Sigma}^{-1}\hat{S}'_{10})'J_K\left(\hat{S}_{11}\odot\hat{\Sigma}^{-1} + (\hat{\Sigma}^{-1}\hat{S}'_{20})\odot I_K - (\hat{S}_{x1}S_{xx}^{-1}\hat{S}_{x1})\odot\hat{\Sigma}^{-1}\right)^{-1} J'_K\,\text{vec}(\hat{\Sigma}^{-1}\hat{S}'_{10}). \tag{15}$$

The results of Theorems 1 and 3 continue to hold in the present case with autocorrelated errors, though the noncentrality parameters are different.

Theorem 4 *Suppose (13) holds and let the LM test statistics be defined by (14) and (15). The results of Theorems 1 and 3 continue to hold with noncentrality parameters defined by*

$$\mathcal{I} = \frac{\pi^2 K}{6} - \operatorname{tr}\left(\Phi' \Gamma^{-1} \Phi \Sigma\right), \tag{16}$$

$$\mathcal{I}_K = \frac{\pi^2}{6} \Sigma \odot \Sigma^{-1} - \left(\Sigma \Phi' \Gamma^{-1} \Phi \Sigma\right) \odot \Sigma^{-1}, \tag{17}$$

where Γ *is the covariance matrix of* $(e_t', ..., e_{t-p+1}')'$, $\Phi = \left(\Phi_1', ..., \Phi_p'\right)'$, $\Phi_i = \sum_{j=i}^{\infty} j^{-1} B_{j-i}$, *and* B_i *is the coefficient to* z^i *in the moving average polynomial* $B(z)$ *from the Wold representation of* e_t.

As a simple example consider the VAR(1), $e_t = A e_{t-1} + \varepsilon_t = \sum_{j=0}^{\infty} A^j \varepsilon_{t-j}$. In this case, $\mathcal{I}$ and $\mathcal{I}_K$ reduce to $\pi^2 K/6 - \operatorname{tr}\left(\Phi_1 \Gamma^{-1} \Phi_1' \Sigma\right)$ and $\frac{\pi^2}{6} \Sigma \odot \Sigma^{-1} - \left(\Sigma \Phi_1 \Gamma^{-1} \Phi_1' \Sigma\right) \odot \Sigma^{-1}$, respectively, where $\Phi_1 = I_K + \sum_{j=2}^{\infty} j^{-1} A^{j-1}$ and $\Gamma = E\left(e_t e_t'\right)$ can be recovered from the relation $\operatorname{vec} \Gamma = \left(I_{K^2} - A \otimes A\right)^{-1} \operatorname{vec} \Sigma$.

4 Finite Sample Performance

In this section we compare the finite sample properties of the LM test (7), (14), and Breitung & Hassler's (2002) $\Lambda_0(d)$ test (henceforth the BH test) in (5) with allowance for short-run dynamics when relevant, see Breitung & Hassler (2002). The asymptotic local power of the LM test can easily be derived from the previous results as

$$P\left(LM > \chi^2_{1,1-\alpha}\right) = 1 - F_{1,\lambda}\left(\chi^2_{1,1-\alpha}\right), \tag{18}$$

where $\chi^2_{1,1-\alpha}$ is the $100(1-\alpha)\,\%$ point of the central χ^2 distribution with one degree of freedom, and $F_{1,\lambda}$ is the distribution function of the noncentral χ^2 distribution with one degree of freedom and noncentrality parameter λ defined in Theorems 1 and 4. Setting $\delta = \theta\sqrt{n}$ in (18), we can compare the asymptotic local power with the finite sample rejection frequencies for any fixed values of θ and n.

The models we consider for the simulation study are

$$\text{Model A} \quad : \quad \begin{bmatrix} (1-L)^{1+\theta} & 0 \\ 0 & (1-L)^{1+\theta} \end{bmatrix} y_t = \varepsilon_t \mathbb{I}(t \geq 1),$$

$$\text{Model B} \quad : \quad (I_2 - AL) \begin{bmatrix} (1-L)^{1+\theta} & 0 \\ 0 & (1-L)^{1+\theta} \end{bmatrix} y_t = \varepsilon_t \mathbb{I}(t \geq 1), \; A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix},$$

$$\text{Model C} \quad : \quad y_{1t} = \beta y_{2t} + u_{1t}, \; (I_2 - AL) \begin{bmatrix} (1-L)^{1-\theta} & 0 \\ 0 & (1-L) \end{bmatrix} \begin{bmatrix} u_{1t} \\ y_{2t} \end{bmatrix} = \varepsilon_t \mathbb{I}(t \geq 1),$$

where the ε_t are *i.i.d.* $N(0, \Sigma)$. The contemporaneous covariance matrix Σ is normalized such that the diagonal elements equal unity and the correlation coefficient ρ is 0 or 0.6. Models A and B are noncointegrated and the alternatives are of the form considered in Theorem 1, i.e. with the same θ for each variate. The cointegrated alternatives of Corollary 2 are considered in Model C, where y_{1t} and y_{2t} are fractionally cointegrated if $\theta > 0$ and noncointegrated under the null hypothesis, $\theta = 0$. To generate data we used $\beta = 1$.

All calculations were made in Ox version 3.20 including the Arfima package version 1.01, see Doornik (2001) and Doornik & Ooms (2001). To calculate the BH test, we adapted the Gauss code available on Jörg Breitung's homepage. Throughout, the nominal size (type I error) of the tests is fixed at 5%, and the number of replications at $10,000$.

Table 1 about here

In Table 1 the finite sample rejection frequencies of the LM and BH tests for the case with *i.i.d.* errors are presented, i.e. for Model A. Under the heading 'Limit', we give the asymptotic local power calculated from (18) with $\delta = \theta\sqrt{n}$. Size corrected rejection frequencies have also been computed and are reported as LMsc and BHsc.

The finite sample sizes of both tests are close to the nominal 5% level, but the LM test is the more powerful test for Model A, except against $\theta > 0$ with $n = 100$ in which case the BH test appears slightly more powerful. Furthermore, the finite sample power of the LM test is close to the corresponding asymptotic local power.

Unreported simulations show that the BH test is robust to the case where the θ's in Model A are allowed to be different, i.e. as in the model of section 3.2. However, the LM_K test is designed for that DGP and directed against alternatives where the θ's are different. Hence, the LM_K test is clearly superior to the BH test in that model.

Table 2 about here

Table 2 presents the simulation results for Model B with $a = 0.4$. For the small sample size, $n = 100$, the BH test is slightly size distorted, with finite sample sizes of 0.0745 and 0.0740 for $\rho = 0$ and $\rho = 0.6$, respectively. When $n = 100$, the BH test has slightly higher power against $\theta < 0$ (opposite the case in Table 1), but against $\theta > 0$ the LM test has much higher power than the BH test. When considering the larger sample size, $n = 250$, or the size corrected tests, the LM test is clearly the superior test for Model B. It is worth noting that, for both $n = 100$ and $n = 250$ and for both values of ρ, the BH test has lower power against $\theta = 0.3$ than against $\theta = 0.2$.

Table 3 about here

To evaluate the sensitivity to the particular value of the coefficient matrix (i.e. $a = 0.4$) in the autoregressive specification in Model B, Table 3 presents the finite sample sizes of the LM and BH tests for different specifications of the coefficient matrix A in Model B. In particular, the values $a = -0.75, -0.5, -0.25, 0, 0.25, 0.5, 0.75$ and sample sizes $n = 100$, $n = 250$, and $n = 500$ are considered. Notice that the column $a = 0$ corresponds to the case where a VAR(1) is estimated for e_t even though it is really an *i.i.d.* process.

For all specifications the size distortions of both tests are small. For samples of $n = 100$ the finite sample size of the LM test ranges from 0.0513 to 0.0762 when $a < 0.75$. However, when $a = 0.75$ the finite sample size of the LM test is approx. 13% for a nominal 5% test. When larger samples of $n = 250$ and $n = 500$ are considered, the finite sample size distortions for $a = 0.75$ are smaller. Overall, Table 3 shows that the size of the LM test is close to the nominal 5% level.

Table 4 about here

Table 4 shows finite sample rejection frequencies of the LM and BH tests for Model C with $a = 0.4$, i.e. against fractionally cointegrated alternatives with short-run dynamics. The column $\theta = 0$ corresponds to the null of $I(1)$ noncointegrated data, the column $\theta = 1$ to standard $I(1) - I(0)$ cointegration, and $0 < \theta < 1$ corresponds to fractional cointegration with $I(1-\theta)$ cointegration errors. Thus, the degree of cointegration is determined by the magnitude of θ. In this model, both tests exhibit finite sample sizes very close to the nominal 5% level. When $\rho = 0$, the finite sample rejection frequencies of the two tests are close. When the errors are contemporaneously correlated, $\rho = 0.6$, both tests have increased power, but the gain in power of the BH test is larger than that of the LM test, as expected, since the BH test is specifically directed towards these alternatives.

Overall, the Monte Carlo study has shown that the LM test has higher finite sample power than the BH test in the noncointegrated model, although both tests can be slightly size distorted when the errors exhibit positive autocorrelation. In addition, Table 4 shows that the LM test is nearly as powerful as the BH test against cointegrated alternatives (which the latter was developed for) when the errors are contemporaneously uncorrelated.

5 Empirical Application

In this section we apply our tests to the data examined previously by Kugler & Neusser (1993) and Choi & Ahn (1999). The data are monthly observations on real interest rates for the USA, Japan, the UK, (West) Germany, France, and Switzerland from January 1980 to October 1991, i.e. 142 observations on six time series. A more detailed description is available in Kugler & Neusser (1993) or Choi & Ahn (1999).

Kugler & Neusser (1993) tested the real interest parity hypothesis by a co-dependence approach, which requires the vector time series in question to be stationary. In order to establish stationarity of the data, Kugler & Neusser (1993) conducted a series of univariate unit root tests, which rejected the unit root null hypothesis for most of the series. They found some sensitivity to the choice of lag length for the augmented Dickey-Fuller tests, while the Phillips-Perron tests all rejected the null. Choi & Ahn (1999) reversed the null and alternative hypotheses, and tested the null hypothesis of level-stationarity against the alternative of a unit root, which seems to be a more natural testing strategy in the present case. It was found that one of the univariate stationarity tests (their LM_I test) rejected the null at 5% level for France, and that two univariate stationarity tests (their $SBDH_T$ and $SBDH_B$ tests) rejected the null at 10% level for the USA. However, none of their multivariate tests rejected the null at 10% level, thus providing more certain evidence than the univariate tests.

We apply our LM and LM_K tests and the BH test of Breitung & Hassler (2002) to the real interest rate data to test the hypothesis that $d = 0$, i.e. that the data are $I(0)$, against fractionally integrated alternatives. We allow for a non-zero mean by setting $z_t = 1$ as in section 3.1, and report the tests without allowing short-run dynamics ($p = 0$) and allowing VAR(p) dynamics with $p = 1$ and $p = 4$.

Table 5 about here

In part (a) of Table 5 we report the results from applying the LM and BH tests to each univariate time series. When $p = 0$ both tests reject clearly for all the time series. However, when $p > 0$ the LM test rejects at 1% level in two of the twelve cases (Germany and Switzerland with $p = 1$), and similarly the BH test rejects at 5% level in one case (Germany with $p = 1$) and at 1% level in one case (France with $p = 4$).

The results from applying the multivariate LM, LM_K, and BH tests are reported in part (b) of Table 5. Again, the null is soundly rejected when no short-run dynamics is allowed, i.e. when $p = 0$, and also when $p = 4$ for the BH test. However, when allowing short-run dynamics with either $p = 1$ or $p = 4$, the LM and LM_K tests unanimously do not reject the null. Thus, the empirical results provide strong evidence that the data are indeed $I(0)$ with non-zero means, when allowance is made for short-run dynamics, and hence support the unit-root tests in Kugler & Neusser (1993) and the stationarity tests in Choi & Ahn (1999).

6 Conclusion

We have introduced a multivariate LM test for fractional integration, generalizing the univariate tests developed recently by Robinson (1994) and Tanaka (1999), among others. We have shown that the regression variant of the LM test derived by Breitung & Hassler (2002) is not equivalent to the LM test in the multivariate case, and indeed, that the two tests have different degrees of freedom in their asymptotic chi-squared distributions.

Desirable distributional properties and optimality properties of the LM test have been established. In particular, the test statistic is asymptotically chi-squared distributed under local alternatives, where the degrees of freedom equal the number of restrictions tested. Under Gaussianity, the LM test is asymptotically efficient against local alternatives. An extension of the LM test statistic, explicitly allowing different integration orders for each variate, was also introduced.

Finite sample properties have been evaluated by Monte Carlo experiments, which show that the LM test compares favorably with the Breitung & Hassler (2002) test. The tests were applied to a multivariate time series of real interest rates for six countries, and more clear-cut evidence were obtained compared to applying univariate tests. The results indicate that, when allowing for short-run dynamics, the real interest rates are jointly $I(0)$ with non-zero means.

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Appendix: Proofs

Proof of Theorem 1. Breitung & Hassler (2002) show that, under $\theta = 0$,

$$\frac{1}{\sqrt{n}} \operatorname{vec}\left(\hat{\Sigma}^{-1/2} S_{10}\right) \to_d N\left(0, I_K \otimes \Omega\right), \tag{19}$$

and by slight modification of the arguments of Breitung & Hassler (2002, p. 180), it follows that

$$n^{-1} S_{11} \to_p \Omega, n^{-1} S_{20} \to_p 0, n^{-1} M_{11} \to_p \Omega, \tag{20}$$

where

$$\Omega = \lim_{n\to\infty} n^{-1} \sum_{t=1}^{n} E\left(x_t^* x_t^{*\prime}\right) = \frac{\pi^2}{6} \Sigma. \tag{21}$$

The distribution under the null follows immediately using $\mathrm{tr}\left(A'B\right) = \mathrm{vec}\left(A\right)'\mathrm{vec}\left(B\right)$ and consistency of $\hat{\Sigma}$.

Consider next the case $\theta = \delta/\sqrt{n}$. Then

$$\mathrm{tr}\left(\Sigma^{-1}S_{10}\right) = \mathrm{tr}\left(\Sigma^{-1}\sum_{t=2}^{n} e_{t-1}^{*}e_{t}'\right) + \frac{\delta}{\sqrt{n}}\,\mathrm{tr}\left(\Sigma^{-1}\sum_{t=2}^{n} e_{t-1}^{*}e_{t-1}^{*\prime}\right) + O_{p}\left(n^{-1}\right), \tag{22}$$

following the arguments of Tanaka (1999, p. 579). Applying (19) and (20) to the second-moment matrices of e_t, the desired result follows.

By uncorrelatedness of x_t,

$$\mathcal{I} = -E_0 \frac{1}{n}\left.\frac{\partial^2 L\left(\theta,\Sigma\right)}{\partial\theta^2}\right|_{\theta=0,\Sigma=\hat{\Sigma}} = \mathrm{tr}\left(\Sigma^{-1}\Omega\right) = \frac{\pi^2 K}{6},$$

which is the Fisher information for θ under Gaussianity. Hence, the noncentrality parameter is maximal, and the test is efficient against local alternatives. ■

Proof of Corollary 2. Since the LM test is invariant to non-singular linear transformations, we equivalently consider $\bar{x}_t = Dx_t$ (corresponding to z_t in Breitung & Hassler (2002)), where

$$D = \begin{pmatrix} \left(\gamma'\Sigma\gamma\right)^{-1/2}\gamma' \\ \beta' - \beta'\Sigma\gamma\left(\gamma'\Sigma\gamma\right)^{-1}\gamma' \end{pmatrix}$$

such that the $(K-r)$-vector $\bar{x}_{1t}$ is *i.i.d.* $(0, I_{K-r})$ and the r-vector $\bar{x}_{2t}$ is uncorrelated with $\bar{x}_{1t}$. The LM test is proportional to $\left(\sum_{k=1}^{K}\lambda_k\right)^2$, where the λ_k are eigenvalues of $\left|\lambda\hat{\Sigma} - n^{-1/2}S_{10}\right| = 0$, or equivalently

$$\left|\lambda n^{-1}\bar{X}'\bar{X} - n^{-1/2}\bar{X}'\bar{X}^{*}\right| = 0, \tag{23}$$

with capital letters denoting matrices of observations, i.e. $\bar{X} = (\bar{x}_1, ..., \bar{x}_n)$ and $\bar{X}^{*} = (\bar{x}_1^{*}, ..., \bar{x}_n^{*})$.

By Lemma A.1 of Breitung & Hassler (2002),

$$\frac{1}{\sqrt{n}}\bar{X}'\bar{X}^{*} = \frac{1}{\sqrt{n}}\begin{pmatrix} \bar{X}_1' & \bar{X}_2' \end{pmatrix}\begin{pmatrix} \bar{X}_1^{*} \\ \bar{X}_2^{*} \end{pmatrix} = \begin{pmatrix} O_p\left(1\right) & O_p\left(1\right) \\ O_p\left(1\right) & O_p\left(\sqrt{n}\right) \end{pmatrix},$$

and thus (23) has $K-r$ eigenvalues that are $O_p\left(1\right)$ and r eigenvalues that are $O_p\left(n^{1/2}\right)$. ■

In the following we need a lemma on some properties of the Hadamard product, which is defined for two $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ as

$$A \odot B = \left(a_{ij}b_{ij}\right),$$

see e.g. Magnus & Neudecker (1999, Chapter 3.6) for more details. The proof of the lemma is easy and is omitted.

Lemma 1 *Property 1. There exists a $K^2 \times K$ matrix $J_K := (\operatorname{vec} E_{11}, ..., \operatorname{vec} E_{KK})$, $E_{ii} = e_i e_i'$ where e_i is the i'th unit K-vector, such that for any $K \times K$ matrix A,*

$$J_K' \operatorname{vec} A = a,$$

where a is the K-vector holding the diagonal of A. If $A_d := I_K \odot A$ is the diagonal matrix obtained from A then

$$\operatorname{vec} A_d = J_K a.$$

Property 2. Connection with the Kronecker product. For all $K \times K$ matrices A and B,

$$J_K' (A \otimes B) J_K = A \odot B,$$

where J_K is defined as in property 1.

Property 3. Let A and B be $K \times K$ matrices such that A is diagonal and B is symmetric. Then

$$ABA = aa' \odot B,$$

where a is defined as in property 1.

Proof of Theorem 3. It follows from (19), application of $\operatorname{vec}(ABC) = (C' \otimes A) \operatorname{vec} B$, and property 2 of Lemma 1 that

$$\frac{1}{\sqrt{n}} J_K' \operatorname{vec}\left(\hat{\Sigma}^{-1} S_{10}'\right) \rightarrow_d N\left(0, \frac{\pi^2}{6} \Sigma \odot \Sigma^{-1}\right).$$

By (20) and consistency of $\hat{\Sigma}$, the distribution under the null follows. Under $\theta = \delta/\sqrt{n}$ the expansion corresponding to (22) is

$$J_K' \operatorname{vec}\left(\Sigma^{-1} S_{10}'\right) = \sum_{t=2}^{n} \operatorname{diag}\left(e_{t-1}^*\right) \hat{\Sigma}^{-1} e_t + \sum_{t=2}^{n} \operatorname{diag}\left(e_{t-1}^*\right) \hat{\Sigma}^{-1} \operatorname{diag}\left(e_{t-1}^*\right) \frac{\delta}{\sqrt{n}} + O_p\left(n^{-1}\right), \tag{24}$$

and the result follows as above. ■

Proof of Theorem 4. Consider first $\theta = 0$. For a fixed $m > p$, define the K^2m-vector $\hat{C}_m = ((\mathrm{vec}\,\hat{C}\,(1))', ..., (\mathrm{vec}\,\hat{C}\,(m))')'$, where $\hat{C}\,(j) = n^{-1}\sum_{t=j+1}^{n}\hat{\varepsilon}_t\hat{\varepsilon}_{t-j}'$ is the j'th residual autocovariance. Hosking (1980) showed that

$$\sqrt{n}\hat{C}_m \to_d N\left(0, I_m \otimes \Sigma \otimes \Sigma - K_m\left(\Gamma^{-1}\otimes\Sigma\right)K_m'\right),$$

where $\Gamma^{-1}\otimes\Sigma$ is the inverse Fisher information for the parameters in $A\,(z)$ and

$$K_m = \begin{bmatrix} \Sigma & & & & 0 \\ \Sigma B_1' & \Sigma & & & \\ \vdots & \vdots & \ddots & & \\ \Sigma B_{m-1}' & \Sigma B_{m-2}' & \cdots & \cdots & \Sigma B_{m-p}' \end{bmatrix} \otimes I_K.$$

Thus,

$$\sqrt{n}\sum_{j=1}^{m} j^{-1}\,\mathrm{vec}\,\hat{C}\,(j) \to_d N\,(0, \Psi_m)$$

with $\Psi_m = \sum_{j=1}^{m} j^{-2}\Sigma\otimes\Sigma - (\Sigma(\Phi_1'^{(m)}, ..., \Phi_p'^{(m)})\Gamma^{-1}(\Phi_1'^{(m)}, ..., \Phi_p'^{(m)})'\Sigma)\otimes\Sigma$ and the $\Phi_i^{(m)}$ truncated at m. It now follows by application of Bernstein's Lemma, see e.g. Hall & Heyde (1980, pp. 191-192), that

$$\sqrt{n}\sum_{j=1}^{n-1} j^{-1}\,\mathrm{vec}\,\hat{C}\,(j) \to_d N\,(0, \Psi)\,,$$

where $\Psi = \lim_{m\to\infty}\Psi_m$. The limiting distributions of LM and LM_K in (14) and (15), when $\theta = 0$, now follow by recalling that $n^{-1}\hat{S}_{10} = \sum_{j=1}^{n-1} j^{-1}\hat{C}\,(j)$, and using that $n^{-1}\hat{S}_{x1} \to_p \Phi\Sigma$ and $n^{-1}S_{xx} \to_p \Gamma$ along with (20).

When $\theta = \delta/\sqrt{n}$, the desired results follow by combining the arguments of the previous theorems, and using expansions like (22) and (24). ■

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Table 1: Finite sample rejection frequencies for Model A

	$\rho = 0$					$\rho = 0.6$				
θ	Limit	LM	BH	LMsc	BHsc	Limit	LM	BH	LMsc	BHsc
$n = 100$										
-0.3	0.9998	0.9945	0.9767	0.9950	0.9788	0.9998	0.9966	0.9779	0.9967	0.9775
-0.2	0.9523	0.8914	0.7161	0.8977	0.7328	0.9523	0.8923	0.7064	0.8947	0.7057
-0.1	0.4420	0.3864	0.1998	0.4000	0.2156	0.4420	0.3899	0.2038	0.3937	0.2032
0	0.0500	0.0457	0.0444	0.0500	0.0500	0.0500	0.0489	0.0501	0.0500	0.0500
0.1	0.4420	0.1855	0.2616	0.1906	0.2726	0.4420	0.1879	0.2609	0.1894	0.2606
0.2	0.9523	0.7159	0.8056	0.7234	0.8152	0.9523	0.7171	0.8029	0.7191	0.8022
0.3	0.9998	0.9667	0.9872	0.9677	0.9881	0.9998	0.9666	0.9884	0.9670	0.9884
$n = 250$										
-0.3	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
-0.2	0.9999	0.9999	0.9951	0.9999	0.9952	0.9999	0.9998	0.9965	0.9998	0.9965
-0.1	0.8180	0.7882	0.5324	0.7867	0.5377	0.8180	0.7876	0.5400	0.7832	0.5458
0	0.0500	0.0504	0.0482	0.0500	0.0500	0.0500	0.0519	0.0477	0.0500	0.0500
0.1	0.8180	0.6241	0.6166	0.6234	0.6203	0.8180	0.6380	0.6278	0.6339	0.6326
0.2	0.9999	0.9964	0.9973	0.9964	0.9974	0.9999	0.9973	0.9966	0.9972	0.9968
0.3	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 2: Finite sample rejection frequencies for Model B with $a = 0.4$

	$\rho = 0$					$\rho = 0.6$				
θ	Limit	LM	BH	LMsc	BHsc	Limit	LM	BH	LMsc	BHsc
$n = 100$										
-0.3	0.6044	0.2308	0.3211	0.1923	0.2513	0.6044	0.2157	0.3137	0.1766	0.2405
-0.2	0.3171	0.0892	0.1639	0.0702	0.1220	0.3171	0.0897	0.1638	0.0677	0.1210
-0.1	0.1150	0.0441	0.0885	0.0331	0.0614	0.1150	0.0376	0.0860	0.0292	0.0579
0	0.0500	0.0591	0.0745	0.0500	0.0500	0.0500	0.0598	0.0740	0.0500	0.0500
0.1	0.1150	0.1450	0.0973	0.1304	0.0689	0.1150	0.1577	0.0982	0.1402	0.0711
0.2	0.3171	0.3130	0.1232	0.2924	0.0882	0.3171	0.3034	0.1155	0.2842	0.0839
0.3	0.6044	0.3959	0.1143	0.3728	0.0818	0.6044	0.3990	0.1089	0.3765	0.0797
$n = 250$										
-0.3	0.9404	0.8209	0.7818	0.8520	0.7560	0.9404	0.8130	0.7797	0.8330	0.7611
-0.2	0.6500	0.4126	0.3943	0.4651	0.3655	0.6500	0.4032	0.3923	0.4346	0.3652
-0.1	0.2164	0.1087	0.1288	0.1366	0.1146	0.2164	0.1069	0.1241	0.1230	0.1099
0	0.0500	0.0394	0.0570	0.0500	0.0500	0.0500	0.0425	0.0576	0.0500	0.0500
0.1	0.2164	0.1081	0.1161	0.1217	0.1051	0.2164	0.1083	0.1217	0.1172	0.1104
0.2	0.6500	0.3029	0.2367	0.3213	0.2185	0.6500	0.3002	0.2290	0.3093	0.2120
0.3	0.9404	0.5084	0.2118	0.5287	0.1949	0.9404	0.5009	0.2226	0.5132	0.2032

Table 3: Simulated size of nominal 5% test for Model B

Test\\a	$\rho = 0$							$\rho = 0.6$						
	−0.75	−0.5	−0.25	0	0.25	0.5	0.75	−0.75	−0.5	−0.25	0	0.25	0.5	0.75
$n = 100$														
LM	0.0569	0.0614	0.0532	0.0560	0.0527	0.0762	0.1334	0.0587	0.0592	0.0518	0.0521	0.0513	0.0690	0.1271
BH	0.0544	0.0531	0.0554	0.0595	0.0656	0.0667	0.0649	0.0561	0.0564	0.0578	0.0604	0.0669	0.0713	0.0618
$n = 250$														
LM	0.0516	0.0541	0.0519	0.0516	0.0462	0.0505	0.1083	0.0537	0.0535	0.0544	0.0496	0.0474	0.0457	0.1064
BH	0.0509	0.0533	0.0483	0.0528	0.0556	0.0561	0.0506	0.0503	0.0529	0.0493	0.0556	0.0600	0.0611	0.0520
$n = 500$														
LM	0.0510	0.0517	0.0528	0.0511	0.0432	0.0403	0.0870	0.0531	0.0494	0.0556	0.0491	0.0444	0.0407	0.0879
BH	0.0543	0.0498	0.0481	0.0554	0.0554	0.0564	0.0464	0.0520	0.0535	0.0493	0.0535	0.0545	0.0589	0.0515

Table 4: Finite sample rejection frequencies for Model C with $a = 0.4$

Test\\θ	$\rho = 0$						$\rho = 0.6$					
	0	0.2	0.4	0.6	0.8	1.0	0	0.2	0.4	0.6	0.8	1.0
$n = 100$												
LM	0.0622	0.0704	0.1921	0.4606	0.7339	0.9000	0.0609	0.0893	0.2718	0.5731	0.8376	0.9535
BH	0.0691	0.1073	0.2515	0.5294	0.7942	0.9414	0.0711	0.1306	0.3625	0.6574	0.8732	0.9619
$n = 250$												
LM	0.0429	0.1554	0.5948	0.9256	0.9939	0.9998	0.0387	0.1862	0.6743	0.9589	0.9979	1.0000
BH	0.0583	0.1901	0.6791	0.9657	0.9992	1.0000	0.0595	0.2845	0.8556	0.9947	0.9998	1.0000
$n = 500$												
LM	0.0411	0.3409	0.9124	0.9976	1.0000	1.0000	0.0403	0.3873	0.9388	0.9995	1.0000	1.0000
BH	0.0578	0.3942	0.9697	1.0000	1.0000	1.0000	0.0534	0.5824	0.9979	1.0000	1.0000	1.0000

Table 5: Empirical results for the Kugler-Neusser data

(a) Univariate tests of $d = 0$ with non-zero mean

	$p = 0$		$p = 1$		$p = 4$	
	LM(1)	BH(1)	LM(1)	BH(1)	LM(1)	BH(1)
USA	36.48^{**}	25.34^{**}	1.81	0.19	1.69	1.30
Japan	37.56^{**}	26.97^{**}	0.02	0.39	0.45	0.36
UK	51.98^{**}	31.23^{**}	0.85	0.64	0.10	0.32
Germany	26.09^{**}	18.07^{**}	8.99^{**}	4.06^{*}	0.43	2.74
France	42.63^{**}	31.12^{**}	0.92	1.16	1.04	8.93^{**}
Switzerland	44.53^{**}	28.05^{**}	20.17^{**}	2.25	0.14	2.05

(b) Multivariate tests of $d = 0$ with non-zero mean

$p = 0$			$p = 1$			$p = 4$		
LM(1)	BH(36)	$\text{LM}_K(6)$	LM(1)	BH(36)	$\text{LM}_K(6)$	LM(1)	BH(36)	$\text{LM}_K(6)$
136.47^{**}	166.44^{**}	145.09^{**}	0.18	41.11	3.76	2.23	76.76^{**}	6.44

One asterisk denotes significance at 5% level and two asterisks denote significance at 1% level. All test statistics are asymptotically χ^2-distributed, with the appropriate degrees of freedom reported in parenthesis.

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