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Spectral Analysis of Fractionally Cointegrated Systems

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Abstract

Cointegration imposes restrictions on the frequency domain behavior of a time series at the zero-frequency. We derive these restrictions for a multivariate fractionally cointegrated system. In particular, we consider a p-vector time series integrated of order d with r cointegrating relations, given by the rows of $[I_r; \beta']$, where the cointegration errors are integrated of order $d-b$, $d \geq b > 0$. We show that, at the zero-frequency, the spectral density matrix of the d'th differenced series has reduced rank $(p-r)$, the coherence and phase measures (multiple and partial) equal unity and zero, respectively, and the gain is the matrix of cointegrating coefficients. Extensions to noncontemporaneous cointegration, seasonal cointegration, and different fractional values of b for each cointegrating relation are considered.

JEL Classification: C32.

Keywords: Common stochastic trend; fractional cointegration; frequency domain analysis; reduced rank; zero-frequency.

1 Introduction

The concept of cointegration introduced by Granger (1981) has become a standard tool in empirical modelling and estimation of relationships among different variates especially in finance and macroeconomics. Cointegration is the existence of one or more long-run relations among a group of (integrated) variates. Equivalently, these variates are said to share a common stochastic trend, and thus move together in the long run even though they may drift apart in the short run. The vast amount of economic theory predicting such long-run co-movement or long-run equilibria has made cointegration testing and estimation part of any empirical researcher's toolkit.

Since cointegration is a long-run property, it refers to the zero-frequency relationship between two or more series in the frequency domain. Levy (2002) has examined the frequency domain implications of cointegration in a simple setup and has shown that the existence of cointegration between two time series in the time domain imposes restrictions on the series zero-frequency behavior in terms of the squared coherence, phase, and gain. The squared coherence, phase, and gain can be easily interpreted as frequency domain equivalents of correlation coefficient, time-delay (lag), and regression coefficient, respectively. In particular, Levy (2002) considers a bivariate time series integrated of order one (i.e. possessing a unit root in the time domain autoregressive representation, denoted $I(1)$) where the linear combination $[1; \beta]$ makes the resulting series integrated of order zero, $I(0)$. He then shows that the squared coherence between the once-differenced series will equal one, the phase-shift between them will equal zero, and the gain will equal $|\beta|$.

However, Levy (2002) confines his analysis to the bivariate case and extensions to multiple time series exhibiting multiple cointegrating relations seem important, both from a theoretical and an empirical point of view. Furthermore, only variates integrated of order one and cointegrating to order zero (standard cointegration) is considered. Following the original idea by Granger (1981), a natural generalization of the cointegration concept is to assume that the raw series are integrated of a fractional order d, see also Granger & Joyeux (1980) and Hosking (1981), and that certain linear combinations are integrated of a smaller fractional order $d - b$ with $d \geq b > 0$ real numbers. Clearly, this allows the study of co-movement among persistent series much more generally than in the standard $I(1) - I(0)$ cointegration case. As it turns out, the fractional cointegration case is well suited for frequency domain analysis with properties analogous to those found in the standard cointegration case.

In the present paper the analysis in Levy (2002) is extended in several directions. (i) We consider a general multivariate process with potentially more than one cointegrating relation. (ii) The multivariate time series in question is allowed to be fractionally integrated and fractionally cointegrated, i.e. both d

and b can take non-integer values. (iii) The spectral density matrix of the d'th differenced variates at the zero-frequency (G) is shown to have reduced rank. (iv) In our multivariate setup more spectral measures become relevant, e.g. partial coherence and partial phase measures, which we include in our analysis. (v) The dependence on b of some of these spectral measures, and in particular that of the eigenvalues of G, is analyzed in terms of their rates of convergence as a function of b. (vi) We analyze the case of noncontemporaneous (fractional) cointegration, where lagged variates may appear in the cointegrating relation, and examine the influence of the timing of the cointegrating variates on the spectral properties of the d'th differenced variates.

The paper proceeds as follows. In the next section we set up the model of fractional cointegration and derive the spectral properties at the zero-frequency. The spectral density matrix of the d'th differenced variates at the zero-frequency is derived and several measures of coherence, phase, and gain are derived and analyzed within the model. Extensions to noncontemporaneous cointegration, seasonal cointegration, and different fractional values of b for each cointegrating relation are also considered. Section 3 offers a brief discussion of our results and some implications.

2 Fractional Cointegration in the Frequency Domain

Consider the triangular system

$$X_{1t} = \beta' X_{2t} + Z_t, \tag{1}$$

$$\Delta^{d-b} Z_t = u_{1t}\mathbb{I}\left(t \geq 1\right), \tag{2}$$

$$\Delta^{d} X_{2t} = u_{2t}\mathbb{I}\left(t \geq 1\right), \tag{3}$$

where $\mathbb{I}\left(\cdot\right)$ denotes the indicator function and the fractional difference operator $\Delta^d = (1-L)^d$ is defined by its binomial expansion

$$(1-L)^d = \sum_{j=0}^{\infty} \frac{\Gamma\left(j-d\right)}{\Gamma\left(-d\right)\Gamma\left(j+1\right)} L^j, \qquad \Gamma\left(z\right) = \int_0^{\infty} t^{z-1} e^{-t} dt,$$

in the lag operator L ($Lx_t = x_{t-1}$), see Granger & Joyeux (1980) or Hosking (1981) for the details. With this definition, Z_t and X_{2t} are well defined for all d and b and, in particular, they are type II fractionally integrated processes, see Marinucci & Robinson (1999).

We assume that X_{1t} is $r \times 1$ and X_{2t} is $(p-r) \times 1$, with $p > r > 0$ integers, such that the system is generated by $p-r$ common stochastic trends (X_{2t}) and exhibit r cointegrating relations (Z_t). The cointegration vectors are thus given by the rows of $[I_r; -\beta']$. In the standard cointegration case, $d = b = 1$, the triangular model (1)-(3) was studied by e.g. Phillips (1991a).

We assume that the innovations $u_t = (u'_{1t}, u'_{2t})'$ are generated by the linear process

$$u_t = C(L)\,\varepsilon_t = \sum_{j=0}^{\infty} C_j \varepsilon_{t-j}, \qquad \sum_{j=0}^{\infty} \operatorname{tr}\left(C_j C_j'\right) < \infty, \tag{4}$$

where ε_t is an uncorrelated process with $E(\varepsilon_t) = 0$ and $E(\varepsilon_t \varepsilon_t')$ equal to a positive definite matrix, which can thus be chosen, without loss of generality, to be the identity matrix. The spectral density matrix of ε_t is uniform, $f_\varepsilon(\lambda) = I_p/(2\pi)$, and under (4), u_t is a covariance stationary process with spectral density matrix $f_u(\lambda) = C\left(e^{-i\lambda}\right) C\left(e^{-i\lambda}\right)^* / (2\pi)$, see Priestley (1981, p. 671). The asterisk indicates complex conjugation combined with transposition. We denote $f_u(0)$ (also known as the long-run covariance variance matrix of u_t) by $\Omega = C(1)\,C(1)'/(2\pi)$, which is real, symmetric, and non-negative definite.

Applying the fractional difference operator to $X_t = (X'_{1t}, X'_{2t})'$ and defining the d'th differenced variates $x_t = \Delta^d X_t$, we obtain a system of equations in differences,

$$x_t = \begin{bmatrix} \Delta^b I_r & \beta' \\ 0 & I_{p-r} \end{bmatrix} u_t \mathbb{I}(t \geq 1), \tag{5}$$

with spectral density matrix, Priestley (1981, p. 671),

$$f(\lambda) = \begin{bmatrix} \left(1 - e^{-i\lambda}\right)^b I_r & \beta' \\ 0 & I_{p-r} \end{bmatrix} f_u(\lambda) \begin{bmatrix} \left(1 - e^{-i\lambda}\right)^b I_r & \beta' \\ 0 & I_{p-r} \end{bmatrix}^*. \tag{6}$$

Separating the components of the spectral density matrix important for the zero-frequency, we can rewrite (6) as

$$\begin{aligned} f(\lambda) &= \begin{bmatrix} \beta' f_{22}(\lambda)\beta & \beta' f_{22}(\lambda) \\ f_{22}(\lambda)\beta & f_{22}(\lambda) \end{bmatrix} \\ &+ \begin{bmatrix} \left(1 - e^{i\lambda}\right)^b \beta' f_{21}(\lambda) + \left(1 - e^{-i\lambda}\right)^b f_{12}(\lambda)\beta + \left|1 - e^{-i\lambda}\right|^{2b} f_{11}(\lambda) & \left(1 - e^{-i\lambda}\right)^b f_{12}(\lambda) \\ \left(1 - e^{i\lambda}\right)^b f_{21}(\lambda) & 0 \end{bmatrix}, \end{aligned} \tag{7}$$

where $f_{ab}(\lambda)$ is the (a, b)'th block of $f_u(\lambda)$, $a, b = 1, 2$.

Since

$$\begin{aligned} \left|1 - e^{-i\lambda}\right|^{2b} &= 2^b (1 - \cos\lambda)^b \\ &= O(\lambda^{2b}) \qquad \text{as } \lambda \to 0^+, \end{aligned}$$

the second term of (7) is $O(\lambda^b)$. Thus, in the vicinity of the origin, (6) reduces to

$$f(\lambda) = \begin{bmatrix} \beta'\Omega_{22}\beta & \beta'\Omega_{22} \\ \Omega_{22}\beta & \Omega_{22} \end{bmatrix} \left(1 + O\left(\lambda^b\right)\right) \qquad \text{as } \lambda \to 0^+. \tag{8}$$

The spectral density matrix of x_t at the zero-frequency, denoted G, is consequently given by

$$G = \begin{bmatrix} \beta' \\ I_{p-r} \end{bmatrix} \Omega_{22} \begin{bmatrix} \beta & I_{p-r} \end{bmatrix}, \tag{9}$$

which has rank $p - r$, see also Robinson & Yajima (2002, Theorem 1).

From (8) we note that the rank deficiency of G depends on the reduction of the integration order implied by the cointegration property. The rank properties can be restated in terms of the eigenvalues of $f(\lambda)$ as $\lambda \to 0^+$, offering a more precise statement. Thus, as $\lambda \to 0^+$, $f(\lambda)$ has $p - r$ non-zero (i.e. $O(1)$) eigenvalues and r eigenvalues of order $O(\lambda^b)$.

In the remainder of this section we examine the implications of this result for several coherence, phase, and gain measures from multivariate spectral analysis, most of which are defined in e.g. Brillinger (1981, chap. 8) or Priestley (1981, chap. 9).

The matrix squared coherence, Koopmans (1974, chap. 5.6), between x_1 and x_2 at frequency λ is defined as

$$\mathcal{K}^2_{x_1x_2}(\lambda) = f^{-1/2}_{x_1}(\lambda)\, f_{x_1x_2}(\lambda)\, f^{-1}_{x_2}(\lambda)\, f_{x_2x_1}(\lambda)\, f^{-1/2}_{x_1}(\lambda)$$

and indicates the extent of linear dependence between x_1 and x_2 at frequency λ. Usually, this is measured in terms of the eigenvalues of $\mathcal{K}^2_{x_1x_2}(\lambda)$ which are all between zero and one. Thus, complete linear dependence of x_1 on x_2 at frequency λ corresponds to the extreme case where all eigenvalues equal unity. On the other hand, if all eigenvalues equal zero no linear relationship exists between x_1 and x_2 at frequency λ. For the present model we get that, at the zero-frequency,

$$\mathcal{K}^2_{x_1x_2}(0) = \left(\beta'\Omega_{22}\beta\right)^{-1/2} \beta'\Omega_{22}\beta \left(\beta'\Omega_{22}\beta\right)^{-1/2} = I_r$$

and its eigenvalues all equal unity.

The multiple coherence, Priestley (1981, chap. 9.3), between a scalar variate x_1 and a (possibly vector-valued) variate x_2 at frequency λ is

$$\mathcal{K}^m_{x_1x_2}(\lambda) = \frac{f_{x_1x_2}(\lambda)\, f^{-1}_{x_2}(\lambda)\, f_{x_1x_2}(\lambda)^*}{f_{x_1}(\lambda)}.$$

It is comparable to the multiple coefficient of determination (R^2) in time domain regression analysis or the multiple correlation coefficient in analysis of variance, and the interpretation of multiple coherence is the same as that of the R^2-statistic for each frequency λ. Thus, we evaluate the multiple coherence between x_{1a} and x_2, i.e. the a'th equation of (5), at the zero-frequency,

$$\mathcal{K}^m_{x_{1a}x_2}(0) = \frac{\beta'_a\Omega_{22}\Omega^{-1}_{22}\Omega_{22}\beta_a}{\beta'_a\Omega_{22}\beta_a} = 1,$$

where β_a is the a'th column of β.

Next, we consider the partial coherence between the a'th component of x_1 and the b'th component of x_2 while controlling for the common influence of the remaining variates in x_2, denoted $x_{2(-b)}$, Priestley (1981, chap. 9.3). The partial coherence is an obvious spectral analogue (at frequency λ) of a partial correlation coefficient or a partial R^2-statistic. Since the multiple coherence equals unity, all the partial coherences also equal unity. To see this, recall that the partial coherence between x_{1a} and x_{2b} controlling for $x_{2(-b)}$, denoted $\mathcal{K}^p_{x_{1a}x_{2b}\cdot x_{2(-b)}}(\lambda)$, can be written as a function of $\mathcal{K}^m$,

$$\mathcal{K}^p_{x_{1a}x_{2b}\cdot x_{2(-b)}}(\lambda) = \sqrt{\frac{\mathcal{K}^m_{x_{1a}x_2}(\lambda) - \mathcal{K}^m_{x_{1a}x_{2(-b)}}(\lambda)}{1 - \mathcal{K}^m_{x_{1a}x_{2(-b)}}(\lambda)}},$$

which equals unity at $\lambda = 0$ since $\mathcal{K}^m_{x_{1a}x_2}(0) = 1$.

Hence, all the coherence measures at the zero-frequency equal unity, supporting the interpretation of (fractional) cointegration as a type of long-run multicollinearity or long-run equilibrium.

From the transfer function

$$B_{x_1x_2}(\lambda) = f_{x_1x_2}(\lambda) f^{-1}_{x_2}(\lambda)$$

one can derive the phase shift and gain as, Brillinger (1981, p. 307),

$$\begin{aligned}\Phi_{x_1x_2}(\lambda) &= \arg(B_{x_1x_2}(\lambda)) \\ \Gamma_{x_1x_2}(\lambda) &= |B_{x_1x_2}(\lambda)|,\end{aligned}$$

where $\arg(A)$ and $|A|$ returns the argument and the modulus, respectively, element-by-element of the complex-valued matrix A. In the simple scalar case, $b_{x_1x_2}(\lambda) = \gamma_{x_1x_2}(\lambda) e^{i\phi_{x_1x_2}(\lambda)}$, the gain measures the amplification of the spectral density of x_2 to approximate that of x_1 at frequency λ and the phase measures the phase shift (lead-lag relationship) between x_1 and x_2. In the multivariate case the interpretation of gain is like that of a regression coefficient, see Brillinger (1981, p. 307).

Thus, from (8) we get that

$$\begin{aligned}B_{x_1x_2}(\lambda) &= \beta'\Omega_{22}\Omega_{22}^{-1}\left(1 + O\left(\lambda^b\right)\right) \\ &= \beta'\left(1 + O\left(\lambda^b\right)\right) \qquad \text{as } \lambda \to 0^+.\end{aligned}$$

It is apparent, since $B_{x_1x_2}(\lambda)$ is real-valued up to an additive term of order $O(\lambda^b)$ as $\lambda \to 0^+$, that the phase shift as $\lambda \to 0^+$ is at most of order $O(\lambda^b)$, i.e.

$$\Phi_{x_1x_2}(\lambda) = O(\lambda^b) \qquad \text{as } \lambda \to 0^+,$$

and the gain is

$$\Gamma_{x_1x_2}(\lambda) = \left|\beta'\left(1+O\left(\lambda^b\right)\right)\right| \qquad \text{as } \lambda \to 0^+.$$

Similarly, the partial phase shifts, Priestley (1981, chap. 9.3), all vanish at the zero-frequency. Recall that the partial cross-spectral density between x_{1a} and x_{2b} allowing for $x_{2(-b)}$ is

$$f_{x_{1a}x_{2b}\cdot x_{2(-b)}}(\lambda) = f_{x_{1a}x_{2b}}(\lambda) - f_{x_{1a}x_{2(-b)}} f^{-1}_{x_{2(-b)}}(\lambda)\, f_{x_{2(-b)}x_{2b}}(\lambda)$$

and the partial phase shift of x_{1a} relative to x_{2b} allowing for $x_{2(-b)}$ is

$$\Phi^p_{x_{1a}x_{2b}\cdot x_{2(-b)}}(\lambda) = \arg\left(f_{x_{1a}x_{2b}\cdot x_{2(-b)}}(\lambda)\right).$$

Since $f_{x_{1a}x_{2b}\cdot x_{2(-b)}}(0)$ is real-valued, the partial phase shifts at $\lambda = 0$ are zero and have rates of convergence

$$\Phi^p_{x_{1a}x_{2b}\cdot x_{2(-b)}}(\lambda) = O(\lambda^b) \qquad \text{as } \lambda \to 0^+,$$

for all a, b, as above.

Setting $p = 2, r = 1$, and $d = b = 1$ in (1)-(3), we obtain the bivariate, standard $I(1) - I(0)$ cointegrated system in Levy (2002), and all his results appear as the corresponding special cases of our results in the more general model (1)-(3).

The analysis so far has been restricted to contemporaneous cointegration, where the components of X_t only enter the cointegrating relation at time t, i.e. no lagged variates appear. The noncontemporaneous case can be analyzed in the same way. Replace (1) by

$$X_{1t} = \beta' X_{2,t-q} + Z_t, \tag{10}$$

and suppose (2)-(3) continue to hold. Then,

$$x_t = \Delta^d X_t = \begin{bmatrix} \Delta^b I_r & \beta' L^q \\ 0 & I_{p-r} \end{bmatrix} u_t \mathbb{I}(t \geq 1)$$

with spectral density matrix given by (apart from terms irrelevant to the zero-frequency)

$$\begin{aligned} f(\lambda) &= \begin{bmatrix} \beta'\Omega_{22}\beta & e^{-iq\lambda}\beta'\Omega_{22} \\ \Omega_{22}\beta e^{iq\lambda} & \Omega_{22} \end{bmatrix} \left(1+O(\lambda^b)\right) \\ &= \begin{bmatrix} \beta'\Omega_{22}\beta & \beta'\Omega_{22} \\ \Omega_{22}\beta & \Omega_{22} \end{bmatrix} \left(1+O(\lambda)+O(\lambda^b)\right) \qquad \text{as } \lambda \to 0^+, \end{aligned}$$

since $e^{iq\lambda} = 1 + O(\lambda)$ for $q \neq 0$ and $\lambda \to 0^+$.

Thus, as $\lambda \to 0^+$, $f(\lambda)$ has $p-r$ non-zero eigenvalues and r eigenvalues of order $O(\lambda^{\min(1,b)})$ and the timing of the variates in the cointegrating relation only matters when $b > 1$. In the time domain, the same result can be seen by considering $X_{1t} - \beta' X_{2t} = X_{1t} - \beta' X_{2,t-q} - \beta' \sum_{j=0}^{q-1} \Delta X_{2,t-j}$, which is $I(\max(d-b, d-1))$ by (10). Hence, $X_{1t} - \beta' X_{2t}$ is $I(d-b)$ for $b \leq 1$, in which case the timing does not interfere with the cointegration properties, but is $I(d-1)$ for $b > 1$, turning $X_{1t} - \beta' X_{2t} + \beta' \sum_{j=0}^{q-1} \Delta X_{2,t-j}$ into a polynomial cointegrating relation. Returning to the frequency domain, the introduction of lagged variates in the time domain naturally causes a change in the phase shift. As $\lambda \to 0^+$, the transfer function is $B_{x_1x_2}(\lambda) = e^{-iq\lambda}\beta'(1+O(\lambda^b))$ with phase shift $\Phi_{x_1x_2}(\lambda) = q\lambda + O(\lambda^b)$. Thus, the phase shift tends to zero at the same rate as before when $b \leq 1$, but at a slower rate when $b > 1$.

To complete the study of the system (1)-(3) we indicate briefly how the analysis can be generalized to seasonal fractional cointegration and to different values of b for each component of Z_t, and the implications of such extensions. First, the extension to seasonal fractional cointegration is straightforward. If we replace all difference operators by the seasonal difference operator, $\Delta_s = (1 - L^s)$, where s is the number of seasons (e.g. $s = 4$ for quarterly data or $s = 7$ for weekly data), the results of the paper carry through virtually unchanged. The frequency domain results hold as stated when the limits are taken as $\lambda \to \lambda_h = 2\pi h/s, h = 0, ..., s$, and in the noncontemporaneous case the results remain unchanged if the lag is changed to qs. More generally, assume the data are fractionally integrated and cointegrated at a finite set of frequencies Λ. In this case the frequency domain results hold as $\lambda \to \bar{\lambda}$, for any $\bar{\lambda} \in \Lambda$. Second, suppose the integration orders of Z_t are $d-b_1, ..., d-b_r$. Then (5) is replaced by

$$x_t = \begin{bmatrix} \Delta(d-b_1, ..., d-b_r) & \beta' \\ 0 & I_{p-r} \end{bmatrix} u_t \mathbb{I}(t \geq 1),$$

where $\Delta(d-b_1, ..., d-b_r) = \text{diag}\left(\Delta^{d-b_1}, ..., \Delta^{d-b_r}\right)$. From this point, the analysis proceeds as above. As $\lambda \to 0^+$, $f(\lambda)$ has $p-r$ non-zero eigenvalues and r eigenvalues of orders $O(\lambda^{b_1}), ..., O(\lambda^{b_r})$ and coherence, phase shift, and gain follow. Similar extensions are possible for d.

3 Discussion

Cointegration imposes restrictions on the frequency domain behavior of a time series at the zero-frequency. We have examined these restrictions for a p-vector time series integrated of order d with r cointegrating relations, given by the rows of $\left[I_r; \beta'\right]$, where the cointegration errors are integrated of order $d-b$, $d \geq b > 0$. It was shown that, at the zero-frequency, the spectral density matrix of the d'th differenced series has reduced rank $(p - r)$, the coherence and phase measures (multiple and partial) equal unity and zero, respectively, and the gain is the matrix of cointegrating coefficients.

Some concluding remarks are in order. First, the analysis of fractional cointegration fits neatly in the frequency domain where it offers a natural generalization of standard cointegration. On the other hand, the time domain representations of fractionally cointegrated systems are often more cumbersome, and starting from a triangular system such as (1)-(3) it does not appear straightforward to derive, e.g., the reduced rank structures or the error correction representation.

Second, the reduced rank conditions on the spectral density matrix of the d'th differenced variates at the zero-frequency (G) are a natural starting point for establishing methods of determining the cointegrating rank in the frequency domain. In particular, tests based on the eigenvalues of G could be employed, following Phillips & Ouliaris (1988) for the standard cointegration case and Robinson & Yajima (2002) for the $d < 1/2$ case, and are currently under investigation by the author.

Third, methods for estimating spectral density matrices are well known and widely applied. Based on the results in the previous section such methods may be used to estimate the cointegration vector(s), as examined by Phillips (1991*b*) in the standard cointegration case, or to test for (fractional) cointegration through estimation of coherence measures. However, further research is needed to examine the properties of such methods, both theoretical and in practice.

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