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8000 AARHUS C - DENMARK ☎ +45 89 42 11 33 - TELEFAX +45 86 13 63 34

Local Empirical Spectral Measure of Multivariate Processes with Long Range Dependence

Morten Ørregaard Nielsen*
University of Aarhus

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Abstract

We derive a functional central limit theorem for the empirical spectral measure or discretely averaged (integrated) periodogram of a multivariate long range dependent stochastic process in a degenerating neighborhood of the origin. We show that, under certain restrictions on the memory parameters, this local empirical spectral measure converges weakly to a Gaussian process with independent increments. Applications to narrow-band frequency domain estimation in time series regression with long range dependence, and to local (to the origin) goodness-of-fit testing are offered.

Keywords: Brownian Motion; Fractional ARIMA; Functional Central Limit Theorem; Goodness-of-fit Test; Integrated Periodogram; Long Memory; Narrow-band Frequency Domain Least Squares

AMS 2000 Classification: Primary: 60G18, 62M15; Secondary: 60F17, 62G20, 62M10

JEL Classification: C14; C22; C32

1 Introduction

We are concerned with p-vector-valued stochastic processes $(X_t)_{t\in\mathbb{Z}}$ that admit a spectral density matrix of the form

$$f(\lambda) \sim \Lambda^{-1}G\Lambda^{-1} \qquad \text{as } \lambda \to 0^+, \tag{1}$$

where $\Lambda = \text{diag}(\lambda^{H_1-1/2}, ..., \lambda^{H_p-1/2})$, the symbol "$\sim$" means that the ratio of the left- and right-hand sides tends to unity (element-by-element), and G is a $p \times p$ real, symmetric, positive definite matrix.

*Address correspondence to: Morten Ørregaard Nielsen, Department of Economics, Building 322, University of Aarhus, DK-8000 Aarhus C, Denmark; telephone: +45 8942 1584; fax: +45 8613 6334; e-mail: monielsen@econ.au.dk.

Such processes are said to have long range dependence or strong dependence, since the autocorrelations decay hyperbolically in contrast to the much faster exponential rate in the weak dependence case.

The parameters $H_1, ..., H_p$ determine the memory of the process, and the components of (X_t), say (X_{at}), may have separate memory parameters, H_a. If $H_a > 0$, (X_{at}) is invertible and admits a linear representation, and if $H_a < 1$ it is covariance stationary. If $H_a = 1/2$, the spectral density is bounded at the origin, and the process has only weak dependence. Sometimes (X_{at}) is said to have negative memory, short memory, or long memory when $H_a < 1/2$, $H_a = 1/2$, or $H_a > 1/2$, respectively. Throughout this paper we shall be concerned with the long memory case, since this is the dominant case in many applications, including in econometrics, hydrology, finance, and other fields. For a review of the properties of long range dependent processes see, e.g., Robinson (1994*b*) and Beran (1994).

Two well known parametric models satisfying (1) are the fractional Gaussian noise and the fractionally integrated autoregressive moving-average (FARIMA) models, see Mandelbrot & Ness (1968), Adenstedt (1974), Granger & Joyeux (1980), and Hosking (1981).

We are concerned with the weak convergence near the origin of the empirical spectral measure of multivariate long range dependent stochastic processes. The empirical spectral measure over a degenerating frequency band near the origin is an estimate of the spectral distribution function local to the origin, i.e.

$$F(\lambda) = \int_0^\lambda f(\theta)\, d\theta, \qquad \lambda \in (-\lambda_\varepsilon, \lambda_\varepsilon], \tag{2}$$

for some 'small' λ_ε, which will be made precise in the next section. Here, $f(\lambda)$ is the matrix of spectral density functions defined by $\Gamma(j) = \int_{-\pi}^{\pi} e^{ij\lambda} f(\lambda)\, d\lambda$, where $\Gamma(j)$ is the j'th autocovariance matrix of (X_t). The typical element, $F_{ab}(\lambda) = \int_0^\lambda f_{ab}(\theta)\, d\theta$, is the cross spectral distribution function between series (X_{at}) and (X_{bt}).

For the remainder of the paper we distinguish between the terms 'discretely averaged periodogram' and 'empirical spectral measure'. For the integrated periodogram evaluated at a particular frequency, we use the term 'discretely averaged periodogram (DAP)'. When the integrated periodogram is considered as a stochastic process over a band of frequencies, we use 'empirical spectral measure'.

The main goal of the present paper is to derive a functional central limit theorem for the local (to the origin) empirical spectral measure. To this end, we first prove a central limit theorem for the DAP, where the averaging is over a degenerating band of frequencies near the origin. This theorem is of interest in its own right, e.g. to derive the asymptotic distribution of the narrow-band least squares estimator, see section 3. In addition, it is used to prove the functional central limit theorerm, which is the ultimate goal of the analysis. The asymptotic properties of spectral estimates, and the weak convergence of the empirical spectral measure, are well understood for weakly dependent processes, see e.g. Grenander & Rosenblatt (1957), Brillinger (1969), Brillinger (1981), and the references therein. However, the processes that we consider in this paper do not belong to this class, as they exhibit spectral poles at the origin.

Recently, the integrated periodogram of long memory processes has been an object of considerable interest in the literature. For scalar-valued processes with fully parameterized spectral density functions, Kokoszka & Mikosch (1997) derived a functional central limit theorem for the empirical spectral measure. In the case where the spectral density function is only locally parameterized, Lobato & Robinson (1996) derived the limiting distribution of the DAP for Gaussian scalar-valued processes. The consistency of the DAP for multivariate processes was proved by Lobato (1997). In related work, Yajima (1989) derived a central limit theorem for the discrete Fourier transform of long memory processes at finitely many λ, and Robinson (1995*b*) derived the limiting moments of the discrete Fourier transform when allowing for an increasing (to infinity) number of λ on a degenerating interval. Thus, our results are also extensions of their work.

The obvious advantage of specifying the spectral density only in a neighborhood of the origin as in (1) and (2), is the nonparametric treatment of the process at other frequencies, assuming only mild regularity conditions such as integrability implied by covariance stationarity. Thus, in applications one would not have to worry about correct specification of the short-run dynamics of the process, e.g. the autoregressive and moving average orders in a FARIMA model. Previously, this type of local specification, termed semiparametric by Robinson (1994*a*), has been applied in methods for estimation of the memory parameter(s) by Geweke & Porter-Hudak (1983), Robinson (1994*a*, 1995*a*, 1995*b*), Lobato & Robinson (1996), and Lobato (1999) among others.

We present two applications of our results. First, we apply the central limit theorem for the DAP to narrow-band least squares estimation in time series regression with long range dependent regressors and errors. The narrow-band estimator was first suggested by Robinson (1994*a*), who showed that, even when the regressors and errors are correlated and long range dependent, consistent estimates can be obtained if the estimation is carried out in the frequency domain using a degenerating band of frequencies around the origin. This type of estimate thus enjoys the advantages of local specification, as discussed above. With our new limiting theory, we are easily able to derive the asymptotic distribution of the narrow-band least squares estimator and show that it is normal.

Our second application is to goodness-of-fit testing in the frequency domain based on the empirical spectral measure. The goodness-of-fit tests are modelled after the corresponding tests in empirical process theory, e.g. Shorack & Wellner (1986). This idea was explored by Bartlett (1955) and Grenander & Rosenblatt (1957). Some recent treatments are Anderson (1993) and Kokoszka & Mikosch (1997), who considered stationary processes with weak dependence and long range dependence, respectively. In particular, we consider local (to the origin) versions of the popular Kolmogorov-Smirnov and Cramér-von Mises testing procedures. The tests are derived from the functional central limit theorem for the local empirical spectral measure, and thus the tests can be considered measures of goodness-of-fit for the spectral density near the origin. Indeed, as mentioned above, they do not depend on the form of the spectral density away from the origin under some mild regularity conditions. Since the processes we consider have long memory, their spectral mass is concentrated around a peak at the origin. Hence, goodness-of-fit near the origin should be given preference. We show that the limiting processes are functionals of time-transformed Brownian Motion, unlike the Brownian Bridge limits in the standard theory, e.g. Shorack & Wellner (1986), Anderson (1993), and Kokoszka & Mikosch (1997). This is of course due to the fact that our process is not tied down at frequency π, as is the case in all the above studies, where the spectral density is completely parameterized.

The remainder of the paper is organized as follows. In section 2 we present our assumptions and the main results, the proofs of which are in sections 4 and 5. Section 3 offers a brief discussion with applications of our results to narrow-band estimation in time series regression with long range dependence, and to local Kolmogorov-Smirnov and Cramér-von Mises goodness-of-fit testing. Section 6 contains some auxillary lemmas.

2 Main Results

Suppose we observe a sample of size n from (X_t). Define the normalized discrete Fourier transform (DFT) and periodogram matrix of $(X_t)_{t=1,\ldots,n}$ by

$$w(\lambda) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} X_t e^{it\lambda} \text{ and } I(\lambda) = w(\lambda) w^*(\lambda), \qquad \lambda \in (-\pi, \pi], \tag{3}$$

where the asterisk denotes transposition combined with complex conjugation. The cross-periodogram between series (X_{at}) and (X_{bt}) is thus $I_{ab}(\lambda) = w_a(\lambda) \overline{w_b(\lambda)}$, where the bar is complex conjugation and $w_a(\lambda)$ is the a'th component of $w(\lambda)$.

Our statistic of interest is the discretely averaged periodogram

$$\hat{F}(\lambda_m) = \frac{2\pi}{n} \sum_{j=1}^{m} I(\lambda_j), \tag{4}$$

where $\lambda_j = 2\pi j/n$ are the Fourier frequencies. The (a,b)'th component, denoted $\hat{F}_{ab}(\lambda_m)$, is the discretely averaged cross-periodogram between series (X_{at}) and (X_{bt}). The number $m = m(n)$ is a user-chosen bandwidth parameter, denoting the portion of the periodogram that will be used in the averaging. The zero-frequency is excluded to render the statistic invariant to location shifts. Note that we could also have considered the continuously averaged periodogram $\tilde{F}(\lambda) = \int_0^\lambda I(\theta)\, d\theta$, but we prefer the discrete version because of the location invariance and its computational simplicity.

The basic setup follows that of Lobato (1997), who showed the consistency of (4) as an estimate of (2), when the bandwidth $m = m(n)$ tends to infinity at a slower rate than n. Let $(X_t)_{t\in\mathbb{Z}}$ be a covariance stationary p-vector-valued stochastic process with a'th component (X_{at}), mean μ, and suppose (X_t) admits the linear representation

$$X_t = \mu + \sum_{j=0}^{\infty} A_j \varepsilon_{t-j}, \quad t \in \mathbb{Z}, \tag{5}$$

with square summable coefficients A_j. The innovation sequence (ε_t) satisfies $E(\varepsilon_t | \mathcal{F}_{t-1}) = 0$ and $E(\varepsilon_t \varepsilon_t' | \mathcal{F}_{t-1}) = I_p$, where $\mathcal{F}_t = \sigma(\{e_s, s \le t\})$ and I_p is the p-dimensional identity matrix. Similarly to (3), define the DFT and periodogram of (ε_t), which we denote $v(\lambda)$ and $J(\lambda)$, respectively.

Let $A(\lambda)$ be the Fourier transform of (A_j), i.e. $A(\lambda) = \sum_{j=0}^{\infty} A_j \exp(ij\lambda)$. Then, under (5), the spectral density matrix of (ε_t) is $f_\varepsilon(\lambda) = I_p/2\pi$, and the spectral density matrix of (X_t) is $f(\lambda) = A(\lambda) A^*(\lambda)/2\pi$.

We now state the assumptions used to prove the central limit theorem for the DAP. Our assumptions strengthen those in Lobato (1997), and are a multivariate generalization of those in Robinson (1994*a*) and Robinson (1995*a*), see also Lobato (1999). They are in some respects much weaker than those employed by Lobato & Robinson (1996) in the univariate case. In particular, we avoid their assumption of Gaussianity.

Assumption 1 *The spectral density matrix of* (X_t) *in* (1) *with typical element* $f_{ab}(\lambda)$*, the cross spectral density between* (X_{at}) *and* (X_{bt})*, satisfies*

$$\left| f_{ab}(\lambda) - G_{ab} \lambda^{1-H_a-H_b} \right| = O\left(\lambda^{1+\alpha-H_a-H_b}\right), \text{ as } \lambda \to 0^+, \ a, b = 1, ..., p, \tag{6}$$

for some $\alpha \in (0, 2]$*, where* G_{ab} *is the* (a, b)*'th element of* G*.*

Assumption 2 *The innovations in* (5) *have square summable coefficient matrices,* $\sum_{j=0}^{\infty} \|A_j\|^2 < \infty$*, and satisfy, almost surely,* $E(\varepsilon_t | \mathcal{F}_{t-1}) = 0$*,* $E(\varepsilon_t \varepsilon_t' | \mathcal{F}_{t-1}) = I_p$*, and the matrices* $\mu_3 = E(\varepsilon_t \otimes \varepsilon_t \varepsilon_t' | \mathcal{F}_{t-1})$ *and* $\mu_4 = E(\varepsilon_t \varepsilon_t' \otimes \varepsilon_t \varepsilon_t' | \mathcal{F}_{t-1})$ *are nonstochastic, finite, and do not depend on* t*, with* $\mathcal{F}_t = \sigma(\{\varepsilon_s, s \le t\})$*.*

Assumption 3 *As* $\lambda \to 0^+$

$$\frac{dA_a(\lambda)}{d\lambda} = O\left(\lambda^{-1} \|A_a(\lambda)\|\right), \ a = 1, ..., p,$$

where $A_a(\lambda)$ *is the* a'*th row of* $A(\lambda) = \sum_{j=0}^{\infty} A_j e^{ij\lambda}$*.*

Assumption 4 *The bandwidth parameter* $m = m(n)$ *satisfies*

$$\frac{1}{m} + \frac{m^{1+2\alpha}}{n^{2\alpha}} \to 0, \text{ as } n \to \infty.$$

Some comments on our conditions are in order. Assumptions 1 and 3 specialize (1) by imposing smoothness conditions on the spectral density matrix of (X_t) commonly employed in the literature. They are satisfied with $\alpha = 2$ if, e.g., (X_t) is a vector fractional Gaussian noise or a vector FARIMA process. Assumption 2 is a straightforward multivariate generalization of the corresponding condition in Robinson (1995a), and imposes a linear structure on (X_t) with square summable coefficients and martingale difference innovations with finite fourth moments. It is satisfied, for instance, if (ε_t) form an *i.i.d.* process with finite fourth moments. Finally, assumption 4 restricts the expansion rate of the bandwidth parameter $m = m(n)$, the weakest constraint being implied by $\alpha = 2$ in which case the condition is $m = o(n^{4/5})$.

We now derive the limiting distribution of the DAP, for any fixed fraction $r \in (0, 1]$ of the bandwidth m. Thus, we consider

$$F_n(r) = \sqrt{m}\lambda_m^{-1}\Lambda_m\left(\hat{F}(\lambda_{[mr]}) - F(\lambda_{mr})\right)\Lambda_m, \tag{7}$$

where $\Lambda_m = \text{diag}(\lambda_m^{H_1-1/2}, ..., \lambda_m^{H_p-1/2})$ and $[x]$ denotes the integer part of x.

The most important tool for proving our results below, and indeed most of the results cited above, is the (Bartlett) approximation

$$\hat{F}\left(\lambda_{[mr]}\right) = \frac{2\pi}{n}\sum_{j=1}^{[mr]}\text{Re}\left(I\left(\lambda_j\right)\right) \simeq \frac{2\pi}{n}\sum_{j=1}^{[mr]}\text{Re}\left(A\left(\lambda_j\right)J\left(\lambda_j\right)A^*\left(\lambda_j\right)\right),$$

see e.g. Hannan (1970, p. 248). Following Robinson (1995a), the right-hand side can, when suitably normalized and centered, be further approximated by the sum of a martingale difference array, allowing us to invoke simple martingale central limit theorems. The details of this approximation and its proof is given as Lemma 4 in section 6.

We now provide, for a fixed $r \in (0, 1]$, a central limit theorem for (7) which is proved in section 4.

Theorem 1 *Under Assumptions 1-4 and* $1/2 \leq H_a < 3/4$, $a = 1, ..., p$, *the* $p \times p$ *matrix-valued function in* (7)*, for a fixed* $r \in (0, 1]$*, is asymptotically normal with mean zero and covariance*

$$\lim_{n\to\infty} Cov\left(F_{a_1b_1,n}(r), F_{a_2b_2,n}(r)\right) = \frac{1}{2}\left(G_{a_1a_2}G_{b_2b_1} + G_{a_1b_2}G_{a_2b_1}\right)\int_0^r s^{2-H_{a_1}-H_{a_2}-H_{b_1}-H_{b_2}}ds.$$

The limiting distribution in Theorem 1 is asymptotically normal, paralleling the well known weak dependence case. When the memory parameters violate the condition that $H_a < 3/4$, $a = 1, ..., p$, it is conjectured that the DAP is non-normal and converges to a function of the Rosenblatt process (see

Rosenblatt (1961, 1979) and Taqqu (1975)). This dichotomy may seem strange at first, since in both cases the model is covariance stationary, but is due to the fact that the spectral density in (6) cannot be square integrable in the latter case, even in an arbitrarily small neighborhood of the origin.

The parameters appearing in the limiting distribution in Theorem 1 can be replaced by consistent estimates. For instance, the multivariate log-periodogram estimates in Robinson (1995*b*), or the multivariate local Whittle pseudo-likelihood estimates in Lobato (1999), which both have nice asymptotic properties, are easy to calculate, and local in the same sense as the statistics of the present paper.

Theorem 1 indicates the asymptotic normality of the DAP for any given fraction of the frequency λ_m. We now turn to our main objective, which is the weak convergence of the stochastic process

$$\left(F_n\left(r\right) = \sqrt{m}\lambda_m^{-1}\Lambda_m\left(\hat{F}(\lambda_{[mr]}) - F(\lambda_{mr})\right)\Lambda_m\right)_{0\leq r\leq 1} \tag{8}$$

in the space $D^{p\times p}\left[0,1\right]$, the space of $p\times p$ matrix-valued cadlag functions on the unit interval. This space is isomorphic with $D^{p^2}\left[0,1\right]$, and may be endowed with a metric that makes it complete and separable, see Billingsley (1999, chapter 3). We are able to establish weak convergence of $\left(F_n\left(r\right)\right)_{0\leq r\leq 1}$ under the same conditions as Theorem 1.

Theorem 2 *Under Assumptions 1-4 and* $1/2\leq H_a<3/4$, $a=1,...,p$, *the stochastic process* $\left(F_n\left(r\right)\right)_{0\leq r\leq 1}$ *defined in* (8) *converges weakly in* $D^{p\times p}\left[0,1\right]$ *to a* $p\times p$ *matrix-valued Gaussian process* $\left(Y\left(r\right)\right)_{0\leq r\leq 1}$ *with* $Y\left(0\right)=0$ *a.s., mean zero, and covariance structure*

$$Cov\left(Y_{a_1b_1}\left(r_1\right),Y_{a_2b_2}\left(r_2\right)\right)=\frac{1}{2}\left(G_{a_1a_2}G_{b_2b_1}+G_{a_1b_2}G_{a_2b_1}\right)\int_0^{\min(r_1,r_2)}r^{2-H_{a_1}-H_{a_2}-H_{b_1}-H_{b_2}}dr.$$

The limiting process in Theorem 2 is Gaussian, has mean zero and independent increments, and is easily seen to be a time-transformed matrix-valued Brownian Motion (and hence to have continuous sample paths almost surely). In empirical process theory, the limits are typically functionals of the Brownian Bridge process, e.g. Shorack & Wellner (1986). In previous studies of weak convergence for empirical spectral measures, this property has been found to carry over to the frequency domain, see e.g. Anderson (1993) and Kokoszka & Mikosch (1997). The Brownian Bridge process appears in the limit since the empirical distribution function and empirical spectral measure are tied down at $r=1$, when the full spectral band $(-\pi,\pi]$ is considered and the spectral density is fully parameterized. This is not the case here, and hence we get the time-transformed Brownian Motion process in the limit.

3 Discussion

The DAP is a statistic of significant independent interest. The properties of its univariate counterpart have long been well known for the weak dependence case, and have recently been explored for univariate long range dependent processes in Robinson (1994*a*) and Lobato & Robinson (1996). There, the DAP is applied in semiparametric estimation of H for univariate long range dependent processes. Thus, a distribution theory for the multivariate DAP is essential in extending this estimator to the multivariate case as in (1). However, a full discussion would be lengthy, and superior semiparametric estimators with likelihood interpretations are available, see e.g. Lobato (1999), so we leave this issue for future research.

Another application suggested by Robinson (1994*a*) in the stationary case, and developed by Robinson & Marinucci (1998) in the nonstationary case, is narrow-band frequency domain least squares (FDLS) estimation of β in the model (bivariate for simplicity)

$$y_y = \beta x_t + e_t, \quad t \in \mathbb{Z},$$

where (x_t) and (e_t) have spectral densities $f_x(\lambda) \sim G_x \lambda^{1-2H_x}$ and $f_e(\lambda) \sim G_e \lambda^{1-2H_e}$, respectively. The narrow-band FDLS estimator of β is

$$\hat{\beta}_m = \hat{F}_{xx}^{-1}(\lambda_m)\,\hat{F}_{xy}(\lambda_m), \tag{9}$$

which Robinson (1994*a*, pp. 537-538) proved to be consistent for β. The asymptotic distribution theory for $\hat{\beta}$ is easily derived as a special case of our theorem. Define the vector $w_t' = (x_t, e_t)$, with spectral density $f_w(\lambda)$ given by (1) with $G = \operatorname{diag}(G_x, G_e)$. Diagonality of G can be considered a local version of the usual orthogonality condition from least squares theory. Suppose further that Assumptions 1-4 are satisfied for w_t (with obvious implications for y_t).

With our new distribution theory it is straightforward to derive the asymptotic distribution for the narrow-band FDLS estimator (9). First, by Theorem 1 of Lobato (1997), $\lambda_m^{2H_x-2}\hat{F}_{xx}(\lambda_m) \to_p \lambda_m^{2H_x-2}F_{xx}(\lambda_m) = G_x/(2-2H_x)$. By our Theorem 1

$$\sqrt{m}\lambda_m^{H_e+H_x-2}\hat{F}_{xe}(\lambda_m) \to_d N(0, \Omega_{xe}),$$

where $\Omega_{xe} = G_x G_e/(6-4H_x-4H_e)$. Thus, it follows that

$$\begin{aligned}\sqrt{m}\lambda_m^{H_e-H_x}\left(\hat{\beta}_m - \beta\right) &= \lambda_m^{2-2H_x}\hat{F}_{xx}^{-1}(\lambda_m)\sqrt{m}\lambda_m^{H_e+H_x-2}\hat{F}_{xe}(\lambda_m) \\ &\to_d N\left(0, \frac{2G_e(1-H_x)^2}{G_x(3-2H_x-2H_e)}\right).\end{aligned}$$

If $H_x = H_e = H \in [1/2, 3/4)$ the distribution is particularly simple, as the normalization is free of H (it is in fact $\sqrt{m}$) and the variance is $2G_e (1-H)^2 / G_x (3-4H)$, admitting very simple asymptotic inference on the regression coefficients.

Finally, we consider the statistical implications of Theorem 2 for goodness-of-fit testing. To simplify the discussion, we consider the case of a scalar-valued stochastic process (x_t) with spectral density $f_x(\lambda) \sim G\lambda^{1-2H}$ satisfying the assumptions of Theorem 2. The weak convergence of the local empirical spectral measure $\left(F_{x,n}(r) = \sqrt{m}\lambda_m^{2H-2}\left(\hat{F}_x\left(\lambda_{[mr]}\right) - F_x\left(\lambda_{mr}\right)\right)\right)_{0\leq r\leq 1}$ in the scalar case is stated as a corollary.

Corollary 3 *Under the assumptions of Theorem 2,* $(F_{x,n}(r))_{0\leq r\leq 1}$ *converges weakly in* $D[0,1]$ *to the time-transformed Brownian Motion*

$$y(r) = (3-4H)^{-1/2} GB(\eta_H(r)), \quad 0 \leq r \leq 1,$$

where $\eta_H(r) = r^{3-4H}$ *and* B *is standard Brownian Motion on* $[0,1]$*.*

Thus, we are able to derive the limiting distribution of some standard goodness-of-fit test statistics in this new setting. The following results are obtained from Corollary 3 and the Continuous Mapping Theorem:

i) Local Kolmogorov-Smirnov test: Under the assumptions of Corollary 3,

$$LKS = \sup_{0\leq r\leq 1} |F_{x,n}(r)| \to_d \frac{G}{\sqrt{3-4H}} \sup_{0\leq r\leq 1} |B(\eta_H(r))|.$$

ii) Local Cramér-von Mises test: Under the assumptions of Corollary 3,

$$LCvM = \int_0^1 F_{x,n}(r)^2 \, dr \to_d \frac{G^2}{3-4H} \int_0^1 B(\eta_H(r))^2 \, dr.$$

Note that we could equivalently have considered the discrete versions of these statistics, i.e. $LKS' = \max_{1\leq j\leq m} |F_{x,n}(j/m)|$ and $LCvM' = \sum_{j=1}^m F_{x,n}(j/m)^2$. The limiting distributions are the same as for their continuous counterparts. It is also worth noting that these limiting distributions are not functionals of the Brownian Bridge process as in the usual case, e.g. Shorack & Wellner (1986), Anderson (1993), and Kokoszka & Mikosch (1997). Instead, they are functionals of the Brownian Motion, since the local empirical spectral measure is not tied down at $r = 1$.

4 Proof of Theorem 1

The proof of asymptotic normality of the DAP employs the martingale difference approximation technique of Robinson (1995a) and Lobato (1999). Applying the Cramèr-Wold device, we need to examine the linear combination (η is a p^2-vector)

$$\begin{aligned} & \eta' \sqrt{m} \lambda_m^{-1} \left(\Lambda_m \otimes \Lambda_m\right) \left(\operatorname{vec} \hat{F}\left(\lambda_{[mr]}\right) - \operatorname{vec} F\left(\lambda_{mr}\right)\right) \\ = & \sum_{a=1}^{p} \sum_{b=1}^{p} \eta_{a(p-1)+b} \sqrt{m} \lambda_m^{H_a+H_b-2} \left(\hat{F}_{ab}\left(\lambda_{[mr]}\right) - F_{ab}\left(\lambda_{mr}\right)\right) \\ = & \sum_{a=1}^{p} \sum_{b=1}^{p} \eta_{a(p-1)+b} \sqrt{m} \lambda_m^{H_a+H_b-2} \left(\frac{2\pi}{n} \sum_{j=1}^{[mr]} \operatorname{Re}\left(I_{ab}\left(\lambda_j\right)\right) - F_{ab}\left(\lambda_{mr}\right)\right), \end{aligned}$$

which by Lemma 4 can be approximated by

$$\sum_{t=1}^{n} \varepsilon_t' \sum_{s=1}^{t-1} c_{t-s,n} \varepsilon_s, \tag{10}$$

where

$$\begin{aligned} c_{tn} &= \frac{1}{2\pi n \sqrt{m}} \sum_{j=1}^{[mr]} \theta_j \cos\left(t\lambda_j\right), \\ \theta_j &= \sum_{a=1}^{p} \sum_{b=1}^{p} \eta_{a(p-1)+b} \lambda_m^{H_a+H_b-1} \operatorname{Re}\left(A_a'\left(\lambda_j\right) \bar{A}_b\left(\lambda_j\right) + A_b'\left(\lambda_j\right) \bar{A}_a\left(\lambda_j\right)\right), \end{aligned}$$

and the dependence on r has been suppressed for notational convenience.

Hence, $z_{tn} = \varepsilon_t' \sum_{s=1}^{t-1} c_{t-s,n} \varepsilon_s$ is a martingale difference array with respect to the filtration $\left(\mathcal{F}_t\right)_{t\in\mathbb{Z}}$, $\mathcal{F}_t = \sigma\left(\{\varepsilon_s, s \leq t\}\right)$, and we can apply the CLT if

$$\sum_{t=1}^{n} E\left(z_{tn}^2 \middle| \mathcal{F}_{t-1}\right) - \sum_{a_1=1}^{p} \sum_{b_1=1}^{p} \sum_{a_2=1}^{p} \sum_{b_2=1}^{p} \eta_{a_1(p-1)+b_1} \eta_{a_2(p-1)+b_2} \Omega_{a_1(p-1)+b_1, a_2(p-1)+b_2} \to_p 0, \tag{11}$$

$$\sum_{t=1}^{n} E\left(z_{tn}^2 1\left(|z_{tn}| > \delta\right)\right) \to 0,\ \delta > 0, \tag{12}$$

see Brown (1971) or Hall & Heyde (1980, chp. 3.2). A sufficient condition for (12) is

$$\sum_{t=1}^{n} E\left(z_{tn}^4\right) \to 0. \tag{13}$$

First, to show (11),

$$\begin{aligned} \sum_{t=1}^{n} E\left(z_{tn}^2 \middle| \mathcal{F}_{t-1}\right) &= \sum_{t=1}^{n} E\left(\sum_{s=1}^{t-1} \sum_{r=1}^{t-1} \varepsilon_s' c_{t-s,n}' \varepsilon_t \varepsilon_t' c_{t-r,n} \varepsilon_r \middle| \mathcal{F}_{t-1}\right) \\ &= \sum_{t=1}^{n} \sum_{s=1}^{t-1} \varepsilon_s' c_{t-s,n}' c_{t-s,n} \varepsilon_s \qquad (14) \\ &\quad + \sum_{t=1}^{n} \sum_{s=1}^{t-1} \sum_{u \neq s} \varepsilon_s' c_{t-s,n}' c_{t-u,n} \varepsilon_u, \qquad (15) \end{aligned}$$

where (15) is negligible by Lemma 5. So we need to show that the mean of (14) is asymptotically equivalent to $\sum_{a_1=1}^{p}\sum_{b_1=1}^{p}\sum_{a_2=1}^{p}\sum_{b_2=1}^{p}\eta_{a_1(p-1)+b_1}\eta_{a_2(p-1)+b_2}\Omega_{a_1(p-1)+b_1,a_2(p-1)+b_2}$. Thus,

$$\begin{aligned} E\,(14) &= \sum_{t=1}^{n}\sum_{s=1}^{t-1}\operatorname{tr}\left(c_{t-s,n}'c_{t-s,n}\right) \\ &= \sum_{t=1}^{n}\sum_{s=1}^{t-1}\sum_{j=1}^{[mr]}\frac{1}{4\pi^2n^2m}\operatorname{tr}\left(\theta_j'\theta_j\right)\cos^2\left(\left(t-s\right)\lambda_j\right) \qquad (16) \\ &\quad+\sum_{t=1}^{n}\sum_{s=1}^{t-1}\sum_{j=1}^{[mr]}\sum_{k\neq j}^{[mr]}\frac{1}{4\pi^2n^2m}\operatorname{tr}\left(\theta_j'\theta_k\right)\cos\left(\left(t-s\right)\lambda_j\right)\cos\left(\left(t-s\right)\lambda_k\right). \qquad (17) \end{aligned}$$

Notice that, since $\|\theta_j\|=O\left(1\right)$ by construction, (17) is bounded by

$$O\left(\sum_{t=1}^{n}\sum_{s=1}^{t-1}\sum_{j=1}^{[mr]}\sum_{k\neq j}^{[mr]}\left(n^2m\right)^{-1}\cos\left(\left(t-s\right)\lambda_j\right)\cos\left(\left(t-s\right)\lambda_k\right)\right),$$

and using that $\sum_{t=1}^{n}\sum_{s=1}^{t-1}\cos\left(\left(t-s\right)\lambda_j\right)\cos\left(\left(t-s\right)\lambda_k\right)=-n/2$, (17) is $O\left(\sum_{j=1}^{[mr]}\sum_{k\neq j}^{[mr]}\left(n^2m\right)^{-1}n\right)=O\left(m/n\right)$.

Next, $\operatorname{tr}\left(\theta_j'\theta_j\right)$ equals $\sum_{a_1=1}^{p}\sum_{b_1=1}^{p}\sum_{a_2=1}^{p}\sum_{b_2=1}^{p}\eta_{a_1(p-1)+b_1}\eta_{a_2(p-1)+b_2}\lambda_m^{H_{a_1}+H_{b_1}+H_{a_2}+H_{b_2}-2}$ times

$$\begin{aligned} &\operatorname{tr}\left(\operatorname{Re}\left(A_{b_1}^*\left(\lambda_j\right)A_{a_1}\left(\lambda_j\right)+A_{a_1}^*\left(\lambda_j\right)A_{b_1}\left(\lambda_j\right)\right)\operatorname{Re}\left(A_{a_2}'\left(\lambda_j\right)\bar{A}_{b_2}\left(\lambda_j\right)+A_{b_2}'\left(\lambda_j\right)\bar{A}_{a_2}\left(\lambda_j\right)\right)\right) \\ =\;& 4\pi^2\left(f_{a_1a_2}\left(\lambda_j\right)f_{b_2b_1}\left(\lambda_j\right)+f_{a_1b_2}\left(\lambda_j\right)f_{a_2b_1}\left(\lambda_j\right)+f_{b_1a_2}\left(\lambda_j\right)f_{b_2a_1}\left(\lambda_j\right)+f_{b_1b_2}\left(\lambda_j\right)f_{a_2a_1}\left(\lambda_j\right)\right) \end{aligned}$$

by definition of $f\left(\lambda\right)$. Since $\left(x+\bar{x}\right)/2=\operatorname{Re}\left(x\right)$, for any complex number x, (16) is

$$\begin{aligned} &\sum_{t=1}^{n}\sum_{s=1}^{t-1}\sum_{j=1}^{[mr]}\frac{2}{n^2m}\sum_{a_1=1}^{p}\sum_{b_1=1}^{p}\sum_{a_2=1}^{p}\sum_{b_2=1}^{p}\eta_{a_1(p-1)+b_1}\eta_{a_2(p-1)+b_2}\lambda_m^{H_{a_1}+H_{b_1}+H_{a_2}+H_{b_2}-2} \\ &\qquad\times\operatorname{Re}\left(f_{a_1a_2}\left(\lambda_j\right)f_{b_2b_1}\left(\lambda_j\right)+f_{a_1b_2}\left(\lambda_j\right)f_{a_2b_1}\left(\lambda_j\right)\right)\cos^2\left(\left(t-s\right)\lambda_j\right). \qquad (18) \end{aligned}$$

The summation over λ_j can be replaced by an integral, viz.

$$\frac{2\pi}{n}\sum_{j=1}^{[mr]}\operatorname{Re}\left(f_{a_1a_2}\left(\lambda_j\right)f_{b_2b_1}\left(\lambda_j\right)+f_{a_1b_2}\left(\lambda_j\right)f_{a_2b_1}\left(\lambda_j\right)\right)\sim\int_0^{\lambda_{mr}}\operatorname{Re}\left(f_{a_1a_2}\left(\lambda\right)f_{b_2b_1}\left(\lambda\right)+f_{a_1b_2}\left(\lambda\right)f_{a_2b_1}\left(\lambda\right)\right)d\lambda.$$

Using this approximation and the relation $\sum_{t=1}^{n-1}\sum_{s=1}^{n-t}\cos^2(s\lambda_j) = (n-1)^2/4$, we can rewrite (18) as

$$
\begin{aligned}
(18) \quad \sim \quad & \sum_{a_1=1}^{p}\sum_{b_1=1}^{p}\sum_{a_2=1}^{p}\sum_{b_2=1}^{p}\eta_{a_1(p-1)+b_1}\eta_{a_2(p-1)+b_2}\lambda_m^{H_{a_1}+H_{b_1}+H_{a_2}+H_{b_2}-2}\left(\sum_{t=1}^{n}\sum_{s=1}^{t-1}\cos^2((t-s)\lambda_j)\right) \\
& \times\frac{1}{n\pi m}\int_0^{\lambda_{mr}}\operatorname{Re}\left(f_{a_1a_2}(\lambda)f_{b_2b_1}(\lambda)+f_{a_1b_2}(\lambda)f_{a_2b_1}(\lambda)\right)d\lambda \\
= \quad & \sum_{a_1=1}^{p}\sum_{b_1=1}^{p}\sum_{a_2=1}^{p}\sum_{b_2=1}^{p}\eta_{a_1(p-1)+b_1}\eta_{a_2(p-1)+b_2}\frac{\lambda_m^{H_{a_1}+H_{b_1}+H_{a_2}+H_{b_2}-3}}{2} \\
& \times\int_0^{\lambda_{mr}}\operatorname{Re}\left(f_{a_1a_2}(\lambda)f_{b_2b_1}(\lambda)+f_{a_1b_2}(\lambda)f_{a_2b_1}(\lambda)\right)d\lambda,
\end{aligned}
$$

and we have shown (11).

Thus, we only have to show (13), which is easy from the above analysis since, by Assumption 2,

$$
\begin{aligned}
\sum_{t=1}^{n}E\left(z_{tn}^4\right) &= \sum_{t=1}^{n}E\left(\sum_{s=1}^{t-1}\varepsilon_s'c_{t-s,n}\varepsilon_t\varepsilon_t'\sum_{u=1}^{t-1}c_{t-u,n}\varepsilon_u\sum_{p=1}^{t-1}\varepsilon_p'c_{t-p,n}\varepsilon_t\varepsilon_t'\sum_{q=1}^{t-1}c_{t-q,n}\varepsilon_q\right) \\
&\leq C\left(\sum_{t=1}^{n}\operatorname{tr}\left(\sum_{s=1}^{t-1}c_{t-s,n}'c_{t-s,n}c_{t-s,n}'c_{t-s,n}\right)+\sum_{t=1}^{n}\operatorname{tr}\left(\sum_{s=1}^{t-1}c_{t-s,n}'\sum_{u=1}^{t-1}c_{t-u,n}c_{t-u,n}'c_{t-s,n}\right)\right)
\end{aligned}
$$

for some constant $C>0$. As in the proof of Lemma 5, this expression can be bounded by $O\left(n\left(\sum_{t=1}^{n}\left\|c_{tn}^2\right\|\right)^2\right) = O\left(n^{-1}\right)$, which completes the proof.

5 Proof of Theorem 2

We have already seen in Theorem 1 that the one-dimensional distributions converge as required, i.e. $F_n(r_1)\to_d Y(r_1), 0\leq r_1\leq 1$. Next, consider $(F_n(r_1), F_n(r_2)), 0\leq r_1<r_2\leq 1$, or equivalently $(F_n(r_1), F_n(r_2)-F_n(r_1))$. The proof in section 4 carries through almost unchanged to this case, so we just outline the differences.

Applying the Cramér-Wold device we need to examine $\eta_1'\operatorname{vec}F_n(r_1)+\eta_2'\operatorname{vec}(F_n(r_2)-F_n(r_1))$, for p^2-vectors η_1 and η_2. As in the proof of Theorem 1, and using Lemma 4, this can be approximated by the sum $\sum_{t=1}^{n}z_{tn}$ of the martingale difference array $z_{tn}=\varepsilon_t'\sum_{s=1}^{t-1}c_{t-s,n}\varepsilon_s$, where

$$
\begin{aligned}
c_{tn} &= \frac{1}{2\pi n\sqrt{m}}\left(\sum_{j=1}^{[mr_1]}\theta_{1j}\cos(t\lambda_j)+\sum_{j=[mr_1]+1}^{[mr_2]}\theta_{2j}\cos(t\lambda_j)\right), \\
\theta_{ij} &= \sum_{a=1}^{p}\sum_{b=1}^{p}\eta_{i,a(p-1)+b}\lambda_m^{H_a+H_b-1}\operatorname{Re}\left(A_a'(\lambda_j)\bar{A}_b(\lambda_j)+A_b'(\lambda_j)\bar{A}_a(\lambda_j)\right), \quad i=1,2.
\end{aligned}
$$

Since the terms corresponding to $\lambda_j \neq \lambda_k$ are negligible precisely as before, we examine

$$\sum_{t=1}^{n}\sum_{s=1}^{t-1}\frac{1}{4\pi^2 n^2 m}\sum_{j=1}^{[mr_1]}\operatorname{tr}\left(\theta_{1j}'\theta_{1j}\right)\cos^2\left(\left(t-s\right)\lambda_j\right) \tag{19}$$

$$+\sum_{t=1}^{n}\sum_{s=1}^{t-1}\frac{1}{4\pi^2 n^2 m}\sum_{j=[mr_1]+1}^{[mr_2]}\operatorname{tr}\left(\theta_{2j}'\theta_{2j}\right)\cos^2\left(\left(t-s\right)\lambda_j\right) \tag{20}$$

$$+\sum_{t=1}^{n}\sum_{s=1}^{t-1}\frac{2}{4\pi^2 n^2 m}\sum_{j=1}^{[mr_1]}\sum_{k=[mr_1]+1}^{[mr_2]}\operatorname{tr}\left(\theta_{1j}'\theta_{2k}\right)\cos\left(\left(t-s\right)\lambda_j\right)\cos\left(\left(t-s\right)\lambda_k\right) \tag{21}$$

corresponding to (16).

The sum of (19) and (20) is seen to converge to

$$\sum_{a=1}^{p}\sum_{b=1}^{p}\left(\eta_{1,a(p-1)+b}Y_{ab}\left(r_1\right)+\eta_{2,a(p-1)+b}\left(Y_{ab}\left(r_2\right)-Y_{ab}\left(r_1\right)\right)\right),$$

by approximating the sums $\sum_1^{[mr_1]}$ and $\sum_{[mr_1]+1}^{[mr_2]}$ by the integrals $\int_0^{\lambda_{mr_1}}$ and $\int_{\lambda_{mr_1}}^{\lambda_{mr_2}}$, respectively. We are left with (21), which is

$$O\left(\sum_{t=1}^{n}\sum_{s=1}^{t-1}\sum_{j=1}^{[mr_1]}\sum_{k=[mr_1]+1}^{[mr_2]}\frac{1}{n^2 m}\cos\left(\left(t-s\right)\lambda_j\right)\cos\left(\left(t-s\right)\lambda_k\right)\right)=O\left(\frac{m}{n}r_1\left(r_2-r_1\right)\right),$$

as for (17). This implies, in particular, that $\left(Y\left(r\right)\right)$ has independent increments. By the Continuous Mapping Theorem we have shown that $\left(F_n\left(r_1\right),F_n\left(r_2\right)\right)\to_d\left(Y\left(r_1\right),Y\left(r_2\right)\right)$.

Precisely the same argument applies for $\left(F_n\left(r_1\right),...,F_n\left(r_k\right)\right)$, for any finite partition $0\leq r_1<r_2<...<r_k\leq 1$. Thus, the finite dimensional distributions of $\left(F_n\left(r\right)\right)$ converge to those of $\left(Y\left(r\right)\right)$ as required.

By Prohorov's Theorem, weak convergence follows if we show that $\left(F_n\left(r\right)\right)$ is tight, see Billingsley (1999, section 5). It follows from Billingsley (1999, problem 5.9) that we only need consider the marginal distribution of each coordinate $\left(F_{ab,n}\left(r\right)\right)$, and from Billingsley (1999, Theorem 13.5), this is tight if

$$E\left(\left|F_{ab,n}\left(r\right)-F_{ab,n}\left(r_1\right)\right|^2\left|F_{ab,n}\left(r_2\right)-F_{ab,n}\left(r\right)\right|^2\right)\leq K\left(r_2^{3-2H_a-2H_b}-r_1^{3-2H_a-2H_b}\right)^2 \tag{22}$$

for $0\leq r_1\leq r\leq r_2\leq 1$ and K finite, which recasts the problem into one similar to showing (13). Again, we apply the approximation in Lemma 4, i.e. $F_{ab,n}\left(r_1\right)-F_{ab,n}\left(r_2\right)=\sum_{t=1}^{n}\varepsilon_t'\sum_{s=1}^{t-1}c_{t-s,n}\left(r_2,r_1\right)\varepsilon_s$ with

$$\begin{aligned} c_{tn}\left(r_2,r_1\right) &= \frac{1}{2\pi n\sqrt{m}}\sum_{j=[mr_2]+1}^{[mr_1]}\theta_j\cos\left(t\lambda_j\right), \\ \theta_j &= \lambda_m^{H_a+H_b-1}\operatorname{Re}\left(A_a'\left(\lambda_j\right)\bar{A}_b\left(\lambda_j\right)+A_b'\left(\lambda_j\right)\bar{A}_a\left(\lambda_j\right)\right), \end{aligned}$$

and by Assumption 2 the left hand side of (22) is

$$\begin{aligned}
& E\left(\left|\sum_{t=1}^{n}\varepsilon_t'\sum_{s=1}^{t-1}c_{t-s,n}\left(r_1,r\right)\varepsilon_s\right|^2\left|\sum_{t=1}^{n}\varepsilon_t'\sum_{s=1}^{t-1}c_{t-s,n}\left(r,r_2\right)\varepsilon_s\right|^2\right) \\
\leq\; & C_1\sum_{t=1}^{n}\operatorname{tr}\left(\sum_{s=1}^{t-1}c_{t-s,n}\left(r,r_2\right)'c_{t-s,n}\left(r,r_2\right)c_{t-s,n}\left(r_1,r\right)'c_{t-s,n}\left(r_1,r\right)\right) && (23)\\
& +C_2\sum_{t=1}^{n}\operatorname{tr}\left(\sum_{s=1}^{t-1}\sum_{u=1}^{t-1}c_{t-s,n}\left(r,r_2\right)'c_{t-s,n}\left(r,r_2\right)c_{t-u,n}\left(r_1,r\right)'c_{t-u,n}\left(r_1,r\right)\right) && (24)
\end{aligned}$$

as in the proof of (13). Since the intervals $[r, r_2]$ and $[r_1, r]$ are disjoint, the property of independent increments implies that both (23) and (24) can be written as

$$C\left(r_2^{3-2H_a-2H_b}-r^{3-2H_a-2H_b}\right)\left(r^{3-2H_a-2H_b}-r_1^{3-2H_a-2H_b}\right),$$

by the same analysis as in the proof of Theorem 1. This is obviously not greater than the right hand side of (22). Since r, r_1, and r_2 are arbitrary, tightness follows, and we have shown the theorem.

6 Auxillary Lemmas

In this section we first prove the martingale approximation (10) in the proof of Theorem 1.

Lemma 4 *The approximation* (10) *holds under the assumptions of Theorem 1.*

Proof. We need to examine

$$\begin{aligned}
& \sum_{a=1}^{p}\sum_{b=1}^{p}\eta_{a(p-1)+b}\sqrt{m}\lambda_m^{H_a+H_b-2}\left(\frac{2\pi}{n}\sum_{j=1}^{[mr]}\operatorname{Re}\left(I_{ab}\left(\lambda_j\right)\right)-F_{ab}\left(\lambda_{mr}\right)\right)\\
=\; & \sum_{a=1}^{p}\sum_{b=1}^{p}\eta_{a(p-1)+b}\sqrt{m}\lambda_m^{H_a+H_b-2}\frac{2\pi}{n}\sum_{j=1}^{[mr]}\operatorname{Re}\left(I_{ab}\left(\lambda_j\right)-A_a\left(\lambda_j\right)J\left(\lambda_j\right)A_b^*\left(\lambda_j\right)\right) && (25)\\
& +\sum_{a=1}^{p}\sum_{b=1}^{p}\eta_{a(p-1)+b}\sqrt{m}\lambda_m^{H_a+H_b-2}\frac{2\pi}{n}\sum_{j=1}^{[mr]}\operatorname{Re}\left(A_a\left(\lambda_j\right)J\left(\lambda_j\right)A_b^*\left(\lambda_j\right)-f_{ab}\left(\lambda_j\right)\right) && (26)\\
& +\sum_{a=1}^{p}\sum_{b=1}^{p}\eta_{a(p-1)+b}\sqrt{m}\lambda_m^{H_a+H_b-2}\left(\frac{2\pi}{n}\sum_{j=1}^{[mr]}\operatorname{Re}\left(f_{ab}\left(\lambda_j\right)\right)-F_{ab}\left(\lambda_m\right)\right). && (27)
\end{aligned}$$

By (4.3) and (4.5) of Lobato (1997), (25) + (27) is $o_p\left(\sqrt{m}\left(m^{H_a+H_b-2}+\lambda_m^{\alpha}\right)\right)$, which is negligible by Assumption 4 and the condition that $H_a < 3/4$. Equation (26) can be written

$$\begin{aligned}
&\sum_{a=1}^{p}\sum_{b=1}^{p}\eta_{a(p-1)+b}\sqrt{m}\lambda_m^{H_a+H_b-2}\frac{2\pi}{n}\sum_{j=1}^{[mr]}\operatorname{Re}\left(A_a(\lambda_j)\frac{1}{2\pi n}\left|\sum_{t=1}^{n}\varepsilon_t e^{it\lambda_j}\right|^2 A_b^*(\lambda_j)-\frac{1}{2\pi}A_a(\lambda_j)A_b^*(\lambda_j)\right)\\
=&\sum_{a=1}^{p}\sum_{b=1}^{p}\eta_{a(p-1)+b}\sqrt{m}\lambda_m^{H_a+H_b-2}\frac{2\pi}{n}\sum_{j=1}^{[mr]}\operatorname{Re}\left(A_a(\lambda_j)\left(\frac{1}{n}\sum_{t=1}^{n}\varepsilon_t\varepsilon_t'-I_p\right)A_b^*(\lambda_j)\right) \quad (28)\\
&+\sum_{a=1}^{p}\sum_{b=1}^{p}\eta_{a(p-1)+b}\sqrt{m}\lambda_m^{H_a+H_b-2}\frac{2\pi}{n}\sum_{j=1}^{[mr]}\operatorname{Re}\left(A_a(\lambda_j)\frac{1}{2\pi n}\sum_{t=1}^{n}\sum_{s\neq t}\varepsilon_t\varepsilon_s' e^{i(t-s)\lambda_j}A_b^*(\lambda_j)\right). \quad (29)
\end{aligned}$$

Recalling that $(\varepsilon_t\varepsilon_t'-I_p)$ is a martingale difference sequence with respect to $(\mathcal{F}_t)_{t\in\mathbb{Z}}$, such that $n^{-1}\sum_{t=1}^{n}\varepsilon_t\varepsilon_t'-I_p=O_p(n^{-1/2})$, (28) is bounded by $\sup_{a,b}O_p(\sqrt{m}\lambda_m^{H_a+H_b-2}n^{-3/2}\sum_{j=1}^{m}|f_{ab}(\lambda_j)|)=O_p(\lambda_m^{1/2})\to_p 0$. Equation (29) is

$$\begin{aligned}
&\sum_{a=1}^{p}\sum_{b=1}^{p}\eta_{a(p-1)+b}\sqrt{m}\lambda_m^{H_a+H_b-2}\frac{1}{n^2}\sum_{j=1}^{[mr]}\operatorname{Re}\left(A_a(\lambda_j)\sum_{t=1}^{n}\sum_{s\neq t}\varepsilon_t\varepsilon_s' e^{i(t-s)\lambda_j}A_b^*(\lambda_j)\right)\\
=&\sum_{t=1}^{n}\varepsilon_t'\sum_{s=1}^{t-1}\sum_{a=1}^{p}\sum_{b=1}^{p}\eta_{a(p-1)+b}\frac{\sqrt{m}}{n^2}\lambda_m^{H_a+H_b-2}\sum_{j=1}^{[mr]}\operatorname{Re}\left(A_a'(\lambda_j)e^{i(t-s)\lambda_j}\bar{A}_b(\lambda_j)\right)\varepsilon_s\\
=&\sum_{t=1}^{n}\varepsilon_t'\sum_{s=1}^{t-1}c_{t-s,n}\varepsilon_s,
\end{aligned}$$

with $c_{tn}=\sum_{a=1}^{p}\sum_{b=1}^{p}\eta_{a(p-1)+b}\frac{\sqrt{m}}{n^2}\lambda_m^{H_a+H_b-2}\sum_{j=1}^{[mr]}\operatorname{Re}\left(A_a'(\lambda_j)e^{it\lambda_j}\bar{A}_b(\lambda_j)\right)$. Equation (10) follows.
■

Lemma 5 *Under the assumptions of Theorem 1,*

$$\sum_{t=1}^{n}\sum_{s=1}^{t-1}\sum_{u\neq s}\varepsilon_s' c_{t-s,n}' c_{t-u,n}\varepsilon_u=o_p(1).$$

Proof. We show convergence to zero in mean-square. The left-hand side has mean zero and variance

$$O\left(n\left(\sum_{s=1}^{n}\|c_{sn}\|^2\right)^2+\sum_{t=3}^{n}\sum_{u=2}^{t-1}\left(\sum_{s=1}^{u-1}\|c_{u-s,n}\|^2\sum_{s=1}^{u-1}\|c_{t-s,n}\|^2\right)\right), \quad (30)$$

following the analysis in Robinson (1995*a*, p. 1646). Since $\|\theta_j\|=O(1)$ by construction,

$$\|c_{sn}\|=O\left(\frac{1}{n\sqrt{m}}\sum_{j=1}^{m}\|\theta_j\|\right)=O\left(\frac{\sqrt{m}}{n}\right).$$

Since $\sum_{j=1}^{k} |\cos(s\lambda_j)| = O(n/s)$, another bound is

$$\|c_{sn}\| = O\left(\frac{1}{n\sqrt{m}} \sum_{j=1}^{m} \|\theta_j\| \, |\cos(s\lambda_j)|\right) = O\left(\frac{1}{s\sqrt{m}}\right).$$

This is a better bound for $\|c_{sn}\|$ when $s > n/m$. Thus, we find that

$$\begin{aligned} \sum_{s=1}^{n} \|c_{sn}\|^2 &= O\left(\sum_{s=1}^{[n/m]} \frac{m}{n^2} + \sum_{s=[n/m]+1}^{n} \frac{1}{s^2 m}\right) \\ &= O\left(n^{-1}\right), \end{aligned}$$

implying that the first term of (30) is $O\left(n^{-1}\right)$. The second term of (30) is bounded by

$$O\left(n\left(\sum_{s=1}^{n} \|c_{sn}\|^2\right)\left(\sum_{s=1}^{[n/2]} s\,\|c_{sn}\|^2\right)\right),$$

see Robinson (1995*a*, p. 1646-1647). The summand in the second sum is $O\left(sm/n^2 + (sm)^{-1}\right)$. Choosing the first bound when $s \le \left[n/m^{2/3}\right]$, the second sum is

$$O\left(\sum_{s=1}^{\left[n/m^{2/3}\right]} \frac{sm}{n^2} + \sum_{s=\left[n/m^{2/3}\right]+1}^{[n/2]} \frac{1}{sm}\right) = O\left(\frac{1}{m^{1/3}}\right),$$

and (30) is $O(n^{-1} + m^{-1/3})$. ∎

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