DEPARTMENT OF ECONOMICS

Working Paper

Semiparametric Estimation in Time Series Regression with Long Range Dependence

Morten Ørregaard Nielsen

Working Paper No. 2002-17

ISSN 1396-2426

UNIVERSITY OF AARHUS • DENMARK

INSTITUT FOR ØKONOMI

AFDELING FOR NATIONALØKONOMI - AARHUS UNIVERSITET - BYGNING 350
8000 AARHUS C - ☎ 89 42 11 33 - TELEFAX 86 13 63 34

WORKING PAPER

Semiparametric Estimation in Time Series Regression with Long Range Dependence

Morten Ørregaard Nielsen

Working Paper No. 2002-17

DEPARTMENT OF ECONOMICS

SCHOOL OF ECONOMICS AND MANAGEMENT - UNIVERSITY OF AARHUS - BUILDING 350
8000 AARHUS C - DENMARK ☎ +45 89 42 11 33 - TELEFAX +45 86 13 63 34

Semiparametric Estimation in Time Series Regression with Long Range Dependence*

Morten Ørregaard Nielsen
University of Aarhus

November 12, 2002

Abstract

We consider semiparametric estimation in time series regression in the presence of long range dependence in both the errors and the stochastic regressors. A central limit theorem is established for a class of semiparametric frequency domain weighted least squares estimates, which includes both narrow band ordinary least squares and narrow band generalized least squares as special cases. The estimates are semiparametric in the sense that focus is on the neighborhood of the origin, and only periodogram ordinates in a degenerating band around the origin are used. This setting differs from earlier work on time series regression with long range dependence where a fully parametric approach has been employed. The generalized least squares estimate is infeasible when the degree of long range dependence is unknown and must be estimated in an initial step. In that case, we show that a feasible estimate exists, which has the same asymptotic properties as the infeasible estimate. By Monte Carlo simulation, we evaluate the finite-sample performance of the generalized least squares estimate and the feasible estimate.

Proposed Running Head: Long Range Dependent Regression

---

*I am grateful to Peter Phillips and seminar participants at Yale University for constructive comments and suggestions as well as Yale University and the Cowles Foundation for their hospitality during my visit when this research was initiated. Address correspondence to: Morten Ørregaard Nielsen, Department of Economics, Building 322, University of Aarhus, DK-8000 Aarhus C, Denmark; e-mail: monielsen@econ.au.dk; telephone: +45 8942 1584.

AMS 2000 Classification: Primary: 62M10; secondary: 60G18, 62J05

JEL Classification: C14, C22

Keywords: Fractional integration, generalized least squares, linear regression, long range dependence, semiparametric estimation, Whittle likelihood

1 Introduction

In this paper we derive central limit theorems for semiparametric estimates of the coefficient vector β in the multiple linear time series regression model

$$y_t = \alpha + \beta' x_t + u_t, \qquad t = 1, 2, ..., \tag{1}$$

where both the $(p-1)$-vector of stochastic regressors x_t and the scalar errors u_t are allowed to have long range dependence.

It is well known that, under a wide variety of regularity conditions, the ordinary least squares and generalized least squares estimates of β are asymptotically normal, see e.g. Hannan (1979). However, as discussed by Robinson (1994*a*, 1994*b*) and Robinson & Hidalgo (1997), this fails to hold when x_t and u_t have sufficient collective long range dependence. To account for this, Robinson (1994*a*) suggested a narrow band (semiparametric) frequency domain least squares estimate, where the estimation is conducted over a degenerating band of frequencies near the origin, and proved its consistency for arbitrary short-run dynamics. As an alternative, Robinson & Hidalgo (1997) introduced a parametric class of (full band) weighted least squares estimates (including generalized least squares as a special case), and proved root-n-consistency and asymptotic normality for these estimates, assuming correct specification of the dynamics at any frequency.

We consider a semiparametric version of the class of weighted least squares estimates in Robinson & Hidalgo (1997). The advantage of the semiparametric approach is that consistency and asymptotic normality are retained without the need for correct specification of the short-run dynamics. Suppose the spectral density matrix of the p-vector $w_t = (x_t', u_t)'$ exists and satisfies

$$f_w(\lambda) \sim \Lambda^{-1} G \Lambda^{-1} \qquad \text{as } \lambda \to 0^+, \tag{2}$$

where the symbol "$\sim$" means that the ratio of the left- and right-hand sides tends to one (elementwise), $\Lambda = \mathrm{diag}(\lambda^{d_1}, ..., \lambda^{d_p})$, and G is a $p \times p$ real, symmetric, positive definite matrix. Then the process w_t is said to have long range dependence or strong dependence since the autocorrelations decay hyperbolically. The parameters $d_1, ..., d_p$ determine the memory of the process, i.e. each component of w_t, say w_{at}, is associated with one memory parameter, d_a. If $d_a > -1/2$, w_{at} is invertible and admits a linear representation, and if $d_a < 1/2$, w_{at} is covariance stationary. If $d_a = 0$, the spectral density is bounded at the origin and w_{at} has only weak dependence. Sometimes w_{at} is said to have negative, short, or long memory when $d_a < 0$, $d_a = 0$, or $d_a > 0$, respectively. Note that the memory parameter of $u_t = w_{pt}$ is d_p in this notation. Throughout this paper we shall be concerned with the case $0 \leq d_a < 1/2$, $a = 1, ..., p$, since this is the dominant case in empirical research, see Robinson (1994*b*) and Beran (1994) for a review of long range dependent processes.

The most well known parametric models satisfying (2) are the fractional Gaussian noise and the fractional ARIMA models, see Mandelbrot & Ness (1968), Adenstedt (1974), Granger & Joyeux (1980), and Hosking (1981). The obvious advantage of specifying the spectral density only in a neighborhood of the origin as in (2), is that it allows treating the spectral density away from the origin nonparametrically, assuming only mild regularity conditions. Thus, in applications we need not worry about correct specification of the short-run dynamics of the process, such as the autoregressive and moving average orders in the fractional ARIMA model. Previously, this type of specification, termed semiparametric by Robinson (1994*a*), has been applied for estimation of the memory parameters by Geweke & Porter-Hudak (1983), Robinson (1994*a*, 1995*a*, 1995*b*), Lobato & Robinson (1996), and Lobato (1997, 1999), among others.

Based on observations (y_t, x_t), $t = 1, ..., n$, we consider the class of semiparametric weighted least squares estimates

$$\hat{\beta}_{\delta,m} = \left(\frac{1}{m} \sum_{j=1}^{m} \lambda_j^{2\delta} \,\mathrm{Re}\left(I_{xx}(\lambda_j)\right) \right)^{-1} \frac{1}{m} \sum_{j=1}^{m} \lambda_j^{2\delta} \,\mathrm{Re}\left(I_{xy}(\lambda_j)\right), \tag{3}$$

where

$$I_{ab}(\lambda) = w_a(\lambda)\, w_b^*(\lambda) \text{ and } w_a(\lambda) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} a_t e^{it\lambda} \tag{4}$$

are the cross-periodogram matrix between a_t and b_t and the discrete Fourier transform of a_t, respectively, $\lambda_j = 2\pi j/n$ are the harmonic frequencies, $m = m(n)$ is a bandwidth parameter, and the asterisk denotes complex conjugation combined with transposition.

Our estimates are semiparametric in the sense that they assume only (2) on the spectral density matrix of w_t, except for weak regularity conditions (see below). Thus, we shall need the bandwidth parameter $m = m(n)$ to tend to infinity at a slower rate than n, such that we remain in a neighborhood of the origin where the functional form of the spectral density (2) is assumed. This has the advantage that our estimate is invariant to the short-run dynamics of the processes x_t and u_t (it is also location invariant since $\lambda_0 = 0$ is left out of the summations in (3)). In contrast, the estimates in Robinson & Hidalgo (1997) use all available periodogram ordinates (i.e. $m = n$) and replace our weights $\lambda_j^{2\delta}$ by weight functions $\phi(\lambda)$, $\pi \leq \lambda < \pi$. Thus, $\phi(\lambda) = 1$ and $\phi(\lambda) = f_u^{-1}(\lambda)$ correspond to ordinary least squares and generalized least squares, respectively, and correct specification of the dynamics of the model at any frequency is assumed.

In our setting, (3) with $\delta = 0$ (i.e. $\hat{\beta}_{0,m}$) is termed the narrow band frequency domain least squares (FDLS) estimate (see Robinson (1994*a*) and Robinson & Marinucci (1998)). Henceforth, we shall term (3) with $\delta = d_p$ (i.e. $\hat{\beta}_{d_p,m}$) the narrow band frequency domain generalized least squares (FDGLS) estimate. The latter case also corresponds to the local Whittle QMLE of β. To see this, consider the local frequency domain Whittle QML objective function for (1),

$$W(\beta, G_{pp}) = \frac{1}{m}\sum_{j=1}^{m}\left(\log f_{pp}(\lambda_j) + \frac{I_{pp}(\lambda_j)}{f_{pp}(\lambda_j)}\right). \tag{5}$$

Concentrate G_{pp} out of the likelihood by setting $\hat{G}_{pp}(\beta) = m^{-1}\sum_{j=1}^{m}\lambda_j^{2d_p} I_{pp}(\lambda_j)$, then the concentrated likelihood is $W_c(\beta) = \log \hat{G}_{pp}(\beta)$ apart from constant terms. The derivative, using that $I_{pp}(\lambda_j) = I_{yy}(\lambda_j) - \mathrm{Re}(\beta' I_{xy}(\lambda_j) + I_{yx}(\lambda_j)\beta - \beta' I_{xx}(\lambda_j)\beta)$, is

$$W_c'(\beta) = 2\hat{G}_{pp}(\beta)^{-1}\frac{1}{m}\sum_{j=1}^{m}\lambda_j^{2d_p}\,\mathrm{Re}\left(I_{xx}(\lambda_j)\beta - I_{xy}(\lambda_j)\right),$$

and setting this equal to zero produces (3) with $\delta = d_p$.

In the next section, we shall give the conditions necessary to prove central limit theorems of the type

$$\sqrt{m}\lambda_m^{d_p}\Lambda_m^{-1}\left(\hat{\beta}_{\delta,m}-\beta\right)\rightarrow_d N\left(0,E^{-1}FE^{-1}\right), \tag{6}$$

where $\Lambda_m = \text{diag}(\lambda_m^{d_1},...,\lambda_m^{d_{p-1}})$ and E, F will be defined later. As mentioned above, the fully parametric version of this class of estimates has been examined by Robinson & Hidalgo (1997), who derived a parametric version of (6) in the case of long range dependent regressors and errors.

For the case with long range dependent errors and fixed (nonstochastic) regressors, Yajima (1988, 1991) derived central limit theorems for the ordinary least squares and generalized least squares estimates under conditions on the cumulants of all orders, and gave conditions for the ordinary least squares estimate to achieve the efficiency of the generalized least squares estimate. Dahlhaus (1995) considered an efficient weighted least squares estimate, and proved asymptotic normality under Gaussianity of the errors. Robinson (1997) gave a central limit theorem for nonparametric regression with fixed regressors assuming that the errors are linear in martingale differences. For a detailed discussion of the fixed regressor case, see Robinson & Hidalgo (1997) and the references therein.

Our emphasis on stochastic long range dependent regressors reflects recent empirical research. Thus, we also cover the case of cointegration where, if $d_p < \min_{1\leq a\leq p-1} d_a$, y_t and x_t are termed (fractionally) cointegrated. Cointegration is essentially the necessary condition to avoid spurious regression effects when data is trended, i.e. when d_a is high, see Phillips (1986) and Tsay & Chung (2000). Since we impose only the condition $d_a \in [0, 1/2)$, for all a, on the memory parameters, our framework provides a unified treatment of cointegration and regression with fractionally integrated regressors and errors.

The paper proceeds as follows. In the next section we present the central limit theorem for (3), and discuss its implications for the FDLS and FDGLS estimates. Section 3 discusses feasible versions of these estimates, and it is shown that the central limit theorem continues to hold for the feasible estimates. Section 4 reports the results of a Monte Carlo investigation of our estimates. The proofs of our theorems appear in sections 5 and 6, and section 7 contains some auxiliary lemmas and propositions.

2 Asymptotic Distribution of Estimates

We shall need the following assumptions on w_t and the spectral density matrix $f_w(\lambda)$ (with obvious implications for y_t).

Assumption 1 *The spectral density matrix of w_t in (2) with typical element $f_{ab}(\lambda)$, the cross spectral density between w_{at} and w_{bt}, satisfies*

$$\left| f_{ab}(\lambda) - G_{ab}\lambda^{-d_a-d_b} \right| = O\left(\lambda^{\alpha-d_a-d_b}\right) \text{ as } \lambda \to 0^+, \; a, b = 1, ..., p,$$

for some $\alpha \in (0, 2]$ and $0 \leq d_a < 1/2$, $a = 1, ..., p$. The matrix G satisfies $G_{ap} = G_{pa} = 0$ for $a = 1, ..., p-1$, and the leading $(p-1) \times (p-1)$ submatrix of G, denoted G_x, is positive definite.

Assumption 2 *w_t is a linear process, $w_t = \mu + \sum_{j=0}^{\infty} A_j \varepsilon_{t-j}$, where the coefficient matrices are square summable, $\sum_{j=0}^{\infty} \|A_j\|^2 < \infty$. The innovations satisfy, almost surely, $E(\varepsilon_t | \mathcal{F}_{t-1}) = 0$, $E(\varepsilon_t \varepsilon_t' | \mathcal{F}_{t-1}) = I_p$, and the matrices $\mu_3 = E(\varepsilon_t \otimes \varepsilon_t \varepsilon_t' | \mathcal{F}_{t-1})$ and $\mu_4 = E(\varepsilon_t \varepsilon_t' \otimes \varepsilon_t \varepsilon_t' | \mathcal{F}_{t-1})$ are nonstochastic, finite, and do not depend on t, with $\mathcal{F}_t = \sigma(\{\varepsilon_s, s \leq t\})$.*

Assumption 3 *As $\lambda \to 0^+$*

$$\frac{dA_a(\lambda)}{d\lambda} = O\left(\lambda^{-1} \|A_a(\lambda)\|\right), \; a = 1, ..., p,$$

where $A_a(\lambda)$ is the a'th row of $A(\lambda) = \sum_{j=0}^{\infty} A_j e^{ij\lambda}$.

We also need a restriction on the expansion rate of the bandwidth parameter m.

Assumption 4 *The bandwidth parameter $m = m(n)$ satisfies, as $n \to \infty$,*

$$\frac{1}{m} + \frac{m^{1+2\alpha}}{n^{2\alpha}} \to 0.$$

Finally, we need to restrict the weighting parameter depending on the memory parameters as follows.

Assumption 5 *The weighting parameter δ satisfies*

$$\max_{1 \leq a \leq p-1} (2d_a + 2d_p - 1)/4 \leq \delta \leq d_p.$$

Our assumptions are a multivariate generalization of those in Robinson (1994a, 1995a), see also Lobato (1997, 1999). They are in some respects much weaker than those employed by Robinson & Hidalgo (1997) in their parametric setup. In particular, we avoid their assumptions of independence between x_t and u_t and complete specification of $f(\lambda)$.

Assumptions 1 and 3 specialize (2) by imposing smoothness conditions on the spectral density matrix of w_t commonly employed in the literature. They are satisfied with $\alpha = 2$ if, e.g., w_t is a vector fractional Gaussian noise or a vector fractional ARIMA process. The condition that G_x must be positive definite is a no multicollinearity condition for the components of x_t. The extra condition that $G_{ap} = G_{pa} = 0$ for $a = 1, ..., p-1$ ensures that the coherence at $\lambda = 0$ between the regressors and the error process is of smaller order, and can be thought of as a local version of the usual orthogonality condition from least squares theory. In particular, it relaxes the independence assumption employed by Robinson & Hidalgo (1997). Assumption 2 is a straightforward multivariate generalization of the corresponding condition in Robinson (1995a), following Lobato (1999), and imposes a linear structure on w_t with square summable coefficients and martingale difference innovations with finite fourth moments. It is satisfied, for instance, if ε_t form an *i.i.d.* process with finite fourth moments. Under Assumption 2 we can write the spectral density matrix of w_t as

$$f(\lambda) = \frac{1}{2\pi} A(\lambda) A^*(\lambda). \tag{7}$$

Assumption 4 restricts the expansion rate of the bandwidth parameter $m = m(n)$. The bandwidth is required to tend to infinity for consistency, but at a slower rate than n to remain in a neighborhood of the origin, where some knowledge of the form of the spectral density is assumed. The maximal rate depends on the adequacy of the approximation (2) to (7), i.e. on the parameter α from Assumption 1, and the weakest constraint is implied by $\alpha = 2$ in which case the condition is $m = o(n^{4/5})$.

Finally, Assumption 5 states the required restrictions on the weighting parameter. Reversing Assumption 5 effectively gives a restriction on the memory parameters for the narrow band FDLS estimate (i.e. $\delta = 0$) to be covered by our theory. Thus, for $\max_{1 \leq a \leq p-1} d_a + d_p < 1/2$, the narrow band FDLS estimate satisfies Assumption 5.

We now state the following central limit theorem for $\hat{\beta}_{\delta,m}$, which is proved in section 5.

Theorem 1 *Under* (1) *and Assumptions 1-5, the estimator defined by* (3) *satisfies*

$$\sqrt{m}\lambda_m^{d_p}\Lambda_m^{-1}\left(\hat{\beta}_{\delta,m}-\beta\right)\to_d N\left(0,E^{-1}FE^{-1}\right) \tag{8}$$

with

$$E_{ab} = \frac{G_{ab}}{1-d_a-d_b+2\delta}, \tag{9}$$

$$F_{ab} = \frac{G_{ab}G_{pp}}{2\left(1-d_a-d_b-2d_p+4\delta\right)}. \tag{10}$$

If the memory parameters of x_t and u_t are all equal, i.e. $d_a = d$, $a = 1, ..., p$, inference is particularly simple since the memory parameter does not appear in the convergence rate and E, F are scalar multiples of G_x. We state this special case as a corollary.

Corollary 1 *Under* (1), *Assumptions 1-5, and* $d_a = d \in [0, 1/2)$, $a = 1, ..., p$, *the estimator defined by* (3) *satisfies*

$$\sqrt{m}\left(\hat{\beta}_{\delta,m}-\beta\right)\to_d N\left(0,\frac{\left(1-2d+2\delta\right)^2}{2\left(1-4d+4\delta\right)}G_{pp}G_x^{-1}\right).$$

Let us focus briefly on the case of scalar x_t. Suppose $f_w(\lambda) = \mathrm{diag}\left(f_x(\lambda), f_u(\lambda)\right)$ with $f_x(\lambda) \sim G_x\lambda^{-2d_x}$ and $f_u(\lambda) \sim G_u\lambda^{-2d_u}$. When $d_x + d_u < 1/2$, the FDLS estimate satisfies

$$\sqrt{m}\lambda_m^{d_u-d_x}\left(\hat{\beta}_{0,m}-\beta\right)\to_d N\left(0,\frac{G_u}{G_x}\frac{\left(1/2-d_x\right)^2}{1/2-d_x-d_u}\right). \tag{11}$$

However, the FDGLS estimate satisfies

$$\sqrt{m}\lambda_m^{d_u-d_x}\left(\hat{\beta}_{d_u,m}-\beta\right)\to_d N\left(0,\frac{G_u}{G_x}\left(1/2-d_x+d_u\right)\right) \tag{12}$$

for the entire stationary region of d_x and d_u, unlike the FDLS estimate. Furthermore, the asymptotic relative efficiency of $\hat{\beta}_{d_u,m}$ with respect to $\hat{\beta}_{0,m}$ (when both are asymptotically normal) is

$$\frac{V(\hat{\beta}_{0,m})}{V(\hat{\beta}_{d_u,m})} = \frac{\left(1/2-d_x\right)^2}{\left(1/2-d_x\right)^2-d_u^2},$$

which equals unity if and only if $d_u = 0$, and exceeds unity otherwise. Hence, as expected, the FDGLS estimate is more efficient and applies for a wider range of (d_x, d_u) than the FDLS estimate.

We end this section by remarking that the location of the spectral pole at the origin is not critical as long as the location is known. If instead the pole was located at $\lambda = \bar{\lambda} \neq 0$, we assume (2) as $\lambda \to \bar{\lambda}$ and use periodogram ordinates close to $\bar{\lambda}$ in the summations in (3). However, the case with a pole at the origin dominates both theoretical and empirical research, so we shall not consider this extension further.

3 Feasible Estimates

For the FDGLS estimate the correct δ is usually not known a priori, and hence this estimate is infeasible in practice. However, δ can obviously be estimated in any given situation by $\hat{\delta} = \hat{d}_p$, where $\hat{d}_p$ is an estimate of d_p based on residuals $\hat{u}_t$ from (1). These residuals can be obtained by e.g. FDLS, which does not require any knowledge of the memory parameters. Although the FDLS estimate is not asymptotically normal for all d, it is consistent, see Robinson (1994*a*) and Lobato (1997), and is thus useful as a preliminary estimate. We assume the following for $\hat{d}_p$.

Assumption 6 *The estimate of d_p satisfies, as $n \to \infty$,*

$$(\log n)\left(\hat{d}_p - d_p\right) \to_p 0.$$

In practice, the estimate can be obtained from residuals $\hat{u}_t$ as mentioned above. Hassler, Marmol & Velasco (2000) and Velasco (2001) provide some evidence that the log-periodogram and Gaussian semiparametric procedures of Robinson (1995*a*, 1995*b*), with carefully chosen bandwidth parameters, satisfy Assumption 6 with $\hat{d}_p - d_p = O_p(m^{-1/2})$.

Denote the feasible estimate $\hat{\beta}_{\hat{d}_p,m}$. The asymptotic distribution is given by the following theorem which is proved in section 6.

Theorem 2 *Under* (1) *and Assumptions 1-4 and 6 the results of Theorem 1 hold with δ replaced by $\hat{d}_p$.*

Thus, under the additional Assumption 6, the initial estimation of the memory parameter of the error process does not influence the asymptotic distribution theory for the regression coefficients obtained in the previous section.

4 Finite Sample Performance

We proceed to investigate the finite sample properties of the FDGLS (henceforth GLS) and feasible FDGLS (henceforth FGLS) estimates in a Monte Carlo study with two different generating mechanisms for x_t and u_t. In particular, we generated $10,000$ replications of x_t and u_t of length $n = 256$, 512, and $1,024$. Both were Gaussian fractional ARIMAs with spectra given by the two models

$$\begin{aligned} \textit{Model A} \quad &: \quad f_a(\lambda) = \frac{1}{2\pi}\left|1 - e^{i\lambda}\right|^{-2d_a}, \quad a = x, u, \\ \textit{Model B} \quad &: \quad f_a(\lambda) = \frac{1}{2\pi}\left|\frac{1 + 0.4e^{i\lambda}}{1 - 0.6e^{i\lambda}}\right|^2 \left|1 - e^{i\lambda}\right|^{-2d_a}, \quad a = x, u, \end{aligned}$$

for the grid of values $d_x = 0.0(0.1)0.4$ and $d_u = 0.0(0.1)d_x$, i.e. $d_u \leq d_x$ to avoid any spurious regression effects, see Phillips (1986) and Tsay & Chung (2000). These models both satisfy Assumptions 1-3 with $\alpha = 2$. From the linear model (1), we then generated y_t by $\alpha = 0$ and $\beta = 1$; the results are not sensitive to the choice of α and β. All calculations were performed in Ox version 3.10, see Doornik (2001) and Doornik & Ooms (2001).

In each model we computed $\hat{\beta}_{d_u,m}$ and $\hat{\beta}_{\hat{d}_u,m}$ with bandwidth parameters $m = \left[n^{0.4}\right]$ and $m = \left[n^{0.5}\right]$, where $[z]$ denotes the integer part of z. The first bandwidth is more conservative, and is expected to be more robust under more complicated generating mechanisms such as Model B. The FGLS estimate was computed by first obtaining the residual process $\hat{u}_t$ from FDLS estimation of β, and then estimating d_u by the Gaussian semiparametric estimator of Robinson (1995a) using the same bandwidth parameter for the entire estimation procedure.

Tables 1-4 about here

In Tables 1-4 we present the results of the simulation study for Model A. Tables 1 and 2 display the Monte Carlo bias of the GLS and FGLS estimates, respectively. The bias is universally lower than 0.008 in absolute value, and there are no clear trends. Tables 3 and 4 display the ratio (henceforth MSE ratio) of the asymptotic variance of $\hat{\beta}_{d_u,m}$ (from Theorem 1) to the simulated mean-squared errors of the GLS and FGLS estimates. In both tables the estimates with the higher bandwidth parameter are superior, their MSE ratios being closer to unity and in some cases up to 20% higher than those with the lower bandwidth parameter. Comparing the results of Tables 3 and 4, the mean-squared errors of the FGLS estimates are in most cases approximately 5% higher than those of the GLS estimates, the difference of course being due to the estimation of d_u. Furthermore, we note a clear monotonicity in the MSE ratios for both estimates. Thus, the ratios tend to be decreasing when $d_x - d_u$ increases. When $d_x = d_u$, i.e. on the diagonals, the asymptotic theory performs very well with MSE ratios around 0.9 for the GLS estimate and 0.85 for the FGLS estimate. The MSE ratios for the fully parametric estimates in Robinson & Hidalgo (1997) display similar magnitudes and patterns across d_x and d_u (c and d, respectively, in their notation).

Tables 5-8 about here

Tables 5-8 present the corresponding simulation results for Model B. Again the bias is negligible, and the pattern of MSE ratios from Tables 3 and 4 is repeated. Naturally, the MSE ratios tend to be lower under this more complicated generating mechanism, but only slightly so. Robinson & Hidalgo (1997) considered only Model A as generating mechanism, but do conjecture that their MSE ratios 'could deteriorate if a richer model of $f(\lambda)$ were estimated.'

Unreported simulations have shown that the highest possible expansion rate for the bandwidth under Assumption 4, $m = \left[n^{0.8}\right]$, generally results in an MSE ratio smaller than 0.6 for the GLS estimate for Model B, and thus appears too high for the sample sizes considered here.

Overall, the asymptotic theory seems to perform well, and the results of the Monte Carlo study are very similar to those obtained by Robinson & Hidalgo (1997) for their fully parametric estimates. However, in contrast to the estimates of Robinson & Hidalgo (1997), ours can be obtained without any prior knowledge of the generating mechanism of x_t and u_t. In particular, we do not need to know if x_t and u_t are generated by Model A or Model B in order to calculate our semiparametric estimates. The simulated bias is negligible in all our specifications and the MSE ratio is high when d_x and d_u are not too far apart. However, when d_x is much larger than d_u, the asymptotic variance is quite small compared to the Monte Carlo result, and consequently asymptotic confidence intervals tend to be too narrow.

5 Proof of Theorem 1

We prove Theorem 1 using the auxiliary results in section 7. The basic technique is the martingale difference approximation method of Robinson (1995*a*). The left-hand side of (8) is

$$\left(\Lambda_m \lambda_m^{-2\delta} \frac{1}{m} \sum_{j=1}^{m} \lambda_j^{2\delta} \operatorname{Re}\left(I_{xx}\left(\lambda_j\right)\right) \Lambda_m\right)^{-1} \Lambda_m \lambda_m^{d_p-2\delta} \frac{1}{\sqrt{m}} \sum_{j=1}^{m} \lambda_j^{2\delta} \operatorname{Re}\left(I_{xp}\left(\lambda_j\right)\right).$$

From Proposition 1 of section 7, the first term on the right-hand side satisfies

$$\Lambda_m \lambda_m^{-2\delta} \frac{1}{m} \sum_{j=1}^{m} \lambda_j^{2\delta} \operatorname{Re}\left(I_{xx}\left(\lambda_j\right)\right) \Lambda_m \to_p E,$$

where E is defined in (9). Note that G_x (and thus E) is invertible by Assumption 1.

For the second term we show that

$$\frac{1}{\sqrt{m}} \lambda_m^{d_p-2\delta} \Lambda_m \sum_{j=1}^{m} \lambda_j^{2\delta} \operatorname{Re}\left(I_{xp}\left(\lambda_j\right)\right) \to_d N\left(0, F\right).$$

By application of the Cramèr-Wold device, we need to examine (η is a $(p-1)$-vector)

$$\begin{aligned}
&\sum_{a=1}^{p-1} \eta_a \frac{1}{\sqrt{m}} \lambda_m^{d_a+d_p-2\delta} \sum_{j=1}^{m} \lambda_j^{2\delta} \operatorname{Re}\left(I_{ap}\left(\lambda_j\right)\right) \\
= &\sum_{a=1}^{p-1} \eta_a \frac{1}{\sqrt{m}} \lambda_m^{d_a+d_p-2\delta} \sum_{j=1}^{m} \lambda_j^{2\delta} \operatorname{Re}\left(I_{ap}\left(\lambda_j\right) - A_a\left(\lambda_j\right) J\left(\lambda_j\right) A_p^*\left(\lambda_j\right)\right) \qquad (13)\\
&+\sum_{a=1}^{p-1} \eta_a \frac{1}{\sqrt{m}} \lambda_m^{d_a+d_p-2\delta} \sum_{j=1}^{m} \lambda_j^{2\delta} \operatorname{Re}\left(A_a\left(\lambda_j\right) \frac{1}{2\pi n} \sum_{t=1}^{n} \varepsilon_t \varepsilon_t' A_p^*\left(\lambda_j\right)\right) \qquad (14)\\
&+\sum_{a=1}^{p-1} \eta_a \frac{1}{\sqrt{m}} \lambda_m^{d_a+d_p-2\delta} \sum_{j=1}^{m} \lambda_j^{2\delta} \operatorname{Re}\left(A_a\left(\lambda_j\right) \frac{1}{2\pi n} \sum_{t=1}^{n} \sum_{s\neq t} \varepsilon_t \varepsilon_s' e^{i(t-s)\lambda_j} A_p^*\left(\lambda_j\right)\right), \qquad (15)
\end{aligned}$$

where $J\left(\lambda\right)$ is the periodogram matrix of the innovations ε_t. Lemma 2 of section 7 proves that (13) is $o_p\left(1\right)$, while Lemma 3 in conjunction with Assumptions 1 and 4 proves that (14) is $o_p\left(1\right)$ since $m^{-1/2} \lambda_m^{d_a+d_p-2\delta} \sum_{j=1}^{m} \lambda_j^{2\delta} \operatorname{Re}\left(f_{ap}\left(\lambda_j\right)\right) = O\left(m^{1+2\alpha} n^{-2\alpha}\right)$.

We are left with (15), which can be written as $\sum_{t=1}^{n} z_{tn}$, where

$$
\begin{aligned}
z_{tn} &= \varepsilon_t' \sum_{s=1}^{t-1} c_{t-s,n} \varepsilon_s, \\
c_{tn} &= \frac{1}{2\pi n \sqrt{m}} \sum_{j=1}^{m} \theta_j \cos(t\lambda_j), \\
\theta_j &= \sum_{a=1}^{p-1} \eta_a \lambda_m^{d_a+d_p-2\delta} \lambda_j^{2\delta} \operatorname{Re}\left(A_a'(\lambda_j) \bar{A}_p(\lambda_j) + A_p'(\lambda_j) \bar{A}_a(\lambda_j)\right).
\end{aligned}
$$

Since z_{tn} is a martingale difference array with respect to the filtration $(\mathcal{F}_t)_{t\in\mathbb{Z}}$, $\mathcal{F}_t = \sigma(\{\varepsilon_s, s \le t\})$, we can apply the CLT of Brown (1971) and Hall & Heyde (1980, chp. 3.2) if

$$\sum_{t=1}^{n} E\left(z_{tn}^2 \middle| \mathcal{F}_{t-1}\right) - \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \eta_a \eta_b F_{ab} \to_p 0, \tag{16}$$

$$\sum_{t=1}^{n} E\left(z_{tn}^2 1(|z_{tn}| > \kappa)\right) \to 0, \ \kappa > 0. \tag{17}$$

A sufficient condition for (17) is

$$\sum_{t=1}^{n} E\left(z_{tn}^4\right) \to 0. \tag{18}$$

First we show (16). The first term on the left-hand side is

$$\sum_{t=1}^{n} E\left(\sum_{s=1}^{t-1}\sum_{r=1}^{t-1} \varepsilon_s' c_{t-s,n}' \varepsilon_t \varepsilon_t' c_{t-r,n} \varepsilon_r \middle| \mathcal{F}_{t-1}\right) = \sum_{t=1}^{n}\sum_{s=1}^{t-1} \varepsilon_s' c_{t-s,n}' c_{t-s,n} \varepsilon_s + o_p(1) \tag{19}$$

by Lemma 4. We need to show that the mean of the first term on the right-hand side of (19) is asymptotically equal to $\sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \eta_a \eta_b F_{ab}$. Thus,

$$
\begin{aligned}
& \sum_{t=1}^{n}\sum_{s=1}^{t-1} E \operatorname{tr}\left(c_{t-s,n}' c_{t-s,n} \varepsilon_s \varepsilon_s'\right) \\
=& \sum_{t=1}^{n}\sum_{s=1}^{t-1} \operatorname{tr}\left(c_{t-s,n}' c_{t-s,n}\right) \\
=& \sum_{t=1}^{n}\sum_{s=1}^{t-1}\sum_{j=1}^{m} \frac{1}{4\pi^2 n^2 m} \operatorname{tr}\left(\theta_j'\theta_j\right) \cos^2((t-s)\lambda_j) && (20) \\
& + \sum_{t=1}^{n}\sum_{s=1}^{t-1}\sum_{j=1}^{m}\sum_{k\neq j} \frac{1}{4\pi^2 n^2 m} \operatorname{tr}\left(\theta_j'\theta_k\right) \cos((t-s)\lambda_j) \cos((t-s)\lambda_k). && (21)
\end{aligned}
$$

Notice that, since $\|\theta_j\| = O(1)$ by Theorem 2 of Robinson (1995*b*), we have

$$(21) = O\left(\sum_{t=1}^{n}\sum_{s=1}^{t-1}\sum_{j=1}^{m}\sum_{k\neq j}\frac{1}{n^2 m}\cos\left(\left(t-s\right)\lambda_j\right)\cos\left(\left(t-s\right)\lambda_k\right)\right)$$

and, using that $\sum_{t=1}^{n}\sum_{s=1}^{t-1}\cos\left(\left(t-s\right)\lambda_j\right)\cos\left(\left(t-s\right)\lambda_k\right) = -n/2$, (21) is $O\left(\sum_{j=1}^{m}\sum_{k\neq j}\left(n^2 m\right)^{-1} n\right) = O\left(m/n\right)$. Now, $\operatorname{tr}\left(\theta_j'\theta_j\right)$ equals $\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\eta_a\eta_b\lambda_m^{d_a+d_b+2d_p-4\delta}\lambda_j^{4\delta}$ times

$$\begin{aligned}
& \operatorname{tr}\left(\operatorname{Re}\left(A_p^*\left(\lambda_j\right)A_a\left(\lambda_j\right)+A_a^*\left(\lambda_j\right)A_p\left(\lambda_j\right)\right)\operatorname{Re}\left(A_b'\left(\lambda_j\right)\bar{A}_p\left(\lambda_j\right)+A_p'\left(\lambda_j\right)\bar{A}_b\left(\lambda_j\right)\right)\right)\\
= \; & 4\pi^2\left(f_{ab}\left(\lambda_j\right)f_{pp}\left(\lambda_j\right)+f_{ap}\left(\lambda_j\right)f_{bp}\left(\lambda_j\right)+f_{pb}\left(\lambda_j\right)f_{pa}\left(\lambda_j\right)+f_{pp}\left(\lambda_j\right)f_{ba}\left(\lambda_j\right)\right)
\end{aligned}$$

by definition of $f(\lambda)$, see (7). By Assumption 1 the second and third terms are of smaller order, and since $(x+\bar{x}) = 2\operatorname{Re}(x)$ for any complex number x, we can thus rewrite (20) as

$$\sum_{t=1}^{n}\sum_{s=1}^{t-1}\sum_{j=1}^{m}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\frac{2\eta_a\eta_b}{n^2 m}\lambda_m^{d_a+d_b+2d_p-4\delta}\lambda_j^{4\delta}\operatorname{Re}\left(f_{ab}\left(\lambda_j\right)f_{pp}\left(\lambda_j\right)\right)\cos^2\left(\left(t-s\right)\lambda_j\right). \tag{22}$$

Using Lemma 1 to approximate the sum $\sum_{j=1}^{m}$ by an integral, and since $\sum_{t=1}^{n-1}\sum_{s=1}^{n-t}\cos^2\left(s\lambda_j\right) = \left(n-1\right)^2/4$, we have that (22) is

$$\begin{aligned}
& \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\frac{\eta_a\eta_b}{n\pi m}\lambda_m^{d_a+d_b+2d_p-4\delta}\left(\sum_{t=1}^{n}\sum_{s=1}^{t-1}\cos^2\left(\left(t-s\right)\lambda_j\right)\right)\int_0^{\lambda_m}\lambda^{4\delta}\operatorname{Re}\left(f_{ab}\left(\lambda\right)f_{pp}\left(\lambda\right)\right)d\lambda\\
= \; & \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\eta_a\eta_b\frac{\lambda_m^{d_a+d_b+2d_p-4\delta}}{2}\int_0^{\lambda_m}\lambda^{4\delta}\operatorname{Re}\left(f_{ab}\left(\lambda\right)f_{pp}\left(\lambda\right)\right)d\lambda,
\end{aligned}$$

and we have shown (16).

Hence, we have to show (18),

$$\begin{aligned}
\sum_{t=1}^{n}E\left(z_{tn}^4\right) &= \sum_{t=1}^{n}E\left(\sum_{s=1}^{t-1}\varepsilon_s'c_{t-s,n}\varepsilon_t\varepsilon_t'\sum_{r=1}^{t-1}c_{t-r,n}\varepsilon_r\sum_{p=1}^{t-1}\varepsilon_p'c_{t-p,n}\varepsilon_t\varepsilon_t'\sum_{q=1}^{t-1}c_{t-q,n}\varepsilon_q\right)\\
&\leq C\left(\sum_{t=1}^{n}\operatorname{tr}\left(\sum_{s=1}^{t-1}c_{t-s,n}'c_{t-s,n}c_{t-s,n}'c_{t-s,n}\right)+\sum_{t=1}^{n}\operatorname{tr}\left(\sum_{s=1}^{t-1}c_{t-s,n}'\sum_{r=1}^{t-1}c_{t-r,n}c_{t-r,n}'c_{t-s,n}\right)\right)
\end{aligned}$$

for some constant $C > 0$ by Assumption 2. Using the arguments in Lemma 4, this expression can be bounded by $O\left(n\left(\sum_{t=1}^{n}\left\|c_{tn}^2\right\|\right)^2\right) = O\left(n^{-1}\right)$, which completes the proof.

6 Proof of Theorem 2

We show that

$$\sqrt{m}\lambda_m^{d_p}\Lambda_m^{-1}\left(\left(\hat{\beta}_{\hat{d}_p,m}-\beta\right)-\left(\hat{\beta}_{d_p,m}-\beta\right)\right)\rightarrow_p 0.$$

By definition of $\hat{\beta}_{\hat{d}_p,m}$ and $\hat{\beta}_{d_p,m}$, this amounts to showing that

$$\lambda_m^{-2d_p}\Lambda_m\left(\frac{1}{m}\sum_{j=1}^m\left(\lambda_j^{2\hat{d}_p}-\lambda_j^{2d_p}\right)\operatorname{Re}\left(I_{xx}\left(\lambda_j\right)\right)\right)\Lambda_m\rightarrow_p 0, \tag{23}$$

$$\sqrt{m}\lambda_m^{-d_p}\Lambda_m\left(\frac{1}{m}\sum_{j=1}^m\left(\lambda_j^{2\hat{d}_p}-\lambda_j^{2d_p}\right)\operatorname{Re}\left(I_{xp}\left(\lambda_j\right)\right)\right)\rightarrow_p 0. \tag{24}$$

Since $\left|\max_{1\le j\le m}\lambda_j^{2\hat{d}_p-2d_p}-1\right|=O_p\left(\left|\hat{d}_p-d_p\right|\log n\right)$, we have

$$\begin{aligned}
&\lambda_m^{d_a+d_b-2d_p}\frac{1}{m}\sum_{j=1}^m\left(\lambda_j^{2\hat{d}_p}-\lambda_j^{2d_p}\right)\operatorname{Re}\left(I_{ab}\left(\lambda_j\right)\right)\\
=\;& O_p\left(\lambda_m^{d_a+d_b-2d_p}\frac{1}{m}\left|\max_{1\le j\le m}\lambda_j^{2\hat{d}_p-2d_p}-1\right|\sum_{j=1}^m\lambda_j^{2d_p}\operatorname{Re}\left(I_{ab}\left(\lambda_j\right)\right)\right)\\
=\;& O_p\left(\lambda_m^{d_a+d_b-2d_p}\frac{1}{m}\left|\hat{d}_p-d_p\right|(\log n)\sum_{j=1}^m\lambda_j^{2d_p-d_a-d_b}\right)\\
=\;& O_p\left(\left|\hat{d}_p-d_p\right|\log n\right)
\end{aligned}$$

and

$$\begin{aligned}
&\sqrt{m}\lambda_m^{d_a-d_p}\frac{1}{m}\sum_{j=1}^m\left(\lambda_j^{2\hat{d}_p}-\lambda_j^{2d_p}\right)\operatorname{Re}\left(I_{ap}\left(\lambda_j\right)\right)\\
=\;& O_p\left(\sqrt{m}\lambda_m^{d_a-d_p}\frac{1}{m}\left(\left|\hat{d}_p-d_p\right|\log n\right)\sum_{j=1}^m\lambda_j^{\alpha+d_p-d_a}\right)\\
=\;& o_p\left(\sqrt{m}\lambda_m^{\alpha}\left|\hat{d}_p-d_p\right|\log n\right)
\end{aligned}$$

by Assumption 1. In view of Assumptions 4 and 6, this proves (23) and (24).

7 Auxiliary Propositions and Lemmas

Here we provide a series of auxiliary results used to prove our main theorems. First, we provide an extension of the consistency result of Lobato (1997, Theorem 1) for the discretely averaged cross-periodogram, showing that the result is equally valid for our weighted cross-periodogram.

Proposition 1 *Under Assumptions 1, 2, 5, and* $m^{-1}+m/n\rightarrow 0$,

$$\lambda_m^{d_a+d_b-2\delta}\frac{1}{m}\sum_{j=1}^{m}\lambda_j^{2\delta}\operatorname{Re}\left(I_{ab}\left(\lambda_j\right)\right)-\frac{G_{ab}}{1-d_a-d_b+2\delta}\rightarrow_p 0,\quad a,b=1,...,p. \tag{25}$$

Proof. Decompose the left-hand side of (25) as

$$\lambda_m^{d_a+d_b-2\delta}\frac{1}{m}\sum_{j=1}^{m}\lambda_j^{2\delta}\operatorname{Re}\left(I_{ab}\left(\lambda_j\right)-A_a\left(\lambda_j\right)J\left(\lambda_j\right)A_b^*\left(\lambda_j\right)\right) \tag{26}$$

$$+\lambda_m^{d_a+d_b-2\delta}\frac{1}{m}\sum_{j=1}^{m}\lambda_j^{2\delta}\operatorname{Re}\left(A_a\left(\lambda_j\right)\frac{1}{2\pi n}\sum_{t=1}^{n}\varepsilon_t\varepsilon_t'A_b^*\left(\lambda_j\right)-f_{ab}\left(\lambda_j\right)\right) \tag{27}$$

$$+\lambda_m^{d_a+d_b-2\delta}\frac{1}{m}\sum_{j=1}^{m}\lambda_j^{2\delta}\operatorname{Re}\left(A_a\left(\lambda_j\right)\frac{1}{2\pi n}\sum_{t=1}^{n}\sum_{s\neq t}\varepsilon_t\varepsilon_s'e^{i(t-s)\lambda_j}A_b^*\left(\lambda_j\right)\right) \tag{28}$$

$$+\lambda_m^{d_a+d_b-2\delta}\frac{1}{m}\sum_{j=1}^{m}\lambda_j^{2\delta}\operatorname{Re}\left(f_{ab}\left(\lambda_j\right)\right)-\frac{G_{ab}}{1-d_a-d_b+2\delta}. \tag{29}$$

By Lemmas 2 and 3 and the analysis of (15) in the proof of Theorem 1, (26) – (28) are all $o_p(1)$. Applying Lemma 1 to (29) we get that

$$\lambda_m^{d_a+d_b-2\delta}\frac{1}{m}\sum_{j=1}^{m}\lambda_j^{2\delta}G_{ab}\lambda_j^{-d_a-d_b}-\frac{G_{ab}}{1-d_a-d_b+2\delta}=o\left(\lambda_m\right),$$

thus completing the proof. ■

The first lemma is undoubtedly well known and is provided for reference.

Lemma 1 *For* $m^{-1}+m/n\rightarrow 0$ *and any* $c\in(-1,1]$,

$$\frac{2\pi}{n}\sum_{j=1}^{m}\lambda_j^{c}-\int_0^{\lambda_m}\lambda^c d\lambda=o\left(\lambda_m^{c+1}\right)$$

as $n\rightarrow\infty$.

Proof. For n sufficiently large, the left-hand side is

$$\sum_{j=2}^{m}\int_{\lambda_{j-1}}^{\lambda_j}\left(\lambda_j^c-\lambda^c\right)d\lambda=\sum_{j=2}^{m}\lambda_j^{c-1}\int_{\lambda_{j-1}}^{\lambda_j}\left(\lambda_j-\left(\frac{\lambda}{\lambda_j}\right)^{c-1}\lambda\right)d\lambda+o(\lambda_m^{c+1}).$$

As $\left|\lambda_j-(\lambda/\lambda_j)^a\,\lambda\right|\le|\lambda_j-\lambda|$ for $\lambda\in(\lambda_{j-1},\lambda_j)$ and $a\le 0$, the first term on the right-hand side is

$$O\left(\sum_{j=1}^{m}\lambda_j^{c-1}\int_{\lambda_{j-1}}^{\lambda_j}\left|\lambda_j-\left(\frac{\lambda}{\lambda_j}\right)^{c-1}\lambda\right|d\lambda\right)=O\left(\left|\sum_{j=1}^{m}\lambda_j^{c-1}\int_{\lambda_{j-1}}^{\lambda_j}(\lambda_j-\lambda)\,d\lambda\right|\right)\qquad(30)$$

and, since $\int_{\lambda_{j-1}}^{\lambda_j}(\lambda_j-\lambda)\,d\lambda=\pi^2/2n^2$, it follows that (30) is $O\left(n^{-2}\sum_{j=1}^{m}\lambda_j^{c-1}\right)=O\left(m^{-1}\lambda_m^{c+1}\right)$.

■

The remaining lemmas are straightforward extensions (to incorporate our weights) and variants of previous results appearing in Robinson (1995*a*) and Lobato (1997, 1999).

Lemma 2 *Under the conditions of Proposition 1, for $a,b=1,...,p$,*

$$\lambda_m^{d_a+d_b-2\delta}\frac{1}{m}\sum_{j=1}^{m}\lambda_j^{2\delta}\,\mathrm{Re}\left(I_{ab}(\lambda_j)-A_a(\lambda_j)\,J(\lambda_j)\,A_b^*(\lambda_j)\right)=o_p(1),\qquad(31)$$

and under the conditions of Theorem 1, for $a=1,...,p-1$,

$$\lambda_m^{d_a+d_p-2\delta}\frac{1}{\sqrt{m}}\sum_{j=1}^{m}\lambda_j^{2\delta}\,\mathrm{Re}\left(I_{ap}(\lambda_j)-A_a(\lambda_j)\,J(\lambda_j)\,A_p^*(\lambda_j)\right)=o_p(1).\qquad(32)$$

Proof. By the same arguments as in Lobato (1997, pp. 143-144), we have that

$$\sum_{j=1}^{m}\lambda_j^{2\delta}\,\mathrm{Re}\left(I_{ab}(\lambda_j)-A_a(\lambda_j)\,J(\lambda_j)\,A_b^*(\lambda_j)\right)=o_p\left(n\lambda_m^{1-d_a-d_b+2\delta}\right)$$

such that (31) is $o_p(1)$.

Under the conditions of Theorem 1, we can use eq. (C.2) in Lobato (1999) to conclude that

$$\sum_{j=1}^{m}\lambda_j^{2\delta}\,\mathrm{Re}\left(I_{ap}(\lambda_j)-A_a(\lambda_j)\,J(\lambda_j)\,A_p^*(\lambda_j)\right)=O_p\left(\lambda_m^{2\delta-d_a-d_p}\left[m^{1/3}(\log m)^{2/3}+\log m+\frac{m^{1/2}}{n^{1/4}}\right]\right),$$

and thus (32) is $O_p\left(m^{-2/3}(\log m)^{2/3}+m^{-1/2}(\log m)+n^{-1/4}\right)=o_p(1)$. ■

Lemma 3 *Under the conditions of Proposition 1, for $a, b = 1, ..., p$,*

$$\frac{1}{\sqrt{m}}\lambda_m^{d_a+d_b-2\delta}\sum_{j=1}^{m}\lambda_j^{2\delta}\,\mathrm{Re}\left(A_a(\lambda_j)\frac{1}{2\pi n}\sum_{t=1}^{n}\varepsilon_t\varepsilon_t' A_b^*(\lambda_j) - f_{ab}(\lambda_j)\right) = o_p(1).$$

Proof. The proof follows parts of the proof of Lobato (1997, Proposition 3). By definition of $f(\lambda)$, the left-hand side is bounded by

$$\left|\frac{1}{\sqrt{m}}\lambda_m^{d_a+d_b-2\delta}\sum_{j=1}^{m}\lambda_j^{2\delta}A_a(\lambda_j)\frac{1}{2\pi}DA_b^*(\lambda_j)\right|, \tag{33}$$

where $D = n^{-1}\sum_{t=1}^{n}\varepsilon_t\varepsilon_t' - I_p$ satisfies $\|D\| = O_p(n^{-1/2})$, since by Assumption 2, $\varepsilon_t\varepsilon_t' - I_p$ is a martingale difference sequence with respect to the filtration $(\mathcal{F}_t)_{t\in\mathbb{Z}}$. Then, since $\|A_i(\lambda_j)\| = O(f_{ii}(\lambda_j)^{1/2})$, $i = a, b$, (33) is bounded by

$$\begin{aligned}
&\frac{1}{4\pi^2\sqrt{m}}\lambda_m^{d_a+d_b-2\delta}\left(\sum_{j=1}^{m}\lambda_j^{4\delta}\|A_a(\lambda_j)\|^2\|D\|^2\|A_b(\lambda_j)\|^2\right)^{1/2}\\
=\;& O_p\left(m^{-1/2}\lambda_m^{d_a+d_b-2\delta}\|D\|\left(\sum_{j=1}^{m}\lambda_j^{4\delta}f_{aa}(\lambda_j)f_{bb}(\lambda_j)\right)^{1/2}\right)\\
=\;& O_p\left(m^{-1/2}\lambda_m^{d_a+d_b-2\delta}\|D\|\left(\sum_{j=1}^{m}\lambda_j^{2\delta}f_{aa}(\lambda_j)\right)^{1/2}\left(\sum_{j=1}^{m}\lambda_j^{2\delta}f_{bb}(\lambda_j)\right)^{1/2}\right),
\end{aligned}$$

which is $O_p(\lambda_m^{1/2}) = o_p(1)$ as required. ■

Lemma 4 *Under the conditions of Theorem 1,*

$$\sum_{t=1}^{n}E\left(\sum_{s=1}^{t-1}\sum_{r=1}^{t-1}\varepsilon_s' c_{t-s,n}'\varepsilon_t\varepsilon_t' c_{t-r,n}\varepsilon_r\middle|\mathcal{F}_{t-1}\right) - \sum_{t=1}^{n}\sum_{s=1}^{t-1}\varepsilon_s' c_{t-s,n}' c_{t-s,n}\varepsilon_s = o_p(1).$$

Proof. We prove convergence in mean-square. The left-hand side is $\sum_{t=1}^{n}\sum_{s=1}^{t-1}\sum_{r\neq s}\varepsilon_s' c_{t-s,n}' c_{t-r,n}\varepsilon_r$, which has mean zero and variance

$$O\left(n\left(\sum_{s=1}^{n}\|c_{sn}\|^2\right)^2 + \sum_{t=3}^{n}\sum_{u=2}^{t-1}\left(\sum_{s=1}^{u-1}\|c_{u-s,n}\|^2\sum_{s=1}^{u-1}\|c_{t-s,n}\|^2\right)\right), \tag{34}$$

following the analysis in Robinson (1995a, p. 1646) and Lobato (1999, pp. 150-151). By Theorem 2 of Robinson (1995b), $\|\theta_j\| = O(1)$ such that $\|c_{sn}\|$ is bounded by

$$\|c_{sn}\| = O\left(\frac{1}{n\sqrt{m}}\sum_{j=1}^{m}\|\theta_j\|\right) = O\left(\frac{\sqrt{m}}{n}\right).$$

Since $\sum_{j=1}^{k}|\cos(s\lambda_j)| = O(n/s)$, another bound is

$$\|c_{sn}\| = O\left(\frac{1}{n\sqrt{m}}\sum_{j=1}^{m}|\cos(s\lambda_j)|\right) = O\left(\frac{1}{s\sqrt{m}}\right),$$

which is a better bound for $\|c_{sn}\|$ when $s > n/m$. Thus, we find that

$$\begin{aligned}\sum_{s=1}^{n}\|c_{sn}\|^2 &= O\left(\sum_{s=1}^{[n/m]}\frac{m}{n^2} + \sum_{s=[n/m]+1}^{n}\frac{1}{s^2 m}\right)\\ &= O\left(n^{-1}\right),\end{aligned}$$

implying that the first term of (34) is $O\left(n^{-1}\right)$. The second term of (34) is bounded by

$$O\left(n\left(\sum_{s=1}^{n}\|c_{sn}\|^2\right)\left(\sum_{s=1}^{[n/2]}s\|c_{sn}\|^2\right)\right),$$

see Robinson (1995a, pp. 1646-1647). The summand in the last sum is $O(sm/n^2 + (sm)^{-1})$. Choose the first bound when $s \leq \left[n/m^{2/3}\right]$, then the last sum is

$$O\left(\sum_{s=1}^{[n/m^{2/3}]}\frac{sm}{n^2} + \sum_{s=\left[n/m^{2/3}\right]+1}^{[n/2]}\frac{1}{sm}\right) = O\left(\frac{1}{m^{1/3}}\right),$$

and (34) $= O\left(n^{-1} + m^{-1/3}\right)$. ■

References

Adenstedt, R. K. (1974), 'On large-sample estimation of the mean of a stationary random sequence', *Annals of Statistics* **2**, 1095–1107.

Beran, J. (1994), *Statistics for Long-Memory Processes*, Chapman-Hall.

Brown, B. M. (1971), 'Martingale central limit theorems', *Annals of Mathematical Statistics* **42**, 59–66.

Dahlhaus, R. (1995), 'Efficient location and regression estimation for long range dependent regression models', *Annals of Statistics* **23**, 1029–1047.

Doornik, J. A. (2001), *Ox: An Object-Oriented Matrix Language*, 4th edn, Timberlake Consultants Press, London.

Doornik, J. A. & Ooms, M. (2001), 'A package for estimating, forecasting and simulating arfima models: Arfima package 1.01 for Ox', *Working Paper, Nuffield College, Oxford* .

Geweke, J. & Porter-Hudak, S. (1983), 'The estimation and application of long memory time series models', *Journal of Time Series Analysis* **4**, 221–238.

Granger, C. W. J. & Joyeux, R. (1980), 'An introduction to long memory time series models and fractional differencing', *Journal of Time Series Analysis* **1**, 15–39.

Hall, P. & Heyde, C. C. (1980), *Martingale Limit Theory and its Application*, Academic Press, New York.

Hannan, E. J. (1979), 'The central limit theorem for time series regression', *Stochastic Processes and their Applications* **9**, 281–289.

Hassler, U., Marmol, F. & Velasco, C. (2000), 'Residual log-periodogram inference for long-run relationships', *Preprint, Universidad Carlos III de Madrid* .

Hosking, J. R. M. (1981), 'Fractional differencing', *Biometrika* **68**, 165–176.

Lobato, I. N. (1997), 'Consistency of the averaged cross-periodogram in long memory series', *Journal of Time Series Analysis* **18**, 137–155.

Lobato, I. N. (1999), 'A semiparametric two-step estimator in a multivariate long memory model', *Journal of Econometrics* **90**, 129–153.

Lobato, I. N. & Robinson, P. M. (1996), 'Averaged periodogram estimation of long memory', *Journal of Econometrics* **73**, 303–324.

Mandelbrot, B. B. & Ness, J. W. V. (1968), 'Fractional brownian motions, fractional noises and applications', *SIAM Review* **10**, 422–437.

Phillips, P. C. B. (1986), 'Understanding spurious regressions in econometrics', *Journal of Econometrics* **33**, 311–340.

Robinson, P. M. (1994*a*), 'Semiparametric analysis of long-memory time series', *Annals of Statistics* **22**, 515–539.

Robinson, P. M. (1994*b*), Time series with strong dependence, *in* C. Sims, ed., 'Advances in Econometrics', Cambridge University Press, Cambridge, pp. 47–95.

Robinson, P. M. (1995*a*), 'Gaussian semiparametric estimation of long range dependence', *Annals of Statistics* **23**, 1630–1661.

Robinson, P. M. (1995*b*), 'Log-periodogram regression of time series with long range dependence', *Annals of Statistics* **23**, 1048–1072.

Robinson, P. M. (1997), 'Large-sample inference for nonparametric regression with dependent errors', *Annals of Statistics* **25**, 2054–2083.

Robinson, P. M. & Hidalgo, F. J. (1997), 'Time series regression with long-range dependence', *Annals of Statistics* **25**, 77–104.

Robinson, P. M. & Marinucci, D. (1998), 'Semiparametric frequency domain analysis of fractional cointegration', *LSE Sticerd Working Paper 348* .

Tsay, W. J. & Chung, C. F. (2000), 'The spurious regression of fractionally integrated processes', *Journal of Econometrics* **96**, 155–182.

Velasco, C. (2001), 'Gaussian semiparametric estimation of fractional cointegration', *Preprint, Universidad Carlos III de Madrid* .

Yajima, Y. (1988), 'On estimation of a regression model with long-memory stationary errors', *Annals of Statistics* **16**, 791–807.

Yajima, Y. (1991), 'Asymptotic properties of the LSE in a regression model with long-memory stationary errors', *Annals of Statistics* **19**, 158–177.

Table 1: Bias (x100) of GLS estimate for Model A.

					$n = 256$					
	$m = [n^{0.4}] = 9$					$m = [n^{0.5}] = 16$				
$d_u \backslash d_x$	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4
0	$-.7733$	$-.0060$	.1793	.0591	.0591	.2953	.0263	$-.1773$	.1337	.0020
0.1	$-$	.4049	.1212	$-.3895$	.0629	$-$	$-.0518$	.1541	$-.0450$	$-.0566$
0.2	$-$	$-$	.2961	.0829	.0920	$-$	$-$	.0927	.0666	$-.0435$
0.3	$-$	$-$	$-$	.1227	.1817	$-$	$-$	$-$	.0648	.1498
0.4	$-$	$-$	$-$	$-$	.0347	$-$	$-$	$-$	$-$	.0270
					$n = 512$					
	$m = [n^{0.4}] = 12$					$m = [n^{0.5}] = 22$				
$d_u \backslash d_x$	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4
0	$-.2024$	$-.1835$	$-.0131$	$-.0095$	.0542	$-.0900$	$-.0369$	$-.0850$	$-.0244$	$-.0257$
0.1	$-$	.0960	$-.0028$	$-.1694$	.1098	$-$	.0169	.0113	.0200	.1260
0.2	$-$	$-$	.3121	$-.0062$	.2058	$-$	$-$	.0622	.1402	$-.1765$
0.3	$-$	$-$	$-$	.1069	.0496	$-$	$-$	$-$	.0828	.0957
0.4	$-$	$-$	$-$	$-$	.1913	$-$	$-$	$-$	$-$	$-.1440$
					$n = 1024$					
	$m = [n^{0.4}] = 16$					$m = [n^{0.5}] = 32$				
$d_u \backslash d_x$	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4
0	$-.3978$	$-.0492$	$-.0090$	$-.0245$	.0597	$-.1952$	$-.1215$	$-.0572$	.0241	$-.0327$
0.1	$-$	$-.0727$	$-.0642$	$-.0984$	$-.0384$	$-$	$-.1049$	.0279	$-.0138$	.0683
0.2	$-$	$-$	.1535	$-.1370$	.0691	$-$	$-$	$-.1881$	$-.0478$	$-.0704$
0.3	$-$	$-$	$-$	.0975	.0183	$-$	$-$	$-$	$-.0736$	.0793
0.4	$-$	$-$	$-$	$-$	.0591	$-$	$-$	$-$	$-$	$-.1162$

Table 2: Bias (x100) of FGLS estimate for Model A.

	$n = 256$									
	$m = [n^{0.4}] = 9$					$m = [n^{0.5}] = 16$				
$d_u \backslash d_x$	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4
0	−.7944	−.0014	.1751	.0578	.0273	.3206	−.0159	−.1937	.1089	−.0080
0.1	−	.3735	.0490	−.3513	.0506	−	−.1036	.1480	−.0222	−.0425
0.2	−	−	.3707	.0799	.1320	−	−	.1950	.0565	−.0557
0.3	−	−	−	.1947	.2056	−	−	−	.0201	.1935
0.4	−	−	−	−	.2135	−	−	−	−	.0475
	$n = 512$									
	$m = [n^{0.4}] = 12$					$m = [n^{0.5}] = 22$				
$d_u \backslash d_x$	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4
0	−.1949	−.1477	−.0034	−.0076	.0600	−.1094	−.0205	−.0849	−.0301	−.0145
0.1	−	.0626	−.0300	−.1943	.1075	−	−.0064	−.0021	.0200	.1476
0.2	−	−	.3309	.0133	.2500	−	−	.1177	.1292	−.1904
0.3	−	−	−	.0595	.0487	−	−	−	.0069	.1094
0.4	−	−	−	−	.1509	−	−	−	−	−.1144
	$n = 1024$									
	$m = [n^{0.4}] = 16$					$m = [n^{0.5}] = 32$				
$d_u \backslash d_x$	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4
0	−.3073	−.0163	−.0188	−.0242	.0831	−.2009	−.1301	−.0563	.0063	−.0486
0.1	−	.0170	−.0924	−.1709	−.0285	−	−.0840	.0204	.0025	.0695
0.2	−	−	.1925	−.1273	.0592	−	−	−.1846	−.0452	−.0648
0.3	−	−	−	.1410	.0204	−	−	−	−.0700	.1098
0.4	−	−	−	−	.0024	−	−	−	−	−.0464

Table 3: MSE ratio of GLS estimate for Model A.

	$n = 256$									
	$m = [n^{0.4}] = 9$					$m = [n^{0.5}] = 16$				
$d_u \backslash d_x$	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4
0	.8858	.8286	.7605	.6042	.4021	.9398	.8692	.8281	.7038	.4553
0.1	−	.9064	.8340	.7468	.6343	−	.9417	.8997	.8580	.7225
0.2	−	−	.8751	.8227	.7672	−	−	.9527	.8950	.8200
0.3	−	−	−	.8532	.8581	−	−	−	.9185	.9092
0.4	−	−	−	−	.8578	−	−	−	−	.8982
	$n = 512$									
	$m = [n^{0.4}] = 12$					$m = [n^{0.5}] = 22$				
$d_u \backslash d_x$	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4
0	.9212	.8944	.7923	.6594	.4268	.9459	.9099	.8538	.7457	.4942
0.1	−	.9050	.9000	.7855	.6714	−	.9587	.9071	.8954	.7519
0.2	−	−	.9035	.8934	.8185	−	−	.9674	.9200	.8681
0.3	−	−	−	.8877	.8900	−	−	−	.9286	.9258
0.4	−	−	−	−	.8817	−	−	−	−	.9184
	$n = 1024$									
	$m = [n^{0.4}] = 16$					$m = [n^{0.5}] = 32$				
$d_u \backslash d_x$	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4
0	.9550	.9079	.8373	.7032	.4752	.9827	.9541	.9007	.7810	.5350
0.1	−	.9494	.9050	.8277	.7218	−	.9770	.9386	.9251	.8089
0.2	−	−	.93425	.9159	.8410	−	−	.9710	.9481	.9023
0.3	−	−	−	.8995	.9016	−	−	−	.9500	.9336
0.4	−	−	−	−	.8820	−	−	−	−	.9117

Table 4: MSE ratio of FGLS estimate for Model A.

					$n = 256$					
	$m = [n^{0.4}] = 9$					$m = [n^{0.5}] = 16$				
$d_u \backslash d_x$	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4
0	.8287	.7737	.7085	.5597	.37413	.8930	.8236	.7826	.6639	.4314
0.1	−	.8448	.7725	.6939	.5829	−	.8910	.8535	.8089	.6755
0.2	−	−	.8050	.7620	.7052	−	−	.9035	.8456	.7709
0.3	−	−	−	.7793	.7882	−	−	−	.8537	.8428
0.4	−	−	−	−	.7608	−	−	−	−	.8314
					$n = 512$					
	$m = [n^{0.4}] = 12$					$m = [n^{0.5}] = 22$				
$d_u \backslash d_x$	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4
0	.8693	.8395	.7441	.6148	.3996	.9131	.8742	.8179	.7108	.4693
0.1	−	.8448	.8372	.7371	.6239	−	.9278	.8700	.8503	.7104
0.2	−	−	.8479	.8266	.7529	−	−	.9212	.8808	.8195
0.3	−	−	−	.8264	.8093	−	−	−	.8847	.8723
0.4	−	−	−	−	.7972	−	−	−	−	.8683
					$n = 1024$					
	$m = [n^{0.4}] = 16$					$m = [n^{0.5}] = 32$				
$d_u \backslash d_x$	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4
0	.9127	.8650	.7866	.6617	.4474	.9561	.9231	.8760	.7494	.5104
0.1	−	.8952	.8585	.7873	.6749	−	.9479	.9090	.8855	.7753
0.2	−	−	.8848	.8574	.7866	−	−	.9405	.9173	.8678
0.3	−	−	−	.8469	.8367	−	−	−	.9202	.8947
0.4	−	−	−	−	.8203	−	−	−	−	.8746

Table 5: Bias (x100) of GLS estimate for Model B.

					$n = 256$					
	$m = [n^{0.4}] = 9$					$m = [n^{0.5}] = 16$				
$d_u \backslash d_x$	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4
0	$-.2044$	.1635	.0611	$-.1900$	.0728	.4680	$-.1765$	$-.1787$	.1638	.0220
0.1	$-$	$-.1195$	$-.1304$	.1619	.0522	$-$	.0037	$-.0905$	$-.0576$	.1679
0.2	$-$	$-$	$-.4386$	.0352	$-.1474$	$-$	$-$	$-.0469$	.2834	$-.2454$
0.3	$-$	$-$	$-$	$-.1338$	.2467	$-$	$-$	$-$	.1265	$-.0978$
0.4	$-$	$-$	$-$	$-$	$-.0978$	$-$	$-$	$-$	$-$	$-.2504$
					$n = 512$					
	$m = [n^{0.4}] = 12$					$m = [n^{0.5}] = 22$				
$d_u \backslash d_x$	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4
0	.0457	$-.0221$	.1051	$-.1531$	.0083	.0678	$-.2716$	$-.0721$	.0289	.0365
0.1	$-$	.0220	$-.0913$	.0911	.0530	$-$	.1370	.0143	$-.0653$	.1247
0.2	$-$	$-$	$-.3577$	$-.2223$	$-.1993$	$-$	$-$	$-.1206$	$-.0003$	$-.0861$
0.3	$-$	$-$	$-$	$-.0351$	.3337	$-$	$-$	$-$	.1794	$-.0876$
0.4	$-$	$-$	$-$	$-$	$-.0613$	$-$	$-$	$-$	$-$	$-.5301$
					$n = 1024$					
	$m = [n^{0.4}] = 16$					$m = [n^{0.5}] = 32$				
$d_u \backslash d_x$	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4
0	.0908	$-.0696$	$-.0998$	$-.0120$	.0203	.0491	$-.0093$	$-.0748$	$-.0161$	$-.0135$
0.1	$-$	$-.0906$	$-.0527$	$-.0502$	.0251	$-$	.1616	.0710	$-.0687$	.0174
0.2	$-$	$-$	$-.3233$	$-.0296$	$-.1125$	$-$	$-$	$-.2311$	$-.1081$	$-.0775$
0.3	$-$	$-$	$-$	.0690	.2212	$-$	$-$	$-$	.1561	$-.0833$
0.4	$-$	$-$	$-$	$-$	.2007	$-$	$-$	$-$	$-$	$-.3223$

Table 6: Bias (x100) of FGLS estimate for Model B.

$n = 256$										
	$m = [n^{0.4}] = 9$					$m = [n^{0.5}] = 16$				
$d_u \backslash d_x$	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4
0	$-.2263$	.0342	.0154	$-.2082$	.0748	.4661	$-.2301$	$-.1878$	.1502	.0067
0.1	–	$-.1463$	$-.1412$	.2009	.0675	–	$-.0124$	$-.0525$	$-.0417$	.1743
0.2	–	–	$-.4286$	$-.0571$	$-.2076$	–	–	.0460	.3203	$-.2713$
0.3	–	–	–	$-.1530$	.3159	–	–	–	.2171	$-.1449$
0.4	–	–	–	–	$-.2354$	–	–	–	–	$-.1875$
$n = 512$										
	$m = [n^{0.4}] = 12$					$m = [n^{0.5}] = 22$				
$d_u \backslash d_x$	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4
0	.2877	$-.0495$	.1330	$-.1251$	$-.0045$	.0552	$-.2613$	$-.0942$	$-.0040$	.0202
0.1	–	.0223	$-.0824$	.1044	.0610	–	.1756	.0484	$-.0707$	.1187
0.2	–	–	$-.3494$	$-.1992$	$-.2081$	–	–	$-.0588$	$-.0229$	$-.0889$
0.3	–	–	–	.0305	.4237	–	–	–	.1858	$-.1330$
0.4	–	–	–	–	$-.0911$	–	–	–	–	$-.5193$
$n = 1024$										
	$m = [n^{0.4}] = 16$					$m = [n^{0.5}] = 32$				
$d_u \backslash d_x$	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4
0	.1009	$-.04255$	$-.0984$	$-.0057$	.0297	.0297	.0260	$-.1000$	$-.0278$	$-.0037$
0.1	–	$-.0780$	$-.0156$	$-.0353$	.0172	–	.1708	.0915	$-.0481$	.0172
0.2	–	–	$-.4002$	$-.0451$	$-.1011$	–	–	$-.2167$	$-.1124$	$-.0743$
0.3	–	–	–	$-.0015$	.2307	–	–	–	.1570	$-.0920$
0.4	–	–	–	–	.1773	–	–	–	–	$-.3276$

Table 7: MSE ratio of GLS estimate for Model B.

	$n = 256$									
	$m = [n^{0.4}] = 9$					$m = [n^{0.5}] = 16$				
$d_u \backslash d_x$	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4
0	.8782	.8500	.7665	.6340	.3966	.9135	.8589	.8212	.7026	.4690
0.1	–	.8903	.8317	.7777	.6326	–	.9240	.8744	.8403	.7057
0.2	–	–	.8817	.8305	.7646	–	–	.9238	.8859	.8334
0.3	–	–	–	.8598	.8311	–	–	–	.8970	.8553
0.4	–	–	–	–	.8540	–	–	–	–	.8810
	$n = 512$									
	$m = [n^{0.4}] = 12$					$m = [n^{0.5}] = 22$				
$d_u \backslash d_x$	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4
0	.8863	.8912	.8051	.6805	.4308	.9531	.8966	.8478	.7550	.5085
0.1	–	.9139	.8746	.8092	.6478	–	.9336	.9048	.8557	.7446
0.2	–	–	.9220	.8612	.8125	–	–	.9464	.9016	.8787
0.3	–	–	–	.9272	.8929	–	–	–	.9380	.8961
0.4	–	–	–	–	.8676	–	–	–	–	.8992
	$n = 1024$									
	$m = [n^{0.4}] = 16$					$m = [n^{0.5}] = 32$				
$d_u \backslash d_x$	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4
0	.9217	.8971	.8479	.7196	.4695	.9826	.9504	.8963	.8092	.5561
0.1	–	.9292	.8889	.8366	.7031	–	.9767	.9378	.8871	.7675
0.2	–	–	.9143	.8963	.8374	–	–	.9761	.9410	.9026
0.3	–	–	–	.9386	.9052	–	–	–	.9518	.9313
0.4	–	–	–	–	.9238	–	–	–	–	.9551

Table 8: MSE ratio of FGLS estimate for Model B.

					$n = 256$					
	$m = [n^{0.4}] = 9$					$m = [n^{0.5}] = 16$				
$d_u \backslash d_x$	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4
0	.8266	.7917	.7093	.5863	.3676	.8857	.8287	.7889	.6725	.4469
0.1	–	.8356	.7706	.7201	.5899	–	.8944	.8344	.8002	.6737
0.2	–	–	.8134	.7671	.7069	–	–	.8864	.8474	.7994
0.3	–	–	–	.7843	.7501	–	–	–	.8544	.8112
0.4	–	–	–	–	.7621	–	–	–	–	.8265
					$n = 512$					
	$m = [n^{0.4}] = 12$					$m = [n^{0.5}] = 22$				
$d_u \backslash d_x$	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4
0	.8395	.8340	.7511	.6301	.4037	.9164	.8590	.8151	.7277	.4845
0.1	–	.8625	.8172	.7550	.6062	–	.9071	.8626	.8265	.7140
0.2	–	–	.8586	.8037	.7484	–	–	.9143	.8622	.8404
0.3	–	–	–	.8547	.8305	–	–	–	.9022	.8520
0.4	–	–	–	–	.7897	–	–	–	–	.8543
					$n = 1024$					
	$m = [n^{0.4}] = 16$					$m = [n^{0.5}] = 32$				
$d_u \backslash d_x$	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4
0	.8783	.8570	.7953	.6714	.4400	.9547	.9216	.8631	.7750	.5352
0.1	–	.8858	.8346	.7891	.6570	–	.9539	.9043	.8531	.7366
0.2	–	–	.8614	.8441	.7830	–	–	.9509	.9109	.8623
0.3	–	–	–	.8773	.8370	–	–	–	.9219	.8994
0.4	–	–	–	–	.8542	–	–	–	–	.9166

Working Paper

2002-5: Anna Christina D'Addio and Michael Rosholm: Left-Censoring in Duration Data: Theory and Applications.

2002-6: Morten Ørregaard Nielsen: Efficient Inference in Multivariate Fractionally Integration Models.

2002-7: Morten Ørregaard Nielsen: Optimal Residual Based Tests for Fractional Cointegration and Exchange Rate Dynamics.

2002-8: Morten Ørregaard Nielsen: Local Whittle Analysis of Stationary Fractional Cointegration.

2002-9: Effrosyni Diamantoudi and Licun Xue: Coalitions, Agreements and Efficiency.

2002-10: Effrosyni Diamantoudi and Eftichios S. Sartzetakis: International Environmental Agreements - The Role of Foresight.

2002-11: N.E. Savin and Allan H. Würtz: Testing the Semiparametric Box–Cox Model with the Bootstrap.

2002-12: Morten Ø. Nielsen, Spectral Analysis of Fractionally Cointegrated Systems

2002-13: Anna Christina D'Addio and Bo E. Honoré, Duration Dependence and Timevarying Variables in Discrete Time Duration Models.

2002-14: Anna Christina D'Addio and Michael Rosholm, Labour Market Transitions of French Youth.

2002-15: Boriss Siliverstovs, Tom Engsted and Niels Haldrup, Long-run forecasting in multicointegrated systems.

2002-16: Morten Ørregaard Nielsen, Local Empirical Spectral Measure of Multivariate Processes with Long Range Dependence.

2002-17: Morten Ørregaard Nielsen, Semiparametric Estimation in Time Series Regression with Long Range Dependence