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Range-based covariance estimation with a view to foreign exchange rates

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Abstract

An estimator of the covariance, termed **co-range**, is proposed. It is based on high and low prices, in contrast to conventional covariance estimators based on open/close prices. The main properties of the new estimator are derived. The co-range appears to be superior to the conventional covariance estimator based on open/close prices in terms of mean squared error and mean absolute deviation. Assuming triangular arbitrage in the foreign exchange market, the co-range may be computed from daily high and low prices.

Keywords: Co-range, foreign exchange rates, price range, high and low prices, covariance estimation.

JEL Classification: C32, C49, G15.

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1 Introduction

The difference between the daily high and low of the log price, the price range, has been used in the academic literature to measure volatility, see e.g. Alizadeh, Brandt & Diebold (2002), Yang & Zhang (2000) and Beckers (1983) among others. Several advantages of the price range as a volatility proxy have been proposed. First of all, data on the price range are available for a variety of assets and dates a long way back. Second, under certain assumptions, it is more efficient than the squared daily return, see Parkinson (1980). Finally, it appears to be robust to some kinds of market microstructure effects, see Alizadeh et al. (2002).

The basic idea of the range boils down to measuring volatility by a combination of extremum estimators (the difference between the high and the low). In this paper, we propose the co-range as a proxy for the covariance between the returns on two assets. The co-range is similar to the price range in the sense that it makes use of "the difference between the high and the low". But, in contrast to the price-range, high-frequency or tick-by-tick data are required for the computation of the co-range in most cases. However, using triangular arbitrage in the foreign exchange market, the co-range may be readily constructed from individual price ranges[1].

By simulations, we show that the co-range does provide a superior estimator of the covariance, compared to conventional estimators based on open/close prices.

In section 2, we give an introduction to the area of range-based volatility estimation and provide a brief overview of the literature. Section 3 presents the co-range in a general setting, and section 4 presents the co-range in the context of foreign exchange rates. The performance of the co-range is explored by simulations in section 5. Section 6 shows how the model underlying the range-based volatility estimators may be extended to a more realistic setting with dependent returns. Section 7 concludes. Appendix is in section 8.

2 Range-based volatility estimation

The theoretical framework employed in this paper draws on Feller (1951) and Parkinson (1980). In Feller (1951), the density function of the range of a Brownian Motion is derived and an unbiased estimator of the diffusion constant is proposed. Parkinson (1980) develops this result to estimate the diffusion constant for a financial asset.

More formally, it is assumed that the $\ln(\cdot)$ of the asset price follows an arithmetic Brownian Motion with zero drift and volatility σ_P

$$dP = \sigma_P dW \tag{1}$$

$P(t)$ is the natural logarithm of the asset price, $W(t)$ is a standard Wiener process. Equation (1) describes the evolution of the log-price within a time interval, $0 \leq t \leq T$, of length T. The range can be expressed as

$$l_P = \left(\sup_{0 \leq t \leq T} P(t) - \inf_{0 \leq t \leq T} P(t) \right) \tag{2}$$

We adopt the notation that $\sup_{0 \leq t \leq T} P(t)$ refers to supremum of the log-price process over the interval, $0 \leq t \leq T$, and $\inf_{0 \leq t \leq T} P(t)$ refers to infimum of the log-price process over the

[1] We are indebted with Frank Diebold for this suggestion.

same interval. In this setup, the range represents the difference between the natural logarithm of the daily high and low prices. Parkinson (1980), drawing on the results of Feller (1951), proves that

$$E\left(\frac{1}{T}\frac{l_P^2}{4\ln(2)}\right) = E\left(\widehat{\sigma}_P^2\right) = \sigma_P^2 \tag{3}$$

$$E\left(\frac{1}{\sqrt{T}}\sqrt{\frac{\pi}{8}}l_P\right) = E\left(\widehat{\sigma}_P\right) = \sigma_P \tag{4}$$

Hence, properly scaled versions of the squared range and the range, provides unbiased estimators of the standard deviation and the variance of the price process, respectively. Parkinson (1980) compares the high/low estimator to the conventional estimator based on open/close prices

$$\widetilde{\sigma}_P^2 = \frac{(P(T) - P(0))^2}{T},$$

and he shows that the relative efficiency of the estimators is

$$\frac{MSE(\widehat{\sigma}_P^2)}{MSE(\widetilde{\sigma}_P^2)} = 0.20367 \tag{5}$$

Combinations of the high/low estimator with the open/close estimator have been proposed by Garman & Klass (1980), Ball & Tourus (1984), and Yang & Zhang (2000).

The range-based volatility estimator (3) is biased in the presence of a non-zero drift term in the Brownian Motion for dP. Rogers & Satchell (1991), Kunitomo (1992), and Yang & Zhang (2000) introduce range-based volatility estimators in the presence of a drift term.

In practice, it is not possible to extract the true $\sup_{0\leq t\leq T} P(t)$ and $\inf_{0\leq t\leq T} P(t)$, due to discrete price observations, and the range adopted in empirical applications of (3) and (4) is

$$l_P = \max_{t\in\{0,1,2,\ldots T\}} P(t) - \min_{t\in\{0,1,2,\ldots,T\}} P(t)$$

Because

$$\sup_{0\leq t\leq T} P(t) \geq \max_{t\in\{0,1,2,\ldots T\}} P(t) \qquad \text{and} \qquad \inf_{0\leq t\leq T} P(t) \leq \min_{t\in\{0,1,2,\ldots,T\}} P(t)$$

it follows that, in practice, (3) is a downward biased estimator for σ_P^2, see Beckers (1983). Adjustments for this discretization bias have been suggested, see Rogers & Satchell (1991) and Rogers (1998).

We would like to emphasize that the constant volatility model for the log price in (1) is not, in general, an appropriate model for asset prices. In fact, there seems to be sound empirical and theoretical evidence supporting a non-constant or time-varying volatility of asset returns, see e.g. Andersen & Bollerslev (1997). However, it is not clear how frequently the volatility changes, but a fundamental assumption of this paper is that the variance-covariance matrix of asset returns is constant over some finite period of time, defined as the interval $0 \leq t \leq T$. We think of the interval as one trading day. Several theoretical and empirical volatility models adopt this assumption, and some stochastic evolution of the diffusion constant in (1) may be postulated to make the volatility time-varying, see e.g. Taylor (1986). Recently, Alizadeh et al. (2002) used the daily price range to estimate a stochastic volatility model where the volatility during the day is constant but the daily volatility evolves stochastically from day to day.

Data on the price range is available for many assets and for a long period of time. Combined with Parkinson (1980)'s result on the relative efficiency of the price range relative to the squared daily return, this has made the price range an attractive volatility estimator. However, compared to the realized volatility estimator proposed by Andersen, Bollerslev, Diebold & Labys (2001a), the price range is an inefficient estimator of the volatility under the assumption of a Brownian Motion process, see Andersen & Bollerslev (1998, p. 898, note 20). Hence if we have access to high-frequency data and if the Brownian motion is the maintained model, then the realized volatility estimator is superior to alternative estimators.

In contrast, the ranking of the volatility estimators is unknown in the presence of market microstructure effects. Recent work look into the potential microstructure-distortions of realized volatility estimators, see Andersen, Bollerslev, Diebold & Labys (2000), Andreou & Ghysels (2001) and Bai, Russell & Tiao (2000). It is far from clear which volatility estimator should be used in the presence of microstructure noise, because the problem is difficult to handle theoretically and because micro structure noise is not well-defined.

In related work, Brandt & Diebold (2001) develop multivariate range-based variance and covariance estimators and compare them to realized volatility estimates in the presence of market micro structure noise.

3 Co-range: General case

In contrast to the conventional setup in the literature on the price range, we employ a multivariate framework. We consider two assets, where the log of the asset prices follow a bivariate Brownian Motion[2], and we allow for the possibility that asset returns are correlated.

$$dP = \sigma_P dW \tag{6}$$

$$dQ = \sigma_Q dZ \tag{7}$$

$$E\left(\begin{bmatrix} dW \\ dZ \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{8}$$

$$E\left(\begin{bmatrix} (dW)^2 & dW dZ \\ dW dZ & (dZ)^2 \end{bmatrix}\right) = \begin{bmatrix} dt & \rho_{P,Q} dt \\ \rho_{P,Q} dt & dt \end{bmatrix} \qquad \sigma_{P,Q} = \sigma_P \cdot \sigma_Q \cdot \rho_{P,Q} \tag{9}$$

P and Q denote log-prices of asset "P" and "Q", respectively, in terms of the same numeraire, e.g. US Dollars. Hence we can interpret dP and dQ as continuously compounded returns. The parameters of the model above, σ_P, σ_Q and $\rho_{P,Q}$, stay constant during the trading day, but may vary from day to day[3].

[2]In the set up of the paper, we are assuming a zero drift in the Brownian Motions. Moreover, we do not introduce any correction for the discretization bias. The motivation for this choise is twofold. Firstly, the notation and the computation of the co-range is more tractable and perhaps more elegant when assuming a zero drift. Secondly, for many financial assets only daily high, low and close prices are available. Therefore, it is not possible to introduce any correction for the drift. See Rogers & Satchell (1991), Kunitomo (1992), and Yang & Zhang (2000).

[3]For simplicity assume that the close price on day $d-1$ is equal to the open price on day d.

Suppose we would like to estimate the variance-covariance matrix between the returns on "P" and "Q" on a daily basis over the last years. Many databases contain intradaily high and low prices on financial assets in addition to open and/or close prices. Conventional estimates based on open and close prices are given by

$$\widetilde{\sigma}_P^2 = \frac{(P(T) - P(0))^2}{T} \qquad \widetilde{\sigma}_Q^2 = \frac{(Q(T) - Q(0))^2}{T} \tag{10}$$

$$\widetilde{\sigma}_{P,Q} = \frac{(P(T) - P(0))(Q(T) - Q(0))}{T} \tag{11}$$

where $\widetilde{}$ denotes estimates based on open/close prices. For the model above these estimates correspond to maximum likelihood estimates.

In contrast, estimates of the variances based on high and low prices, denoted by $\widehat{}$, are given as

$$l_P = \left(\sup_{0 \leq t \leq T} P(t) - \inf_{0 \leq t \leq T} P(t) \right)$$

$$\widehat{\sigma}_P^2 = \frac{1}{T} \frac{l_P^2}{4 \ln(2)} \qquad \widehat{\sigma}_Q^2 = \frac{1}{T} \frac{l_Q^2}{4 \ln(2)} \tag{12}$$

see Parkinson (1980). l_P, the difference between the high and low log-price, corresponds to the (log)-price range. To our knowledge, no range-based estimates have been proposed to estimate $\sigma_{P,Q}$. The contribution of this paper is to propose a range-based estimator of $\sigma_{P,Q}$, which is superior to the conventional covariance estimator based on open/close prices, $\widetilde{\sigma}_{P,Q}$.

Assume that we observe continuous[4] sample paths of P and Q on an interval, $0 \leq t \leq T$, of length T. The variance-covariance matrix of asset returns is defined as

$$\Omega = E\left(\begin{bmatrix} (dP)^2 & dPdQ \\ dPdQ & (dQ)^2 \end{bmatrix} \right) = \begin{bmatrix} \sigma_P^2 & \sigma_{P,Q} \\ \sigma_{P,Q} & \sigma_Q^2 \end{bmatrix} dt$$

The following theorem proposes a new estimator of the covariance, termed co-range.

Theorem 1 Assume that two continuous time processes P and Q are governed by the processes from (6)-(9). The processes are observed on an interval $0 \leq t \leq T$, of length T. Choose a beta-vector

$$(\beta_P, \beta_Q) \qquad \beta_P \in \mathrm{R} \backslash \{0\}, \quad \beta_Q \in \mathrm{R} \backslash \{0\}.$$

Define the co-range, with the associated beta-vector (β_P, β_Q), of two continuous-time processes as

$$\widehat{\sigma}_{P,Q(\beta_P, \beta_Q)} = \frac{1}{T} \frac{l^2_{\beta_P P + \beta_Q Q} - l^2_{\beta_P P} - l^2_{\beta_Q Q}}{8 \ln(2) \beta_P \beta_Q} \tag{13}$$

The co-range is an unbiased estimator of $\sigma_{P,Q} = \sigma_P \cdot \sigma_Q \cdot \rho_{P,Q}$.

[4]This is clearly an unrealistic assumption due to the discreteness of financial markets, but we come back to this issue in section 5.

Proof. Consider a process Y, defined by

$$Y = \beta_P P + \beta_Q Q \qquad \beta_P \neq 0, \quad \beta_Q \neq 0 \tag{14}$$

Y is governed by a Brownian Motion:

$$dY = d(\beta_P P + \beta_Q Q) = \beta_P \sigma_P dW + \beta_Q \sigma_Q dZ$$

It follows that the first two moments of dY may be expressed as

$$E\left[d(Y)\right] = 0 \tag{15}$$

$$\begin{aligned} E\left[d(Y)^2\right] &= \beta_P^2 \sigma_P^2 dt + \beta_Q^2 \sigma_Q^2 dt + 2\beta_P \beta_Q \sigma_P \sigma_Q \rho_{P,Q} dt \\ &= \left(\beta_P^2 \sigma_P^2 + \beta_Q^2 \sigma_Q^2 + 2\beta_P \beta_Q \sigma_P \sigma_Q \rho_{P,Q}\right) dt \end{aligned} \tag{16}$$

Applying the results of Parkinson (1980), it follows that l_Y^2, the squared range of Y, provides an unbiased estimator of the variance

$$\frac{1}{4\ln(2)} E(l_Y^2) = T\beta_P^2 \sigma_P^2 + T\beta_Q^2 \sigma_Q^2 + T2\beta_P \beta_Q \sigma_P \sigma_Q \rho_{P,Q}$$

where

$$l_Y = \sup\left(\beta_P P + \beta_Q Q\right) - \inf\left(\beta_P P + \beta_Q Q\right)$$

Taking the expected value of (13),

$$\begin{aligned} & E\left[\frac{1}{T} \frac{l^2_{\beta_P P + \beta_Q Q} - l^2_{\beta_P P} - l^2_{\beta_Q Q}}{8\ln(2)\beta_P \beta_Q}\right] \\ = \;& \frac{\beta_P^2 \sigma_P^2 + \beta_Q^2 \sigma_Q^2 + 2\beta_P \beta_Q \sigma_P \sigma_Q \rho_{P,Q} - \beta_P^2 \sigma_P^2 - \beta_Q^2 \sigma_Q^2}{2\beta_P \beta_Q} \\ = \;& \sigma_P \cdot \sigma_Q \cdot \rho_{P,Q}. \end{aligned}$$

■

The co-range is a linear combination of three squared ranges: The range of $\beta_P P$, the range of $\beta_Q Q$ and the range of $\beta_P P + \beta_Q Q$. Notice that the range of $\beta_P P + \beta_Q Q$ cannot be inferred from the range of $\beta_P P$ and the range of $\beta_Q Q$. The sample path for $\beta_P P + \beta_Q Q$ is needed in order to compute price range for $\beta_P P + \beta_Q Q$, so in other words, tick-by-tick data are required for computation of the co-range in equation (13).

Defined in terms of clean prices (not log-prices), Y corresponds to

$$e^Y = y = p^{\beta_P} q^{\beta_Q}$$

which is a geometric average of the prices p and q with weights β_P and β_Q, respectively. It does not have an exact economic interpretation. However, we know that[5]

$$\ln\left[\beta_P \frac{p(t)}{p(t-1)} + \beta_Q \frac{q(t)}{q(t-1)}\right] \approx \beta_P \ln\left[\frac{p(t)}{p(t-1)}\right] + \beta_Q \ln\left[\frac{q(t)}{q(t-1)}\right] = \Delta Y, \qquad \beta_P + \beta_Q = 1$$

Therefore, ΔY may be interpreted as the (approximate) log-return on the portfolio consisting of assets "P" and "Q" with corresponding weights β_P and β_Q.

[5]See e.g. Campbell, Lo & MacKinlay (1997, p. 12). For simplicity we are using discrete time notation.

Definition 1 The range-based estimator of Ω is defined as

$$\widehat{\Omega} = \begin{pmatrix} \widehat{\sigma}_P^2 & \widehat{\sigma}_{QP}^2 \\ \widehat{\sigma}_{PQ}^2 & \widehat{\sigma}_Q^2 \end{pmatrix}$$

where $\widehat{\sigma}_P^2$ and $\widehat{\sigma}_Q^2$ are given in (12) and the co-range, $\widehat{\sigma}_{PQ}^2$, is defined in Theorem 1.

The co-range satisfies the properties described in the Proposition below.

Proposition 1 The co-range is symmetric in the sense that

$$\widehat{\sigma}_{P,Q(\beta_P,\beta_Q)}^2 = \sigma_{Q,P(\beta_Q,\beta_P)}^2$$

The co-range is a generalization of the range-based volatility estimator

$$\widehat{\sigma}_{P,P(\beta_P,\beta_Q)}^2 = \widehat{\sigma}_P^2$$

Proof. Symmetry follows straightforwardly because the range of $\beta_P P + \beta_Q Q$ is equal to the range of $\beta_Q Q + \beta_P P$:

$$l_{\beta_P P+\beta_Q Q} = l_{\beta_Q Q+\beta_P P}$$

The generalization follows from

$$\begin{aligned}
\widehat{\sigma}_{P,P(\beta_P,\beta_Q)}^2 &= \frac{1}{T}\frac{l_{\beta_P P+\beta_Q P}^2 - l_{\beta_P P}^2 - l_{\beta_Q P}^2}{8\ln(2)\beta_P\beta_Q} \\
&= \frac{1}{T}\frac{\left(\left(\beta_P+\beta_Q\right)l_P\right)^2 - \beta_P^2 l_P^2 - \beta_Q^2 l_P^2}{4\ln(2)2\beta_P\beta_Q} \\
&= \frac{1}{T}\frac{l_P^2}{4\ln(2)} = \widehat{\sigma}_P^2
\end{aligned}$$

■

Theorem 2 In the bivariate case, the estimated variance-covariance matrix, $\widehat{\Omega}$, is by construction positive semi-definite.

Proof. See section 8.1. ■

Theorem 3 In cases with more than two assets, the estimated variance-covariance matrix, $\widehat{\Omega}$, is not by construction positive semi-definite.

Proof. This is shown by simulation (see below). ■

It is rather puzzling that in the bivariate case, the estimated variance-covariance matrix is positive semi-definite, whereas in cases with more than two assets, the estimated variance-covariance matrix is not positive semi-definite by construction.

Consider a portfolio with three assets and define the estimated variance-covariance matrix as follows

$$\widehat{\Omega} = \begin{pmatrix} \hat{\sigma}_1^2 & \hat{\sigma}_{12} & \hat{\sigma}_{13} \\ \hat{\sigma}_{12} & \hat{\sigma}_2^2 & \hat{\sigma}_{23} \\ \hat{\sigma}_{13} & \hat{\sigma}_{23} & \hat{\sigma}_3^2 \end{pmatrix}$$

We use the following Proposition:

Proposition 2 Let A be an $n \times n$ symmetric matrix. Then A is positive definite if and only if all the principal minors of A are non-negative.

$\widehat{\Omega}$ has 3 first-order principal minors, the elements on the main diagonal, which are non-negative by construction. The matrix has also 3 second-order principal minors

$$\left|\begin{pmatrix} \hat{\sigma}_1^2 & \hat{\sigma}_{12} \\ \hat{\sigma}_{12} & \hat{\sigma}_2^2 \end{pmatrix}\right|, \left|\begin{pmatrix} \hat{\sigma}_1^2 & \hat{\sigma}_{13} \\ \hat{\sigma}_{13} & \hat{\sigma}_3^2 \end{pmatrix}\right| \text{ and } \left|\begin{pmatrix} \hat{\sigma}_2^2 & \hat{\sigma}_{23} \\ \hat{\sigma}_{23} & \hat{\sigma}_3^2 \end{pmatrix}\right|$$

but we know from Theorem 2 that these principal minors are non-negative. Finally, $\widehat{\Omega}$ has one third-order principal minor which is

$$\left|\widehat{\Omega}\right| = \hat{\sigma}_1^2\hat{\sigma}_2^2\hat{\sigma}_3^2 + 2\hat{\sigma}_{12}\hat{\sigma}_{13}\hat{\sigma}_{23} - (\hat{\sigma}_{13})^2\hat{\sigma}_2^2 - \hat{\sigma}_1^2(\hat{\sigma}_{23})^2 - \hat{\sigma}_3^2(\hat{\sigma}_{12})^2 \tag{17}$$

From (the proof of) Theorem 2, we learned that the implied correlations

$$\rho_{ab}^{implied} = \frac{\hat{\sigma}_{ab}}{\sqrt{\hat{\sigma}_a^2\hat{\sigma}_b^2}} = \frac{\hat{\sigma}_{ab}}{\hat{\sigma}_a\hat{\sigma}_b}, \qquad b \neq a \tag{18}$$

obey

$$-1 \leq \rho_{ab}^{implied} \leq 1 \tag{19}$$

We substitute the expression for the implied correlations (18) into (17):

$$\begin{aligned} \left|\widehat{\Omega}\right| &= \hat{\sigma}_1^2\hat{\sigma}_2^2\hat{\sigma}_3^2 + 2\rho_{12}^{implied}\hat{\sigma}_1\hat{\sigma}_2\rho_{13}^{implied}\hat{\sigma}_1\hat{\sigma}_3\rho_{23}^{implied}\hat{\sigma}_2\hat{\sigma}_3 - \\ &\quad \left(\rho_{13}^{implied}\hat{\sigma}_1\hat{\sigma}_3\right)^2\hat{\sigma}_2^2 - \hat{\sigma}_1^2\left(\rho_{23}^{implied}\hat{\sigma}_2\hat{\sigma}_3\right)^2 - \hat{\sigma}_3^2\left(\rho_{12}^{implied}\hat{\sigma}_1\hat{\sigma}_2\right)^2 \\ &= \hat{\sigma}_1^2\hat{\sigma}_2^2\hat{\sigma}_3^2\left(1 + 2\rho_{12}^{implied}\rho_{13}^{implied}\rho_{23}^{implied} - \left(\rho_{13}^{implied}\right)^2 - \left(\rho_{23}^{implied}\right)^2 - \left(\rho_{12}^{implied}\right)^2\right) \end{aligned}$$

So the sign of $\left|\widehat{\Omega}\right|$ is equal to the sign of

$$\left(1 + 2\rho_{12}^{implied}\rho_{13}^{implied}\rho_{23}^{implied} - \left(\rho_{13}^{implied}\right)^2 - \left(\rho_{23}^{implied}\right)^2 - \left(\rho_{12}^{implied}\right)^2\right) \tag{20}$$

There are no restrictions on the implied correlations except from (19). Hence the sign of $\left|\widehat{\Omega}\right|$ may become negative if e.g. $\rho_{13}^{implied} = \rho_{12}^{implied} \approx 1, \rho_{23}^{implied} \approx -1$. In Table 4, see section 9, we give a simple numerical example of a violation of $\left|\widehat{\Omega}\right| \geq 0$ in discrete time[6]. We consider three assets and the corresponding log-prices. Three prices are observed per day and all transactions occur simultaneously. In this example

$$\left(1 + 2\rho_{12}^{implied}\rho_{13}^{implied}\rho_{23}^{implied} - \left(\rho_{13}^{implied}\right)^2 - \left(\rho_{23}^{implied}\right)^2 - \left(\rho_{12}^{implied}\right)^2\right) < 0,$$

and hence $\widehat{\Omega}$ is not positive semi-definite. The example in Table 3 serves as a proof of Theorem 3.

[6]The example can easily be extended to continuous time by interpolating between prices.

Intuitive remarks on the co-range. Below, we describe why the co-range may be a superior covariance-estimator compared to the estimator based on open/close prices.

Consider two sample paths of log prices P and Q in panel a) of Figure 1. The returns (the first difference of the log-prices) seem to be perfectly negatively correlated, but the ordinary covariance estimator based on open/close prices (that is the price corresponding to time 1 and 9) yields an estimate of

$$\widetilde{\sigma}_{P,Q} = \frac{(P(9) - P(1))\,(Q(9) - Q(1))}{8} = 0$$

whereas the co-range, using the beta-vector of $(\beta_P, \beta_Q) = (1,1)$, yields an estimate of

$$\widehat{\sigma}_{P,Q(1,1)} = \frac{1}{8}\frac{l_{P+Q}^2 - l_P^2 - l_Q^2}{8\ln(2)} = \frac{1}{8}\frac{0^2 - 4^2 - 4^2}{8\ln(2)} = -0.72$$

In this case, the co-range estimator seems to be superior to the return-based estimator, because it captures the negative comovement. Consider panel b) in Figure 1. The returns from the two

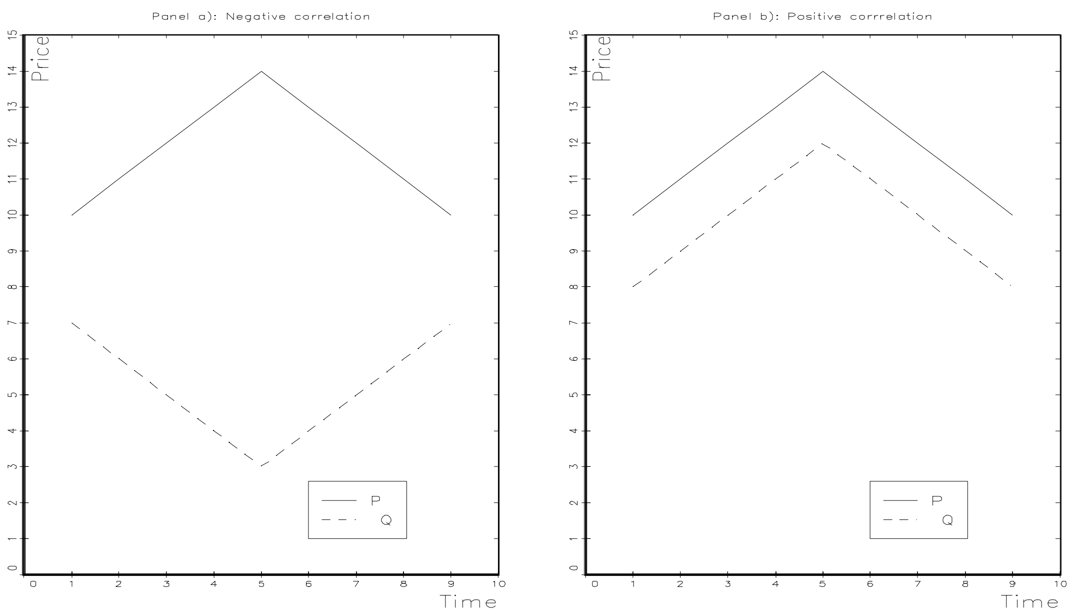

Figure 1: Intuition of the co-range.

assets seem to be positively correlated. The covariance-estimator based on open-close prices is

$$\widetilde{\sigma}_{P,Q} = \frac{\left(P_{A/B}(9) - P_{A/B}(1)\right)\left(P_{B/C}(9) - P_{B/C}(1)\right)}{8} = 0$$

and the co-range estimator yields

$$\begin{aligned}\widehat{\sigma}_{P,Q(1,1)} &= \frac{1}{8}\frac{l_{P+Q}^2 - l_P^2 - l_Q^2}{8\ln(2)} \\ &= \frac{1}{8}\frac{\left((12+14) - (8+10)\right)^2 - (14-10)^2 - (12-8)^2}{8\ln(2)} = 0.72\end{aligned}$$

The two stylized examples clearly illustrate that the co-range is indeed able to capture the daily comovements of the two assets: On days with high intradaily co-variation and where daily returns are low, the co-range is superior to the covariance-estimator based on open/close prices.

4 Co-range: Foreign Exchange market

As noted below Theorem 1, computation of the co-range requires the use of tick-by-tick data. However, as suggested by Frank Diebold, the co-range may be computed without the use of tick-by-tick data in the case of foreign exchange rates. This idea is developed in Brandt & Diebold (2001). This is important because tick-by-tick data are not available for that many assets, and if they are available, data typically span only a few years. In contrast, daily high and low prices are available for most foreign exchange rates (FX) for a long time period. We recast the model from section 3 in terms of foreign exchange rates. $P_{A/B}$ and $P_{C/B}$ denote log-exchange rates of currency A and C, respectively, in terms of currency B. Hence we can interpret $dP_{A/B}$ and $dP_{C/B}$ as continuously compounded returns in units of currency B.

$$dP_{A/B} = \sigma_{A/B} dW \tag{21}$$

$$dP_{C/B} = \sigma_{C/B} dZ \tag{22}$$

$$E\left(\begin{bmatrix} dW \\ dZ \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \sigma_{A/B,C/B} = \rho_{A/B,C/B} \cdot \sigma_{A/B} \cdot \sigma_{C/B} \tag{23}$$

$$E\left(\begin{bmatrix} (dW)^2 & dW dZ \\ dW dZ & (dZ)^2 \end{bmatrix}\right) = \begin{bmatrix} dt & \rho_{A/B,C/B} dt \\ \rho_{A/B,C/B} dt & dt \end{bmatrix} \tag{24}$$

Further, we assume that triangular arbitrage holds perfectly:

$$\frac{p_{A/B}(t)}{p_{C/B}(t)} = p_{A/C}(t) \qquad \forall t$$

At every point in time, we can infer the price $p_{A/C}$ from the two prices $p_{A/B}$ and $p_{C/B}$. Formulated in log-prices, the triangular arbitrage condition above, reads

$$P_{A/B}(t) - P_{C/B}(t) = P_{A/C}(t) \qquad \forall t \tag{25}$$

Corollary 1 describes a version of the co-range that does not require tick-by-tick data.

Corollary 1 Assume that two continuous time processes (21)-(24) describe the evolution of the log-exchange rates $P_{A/B}$ and $P_{C/B}$. Further, assume that triangular arbitrage holds perfectly in the sense of equation (25). The co-range, with the beta-vector of $(1, -1)$, see Theorem 1, is

$$\widehat{\sigma}_{A/B,C/B(1,-1)} = \frac{1}{T} \frac{l^2_{P_{A/C}} - l^2_{P_{A/B}} - l^2_{P_{C/B}}}{-8\ln(2)}$$

The co-range is an unbiased estimator of $\sigma_{A/B,C/B} = \sigma_{A/B} \cdot \sigma_{C/B} \cdot \rho_{A/B,C/B}$.

Proof. Apply the co-range (see Theorem 1) with beta-vector $(1, -1)$ to $P_{A/B}$ and $P_{C/B}$:

$$\widehat{\sigma}_{A/B,C/B(1,-1)} = \frac{1}{T} \frac{l^2_{P_{A/B}-P_{C/B}} - l^2_{P_{A/B}} - l^2_{P_{C/B}}}{(-1)8\ln(2)} \tag{26}$$

Precluding triangular arbitrage

$$P_{A/B} - P_{C/B} = P_{A/C}$$

reduces (26) to

$$\widehat{\sigma}_{A/B,C/B(1,-1)} = \frac{1}{T}\frac{l^2_{P_{A/C}} - l^2_{P_{A/B}} - l^2_{P_{C/B}}}{-8\ln(2)}$$

and the Corollary follows immediately. ■

In contrast to the general formula for the co-range, see Theorem 1, the assumption of absence of triangular arbitrage possibilities in Corollary 1 circumvents the use of tick-by-tick data. This idea of using triangular arbitrage in order to back out covariances was developed in Andersen, Bollerslev, Diebold & Labys (2001b, p. 15). In the context of option prices, Lopez & Walter (2001, p. 7) use the idea to compute implied covariances between exchange rates from implied variances on individual exchange rates.

The assumption of absence of triangular arbitrage opportunities does not hold perfectly, due to transactions costs and other frictions in the market. However, we believe the assumption is rather robust. In fact, precluding triangular arbitrage boils down to the law of one price which is one of the most fundamental valuation principles in finance.

5 Simulation evidence on the co-range

To assess the properties of the co-range as a covariance estimator, we perform an extensive simulation analysis. In the design of the simulation experiment, we aim at replicating foreign exchange rates.

For the purpose of performing simulations, we discretize the bivariate Brownian Motion model, (21)-(24).

Design of simulation experiment Consider two log exchange rates, $P_{A/B}$ and $P_{C/B}$, that follow a bivariate random walk with homoskedastic and contemporaneously correlated innovations

$$\begin{aligned} P_{A/B}(t) &= P_{A/B}(t-1) + e(t) \qquad t = 1,2,3,...,T \qquad P_{A/B}(0) = \overline{P}_{A/B} \\ P_{C/B}(t) &= P_{C/B}(t-1) + v(t) \qquad t = 1,2,3,...,T \qquad P_{C/B}(0) = \overline{P}_{C/B} \\ \begin{bmatrix} e(t) \\ v(t) \end{bmatrix} &\sim nid\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^2_{A/B} & \sigma_{A/B,C/B} \\ \sigma_{A/B,C/B} & \sigma^2_{C/B} \end{bmatrix} \right) \\ \rho_{A/B,C/B} &= \frac{\sigma_{A/B,C/B}}{\sigma_{A/B}\sigma_{C/B}} \end{aligned} \tag{27}$$

where[7]

$$\begin{aligned} P_{A/B}(t) &= \ln\left[p_{A/B}(t)\right] \\ P_{C/B}(t) &= \ln\left[p_{C/B}(t)\right] \end{aligned}$$

[7] The random walk process (discrete time version of Brownian motion) for the log-prices follows from the assumption that prices follow a geometric Brownian motion. Strictly speaking, this would imply that the random walk process contains a drift, $(\mu - \frac{1}{2}\sigma^2)$, but we abstain from this fact here. The drift is probably negligible.

$\sigma^2_{A/B}$ and $\sigma^2_{C/B}$ denote the variance of the log-returns for positions in "A/B" and "C/B", respectively. T denotes the number of returns observed per day.

The innovations are simulated by a Choleski decomposition of the covariance matrix in (27):

$$\begin{bmatrix} e(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} \sigma_{A/B} & 0 \\ \frac{\sigma_{A/B,C/B}}{\sigma_{A/B}} & \sqrt{\sigma^2_{C/B} - \frac{\sigma^2_{A/B,C/B}}{\sigma^2_{A/B}}} \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$
$$= \begin{bmatrix} \sigma_{A/B}x(t) \\ \sigma_{C/B}\rho_{A/B,C/B}x(t) + \sigma_{C/B}\sqrt{1-\rho^2_{A/B,C/B}}y(t) \end{bmatrix}$$

where

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \sim nid\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

The general set up of the simulation experiment is[8]:

- $T \in \{2880, 1440, 480\}$. This is approximately equivalent to observing the price every {30 sec, 60 sec, 180 sec}, if we assume that the market is open 24 hours;
- The number of simulated trading days (or Monte Carlo replications) equals 100.000;
- $\sigma_{A/B}$ is chosen such that a position in A/B has an annual (250 days per year) return volatility (standard deviation) of 9.52% per year;
- $\sigma_{C/B}$ is chosen such that a position in C/B has an annual (250 days per year) return volatility (standard deviation) of 15.87% per year;
- $\rho_{A/B,C/B} \in \{\pm 0.99, \pm 0.8, \pm 0.5, \pm 0.2, 0\}$.

We believe, the chosen parameter values are "representative" for the foreign exchange markets. As the focus of this simulation experiment is on the behavior of the co-range, we consider a wide range of parameters for the correlation coefficient, $\rho_{A/B,C/B}$, ranging from -0.99 to 0.99.

We compare two types of variance and covariance-estimators based on ranges and open/close prices, respectively. In Table 1, range-based estimators appear in column 2 and return-based estimators appear in column 3. The estimators correspond to equations (10) - (12) and The-

Moment	Range-based estimator	Open/close-based estimator
$\sigma^2_{A/B}$	$\widehat{\sigma}^2_{A/B} = \frac{1}{T}\frac{l^2_{A/B}}{4\ln(2)}$	$\widetilde{\sigma}^2_{A/B} = \frac{(P_{A/B}(T)-P_{A/B}(0))^2}{T}$
$\sigma^2_{C/B}$	$\widehat{\sigma}^2_{A/B} = \frac{1}{T}\frac{l^2_{C/B}}{4\ln(2)}$	$\widetilde{\sigma}^2_{C/B} = \frac{(P_{C/B}(T)-P_{C/B}(0))^2}{T}$
$\sigma_{A/B,C/B}$	$\widehat{\sigma}_{A/B,C/B}(1,-1) = \frac{1}{T}\frac{l^2_{A/B-C/B}-l^2_{A/B}-l^2_{C/B}}{-8\ln(2)}$	$\widetilde{\sigma}_{A/B,C/B} = \frac{(P_{A/B}(T)-P_{A/B}(0))(P_{C/B}(T)-P_{C/B}(0))}{T}$

Table 1: Estimators used in the simulation experiment

[8]We do not specify starting prices for each day, $\bar{P}$ and $\bar{Q}$, because they do not affect the simulation results.

orem 1. The only difference is that the estimators above are based on a discrete sample of prices. Hence, in this section, T refers to the number of return observations per day. Only results for the estimates of $\sigma_{A/B,C/B}$ are reported. The simulations for the variance estimates are completely in line with the theoretical findings of Parkinson (1980) and may be obtained from the authors.

For each estimator we compute the $Bias$, the Mean Squared Error (MSE) and the Mean Absolute Deviation (MAD). We also define three relative efficiency measures where the benchmark is the returns-based estimates:

$$\text{Relative } MSE = \frac{MSE(\text{Range-based estimator})}{MSE(\text{Daily returns estimator})} \tag{28}$$

$$\text{Relative } MAD = \frac{MAD(\text{Range-based estimator})}{MAD(\text{Daily returns estimator})} \tag{29}$$

$$\text{Relative } Bias = \frac{Bias(\text{Range-based estimator})}{Bias(\text{Daily returns estimator})} \tag{30}$$

Results Table 5-7 contain the simulation results for the covariance-estimates. Each table summarizes the outcome of 9 Monte Carlo experiments where all parameters are constant except from $\rho_{A/B,C/B}$. The number of return observations per day is 2880 in Table 5, 1440 per day in Table 6, and 480 per day in Table 7.

In contrast to range-based variance estimator, we are not able to decide a priori whether we expect a positive or negative bias for the co-range. The reason is that three ranges enter the co-range with different signs and hence there are two opposing forces affecting the bias:

$$\widehat{\sigma}_{A/B,C/B(1,-1)} = \frac{1}{T}\frac{l^2_{A/B-C/B} - l^2_{A/B} - l^2_{C/B}}{-8\ln(2)} = \frac{1}{T}\frac{\overbrace{-l^2_{A/B-C/B}}^{\text{Positive bias}} + \overbrace{\left(l^2_{A/B} + l^2_{C/B}\right)}^{\text{Negative bias}}}{8\ln(2)} \tag{31}$$

On the other hand, we have a strong prior on the shape of the correspondence between the correlation coefficient between asset returns, $\rho_{A/B,C/B}$, and the bias of the co-range. A fundamental assumption underlying this prior is that the bias of the range-based volatility estimator is decreasing in the true volatility[9].

$$\frac{\partial\left[Bias\left(\widehat{\sigma}^2_{A/B}\right)\right]}{\partial\sigma^2_{A/B}} < 0, \qquad Bias\left(\widehat{\sigma}^2_{A/B}\right) = E\left(\widehat{\sigma}^2_{A/B} - \sigma^2_{A/B}\right)$$

Assume that we keep everything constant except from $\rho_{A/B,C/B}$ (which is the case in the simulation experiments). The source of the negative bias, $\left(l^2_{A/B} + l^2_{C/B}\right)$, is constant, whereas the source of the positive bias, $-l^2_{A/B+C/B}$, decreases with $\rho_{A/B,C/B}$ because the variance of sum of log-returns

$$Var(\left(P_{A/B}(t) - P_{C/B}(t)\right)\left(P_{A/B}(t-1) - P_{C/B}(t-1)\right)) = \sigma^2_{A/B} + \sigma^2_{C/B} - 2\rho_{A/B,C/B}\sigma_{A/B}\sigma_{C/B}$$

[9]Remember that $Bias\left(\widehat{\sigma}^2_{A/B}\right) < 0$, so $\frac{\partial\left[\left|Bias\left(\widehat{\sigma}^2_{A/B}\right)\right|\right]}{\partial\sigma^2_{A/B}} > 0$.

is decreasing in $\rho_{A/B,C/B}$ (the role of the term, $l^2_{A/B+C/B}$, is to estimate the sum of the log-returns above). Hence we expect that the bias of the co-range is inversely[10] related to the correlation coefficient $\rho_{A/B,C/B}$. Table 5, 6 and 7 contain the bias for the co-range and the inverse relationship between $\rho_{A/B,C/B}$ and the bias appears clearly.

Second, and most importantly, it appears from Table 5, 6 and 7 that the co-range dominates the estimator based on daily returns in terms of relative MSE and relative MAD. The relative MSE stays around 0.2, which indicates that the efficiency result from Parkinson (1980) holds also for the co-range. It is important to note that this result holds for the general case ie. with and withour triangular arbitrage conditions.

6 Range-based volatility estimation in the presence of dependent returns

Parkinson (1980) adopts a Brownian Motion process for the log-price of a financial asset. This process has been widely applied in theoretical finance but it is inconsistent with several empirical facts on financial returns. One implication of the model is that volatility of returns is constant over time. As already mentioned, this assumption is at odds with the empirical evidence of time-varying volatility. Another implication is that high-frequency returns are normally distributed, which is in sharp contrast to the empirical evidence of more fat-tailed distributions. This section explores the potential for extending the univariate range-based volatility estimator to a setting with time-varying volatility and heavy-tailed returns.[11]

It is noteworthy that the entire literature on range-based volatility estimation adopts the Brownian Motion model even though the seminal paper by Feller (1951) did **not** adopt this model. The following quotation from Campbell et al. (1997) expresses the widely held belief that range-based volatility estimation only carries through in a constant volatility environment:

> "Other researchers have used the difference between high and low prices on a given day to estimate volatility for that day (Garman & Klass (1980), Parkinson (1980)). Such methods implicitly assume that volatility is constant over some interval of time" From Campbell et al. (1997, p. 481).

Below we show by asymptotic means, that range-based volatility estimation may be unified with time-varying volatility. Feller (1951) defined

$$p_n = r_1 + r_2 + r_3 + ... + r_n$$

where r_k's is a sequence of independent identically distributed random variables with $E(r_k) = 0$ and $Var(r_k) = 1$. The asymptotic ($n \to \infty$) distribution of the range of p_n was derived by Feller (1951) by use of the property that p_n obeys a FCLT[12]

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{[zn]} r_j = \frac{1}{\sqrt{n}} p_{[zn]} \to W(z)$$

[10]In fact, one can show that this applies for any beta-vector (with non-zero elements) associated with the co-range.

[11]For simplicity, we concentrate on the univariate case.

[12]Feller(1951) did not phrase it like this, because the FCLT was not named in 1951. $[n]$ denotes the largest integer that is less than or equal to n.

where $W(z)$ is a standard Brownian Motion.

Hence it is possible to relax the assumptions on r_k as long as p_n obeys a FCLT. In particular, we would like to point out that the empirical regularities of financial returns (volatility-dependence and fat tails), may be allowed for in the process driving financial returns[13]. Theoretically a FCLT must hold and, for the range-based volatility estimator to work in practice, the number of return observations used in the computation of the range-based volatility estimate must be sufficiently large such that $\frac{1}{\sqrt{n}}p_{[zn]}$ provides a reasonable approximation to the Wiener process $W(z)$. As an example we consider a GARCH(1,1) process with normally distributed innovations:

$$
\begin{aligned}
P_{t+1} - P_t &= r_t \\
r_t &= \sigma_t u_t \qquad u_t \sim nid(0,1) && (32) \\
\sigma_t^2 &= \omega + \alpha r_{t-1}^2 + \beta \sigma_{t-1}^2 \qquad , \omega > 0,\ \alpha + \beta < 1 && (33) \\
\sigma_p^2 &= \frac{\omega \cdot (\text{Return obs})}{(1-\alpha-\beta)} = 3.6e^{-5} && (34)
\end{aligned}
$$

From Proposition 2.2 in Davidsson (2002), it follows that r_t obeys a FCLT and hence

$$\frac{1}{\sqrt{n}} p_{[zn]} = \frac{1}{\sqrt{n}} \sum_{j=1}^{[zn]} r_j \rightarrow E(\sigma_t) W(z) \tag{35}$$

In other words, in the case of GARCH(1,1) returns, if n is sufficiently large, Feller (1951)'s range-based volatility estimator provides an estimator of the unconditional variance $\sigma_p^2 = E(\sigma_t^2) = \frac{\omega}{1-\alpha-\beta}$. We perform a simulation experiment to explore how the range-based volatility estimator behaves in the presence GARCH(1,1)-returns. The constant term in the conditional variance equation[14] for the GARCH(1,1) process, ω, is chosen according to

$$\sigma_p^2 = \frac{\omega \cdot (\text{Return obs})}{(1-\alpha-\beta)} = 3.6e^{-5} \tag{36}$$

so the unconditional daily variance of the return, σ_p^2, is equal to $3.6e^{-5}$. In this way, σ_p^2 is kept constant such that we are able to focus exclusively on the impact of the number of price observations per day. Table 2 compares the performance of the range-based estimator of σ_p^2 to the open/close-estimator based on the daily squared return. (Return obs) denotes the number of return observations per day. The bias of the daily estimator in Table 2 is, as expected, negligible compared to the range-based estimator. However, the relative MSE amounts to 0.2604 for 50 price observations per day and decreases towards the theoretical asymptotic value[15] of 0.20367. The message from this experiment is that the efficiency result for the range-based volatility estimator, see Parkinson (1980), may also apply to more realistic settings. Equation (35) holds if n, the number of price observation per day, is very large (theoretically n approaches infinity). However, the simulations indicate that the result seems to hold even for moderate numbers of return observations per day.

[13] Siddiqui (1976) points out that the range-based volatility estimator of Feller (1951) can be extended to allow for correlation among r_k's.

[14] In the lower part of Table 2, the GARCH process is specified.

[15] See equation (5).

Monte Carlo Experiment

Return obs	50	250	600	1000	5000
$Bias(\widehat{\sigma}_P^2)$	$-8.464e^{-6}$	$-4.276e^{-6}$	$-2.933e^{-6}$	$-2.326e^{-6}$	$-1.126e^{-6}$
$Bias(\widetilde{\sigma}_P^2)$	$5.515e^{-8}$	$-2.580e^{-8}$	$-7.908e^{-8}$	$-3.953e^{-8}$	$-4.844e^{-8}$
Relative $Bias$	$-1.535e^2$	$1.658e^2$	$3.710e^1$	$5.884e^1$	$2.325e^1$
Relative MSE	0.2604	0.2378	0.2213	0.2139	0.2043
Relative MAD	0.5980	0.5403	0.5140	0.5036	0.4872

$P_{t+1} - P_t = r_t$

r_t is generated from the GARCH(1,1) model in (32)-(36) with

$\alpha = 0.15, \quad \beta = 0.8, \quad \omega = \frac{\sigma_p^2(1-\alpha-\beta)}{\text{(Return obs)}}, \quad \sigma_p^2 = 3.6e^{-5}$

Monte Carlo replications = 1000000

Table 2: Monte Carlo experiment for the range in the presence of dependent returns (Normal distribution)

GARCH processes are able to capture the time-varying volatility, which is typically found in financial returns. However, a fat-tailed distribution for the innovations u_t in (32) is often required to account for excess kurtosis in data. Hence, we change the simulation experiment accordingly: The innovations in the GARCH model follow a standardized t-distribution with 5 degrees of freedom (other things stay the same)

$$r_t = \sigma_t u_t \qquad u_t \sim \text{Standardized } t(df = 5) \tag{37}$$

By Proposition 2.2 in Davidsson (2002), a FCLT holds for this process, and hence use of the range-based volatility estimator is justified. Table 3 shows the results of the simulation

Monte Carlo Experiment

Return obs	50	250	600	1000	5000
$Bias(\widehat{\sigma}_P^2)$	$-9.476e^{-6}$	$-5.134e^{-6}$	$-3.448e^{-6}$	$-3.138e^{-6}$	$-1.576e^{-6}$
$Bias(\widetilde{\sigma}_P^2)$	$6.792e^{-8}$	$6.109e^{-8}$	$5.018e^{-7}$	$-1.426e^{-7}$	$5.205e^{-9}$
Relative $Bias$	$-1.395e^2$	$-8.404e^1$	$-0.6870e^1$	$2.200e^1$	$-3.027e^2$
Relative MSE	0.4358	0.3717	0.2363	0.2846	0.2495
Relative MAD	$6.640e^{-1}$	$6.013e^{-1}$	$5.622e^{-1}$	$5.441e^{-1}$	$5.066e^{-1}$

$P_{t+1} - P_t = r_t$

r_t is generated from the GARCH(1,1) model in (37) and (33)-(36) with

$\alpha = 0.15, \quad \beta = 0.8, \quad \omega = \frac{\sigma_p^2(1-\alpha-\beta)}{\text{(Return obs)}}, \quad \sigma_p^2 = 3.6e^{-5}$

Monte Carlo replications = 1000000

Table 3: Monte Carlo experiment for the range in the presence of dependent returns (t-distribution)

experiment. Introduction of the t-distribution slightly worsened the performance of the range-based volatility estimator, but even with 600 price observations, the relative MSE is close to the theoretical value of 0.20367. Relative MSE never gets below 0.5, indicating that even in this setting, the range-based volatility estimator is superior to squared returns.

This section illustrated that the empirical facts of dependent and fat-tailed returns may be incorporated into the theoretical model underlying range-based volatility estimators.

7 Concluding remarks

We propose a new estimator of the covariance of financial returns based on high/low prices, termed co-range. For the foreign exchange market, the co-range may be computed from individual high and low prices. Simulations indicate that the co-range is approximately 5 times more efficient than the estimator based on open/close prices, paralleling the efficiency result from Parkinson (1980) in the univariate case. The empirical properties of range-based variance and covariance proxies are explored in Brunetti & Lildholdt (2001) for foreign exchange rates.

In univariate GARCH models, the range-based volatility estimator has been found to enter the conditional variance equation significantly, see Lin & Rozeff (1994) and Chen (1997). Similarly, the explanatory power of the co-range in the conditional variance/covariance equation of multivariate GARCH models is an interesting topic for future research[16].

The range-based estimate of the bivariate variance-covariance matrix may be used in the context of stochastic volatility models. Alizadeh et al. (2002) estimates a univariate range-based stochastic volatility model for foreign exchange rates, and the co-range could be an important ingredient in multivariate extensions. For that purpose, it would be relevant to construct a positive semi-definite range-based estimate of the variance-covariance matrix in cases with more than two assets. However, that would probably be difficult to unify with unbiased estimation of the covariances.

Indirectly, the co-range provides an estimate of the correlation coefficient. In the notation of section 4, a (co)-range-based estimate of the correlation coefficient is

$$\widehat{\rho}_{A/B,C/B} = \frac{\widehat{\sigma}_{A/B,C/B(1,-1)}}{\sqrt{\widehat{\sigma}^2_{A/B}\widehat{\sigma}^2_{C/B}}}$$

The properties of this correlation estimator should be explored.

8 Appendix

8.1 Proof of Theorem 2

We make use of two properties

$$\widehat{\Omega} \text{ is positive semidefinite} \Leftrightarrow \text{the eigenvalues of } \widehat{\Omega},\ \lambda_1 \text{ and } \lambda_2, \text{ are non-negative}$$

and

$$\left|\widehat{\Omega}\right| = \lambda_1 \lambda_2$$

[16]The authors are currently working in this direction.

If we show that $\lambda_1 > 0$ and $\left|\widehat{\Omega}\right| \geq 0$, then it follows that $\widehat{\Omega}$ is positive semidefinite.
First assume that the sample paths for P and Q are non-constant. Then the eigenvalues are

$$\lambda_1 = \frac{\widehat{\sigma}_P^2 + \widehat{\sigma}_Q^2 + \sqrt{\left(\widehat{\sigma}_P^2 - \widehat{\sigma}_Q^2\right)^2 + 4\left(\widehat{\sigma}_{P,Q(\beta_\mathsf{P},\beta_\mathsf{Q})}\right)^2}}{2}$$

$$\lambda_2 = \frac{\widehat{\sigma}_P^2 + \widehat{\sigma}_Q^2 - \sqrt{\left(\widehat{\sigma}_P^2 - \widehat{\sigma}_Q^2\right)^2 + 4\left(\widehat{\sigma}_{P,Q(\beta_\mathsf{P},\beta_\mathsf{Q})}\right)^2}}{2}$$

The first eigenvalue

$$\lambda_1 = \frac{\widehat{\sigma}_P^2 + \widehat{\sigma}_Q^2 + \sqrt{\left(\widehat{\sigma}_P^2 - \widehat{\sigma}_Q^2\right)^2 + 4\left(\widehat{\sigma}_{P,Q(\beta_\mathsf{P},\beta_\mathsf{Q})}\right)^2}}{2} \geq \frac{\widehat{\sigma}_P^2 + \widehat{\sigma}_Q^2 + \sqrt{\left(\widehat{\sigma}_P^2 - \widehat{\sigma}_Q^2\right)^2}}{2} = \widehat{\sigma}_P^2 > 0$$

and hence

$$\lambda_1 > 0$$

because of the assumption of non-constant sample paths. Now we are proving that the determinant is non-negative

$$\left|\widehat{\Omega}\right| = \left|\begin{pmatrix} \widehat{\sigma}_P^2 & \widehat{\sigma}_{P,Q(\beta_\mathsf{P},\beta_\mathsf{Q})} \\ \widehat{\sigma}_{P,Q(\beta_\mathsf{P},\beta_\mathsf{Q})} & \widehat{\sigma}_Q^2 \end{pmatrix}\right| = \widehat{\sigma}_P^2\widehat{\sigma}_Q^2 - \left(\widehat{\sigma}_{P,Q(\beta_\mathsf{P},\beta_\mathsf{Q})}\right)^2 \geq 0$$

This is equivalent to

$$\widehat{\sigma}_P^2\widehat{\sigma}_Q^2 - \left(\widehat{\sigma}_{P,Q(\beta_\mathsf{P},\beta_\mathsf{Q})}\right)^2 \geq 0$$

or

$$1 \geq \frac{\left(\widehat{\sigma}_{P,Q(\beta_\mathsf{P},\beta_\mathsf{Q})}\right)^2}{\widehat{\sigma_P^2}\widehat{\sigma_Q^2}} = \rho_{implied}^2$$

Showing that $\left|\widehat{\Omega}\right|$ is non-negative amounts to showing that the implied squared correlation coefficient between the returns of the two assets is less than or equal to unity, which is equivalent to

$$-1 \leq \rho_{implied} = \frac{\widehat{\sigma}_{P,Q(\beta_\mathsf{P},\beta_\mathsf{Q})}}{\sqrt{\widehat{\sigma}_P^2\widehat{\sigma}_Q^2}} = \frac{l_{\beta_\mathsf{P}P+\beta_\mathsf{Q}Q}^2 - l_{\beta_\mathsf{P}P}^2 - l_{\beta_\mathsf{Q}Q}^2}{2\beta_P\beta_Q\sqrt{l_P^2 l_Q^2}} \leq 1$$

or

$$-1 \underbrace{\leq}_{A} \rho_{implied} = \frac{\widehat{\sigma}_{P,Q(\beta_\mathsf{P},\beta_\mathsf{Q})}}{\sqrt{\widehat{\sigma}_P^2\widehat{\sigma}_Q^2}} = \frac{l_{\beta_\mathsf{P}P+\beta_\mathsf{Q}Q}^2 - l_{\beta_\mathsf{P}P}^2 - l_{\beta_\mathsf{Q}Q}^2}{2\sqrt{l_{\beta_\mathsf{P}P}^2 l_{\beta_\mathsf{Q}Q}^2}} \underbrace{\leq}_{B} 1$$

Some notation is needed now. Define

$$\sup(\beta_P P + \beta_Q Q) = (P^{h(\beta_\mathsf{P})} + Q^{h(\beta_\mathsf{Q})}) \tag{38}$$
$$\inf(\beta_P P + \beta_Q Q) = (P^{l(\beta_\mathsf{P})} + Q^{l(\beta_\mathsf{Q})}) \tag{39}$$
$$\sup \beta_P P = P^{high(\beta_\mathsf{P})} \tag{40}$$
$$\inf \beta_P P = P^{low(\beta_\mathsf{P})} \tag{41}$$
$$\sup \beta_Q Q = Q^{high(\beta_\mathsf{Q})} \tag{42}$$
$$\inf \beta_Q Q = Q^{low(\beta_\mathsf{Q})} \tag{43}$$

Note the difference between e.g. $P^{h(\beta_\mathsf{P})}$ and $P^{high(\beta_\mathsf{P})}$: $P^{h(\beta_\mathsf{P})}$ denotes the value of $\beta_P P$, where $\sup(\beta_P P + \beta_Q Q)$ is attained, whereas $P^{high(\beta_\mathsf{P})}$denotes the value of $\beta_P P$ where $\sup \beta_P P$ is attained. First we prove A.

Proof of A We need the following properties

$$P^{h(\beta_\mathsf{P})} + Q^{h(\beta_\mathsf{Q})} \geq P^{high(\beta_\mathsf{P})} + Q^{low(\beta_\mathsf{Q})} \tag{44}$$
$$P^{h(\beta_\mathsf{P})} + Q^{h(\beta_\mathsf{Q})} \geq P^{low(\beta_\mathsf{P})} + Q^{high(\beta_\mathsf{Q})} \tag{45}$$
$$P^{l(\beta_\mathsf{P})} + Q^{l(\beta_\mathsf{Q})} \leq P^{low(\beta_\mathsf{P})} + Q^{high(\beta_\mathsf{Q})} \tag{46}$$
$$P^{l(\beta_\mathsf{P})} + Q^{l(\beta_\mathsf{Q})} \leq P^{high(\beta_\mathsf{P})} + Q^{low(\beta_\mathsf{Q})} \tag{47}$$

To clarify (44), note that by definition

$$P^{h(\beta_\mathsf{P})} + Q^{h(\beta_\mathsf{Q})} \geq \beta_P P_t + \beta_Q Q_t \qquad \forall t$$

Let us choose t such that $\beta_P P_t = P^{high(\beta_\mathsf{P})}$. Then

$$P^{h(\beta_\mathsf{P})} + Q^{h(\beta_\mathsf{Q})} \geq P^{high(\beta_\mathsf{P})} + \beta_Q Q_t \qquad \forall t \tag{48}$$

By definition we know that

$$\beta_Q Q_t \geq Q^{low(\beta_\mathsf{Q})} \tag{49}$$

Combining (48) and (49), we get

$$P^{h(\beta_\mathsf{P})} + Q^{h(\beta_\mathsf{Q})} \geq P^{high} + \beta_Q Q_t \geq P^{high(\beta_\mathsf{P})} + Q^{low(\beta_\mathsf{Q})}$$

which yields equation (44). In a similar way (45)-(47) follow straightforwardly. By (44) and (46) it follows that

$$\begin{aligned}(\beta_P P + \beta_Q Q)^{range} &= P^{h(\beta_\mathsf{P})} + Q^{h(\beta_\mathsf{Q})} - \left(P^{l(\beta_\mathsf{P})} + Q^{l(\beta_\mathsf{Q})}\right) \\ &\geq P^{high(\beta_\mathsf{P})} + Q^{low(\beta_\mathsf{Q})} - \left(P^{low(\beta_\mathsf{P})} + Q^{high(\beta_\mathsf{Q})}\right)\end{aligned}$$

which reduces to

$$(\beta_P P + \beta_Q Q)^{range} \geq (\beta_P P)^{range} - \left(\beta_Q Q\right)^{range} \tag{50}$$

when

$$(\beta_P P)^{range} = P^{high(\beta_P)} - P^{low(\beta_P)}$$
$$\left(\beta_Q Q\right)^{range} = Q^{high(\beta_Q)} - Q^{low(\beta_Q)}$$

Similarly using (45) and (47)

$$\begin{aligned}(\beta_P P + \beta_Q Q)^{range} &= P^{h(\beta_P)} + Q^{h(\beta_Q)} - \left(P^{l(\beta_P)} + Q^{l(\beta_Q)}\right) \\ &\geq P^{low(\beta_P)} + Q^{high(\beta_Q)} - \left(P^{high(\beta_P)} + Q^{low(\beta_Q)}\right)\end{aligned}$$

which implies

$$(\beta_P P + \beta_Q Q)^{range} \geq \left(\beta_Q Q\right)^{range} - (\beta_P P)^{range} \tag{51}$$

Combining equations (50) and (51), yield

$$\begin{aligned}(\beta_P P + \beta_Q Q)^{range} &\geq \left|\left(\beta_Q Q\right)^{range} - (\beta_P P)^{range}\right| \Leftrightarrow \\ \left[(\beta_P P + \beta_Q Q)^{range}\right]^2 &\geq \left|\left(\beta_Q Q\right)^{range} - (\beta_P P)^{range}\right|^2\end{aligned} \tag{52}$$

We would like to prove that

$$-1 \leq \frac{1}{2}\frac{\left[(\beta_P P + \beta_Q Q)^{range}\right]^2 - \left((\beta_P P)^{range}\right)^2 - \left(\left(\beta_Q Q\right)^{range}\right)^2}{\sqrt{\left((\beta_P P)^{range}\right)^2 \left(\left(\beta_Q Q\right)^{range}\right)^2}} \tag{53}$$

which can be rearranged to

$$\begin{aligned}-2\,(\beta_P P)^{range}\left(\beta_Q Q\right)^{range} + \left((\beta_P P)^{range}\right)^2 + \left(\left(\beta_Q Q\right)^{range}\right)^2 &\leq \left[(\beta_P P + \beta_Q Q)^{range}\right]^2 \\ \left|(\beta_P P)^{range} - \left(\beta_Q Q\right)^{range}\right|^2 &\leq \left[(\beta_P P + \beta_Q Q)^{range}\right]^2\end{aligned}$$

This condition is satisfied by Equation (52), and hence equation (53) has been proved.

Finally, we need to address the situation where one or both sample paths are constant. If one sample path is constant and the other one is non-constant, it follows immediately that

$$\widehat{\sigma}^2_{PQ} = 0$$

and

$$\widehat{\sigma}^2_P = 0 \text{ and } \widehat{\sigma}^2_Q > 0$$

and hence

$$\left|\widehat{\Omega}\right| = 0 \qquad \lambda_1 = 0 \qquad \lambda_2 = \widehat{\sigma^2_Q}$$

and in this case, the variance-covariance matrix is positive semi-definite.

If both sample paths are constant

$$\widehat{\sigma}^2_{PQ} = 0$$

and

$$\widehat{\sigma}^2_P = \widehat{\sigma}^2_Q = 0$$

and hence

$$\left|\widehat{\Omega}\right| = 0 \qquad \lambda_1 = \lambda_2 = 0$$

In this case, too, the variance-covariance matrix is positive semi-definite.

Proof of B Rewrite $\rho_{implied}$ as

$$\begin{aligned}
&\rho_{implied} \\
&= \frac{l^2_{\beta_P P + \beta_Q Q} - l^2_{\beta_P P} - l^2_{\beta_Q Q}}{2\sqrt{l^2_{\beta_P P} l^2_{\beta_Q Q}}} \\
&= \frac{1}{2}\frac{\left(\left(P^{h(\beta_P)} - P^{l(\beta_P)}\right) + \left(Q^{h(\beta_Q)} - Q^{l(\beta_Q)}\right)\right)^2 - \left(P^{high(\beta_P)} - P^{low(\beta_P)}\right)^2 - \left(Q^{high(\beta_Q)} - Q^{low(\beta_Q)}\right)^2}{\left(P^{high(\beta_P)} - P^{low(\beta_P)}\right)\left(Q^{high(\beta_Q)} - Q^{low(\beta_Q)}\right)} \\
&= \frac{1}{2}\frac{(a+b)^2 - c^2 - d^2}{cd}
\end{aligned}$$

We are going to make use of the following properties, which follow immediately from the definitions in (38)-(43).

$$\begin{aligned}
0 \le a = P^{h(\beta_P)} - P^{l(\beta_P)} \le P^{high(\beta_P)} - P^{low(\beta_P)} = c \\
0 \le b = Q^{h(\beta_Q)} - Q^{l(\beta_Q)} \le Q^{high(\beta_Q)} - Q^{low(\beta_Q)} = d
\end{aligned}$$

In particular, we would like to prove that

$$\max \rho_{implied} = 1 \tag{54}$$

$$\text{st. } 0 \le a \le c \tag{55}$$

$$0 \le b \le d \tag{56}$$

This constrained maximization problem may be solved by forming the Lagrange-function, writing up the Kuhn-Tucker Conditions etc. However the solution of the maximization problem is straightforward. If the maximization problem has a solution it must belong to one of the four cases:

1. $(a < c, b < d)$
2. $(a = c, b < d)$
3. $(a = c, b < d)$
4. $(a = c, b = d)$

Suppose that the solution belongs to the case of 1). Then it is clear that the objective function may be increased by increasing a and/or b due to the fact that,

$$\frac{\partial \rho_{implied}}{\partial a} = \frac{\partial \rho_{implied}}{\partial b} = \frac{a+b}{cd} \ge 0$$

and this is a contradiction to the claim in 1).

Analogously, solutions belonging to case 2) and 3) may be ruled out.

Now, if a solution to the maximization problem exists, it must belong to 4). The objective function turns out to be constant subject to the restriction from 4)

$$f(a,b)\big|_{(a,b)=(c,d)} = \frac{1}{2}\frac{c^2 + d^2 + 2cd - c^2 - d^2}{cd} = 1$$

and hence we have proved (54)-(56), which completes the proof of B.

This completes the proof of Theorem 2.

References

Alizadeh, S., Brandt, M. & Diebold, F. (2002), 'High and low frequency exchange rate volatility dynamics: Range-based estimation of stochastic volatility models', Forthcoming Journal of Finance 57, f.

Andersen, T. & Bollerslev, T. (1997), 'Heterogenous information arrivals and return volatility dynamics: Uncovering the long-run in high frequency returns', Journal of Finance 52, 975–1005.

Andersen, T. & Bollerslev, T. (1998), 'Answering the skeptics: Yes, standard volatility models do provide accurate forecasts', International Economic Review 39(4), 885–905.

Andersen, T., Bollerslev, T., Diebold, F. & Labys, P. (2000), 'Great realizations', Risk 13, 105–108.

Andersen, T. G., Bollerslev, T., Diebold, F. & Labys, P. (2001a), 'The distribution of exchange rate volatility', Journal of the American Statistical Association 96, 42–55.

Andersen, T. G., Bollerslev, T., Diebold, F. X. & Labys, P. (2001b), 'Modelling and forecasting realized volatility', Manuscript .

Andreou, E. & Ghysels, E. (2001), 'When does microstructure noise affect the estimation of asset pricing models', Manuscript .

Bai, X., Russell, J. & Tiao, G. (2000), 'Beyond merton's utopia: Effects of non-normality and dependence on the precision of variance estimates using high-frequency financial data', Unpublished paper, University of Chicago .

Ball, C. & Tourus, W. (1984), 'The maximum likelihood estimation of security price volatility: Theory, evidence, and application to option pricing', Journal of Business 57(1), 97–112.

Beckers, S. (1983), 'Variance of security price returns based on high, low and closing prices', Journal of Business 56, 97–112.

Brandt, M. & Diebold, F. (2001), Multivariate range-based volatility. Manuscript.

Brunetti, C. & Lildholdt, P. (2001), 'Volatility dynamics in the foreign exchange market measured by daily returns and price ranges', Manuscript .

Campbell, J., Lo, A. & MacKinlay, A. (1997), The Econometrics of Financial Markets, Princeton University Press, New Jersey.

Chen, A. (1997), 'Volatility of exchange rate futures and high-low price spreads', Journal of Economics and Finance 21(1), 33–42.

Davidsson, J. (2002), 'Establishing conditions for the functional central limit theorem in nonlinear and semiparametric time series processes', Journal of Econometrics 106(2), 243–269.

Feller, W. (1951), 'The asymptotic distribution of the range of sums of independent random variables', Annals of Mathematical Statistics 22, 427–432.

Garman, M. & Klass, M. (1980), 'On the estimation of security price volatilities from historical data', Journal of Business 53(1), 67–78.

Kunitomo, N. (1992), 'Improving the parkinson method of estimating security price volatilities', Journal of Business 65, 295–302.

Lin, J. & Rozeff, M. (1994), 'Variance, return, and high-low price spreads', Journal of Financial Research 17, 301–319.

Lopez, J. A. & Walter, C. (2001), 'Evaluating covariance matrix forecasts in a value-at-risk framework', Forthcoming in The Journal of Risk .

Parkinson, M. (1980), 'The extreme value method for estimating the variance of the rate of return', Journal of Business 53(1), 61–65.

Rogers, L. (1998), 'Volatility estimation with price quanta', Mathematical Finance 8(3), 277–290.

Rogers, L. & Satchell, S. (1991), 'Estimating variance from high, low, and closing prices', Annals of Applied Probability 1, 504–512.

Siddiqui, M. M. (1976), 'The asymptotic distribution of the range and other functions of partial sums of stationary processes', Water Resources Research 12(6), 1271–1276.

Taylor, S. (1986), Modelling Financial Time Series, Wiley.

Yang, D. & Zhang, Q. (2000), 'Drift-independent volatility estimation based on high, low, open and close prices', Journal of Business 73, 477–491.

9 Tables

Log-prices of individual assets

Asset No.	1	2	3
Time 1	0.051682210	1.2096253	−0.16374717
Time 2	−1.6755751	−0.23525380	0.32486793
Time 3	0.97597537	−0.29329586	−2.5099104
Min	−1.6755751	−0.29329586	−2.5099104
Max	0.97597537	1.2096253	0.32486793

Sums of log-prices

Asset No	$1+2$	$1+3$	$2+3$
Time 1	1.2613075	−0.11206496	1.0458781
Time 2	−1.9108289	−1.3507071	0.089614123
Time 3	0.68267951	−1.5339350	−2.8032062
Min	−1.9108289	−1.5339350	−2.8032062
Max	1.2613075	−0.11206496	1.0458781

Implied correlations

$\rho_{12}^{implied} = 0.096981634$	$\rho_{13}^{implied} = -0.86774917$	$\rho_{23}^{implied} = 0.53054398$

Expression in equation (20)

$$\left(1 + 2\rho_{12}^{implied}\rho_{13}^{implied}\rho_{23}^{implied} - \left(\rho_{13}^{implied}\right)^2 - \left(\rho_{23}^{implied}\right)^2 - \left(\rho_{12}^{implied}\right)^2\right) = -0.13316762$$

Table 4: Numerical example illustrating the properties of the estimated variance-covariance matrix in the trivariate case. The beta-vector used for the computation of the co-range is $(\beta_a, \beta_b) = (1, 1)$.

Monte Carlo Experiment

ρ_{zw}	-0.99	-0.80	-0.50	-0.20	0.00	0.20	0.50	0.80	0.99
$Bias(\widehat{\sigma}^2_{A/B,C/B(1,-1)})$	1.6e^{-6}	1.1e^{-6}	8.0e^{-7}	1.2e^{-7}	3.4e^{-8}	-4.4e^{-7}	-8.2e^{-7}	-1.4e^{-6}	-1.6e^{-6}
$Bias(\widetilde{\sigma}^2_{A/B,C/B})$	9.8e^{-8}	8.7e^{-8}	4.7e^{-8}	-2.8e^{-7}	-4.9e^{-9}	-2.6e^{-7}	-2.1e^{-7}	-2.7e^{-7}	-4.0e^{-7}
Relative $Bias$	1.7e^{1}	1.3e^{1}	1.7e^{1}	-4.2e^{-1}	-6.9	1.7	3.9	5.2	4.0
Relative MSE	2.0e^{-1}	2.1e^{-1}	2.1e^{-1}	2.1e^{-1}	2.1e^{-1}	2.1e^{-1}	2.0e^{-1}	2.0e^{-1}	2.0e^{-1}
Relative MAD	4.8e^{-1}	4.9e^{-1}	5.0e^{-1}	5.1e^{-1}	5.2e^{-1}	5.1e^{-1}	4.9e^{-1}	4.8e^{-1}	4.8e^{-1}

$\sigma^2_{A/B} = 3.6e^{-5}$
$\sigma^2_{C/B} = 1.0e^{-4}$
Monte Carlo replications $= 100,000$
Return observations per day $= 2880$

Table 5: Monte Carlo experiment for the co-range

Monte Carlo Experiment

ρ_{zw}	-0.99	-0.80	-0.50	-0.20	0.00	0.20	0.50	0.80	0.99
$Bias(\widehat{\sigma}^2_{A/B,C/B(1,-1)})$	2.2e^{-6}	1.7e^{-6}	1.1e^{-6}	3.8e^{-7}	6.1e^{-8}	-6.0e^{-7}	-1.2e^{-6}	-1.6e^{-6}	-2.3e^{-6}
$Bias(\widetilde{\sigma}^2_{A/B,C/B})$	4.9e^{-8}	3.0e^{-7}	8.0e^{-9}	-1.8e^{-7}	-3.9e^{-8}	-4.5e^{-7}	-3.5e^{-7}	1.8e^{-7}	-7.7e^{-7}
Relative $Bias$	4.5e^{1}	5.6	1.4e^{2}	-2.2	-1.6	1.3e	3.4e	-9.1e^{1}	3
Relative MSE	2.0e^{-1}	2.1e^{-1}	2.1e^{-1}	2.1e^{-1}	2.1e^{-1}	2.1e^{-1}	2.0e^{-1}	2.0e^{-1}	2.0e^{-1}
Relative MAD	4.8e^{-1}	4.9e^{-1}	5.0e^{-1}	5.1e^{-1}	5.2e^{-1}	5.1e^{-1}	4.9e^{-1}	4.8e^{-1}	4.9e^{-1}

$\sigma^2_{A/B} = 3.6e^{-5}$
$\sigma^2_{C/B} = 1.0e^{-4}$
Monte Carlo replications $= 100,000$
Return observations per day $= 1440$

Table 6: Monte Carlo experiment for the co-range

Monte Carlo Experiment

ρ_{zw}	-0.99	-0.80	-0.50	-0.20	0.00	0.20	0.50	0.80	0.99
$Bias(\widehat{\sigma}^2_{A/B,C/B(1,-1)})$	3.6e^{-6}	3.0e^{-6}	1.8e^{-6}	6.4e^{-7}	9.7e^{-8}	-6.7e^{-7}	-1.9e^{-6}	-2.9e^{-6}	-3.8e^{-6}
$Bias(\widetilde{\sigma}^2_{A/B,C/B})$	3.3e^{-9}	-5.7e^{-8}	1.7e^{-7}	-1.1e^{-7}	1.5e^{-7}	-1.6e^{-7}	-1.1e^{-7}	-1.8e^{-7}	-4.4e^{-7}
Relative $Bias$	1.1e^{3}	-5.2e^{1}	1.1e^{1}	-6.0	6.7e^{-1}	4.3	1.7e^{1}	1.7e^{1}	8.6
Relative MSE	2.0e^{-1}	2.0e^{-1}	2.1e^{-1}	2.1e^{-1}	2.1e^{-1}	2.0e^{-1}	2.0e^{-1}	2.0e^{-1}	2.0e^{-1}
Relative MAD	4.9e^{-1}	4.9e^{-1}	5.0e^{-1}	5.1e^{-1}	5.1e^{-1}	5.0e^{-1}	4.8e^{-1}	4.8e^{-1}	4.9e^{-1}

$\sigma^2_{A/B} = 3.6e^{-5}$
$\sigma^2_{C/B} = 1.0e^{-4}$
Monte Carlo replications $= 100,000$
Return observations per day $= 480$

Table 7: Monte Carlo experiment for co-range

Working Paper

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